Asset purchase policy at the effective lower bound for interest rates

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Abstract

This paper extends the canonical New Keynesian model to incorporate imperfect substitutability between short-term and long-term bonds. This stylised modification provides a channel through which asset purchases by the policymaker can affect aggregate demand. Because assets are imperfect substitutes, central bank asset purchases that alter the relative supplies of assets influence asset prices. In the model, aggregate demand depends on the prices (or interest rates) of both long-term and short-term bonds. To the extent that central bank asset purchases reduce long-term interest rates (over and above the effect of expected future short rates), aggregate demand can be stimulated, leading to higher inflation through a conventional New Keynesian Phillips curve.

An optimal commitment policy in which the policymaker uses asset purchases as an additional policy instrument can improve economic outcomes in the face of a negative demand shock that drives the short-term policy rate to its lower bound. This is true even if asset purchases policies are also subject to (both upper and lower) bounds. However, the imperfect substitutability between bonds that gives asset purchases their traction also makes it more difficult for conventional monetary policy to stabilise the economy in the event of a large negative demand shock. If asset purchases are also subject to bounds, then it may not be possible for the policymaker to stabilise the output gap and inflation as well as is possible using only interest rate policy in the canonical New Keynesian model.

PRELIMINARY AND INCOMPLETE
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1 Introduction

The recent global slowdown has been particularly severe and synchronised. Monetary and fiscal policies have been substantially loosened to support aggregate demand. Short-term nominal policy rates in a number of countries have reached historically low levels and in some cases have been reduced to their effective lower bound (ELB).[1] In addition, a number of central banks have deployed a broader range of policy tools than usual. Some have provided 'forward guidance' to financial markets by making statements about the possible path of future policy rates. And some central banks have engaged in so-called 'unconventional' monetary policies that involve the purchase of assets by the central bank.[2] There are several potential approaches to unconventional monetary policy depending, among other things, on whether the assets are purchased from the government or the private sector and whether the purchases are associated with an expansion of the monetary base.[3] In general, however, these policy actions affect the size and/or composition of the central bank's balance sheet.

Figure 1 presents policy rates and central bank balance sheet data for the United States, Euro area and United Kingdom since the start of 2008. Sharp declines in policy rates (bottom panel) were accompanied by expansions of central bank balance sheets. This simple observation obscures differences in the composition of central bank balance sheets across countries, reflecting differences in the specific policy measures adopted (Meier (2009)). Nevertheless, it is clear that recent events have led to a substantial loosening of the monetary policy stance in many countries, using a range of instruments.

As noted by Meier (2009), different approaches to unconventional monetary policy can be motivated by alternative views of the transmission channels through which they affect activity and inflation. For example, Benford *et al.* (2009) note that there are several channels through which which the Bank of England's 'quantitative easing' policy may affect the economy. First, purchases of assets (bonds) held by the private sector could increase the prices of those assets. As bond prices increase, yields fall and private sector borrowing costs are reduced, stimulating aggregate demand. Second, because asset purchases are financed by the creation of central bank money, they lead to an increase in reserve balances held by banks at the central banks.[4] The increase in reserve balances may facilitate an expan-

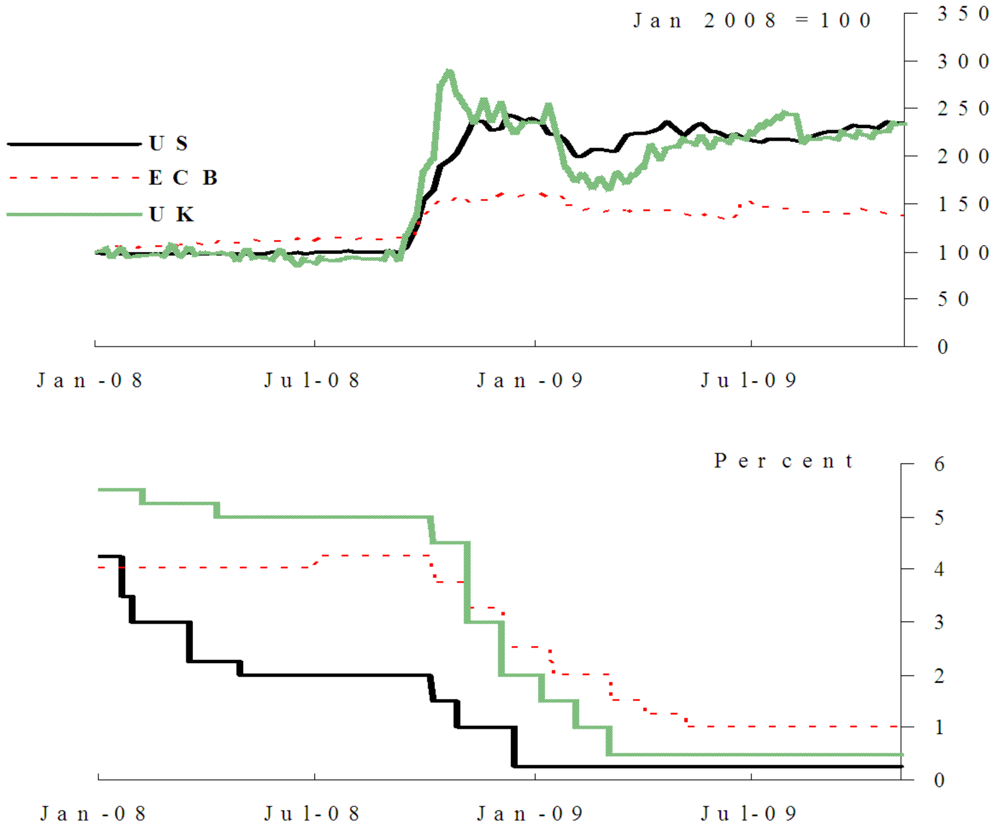
[1] In principle, the ELB may be lower than zero if there are transactions costs associated with holding money (see Yates (2003)). But in practice, the ELB may be positive for a number of reasons. For example, low levels of policy rates may cause difficulties for the functioning of financial intermediaries that maintain a spread between deposit and lending rates to cover the costs of providing banking services and to make a return on capital (see Bank of England (2009)).

[2] Bean (2009) notes that there is nothing unusual about such asset purchases *per se* – they are "just a return to the classic policy operation of the textbook: an open market operation. The only things that distinguish the present operations ... are the circumstances under which they are taking place and their scale".

[3] Benford *et al.* (2009) discuss the approaches taken by a number of central banks.

[4] As explained by Benford *et al.* (2009), when the central bank purchases an asset from a non-bank asset holder, the central bank credits the seller's bank's reserve account at the central bank and the seller's bank credits the asset seller with a deposit.

Figure 1: Balance sheet (top panel) and policy rates (bottom panel) for US, ECB and UK



sion in bank lending. Third, asset purchases may influence inflation expectations by demonstrating policymakers' resolve to return inflation to target by any means necessary.

None of these channels are present in the canonical 'New Keynesian' model often used to analyse monetary policy.[5] In that framework, the only monetary policy instrument available is the short-term nominal interest rate. A number of papers have examined optimal policy for the canonical model in the presence of a lower bound on the short-term nominal interest rate. Eggertsson & Woodford (2003) and Jung *et al.* (2005) show that, with perfect foresight, the optimal policy under commitment involves keeping the nominal interest rate lower for a longer period than would be implied by a discretionary policy (which would set the nominal

[5]See Clarida *et al.* (1999) for an early review and Woodford (2003) and Galí (2008) for recent comprehensive treatments. To the extent that unconventional monetary policy is viewed purely as a signal of a commitment to holding the policy rate low for a prolonged period, the 'expectations' channel is arguably captured to some extent (see Eggertsson & Woodford (2003)).

interest rate equal to the natural real rate as soon as it is feasible to do so). The credible commitment to maintaining an expansionary monetary policy stance for a 'prolonged' period generates an increase in expected inflation, reduces current real interest rates and stimulates aggregate demand. This confirms the earlier argument put forward by Krugman (1998). A completely stochastic treatment of the optimal policy problem in the canonical model tends to suggest that the policy rate remains at its lower bound for longer than in the perfect foresight case (see Adam & Billi (2006) and Nakov (2008)). This is because the policymaker recognises the risk of further negative demand shocks (to which interest rate policy could not respond).

So the conventional monetary policy response to a large negative demand shock that forces the policy rate to its lower bound is to engineer a prolonged period of low nominal interest rates and to communicate this policy to the private sector. In many parameterisations of the canonical New Keynesian model, the optimal commitment policy generates relatively good outcomes, suggesting that the economic effects of a lower bound to the policy rate are relatively mild. Levin *et al.* (2009) show that for calibrations of the canonical model in which aggregate demand is sensitive to real interest rates, such 'forward guidance' is not able to prevent large and persistent falls in the natural real interest rate from generating significant effects on activity and inflation. The baseline parameter values for the model used in this paper are in line with this type of calibration. This means that large negative demand shocks can have material effects on activity and inflation, providing scope for the use of asset purchases as an additional policy instrument.

For asset purchases to have an effect on activity and inflation requires a deviation from the canonical New Keynesian assumptions. A number of recent papers have examined extensions to the canonical model that provide a role for unconventional monetary policies. Gertler & Karadi (2009) construct a model with a banking sector that is subject to financial frictions that resemble the financial accelerator mechanism developed for firms by Bernanke *et al.* (1999). Cúrdia & Woodford (2009) construct a model with heterogeneous households that differ in their intertemporal preferences over consumption. This gives rise to an endogenous division between households into savers and borrowers. Financial intermediation between households creates a wedge between borrowing and lending rates that affects aggregate activity and welfare. By intervening in the market for loans, monetary policy can improve welfare.

This paper incorporates imperfect asset substitutability using an approach similar to that of Andrés *et al.* (2004). Portfolio adjustment costs are introduced into households' utility functions such that the larger their holdings of short-term bonds, the more they value *long-term* bonds. This assumption is motivated by the notion that agents are more willing to hold less liquid assets if they have ample holdings of more liquid assets. The assumption creates a wedge between the market rates of return on long and short bonds. This approach is a simple way to capture the notion that relative asset prices depend on their relative supply. This idea was part of the monetary theory put forward by Tobin (1969), among others. Empirical evidence for the effects of relative asset supplies on relative returns is presented by Greenwood & Vayanos (2008).

This stylised modification to the canonical New Keynesian model provides a channel through which asset purchases by the policymaker can affect aggregate demand. Because assets are imperfect substitutes, the policymaker can use asset purchases to alter the relative supplies of assets and hence bond returns. In the model, aggregate demand depends on both long-term and short-term bond returns. To the extent that central bank asset purchases reduce long-term interest rates (over and above the effect of expected future short rates), aggregate demand can be stimulated, leading to higher inflation through a conventional New Keynesian Phillips curve.

Of course, compared with the case in which only the short-term nominal interest rate is used, an optimal commitment policy in which the policymaker uses asset purchases as an additional policy instrument can improve economic outcomes in the face of a negative demand shock that drives the short term policy rate to its lower bound. But an important implication of the modelling approach is that the welfare-based loss function that the policymaker should minimise is a function of the variance of the 'portfolio mix' (households' relative holdings of short-term and long-term bonds) as well as the output gap and inflation. So while the introduction of imperfect asset substitutability provides a channel through which the policymaker may stabilise the economy, it also creates a distortion that the policymaker should aim to offset. This is analogous to the sticky price friction of the canonical New Keynesian model: staggered price setting creates a distortion that should be offset (relative price dispersion), but also provides the policymaker with a tool that can be used to offset it (changes in nominal interest rates can affect activity).

Indeed, the imperfect asset substitutability channel that gives asset purchases their traction as a policy instrument reduces the efficacy of conventional monetary policy (operating via the short-term nominal interest rate). As explained later, a reduction in the short-term nominal interest rate reduces household liquidity. (One way to think about this is that the policymaker pushes the short-term nominal interest rate down through conventional open market operations, buying short-term bonds with money). The reduction in liquidity increases the premium on long-term bonds, so that long-term rates fall by less than the cumulative fall in expected future short rates. The reduced effectiveness of the conventional short-term policy rate means that the effective lower bound for nominal interest rates becomes a more costly constraint on conventional policy.

2 The model

This section provides an overview of the model. More details of the derivation are presented in Appendix A. Section 2.1 outlines the government budget constraint and asset markets. Section 2.2 discusses how household behaviour influences (and is influenced by) relative bond yields. Section 2.3 presents a brief summary of the supply side of the model (which is standard) and Section 2.4 discusses the baseline parameter values used in the simulations.

2.1 The government budget constraint and asset markets

As is common in many models of this type, fiscal policy does not play an important role in the economy: there is no government spending and net transfers are made to households on a lump sum basis (so taxation does not distort any economic decisions). This simplification is made to focus attention on the extent to which monetary policy can combat large negative demand shocks when short-term interest rates are subject to a lower bound. So this analysis abstracts from the debate on fiscal policy when interest rates are very low (for a recent contribution, see Christiano *et al.* (2009)).

The government budget constraint is:

$$\frac{V_t B_{c,t}}{P_t} + \frac{B_t}{P_t} - \frac{[1 + V_t] B_{c,t-1}}{P_t} - \frac{R_{t-1} B_{t-1}}{P_t} + \frac{\Delta_t}{P_t} = \frac{T_t}{P_t}$$

which states that issuance of bonds (B and B_c, discussed below) plus the change in the central bank balance sheet (Δ, discussed below) finances net transfers to households (T). All items in the budget constraint are deflated by the aggregate price index P (the price of a Dixit-Stiglitz consumption bundle described below).

The left hand side of the budget constraint represents the government's net issuance of liabilities. The government issues two types of bonds: one period bonds (B) and consols (B_c). One period bonds sell at a unit price and are redeemed at price R in the following period (R is the nominal interest rate on one period bonds). Consols yield one unit of currency each period for the infinite future. The value (ie price) of a consol is denoted V. Consols are infinitely-lived instruments and do not have a redemption date.[6]

Modelling long-term bonds as consols is a useful alternative to the assumption in Andrés *et al.* (2004), who assume that the long-term bond is a zero coupon fixed maturity bond. The authors also assume that there is no secondary market for long-term bonds so that agents who buy long-term government debt must hold it until maturity. As Andrés *et al.* (2004) point out, ruling out trades in long-term debt on a secondary market reduces the number of variables in the model, which is a useful simplification.[7] The use of consols as the long-term bond permits the assumption that they can be traded each period so that the optimal long-bond holdings depend on the *one-period* return on consols. This may be viewed as a similar simplification. Nevertheless, as will be demonstrated, with imperfect substitutability between assets this approach still creates a wedge between market rates of return on those assets.

It is convenient to write the government budget constraint in terms of the one period return on consols. To do so, we define:

$$B^g_{L,t} \equiv V_t B_{c,t}$$

[6] Of course, the government may withdraw existing consols from circulation by purchasing them from private agents at the market price.

[7] Suppose the maturity of long-term debt is L periods. Allowing trade on secondary markets would require keeping track of household holdings of debt with $L, L-1, L-2, ..., 1$ periods to maturity.

and rewrite the budget constraint as

$$\frac{B_{L,t}^{g}}{P_t}+\frac{B_t}{P_t}-\frac{[1+V_t]\,B_{L,t-1}^{g}}{V_{t-1}P_t}-\frac{R_{t-1}B_{t-1}}{P_t}+\frac{\Delta_t}{P_t}=\frac{T_t}{P_t}$$

If we now define

$$R_{L,t}\equiv\frac{1+V_t}{V_{t-1}}$$

as the ex post nominal return on consols, then the government budget constraint is

$$\frac{B_{L,t}^{g}}{P_t}+\frac{B_t}{P_t}-\frac{R_{L,t}B_{L,t-1}}{P_t}-\frac{R_{t-1}B_{t-1}}{P_t}+\frac{\Delta_t}{P_t}=\frac{T_t}{P_t}$$

The change in the central bank balance sheet is equal to money creation and net asset purchases:

$$\frac{\Delta_t}{P_t}=\frac{M_t-M_{t-1}}{P_t}-\left[\frac{Q_t}{P_t}-\frac{R_{L,t}Q_{t-1}}{P_t}\right]$$

where the second term records the net increase in the central bank's holdings of long term government debt, which are denoted by Q. This setup assumes that asset purchases are concentrated in long-term bonds, in line with the focus of asset purchase schemes recently introduced by central banks.[8] In this simple model, if the central bank makes net positive purchases, money creation must be greater than otherwise in order for a given level of transfers to households to be achieved with the existing portfolio of government debt.

The asset purchase policy is operated by varying the fraction of bonds held on the central bank balance sheet:

$$Q_t=q_tB_{L,t}^{g}$$

which means that the consolidated government budget constraint is

$$b_t+m_t+(1-q_t)\,b_{L,t}^{g}=\pi_t^{-1}\left[m_{t-1}+R_{t-1}b_{t-1}+R_{L,t}\,(1-q_{t-1})\,b_{L,t-1}^{g}\right]+\tau_t$$

where lower case letters denote nominal quantities deflated by the price index,

$$\pi_t\equiv\frac{P_t}{P_{t-1}}$$

is the inflation rate and

$$\tau_t\equiv\frac{T_t}{P_t}$$

is the real net transfer to/from households.

The choice variables for the government are net transfers to households and debt issuance. The real stock of consols is assumed to be held fixed so that the value of long-term bonds is given by:

$$b_{L,t}^{g}=\bar{b}_CV_t$$

[8]For example, the maturity range for gilts eligible for the Bank of England's Asset Purchase Facility (APF) was initially set at five to twenty-five years.

where it should be noted that the total value of consols depends on the price V_t, and therefore responds to developments in the economy.

Net transfers to households are set according to a simple rule designed to stabilise the total debt stock:

$$\frac{\tau}{b}\hat{\tau}_t = -\beta^{-1}\hat{R}_{t-1} - \theta\hat{b}_{t-1}$$

where the notation $\hat{x}_t \equiv \ln(x_t/x)$ denotes the log deviation of variable x_t from its steady state value x. The transfer rule responds to the lagged debt stock in a way that ensures that debt issuance is a stable process. The transfer rule also adjusts payments to/from households to offset the cost of financing the previously issued short-term debt. This reduces the feedback from debt financing costs to the debt stock and can have important implications for model responses. Section 4 examines the sensitivity of results to the transfer rule.

Monetary policy is conducted in terms of the short term nominal interest rate (R) and the fraction of long-term bonds held on the central bank's balance sheet (q). Section 3 examines a range of cases in which monetary policy is set optimally according to a welfare-based loss function.

2.2 Households

For purchases of assets by the central bank to have an effect on relative bond yields, there must be impediments to arbitrage behaviour that will equalise asset returns. This impediment is introduced in a manner similar to that used by Andrés *et al.* (2004). Households hold both long-term bonds and short-term (one period) bonds. However, households perceive that longer term bonds are less liquid than short-term bonds. This perception is not modelled formally in terms of specific assumptions about the liquidity conditions in the two asset markets. Instead, it is captured by the assumption that unrestricted households demand additional holdings of short-term bonds when their holdings of long-term bonds increase. As Andrés *et al.* (2004) argue, this assumption captures Tobin's assertion that the relative returns of different assets will be influenced by their relative supplies.

This setup means that (a) long-term bond yields are influenced by the relative supplies of short- and long-term bonds; and (b) the wedge between long-term and short-term bond yields has implications for aggregate demand. The mechanism works as follows. For households to be willing to hold long-term bonds, the rates of return on long-term and short-term bonds must be equated. Because of the 'liquidity cost' associated with holding long-term bonds, there is a wedge between the *market* rates of return on short-term and long-term bonds. This wedge is such that the 'effective' rates of return to households (ie adjusted for the 'liquidity cost') are equal.

An important difference between the model of Andrés *et al.* (2004) and the model presented here is that Andrés *et al.* (2004) assume that the 'liquidity concern' of households is a function of the ratio of money balances to long-term bond holdings. In the present model, as noted above, the liquidity concern is assumed to be a function of the ratio of short-term bonds to long-term bonds. This assumption focuses attention on the relative supplies of interest bearing assets that would otherwise (ie in a canonical New Keynesian model) be perfect substitutes. Moreover, it means that relative asset supplies have implications for relative asset prices and economic activity without the need to introduce a limited participation assumption. Of course, the behaviour of monetary aggregates can be an important part of the way that asset purchase policies affect the economy as discussed in the Introduction. But the aim of this paper is to try to isolate the 'portfolio balance' channel of asset purchase policies from other channels.

Based on the previous discussion the optimisation problem of the representative household is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \phi_t \left[\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} + \frac{\chi_m^{-1}}{1-\sigma_m^{-1}} \left(\frac{M_t}{P_t} \right)^{1-1/\sigma_m} - \frac{\tilde{\nu}}{2} \left[\delta \frac{B_t}{B_{L,t}} - 1 \right]^2 \right]$$

where c is consumption (of a Dixit-Stiglitz consumption bundle, described below), n is hours worked, M/P are real money balances and B/B_L is the ratio of short-term to long-term bond holdings. The sub-utility function chosen for real money balances does not include a satiation level of real money balances: as nominal returns on interest bearing assets approaches zero, the desired level of real money balances approaches infinity. In the analysis that follows, it is assumed that the effective lower bound on nominal asset returns in slightly positive, so that demand for real money balances remains finite.

The inclusion of the final term in the utility function reflects the assumption that portfolio decisions are influenced by relative asset holdings, as discussed above. Following Andrés *et al.* (2004), we assume that δ is the steady-state ratio of long-term bond holdings relative to short term bond holdings. This means that portfolio costs are zero in the steady state. A preference shock ϕ_t is included and will serve as the 'demand shock' that generates a persistent decline in the natural real interest rate considered in the simulation experiments examined below.

Of course, the strength of the microfoundations of the 'asset adjustment cost' in the utility function can be questioned: why should households care about their portfolio allocation? One response is that amending the utility function is a short-cut to modelling a more structural financial friction. Appendix D sketches a model in which financial intermediaries finance short-term lending to households using a mixture of long-term and short-term government bonds. That setup leads to identical behavioural equations and an identical welfare-based loss function. That model merely relocates a relatively ad hoc friction from household's utility functions to financial intermediaries' cost functions. But it suggests that better articulated models of financial frictions could give rise to very similar results to those presented in this paper.

Maximisation is subject to a nominal budget constraint given by:

$$B_{L,t} + B_t + M_t = R_{L,t} B_{L,t-1} + R_{t-1} B_{t-1} + M_{t-1} + W_t n_t + T_t + D_t - P_t c_t \quad (1)$$

The left hand side of the budget constraint represents the household's holdings of nominal assets. These consist of one period bonds (B), consols (B_L) and money

(M). The existing asset holdings of the household can be liquidated to purchase new assets. The existing holdings have value $R_{L,t}B_{L,t-1} + R_{t-1}B_{t-1} + M_{t-1}$ which captures the ex post returns on short and long term bonds. The remaining terms in the budget constraint capture the household's net income. This is wage income from supplying n_t units of labour at nominal wage rate W_t and lump sum (net) fiscal (T_t) and dividend (D_t) transfers from the government and firms respectively less expenditure on consumption (c_t).

As shown in Appendix A, the key log-linearised first order conditions are an Euler equation for the output gap (x), a no-arbitrage relationship betwen long-term and short-term bond returns and a money demand function:

$$\hat{x}_t = E_t\hat{x}_{t+1} - \sigma\left[\frac{1}{1+\delta}\hat{R}_t + \frac{\delta}{1+\delta}\hat{R}^e_{L,t} - E_t\hat{\pi}_{t+1} - r^*_t\right] \tag{2}$$

$$\hat{R}^e_{L,t} = \hat{R}_t - \nu\left[\hat{b}_t - \hat{b}_{L,t}\right] \tag{3}$$

$$\hat{m}_t = \frac{\sigma_m}{\sigma}\hat{x}_t - \frac{\beta\sigma_m}{1-\beta}\hat{R}_t + \frac{\beta\sigma_m}{1-\beta}\nu\frac{\delta}{1+\delta}\left[\hat{b}_t - \hat{b}_{L,t}\right]$$

where

$$\nu \equiv (1+\delta)\,\tilde{\nu}c^{1/\sigma}\left(\bar{b}_L\right)^{-1}$$

and

$$\hat{R}^e_{L,t} \equiv E_t\hat{R}_{L,t+1}$$

The 'natural real rate of interest' is defined as[9]

$$r^*_t \equiv -E_t\left(\hat{\phi}_{t+1} - \hat{\phi}_t\right) \tag{4}$$

and is assumed to follow the exogenous process

$$r^*_t = \rho r^*_{t-1} + \varepsilon_t \tag{5}$$

The Euler equation (2) demonstrates that aggregate demand is driven by a weighted average of the interest rates on short-term and long-term bonds. The pricing equation for long-term bonds (3) indicates that aggregate demand therefore also depends on the households's relative holdings of short-term and long-term bonds. An increase in the household's relative holdings of short-term bonds acts like a reduction in the short term real interest rate and boosts demand. An increase in relative holdings of short term bonds represents an increase in unrestricted household's (marginal) liquidity. This effect means that a shift towards short term bond holdings, reduces the wedge between the rates of return on long-term and short-term bonds, as shown in equation (3).

Bond market clearing requires that the supply of bonds available to private agents is taken up by households

$$b_{L,t} = (1-q_t)\,b^g_{L,t} = (1-q_t)\,\bar{b}_C V_t$$

[9]If preference shocks (ϕ) are the only shocks then it is readily verified that, in the absence of sticky prices or bond market imperfections, the real interest rate satifies equation (4).

which can be log-linearised to give[10]

$$-q_t + \hat{V}_t = \hat{b}_{L,t} \tag{6}$$

Equation (6) shows that asset purchases (q) influence the quantity of long bonds available to households and hence long-term bond yields via (3).

2.3 Firms

There is a set of monopolistically competitive producers indexed by $j \in (0,1)$ that produce differentiated products that form a Dixit-Stiglitz consumption bundle that is purchased by households. The consumption bundle is given by

$$c_t = \left[\int_0^1 c_{j,t}^{1-\eta^{-1}} dj \right]^{\frac{1}{1-\eta^{-1}}}$$

where c_j is consumption of firm j's output.

Firms produce using a constant returns production function in the single input (labour):

$$c_{j,t} = A n_{j,t}$$

where A is productivity parameter.

The real profit of producer j is:

$$\frac{P_{jt}}{P_t} c_{jt} - w_t n_{j,t} = \left((1+s) \frac{P_{jt}}{P_t} - \frac{w_t}{A} \right) \left(\frac{P_{j,t}}{P_t} \right)^{-\eta} c_t$$

where s is a subsidy paid to producers in order to ensure that the steady-state level of output is efficient. This assumption permits the use of a quadratic approximation of the household utility function as the appropriate welfare criterion (see Benigno & Woodford (2006)).

Under a Calvo (1983) pricing scheme, the objective function for a producer that is able to reset prices is thus:

$$\max E_t \sum_{k=t}^{\infty} \Lambda_k (\beta\alpha)^{k-t} \left((1+s) \frac{P_{jt}}{P_k} - \frac{w_k}{A} \right) \left(\frac{P_{j,t}}{P_k} \right)^{-\eta} c_k$$

where Λ represents the household's stochastic discount factor and $0 \leq \alpha < 1$ is the probability that the producer is *not* allowed to reset its price each period. Well-known manipulations lead to a New Keynesian Phillips curve:

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa \hat{x}_t$$

where

$$\kappa = \frac{(1-\alpha)(1-\beta\alpha)}{\alpha} \left(\psi + \sigma^{-1} \right)$$

[10] A linear (rather than log-linear) approximation is applied to q since the steady-state level of q is assumed to be zero, as discussed later.

Table 1: Parameter values

	Description	Value
σ	Elasticity of intertemporal substitution	6
β	Discount factor	0.9925
κ	Slope of Phillips curve	0.024
ρ	Autocorrelation of natural real interest rate	0.85
η	Elasticity of substitution in consumption bundle	5
σ_m	Money demand elasticity	6
α	Calvo probability of *not* changing price	0.75
ψ	Labour supply elasticity	0.11
δ	Steady state ratio of long-term bonds to short-term bonds	3
ν	Elasticity of long-term bond rate with respect to portfolio mix	0.09
θ	Feedback parameter in tax/transfer rule	0.01

2.4 Parameter values

A number of parameters are set in order to pin down the steady state of the model. The productivity parameter A is chosen to normalise output to unity in the steady state. The parameter χ_m is set to ensure that real money balances are a small fraction (0.001) of output in steady state. The steady-state inflation rate is normalised to zero ($\pi = 1$). The level of asset purchases is also zero in steady state ($q = 0$) in order to implement the efficient equilibrium.

Table 1 shows the baseline parameter values used in the policy simulations below.

Parameters σ, κ and ρ are set in line with Levin *et al.* (2009), who use these values to show that large negative real interest rate shocks can have significant effects on activity even under optimal commitment policy in a canonical New Keynesian model. The value for β is slightly lower, implying a higher steady-state nominal interest rate (since we assume a small positive effective lower bound for nominal rates as discussed in Section 2.2). The elasticity of money demand is set to ensure a unit income elasticity of money demand. The value of $\eta = 5$ is commonly used in the canonical model. The assumption about κ is sufficient to pin down the slope of the Phillips curve. Under the assumption that firms change prices on average once a year ($\alpha = 0.75$), the implied value for the elasticity of disutility of labour supply is $\psi = 0.11$.

The steady state ratio of long-term to short-term bonds (δ) is set to 3 in light of the US data presented in Kuttner (2006). The elasticity of long-term bond rate with respect to household's portfolio mix is set to $\nu = 0.09$. There is little guidance in the literature on the appropriate range of values for this parameter. However, Andrés *et al.* (2004) estimate a similar parameter (relating the long-term bond premium to household's relative holdings of money and long bonds) to be around 0.045 using US data. The evidence presented in Bernanke *et al.* (2004) suggests that a 10% reduction in the stock of long-term bonds (associated with US Treasury buy-backs) reduced long yields by around 100 basis points. This suggests a value for ν of around 0.25. The value chosen here lies between these estimates. Finally, the feedback parameter in the transfer rule is set to $\theta = 0.01$ which implies that the stock of short debt moves persistently in response to shocks. Section 4 examines the sensitivity of optimal policy responses to alternative assumptions about these parameters.

3 Policy responses to a large negative demand shock

This section analyses optimal policy responses to a large negative demand shock when the policy rate is constrained by a lower bound. The monetary policymaker is assumed to minimise a discounted loss function consistent with the household's utility function. Appendix C shows that the utility-based loss function is given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\hat{x}_t^2 + \frac{\eta}{\kappa} \hat{\pi}_t^2 + \frac{\nu}{(1+\delta)\left(\sigma^{-1}+\psi\right)} \frac{\bar{b}_L}{c} \left[\hat{b}_t - \hat{b}_{L,t} \right]^2 \right] \tag{7}$$

The loss function specifies that the policymaker is concerned about stabilising the output gap, inflation and the relative supplies of short-term and long-term bonds. The first two terms in parentheses appear in the welfare-based loss function of the canonical New Keynesian model.[11] The third term appears because of the introduction of imperfect substitutability between assets. This additional friction can be eliminated by stabilising the relative supplies of assets.

With a single instrument (the short-term nominal interest rate, $\hat{R}$) the policymaker is, in general, unable to offset both frictions (sticky prices and imperfect substitutability of assets). This observation implies that monetary policy will in general be conducted using a combination of short-term interest rates and asset purchases, not just in 'exceptional circumstances' in which the policy rate has been driven to its lower bound. Casual observation of recent events reveals that policymakers have in practice tended to use asset purchases as a policy tool only in such exceptional circumstances. One practical consideration is that in 'normal times' the implications of imperfect substitutability of assets are simply taken into account in the setting of the short-term policy rate if the effect of imperfect asset subsitutability is sufficiently small (Borio & Disyatat (2009)).

The policymaker minimises the loss function (7) subject to:

[11] See Woodford (2003).

1. The log-linearised model equations:[12]

$$
\begin{aligned}
\hat{x}_t &= E_t\hat{x}_{t+1} - \sigma\left[\frac{1}{1+\delta}\hat{R}_t + \frac{\delta}{1+\delta}\hat{R}^e_{L,t} - E_t\hat{\pi}_{t+1} - r^*_t\right] \\
\hat{R}_t &= \hat{R}^e_{L,t} + \nu\left[\hat{b}_t - \hat{b}_{L,t}\right] \\
\hat{m}_t &= \frac{\sigma_m}{\sigma}\hat{x}_t - \frac{\beta\sigma_m}{1-\beta}\hat{R}_t + \frac{\beta\sigma_m}{1-\beta}\nu\frac{\delta}{1+\delta}\left[\hat{b}_t - \hat{b}_{L,t}\right] \\
\hat{\pi}_t &= \beta E_t\hat{\pi}_{t+1} + \kappa\hat{x}_t \\
\hat{b}_t + \frac{m}{b}\hat{m}_t - \delta q_t &= -\left[\frac{m}{b} + \beta^{-1}(1+\delta)\right]\hat{\pi}_t + \frac{m}{b}\hat{m}_{t-1} \\
&\quad + \left(\beta^{-1} - \theta\right)\hat{b}_{t-1} - \beta^{-1}\delta q_{t-1} \\
-q_t + \hat{V}_t &= \hat{b}_{L,t} \\
\hat{R}^e_{L,t} &= \beta E_t\hat{V}_{t+1} - \hat{V}_t
\end{aligned}
$$

2. $\hat{R}_t \geq \underline{R}$

3. $\underline{q} \leq q_t \leq \bar{q}$

Minimisation of the loss function is subject to a lower bound on the short-term interest rate ($\hat{R}$) and upper and lower bounds on the scale of assets held on the central bank's balance sheet. The lower bound on the short-term nominal interest rate is assumed to be 25 basis points measured at an annual rate. The baseline values for the bounds on asset purchases are set at their theoretical extrema ($\underline{q}$=0 and $\bar{q} = 1$),[13] though in practice most asset purchase schemes have either legislative or practical limits. Finally, the one-period return on long-term bonds ($\hat{R}^e_{L,t}$) should also be bounded (if the return falls below zero, households will prefer money to long-term bonds and the demand for these assets will fall to zero). This constraint is not imposed on the optimisation problem but it is verified that it is satisfied along the equilibrium paths studied below.

A number of assumptions are made to facilitate the analysis. First, before the shock to the natural real interest rate arrives, the model is in steady state. This assumption eliminates the distinction between optimal policy viewed from a timeless perspective and the Ramsey optimal policy. Second, the solution is computed under the assumption of perfect foresight. After the shock to the natural real interest rate, its future path is known with certainty. This permits the use of a 'piecewise linear' solution approach similar to that used by Eggertsson & Woodford (2003), Jung *et al.* (2005) and Levin *et al.* (2009). However, the presence of bounds on multiple instruments complicates the algorithm somewhat. Appendix E provides some details.

The rest of this section considers optimal policy in two cases. In the first case, neither nominal interest rates nor asset purchases are constrained. In the second

[12] The equations are derived in Appendix A.

[13] In the model, the central bank cannot issue its own long-term bonds, so $q_t \geq 0$, and it cannot purchase more than 100% of the outstanding stock, so $q_t \leq 1$.

case, nominal interest rates are constrained by the effective lower bound and asset purchases are subject to both upper and lower bounds. For each case, two scenarios are considered. In the first scenario, asset purchases are assumed not to be available as an instrument. In the second scenario, asset purchases can be used alongside interest rate policy. Finally, outcomes under optimal policy (using both the short-term nominal interest rate and asset purchases) are compared with outcomes from a canonical New Keynesian model with perfect asset substitutability.

In each case, the natural rate of interest unexpectedly falls to –3% after which it follows the simple autoregressive process (5).

3.1 Optimal policy when policy instruments are not bounded

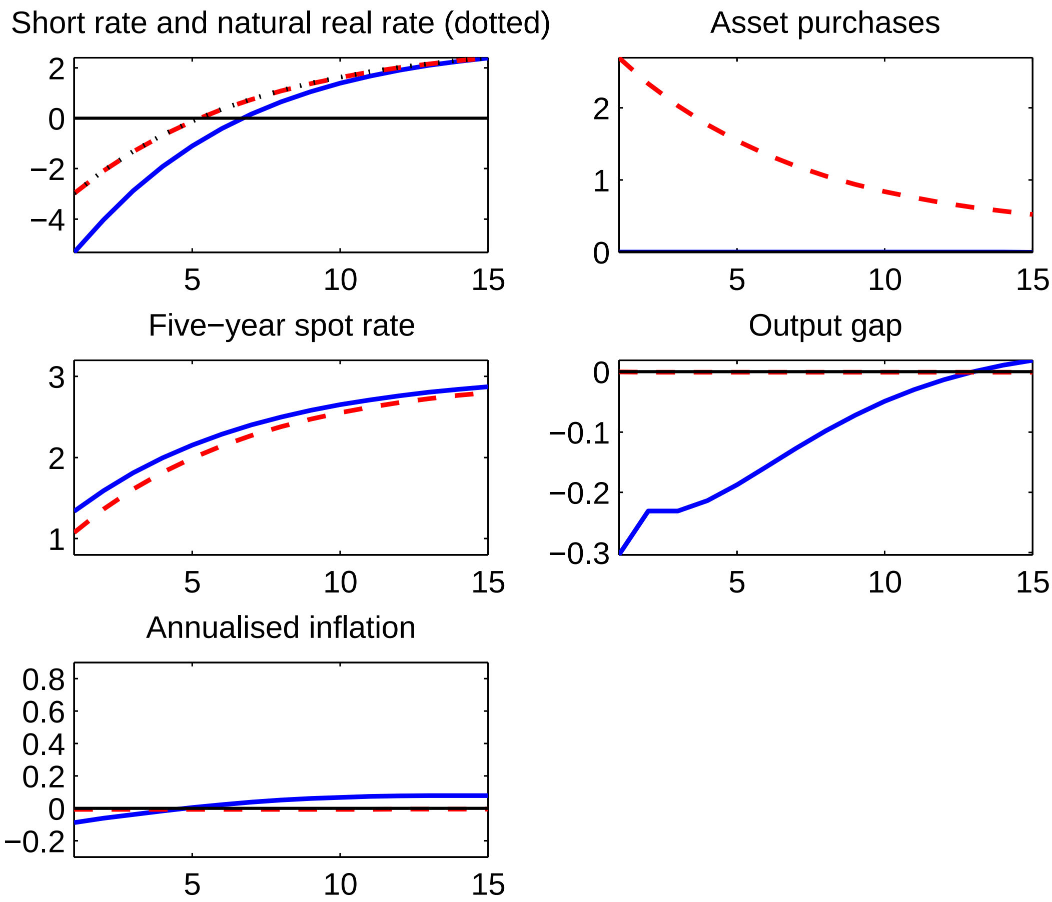
The purpose of this section is to characterise optimal interest rate policy in response to a large negative demand shock when the ELB does not bind. Of course, this is not a realistic scenario. But it is a useful thought experiment to shed light on some of the mechanisms in the model (and the differences with the canonical New Keynesian model) without dealing with the complexities of the bounds on policy instruments. These are considered in the next section.

Recall that in the canonical New Keynesian model, with perfect asset substitutability, the optimal policy is to set the nominal policy rate to perfectly track movements in the natural real interest rate.[14] This policy completely stabilises the output gap and inflation. However, in the present model, this policy prescription no longer holds. Since assets are imperfect substitutes, conventional monetary policy has an effect on the endogenous premium between long and short bonds. In this model, implementing a lower policy rate (in response to a negative demand shock) leads to a reduction in debt financing costs for the government. Since the supply of long-term government debt is held fixed, the reduction in debt financing costs induces a reduction in the supply of short-term debt. However, as households reduce their holdings of short-term debt, their portfolio mix shifts towards long-term bonds. The premium on long-term bonds is a decreasing function of the ratio of short-term to long-term goverment debt, so the premium rises.

Figure 2 compares optimal monetary policy when no bounds on the policy instruments are imposed. In the first case (solid blue lines) an additional assumption is imposed that it is only possible to use the short-term nominal interest rate. This can be interpreted as a special case in which there is no lower bound on the short-term nominal interest rate ($\underline{R} = -\infty$) but asset purchases are prohibited ($\underline{q} = \bar{q} = 0$.) As predicted, with only one instrument available, the policymaker does not perfectly stabilise the output gap and inflation, even by cutting the short-term nominal interest rate to –5%. Imperfect substitutability between financial assets means that the nominal interest rate has to be cut by much more than the fall in the natural real interest rate. This is required in order to (partially) offset the rise in the premium on long-term bonds on the effective real interest rate faced by households. The policy is relatively effective at stabilising the output gap and inflation: the initial

[14] In a model with a non-zero inflation target, the nominal interest will be equal to the natural real interest rate plus the inflation target.

Figure 2: Optimal policy when there are no constraints on instruments: short-term policy rate only (blue, solid), short-term policy rate and asset purchases (red, dashed)



fall in the output gap is only around 0.2 percentage points. Inflation falls initially before rising above target in response to a prolonged (but small) positive output gap beyond the horizon plotted in the charts.

A sufficiently aggressive interest rate response (that is, an even larger cut in the nominal interest rate than shown in Figure 2) would stabilise the output gap and inflation. But because the loss function places weight on households' portfolio mix (see equation (7)), it is not optimal to stabilise the output gap and inflation unless the portfolio mix is also stabilised at the desired level.

The case in which policymakers are also permitted to use asset purchases is depicted by the red dashed lines. This experiment corresponds to a setup in which $\underline{R} = -\infty$, $\underline{q} = -\infty$ and $\bar{q} = \infty$. In this case, the output gap and inflation are perfectly stabilised. This is brought about by using asset purchases to eliminate the premium on long-term bonds, so that the one-period returns on long-term and short-term bonds are equalised. Then by setting the short-term nominal interest rate to track the natural real interest rate, it is also possible to stabilise the output gap and inflation.

Two points are evident from Figure 2. First, the effects of the premium on long-term bonds can be significant. When asset purchases are *not* permitted, five-year spot rates are higher than when asset purchases are used, despite the fact that short-term rates are significantly *lower*.[15]

The second point is that the bounds on the policy instruments considered in the next section are likely to place significant constraints on the policymaker's ability to stabilise the economy. This can be seen in the case in the use of asset purchases is permitted (top right panel, dashed red line). To eliminate the premium on long-term bonds, the policymaker would need to purchase more than twice the existing stock of long-term bonds. Of course, this is infeasible: the central bank can purchase at most 100% of the stock of long-term bonds. So in response to a large negative demand shock, the upper bound on asset purchases is very likely to bind: inhibiting the policymaker's ability to stabilise the output gap and inflation. We investigate that case in the next section.

3.2 Optimal policy with bounded policy instruments

This section examines the more interesting case in which both the nominal interest rate and asset purchases are subject to the bounds described above. In particular, the short-term nominal interest rate is constrained by a (small positive) effective lower bound (25 basis points on the annualised short-term rate) and asset purchases (q) are restricted to be non-negative and to be no greater than 100% of the stock of long-term bonds in circulation. As noted above, these are the least restrictive bounds, given the assumptions of the model.

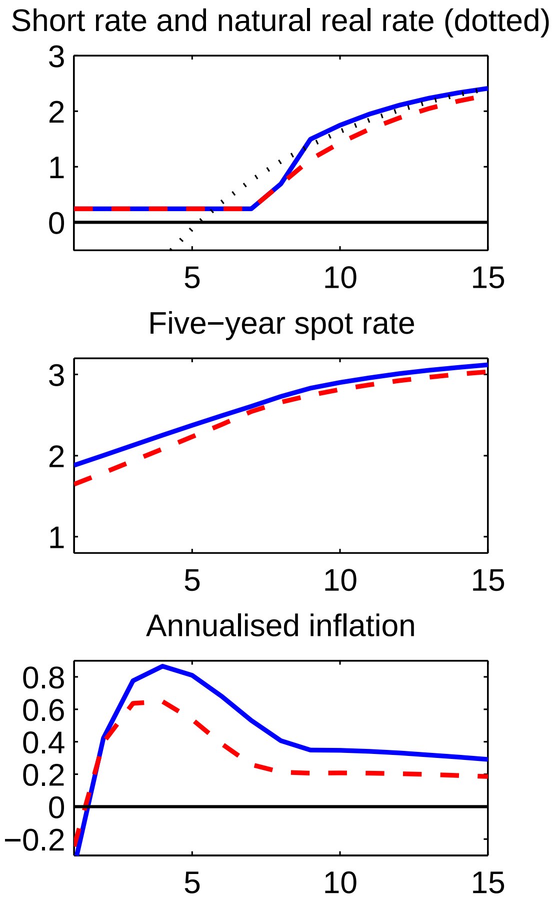
3.2.1 The effects of asset purchases

Figure 3 plots the responses when policy instruments are bounded. Again, two scenarios are depicted. In the first case (solid blue lines) asset purchases are not permitted (which again can be interpreted as the assumption that $\underline{q} = \bar{q} = 0$.). The second case (dashed red lines) shows the case in which asset purchases are permitted subject to the maximum bounds ($\underline{q} = 0$ and $\bar{q} = 1$).

It is evident that the use of asset purchases helps stabilise the output gap and inflation: the effect of using asset purchases as an additional policy tool are intuitive. When asset purchases are used alongside nominal interest rate policy, the impact effect on the output gap is around 1.5 percentage points smaller. Thereafter, the output gap returns more quickly to zero, which results in a more muted response of inflation.

[15]That is, the solid blue line in the centre-left panel of the figure is higher than the dashed red line.

Figure 3: Optimal policy when instruments are bounded: short-term policy rate only (blue, solid), short-term policy rate and asset purchases (red, dashed)



Nevertheless, even with asset purchases as an additional policy instrument, the effect of the shock on activity is significant, because the upper bound on asset purchases binds immediately. As explained in Section 3.1, the fall in nominal interest rates in response to the negative demand shock generates a reduction in the issuance of short-term debt and hence (other things equal) a shift in household asset portfolios towards long-term bonds. In the absence of asset purchases by the central bank, the premium on long-term bonds therefore rises. In Section 3.1, it was shown that in the absence of bounds on both instruments, the optimal asset purchase policy would be to purchase more than 2.5 times the available stock of assets (top right panel of figure 2). When the nominal interest rate is bounded by the effective lower bound, the optimal *unconstrained* level of asset purchases would be even larger, because when one instrument is bounded, it is optimal to rely more on the unconstrained instrument.[16]

Of course, in Figure 3, both the nominal interest rate and asset purchases are bounded. This significantly reduces the scope for stabilisation of activity and inflation through asset purchases. So the effect on long-term yields from the use of asset purchases is relatively small, though long-term rates are indeed lower than in the absence of asset purchases (middle left panel). In part, this reflects the fact that the short-term nominal interest rate rises more slowly away from the effective lower bound.

A striking feature of figure 3 is that inflation remains above trend for a prolonged period. Inflation is above trend because of an expected future sequence of positive (though small) output gaps.[17] The output gap remains positive for a prolonged period because policy continues to provide stimulus to the economy. For the first few quarters following the shock, the optimal policy is to reduce the effective rate of interest relevant for household decisions by lowering the short-term policy rate and (if permitted) engaging in asset purchases. But as the natural real interest rate moves back towards its long-run level, it is necessary to unwind the initial policy stimulus and tighten policy. This is achieved by increasing the short-term policy rate and (in the case in which asset purchase policies are permitted) reducing the quantity of long-term debt held on the central bank balance sheet. Raising the short-term nominal interest rate will, other things equal, increase the short-term real interest rate and therefore reduce aggregate demand, the output gap and inflation. But increasing the short-term nominal interest rate also increases the government's cost of financing transfers to households, leading to an increase in short-term bond issuance, and a fall on the premium on long-term bonds. The reduction in the bond premium therefore partially offsetting the tightening delivered by the increase in the policy rate. This indicates that, for the period over which the initial policy stimulus is being unwound, the *lower* bound on asset purchases is binding. That is, the policymaker would, if it were feasible, hold negative quantities of long-term debt (or equivalently would prefer to issue long-term bonds to the private sector).

3.2.2 Comparison with the canonical New Keynesian model

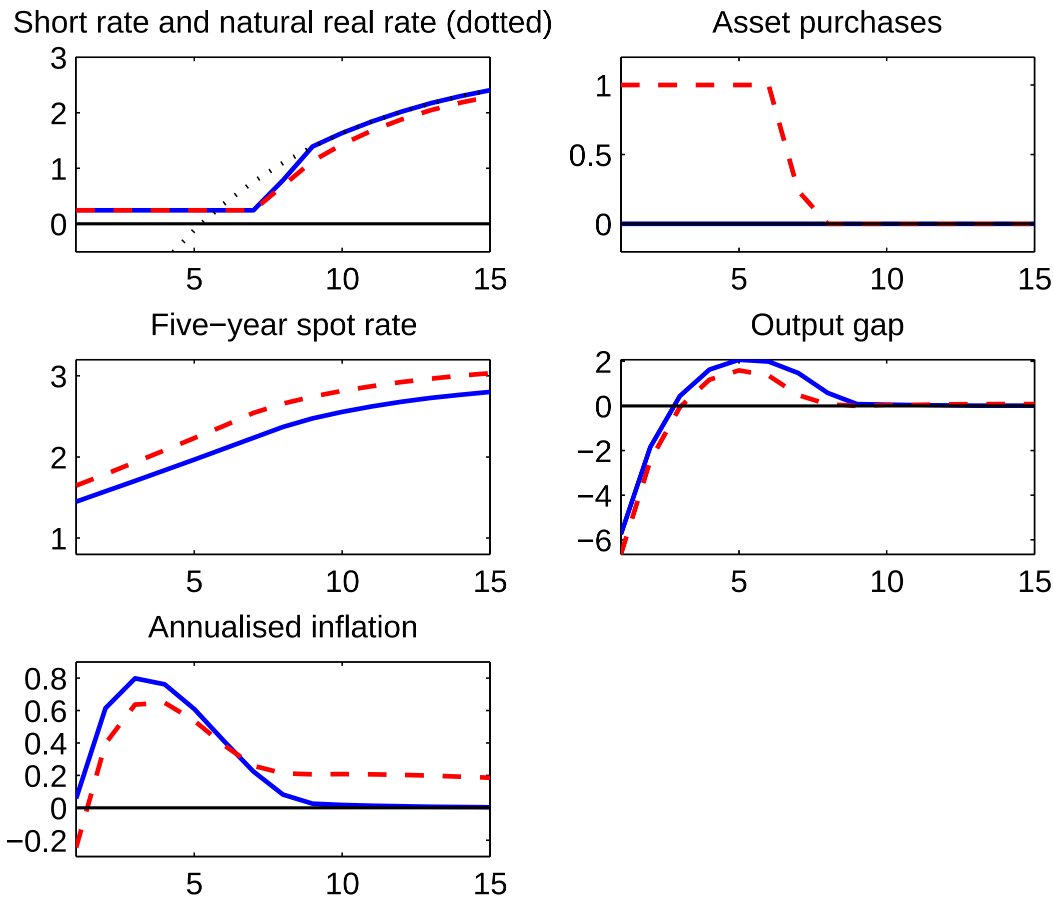
The purpose of this section is to examine the extent to which the presence of an additional policy instrument is beneficial for stabilisation policy relative to the canonical New Keynesian model. When assets are imperfectly substitutable, there is an additional friction that policy must attempt to counter. Of course, this friction also brings into play a new policy instrument (asset purchases) which, in an unconstrained case, can be used to completely offset the friction as shown in Section 3.1.

Figure 4 suggests that the output gap and inflation are better stabilised in the canonical New Keynesian model than in the present model, even when policymakers

[16] In such a simulation (not shown) asset purchases would amount to more than four times the available stock of long-term bonds.

[17] This is difficult to see in the middle left panel of figure 3 because of the scale of the initial fall in the output gap. The average output gap between periods 10 and 15 is 0.067% for the case in which asset purchases are permitted (dashed red lines).

Figure 4: Optimal policy outcomes: canonical New Keynesian case (blue, solid) vs present model with asset purchase policies (red, dashed)



use asset purchases as a policy instrument. It is difficult to judge this unambiguously from Figure 4, as although the initial impact on the output gap in the present model is larger than in the canonical New Keynesian model, the subsequent positive output gap appears smaller. It is clear from the relative inflation responses, however, that with imperfectly substitutable assets, a very small output gap persists for many periods. As discussed in the previous section, this effect arises because the tightening in the short-term nominal interest rate partly offset by a reduction in the premium on long-term bonds.

An important reason for this result is that the bounds on asset purchases are such that the maximum possible level of asset purchases is not sufficient to offset the additional friction introduced by imperfect asset substitutability. So even though asset purchases help to reduce the premium on long-term interest rates, they are not sufficient to fully offset it. This can be seen from the fact that long-term rates are *higher* in the presence of asset purchases despite the fact that short rates are lower.

Of course, another difference between the model considered here and the canonical New Keynesian model is the fact that the loss function depends on the private sector's relative holdings of short-term and long-term bonds. The policymaker will equalise the marginal benefit from better stabilisation of the 'asset gap' with the marginal cost of worse stabilisation of the output gap and inflation. Figure 5 considers the case in which the policymaker minimises the loss function relevant for the canonical New Keynesian model. This experiment sheds some light on the extent to which the structure of the economy and the policymaker's objective function contribute to the results in Figure 4. Figure 5 shows the *canonical New Keynesian model responses* (red dashed lines) and the case in which assets are imperfectly substitutable, but policymakers aim to stabilise only the output gap and inflation (solid blue lines). That is, the loss function (7) is replaced by:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\hat{x}_t^2 + \frac{\eta}{\kappa} \hat{\pi}_t^2 \right]$$

which is the loss function of the canonical New Keynesian model.[18]

Figure 5 places the canonical New Keynesian model and the present model on a more equal footing, since in both cases the policymaker is aiming to stabilise the same loss function. It is apparent that, the combination of interest rate policy and asset purchases (solid blue lines) is better to stabilise the output gap and inflation in the early part of the simulation, compared with performance of interest rate policy alone in the canonical New Keynesian model (dashed red lines). So despite the fact that imperfect substitutability of assets damages the efficacy of conventional (interest rate) policy, the use of asset purchases is sufficient to offset this and deliver better outcomes for the output gap and inflation in the early part of the simulation. Of course, this improved performance is partially offset by worse performance later in the simulation.[19] However, computing the discounted loss for the squared deviations of the output gap and inflation in the two cases (for 150 periods) indicates that the loss in the canonical New Keynesian model is around 4% higher than in the model with imperfect asset substitutability. So when assets are imperfect substitutes, the

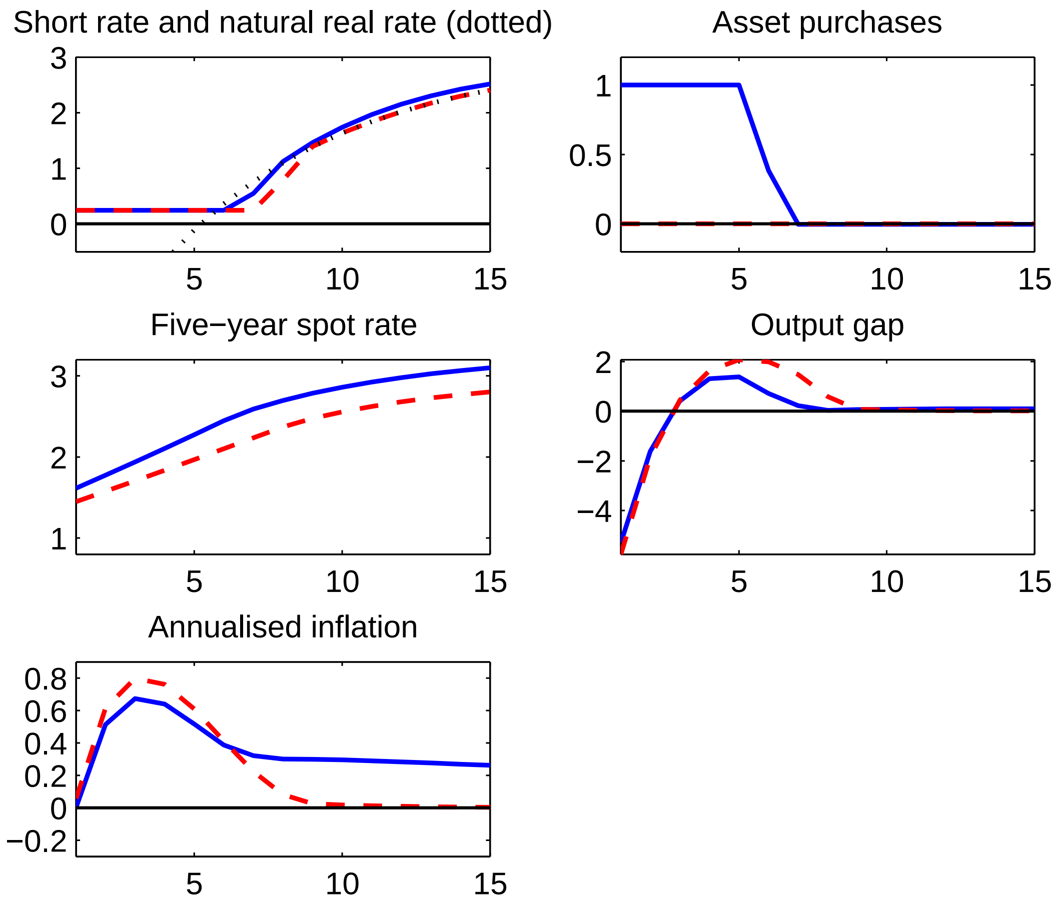
[18]In fact, the loss function includes a very small weight on the asset purchase instrument:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\hat{x}_t^2 + \frac{\eta}{\kappa} \hat{\pi}_t^2 + \zeta q_t^2 \right]$$

where $\zeta = 0.01$. This is required because in periods during which the effective bound on the nominal interest rate does not bind, the optimal policy mix between the short-term policy rate and asset purchases is indeterminate. This is because the policymaker is able to perfectly stabilise the output gap and inflation using either the nominal interest rate or asset purchases. The loss function used here ensures that the policymaker prefers to use the nominal interest rate when it is not constrained by the ELB.

[19]The solid blue line for the output gap is slightly higher than the dashed red line from period 10 onwards. This persistent positive expected output gap generates higher inflation from period 7 onwards (solid blue lines, bottom left panel).

Figure 5: Responses in canonical New Keynesian model (red, dashed) and present model when policymaker minimises New Keynesian loss function (solid, blue)

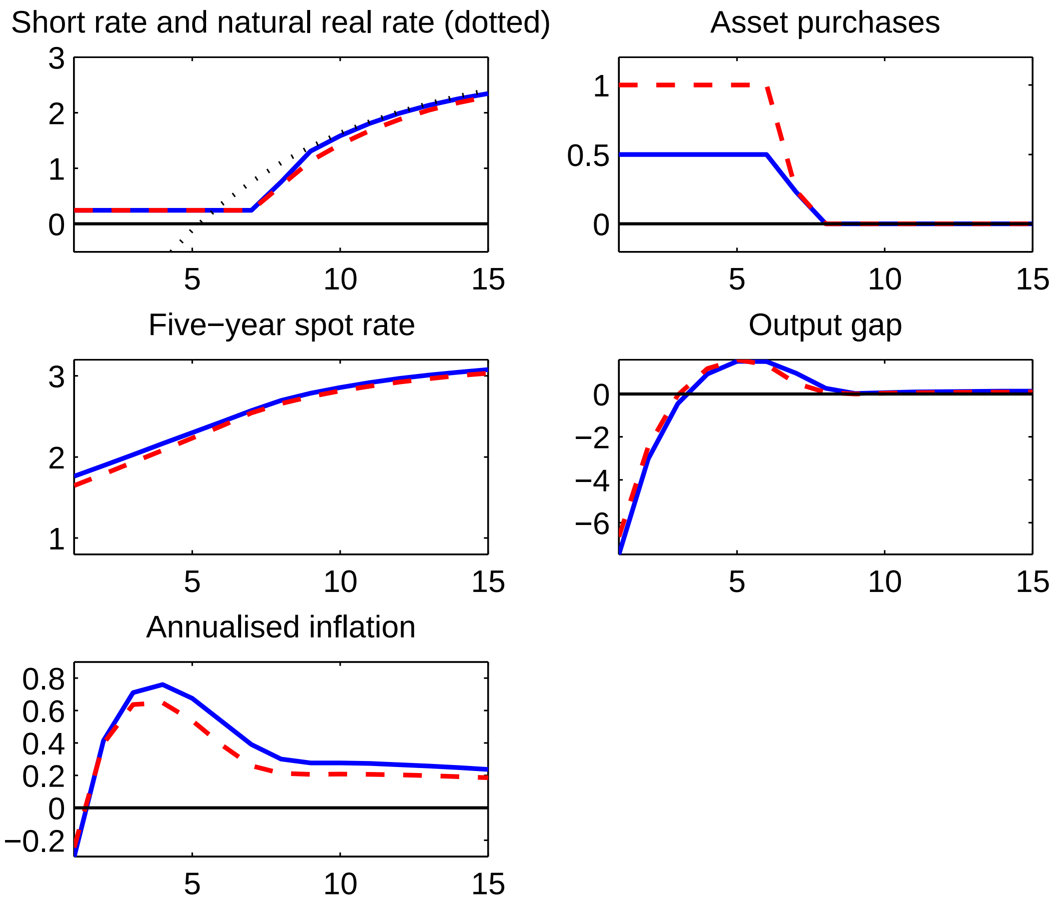


policymaker is able to trade off better stabilisation of the output gap and inflation in the short run for slightly worse performance later on.

4 Sensitivity analysis

This section considers how the optimal responses to a negative demand shock change under alternative parameterisations of the model. The following cases are considered in turn. In Section 4.1 the upper bound on asset purchases is reduced, to reflect the assumption that the central bank may not be permitted to purchase the entire stock of long-term bonds. In Section 4.2, the responsiveness of bond returns to the relative supplies of short-term and long-term bonds is reduced. In Section 4.3, the rule used to determine the scale of transfers to households is investigated.

Figure 6: Responses in baseline case (red, dashed) and smaller upper bound for asset purchases (solid, blue)



4.1 Smaller upper bound for asset purchases

This section examines the case in which the upper bound for asset purchases is set at $\bar{q} = 0.5$ corresponding to an assumption that asset purchases can total at most half of the available long-term bonds. Figure 6 shows the equilibrium in this case (solid blue lines) against the baseline responses (dashed red lines). The charts show that, unsurprisingly, the smaller upper bound for asset purchases inhibits the policymaker's ability to stabilise the output gap and inflation. Asset purchases are unwound as quickly as the baseline case because, as explained in Section 3.2.1 above, the intial policy stimulus must be unwound as the natural real interest rate moves back towards its long-run level. The short-term nominal interest rate is tightened more quickly than in the baseline case because the 'asset gap' that the policymaker places weight on stabilising is smaller, given the smaller quantity of long-term asset purchases undertaken in the first few periods.

Recall that, as discussed in Section 3.2.1, imperfect asset market substitutability means that increasing the short-term nominal interest rate reduces the premium on long-term bonds, partially offsetting the effect of higher short rates on the effective rate of interest paid by households. When the maximum scale of asset purchases is contrained to be smaller (since $\bar{q} = 0.5$) the effect of asset purchases on the relative supply of long-term bonds is reduced (relative to the baseline case). This means that the initial effect of asset purchases on long-term nominal rates is less than in the baseline case (centre-left panel of Figure 6). While this reduces the amount of stimulus available in the early quarters of the simulation, it permits a more rapid increase in the short-term nominal interest rate when policy needs to be tighter. So a sharper increase in the short-term nominal interest rate delivers a path for the five-year spot rate that is very similar to the baseline case (over the latter part of the simulation, when policy is being tightened).

4.2 Reduced elasticity of long-term bond rates to relative asset supplies

This section considers the case in which the sensitivitiy of long-term bond yields to asset purchases is reduced. Specifically the elasticity is set to $\nu = 0.045$ relative to the baseline assumption of $\nu = 0.09$. As noted in Section 2.4, the baseline parameterisation lies between the empirical estimates of such elasticities of Andrés *et al.* (2004) and Bernanke *et al.* (2004). The lower elasticity used in this section is in line with the estimate of Andrés *et al.* (2004).

Figure 7 depicts equilbrium paths with a lower elasticity of long-term bond rates with respect to asset purchases (solid blue lines) alongside the baseline version of the model (red dashed lines). As expected, asset purchases are less effective at stabilising the output gap and inflation in the short run. The profile for asset purchases appears to be 'looser' in the sense that maximum asset purchases are maintained for longer and then unwound more slowly. But because the elasticity on long-term bond yields from such purchases is lower, this path of asset purchases is not sufficient to stabilise the output gap and inflation as well as the baseline case. The loss function is again important since reducing the elasticity ν also reduces the weight placed on the portfolio mix in the loss function 7. This enables asset purchases to be unwound more slowly since the policymaker is less concerned with stabilising households' portfolio mix relative to the output gap and inflation.

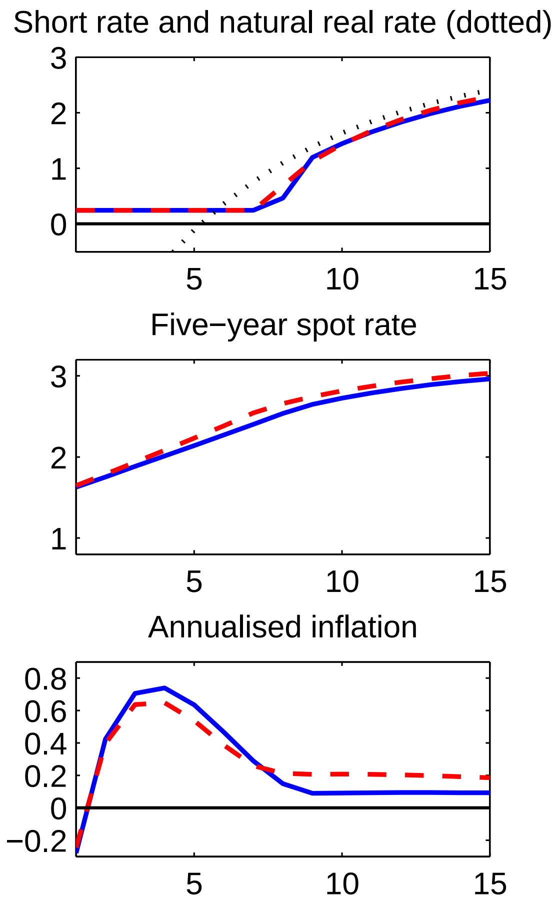
4.3 More responsive transfer rule

[TO BE COMPLETED]

5 Conclusion

This paper has explored asset purchase policies in a model in which long-term and short-term bonds are imperfect substitutes. Imperfect substitutability is introduced by the assumption that households have a preferred portfolio allocation between

Figure 7: Responses in baseline case (red, dashed) and reduced elasticity of long-term bond rates to relative asset supplies (solid, blue)



short-term and long-term bonds. Deviations of the portfolio mix from the desired allocation is assumed to be costly to the household and is model by the addition of a term in the household utility function. This approach means that households equate the effective rates of return on short-term and long-term bonds. The effective rates of return consist of the market rates of return adjusted for the costs of deviating from the desired portfolio allocation. A further implication is that long-term interest rates are a function of both the expected path of short-term rates and the expected deviations of bond holdings from the desired portfolio: long-term interest rates depend on the household's relative holdings of short-term and long-term debt.

Modelled in this way, imperfect asset substitutability has two implications for monetary policy. First, in addition to the short-term policy rate, the policymaker has an additional instrument to affect market interest rates and hence aggregate demand. The private sector's relative holdings of short-term and long-term debt can be influenced by purchases and sales of these debt instruments by the central bank. Second, the welfare function that the policymaker should aim to stabilise includes not only the output gap and inflation (as in the canonical New Keynesian model), but also the deviations of household's relative holdings of short-term and long-term bonds from their desired portfolio mix.

Relative to the canonical New Keynesian model, the additional policy instrument (asset purchases) creates the possibility of improving the stabilisation of the output gap and inflation in response to a negative demand shock that drives the short-term nominal interest rate to its lower bound. But two factors can act to partially offset this benefit. First, asset purchases are themselves subject to bounds. A central bank certainly cannot purchase more than 100% of the entire stock of any asset and in practice there are are likely to be smaller upper bounds on the level of purchases that would be desirable. And in general a central bank cannot issue interest bearing liabilities that have identical characteristics to those already in circulation. The second fact is the fact that optimal policy should place some weight on the stabilisation of portfolios around the desired level. This means that the policymaker will trade off the benefits of better stabilising the output gap and inflation against the benefits of better stabilising portfolios around the desired level.

For the model presented in this paper, the constraints on asset purchases and the fact that the loss function also contains deviations of portfolios from the desired level mean that policy is unable to stabilise the output gap and inflation as well as the canonical New Keynesian model, even with asset purchases as a second instrument. But when the policymaker minimises the loss function associated with the canonical New Keynesian model it is possible to improve upon the outcomes in the canonical model. So even though the presence of imperfectly substitutable assets reduces the efficacy of conventional (short-term interest rate) policy, it also gives asset purchase policies traction over the real economy. And in this case using asset purchase policies can lead to improved outcomes, even if the purchases are bounded.

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A Model derivation

A.1 Households

The optimisation problem considered in Section ?? is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \phi_t \left[\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} + \frac{\chi_m^{-1}}{1-\sigma_m^{-1}} \left(\frac{M_t}{P_t} \right)^{1-1/\sigma_m} - \frac{\tilde{\nu}}{2} \left[\delta \frac{B_t}{B_{L,t}} - 1 \right]^2 \right]$$

subject to

$$B_{L,t} + B_t + M_t = R_{L,t} B_{L,t-1} + R_{t-1} B_{t-1} + M_{t-1} + W_t n_t + T_t + D_t - P_t c_t \tag{8}$$

The first order conditions for the optimisation problem are:

$$\frac{\phi_t}{c_t^{1/\sigma}} = \mu_t P_t \tag{9}$$

$$\phi_t n_t^{\psi} = W_t \mu_t \tag{10}$$

$$\phi_t \chi_m^{-1} \left[\frac{M_t}{P_t} \right]^{-1/\sigma_m} \frac{1}{P_t} - \mu_t + \beta E_t \mu_{t+1} = 0 \tag{11}$$

$$-\mu_t + \phi_t \tilde{\nu} \left[\delta \frac{B_t}{B_{L,t}} - 1 \right] \frac{\delta B_t}{B_{L,t}^2} + \beta E_t \mu_{t+1} R_{L,t+1} = 0 \tag{12}$$

$$-\phi_t \tilde{\nu} \left[\delta \frac{B_t}{B_{L,t}} - 1 \right] \frac{\delta}{B_{L,t}} - \mu_t + \beta R_t E_t \mu_{t+1} = 0 \tag{13}$$

where μ is the Lagrange multiplier on the nominal budget constraint (8).

Last equation is

$$-\phi_t \tilde{\nu} \left[\delta \frac{b_t}{b_{L,t}} - 1 \right] \frac{\delta}{b_{L,t}} - \Lambda_t + \beta R_t E_t \pi_{t+1}^{-1} \Lambda_{t+1} = 0$$

implies that Euler equation is

$$-\tilde{\nu} \left[\delta \frac{b_t}{b_{L,t}} - 1 \right] \frac{\delta}{b_{L,t}} - \frac{1}{c_t^{1/\sigma}} + \beta R_t E_t \pi_{t+1}^{-1} \frac{\phi_{t+1}/\phi_t}{c_{t+1}^{1/\sigma}} = 0$$

We define the real Lagrange multiplier as:

$$\Lambda_t \equiv P_t \mu_t$$

and real money balances and bond holdings as

$$\begin{aligned} m_t &\equiv \frac{M_t}{P_t} \\ b_t &\equiv \frac{B_t}{P_t} \\ b_{L,t} &\equiv \frac{B_{L,t}}{P_t} \end{aligned}$$

and inflation as

$$\frac{P_t}{P_{t-1}} \equiv \pi_t$$

We can combine (9) and (13) to create an Euler equation for consumption:

$$-\tilde{\nu}\left[\delta\frac{b_t}{b_{L,t}}-1\right]\frac{\delta}{b_{L,t}}-\frac{1}{c_t^{1/\sigma}}+\beta R_t E_t \pi_{t+1}^{-1}\frac{\phi_{t+1}/\phi_t}{c_{t+1}^{1/\sigma}}=0$$

which can be log-linearised to give:

$$\begin{aligned} 0 &= -\frac{\tilde{\nu}\delta}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]+\frac{\sigma^{-1}}{c^{1/\sigma}}\hat{c}_t-\frac{\sigma^{-1}}{c^{1/\sigma}}E_t\hat{c}_{t+1} \\ &\quad +\frac{1}{c^{1/\sigma}}\left[\hat{R}_t-E_t\hat{\pi}_{t+1}+E_t\left(\phi_{t+1}-\phi_t\right)\right] \end{aligned}$$

or

$$\begin{aligned} 0 &= -\frac{\tilde{\nu}\delta c\sigma}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]+\hat{c}_t-E_t\hat{c}_{t+1} \\ &\quad +\sigma\left[\hat{R}_t-E_t\hat{\pi}_{t+1}+E_t\left(\phi_{t+1}-\phi_t\right)\right] \end{aligned}$$

or

$$\begin{aligned} \hat{c}_t &= E_t\hat{c}_{t+1}-\sigma\left[\hat{R}_t-E_t\hat{\pi}_{t+1}+E_t\left(\phi_{t+1}-\phi_t\right)\right] \\ &\quad +\frac{\tilde{\nu}\delta c^{1/\sigma}\sigma}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right] \end{aligned}$$

The labour supply condition (10) can be log-linearised to give

$$\psi\hat{n}_t=\hat{w}_t-\sigma^{-1}\hat{c}_t$$

The first order condition for long-term bonds (12) can be written as:

$$-\Lambda_t^u+\phi_t\tilde{\nu}\left[\delta\frac{b_t}{b_{L,t}}-1\right]\frac{\delta b_t}{b_{L,t}^2}+\beta E_t\left[\frac{\Lambda_{t+1}}{\pi_{t+1}}R_{L,t+1}\right]=0$$

Log-linearising gives:

$$\hat{\Lambda}_t=\frac{\tilde{\nu}}{\Lambda b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]+E_t\left[\hat{\Lambda}_{t+1}-\hat{\pi}_{t+1}+\hat{R}_{L,t+1}\right]$$

and since

$$\Lambda=\frac{1}{c^{1/\sigma}}$$

we get

$$\hat{\Lambda}_t=\frac{\tilde{\nu}c^{1/\sigma}}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]+E_t\left[\hat{\Lambda}_{t+1}-\hat{\pi}_{t+1}+\hat{R}_{L,t+1}\right]$$

The first order condition for one period bonds is

$$-\phi_t\tilde{\nu}\left[\delta\frac{b_t}{b_{L,t}}-1\right]\frac{\delta}{b_{L,t}}-\Lambda_t+\beta R_t E_t\pi_{t+1}^{-1}\Lambda_{t+1}=0$$

which can be log-linearised to give

$$-\tilde{\nu}\frac{\delta}{b_{L,t}}\left[\hat{b}_t-\hat{b}_{L,t}\right]-\Lambda\hat{\Lambda}_t+\Lambda\left[\hat{R}_t-E_t\hat{\pi}_{t+1}+E_t\hat{\Lambda}_{t+1}\right]=0$$

or

$$\begin{aligned}\hat{\Lambda}_t &= E_t\left[\hat{\Lambda}_{t+1}-E_t\hat{\pi}_{t+1}\right]+\hat{R}_t-\tilde{\nu}\frac{\delta}{\Lambda b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]\\ &= E_t\left[\hat{\Lambda}_{t+1}-E_t\hat{\pi}_{t+1}\right]+\hat{R}_t-\tilde{\nu}\frac{c^{1/\sigma}\delta}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]\end{aligned}$$

Equating expressions for $\hat{\Lambda}_t$ gives

$$\frac{\tilde{\nu}c^{1/\sigma}}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]+E_t\hat{R}_{L,t+1}=\hat{R}_t-\tilde{\nu}\frac{c^{1/\sigma}\delta}{b_{L,t}}\left[\hat{b}_t-\hat{b}_{L,t}\right]$$

or

$$E_t\hat{R}_{L,t+1}=\hat{R}_t-(1+\delta)\frac{\tilde{\nu}c^{1/\sigma}}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]$$

We can construct a money demand relationship by noting that

$$\phi_t\chi_m^{-1}m_t^{-1/\sigma_m}-\Lambda_t+\beta E_t\pi_{t+1}^{-1}\Lambda_{t+1}=0$$

so that

$$\chi_m^{-1}m^{-1/\sigma_m}\left[\hat{\phi}_t-\sigma_m^{-1}\hat{m}_t\right]-\Lambda\hat{\Lambda}_t+\beta\Lambda E_t\left[\hat{\Lambda}_{t+1}-\hat{\pi}_{t+1}\right]=0$$

or

$$\chi_m^{-1}m^{-1/\sigma_m}\left[\hat{\phi}_t-\sigma_m^{-1}\hat{m}_t\right]-\Lambda\hat{\Lambda}_t+\beta\Lambda\left[\hat{\Lambda}_t-\hat{R}_t+\tilde{\nu}\frac{c^{1/\sigma}\delta}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]\right]=0$$

$$\chi_m^{-1}m^{-1/\sigma_m}\left[\hat{\phi}_t-\sigma_m^{-1}\hat{m}_t\right]-\Lambda(1-\beta)\hat{\Lambda}_t+\beta\Lambda\left[-\hat{R}_t+\tilde{\nu}\frac{c^{1/\sigma}\delta}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]\right]=0$$

In steady state we have

$$\chi_m^{-1}m^{-1/\sigma_m}=(1-\beta)\Lambda$$

so that

$$\left[\hat{\phi}_t-\sigma_m^{-1}\hat{m}_t\right]-\hat{\Lambda}_t+\frac{\beta}{1-\beta}\left[-\hat{R}_t+\tilde{\nu}\frac{c^{1/\sigma}\delta}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]\right]=0$$

or

$$\hat{m}_t=\sigma_m\left(\hat{\phi}_t-\hat{\Lambda}_t\right)-\frac{\beta\sigma_m}{1-\beta}\hat{R}_t+\frac{\beta\sigma_m}{1-\beta}\tilde{\nu}\frac{c^{1/\sigma}\delta}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]$$

or

$$\hat{m}_t=\frac{\sigma_m}{\sigma}\hat{c}_t-\frac{\beta\sigma_m}{1-\beta}\hat{R}_t+\frac{\beta\sigma_m}{1-\beta}\tilde{\nu}\frac{c^{1/\sigma}\delta}{b_L}\left[\hat{b}_t-\hat{b}_{L,t}\right]$$

Note also that since

$$R_{L,t} = \frac{1+V_t}{V_{t-1}}$$

in steady state we have

$$\beta^{-1}V = 1 + V \implies V = \frac{1}{\beta^{-1} - 1}$$

so log-linearising gives us

$$\begin{aligned} \beta^{-1}\left[\hat{R}_{L,t} + \hat{V}_{t-1}\right] &= \hat{V}_t \\ \hat{R}_{L,t} &= \beta\hat{V}_t - \hat{V}_{t-1} \end{aligned}$$

A.2 Firms

As noted in the text, the first order condition for a producer resetting its price at date t is:

$$E_t \sum_{k=t}^{\infty} \Lambda_k (\beta\alpha)^{k-t} \left((1-\eta)\frac{1+s}{P_k} + \eta\frac{w_k}{P_{j,t}A} \right) \left(\frac{P_{j,t}}{P_k}\right)^{-\eta} c_k = 0$$

or

$$E_t \sum_{k=t}^{\infty} \Lambda_k (\beta\alpha)^{s-t} \left((1-\eta)\frac{(1+s)\,p_{j,t}}{\Pi_{t,k}} + \eta\frac{w_k}{A} \right) \left(\frac{p_{j,k}}{\Pi_{t,k}}\right)^{-\eta} c_k = 0 \tag{14}$$

if we define the price set by firm j relative aggregate price level as:

$$p_{j,t} \equiv \frac{P_{j,t}}{P_t}$$

and the relative inflation factor

$$\begin{aligned} \Pi_{t,k} &\equiv \frac{P_k}{P_{t-1}} = \Pi_k \times \Pi_{k-1} \times ... \times \Pi_{t+1} \text{ for } k \geq t+1 \\ &\equiv 1 \text{ for } k = t \end{aligned}$$

where we normalise by the aggregate price level from the *previous* period because this is contained in firms' information set.

Since all firms are identical in terms of their information and production constraints, all firms that are able to change prices at date t will choose the same price, which we denote as p_t^*. Thus

$$E_t \sum_{s=t}^{\infty} \Lambda_k (\beta\alpha)^{k-t} \left((1-\eta)\frac{(1+s)\,p_t^*}{\Pi_{t,k}} + \eta\frac{w_k}{A} \right) \left(\frac{p_t^*}{\Pi_{t,k}}\right)^{-\eta} c_k = 0$$

The retailer's price is:

$$\begin{aligned} P_t &= \left[\int_0^1 P_{j,t}^{1-\eta} dj\right]^{\frac{1}{1-\eta}} \\ &= \left[\sum_{k=0}^{\infty} (1-\alpha)\,\alpha^k \left(P_{t-k}^*\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \end{aligned}$$

where the equality follows from grouping the firms into cohorts according to the date at which they last reset their price and noting that the mass of firms that have not reset their price since date $t-k$ is $(1-\alpha)\,\alpha^k$. This means that the aggregate price level can be written as

$$P_t = \left[\alpha P_{t-1}^{1-\eta} + (1-\alpha)\,(P_t^*)^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

so that

$$1 = \alpha\left(\frac{1}{\pi_t}\right)^{1-\eta} + (1-\alpha)\,(p_t^*)^{1-\eta} \tag{15}$$

Log-linearising the pricing equation gives

$$E_t \sum_{k=t}^{\infty} (\beta\alpha)^{s-t}\left[\hat{p}_t^* - \hat{\Pi}_{t,k} - \hat{w}_k\right] = 0$$

or

$$\begin{aligned}
\hat{p}_t^* &= (1-\beta\alpha)\,E_t \sum_{k=t}^{\infty} (\beta\alpha)^{k-t}\left(\hat{w}_k + \hat{\Pi}_{t,k}\right) \\
&= (1-\beta\alpha)\,\hat{w}_t + (1-\beta\alpha)\,E_t \sum_{k=t+1}^{\infty} (\beta\alpha)^{k-t}\left(\hat{w}_k + \hat{\Pi}_{t,k}\right) \\
&= (1-\beta\alpha)\,\hat{w}_t + \beta\alpha\,(1-\beta\alpha)\,E_t \sum_{k=t+1}^{\infty} (\beta\alpha)^{k-(t+1)}\left(\hat{w}_k + \hat{\Pi}_{t+1,k} + \hat{\pi}_{t+1}\right) \\
&= (1-\beta\alpha)\,\hat{w}_t + \beta\alpha E_t\hat{\pi}_{t+1} + \beta\alpha E_t\hat{p}_{t+1}^*
\end{aligned}$$

where the final equality makes use of the law of iterated conditional expectations. Linearising the expression for the aggregate price level (15) gives:

$$0 = -\alpha\hat{\pi}_t + (1-\alpha)\,\hat{p}_t^*$$

so that

$$\hat{p}_t^* = \frac{\alpha}{1-\alpha}\hat{\pi}_t$$

Using this information in the log-linearised pricing equation gives:

$$\frac{\alpha}{1-\alpha}\hat{\pi}_t = (1-\beta\alpha)\,\hat{w}_t + \beta\alpha E_t\hat{\pi}_{t+1} + \beta\alpha E_t\frac{\alpha}{1-\alpha}\hat{\pi}_{t+1}$$

or

$$\hat{\pi}_t = \frac{(1-\beta\alpha)\,(1-\alpha)}{\alpha}\hat{w}_t + \beta E_t\hat{\pi}_{t+1} \tag{16}$$

Given the aggregate labour supply equation and market clearing we have

$$\hat{\pi}_t = \frac{(1-\alpha)\,(1-\beta\alpha)}{\alpha}\left[\psi + \frac{1}{\sigma}\right]\hat{x}_t + \beta E_t\hat{\pi}_{t+1}$$

A.3 The government budget constraint

The government's budget constraint (in real terms) is

$$\frac{B_t}{P_t}+\frac{B^g_{L,t}}{P_t}-\frac{R_{t-1}B_{t-1}}{P_t}-\frac{R_{L,t}B_{L,t-1}}{P_t}+\frac{\Delta_t}{P_t}=\frac{T_t}{P_t}$$

which states that issuance of short term debt (B) and long term debt (B^g_L) plus the change in the central bank's balance sheet (Δ) is used to finance net transfers to households (T).

Seigniorage is given by

$$\frac{\Delta_t}{P_t}=\frac{M_t-M_{t-1}}{P_t}-\left[\frac{Q_t}{P_t}-\frac{R_{L,t-1}Q_{t-1}}{P_t}\right]$$

where the second term represents purchases of long-term bonds by the central bank. Specifically, the second term records the net increase in the central bank's holdings of long term government debt. If the central bank makes net positive purchases, money creation must be greater than otherwise in order for the fiscal commitments to households to be met with the existing portfolio of government debt.

The asset purchase policy is operated by varying the fraction of bonds held on the central bank balance sheet:

$$Q_t=q_tB^g_{L,t}$$

which allows us to write the budget constraint as

$$b_t+m_t+(1-q_t)\,b^g_{L,t}=\pi_t^{-1}\left[m_{t-1}+R_{t-1}b_{t-1}+R_{L,t}\,(1-q_{t-1})\,b^g_{L,t-1}\right]+\tau_t$$

where

$$\tau_t\equiv\frac{T_t}{P_t}$$

We assume that the fiscal rule can be expressed as

$$\frac{\tau}{b}\hat{\tau}_t=-\theta\hat{b}_{t-1}$$

In steady state we have

$$\left(1-\beta^{-1}\right)\left(b+b^g_L\right)=\tau$$

Suppose also that quantity of long term bonds held fixed in real terms

$$b^g_{L,t}=\bar{b}_CV_t$$

Log-linearising implies that

$$\begin{aligned}b\hat{b}_t+m\hat{m}_t-\bar{b}_Lq_t \quad &= \quad -\pi^{-1}\left[m+Rb+R_L\bar{b}_L\right]\hat{\pi}_t+\frac{m}{\pi}\hat{m}_{t-1}+\frac{Rb}{\pi}\left[\hat{R}_{t-1}+\hat{b}_{t-1}\right]\\&\quad +\frac{R_L\bar{b}_L}{\pi}\left[\hat{R}_{L,t}-q_t\right]-\theta b\hat{b}_{t-1}\end{aligned}$$

Thus

$$b\hat{b}_t + m\hat{m}_t - \bar{b}_L q_t = \left[\frac{Rb}{\pi} - \theta\tau\right]\hat{b}_{t-1} - \pi^{-1}\left[m + Rb + R_L\bar{b}_L\right]\hat{\pi}_t + \frac{m}{\pi}\hat{m}_{t-1} + \frac{Rb}{\pi}\hat{R}_{t-1} + \frac{R_L\bar{b}_L}{\pi}\hat{R}_{L,t} - \frac{R_L\bar{b}_L}{\pi}q_{t-1}$$

or

$$\hat{b}_t + \frac{m}{b}\hat{m}_t + \left[\frac{m}{b} + R + \frac{R_L\bar{b}_L}{b}\right]\hat{\pi}_t - \frac{R_L\bar{b}_L}{b}\hat{R}_{L,t} - \bar{b}_L q_t = [R-\theta]\hat{b}_{t-1} + \frac{m}{b}\hat{m}_{t-1} + R\hat{R}_{t-1} - R_L\bar{b}_L q_{t-1}$$

A.4 Market clearing

Goods market clearing requires:

$$c_t = \mathcal{D}_t^{-1} y_t$$

where $\mathcal{D}_t$ is a measure of price dispersion.

It is straightforward to show that in the absence of price-setting and imperfect asset substitutability frictions, the preference shock has no impact on activity. Under the assumption that the only shock hitting the model is the preference shock, then the efficient level of output is constant. This means that, to a first order approximation, the log-deviations of consumption and output from steady state are equal to the output gap:

$$\hat{c}_t = \hat{y}_t = \hat{x}_t$$

B Equation listing

$$\hat{c}_t = E_t\hat{c}_{t+1} - \sigma\left[\frac{1}{1+\delta}\hat{R}_t + \frac{\delta}{1+\delta}\hat{R}^e_{L,t} - E_t\hat{\pi}_{t+1} - r^*_t\right]$$

$$\hat{R}_t = \hat{R}^e_{L,t} + \nu\left[\hat{b}_t - \hat{b}_{L,t}\right]$$

$$\hat{m}_t = \frac{\sigma_m}{\sigma}\hat{c}_t - \frac{\beta\sigma_m}{1-\beta}\hat{R}_t + \frac{\beta\sigma_m}{1-\beta}\nu\frac{\delta}{1+\delta}\left[\hat{b}_t - \hat{b}_{L,t}\right]$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \kappa\hat{c}_t$$

$$\hat{b}_t + \frac{m}{b}\hat{m}_t - \delta q_t = -\left[\frac{m}{b} + \beta^{-1}(1+\delta)\right]\hat{\pi}_t + \frac{m}{b}\hat{m}_{t-1} + \left(\beta^{-1} - \theta\right)\hat{b}_{t-1} - \beta^{-1}\delta q_{t-1}$$

$$-q_t + \hat{V}_t = \hat{b}_{L,t}$$

$$\hat{R}^e_{L,t} = \beta E_t\hat{V}_{t+1} - \hat{V}_t$$

where $\nu \equiv (1+\delta)\,\tilde{\nu}c^{1/\sigma}\left(\bar{b}_L\right)^{-1}$.

C Utility-based loss function

The period utility function is:

$$U_t = \phi_t \left[\frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} + \frac{\chi_m^{-1}}{1-\sigma_m^{-1}} \left(\frac{M_t}{P_t} \right)^{1-1/\sigma_m} - \frac{\tilde{\nu}}{2} \left[\delta \frac{B_t}{B_{L,t}} - 1 \right]^2 \right]$$

Since the preference shock is exogenous to policy and the model is calibrated to ensure that the quantity of money in circulation is negligible (the 'cashless limit'), the utility function used to construct the loss function is

$$U_t \approx \frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{n_t^{1+\psi}}{1+\psi} - \frac{\tilde{\nu}}{2} \left[\delta \frac{B_t}{B_{L,t}} - 1 \right]^2$$

The second order approximation to the utility function is

$$U_t \approx c^{1-\frac{1}{\sigma}} \left[\hat{c}_t - \frac{1}{2\sigma} \hat{c}_t^2 \right] - n^{1+\psi} \left[\hat{n}_t + \frac{\psi}{2} \hat{n}_t^2 \right] - \frac{\tilde{\nu}}{2} \left[\hat{b}_t - \hat{b}_{L,t} \right]^2$$

The derivation of the final representation of the loss function is standard. The market clearing condition for goods is

$$c_t = \mathcal{D}_t^{-1} y_t$$

where $\mathcal{D}_t \equiv \int_0^1 \left(P_t(i) / P_t \right)^{-\eta} di$ is the price dispersion term associated with staggered pricing. We see that

$$\hat{c}_t = \hat{y}_t - \hat{\mathcal{D}}_t$$

which means that the utility function can be written as

$$\begin{aligned} U_t &\approx c^{1-\frac{1}{\sigma}} \left[\hat{y}_t - \hat{\mathcal{D}}_t - \frac{1}{2\sigma} \left(\hat{y}_t - \hat{\mathcal{D}}_t \right)^2 \right] - n^{1+\psi} \left[\hat{n}_t + \frac{\psi}{2} \hat{n}_t^2 \right] - \frac{\tilde{\nu}}{2} \left[\hat{b}_t - \hat{b}_{L,t} \right]^2 \\ &= c^{1-\frac{1}{\sigma}} \left[\hat{y}_t - \hat{\mathcal{D}}_t - \frac{1}{2\sigma} \hat{y}_t^2 \right] - n^{1+\psi} \left[\hat{n}_t + \frac{\psi}{2} \hat{n}_t^2 \right] - \frac{\tilde{\nu}}{2} \left[\hat{b}_t - \hat{b}_{L,t} \right]^2 \end{aligned}$$

which follows because the price dispersion term $\mathcal{D}_t$ is a second-order term. The production function implies that

$$\hat{y}_t = \hat{n}_t$$

so that the utility function is

$$\begin{aligned} U_t &= c^{1-\frac{1}{\sigma}} \left[\hat{y}_t - \hat{\mathcal{D}}_t - \frac{1}{2\sigma} \hat{y}_t^2 \right] - n^{1+\psi} \left[\hat{y}_t + \frac{\psi}{2} \hat{y}_t^2 \right] - \frac{\tilde{\nu}}{2} \left[\hat{b}_t - \hat{b}_{L,t} \right]^2 \\ &= \left(c^{1-\frac{1}{\sigma}} - n^{1+\psi} \right) \hat{y}_t - \frac{1}{2} \left(\frac{c^{1-\frac{1}{\sigma}}}{\sigma} + \psi n^{1+\psi} \right) \hat{y}_t^2 - c^{1-\frac{1}{\sigma}} \hat{\mathcal{D}}_t - \frac{\tilde{\nu}}{2} \left[\hat{b}_t - \hat{b}_{L,t} \right]^2 \end{aligned}$$

The steady state labour supply relationship is

$$\begin{aligned} n^{\psi} &= w c^{-1/\sigma} \\ &= A c^{-1/\sigma} \end{aligned}$$

which follows from the assumption that susidies to firms are set to eliminate the distortion from monopolistic competition. Steady-state market clearing is

$$c = y = An$$

since steady state dispersion is $\mathcal{D} = 1$.

This implies that

$$n^{1+\psi} = c^{1-1/\sigma}$$

so that the utility function can be written as

$$U_t = -\frac{1}{2}c^{1-\frac{1}{\sigma}}\left(\frac{1}{\sigma}+\psi\right)\hat{y}_t^2 - c^{1-\frac{1}{\sigma}}\hat{\mathcal{D}}_t - \frac{\tilde{\nu}}{2}\left[\hat{b}_t - \hat{b}_{L,t}\right]^2$$

Recall that the price dispersion term is

$$\mathcal{D}_t = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\eta} di$$

which in equilibrium is given by

$$\mathcal{D}_t = \alpha \mathcal{D}_{t-1}\pi_t^{\eta} + (1-\alpha)(p_t^*)^{-\eta}$$

Using the price index (15), the optimal price can be written as

$$p_t^* = \left[\frac{1-\alpha\pi_t^{\eta-1}}{1-\alpha}\right]^{\frac{1}{1-\eta}}$$

so the price dispersion is

$$\mathcal{D}_t = \alpha \mathcal{D}_{t-1}\pi_t^{\eta} + (1-\alpha)\left[\frac{1-\alpha\pi_t^{\eta-1}}{1-\alpha}\right]^{\frac{\eta}{\eta-1}}$$

Taking a second order Taylor expansion gives

$$\begin{aligned}
\hat{\mathcal{D}}_t &\approx \alpha\left(\hat{\mathcal{D}}_{t-1} + \eta\hat{\pi}_t\right) + (1-\alpha)\left[\frac{-\alpha\eta\hat{\pi}_t}{1-\alpha}\right] \\
&\quad + \frac{\alpha\eta(\eta-1)}{2}\hat{\pi}_t^2 + \frac{1}{2}\left[\frac{\alpha^2\eta}{1-\alpha} - \alpha\eta(\eta-2)\right]\hat{\pi}_t^2 \\
&\approx \alpha\hat{\mathcal{D}}_{t-1} + \frac{\alpha\eta}{2(1-\alpha)}\hat{\pi}_t^2
\end{aligned}$$

The loss function to be minimised can be defined as

$$\begin{aligned}
\mathcal{L} &= -2\sum_{t=0}^{\infty}\beta^t U_t \\
&= \sum_{t=0}^{\infty}\beta^t\left[c^{1-\frac{1}{\sigma}}\left(\frac{1}{\sigma}+\psi\right)\hat{y}_t^2 + 2c^{1-\frac{1}{\sigma}}\hat{\mathcal{D}}_t + \tilde{\nu}\left[\hat{b}_t - \hat{b}_{L,t}\right]^2\right]
\end{aligned}$$

If we note that

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{D}}_t &= \alpha\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{D}}_{t-1}+\sum_{t=0}^{\infty}\beta^t\frac{\alpha\eta}{2\left(1-\alpha\right)}\hat{\pi}_t^2 \\ &= \alpha\hat{\mathcal{D}}_{t-1}+\alpha\beta\sum_{t=1}^{\infty}\beta^{t-1}\hat{\mathcal{D}}_{t-1}+\sum_{t=0}^{\infty}\beta^t\frac{\alpha\eta}{2\left(1-\alpha\right)}\hat{\pi}_t^2 \\ &= \alpha\hat{\mathcal{D}}_{t-1}+\alpha\beta\sum_{t=0}^{\infty}\beta^{t}\hat{\mathcal{D}}_{t}+\sum_{t=0}^{\infty}\beta^t\frac{\alpha\eta}{2\left(1-\alpha\right)}\hat{\pi}_t^2\end{aligned}$$

then we can see that

$$\sum_{t=0}^{\infty}\beta^t\hat{\mathcal{D}}_t=\frac{\alpha}{1-\alpha\beta}\hat{\mathcal{D}}_{t-1}+\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\frac{\alpha\eta}{\left(1-\alpha\beta\right)\left(1-\alpha\right)}\hat{\pi}_t^2$$

Using this information in the definition of the loss function gives

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty}\beta^t\left[c^{1-\frac{1}{\sigma}}\left(\frac{1}{\sigma}+\psi\right)\hat{y}_t^2+c^{1-\frac{1}{\sigma}}\frac{\alpha\eta}{\left(1-\alpha\beta\right)\left(1-\alpha\right)}\hat{\pi}_t^2+\tilde{\nu}\left[\hat{b}_t-\hat{b}_{L,t}\right]^2\right] \\ &\quad +\frac{\alpha}{\left(1-\alpha\beta\right)\left(1-\beta\right)}\hat{\mathcal{D}}_{t-1}\end{aligned}$$

The term in $\hat{\mathcal{D}}_{t-1}$ is independent of policy and can be ignored. Normalising the loss function so that the weight on output gap deviations is unity gives:

$$\mathcal{L}\propto\sum_{t=0}^{\infty}\beta^t\left[\hat{y}_t^2+\frac{\eta}{\kappa}\hat{\pi}_t^2+\frac{\nu}{\left(1+\delta\right)\left(\sigma^{-1}+\psi\right)}\frac{\bar{b}_L}{c}\left[\hat{b}_t-\hat{b}_{L,t}\right]^2\right]$$

which follows from the definition of κ and the fact that:

$$\tilde{\nu}=\frac{\nu\bar{b}_Lc^{-1/\sigma}}{1+\delta}$$

D Imperfectly substitutable assets and financial intermediation

[TO BE UPDATED]

Suppose there is a financial intermediary that finances one-period loans to households at rate R^c, financed by investments in short and long-term government debt. The maximisation problem of the financial intermdiary is

$$\max E_t\left[R_tB_t+R_{L,t+1}B_{L,t}-\left(R_t^c+\frac{\nu}{2}\left(\delta\frac{B_t}{B_{L,t}}-1\right)^2\right)A_t\right]$$

subject to

$$A_t=B_t+B_{L,t}$$

The maximisation problem incorporates the assumption that financial intermediation entails a cost related to the mix of short and long-term bonds held in the intermediary's portfolio.

This maximisation problem can be represented as

$$\max E_t\left[R_tB_t + R_{L,t+1}B_{L,t} - \left(R_t^c + \frac{\nu}{2}\left(\delta\frac{B_t}{B_{L,t}} - 1\right)^2\right)[B_t + B_{L,t}]\right]$$

The first order conditions are

$$\begin{aligned} R_t - \left(R_t^c + \frac{\nu}{2}\left(\delta\frac{B_t}{B_{L,t}} - 1\right)^2\right) - \nu\delta\left(\delta\frac{B_t}{B_{L,t}} - 1\right)\frac{B_t + B_{L,t}}{B_{L,t}} &= 0 \\ E_tR_{L,t+1} - \left(R_t^c + \frac{\nu}{2}\left(\delta\frac{B_t}{B_{L,t}} - 1\right)^2\right) + \nu\delta\left(\delta\frac{B_t}{B_{L,t}} - 1\right)\frac{B_t}{B_{L,t}}\frac{B_t + B_{L,t}}{B_{L,t}} &= 0 \end{aligned}$$

Linearising gives

$$\begin{aligned} R\hat{R}_t - R\hat{R}_t^c - \nu\delta\frac{B + B_L}{B_L}\delta\frac{B}{B_L}\left(\hat{B}_t - \hat{B}_{L,t}\right) &= 0 \\ RE_t\hat{R}_{L,t+1} - R\hat{R}_t^c + \nu\delta\frac{B}{B_L}\frac{B + B_L}{B_L}\delta\frac{B}{B_L}\left(\hat{B}_t - \hat{B}_{L,t}\right) &= 0 \end{aligned}$$

and noting that

$$\frac{B_L}{B} = \delta \quad , \quad \frac{B + B_L}{B_L} = 1 + \delta^{-1}$$

gives

$$\begin{aligned} R\hat{R}_t - R\hat{R}_t^c - \nu\delta\left(1 + \delta^{-1}\right)\left(\hat{B}_t - \hat{B}_{L,t}\right) &= 0 \\ RE_t\hat{R}_{L,t+1} - R\hat{R}_t^c + \nu\left(1 + \delta^{-1}\right)\left(\hat{B}_t - \hat{B}_{L,t}\right) &= 0 \end{aligned}$$

Rearranging and noting that $R = \beta^{-1}$ in a zero inflation steady state gives

$$\begin{aligned} \hat{R}_t^c &= \hat{R}_t - \beta\nu\delta\left(1 + \delta^{-1}\right)\left(\hat{B}_t - \hat{B}_{L,t}\right) \\ \hat{R}_t^c &= E_t\hat{R}_{L,t+1} + \beta\nu\left(1 + \delta^{-1}\right)\left(\hat{B}_t - \hat{B}_{L,t}\right) \end{aligned}$$

Suppose also that the price setting friction is a quadratic adjustment cost. A firm's profit maximisation problem is given by:

$$\max E_0\sum_{t=0}^{\infty}\beta^t\left[(1+s)\,P_t(i)\,Y_t(i) - W_tn_t(i) - \frac{\chi_p}{2}\left[\frac{P_t(i)}{P_{t-1}(i)} - 1\right]^2 P_tY_t\right]$$

subject to

$$\begin{aligned} Y_t(i) &= \left(\frac{P_t(i)}{P_t}\right)^{-\eta} Y_t \\ Y_t(i) &= n_t(i) \end{aligned}$$

Or

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\left((1+s) P_t(i) - W_t \right) \left(\frac{P_t(i)}{P_t} \right)^{-\eta} - \frac{\chi_p}{2} \left[\frac{P_t(i)}{P_{t-1}(i)} - 1 \right]^2 P_t \right] Y_t$$

which has first order condition

$$\begin{aligned} & Y_t \left[(1-\eta)(1+s) \left(\frac{P_t(i)}{P_t} \right)^{-\eta} + \eta W_t \left(\frac{P_t(i)}{P_t} \right)^{-\eta} \frac{1}{P_t(i)} - \chi_p \left[\frac{P_t(i)}{P_{t-1}(i)} - 1 \right] \frac{P_t}{P_{t-1}(i)} \right] \\ = & -\beta \chi_p E_t \left[\frac{P_{t+1}(i)}{P_t(i)} - 1 \right] \frac{P_{t+1} P_{t+1}(i)}{P_t(i)^2} Y_{t+1} \end{aligned}$$

In a symmetric equilibrium, this becomes

$$(1-\eta)(1+s) + \eta w_t - \chi_p [\pi_t - 1] \pi_t = -\beta \chi_p E_t [\pi_{t+1} - 1] \pi_{t+1}^2 \frac{Y_{t+1}}{Y_t}$$

Log-linearising gives:

$$\eta w \hat{w}_t - \chi_p \hat{\pi}_t = -\beta \chi_p E_t \hat{\pi}_{t+1}$$

Assuming that profits from firms and financial intermediaries are transferred to households, the resource constraint in a symmetric equilibrium is

$$y_t = c_t - \frac{\chi_p}{2} [\pi_t - 1]^2 y_t - \frac{\nu}{2} \left(\delta \frac{B_t}{B_{L,t}} - 1 \right)^2 (B_t + B_{L,t})$$

The analysis above can be used to show that this model delivers identical behavioural equations as that considered in the main text. An extension of the analysis in Nisticó (2007) can be used to show that the welfare-based loss function is also identical.

E Solving for optimal commitment policy with bounded instruments

Our approach to solving for optimal commitment policies in the presence of bounds on the policy instruments is a simple generalisation of the approach outlined in Dennis (2007).

We suppose that the policymaker solves the following problem:

$$\min E_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{y}_t' \mathbf{W} \mathbf{y}_t + (\mathbf{x}_t - \delta \mathbf{x}_{t-1})' \mathbf{Q} (\mathbf{x}_t - \delta \mathbf{x}_{t-1}) \right] \tag{17}$$

subject to

$$\mathbf{A}_0 \mathbf{y}_t = \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 E_t \mathbf{y}_{t+1} + \mathbf{A}_3 \mathbf{x}_t + \mathbf{A}_4 E_t \mathbf{x}_{t+1} + \mathbf{A}_6 \mathbf{x}_{t-1} + \mathbf{A}_5 \mathbf{v}_t \tag{18}$$

and

$$\mathbf{S}\mathbf{x}_t \geq \mathbf{s} \tag{19}$$

where $\mathbf{x}$ are the policy instruments, $\mathbf{y}$ are the remaining endogenous variables and $\mathbf{v}$ are iid shocks. All variables are measured as log-deviations from steady state. The notation is based on that of Dennis (2007) and there are only two minor differences from his setup. The first is that the loss function (17) is defined in terms of quasi-differences in the policy instruments, whereas in Dennis's formulation this term is defined in terms of deviations of the policy instrument from steady state ($\mathbf{x}_t'\mathbf{Q}\mathbf{x}_t$). The quasi-difference parameter $\delta \in [0, 1]$ is a simple device to allow us to consider cases when there are costs of keeping instruments away from their steady state levels ($\delta = 0$) and the case in which changes in the policy instrument are deemed costly ($\delta = 1$). The second difference is the inclusion of inequality constraints on the instruments: (19). This constraint allows us to take account of the effective lower bound on the nominal interest rate and the bounds on asset purchases. In our model we have

$$\mathbf{x}_t \equiv \begin{bmatrix} R_t \\ q_t \end{bmatrix}$$

with

$$\begin{aligned} \mathbf{S} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \\ \mathbf{s} &= \begin{bmatrix} b \\ -\bar{q} \\ \underline{q} \end{bmatrix} \end{aligned}$$

The Lagrangean is:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \mathbf{y}_t'\mathbf{W}\mathbf{y}_t + (\mathbf{x}_t - \delta\mathbf{x}_{t-1})' \mathbf{Q} (\mathbf{x}_t - \delta\mathbf{x}_{t-1}) \\ +2\lambda_t' \begin{pmatrix} \mathbf{A}_0\mathbf{y}_t - \mathbf{A}_1\mathbf{y}_{t-1} - \mathbf{A}_2 E_t\mathbf{y}_{t+1} \\ -\mathbf{A}_3\mathbf{x}_t - \mathbf{A}_4 E_t\mathbf{x}_{t+1} - \mathbf{A}_6\mathbf{x}_{t-1} - \mathbf{A}_5\mathbf{v}_t \end{pmatrix} \\ +2\mu_t' (\mathbf{S}\mathbf{x}_t - \mathbf{s}) \end{array} \right]$$

and the first order conditions with respect to $\mathbf{x}$, $\mathbf{y}$, λ and μ are:[20]

$$\begin{aligned} 0 &= \mathbf{Q}(\mathbf{x}_t - \delta\mathbf{x}_{t-1}) - \beta\delta\mathbf{Q}(E_t\mathbf{x}_{t+1} - \delta\mathbf{x}_t) - \mathbf{A}_3'\lambda_t \\ &\quad -\beta^{-1}\mathbf{A}_4'\lambda_{t-1} - \beta\mathbf{A}_6' E_t\lambda_{t+1} + \mathbf{S}'\mu_t \\ 0 &= \mathbf{W}\mathbf{y}_t + \mathbf{A}_0'\lambda_t - \beta^{-1}\mathbf{A}_2'\lambda_{t-1} - \beta\mathbf{A}_1' E_t\lambda_{t+1} \\ 0 &= \mathbf{A}_0\mathbf{y}_t - \mathbf{A}_1\mathbf{y}_{t-1} - \mathbf{A}_2 E_t\mathbf{y}_{t+1} - \mathbf{A}_3\mathbf{x}_t - \mathbf{A}_4 E_t\mathbf{x}_{t+1} - \mathbf{A}_6\mathbf{x}_{t-1} - \mathbf{A}_5\mathbf{v}_t \\ 0 &= \mu_t'(\mathbf{S}\mathbf{x}_t - \mathbf{s}) \end{aligned}$$

The final equation in the first order conditions is a representation of the Kuhn-Tucker optimality condition. If the constraint does not bind, then the associated

[20] We ignore the fact that the first order conditions for the endogenous variables and instrument are different in period 0.

Lagrange multiplier in the μ vector is zero. But if the constraint binds, the value of the constrained variable is determined by the constraint. In that case, the Lagrange multiplier is non-zero and will be determined by the first order condition for the instrument (the first equation in the set of first order conditions).

Because the final first order condition is non-linear in μ and $\mathbf{x}$, it is not possible to solve the model directly using linear methods. However, if we think of the status of each constraint ('binding' or 'non-binding') as a particular 'regime', then the evolution of the endogenous variables can be expressed as the solution to a sequence of 'piecewise linear' models. For example, the solution path may be characterised by an initial regime in which only the effective lower bound on the nominal interest rate binds, followed by a regime in which constraints on both the interest rate and asset purchase instrument are binding, followed by a regime in which no constraints bind. Each regime in the solution path can be represented as a set of linear equations. Given a guess about the dates at which the solution moves from regime to regime, we can piece together the linear models that characterise each regime. These models can be solved together for the paths of the endogenous variables. To assess if our guess about the dates at which the solution moves from regime to regime is correct, we check that the resulting paths of endogenous variables satisfy all of the first order conditions.[21]

For our model the 'regimes' are:

Index (k)	Conditions
0	No constraints bind
1	$R_t = b$
2	$R_t = b$, $q_t = \bar{q}$
3	$R_t = b$, $q_t = \underline{q}$
4	$q_t = \bar{q}$
5	$q_t = \underline{q}$

We therefore construct a number of versions of the model, each of which is relevant to a particular 'regime', k. To do so, we stacking the optimality conditions

[21] For example, during phases when the constraints on the instruments do not bind, the equilibrium values of the instruments should not violate the constraints.

to give:

$$\begin{bmatrix} \mathbf{I}-\mathcal{J}_k & \mathbf{0} & \mathbf{0} & \mathcal{J}_k\mathbf{S} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_0 & -\mathbf{A}_3 \\ \mathbf{0} & \mathbf{A}_0' & \mathbf{W} & \mathbf{0} \\ (\mathcal{J}_k\mathbf{S})' & -\mathbf{A}_3' & \mathbf{0} & \mathbf{Q}\left(1+\beta\delta^2\right) \end{bmatrix} \begin{bmatrix} \mu_t \\ \lambda_t \\ \mathbf{y}_t \\ \mathbf{x}_t \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_1 & \mathbf{A}_6 \\ \mathbf{0} & \beta^{-1}\mathbf{A}_2' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta^{-1}\mathbf{A}_4' & \mathbf{0} & \delta\mathbf{Q} \end{bmatrix} \begin{bmatrix} \mu_{t-1} \\ \lambda_{t-1} \\ \mathbf{y}_{t-1} \\ \mathbf{x}_{t-1} \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_4 \\ \mathbf{0} & \beta\mathbf{A}_1' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{A}_6' & \mathbf{0} & \beta\delta\mathbf{Q} \end{bmatrix} E_t \begin{bmatrix} \mu_{t+1} \\ \lambda_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathcal{J}_k \\ \mathbf{A}_5 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_t \\ \mathbf{s} \end{bmatrix}$$

where $\mathcal{J}_k$ denotes an indicator matrix that defines the 'regime'. For example, we set $\mathcal{J}_k = \mathbf{0}$ when none of the constraints bind.

So for each regime k, we can write the system above as

$$\mathbf{H}\mathbf{z}_{t+1} + \mathbf{G}_k\mathbf{z}_t + \mathbf{F}\mathbf{z}_{t-1} = \mathbf{\Psi}_k\mathbf{u}_t$$

where

$$\mathbf{z}_t \equiv \begin{bmatrix} \mu_t \\ \lambda_t \\ \mathbf{y}_t \\ \mathbf{x}_t \end{bmatrix}$$
$$\mathbf{u}_t \equiv \begin{bmatrix} \mathbf{v}_t \\ \mathbf{s} \end{bmatrix}$$

So the solution of the model over a sequence of regimes $k = \{k_1, ..., k_n\}$ can be written as:

$$\mathbf{JZ} = \mathbf{M}$$

where

$$\mathbf{J} = \begin{bmatrix} \mathbf{G}_{k_1} & \mathbf{H} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{F} & \mathbf{G}_{k_1} & \mathbf{H} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & .. & & & & & .. & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{F} & \mathbf{G}_{k_i} & \mathbf{H} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{F} & \mathbf{G}_{k_{i+1}} & \mathbf{H} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & .. & & & & & .. & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{F} & \mathbf{G}_{k_{n-1}} & \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{F} & \mathbf{G}_{k_n} & \mathbf{H} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{F}_0 & \mathbf{G}_0 \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \dots \\ \mathbf{z}_{p_i} \\ \mathbf{z}_{p_i+1} \\ \dots \\ \mathbf{z}_{p_n-1} \\ \mathbf{z}_{p_n} \\ \mathbf{z}_{p_n+1} \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} -\mathbf{F}_{k_1}\mathbf{z}_0 + \mathbf{\Psi}_{k_1}\mathbf{u}_1 \\ \mathbf{\Psi}_{k_1}\mathbf{u}_2 \\ \dots \\ \dots \\ \mathbf{\Psi}_{k_i}\mathbf{u}_{p_i} \\ \mathbf{\Psi}_{k_i+1}\mathbf{u}_{p_i+1} \\ \dots \\ \mathbf{\Psi}_{k_{n-1}}\mathbf{u}_{p_n-1} \\ \mathbf{\Psi}_{k_n}\mathbf{u}_{p_n} \\ \mathbf{\Psi}_0\mathbf{u}_{p_n+1} - \mathbf{H}_0\mathbf{z}_{p_n+1} \end{bmatrix}$$

The solution path for the endogenous variables can therefore be computed as

$$\mathbf{Z} = \mathbf{J}^{-1}\mathbf{M}$$

and the solutions can be examined to check whether they satisfy the first order conditions.

Two aspects of the approach above are worthy of note. The first is that we use the notation p_{k_i} to mark the end of regime k_i. The second point is that the solution path includes periods beyond the end of the 'final' regime. We make the assumption that the model eventually returns to regime 0 (no constraints are binding). So the solution beyond the end of regime k_n is characterised by

$$\mathbf{H}\mathbf{z}_{t+1} + \mathbf{G}_0\mathbf{z}_t + \mathbf{F}\mathbf{z}_{t-1} = \mathbf{\Psi}_0\mathbf{u}_t$$

which can be solved using standard methods to deliver a solution of the form

$$\mathbf{z}_t = \mathbf{P}_0\mathbf{z}_{t-1} + \mathbf{\Gamma}_0\mathbf{u}_t$$

which suggests that we can modify the $\mathbf{J}$ and $\mathbf{M}$ matrices to become:

$$\mathbf{J} = \begin{bmatrix} \mathbf{G}_{k_1} & \mathbf{H} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{F} & \mathbf{G}_{k_1} & \mathbf{H} & .. & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & .. & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{F} & \mathbf{G}_{k_{n-1}} & \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{F} & \mathbf{G}_{k_n} & \mathbf{H} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & .. & \mathbf{0} & \mathbf{0} & \mathbf{F}_0 & \mathbf{G}_0 + \mathbf{H}_0\mathbf{P}_0 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} -\mathbf{F}_{k_1}\mathbf{z}_0 + \mathbf{\Psi}_{k_1}\mathbf{u}_1 \\ \mathbf{\Psi}_{k_1}\mathbf{u}_2 \\ \cdots \\ \mathbf{\Psi}_{k_{n-1}}\mathbf{u}_{p_n-1} \\ \mathbf{\Psi}_{k_n}\mathbf{u}_{p_n} \\ \mathbf{\Psi}_0\mathbf{u}_{p_n+1} \end{bmatrix}$$

E.1 Solving the model in the paper

Because the number of potential sequences of regimes is very large, we use some information about the particular model to guide the solution approach. The structure of the model suggests the following sequence of regimes is the most likely to support an equilibrium.

Regime	Index	Conditions	Start date	End date
k_1	1	$R_t = b$, $q_t = \bar{q}$	1	p_1
k_2	2	$R_t = b$	$p_1 + 1$	p_2
k_3	0	Unconstrained	$p_2 + 1$	p_3
k_4	4	$q_t = \underline{q}$	$p_3 + 1$	p_4

This is a plausible conjecture based on inspection of equilibria in which no constraints are placed on q. In these cases, it is optimal to make full use of the asset purchase instrument in the initial periods following the shock. In simulations that do not impose an upper bound on q equilibrium paths in which $q_t > \bar{q}$ are observed. As the shock subsides, policy stimuluse is removed and finally, as the period during which the effective lower bound on the nominal rate comes to an end, policy begins to tighten to partially offset the initial stimulus. In simulations with unconstrained q part of this tightening comes about through the policymaker choosing $q_t < \underline{q}$.

The unknowns to be solved for are the dates p_1 to p_4.[22] The algorithm for finding possible equilibria is simply to search across the relevant dates:

1. Set an upper bound for the possible values of p_4 that will be considered. Ensure that this upper bound, denoted $\bar{p}$, is large.
2. Construct the following loops:

```
For c1 = 0,....,p̄
    For c2 = c1,....,p̄
        For c3 = c2,....,p̄
            For c4 = p̄,...,c3
                Construct J and M with [p1 p2 p3] = [c1 c2 c3]
                Solve model Z = J^-1 M
                Check validity of solution:
               paths for instruments must satisfy the bounds;
```

[22] Of course, if our conjecture about the sequence of regimes is incorrect, we will not find any dates for which the solutions for endogenous variables satisfy the optimality conditions.

```
                    Lagrange multipliers on instruments negative when constraints
bind
                End
            End
        End
    End
```