Intertemporal Optimization Under Uncertainty

Petra M. Geraats

University of Cambridge

This handout applies a formal method for intertemporal optimization under uncertainty to derive the Euler equations that describe the social optimum of a simple real business cycle model. It adopts the approach of *dynamic programming* based on Bellman's *principle of optimality*. This is a useful tool that provides a general framework for dynamic optimization for deterministic and stochastic problems in discrete or continuous time.

1 Model

The economy consists of a representative consumer and a representative firm under perfect competition. The consumer is endowed with 1 unit of time each period, which can be devoted to work L or leisure $1-L$. The consumer exhibits positive, diminishing marginal utility from consumption C and leisure $1-L$ and has a subjective intertemporal discount factor β, where $0<\beta<1$. The expected life-time utility of the consumer equals

$$U_t = \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} u\left(C_s, 1-L_s\right) \tag{1}$$

where $u(C, 1-L)$ denotes the utility function which satisfies $u_C>0$, $u_{CC}<0$, $u_{1-L}>0$ and $u_{1-L,1-L}<0$. Subscripts of variables indicate time, subscripts of functions partial derivatives.

The firm produces output Y using capital K and labor L. The production technology exhibits constant returns to scale and positive, diminishing marginal product of capital K and effective labor AL. Technology A is subject to random shocks. Production can be described by

$$Y_t = F\left(K_t, A_t L_t\right) \tag{2}$$

where $F(K, AL)$ denotes the production function which satisfies $F_K>0$, $F_{KK}<0$, $F_{AL}>0$, $F_{AL,AL}<0$, and homogeneity of degree one.

Furthermore, capital depreciates completely ($\delta = 1$) and there is no government ($G_t = 0$). So, the capital stock is equal to savings in the previous period:

$$K_{t+1} = Y_t - C_t \tag{3}$$

The social planner maximizes the expected life-time utility of the consumer subject to these technological constraints. Notice that this is a model without market failures. By the first welfare theorem, the solution for the social optimum corresponds to the outcome for a market economy.

2 Solution

Formally, the problem faced by the social planner is to find the levels of consumption, $\{C_s\}_{s=t}^{\infty}$, and labor, $\{L_s\}_{s=t}^{\infty}$, that maximize (1) subject to (2) and (3). The highest expected life-time utility U_t that can be achieved with optimal levels of consumption and labor depends on the current capital stock K_t and technology A_t, and it is described by the so called *value function*:

$$\begin{aligned} V\left(K_t; A_t\right) \quad &\equiv \quad \max_{\{C_s, L_s\}_{s=t}^{\infty}} \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} u\left(C_s, 1 - L_s\right) \\ &\qquad \text{subject to } K_{t+1} = F(K_t, A_t L_t) - C_t \end{aligned} \tag{4}$$

An interesting feature of the value function is that it can be written as

$$\begin{aligned} V\left(K_t; A_t\right) \quad &\equiv \quad \max_{C_t, L_t} \left[u\left(C_t, 1 - L_t\right) + \max_{\{C_s, L_s\}_{s=t+1}^{\infty}} \mathrm{E}_t \sum_{s=t+1}^{\infty} \beta^{s-t} u\left(C_s, 1 - L_s\right) \right] \\ &\qquad \text{subject to } K_{t+1} = F(K_t, A_t L_t) - C_t \end{aligned}$$

Notice that the second term of the objective function between brackets equals $\beta \, \mathrm{E}_t \, V\left(K_{t+1}; A_{t+1}\right)$. Hence,

$$\begin{aligned} V\left(K_t; A_t\right) \quad &\equiv \quad \max_{C_t, L_t} \left[u\left(C_t, 1 - L_t\right) + \beta \, \mathrm{E}_t \, V\left(K_{t+1}; A_{t+1}\right) \right] \\ &\qquad \text{subject to } K_{t+1} = F(K_t, A_t L_t) - C_t \end{aligned} \tag{5}$$

This is the *Bellman equation*. It effectively reduces the multi-period optimization to a two-stage problem. First, we optimize current utility taking into account that our choice of C_t and L_t affects future possibilities through K_{t+1}. Second, in the future we also optimize so that our outcome can be summarized by the expected discounted future value function. This important insight in dynamic programming is referred to as the *optimality principle.*

Substituting the constraint on K_{t+1} into the objective function of the Bellman equation (5) we obtain

$$\mathcal{V}\left(C_t, L_t, K_t; A_t\right) \equiv u\left(C_t, 1 - L_t\right) + \beta \, \mathrm{E}_t \, V\left(F(K_t, A_t L_t) - C_t; A_{t+1}\right) \tag{6}$$

Denote the optimal levels for consumption and labor supply which depend on the current capital stock and technology, by $C_t^*(K_t; A_t)$ and $L_t^*(K_t; A_t)$, respectively. So, $V(K_t; A_t) = \mathcal{V}(C_t^*(K_t; A_t), L_t^*(K_t; A_t), K_t; A_t)$.

Then the first-order conditions for (5) are:

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial C_t} &= u_C(C_t^*, 1 - L_t^*) - \beta \, \mathrm{E}_t \, V'(K_{t+1}; A_{t+1}) = 0 \\ &\Rightarrow \; u_C(C_t^*, 1 - L_t^*) = \beta \, \mathrm{E}_t \, V'(K_{t+1}; A_{t+1}) \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial L_t} &= -u_{1-L}(C_t^*, 1 - L_t^*) + \beta \, \mathrm{E}_t \left[V'(K_{t+1}; A_{t+1}) \, A_t F_{AL}(K_t, A_t L_t^*) \right] = 0 \\ &\Rightarrow \; u_{1-L}(C_t^*, 1 - L_t^*) \frac{1}{w_t^*} = \beta \, \mathrm{E}_t \left[V'(K_{t+1}; A_{t+1}) \right] \end{aligned} \tag{8}$$

where $w_t^* = A_t F_{AL}(K_t, A_t L_t^*)$ is the corresponding optimal real wage in a competitive market economy.

Equating (7) and (8) gives the Euler equation for the intratemporal trade-off between consumption and labor supply:

$$u_C(C_t^*, 1 - L_t^*) = u_{1-L}(C_t^*, 1 - L_t^*) \frac{1}{w_t^*}$$

This expression tells us that in the optimum, the utility derived from a marginal unit of consumption equals the utility from the amount of leisure that the consumer must sacrifice to obtain the extra consumption.

The same first order conditions (7) and (8) can be used to derive the intertemporal Euler equations for consumption and leisure. To find the expected value of the marginal value function, $\mathrm{E}_t \, V'(K_{t+1}; A_{t+1})$, we use the Bellman equation (5). First, we take the derivative of the value function,

$$\begin{aligned} V'(K_t; A_t) &= \left. \frac{\partial \mathcal{V}}{\partial C_t} \frac{\partial C_t}{\partial K_t} \right|_{C_t = C_t^*, L_t = L_t^*} + \left. \frac{\partial \mathcal{V}}{\partial L_t} \frac{\partial L_t}{\partial K_t} \right|_{C_t = C_t^*, L_t = L_t^*} + \left. \frac{\partial \mathcal{V}}{\partial K_t} \right|_{C_t = C_t^*, L_t = L_t^*} \\ &= \left. \frac{\partial \mathcal{V}}{\partial K_t} \right|_{C_t = C_t^*, L_t = L_t^*} \end{aligned} \tag{9}$$

using (7) and (8). This is an application of the *envelope theorem*. Starting from the optimum, the total effect $d\mathcal{V}/dK_t$ of a marginal change in the state variable K_t equals the partial effect $\partial \mathcal{V}/\partial K_t$ because the induced changes in the control variables C_t and L_t vanish in the optimum. Using (6), (9) yields

$$\begin{aligned} V'(K_t; A_t) &= \beta \, \mathrm{E}_t \left[V'(K_{t+1}; A_{t+1}) \frac{\partial F(K_t, A_t L_t^*)}{\partial K_t} \right] \\ &= \beta \, \mathrm{E}_t \left[V'(K_{t+1}; A_{t+1}) \right] (1 + r_t^*) \end{aligned} \tag{10}$$

where $r_t^* = \partial F(K_t, A_t L_t^*)/\partial K_t - 1$ is the corresponding optimal real interest rate in a competitive market economy with complete capital depreciation.[1]

Next, we substitute (7) into (10) to obtain

$$V'\left(K_t; A_t\right) = u_C\left(C_t^*, 1 - L_t^*\right)\left(1 + r_t^*\right)$$

Since this equation holds for all periods t, we can write

$$V'\left(K_{t+1}; A_{t+1}\right) = u_C\left(C_{t+1}^*, 1 - L_{t+1}^*\right)\left(1 + r_{t+1}^*\right)$$

Finally, substituting this result into (7) gives

$$u_C\left(C_t^*, 1 - L_t^*\right) = \beta\,\mathrm{E}_t\left[u_C\left(C_{t+1}^*, 1 - L_{t+1}^*\right)\left(1 + r_{t+1}^*\right)\right]$$

This is the stochastic Euler equation for consumption describing the intertemporal trade-off between current and future consumption. In the optimum, the utility derived from a marginal unit of current consumption equals the discounted expected value of the utility from the amount of future consumption that the consumer must sacrifice to obtain the extra consumption today.

Similarly, we can substitute (8) into (10) to obtain

$$V'\left(K_t; A_t\right) = u_{1-L}\left(C_t^*, 1 - L_t^*\right)\frac{\left(1 + r_t^*\right)}{w_t^*}$$

and thereby the stochastic Euler equation for labor supply

$$u_{1-L}\left(C_t^*, 1 - L_t^*\right)\frac{1}{w_t^*} = \beta\,\mathrm{E}_t\left[u_{1-L}\left(C_{t+1}^*, 1 - L_{t+1}^*\right)\frac{\left(1 + r_{t+1}^*\right)}{w_{t+1}^*}\right]$$

It shows that in the optimum, the utility derived from the leisure obtained by giving up a marginal unit of current income is equal to the discounted expected value of the utility from the amount of future leisure that the consumer must sacrifice to enjoy the extra leisure today.

[1] Strictly speaking, this presumes certainty equivalence.