ELSEVIER
Journal of Mathematical Economics 30 (1998) 1–35
JOURNAL OF Mathematical ECONOMICS

Repeated games with *M*-period bounded memory (pure strategies)

Hamid Sabourian [*]

King's College, Cambridge CB2 1ST, UK

Received 28 December 1989; accepted 23 October 1997

Abstract

This paper provides a characterization for the set of outcomes which can be sustained by subgame perfect equilibrium strategies in repeated games with *M*-period bounded memory, pure strategies, no discounting and finite number of action profiles. The characterization indicates that the equilibrium set expands very fast (in an 'exponential' fashion) as the memory increases and slowly (in a 'polynomial' fashion) as the set of action profiles available to the players at each stage expands. © 1998 Elsevier Science S.A. All rights reserved.

JEL classification: C72; C73; C79

Keywords: Repeated games; Bounded rationality; Memory

---

1. Introduction

In any repeated game (RG), a large number of outcome paths can be supported as subgame perfect equilibria because the players can choose history-dependent strategies which punish any player who does not adhere to the appropriate equilibrium outcome path. Therefore the possibility of punishments and thus the existence of a large number of equilibria arises because players can choose history-dependent strategies. Moreover, each player rationally chooses a history-

---

[*] Corresponding author. Tel.: + 44-1223368265; fax: + 44-1223335475; e-mail: hs102@econ.cam.ac.uk.

PII S0304-4068(97)00061-X

dependent strategy because the others condition their actions on the history. Since there are a large number of histories, one can construct many subgame perfect equilibrium (SPE) strategies (some of them extremely complicated).[1]

One research strategy for dealing with the large number of equilibria in RGs is to impose restrictions on the way the strategies depend on the history. One justification for imposing restrictions on RG strategies is in terms of limitations of human computational and storage ability ('bounded rationality'). For example, in recent years many have considered RG strategies which are implementable by finite automata. (See Kalai, 1990 for a survey). In this paper, however, the bound is on the memory of the players (the number of periods they can recall) and not on their computational abilities. There are simply limits to storage and organization of information.

In Sabourian (1989b), a series of sufficiency conditions for an outcome to be a one-period memory SPE is provided. Moreover, the paper demonstrates a Folk-theorem-type result: with no discounting any individually rational profile of actions can be supported as a SPE with one-period memory if either the stage game contains a 'large' number of elements (if it is a connected set) and/or if randomization (public or private) is allowed.

When the number of actions available in the stage game is small, restricting the memory does reduce the size of pure SPE set (and captures some aspects of bounded rationality). This paper provides a characterization for the set of M-period memory pure SPE strategies when the set of actions is finite.[2] The characterization (which is a generalization of the one obtained in Sabourian (1989b) for the one-memory case) indicates that in obtaining Folk-Theorem-type results (without randomization) one faces a trade-off between the number of elements of the action space of the stage game and the length of the memory. One needs either a long memory or an action space with 'many' elements to establish a Folk-Theorem result. Moreover, the results indicate that the size of SPE set expands very fast (in an 'exponential' fashion) as the memory increases and slowly (in a 'polynomial' fashion) as the action space expands.

2. The game

Consider the one-shot game $G=(A_i \cdot \pi_i)_{i=1}^{n}$ where A_i and π_i are the strategy (action) space and the payoff function of player i, respectively. Let $A=\prod_{i=1}^{n} A_i$ and $\pi=(\pi_1, \dots, \pi_n)$. Thus π_i is a function from A to $\mathbb{R}$. I assume that A_i is finite $\forall i$. I define x_{-i} as any n-tuple $x=(x_1, \dots, x_n)$ without its ith element.

---

[1] See Pearce (1992) and Sabourian (1989a) for surveys of the literature.

[2] The sufficiency condition in this paper does not depend on the finiteness assumption and is proved by constructing appropriate Eulerian graphs.

For any action profile $a \varepsilon A$, define the maximum payoff i can obtain by deviating as

$$u_i(a) = \max_{a'_i \in A_i} \pi_i(a'_i, a_{-i}). \tag{1}$$

Consider the infinite repetition of G. Let the set of histories up to period t be H^t. Thus $H^1 = \emptyset$ and $H^t = A^{t-1}$, where $\emptyset$ is the empty set and A^{t-1} is the $(t-1)$-fold cartesian product of A. The set of supergame strategies of i, when there is no bound on the memory, is denoted by F_i with typical element $f_i = \{s_i^t(\cdot)\}_{t=1}^{\infty}$, where $s_i^t: H^t \to A_i$. I denote $(s_1^t(h^t), \dots, s_n^t(h^t))$ by $s^t(h^t)$ and $\prod_{i=1}^n F_i$ by F.

Definition 1: A supergame strategy $f_i = \{s_i^t(\cdot)\}_{t=1}^{\infty}$ has M-period memory (M is an integer) if for all integers t and τ greater than M and for any two histories $h^t = (a^1, \dots, a^{t-1})$ and $\bar{h}^\tau = (\bar{a}^1, \dots, \bar{a}^{\tau-1})$, if $a^{t-m} = \bar{a}^{\tau-m}$ $\forall$ integers $m \le M$ then $s_i^t(h^t) = s_i^\tau(\bar{h}^\tau)$.

Thus the choice of action at any stage t for M-period memory strategies depends only on the last M stages of the supergame and not on t (stationary bounded memory). Denote the set of M-period memory supergame strategies for i by F_i^M and let $F^M = \prod_{i=1}^n F_i^M$.

Given any profile of supergame strategies f, $\sigma^1(f) = s^1(\emptyset)$ denotes the outcome at time $t = 1$ and $\sigma^t(f) = s^t(\sigma^1(f), \dots, \sigma^{t-1}(f))$ denotes the outcome at $t > 1$. Moreover, I shall denote the sequence $(\sigma^t(f))_{t=1}^{\infty}$ by $\sigma(f)$ and refer to it as the outcome path. Any $f \in F$ is said to be stationary if $\sigma^t(f) = \sigma^{t'}(f)$ $\forall t$ and t'.

Given any $f = (s^t(\cdot))_{t=1}^{\infty}$ the supergame strategies determined by f after any history $h^t = (a^1, \dots, a^{t-1})$ is denoted by $f|h^t$.

I shall assume players do not discount the future and use the Limit-of-the-Mean criterion. Thus the supergame payoff of i if the sequence of action profiles $h = (a^1, a^2, \dots)$ occurs is $\pi_i^\infty(h) = \lim_{T\to\infty}\inf \frac{1}{T}\sum_{t=1}^{T}\pi_i(a^t)$ (it makes no difference which limit point of the average payoffs is used).

Definition 2: A supergame strategy $f \in F^M$ is an M-memory SPE if

$$\pi_i^\infty\left(\sigma\left(f|h^t\right)\right) \ge \pi_i^\infty\left(\sigma\left(\tilde{f}_i, f_{-i}|h^t\right)\right), \forall i, \forall h^t, \text{and } \forall \tilde{f}_i \in F_i^M. \tag{2}$$

Since in the above definition $f_{-i}|h^t$ has M-period memory it follows from standard theorems on Markov decision problems that there exists an optimal strategy (in the class of all strategies) for player i, given $f_{-i}|h^t$, which has M-period memory. Thus I could define a M-memory SPE as a profile of strategies $f \in F^M$ such that the inequality in Eq. (2) holds $\forall \tilde{f}_i \in F_i$ (and not only $\forall \tilde{f}_i \in F_i^M$ as in Eq. (2)). Since $F^M \subseteq F^{M'}$ for all $M' > M$, if $f \in F^M$ is an M-period SPE then f is also an M'-period SPE for any $M' > M$.

For any finite path (string) of action profiles $c=(a^1,\ \dots,\ a^R)\in A^R$, denote the mth element $(m\leq R)$ of the path and the average payoff player i receives by c^m and $\overline{\pi}_i(c)$, respectively. Thus $c^m=a^m$ and

$$\overline{\pi}_i(c)=\frac{1}{R}\sum_{r=1}^{R}\pi_i(a^r) \tag{3}$$

Since M-period memory strategies depend only on the previous M-period history, for any action path $c=(a^1,\ \dots,\ a^R)\in A^R$, I need to define the M-period history cycle (path) induced by c (the set of adjacent strings of action profiles of length M in c). Formally, define the M-period history cycle and the M-period history path induced by c as $\hat{c}=(e(1),\ \dots,\ e(R))$ and $\bar{c}=(e(1),\ \dots,\ e(R-M))$ respectively, where

$$e(r)=(a^r,\dots,a^{r+M-1})\ \text{for every integer}\ r\leq R, \tag{4}$$

and $a^{R+\tau}=a^{\tau}$ for any integer $\tau<M$. Following the computer science literature (see Hall, 1968 or Golomb, 1967), I shall refer to the M-period history cycle (path) as the M-string shift register cycle (path) induced by c or more simply by SR cycle (path) induced by c. Clearly, for any adjacent pair of strings of length M in $\hat{c}$ (or in $\bar{c}$), the last $(M-1)$ elements of the first string coincide with the first $(M-1)$ elements of the second string. Notice also that the sum of the payoffs i obtains when any path $c=(a^1,\ \dots,\ a^R)$ is chosen is equal to the sum of the *average* payoffs i obtains when all elements of $\hat{c}=(e(1),\ \dots,\ e(R))$ occur. Thus

$$\sum_{r=1}^{R}\pi_i(a^r)=\frac{1}{M}\sum_{r=1}^{R}\sum_{m=1}^{M}\pi_i(e^m(r))=\sum_{r=1}^{R}\overline{\pi}_i(e(r))\,\cdot\text{[3]} \tag{5}$$

Since A and M are finite, the path $\sigma(f)$ induced by any M-memory strategy profile $f\varepsilon F^M$ is ultimately periodic, and the length of any such cycle does not exceed $|A|^M$.[4] For any $f\in F^M$, I shall refer to the cycle paths generated by f by $\Sigma(f)$. Note that if $\Sigma(f)=(a^1,\ \dots,\ a^R)$ and if $(e(1),\ \dots,\ e(R))$ is the SR cycle induced by $\Sigma(f)$, then $\pi_i^{\infty}(f)=\overline{\pi}_i(\Sigma(f))=\frac{1}{R}\sum_{r=1}^{R}\pi_i(e(r))$ and for every pair of integers r and r' less than R, $e(r)\neq e(r')$ if $r\neq r'$. (Otherwise f enters a cycle before playing all members of $\Sigma(f)$.) I can now define the set of cycle paths that are implementable by M-memory strategy profiles as

$$\begin{aligned}C^M=\Big\{&(a^1,\dots,a^R)\,\varepsilon A^R|e(r)\neq e(r'),\ \forall r\neq r'\leq R\ \text{where}\ e(r)\ \text{and}\\ &\times e(r')\ \text{satisfy equation 4}\Big\}\end{aligned} \tag{6}$$

Finally, note also that any path of action profiles which appear along an outcome path generated by M-memory strategy profile must be such that the elements of its

---

[3] The first equality follows from the fact that any $a^r\in c$ occurs M times in $\hat{c}$. Notice that $e^m(r)$ refers to the mth element of the M-string $e(r)$.

[4] For any set K, $|K|$ refers to the cardinality of that set.

SR path, other than the last $M-1$ elements, do not coincide. Therefore I define the set of such paths as

$$\overline{C}^M = \{(a^1, \dots, a^R) \varepsilon A^R | e(r) \neq e(r'), \forall r \neq r' \leq R - M \text{ where } e(r) \text{ and } e(r') \text{ satisfy equation } 4\} \tag{7}$$

3. Some intuition for the sufficiency condition

To support an outcome path as a SPE of a RG one needs to construct strategies which punish deviations from the equilibrium path and induce punishers to punish the deviator. In some cases it may be more costly for player j to punish a deviator i than to be punished. As a result, in order to induce j to punish i one needs to 'reward' j after punishing i. Therefore, in general (see Sabourian, 1989a) one may need to construct *finite punishment paths* which punish any deviator a finite number of times and return to the 'reward' phase (which may be the original equilibrium path).

To sustain an action profile $\bar{a} \in A$ as a stationary M-memory SPE, I consider a supergame strategy profile $f \in F$ which plays $\bar{a}$ forever on the equilibrium path and punishes any deviator i by first playing a *finite* sequence of action profiles $c(i)$ and then returns to $\bar{a}$ forever. Further deviations, by player j from $c(i)$ are punished by starting $c(j)$ and then reverting to $\bar{a}$ forever (thus the sequence $c(i)$ is the punishment phase for i and $\bar{a}$ is the 'reward phase'). To show that such a strategy profile $f=(f_i, f_{-i})$ is a M-memory SPE note the following two points. Firstly, with the Limit-of-the-Mean criterion one only needs to show that it does not pay i to deviate *infinitely* often from f_i, given f_{-i}, after any history. Since after every history all players who are adhering to f are either playing $\bar{a}$ or $c(j)$ for some j, it follows that f is a SPE if it does not pay i to deviate infinitely often from $\bar{a}$ or from $c(j)$ $\forall j$. But one does not need to consider infinite deviations by i from $c(j)$ for $j \neq i$. This is because if i deviates once from $c(j)$, given that other players follow f_{-i}, $c(i)$ will be invoked and any further deviations will be from either $c(i)$ or $\bar{a}$. Thus given that others follow f_{-i}, i can not deviate infinitely often from $c(j)$ when $j \neq i$. Therefore f is a SPE if for every i it does not pay i to deviate infinitely often either from $\bar{a}$ or from $c(i)$. (See Remark 4 below.) Secondly, with M-period memory to construct strategies which punish a deviator for a finite number of periods and then return to the 'reward phase', the elements of the SR path (M-period histories/strings)) induced by the punishment phase $c(i)$ need to be distinct—$c(i)$ must be an element of C^M (or $\overline{C}^M$); otherwise, the punishment phase enters a cycle and never returns to the 'reward phase' $\bar{a}$.

To find the weakest conditions which ensure that an action profile $\bar{a}$ can be sustained as a M-memory SPE, one needs to find for all i the worst finite credible punishment phase $c(i)$ belonging to $\overline{C}^M$. To do this, consider first any set of distinct action profiles $B(i)=\{b^r(i)\}_{r=1}^{R(i)}\subseteq A$ such that

$$\pi_i(\bar{a})\geq\pi_i(b^r(i)),\forall r\leq R(i) \tag{8}$$

and find the longest *cycle* path which is implementable with *M*-memory and involves action profiles belonging to the set $B(i)$. Clearly, such a path belong to C^M and the elements of its *M*-string SR cycle belong to $B(i)^M$—the *M*-fold cartesian product of $B(i)$. It turns out that there exists a cycle path which can be implementable by a *M*-memory strategy profile and whose SR cycle coincide with the set of *all* *M*-strings belonging to the set $B(i)^M$. Such a maximal cycle path are called de Bruijn sequences and the existence of them was first proved by Good (1946) and de Bruijn (1946). Clearly, the length of such a maximal sequence is the cardinality of $B(i)^M$ $(=R^M)$ and any element of $B(i)$ occurs R^{M-1} times in such a maximal path (this is because every element of $B(i)$ occurs an equal number of times).

Now suppose that the punishment phase $c(i)$ consists of the above maximal sequence and that it starts by playing $b^1(i)$, M times and then plays every $b^\tau(i)$, R^{M-1} times (as suggested above). Clearly, if after playing the last element of $c(i)$ the profile $b^1(i)$ is played M times the cycle restarts. If, on the other hand, after playing the last element of $c(i)$ the profile $b^1(i)$ is played $(M-1)$ times followed by $\bar{a}\notin B(i)$, the cycle does not restart and the path can return to $\bar{a}$. Thus the punishment path $c(i)$ can consist of playing every element of $B(i)$, R^{M-1} times followed by playing $b^1(i)$ an extra $(M-1)$ times.[5] Clearly with the above punishment path it does not pay i to deviate infinitely often from $\bar{a}$ if

$$\pi_i(\bar{a})\geq\frac{1}{M+R^M}\left\{u_i(\bar{a})+(M-1)\pi_i\big(b^1(i)\big)+R^{M-1}\sum_{\tau=1}^{R}\pi_i(b^\tau(i))\right\} \tag{9}$$

Condition 9 says that i prefers $\bar{a}$ to the average payoff associated with deviating from $\bar{a}$ and then facing the punishment sequence $c(i)$ when the latter consists of playing the maximal de Bruijn sequence associated with the set $B(i)$ followed by $b^1(i)$ played $M-1$ times.

To ensure that i does not deviate from $c(i)$ described above (and thus the strategy profile f is SPE), consider first the case of one-memory. In this case $c(i)$ consists of playing each element of $B(i)$ once. Thus if i deviates from say $b^r(i)$ at time t, then the average payoff i obtains between the last time he deviated before

---

[5] In other words, the punishment path $c(i)$ need to belong to the set of *M*-memory paths $\overline{C}^M$ and not necessarily to the set of cycle paths C^M. As a result, to implement $c(i)=(a^1,\ \dots,\ a^K)$ by a *M*-memory strategy profile one needs all the *M*-period adjacent strings $e(k)$ to be distinct for every $k\leq K-M$.

t and t is $\frac{1}{r}\{u_i(b^r(i)) + \sum_{\tau=1}^{r-1}\pi_i(b^\tau(i))\}$. Thus it does not pay to deviate infinitely often from $c(i)$ if

$$\pi_i(\bar{a}) \geq \frac{1}{r}\left\{u_i(b^r(i)) + \sum_{\tau=1}^{r-1}\pi_i(b^\tau(i))\right\} \tag{10}$$

In the M-memory case, the above condition for i not deviating from $c(i)$ can be weakened to $\forall r \leq R$

$$\pi_i(\bar{a}) \geq \frac{1}{M+(r-1)^M}\left\{u_i(b^r(i)) + (M-1)\pi_i(b^1(i)) \right.$$
$$\left. +(r-1)^{M-1}\sum_{\tau=1}^{r-1}\pi_i(b^\tau(i))\right\} \tag{11}$$

where the RHS of the above stands for $u_i(b^1(i))$ if $r=1$. The RHS of condition 11 is the same as that of condition 10 except that every $b^\tau(i)$ $(\tau \leq r-1)$ occurs $(r\text{-}1)^{M-1}$ times on the RHS of condition 11 rather than once as in condition 10 and there is the extra term $(M-1)\pi_i(b^1(i))$. This weakening can be done by playing the de Bruijn sequence for the M-strings belonging to $B(i)^M$ described above in the following sequence of phases: in the first phase play all the M-strings belonging to $(b^1(i))^M$, in the second phase play all the M-strings belonging to $(b^1(i), b^2(i))^M$ not already played and so on until all M-strings of $B(i)^M$ are chosen once (this is done in the $R(i)$th phase); also play $b^1(i)$, $M-1$ times at the end of each phase. Such a punishment phase insures that if a player deviates from $b^r(i)$ at time t, then the average payoff i obtains between the last time before t at which i deviated and t does not exceed the RHS of condition 11. It turns out that if $B(i)$ satisfies condition 8, conditions 9 and 11 are the weakest conditions which ensure that the above strategy profile is a SPE.

Notice that so far I have assumed that i prefers $\bar{a}$ to the elements of $B(i)$ (condition 8). One may be able to make the finite punishment phase longer (and possibly worse from i's point of view), without entering a cycle, by allowing action profiles which make i better off than at $\bar{a}$. For example, suppose for some $a' \varepsilon A$, $\pi_i(a') > \pi_i(\bar{a})$ and $(M-1)\pi_i(b^1) + \pi_i(a') < M\pi_i(\bar{a})$. Then the punishment phase can be made longer (and possibly worse from i's point of view) by playing a new M-string consisting of a' followed by $b^1(i)$, played $(M-1)$ times. Thus it seems that one may be able to weaken conditions 9 and 11 by considering all M-strings which generate an average payoff less than $\pi_i(\bar{a})$. In Section 4, it is shown that there exists M-memory strategy profile f with finite punishment path $c(i)$ such that the sum of the payoffs of the path $c(i)$ is equal to $(M-1)\pi_i(b^1(i))$ plus the sum of the average payoff of *all* M-strings belonging to any set $B(i)^M$ and whose average payoff are less than $\pi_i(\bar{a})$. Formally, define for any i, for any

π_i and for any finite set of action profiles $B \subseteq A$, the set of M-strings belonging to B^M whose average payoff is less than π_i as follows

$$E(\pi_i, B) = \left\{ e \in B^M | \overline{\pi}_i(e) < \pi_i \right\}. \tag{12}$$

The finite punishment path $c(i)$ of length K that is constructed below consists of playing a sequence of action profiles belonging to some subset $B(i) = \{b^{\tau}(i)\}_{\tau=1}^{R(i)}$ of A such that $K\overline{\pi}_i(c(i)) = (M-1)\pi_i(b^1(i)) + \sum_{e \in E(\pi_i, B(i))} \overline{\pi}_i(e)$. To ensure that it does not pay i to deviate from $c(i)$, when a condition similar to condition 11 is satisfied, $c(i)$ is constructed such that it consists of playing $b^1(i)$ for the first $(M-1)$ periods followed by playing a sequence of actions involving $R(i)$ different phases. The sequence of actions in each phase $r \leq R(i)$ is such that the elements of the SR cycle induced by the action profiles in this phase are the same as those of the set $\{e \in E(\pi_i, B_r(i)) | e \notin E(\pi_i, B_{r-1}(i))\}$ where

$$B_r(i) = \{b^{\tau}(i)\}_{\tau=1}^{r}, \forall r \leq R(i) \tag{13}$$

Since $E(\pi_i, B_{r-1}(i)) \subseteq E(\pi_i, B_r(i))$, the above construction of $c(i)$ is such that the sum of the payoff i receives when $c(i)$ is played is $(M-1)\pi_i(b^1(i)) + \sum_{e \in E(\pi_i, B(i))} \overline{\pi}_i(e)$.[6] In Section 4, it is shown that $c(i)$ described above can be implemented by a M-memory strategy profile.

4. Sufficiency conditions for stationary equilibria

Before turning to the formal statement of the Theorem, note that if the punishment paths for different players differ, to implement the above strategy profile with M-memory, at any time the players, by observing the past, must also be able to deduce who, if any, they should punish. Thus the players need to know who was the last deviator. With unbounded memory this is always possible. With bounded memory this is possible if elements of the following phases differ: the equilibrium phase $\bar{a}$, the punishment paths $c(j)$ $(j = 1, \ldots, n)$, and phases involving deviations from $\bar{a}$ or $c(j)$. Formally, let $X(i)$ be the set of action profiles involving deviations by i from the above strategy profile (from $\bar{a}$ or $c(j)$ for all $j = 1, \ldots, n$); thus for any $\bar{a} \varepsilon A$ and for any set $B(i) = \{b^{\tau}(i)\}_{\tau=1}^{R(i)} \subseteq A$ let

$$X(i) = \left\{ a \in A | a_{-i} = \bar{a}_{-i} \text{ and } a_i \neq \bar{a}_i \right\} \cup \overline{X}(i) \tag{14}$$

where $\overline{X}(i) \equiv \{a \in A | a_{-i} = b_{-i}^r(j)$ and $a_i \neq b_i^r(j)$ for some integers $j \leq n$ and $r \leq R(j)\}$. The following condition ensures that players can, when necessary,

---

[6] Note that the set $B(i)$ is the same as the set $B_{R(i)}(i)$.

deduce the current phase of the game and thus they can implement the strategy profile described in Section 3 with bounded memory:

$$\left.\begin{array}{ll} \forall i, j, \tau \text{ and } \tau' \text{ if } b^{\tau}(i) = b^{\tau'}(j) \Rightarrow B(i) = B(j) & \text{(a)} \\ \forall i, j \text{ and } \tau, b^{\tau}(j) \notin X(i) & \text{(b)} \\ \forall i \neq j, \text{ if } X(i) \cap X(j) \text{ is not empty} \Rightarrow B(i) = B(j) & \text{(c)} \end{array}\right\} \tag{15}$$

Condition 15a is assumed because with M-memory strategies if $b^{\tau}(i) = b^{\tau'}(i)$, then the outcome path which can be implemented after observing $b^{\tau}(i)$ M times and after observing $b^{\tau'}(j)$ M times have to be the same. Condition 15b is assumed so that players, by (only) observing the last period action profiles, can determine whether there has been a deviation in the previous period or not (if $b^{\tau}(j) \in X(i)$, then the action profile $b^{\tau}(j)$ is indistinguishable from an outcome which involves player i deviating from the prescribed strategy after some history). Condition 15c is assumed so that if players, by observing the last period's actions, can not determine whether i or j has deviated ($X(i) \cap X(j)$ is not empty), then the punishment path for both players has to be the same—$B(i) = B(j)$ (see Sabourian, 1989b Remark 1 and also Remark 3 below for a further discussion of this condition).

I can now state the first sufficiency result.

Theorem 1: *Let* $\pi = \pi(\bar{a})$ *for some action profile* $\bar{a} \in A$. *Then* $\pi \varepsilon \mathbb{R}^n$ *can be supported as a M-memory SPE if* $\forall i$ *there exists a set of action profiles* $B(i) = \{b^{\tau}(i)\}_{\tau=1}^{R(i)} \subseteq A$ *s.t.* $\bar{a} \notin B(i)$ *and*

$$\text{(i) } \pi_i \geq \frac{1}{M + |E(\pi_i, B(i))|}\left\{u_i(\bar{a}) + (M-1)\pi_i(b^1(i)) + \sum_{e \in E(\pi_i, B(i))} \bar{\pi}_i(e)\right\} \tag{16}$$

$$\text{(ii) } \pi_i \geq u_i(b^1(i)) \tag{17a}$$

$$\pi_i \geq \frac{1}{M + |E(\pi_i, B_{r-1}(i))|}\left\{u_i(b^r(i)) + (M-1)\pi_i(b^1(i)) + \sum_{e \in E(\pi_i, B_{r-1}(i))} \bar{\pi}_i(e)\right\} \forall r > 1 \tag{17b}$$

(iii) Condition 15 holds.

Condition (i) (Eq. (16)) says that the payoff i obtains when $\bar{a}$ is chosen $M + |E(\pi_i, B(i))|$ number of times is no less than the payoff i obtains from deviating from $\bar{a}$ and then receiving the payoff when $b^1(i)$ is chosen $(M-1)$ times plus the sum of the average payoffs of all elements of $E(\pi_i, B(i))$. This condition will ensure that it would not pay i to deviate infinitely often from $\bar{a}$. Similarly, condition (ii) (Eqs. (17a) and (17b)) says the payoff i obtains when $\bar{a}$ is chosen $M+|E(\pi_i,B_{r-1}(i))|$ number of times is no less than the payoff i obtains from deviating from b^r and then receiving the payoff when $b^1(i)$ is chosen $(M-1)$ times plus the sum of the average payoffs of all elements of $E(\pi_i,B_{r-1}(i))$. If the elements of the punishment path $c(i)$ occurs in a certain order (to be described below), this condition will ensure that it would not pay i to deviate infinitely often from the $c(i)$ phase. The reason for making condition (iii) was given in the previous paragraph—namely to make the required strategy profile well-defined.

Point of notation: Henceforth, I shall refer to the set $E(\pi_i,B_r(i))$ and its cardinality by $E(\pi_i,r)$ and $t(i,r)$, respectively. I shall also define $t(i,0)$ to be equal to zero.

As was mentioned before, to ensure that it does not pay i to deviate from the punishment path, $c(i)$ is constructed as follows. For any set $B(i)\subseteq A$, the punishment path for i will involve (for reasons which will become clear shortly) playing a sequence of actions $c(i)=(\sigma^1,\ \dots,\ \sigma^K)\in A^K$ where $K=t(i,R(i))+M-1$ s.t. $c(i)\varepsilon\overline{C}^M$ (it is implementable by a M-memory strategy profile) and for any integer $r\leq R(i)$

$$\text{(a)}\ \sigma^{t(i,r-1)+M}=b^r \text{ and } \sigma^t\varepsilon B_r(i) \text{ for all } t<t(i,r)+M$$

$$\text{(b)}\ \sum_{\mu=1}^{t(i,r)+M-1}\pi_i(\sigma^\mu)=(M-1)\pi_i\big(b^1(i)\big)+\sum_{e\in E(\pi_i,r)}\overline{\pi}_i(e)$$

$$\text{(c) if } \sigma^t=b^r(i) \text{ and } t>t(r-1,i)+M \text{ then } \sum_{\mu=t(i,r-1)+M}^{t-1}\pi_i(\sigma^\mu)$$

$$\leq(t-t(i,r-1)-M)\pi_i.$$

Since $c(i)\varepsilon\overline{C}^M$, condition (a) says that the path $c(i)$ consists of playing b^1 for the first $M-1$ periods followed by $R(i)$ different phases: each phase $r\leq R(i)$ begins by playing b^r at period $t(i,r-1)+M$, it ends at period $(t(i,r)+M-1)$ and it involves playing elements of the set B_r. Given that the path $c(i)$ consists of playing b^1 for the first $M-1$ periods, condition (b) implies that the sum of the payoff that i receives from the beginning of phase 1 (from period M) to the end of phase r is equal to $\sum_{e\in E(\pi_i,B_r)}\overline{\pi}_i(e)$.. Condition (c) says that the average payoff to i between the first time $b^r(i)$ occurs (at period $t(i,r-1)+M$) and any other time it occurs on the path is less than π_i. Informally, the reason I need $c(i)$ to satisfy (a)–(c) is as follows. Given Eqs. (16), (17a) and (17b), $c(i)$ should be such that i receives a payoff not greater than the RHS of Eq. (16) if he deviates (infinitely often) from $\bar{a}$ and a payoff not greater than the RHS of Eqs. (17a) and (17b), if he deviates (infinitely often) from $b^r(i)$. Condition (b) implies that the punishment path is such that after a deviation by i, he receives a payoff $(M-1)\pi_i(b^1(i)) + \sum_{e\in E(\pi_i,B(i))}\overline{\pi}_i(e)$ during the punishment period. Thus given Eq. (16), it does not pay i to deviate infinitely often from $\bar{a}$.

Now, consider the case in which player i deviates infinitely often from say σ^t during the punishment phase $c(i)$. Suppose that $\sigma^t = b^r(i)$, then it follows from condition (a) that $t \geq t(i,r-1)+M$. Denote the maximum payoff that i gets in this punishment phase up to and including the period of deviation t (the maximum payoff i obtains between the last time i deviated and t) by $D \equiv \sum_{\mu=1}^{t-1}\pi_i(\sigma^\mu) + u_i(b^r(i))$. If this sum is less than $t\pi_i$ then it does not pay i to deviate infinitely often from $\sigma^t = b^r(i)$ (and thus from $c(i)$). But

$$\begin{aligned}
D &\equiv \sum_{\mu=1}^{t(i,r-1)+M-1} \pi_i(\sigma^\mu) + \sum_{\mu=t(i,r-1)+M}^{t-1} \pi_i(\sigma^\mu) + u\big(b_i^r(i)\big)\\
&= (M-1)\pi_i\big(b^1(i)\big) + \sum_{e\in E(\pi_i,r-1)} \overline{\pi}_i(e)\\
&\quad + \sum_{\mu=t(i,r-1)+M}^{t-1} \pi_i(\sigma^\mu) + u_i(b^r(i))\\
&\leq (M-1)\pi_i\big(b^1(i)\big) + \sum_{e\in E(\pi_i,r-1)} \overline{\pi}_i(e) + (t-t(i,r-1)-M)\pi_i\\
&\quad + u_i(b^r(i)) \leq t\pi_i
\end{aligned}$$

The equality in the above follows from condition (b) above, the first inequality follows from condition (c) above and the second inequality follows from Eqs. (17a) and (17b). Thus, very informally, (a)–(c) and Eqs. (17a) and (17b) ensure that it does not pay i to deviate during the punishment paths $c(i)$. (Since players do not discount the future and a single deviation from $c(j)$ invokes $c(i)$ phase, I do not need to consider deviations of i from $c(j)$, for $j \neq i$—see Remark 4 below for a further discussion of this point.)

The difficulty in the proof of Theorem 1 is to show that for all i there exists a path $c(i) = (a^1, \ldots, a^K)$ which is implementable by a M-memory strategy profile and satisfies the (a)–(c) above. The next Lemma demonstrates the existence of such a path.

Lemma 1: *For any finite set* $B_R = \{b^t\}_{t=1}^R \subseteq A$ *and for any* $r \leq R$, *let* $B_r = \{b^t\}_{t=1}^r$, $E(\pi_i,r) = \{e \in (B_r)^M | \overline{\pi}_i(e) < \pi_i\}$ *and* $t(r) = |E(\pi_i,r)|$. *Then there exists an outcome path* $c = (\sigma^1, \ldots, \sigma^{t(R)+M-1})$ *such that* $c \in \overline{C}^M$ *and* $\forall r \leq R$ *if* $E(\pi_i,r)$ *is not empty then:*

(a) $\sigma^{t(r-1)+M} = b^r$ *and* $\sigma^t \varepsilon B_r$ *for all* $t < t(r)+M$;

(b) the SR cycle induced by $(\sigma^{t(r-1)+M}, \ldots, \sigma^{t(r)+M-1})$ *is equal to* $G(\pi_i,r)$,[7] *where* $G(\pi_i,r)$ *is defined by*

$$G(\pi_i,P) \equiv \{e\varepsilon E(\pi_i,r) | e \notin E(\pi_i,r-1)\} \tag{18}$$

(c) If $\sigma^t = b^r$ *for some t s.t.* $t(r-1)+M < t < t(R)+M$ *then* $\sum_{\mu=t(r-1)+M}^{t-1} \pi_i(\sigma^{\mu})) \leq (t - t(r-1) - M)\pi_i$.

Remark 1: *Clearly,* $\{G(\pi_i,1), \ldots, G(\pi_i,r)\}$ *is a partition of the set* $E(\pi_i,r)$. Therefore, if conditions (a) and (b) of Lemma 1 are satisfied for all $r' \leq r$ then

$$(\mathrm{b}') \quad \sum_{\tau=1}^{t(i,r)+M-1} \pi_i(\sigma^{\tau}) = (M-1)\pi_i(b^1) + \sum_{e \in G(\pi_i,1)} \overline{\pi}_i(e) + \ldots$$

$$+ \sum_{e \in G(\pi_i,r)} \overline{\pi}_i(e) = (M-1)\pi_i(b^1) + \sum_{e \in E(\pi_i,r)} \overline{\pi}_i(e)$$

The proof of Lemma 1 can be found in Appendix A; here, I shall provide a sketch of the proof.

Sketch of the Proof of Lemma 1: The proof is established by induction. Demonstrating the result when $R=1$ (B_R is a singleton set) is trivial. The next step is to show that if there exists a path $\bar{c} = (\sigma^1, \ldots, \sigma^{t(P-1)+M-1}) \in \overline{C}^M$ which satisfies Lemma 1 for the set $B_{P-1} = \{b^r\}_{r=1}^{P-1}$ for any $P \leq R$, then there exists a path $c \in \overline{C}^M$ which satisfies Lemma 1 for the set $B_P = \{b^r\}_{r=1}^{P}$.

The required path c for the set B_P is obtained by amalgamating of $\bar{c} \in \overline{C}^M$ with an outcome path $d = (a^1, \ldots, a^T) \varepsilon \overline{C}^M$ as follows. The new path c has the same sequence of outcomes as that of the path $\bar{c}$ up to period $t(P-1)+M$ and then plays the sequence $d = (a^1, \ldots, a^T)$. Thus $c = (\bar{c}, d)$. To ensure that c satisfies the conditions of Lemma 1, d is constructed so that it satisfies certain conditions. Firstly, the elements of the SR cycle induced by d, denoted by $\hat{d}$, has to be the same as those of the set $G(\pi_i,P)$:

$$\hat{d} = G(\pi_i,P) \tag{19}$$

Since $\bar{c}$ satisfies conditions (a) and (b) of the Lemma for the set B_{P-1}, the above condition together with setting $a^1 = b^P$ ensure that c satisfies conditions (a) and (b) of the Lemma for the set B_P.

Secondly, to ensure that the new path can be implemented by a M-memory strategy profile ($c\varepsilon\overline{C}^M$), d needs to be implementable by a M-period memory strategy profile and the elements of the SR path induced by $\bar{c}$ and d should not coincide. However, these two conditions (together with $\bar{c}\varepsilon\overline{C}^M$) are not sufficient

---

[7] Two sets are equal if they have identical elements.

to ensure that $c \varepsilon \overline{C}^M$. This is because for some $m < M$ the M-period sequence $(\sigma^{t(P-1)+m}, \ldots, \sigma^{t(P-1)+M-1}, a^1, \ldots, a^m)$ may occur more than once on the SR path induced by $c = (\bar{c}, d)$. Thus c has to be such that if $\hat{d} = (e(1), \ldots, e(T))$ then

$$e(t) \neq (\sigma^{t(P-1)+m}, \ldots, \sigma^{t(P-1)+M-1}, a^1, \ldots, a^m), \forall m < M, \forall t \leq T - M \tag{20}$$

Finally, to ensure that c satisfies condition (c) of Lemma 1 for the set B_P, the path d needs to satisfy

$$\text{if } a^t = b^P \text{ for some } t > 1, \text{ then } \sum_{\mu=1}^{t-1} \pi_i(a^\mu) \leq (t-1)\pi_i \tag{21}$$

Lemma A3 in Appendix A demonstrates the existence of a sequence of action profiles d which is implementable by a M-memory strategy profile and satisfies conditions 19–21. Notice that d satisfies Eq. (19) if the SR path it induces traverses every element of $G(\pi_i, P)$ once and only once. The existence of such a path can be demonstrated by constructing a directed Eulerian graph [8] such that the edges of the graph correspond to elements of $G(\pi_i, P)$. The existence of such a SR path is only sufficient to guarantee condition 19. In order for conditions 20 and 21 to be satisfied, one may need such a SR path of d to traverse the elements of $G(\pi_i, P)$ in a certain order. This is for two independent reasons. Firstly, if $(\sigma^{t(P-1)+m}, \ldots, \sigma^{t(P-1)+M-1}, a^1, \ldots, a^m) \in G(\pi_i, P)$ for some $m < M$ then the sequence d should be such that if $e(t) = (\sigma^{t(P-1)+m}, \ldots, \sigma^{t(P-1)+M-1}, a^1, \ldots, a^m)$ then $t > T - M$ (otherwise, condition 20 will be violated). Thus if $(\sigma^{t(P-1)+m}, \ldots, \sigma^{t(P-1)+M-1}, a^1, \ldots, a^m) \in G(\pi_i, P)$, the SR path of d should traverse elements of $G(\pi_i, P)$ such that the last $(M - m)$ elements of d are $(\sigma^{t(P-1)+m}, \ldots, \sigma^{t(P-1)+M-1})$. Secondly, if t in condition 21 is not an integer multiple of M, d may not satisfy condition 21, unless d traverses $G(\pi_i, P)$ in a certain order. To demonstrate the problem, suppose that τ is the maximum integer s.t. $t = \tau M + k$ for some $k \geq 0$, then

$$\begin{aligned}\sum_{\mu=1}^{t-1} \pi_i(a^\mu) &= \sum_{\mu=1}^{\tau M} \pi_i(a^\mu)) + \sum_{\mu=\tau M+1}^{t-1} \pi_i(a^\mu) \\ &= \sum_{\mu=0}^{\tau-1} M \overline{\pi}_i(e(\mu M + 1)) + \sum_{\mu=\tau M+1}^{t-1} \pi_i(a^\mu).\end{aligned}$$

Since condition 19 implies that $e(t) \varepsilon G(\pi_i, P)\ \forall t \leq T$, the first term on the RHS of the last equation is no more than $\tau M \pi_i$. But the second term, $\sum_{\mu=\tau M+1}^{t-1} \pi_i(a^\mu)$,

---

[8] A directed graph is Eulerian if there is a path which traverses every edge of the graph once and only once.

may not be less than $(t - \tau M - 1)\pi_i$ and thus the RHS of the last expression may not be less than $(t-1)\pi_i$, as is required by condition 21. [9] To overcome this problem (and thus satisfy condition 21), one needs $\hat{d}$ to traverse $G(\pi_i, P)$ in such a way that the worse outcomes (from i's point of view) occur at the first $(M-1)$ period of d. It turns out that this together with condition 19 ensure that condition 21 is satisfied.

To ensure that $\hat{d}$ traverses every element of $G(\pi_i, P)$ once and only once and that the order of the traverse is such that d satisfies conditions 20 and 21, I partition $G(\pi_i, P)$ and then ensure that the traverse goes through the elements of the partition sequentially. Appendix A makes this precise.

Having completed a sketch of the proof of Lemma 1, let me henceforth refer to a path $c \in \overline{C}^M$ which satisfies the conditions of Lemma 1, for any set B_R, as the path which implements Lemma 1 for the set B_R. For any i, consider the set $B(i) = \{b^r(i)\}_{r=1}^{R(i)}$ which satisfy the assumptions of Theorem 1. Denote the path which implements Lemma 1 for the set $B(i)$ by $c(i) = (a^1(i), \dots, a^{K(i)}(i))$ where $K(i) = |E(\pi, B(i))| + M - 1$.

Now define the strategy profile $f = \{s^t(\cdot)\}_{t=0}^{\infty}$ which supports $\bar{a} \in A$ (satisfying the conditions of Theorem 1) as a M-memory SPE as follows. Play $\bar{a}$ if $\bar{a}$ is observed in the previous period. If player i deviates from $\bar{a}$, f dictates playing $c(i)$ (punishing i) for $|E(\pi_i, B(i))| + M - 1$ period and then returns to $\bar{a}$. If player i deviates, when $c(j)$ is to be played, then $c(i)$—the punishment phase for i for $|E(\pi_i, B(i))| + M - 1$ periods—will be invoked. More formally, for any history $h^t = (a^1, \dots, a^{t-1})$,

$$s^t(h^t) = \begin{cases} a^k(i) & \text{if } a^{t-k} \in X(i) \text{ and } (a^{t-k+1}, \dots, a^{t-1}) = (a^1(i), \dots, a^{k-1}(i)) \text{ for some } k \le M \text{ or if} \\ & (a^{t-M}, \dots, a^{t-1}) = (a^{k-M}(i), \dots, a^{k-1}(i)) \text{ for some } k \text{ s.t. } M < k < |E(\pi_i, B(i))| + M \\ \bar{a} & \text{otherwise} \end{cases} \tag{22}$$

where, as before, $X(i)$ defined in Eq. (14) is the set of all action profiles which involve a deviation by i. Finally, note that f supports $\bar{a}$ ($\sigma^t(f) = \bar{a}\ \forall t$) and f is well-defined and needs M-period memory only (this follows from $c(i) \in \overline{C}^M$, $\bar{a} \notin B(i)$ and condition (iii) of Theorem 1).

Lemma 2: The strategy vector $\boldsymbol{f} \in F^M$, defined by Eq. (22), is a SPE.

Proof: Consider any history $h^t = (a^1, \dots, a^{t-1})$. Since $\pi_i^{\infty}(\sigma(\boldsymbol{f}|h^t)) = \pi_i(\bar{a}) = \pi_i$, I need to show that $\pi_i \ge \pi_i^{\infty}(\sigma(\tilde{\boldsymbol{f}}_i, \boldsymbol{f}_{-i}|h^t))\ \forall i$ and $\forall \tilde{\boldsymbol{f}}_i \in F^M$. With the Limit-of-the-Mean criterion, I only need to consider $\tilde{\boldsymbol{f}}_i$ which deviates from $\boldsymbol{f}_i|h^t$ infinitely often. (With finite deviations $\pi_i^{\infty}(\sigma(\tilde{\boldsymbol{f}}_i, \boldsymbol{f}_{-i}|h^t)) = \pi_i$.) Let $\tilde{a}^{\tau} = \sigma^t$

---

[9] Note that if t is an integer multiple of M, then the second term vanishes and thus condition 21 is satisfied irrespective of the order of the traverse.

$(\tilde{\boldsymbol{f}}_i, \boldsymbol{f}_{-i}|h^t)$ $\forall\tau \geq t$. Since $\tilde{\boldsymbol{f}}_i|h^t$ deviates from $\boldsymbol{f}_i|h^t$ infinitely often, there exists a sequence $\{\mu^\tau\}_{\tau=1}^{\infty}$ such that for every integer $\tau \geq 1$ the following holds (the sequence $\{\mu^\tau\}_{\tau=1}^{\infty}$ correspond to periods of deviation by player i).

(i) Either $\tilde{a}_{-i}^{\mu^\tau} = \bar{a}_{-i}$ and $\tilde{a}^{\mu^\tau} \neq \bar{a}_i$
Or $\tilde{a}_{-i}^{\mu^\tau} = b_{-i}^r(i)$ and $\tilde{a}_i^{\mu^\tau} \neq b_i^r(i)$ for some $r \leq R(i)$,

(ii) $\tilde{a}^{\mu^\tau + k} = a^k(i) \in B_r(i)$ if $k < \min\left\{\left|E(\pi_i, r)\right| + M, \mu^{\tau+1} - \mu^\tau\right\}$,

(iii) $\tilde{a}^{\mu^\tau + k} = \bar{a}$ if $\left|E\left(\pi_i, B(i)\right)\right| + M \leq k < \mu^{\tau+1} - \mu^\tau$.

Conditions (i)–(iii) follow immediately from the definition of $\underset{\sim}{\boldsymbol{f}}$ and from deviations taking place at periods μ^τ for all τ. Now, $\pi_i^{\infty}(\sigma(\tilde{\boldsymbol{f}}_i, \boldsymbol{f}_{-i}|h^i) = \lim_{n\to\infty} 1/(\mu^{n+1} - \mu^1)\sum_{\tau=1}^{n}\sum_{\mu=\mu^\tau+1}^{\mu^{\tau+1}}\pi_i(\tilde{a}^\mu)$. Therefore the proof is complete if $\sum_{\mu=\mu^\tau+1}^{\mu^{\tau+1}}\pi_i(\tilde{a}^\mu) \leq (\mu^{\tau+1} - \mu^\tau)\pi_i$ for any integer τ. There are two possible cases to consider.

Case (a): $\mu^{\tau+1} - \mu^\tau \geq |E(\pi_i, R(i))| + M$. By the definition of $\boldsymbol{f}$ the punishment phase $c(i)$ implements Lemma 1 for the set $B(i)$. Therefore, it follows from Remark 1 (condition (b′)) together with (ii) and (iii) above that

$$\sum_{\mu=\mu^\tau+1}^{\mu^{\tau+1}} \pi_i(\tilde{a}^\mu) = (M-1)\pi_i\left(b^1(i)\right) + \sum_{e \in E(\pi_i, B(i))} \overline{\pi}_i(e)$$

$$+\left(\mu^{\tau+1} - \mu^\tau - |E\left(\pi_i, B(i)\right)| - M\right)\pi_i(\bar{a}) + u_i(\bar{a}) \tag{23}$$

But Eq. (16) implies that the RHS of Eq. (23) is less than $(\mu^{\tau+1} - \mu^\tau)\pi_i$.

Case (b) $\mu^{\tau+1} - \mu^\tau < |E(\pi_i, R(i))| + M$. In this case $a_{-i}^{\mu^{\tau+1}} = b_{-i}^r(i)$ for some $r \leq R(i)$. Thus it follows from (ii) above that $\mu^{\tau+1} - \mu^\tau \geq |E(\pi_i, r-1)| + M)$. Therefore, it follows from the definition of $\boldsymbol{f}$ and (b′) of Remark 1 that

$$\sum_{\mu=\mu^\tau+1}^{\mu^{\tau+1}} \pi_i(\tilde{a}^\mu) = (M-1)\pi_i\left(b^1(i)\right) + \sum_{e \in E(\pi_i, r-1)} \overline{\pi}_i(e)$$

$$+ \sum_{\mu=\mu^\tau+|E(\pi_i, r-1)|+M}^{\mu^{\tau+1}-1} \pi_i(\tilde{a}^\mu) + u_i(b^r) \tag{24}$$

Since $c(i)$ implements Lemma 1 for the set $B(i)$, (c) of Lemma 1 implies that the third term on the RHS of Eq. (24) $\leq (\mu^{\tau+1} - \mu^\tau - |E(\pi_i, r-1)| - M)\pi_i$. Therefore, by Eqs. (17a) and (17b) the RHS of Eq. (24) $\leq ((\mu^{\tau+1} - \mu)^\tau)\pi_i$. Q.E.D.

The above Lemma completes the proof of Theorem 1.

Remark 2: The proof of Theorem 1 does not depend on the finiteness of the action set A. What is needed to prove Theorem 1 is that $\forall i$ the set $B(i)$, from which the punishment for i is chosen, has to be finite.

Remark 3: As in Sabourian (1989b), conditions 15b and 15c are satisfied if $\forall a^1$ and $a^2 \in (\bigcup_{i=1}^{n} B(i) \cup \bar{a})$, $a^1 \neq a^2$, the following two conditions hold:

$$a^1 \text{ and } a^2 \text{ differ in at least two components,} \tag{25}$$

$$\text{if } a^1_{-ij} = a^2_{-ij}, \text{ then } B(i) = B(j). \tag{26}$$

It is easy to demonstrate that condition 25 is equivalent to condition 15b and condition 26 is equivalent to condition 15c. Notice also that when $n = 2$, a^l_{-ij} is empty for $l = 1, 2$: thus condition 26 (or condition 15c) implies that $B(i) = B(j)$. Therefore for $n = 2$, Theorem 1 requires that the punishment paths for both players be identical.

Remark 4: As was mentioned before, since I assume that players use the Limit-of-the-Mean criterion, I do not need to consider finite number of deviations by any player. As a result, in the proof of Theorem 1, deviations by any player i from the punishment path $c(j)$ of another player j need not be taken into account (this is because i can deviate from $c(j)$ only once before the punishment path $c(i)$ is invoked). [10] If players use the Over-Taking criterion (see Rubinstein, 1979) or if players discount the future (see Fudenberg and Maskin, 1986) deviations by i from $c(j)$ matter. Thus to extend Theorem 1 to these cases, one needs a further condition to ensure that player i does not prefer the infinite path $\{(a_i, a^k(j)_{-i});\ a^1(i);\ \ldots,\ a^{K(i)};\ \bar{a};\ \bar{a};\ \bar{a};\ \ldots\}$ to the infinite path $\{a^k(j);\ \ldots;\ a^{K(i)}(j);\ \bar{a};\ \bar{a};\ \bar{a};\ \ldots\}$ for any i, for any a_i, for any $j \neq i$ and for any $k \leq K(i)$. For example, in the case of the Over-Taking criterion, if $K(i) \geq K(j)$ then the following condition ensures that it does not pay i to deviate from $c(j)$ for any $j \neq i$

$$\sum_{\tau=k}^{K(j)} \pi_i(a^\tau(j)) + (K(i) - K(j) + k)\pi_i(\bar{a}) \geq u_i(a^k(j)) + \sum_{\tau=1}^{K(i)} \pi_i(a^\tau(i)), \forall k \leq K(j).$$

---

[10] Clearly, if two players could coordinate on such deviations—if i deviates from $c(j)$ and then j deviates from $c(i)$ and so on—then one would need a further condition to ensure that it does not pay i to deviate from $c(j)$. Here, one does not need to consider such coordinated deviations because in this paper the equilibrium concept I use is subgame perfect equilibrium. Clearly, if one uses a different equilibrium concept involving noise (e.g., extensive form of trembling-hand perfect equilibrium) or an equilibrium concept involving deviations by a coalition of players (e.g., strong perfect equilibrium), then one may need to consider the possibility of coordinated deviations and thus one may need to impose additional conditions to ensure that i does not deviate from $c(j)$.

Remark 5: Note that if $M = 1$, then $c(i) = (b_1(i), \ldots, b_R(i))$. Thus, Theorem 1 is the same as Theorem 1 of Sabourian (1989b) for the case of $M = 1$.

Remark 6: Notice that $E(\pi_i, B(i))$ is the set of *all* strings of length M in $B(i)^M$ with average payoff less than π_i. Therefore it could be said, informally, that the punishment path for player i in the proof of Theorem 1 uses all punishment strings of length M that can be imposed on i using elements of the set $B(i)$.

Remark 7: Note that if $B_r(i)$ is such that $\forall a \in B_r(i)$, $\pi_i(a) < \pi_i$ then $E(\pi_i, B_r(i)) = (B_r(i))^M$ and thus $\sum_{e \in E(\pi_i, B_r(i))} \overline{\pi}_i(e) = r^{M-1} \sum_{\tau=1}^{r} \pi_i(b^\tau(i))$. Therefore, if $\pi_i(a) < \pi_i$ $\forall a \in B(i)$ (condition 8), conditions 16 and 17 in Theorem 1 are the same as conditions 9 and 11. More generally, suppose $\overline{B}_r(i) = \{a \in B_r(i) | \pi_i(a) < \pi_i\}$ then

$$\sum_{e \in E(\pi_i, B_r(i))} \overline{\pi}_i(e) = \sum_{\tau=1}^{r} n_\tau \pi_i(b^\tau(i)), \text{ for some } n_\tau \tag{27}$$

where for every τ, $|\overline{B}_r(i)|^{M-1} \le n_\tau \le |B_r(i)|^{M-1} = r^{M-1}$. Thus the number of times any $b^\tau(i) \in \overline{B}(i)$ occurs in the punishment phase $c(i)$ in the proof of Theorem 1, is bounded below and above by $|\overline{B}_R(i)|^{M-1}$ and $|B(i)|^{M-1}$, respectively. Therefore, it could be said, very loosely, that conditions 16 and 17 (or conditions 9 and 11), and the above discussion illustrate that the size of SPE increases very rapidly ('exponentially') as the memory increases, and slowly (in a 'polynomial' fashion) as the cardinality of the punishment set $|B(i)|$ increases.

Finally, one can establish a Folk-Theorem-type result if there is a 'large' number of action profiles (the set A is connected) as in Sabourian (1989b) or if the memory is sufficiently large as follows. Let $\gamma \in \arg\min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} \pi_i(a_i, a_{-i})$. Any action profile $a \varepsilon A$ is said to be an (strictly) individually rational (IR) if for all i, $\pi_i(a) > \pi_i(\gamma)$. Also any payoff vector π is IR if $\pi = \pi(a)$ for some IR action profile $a \varepsilon A$.

Corollary 1: *There exists a finite* $\overline{M}$ *such that any IR outcome* $\overline{a} \in A$ *with its associated IR payoff* $\overline{\pi} = \pi(\overline{a})$ *can be supported as a M-memory SPE for all* $M \ge \overline{M}$.

Proof: Let $\overline{M}$ be such that for every IR outcome $\overline{a} \in A$ and for all $M \ge \overline{M}$

$$\pi_i(\overline{a}) \ge \frac{1}{M}\{u_i(\overline{a}) + (M-1)\pi_i(\gamma^i)\} \tag{28}$$

Since $\pi_i(\overline{a}) > \pi_i(\gamma^i)$ and A is finite, $\overline{M}$ exists. Now, use Theorem 1 to prove the result as follows. Define the set $B(i)$ to be γ^i. Condition 28 implies that condition (i) of Theorem 1 is satisfied. Conditions (ii)–(iii) of Theorem 1 are automatically satisfied because for every i, $B(i)$ contains one element and $\pi_i(\bar{a}) > \pi_i(\gamma^i) = u_i(\gamma^i)$. Q.E.D.

5. Sufficiency conditions for nonstationary M-memory SPE

In Section 4, Theorem 1 was proved by constructing M-memory strategies which play the stationary outcome $\bar{a} \in A$ in equilibrium and punish any deviator by playing a finite punishment path and then return to the $\bar{a}$ forever. Theorem 1 can be weakened in two ways. Firstly, one can allow for strategies which are nonstationary on the equilibrium path. Secondly, to support an outcome path as a SPE, one does not need to consider only strategies which revert to the original equilibrium path after punishing a deviator. What is needed after a punishment phase is to revert to another equilibrium path, which gives the deviator an average payoff no more than that obtained from the original equilibrium path.

Theorem 2: *Any path $d(0) \in C^M$ (see Eq. (6) for the definition of C^M) and the associated payoff vector $\boldsymbol{\pi}^0 = (\bar{\pi}_1(d(0), \dots, \bar{\pi}_n(d(0)))$ can be sustained as a M-memory SPE, if for each player $i = 1, \dots, n$ there exists an outcome path $d(i) = (a^1(i), \dots, a^{Q(i)}) \in C^M$, with an average payoff $\pi_i = \bar{\pi}_i(d(i))$, and a set of action profiles $B(i) = \{b^\tau(i))\}_{\tau=1}^{R(i)} \subset A$ such that $\forall j = 0, 1, \dots, n$ and for any integer $q \leq Q(i)$, the following holds.*

(i) $\bar{\pi}_i(d(j)) \geq \pi_i (= \bar{\pi}_i(d(i)))$.

(ii) $\pi_i \geq \{1/[q + (M-1) + |E(\pi_i, B(i))|]\}\{u_i(a^q(i)) + \sum_{\tau=1}^{q-1} \pi_i(a^\tau(i)) + (M-1)\pi_i(b^1(i)) + \sum_{e \in E(\pi_i, B(i))} \bar{\pi}_i(e)\}$.

(iii) $\pi_i \geq u_i(b^1(i))$ and for any r such that $1 < r \leq R$

$$\pi_i \geq \frac{1}{M + |E(\pi_i, B_{r-1}(i))|}\left\{u_i(b^r(i)) + (M-1)\pi_i(b^1(i)) + \sum_{e \in E(\pi_i, B_{r-1}(i))} \bar{\pi}_i(e)\right\}.$$

(iv) (a) Condition 15a holds and for any $e^r(i)$ and $e^{r'}(j)$ belonging to the SR path generated by $d(i)$ and $d(j)$, respectively, if $e^r(i) = e^{r'}(j)$ then $(a^r(i), \dots, a^{Q(i)}(i)) = (a^{r'}(j), \dots, a^{Q(j)}(j))$.

(b) $\forall r \leq R(j)$, $b^r(j) \notin X^*(i)$ where $X^*(i) = \bar{X}(i) \cup \{a | a_{-i} = a^q(j)_{-i}$ and $a_i \neq a^q(j)_i$ for some j and $q\}$ and $\bar{X}(i)$ is defined as in condition 14.

(c) If $X^*(i) \cap X^*(j)$ is not empty then $B(i) = B(j)$.

In the above Theorem, $d(0)$ is the equilibrium cycle path. As before, $B(i)$ is the set of actions which are used to punish i in the finite punishment phase $c(i)$ and $d(i)$ is the equilibrium cycle path which will be played after the punishment phase

$c(i)$. Condition (i) of Theorem 2 says that the equilibrium payoff $\pi_i^0 = \overline{\pi}_i(d(0))$ is no less that the payoff i receives when the path $d(i)$ is played and that i weakly prefers $d(j)$ to $d(i)$ for any j $(=1, \dots, n)$. Conditions (ii)–(iv) are generalizations of (i)–(iii) of Theorem 1, respectively and are assumed for the same reasons as those use in Section 4. The proof of Theorem 2 is very similar to that of Theorem 1. It involves constructing a strategy profile f which satisfies the following.

(i) It plays the cycle $d(0)$ along the equilibrium path.

(ii) It punishes any deviation by i by playing the same finite punishment phase $c(i)$ as in the proof of Theorem 1 (by playing $c(i)$, the path which implements Lemma 1 for the set $B(i)$ for $M-1+|E(\pi_i, B(i))|$ periods) and then plays the cycle path $d(i)$ instead of the original equilibrium cycle $d(0)$.

(iii) It punishes any deviation by player j from $c(i)$ or $d(i)$ by restarting the punishment phase with j as the player to be punished—$c(j)$ will be played for $M-1+|E(\pi_j, B(j))|$ periods—and then plays the cycle $d(j)$.

Since $d(i) \in C^M$ $\forall i = 0, \dots, n$ and $c(i) \in \overline{C}^M$, the above strategy profile f requires M-period memory. To show that f is also a SPE, follow the proof of Lemma 2 and firstly replace $\bar{a}$ in the text by '$\forall a \in d(i)$ for $i = 0, \dots, n$' and then appeal to (i)–(iii) of Theorem 2, instead of (i) and (ii) of Theorem 1, to show that it does not pay any i to deviate from $d(j)$ $\forall j = 0, \dots, n$. (A formal proof can be found in Sabourian, 1989c).

Using nonstationary strategies, one can trivially extend Corollary 1 to show that there exists a $\overline{M}$ such that any vector of payoffs $\boldsymbol{\pi} = (\pi_i, \dots, \pi_n) \in R^n$ can be sustained as a M-memory SPE for all $M \geq \overline{M}$ if $\forall i$, $\pi_i > \pi_i(\gamma^i)$ and $\pi_i = \overline{\pi}_i(d)$ for some finite sequence of action profiles $d = (a^1, \dots, A^R) \in C^M$.

The sufficiency conditions in Theorem 2, turn out to be (almost) necessary for a path to be a M-memory SPE. Section 6 demonstrates that (i)–(iii) of Theorem 2 are (almost) necessary conditions.

6. Necessary conditions for M-memory SPE

To provide a converse result to Theorem 2, I need to find for each i, the SPE outcome which generates the lowest payoff for player i in the class of SPE outcomes. (See Abreu, 1984.) Let $\nu_i = \min_{f \in P} \pi_i^{\infty}(\sigma(f))$ and $\overline{P}_i = \arg\min_{f \in P} \pi_i^{\infty}(\sigma(f))$, where P is the set of M-memory SPE strategy profiles. Thus υ_i and P_i denote the optimal penal payoff and the set of optimal penal codes for i, respectively. Note that $\overline{P}_i$ is not empty because $|A|$ and M are finite.

Proposition 1: *For each player i there exists an optimal penal code* $\bar{f} \in F^M$ *with cycle path* $\Sigma(\bar{f}) = (\bar{a}^1, \dots, \bar{a}^Q)$ *and a set of punishment action profiles* $B = \{b^\tau\}_{\tau=1}^R \subseteq A$ *such that for any integer* $q \leq Q$*, the following holds.*

(i) $\upsilon_i \geq \{1/[q+M-1+|E(\upsilon_i, B)|]\}\{u_i(\bar{a}^q) + \sum_{\tau=1}^{q-1} \pi_i(\bar{a}^\tau) + (M-1)\pi_i(b^l) + \sum_{e \in E(\upsilon_i, B)} \overline{\pi}_i(e)\}$ for some $b^l \varepsilon B$.

(ii) $v_i \geq u_i(b^1(i))$ and for any r s.t. $1 < r \leq R$, $v_i \geq \{1/[M + |E(v_i,B_{r-1})|]\}\{u_i(b^r) + (M-1)\pi_i(b^l) + \sum_{-e \in E(v_i,B_{r-1})}\overline{\pi}_i(e)\}$ for some $B^l \in B_{r-1}$.

Where, as before, $B_{r-1} \equiv \{b^\tau\}_{\tau=1}^{r-1}$.

See Appendix B for the proof of Proposition 1.

Theorem 3: *Suppose the (cycle) path $d(0)$ and the associated payoff $\pi_i^0 = \overline{\pi}_i(d(0))$, $\forall i$, can be supported as a M-memory SPE, then $d(0) \in C^M$ and $\forall i = 1, \dots, n$ there exists an outcome path $d(i) = (a^1(i), \dots, a^{Q(i)}) \in C^M$, with the average payoff $\pi_i = \overline{\pi}_i(d(i))$ and a set $B(i) = \{b^\tau(i)\}_{\tau=1}^{R(i)} \subseteq A$ s.t. $\forall j = 0, 1, \dots, n$, the following holds.*

(i) $\overline{\pi}_i(d(j)) \geq \pi_i$.

(ii) For any integer $q \leq Q(i)$ and for some $b^l(i) \varepsilon B(i)$, $\pi_i \geq \{1/[q + (M-1) + |E(\pi_i,B(i))|]\}\{u_i(a^q(i)) + \sum_{\tau=1}^{q-1}\pi_i(a^\tau(i)) + (M-1)\pi_i(b^l(i)) + \sum_{e \in E(\pi_i,B(i))}\overline{\pi}_i(e)\}$.

(iii) $\pi_i \geq u_i(b^1(i))$ and for any r s.t. $1 < r \leq R(i)$, $\pi_i \geq \{1/[M + |E(v_i,B_{r-1}(i))|]\}\{u_i(b^r(i)) + M - 1)\pi_i(b^l(i)) + \sum_{e \in E(v_i,B_{r-1}(i))}\overline{\pi}_i(e)\}$ for some $b^l(i) \in B_{r-1}(i) \equiv \{b^\tau(i)\}_{\tau=1}^{r-1}$.

The necessary conditions (i)–(iii) in Theorem 3 are identical to conditions (i)–(iii) in Theorem 2 (the sufficiency result) except that the RHS of the inequalities in conditions (ii) and (iii) in Theorem 3 contain the term $(M-1)\pi_i(b^l(i))$ for some $b^l(i)$, whereas the RHS of the equivalent inequalities in Theorem 2 contain the term $(M-1)\pi_i(b^1(i))$. Also as was mentioned before, condition (iv) in Theorem 2 is assumed to ensure that the strategy profile needed to support $d(0)$ in Theorem 2 is implementable with M-memory.

Proof of Theorem 3: Proposition 1 implies that there exists an optimal penal code $f(i)$ for i which satisfies (i) and (ii) of Proposition 1. For each $i = 1, \dots, n$, denote the cycle path that $f(i)$ generates by $d(i) = (a^1(i), \dots, a^{Q(i)})$ and denote $\overline{\pi}_i(d(i))$ by π_i. Then it follows from conditions (i) and (ii) of Proposition 1 that for each i there exists a set $B(i) = \{b^\tau(i)\}_{\tau=1}^{R(i)} \subseteq A$ s.t. $d(i)$ and $B(i)$ satisfy (ii) and (iii) of Theorem 3. Since $d(0)$ can be supported as a M-memory SPE, it follows that $d(0) \in C^M$ and $\pi_i^0 = \overline{\pi}_i(d(0)) \geq v_i = \overline{\pi}_i(d(i))$. Moreover, since $d(j)$ is the cycle path of the SPE profile $f(j)$ and $d(i)$ is the cycle path of an optimal penal code for i, $\pi_i(d(j)) \geq \pi_i(d(i))$, $\forall i$ and $j \geq 1$. Q.E.D.

Acknowledgements

I would like to thank an anonymous referee, an associate editor, L. Anderlini, D. Canning and R.E. Evans for very helpful comments and discussions. I also like

to thank K. Moody and A. Thomson for some helpful suggestions on Eulerian graphs.

Appendix A. Proof of Lemma 1

First a few definitions and a result from graph theory. A path in a graph is a sequence $(v^1, e^1, v^2, e^2, \ldots, e^{K-1}, v^K)$ of alternating vertices and edges such that for all positive integer $k < K$, e^k connects vertices v^k and v^{k+1}.[11] This path is a cycle if $v_1 = v_K$. A directed graph is said to be *Eulerian* if it has a directed path or a directed cycle which traverses every edge of the graph once and only once. Such a directed path (cycle) is referred to as an Eulerian path (cycle). An undirected graph is *connected* if all pairs of vertices can be joined by a path. A directed graph is *connected* if its underlying undirected graph is connected—if all pairs of vertices can be joined by a path (not necessarily a directed path). Finally, for any vertex v of a graph, denote the set of edges entering and exiting v by $\overrightarrow{\rho}(v)$ and $\overleftarrow{\rho}(v)$, respectively.

Lemma A1: *A connected directed graph has an Eulerian cycle if for every vertex v of the graph*

$$|\overrightarrow{\rho}(v)| = |\overleftarrow{\rho}(v)|. \tag{A1}$$

A connected directed graph has an Eulerian path with initial vertex $\underline{v}$ and terminal vertex $\bar{v}$ if every vertex $v \neq \underline{v}$ or $\bar{v}$ satisfies condition A1, $\overrightarrow{\rho}(\underline{v}) + 1 = \overleftarrow{\rho}(\underline{v})$ and $\overrightarrow{\rho}(\bar{v}) = \overleftarrow{\rho}(\bar{v}) + 1$. (See section 5.4 of Swamy and Thulasiraman, 1981).

As was mentioned in the induction argument in Section 4 on the sketch of the proof of Lemma 1, I need to construct a path d which satisfies conditions 19–21 discussed in the section on sketch of the proof of Lemma 1. The existence of a path which traverses every element of $G(\pi_i, P)$ is only sufficient to guarantee condition 19. In order for conditions 20 and 21 to be satisfied, one needs the elements of $G(\pi_i, P)$ in this traverse to occur in a certain order. For this purpose,

---

[11] In this appendix, I will not explicitly refer to the edges of a path and simply refer to a sequence of vertices $(v^1, \ldots, v^K)$ as a path if all the adjacent vertices v^k and v^{k+1} are connected by an edge, for all $k \leq K$.

for any sequence of action profile $\alpha \equiv (\alpha^1, \ldots, \alpha^{M-1}) \varepsilon A^{M-1}$ let us partition $G(\pi_i, P)$ into

$$\phi(P,\alpha) = \left\{ e \in G(\pi,P) | e \neq \left(\alpha^m, \cdots, \alpha^{M-1}, b^P, \underset{m-1}{\leftrightarrow} \right) \right.$$

$$\left. \text{for some integer } m < M \right\} \tag{A2}$$

$$\text{and } \overline{G}(\pi_i, P, \alpha) = \{ e \in G(\pi_i, P) | e \notin \phi(P) \}, \tag{A3}$$

where $\overset{\leftrightarrow}{m}$ stands for a string of length m consisting of b^ls only (b^l appearing m times) and b^l is such that

$$b^l \in \underset{a \in B_P}{\arg\min}\, \pi_i(a) \tag{A4}$$

Also define $\overline{m}(P,\alpha)$ and $\eta(P)$ as follows

$$\overline{m}(P,\alpha) \equiv \begin{cases} \min\left\{ m < M | \left(\alpha^m, \cdots, \alpha^{M-1}, b^P, \underset{m-1}{\leftrightarrow} \right) \varepsilon G \right\} & \text{if } \phi(P,\alpha) \text{ is not empty} \\ M & \text{otherwise} \end{cases} \tag{A5}$$

$$\eta(P) \in \underset{a \in \{a' \in B_P | a' \neq b^l\}}{\arg\min} \pi_i(a) \tag{A6}$$

Point of Notation: When the meaning is not ambiguous, I shall refer to $G(\pi_i, P)$, $\phi(P,\alpha)$, $\overline{G}(\pi_i, P, \alpha)$, $\overline{m}(P,\alpha)$, and $\eta(P)$ by G, ϕ, $\overline{G}$, $\overline{m}$, and η, respectively. For the next two Lemmas the sequence α is fixed.

Property (γ): G is said to satisfy property (γ) if

$$\pi_i(b^P) + \pi_i(\eta) + (M-2)\pi_i(b^l) < M\pi_i \tag{A7}$$

Lemma A2: *Suppose G satisfies property (γ). Then there exist a M-string path $Q = \{e(1), \ldots, e(|\overline{G}|)\}$ such that $e(r) \varepsilon (B_P)^M$ (the M-fold cartesian product of B_P) $\forall r \leq |\overline{G}|$, $e(r') \neq e(r)$ $\forall r' \neq r$ and*

(i) the elements of Q are the same as those of the set $\overline{G}$;

(ii) $e(1) = (\mathrm{b}^{\mathrm{P}}, \underset{M-2}{\leftrightarrow}, a)$ for some $a \in B_P$;

(iii) if ϕ is nonempty then $e(|\overline{G}|) = (a', \alpha^{\overline{m}}, \cdots, \alpha^{M-1}, b^P, \underset{\overline{m}-2}{\leftrightarrow})$ for some $a' \varepsilon B_{\mathrm{P}}$.

Proof: The above is proved by first constructing a directed graph whose edges correspond to the elements of $\overline{G}$ and then showing (a) if ϕ is not empty then the directed Graph has an Eulerian path with the initial vertex $\underline{v} \equiv (b^P, \underset{M-2}{\leftrightarrow})$ and a final vertex $\bar{v} \equiv (\alpha^{\bar{m}}, \cdots, \alpha^{M-1}, b^P, \underset{\bar{m}-2}{\leftrightarrow})$ and (b) if $\phi(P)$ is empty then the graph has a Eulerian cycle with the initial vertex $\underline{v}$. (The proof is similar to proving the existence of de Bruijn sequences, see Golomb, 1967 or Hall, 1968).

Consider the directed graph (V,E), where V is the vertex set and E is the edge set, defined as

$$V = \left\{ v \in (B_P)^{M-1} \;\middle|\; \begin{array}{ll} \text{(i)} & \sum_{m=1}^{M-1} \pi_i(v^m) + \pi_i(a) < M\pi_i \text{ for some } a \in B_P \\ \text{(ii)} & \text{if } b^P \notin v \text{ then } \sum_{m=1}^{M-1} \pi_i(v^m) + \pi_i(b^P) < M\pi_i \end{array} \right\} \tag{A8}$$

$$E = \left\{ (v,w) \in V^2 \;\middle|\; \begin{array}{ll} \text{(i)} & v^{m+1} = w^m,\ 1 \le m < M-1 \\ \text{(ii)} & b^P \in v \text{ or } b^P \in w \\ \text{(iii)} & \sum_{m=1}^{M-1} \pi_i(v^m) + \pi_i(w^{M-1}) < M\pi_i \\ \text{(iv)} & (v, w^{M-1}) \notin \phi \end{array} \right\} \tag{A9}$$

Each edge $(v,w) \in E$ can thus be labelled by a M-tuple $e = (e^1, \ldots, e^M) \in B_P^M$, where $e^1 = v^1$, $e^m = v^m = w^{m-1}$ $\forall m$ s.t. $1 < m < M$ and $e^M = w^{M-1}$. Moreover, (ii)–(iv) of Eq. (A9) imply that for any such edge $e \in E$, $b^P \in e$, $\frac{1}{M}\Sigma_{m=1}^{M}\pi_i(e^m) < \pi_i$ and $e \notin \phi$, respectively. Since the set $\bar{G}$ is simply $\{e \in (B_P)^M | \frac{1}{M}\Sigma\pi_i(e^m) < \pi_i,\ b^P \in e$ and $e \notin \phi\}$, it follows that we can label the edges of the graph (V,E) uniquely by the elements of $\bar{G}$.

Next I need to show that (V,E) has an Eulerian path with an initial vertex $\underline{v} = (b^P, \underset{M-2}{\leftrightarrow})$ and a terminal vertex $\bar{v} = (\alpha^{\bar{m}}, \cdots, \alpha^{M-1}, b^P, \underset{\bar{m}-2}{\leftrightarrow})$ if ϕ is not empty and it has an Eulerian cycle, otherwise. It follows from Lemma A1 that (V,E) has such a path or cycle [12] if it is connected and the number of edges entering and exiting any vertex v satisfies the conditions of Lemma A1. I shall consider these in turn.

Step 1: Connectedness of (V,E).

It follows from the definition of b^1 in condition A4 that if G is nonempty then $(b^P, \underset{M-1}{\leftrightarrow}) \in \bar{G}$. Therefore $(\underset{M-1}{\leftrightarrow}) \in V$. Now to show that (V,E) is connected, I need to show that for any $v \in V$ such that $v \neq (\underset{M-1}{\leftrightarrow})$ there is a directed path from v to $(\underset{M-1}{\leftrightarrow}) \in A^{M-1}$.. To show the existence of such a path, I consider seven different cases (these cases exhaust all the possibilities).

---

[12] In the rest of this proof, paths and cycles refer to paths and cycles in the graph (V,E).

Case 1: $v=(\overset{\leftrightarrow}{m}, b^P, \underset{M-m-2}{\leftrightarrow})$ for some non-negative integer $m \le M-2$ and $\alpha^{M-1} \neq b^l$. Since $v \varepsilon V$, it follows from the definition of b^l—condition A4—and $\alpha^{M-1} \neq b^l$ that the following path of vertices belong to the graph (V,E)

$$\Big(\overset{\leftrightarrow}{m}, b^P, \underset{M-m-2}{\leftrightarrow}\Big)\Big(\underset{m-1}{\leftrightarrow}, b^P, \underset{M-m-1}{\leftrightarrow}\Big) \cdots \Big(\underset{1}{\leftrightarrow}, b^P, \underset{M-3}{\leftrightarrow}\Big)\Big(b^P, \underset{M-2}{\leftrightarrow}\Big)\Big(\underset{M-1}{\leftrightarrow}\Big) \tag{A10}$$

Therefore $v=(\overset{\leftrightarrow}{m}, b^P, \underset{M-m-2}{\leftrightarrow})$ is connected to $(\underset{M-1}{\leftrightarrow})$.

Case 2: $v=(\overset{\leftrightarrow}{m}, b^P, \underset{M-m-2}{\leftrightarrow})$ for some non-negative integer $m \le M-2$ and $\alpha^{M-1}=b^l$. It follows from definitions of b^l and η—conditions A4 and A6, respectively—and property (γ) that the following path of vertices belong to the graph (V,E):

$$\Big(\underset{m}{\leftrightarrow}, b^P, \underset{M-m-2}{\leftrightarrow}\Big)\Big(\underset{m-1}{\leftrightarrow}, b^P, \underset{M-m-2}{\leftrightarrow}, \eta\Big)\Big(\underset{m-2}{\leftrightarrow}, b^P, \underset{M-m-2}{\leftrightarrow}, \eta, \underset{1}{\leftrightarrow}\Big)$$
$$\cdots \Big(\underset{1}{\leftrightarrow}, b^P, \underset{M-m-2}{\leftrightarrow}, \eta, \underset{m-2}{\leftrightarrow}\Big)\Big(b^P, \underset{M-m-2}{\leftrightarrow}, \eta, \underset{m-1}{\leftrightarrow}\Big) \tag{A11}$$

Moreover, it follows from property (γ) and the definition of (V,E) that the following path belong to the graph (V,E)

$$\Big(\underset{M-m-2}{\leftrightarrow}, \eta, \underset{m-1}{\leftrightarrow}, \eta\Big)\Big(\underset{M-m-3}{\leftrightarrow}, \eta, \underset{m-1}{\leftrightarrow}, \eta, b^P\Big)\Big(\underset{M-m-4}{\leftrightarrow}, \eta, \underset{m-1}{\leftrightarrow}, \eta, b^P, \underset{1}{\leftrightarrow}\Big)$$
$$\ldots \Big(\eta, b^P, \underset{M-3}{\leftrightarrow}\Big)\Big(b^P, \underset{M-2}{\leftrightarrow}\Big)\Big(\underset{M-1}{\leftrightarrow}\Big) \tag{A12}$$

It also follows from property (γ) that the last vertex in (A11) is connected to the first vertex of the path in (A12). Therefore v, the first vertex in (A11), is connected to $(\underset{M-1}{\leftrightarrow})$, the last vertex in (A12). [13]

Case 3: $b^P \notin V$. Since $v \varepsilon V$ and $b^P \notin v$, it follows from the definition of V that there is an edge connecting $(b^P, v^1, \ldots, v^{M-2})$ to $v=(v^1, \ldots, v^{M-1})$. Now define $\tilde{m} \equiv \min\{m<M \mid v^m \neq b^l\}$. Since $v \neq (\underset{M-1}{\leftrightarrow})$, it follows that $\tilde{m}$ is well defined. Then it follows from the definition of b^l—condition A4—that the following path of vertices belong to the underlying undirected graph of (V,E)

$$(v^1, \ldots, v^{M-1})\,(b^P, v^1, \ldots, v^{M-2})\Big(\underset{1}{\leftrightarrow}, b^P, v^1, \ldots, v^{M-3}\Big)$$
$$\ldots \Big(\underset{M-\tilde{m}-2}{\leftrightarrow}, b^P, v^1, \ldots, v^{\tilde{m}}\Big)\Big(\underset{M-\tilde{m}-1}{\leftrightarrow}, b^P, v^1, \ldots, v^{\tilde{m}-1}\Big)$$

(Note that none of the edges defined by the above path belong to ϕ because $v^{\tilde{m}} \neq b^l$.) But it follows from the definition of $\tilde{m}$ that the last vertex in the above

---

[13] It is easy to demonstrate that in this case if Property (γ) does not hold then v may not be connected to $\overset{\leftrightarrow}{M-1}$.

path is $(\underset{M-\tilde{m}-1}{\leftrightarrow},\mathrm{b}^P,\underset{\tilde{m}-1}{\leftrightarrow})$. Thus, by Case 1 or Case 2 the last vertex of the above path is connected to $(\underset{\mathrm{M}-1}{\leftrightarrow})$. Therefore v is connected to $(\underset{\mathrm{M}-1}{\leftrightarrow})$.

Case 4: For some $m \geq 1$, $v^m = b^P$, $v^{m'} \neq b^l$ for some $m' > m$ and $v^{m'} \neq b^P$ $\forall m' > m$. Since $v^{m'} \neq b^l$ for some $m' > m$ and b^l satisfies condition A4, it follows that the following path with the initial vertex v belong to the graph (V,E):

$$(v^1,\ldots,v^{m-1},b^P,v^{m+1},\ldots,v^{M-1})\left(v^2,\ldots,v^{m-1},b^P,v^{m+1},\ldots,v^{M-2},\underset{1}{\leftrightarrow}\right)$$
$$\ldots\left(v^{m-1},b^P,v^{m+1},\ldots,v^{M-2},\underset{m-2}{\leftrightarrow}\right)\left(b^P,v^{m+1},\ldots,v^{M-1},\underset{m-1}{\leftrightarrow}\right)$$
$$\left(v^{m+1},\ldots,v^{M-1},\underset{m}{\leftrightarrow}\right)$$

(Note that none of the edges defined by the above path belong to ϕ because $v^{m'} \neq b^l$ for some $m' > m$.) But b^P does not belong to the last vertex of the above path. Therefore, it follows from Case 3 above that the last vertex of the above path is connected to $(\underset{M-1}{\leftrightarrow})$.. Thus v is connected to $(\underset{M-1}{\leftrightarrow})$..

Case 5: For some $m \geq 1$, $v^m = b^P$, $v^{m'} = b^l$, $\forall m' > m$ and v is connected to $(v^2, \ldots, v^{M-1}, \eta)$. Since v is connected to $(v^2, \ldots, v^{M-1}, \eta)$, it follows from Cases 3 and 4 that v is connected to $(\underset{M-1}{\leftrightarrow})$.. This is because either $b^P \notin (v^2, \ldots, v^{M-1}, \eta)$ in which case $(v^2, \ldots, v^{M-1}, \eta)$ satisfies Case 3 or $b^P \varepsilon (v^2, \ldots, v^{M-1}, \eta)$, in which case $(v^2, \ldots, v^{M-1}, \eta)$ satisfies conditions of Case 4.

Case 6: For some $m \geq 1$, $v^m = b^P$, $v^{m'} \neq b^P$ $\forall m' < m$, $v^{m'} = b^l$ $\forall m' > m$ and v is not connected to $(v^2, \ldots, v^{M-1}, \eta)$. In this case, I shall first try to show

$$(b^l,v^1,\ldots,v^{m-1}) \neq (\alpha^{M-m-1},\ldots,\alpha^{M-1}) \tag{A13}$$

This is because if $\mathrm{v} \equiv (\mathrm{v}^1,\ldots,\mathrm{v}^{m-1},\mathrm{b}^\mathrm{P},\underset{M-m-1}{\leftrightarrow})$ is not connected to $(v^2,\ldots,v^{M-1}, \eta)$ then it must be connected either to $(b^l, v^1, \ldots, v^{M-2})$ or to $(v^2, \ldots, v^{M-1}, b^l)$. (Otherwise, it will not be connected to any vertex, which contradicts $v \varepsilon V$.) Now consider each of the two possibilities. If v is connected to $(b^l, v^1, \ldots, v^{M-2})$ then $(b^1,\mathrm{v}^1,\ldots,\mathrm{v}^{m-1},\mathrm{b}^\mathrm{P},\underset{M-m-1}{\leftrightarrow}) \notin \phi$. Thus condition A13 follows immediately from the definition of ϕ. In the case in which v is connected to $(v^2, \ldots, v^{M-1}, b^l)$ it must be that $(v^1, \ldots, v^{M-1}, b^l) \notin \phi$. But this together with $\mathrm{v} \equiv (\mathrm{v}^1,\ldots,\mathrm{v}^{m-1},\mathrm{b}^\mathrm{P},\underset{M-m-1}{\leftrightarrow})$ and the definition of ϕ implies condition A13.

But condition A13 implies that $(\underset{1}{\leftrightarrow},\mathrm{v}^1,\ldots,\mathrm{v}^{M-1})$, $(\underset{2}{\leftrightarrow},\mathrm{v}^1,\ldots,\mathrm{v}^{M-2})\ldots(\underset{M-1-m}{\leftrightarrow},\mathrm{v}^1,\ldots,\mathrm{v}^\mathrm{m})$ do not belong to ϕ. Hence it follows that the following path belong to the graph (V,E)

$$(v^1,\ldots,v^{M-1})\left(\underset{1}{\leftrightarrow},v^1,\ldots,v^{M-2}\right)\ldots\left(\underset{M-m-1}{\leftrightarrow},v^1,\ldots,v^m\right)$$
$$\left(\underset{M-m}{\leftrightarrow},v^1,\ldots,v^{m-1}\right)$$

Since in this case the last vertex in the above path does not contain b^P (by assumption), it follows from Case 3 that it is connected to $(\underset{M-1}{\leftrightarrow})$.. Thus v is connected to $(\underset{M-1}{\leftrightarrow})$.

Case 7: There exist m and $m' < M$ such that $m' \neq m$, $v^m = b^P$ and $v^{m'} = b^P$. Let $m^1 = \max\{m' < M | v^{m'} = b^P\}$ and $m^2 = \max\{m' < m^1 | v^{m'} = b^P\}$. By assumption m^1 and m^2 exist. Now consider two subcases. If $v^{m''} \neq b^l$ for some $m'' > m^1$ then it follows from Case 4 that v is connected to $(\underset{M-1}{\leftrightarrow})$. If $v^{m''} = b^1$ for all $m'' > m^1$ $(> m^2)$, consider the following path

$$(v^1,\dots,v^{M-1})\left(v^2,\dots,v^{M-1},\underset{1}{\leftrightarrow}\right)\dots\left(v^{m^2},\dots,v^{M-1},\underset{m^2-1}{\leftrightarrow}\right)$$

$$\left(v^{m^2+1},\dots,v^{M-1},\underset{m^2}{\leftrightarrow}\right)$$

This path belongs to the graph (V,E) because every vertex in the above expression, other than the last one, contain b^P twice. Moreover, it follows from Case 5 or Case 6 that the last vertex in the above path, and thus v, is connected to $(\underset{M-1}{\leftrightarrow})$..

Step 2: If ϕ is not empty then $\underline{v} = (\mathrm{b}^{\mathrm{P}}, \underset{M-1}{\leftrightarrow})$ and $\bar{\mathrm{v}} = (\alpha^{\bar{\mathrm{m}}},\dots,\alpha^{\mathrm{M}-1},\mathrm{b}^{\mathrm{P}}, \underset{m-2}{\leftrightarrow})$ satisfy $\vec{\rho}(\underline{v}) + 1 = \overleftarrow{\rho}(\underline{v})$ and $\vec{\rho}(\bar{\mathrm{v}}) = \overleftarrow{\rho}(\bar{\mathrm{v}}) + 1$, respectively.

Clearly $\overleftarrow{\rho}(\underline{v}) = |\{a \in \mathrm{B_P} | (\mathrm{M}-1)\bar{\pi}_i(\underline{v}) + \pi_i(\mathrm{a}) \leq \mathrm{M}\pi_i\}|$ and $\vec{\rho}(\underline{v}) = |\{a \in \mathrm{B_P} | (\mathrm{M}-1)\bar{\pi}_i(\underline{v}) + \pi_i(\mathrm{a}) \leq \mathrm{M}\pi_i$ and $\mathrm{a} \neq \alpha^{\mathrm{M}-1}\}|$. Therefore $\vec{\rho}(\underline{v}) + 1 = \overleftarrow{\rho}(\underline{v})$.

Also it follows from the definition of $\bar{m}$—condition A5—that $\vec{\rho}(\bar{\mathrm{v}}) = |\{a \in \mathrm{B_P} | (\mathrm{M}-1)\bar{\pi}_i(\bar{\mathrm{v}}) + \pi_i(\mathrm{a}) \leq \mathrm{M}\pi_i\}|$ and $\overleftarrow{\rho}(\bar{\mathrm{v}}) = |\{a \in \mathrm{B_P} | (\mathrm{M}-1)\bar{\pi}_i(\bar{\mathrm{v}}) + \pi_i(\mathrm{a}) \leq \mathrm{M}\pi_i$ and $\mathrm{a} \neq \mathrm{b}^l\}|$. Therefore $\vec{\rho}(\bar{\mathrm{v}}) = \overleftarrow{\rho}(\bar{\mathrm{v}}) + 1$.

Step 3: If ϕ is not empty then for any $v \varepsilon V$, different from $\underline{v}$ and $\bar{v}$, $\vec{\rho}(\mathrm{v}) = \overleftarrow{\rho}(\mathrm{v})$.

There are three subcases to consider.

(a) If $b^P \notin v$ then $(v,w) \in E \Rightarrow w^{M-1} = b^P$ and $(w,v) \in E \Rightarrow w^1 = b^P$. Therefore $\vec{\rho}(\mathrm{v}) = \overleftarrow{\rho}(\mathrm{v}) = 1$.

(b) If $b^P \epsilon v$ and $(v,a) \notin \phi$ for all $a \in B_P$ then it follows from v being different from $\underline{v}$ and from $\bar{v}$ that $\vec{\rho}(\mathrm{v}) = |\{a \in \mathrm{B_P} | (\mathrm{M}-1)\bar{\pi}_i(\mathrm{v}) + \pi_i(a) \leq M\pi_i\}| = \overleftarrow{\rho}(v)$.

(c) If $b^P \varepsilon v$ and $(v,a) \varepsilon \phi$ for some $a \in B_P$ then it follows from v being different from $\underline{v}$ and from $\bar{v}$ that $(v,a) = (\alpha^m, \dots, \alpha^{M-1}, b^P, b^l, \dots, b^l)$ for some m such that $\bar{m} < m < M$ $(m > \bar{m}$ because $v \neq \bar{v})$. Therefore $\vec{\rho}(\mathrm{v}) = |\{\mathrm{a} \in \mathrm{B_P} | (\mathrm{M}-1)\bar{\pi}_i(\mathrm{v}) + \pi_i(\mathrm{a}) \leq \mathrm{M}\pi_i$ and $\mathrm{a} \neq \mathrm{b}^{\mathrm{l}}\}|$ and $\overleftarrow{\rho}(\mathrm{v}) = |\{\mathrm{a} \in \mathrm{B_P} | (\mathrm{M}-1)\bar{\pi}_i(\mathrm{v}) + \pi_i(\mathrm{a}) \leq \mathrm{M}\pi_i$ and $\mathrm{a} \neq \alpha^{\mathrm{m}-1}\}|$. Clearly, in this case $\vec{\rho}(\mathrm{v}) = \overleftarrow{\rho}(\mathrm{v})$.

Step 4: If ϕ is empty then for any $v \varepsilon V$ $\vec{\rho}(\mathrm{v}) = \overleftarrow{\rho}(\mathrm{v})$.

This follows from both $\vec{\rho}(\mathrm{v})$ and $\overleftarrow{\rho}(\mathrm{v})$ being equal to one if $b^P \notin v$ and from both being equal to $|\{\mathrm{a} \in \mathrm{B_P} | (\mathrm{M}-1)\bar{\pi}_i(\mathrm{v}) + \pi_i(\mathrm{a}) \leq \mathrm{M}\pi_i\}|$, otherwise.

Steps 1–4 above imply that the graph (V,E) has an Eulerian path or an Eulerian cycle $Q=\{e(1), \ldots, e(|\overline{G}|)\}$ which traverses E according to $\{e(1), \ldots, e(|\overline{G}|)\}$, $e(1)=(\mathrm{b}^{\mathrm{P}},\mathrm{M}-\overset{\leftrightarrow}{2},\mathrm{a})$ for some $a \varepsilon B_P$ and if ϕ is not empty $e(|\overline{G}|)=(d',\alpha^{\overline{m}}, \ldots, \alpha^{M-1}, b^P, b^l, \ldots, b^l)$ for some $d' \in B_P$. Since each edge $e(j)$ corresponds to some unique element of $\overline{G}$, the Eulerian path Q satisfies Lemma A2. Q.E.D.

Lemma A3: *Suppose* G *is not empty and* $\alpha \equiv (\alpha^1, \ldots, \alpha^{M-1}) \varepsilon (B_{P-1})^{M-1}$. *Then there exists a path of action profiles* $d=(a^1, \ldots, a^{|G|}) \varepsilon C^M$ *with SR cycle* $\hat{d}=(e(1), \ldots, e(|G|))$ *such that*

(i) $e(t) \neq (\alpha^m, \ldots, \alpha^{M-1}, a^1, \ldots, a^m)$ $\forall m$ and t such that $1 \leq m < M$ and $t \leq |G|-M$.

(ii) $a^1=b^P$ and the elements of $\hat{d}$ are the same as those of G.

(iii) if $a^t=b^P$ for some $t>1$ then $\sum_{\tau=1}^{t-1}\pi_i(a^\tau) \leq (t-1)\pi_i$.

Proof of Lemma A3: There are four cases to consider.

Case 1: G does not satisfy property (γ) and every $e \varepsilon G$ is a permutation of $(\mathrm{b}^{\mathrm{P}}, \underset{M-1}{\leftrightarrow})$

In this case, let $\mathrm{d}=(\mathrm{b}^{\mathrm{P}}, \underset{M-1}{\leftrightarrow}) \in (\mathrm{B}_{\mathrm{P}})^{\mathrm{M}}$. Clearly, d satisfies conditions of the Lemma.

Case 2: G does not satisfy property (γ) and $\eta \neq b^P$.

Consider the following paths.

$$d(0)=\Big(b^P,\eta,\underset{M-2}{\leftrightarrow},b^P,\underset{1}{\leftrightarrow},\eta,\underset{M-3}{\leftrightarrow},b^P,\underset{2}{\leftrightarrow},\eta,\underset{M-4}{\leftrightarrow},b^P,\ldots,b^P,$$

$$\underset{M-2}{\leftrightarrow},\eta,b^P,\underset{M-1}{\leftrightarrow}\Big)$$

$$d(m)=\Big(b^P,\underset{m}{\leftrightarrow},\eta,\underset{M-m-2}{\leftrightarrow},b^P,\underset{m+1}{\leftrightarrow},\eta,\underset{M-m-3}{\leftrightarrow},b^P,\ldots,\eta,b^P,\underset{M-1}{\leftrightarrow},b^P,\eta,$$

$$\underset{M-2}{\leftrightarrow},b^P,\underset{2}{\leftrightarrow},\eta,\underset{M-3}{\leftrightarrow},\ldots,b^P,\underset{m-1}{\leftrightarrow},\eta,\underset{M-m-1}{\leftrightarrow}\Big), \forall m \text{ s.t. } 0<m<M$$

It is clear by inspection that in this case that for any m $(=0, 1, \ldots, M-1)$, b^P occurs every M periods along the path $d(m)$ and that the elements of the SR cycle $\hat{d}(m)$ generated by $d(m)$ are the same as the elements of G. The difference between $\hat{d}(0)$, $\hat{d}(1)$, $\ldots$ and $\hat{d}(M-1)$ is simply that the order the elements of G appear along these paths are different. The first element of $\hat{d}(0)$ is $(\mathrm{b}^{\mathrm{P}},\eta,\underset{M-2}{\leftrightarrow})$ whereas the first element of $\hat{d}(m)$ is $(\mathrm{b}^{\mathrm{P}},\underset{m}{\leftrightarrow},\eta,\underset{M-m-2}{\leftrightarrow})$. Now define the required path d as follows

$$d=\begin{cases} d(m) & \text{if } (\alpha^m, \ldots, \alpha^{M-1})=\Big(\eta,\underset{M-m-1}{\leftrightarrow}\Big) \\ d(0) & \text{otherwise} \end{cases}$$

It is clear by inspection that d satisfies conditions (i) and (ii) of the Lemma. To show that condition (iii) of the Lemma is satisfied denote the elements of d and its SR path by $(a^1, \ldots, a^G)$ and $(e(1), \ldots, e(G))$, respectively. Now if $a^t = b^P$ and $t > 1$, then it is clear by inspection that $t = kM + 1$ for some integer k and

$$\sum_{\tau=1}^{t-1} \pi_i(a^\tau) = M\{\overline{\pi}_i(e(1)) + \overline{\pi}_i(e(M+1) + \ldots + \overline{\pi}_i(e(M(k-1)+1))\}$$

Since $e(\tau) \in$ G, $\overline{\pi}_i(e(\tau)) < \pi_i$ for every τ; thus the RHS of the last equality is less than $Mk\pi_i = (t-1)\pi_i$.

Case 3: $\overline{G}$ does not satisfy property (γ), $(\mathrm{b}^{\mathrm{P}},\eta,\underset{M-2}{\leftrightarrow}) \in G$ and $\eta = b^P$.

In this case any element of G is either a permutation of $(\mathrm{b}^{\mathrm{P}},\underset{M-1}{\leftrightarrow}) \in \mathrm{A}^{\mathrm{M}}$ or a permutation of $(\mathrm{b}^{\mathrm{P}},\mathrm{b}^{\mathrm{P}},\underset{M-2}{\leftrightarrow}) \in \mathrm{A}^{\mathrm{M}}$. A simple permutation exercise demonstrates that $|G| = (1/2)(M^2 + M)$. Suppose that M is even. Let $k = M/2$. Now consider the following path of actions.

$$d = b^P \underset{k-1}{\leftrightarrow} b^P \underset{k}{\leftrightarrow} b^P \underset{k-2}{\leftrightarrow} b^P \underset{k+1}{\leftrightarrow} b^P \underset{k-3}{\leftrightarrow} b^P \underset{k+2}{\leftrightarrow}, \ldots, b^P \overset{\leftrightarrow}{1}\, b^P \underset{2k-2}{\leftrightarrow} b^P b^P \underset{M-1}{\leftrightarrow}$$

If we denote the elements of d by $(a^1, \ldots, a^{\mathrm{T}})$ then for every integer $t \le T$.

$$a^t = \begin{cases} b^P & \text{if either } t = 1 + \tau(M+1) \text{ or } t = 1 + \tau(M+1) + k - \tau \\ & \text{for some } \tau \text{ s.t. } 0 \le \tau \le k \\ b^l & \text{otherwise} \end{cases} \tag{A14}$$

A simple inspection of the sequence confirms that (when M is even) the length of d is $T \equiv (M^2 + M)/2 = |G|$, no string of length M occurs more than once in d and every string of length M of d is either a permutation of $(\mathrm{b}^{\mathrm{P}},\underset{M-1}{\leftrightarrow}) \in \mathrm{A}^{\mathrm{M}}$ or a permutation of $(\mathrm{b}^{\mathrm{P}},\mathrm{b}^{\mathrm{P}},\underset{M-2}{\leftrightarrow}) \in \mathrm{A}^{\mathrm{M}}$. Therefore the SR path induced by d coincides with the set G. Thus, given that $a^1 = b^P$, d satisfies condition (ii) of the Lemma. To show that d satisfies condition (i) denote the SR path of d by $(e(1), \ldots, e(T))$. Now I need to show that $(\alpha^m, \ldots, \alpha^{M-1}, a^1, \ldots, a^m) \ne e(t) \forall m < M$ and $\forall t \le T - M$. Let $\tilde{m} = \max\{m < M | \alpha^m \ne b^l\}$. Then it is clear by inspection that if $m > \tilde{m}$ or if $\tilde{m}$ does not exist then $(\alpha^m, \ldots, \alpha^{M-1}, a^1, \ldots, a^m) = (\underset{M-m}{\leftrightarrow},\mathrm{a}^1,\ldots,\mathrm{a}^m) = e(T - M + m + 1) \ne e(t)$ $\forall t \le T - M$. Moreover, since $\alpha^m \ne b^P$ $\forall m < M$, $a^1 = b^P$ and $e(t)$ is either a permutation of $(b^P, b^l, \ldots, b^l)$ or $(\mathrm{b}^{\mathrm{P}},\mathrm{b}^{\mathrm{P}},\underset{M-2}{\leftrightarrow})$, it follows that $(\alpha^m, \ldots, \alpha^{M-1}, a^1, \ldots, a^m) \ne e(t)$ $\forall m \le \tilde{m}$ and for all t.

To demonstrate that c satisfies condition (iii) of the Lemma, note that if $a^t = b^P$ then it follows from condition A14 that either $t = 1 + \tau(M+1)$ or

$t = 1 + \tau(M+1) + k - \tau$ for some non-negative integer $\tau \leq k$. In the case in which $t = 1 + \tau(M+1)$

$$\sum_{\mu=1}^{t-1} \pi_i(a^{\mu}) = \tau\{(2k-1)\pi_i(b^l) + 2\pi_i(b^P)\}$$

$$= \tau\{(M-1)\pi_i(b^l) + 2\pi_i(b^P)\} \tag{A15}$$

But $\pi_i(b^l) \leq \pi_i$ and $(M-2)\pi_i(b^l) + 2\pi_i(b^P) \leq M\pi_i$ (this follows from $(b^P, b^P, \underset{M-2}{\leftrightarrow}) \in G$)). Therefore the RHS of Eq. (A15) is less or equal to $\tau(M+1)\pi_i = (t-1)\pi_i$.

If, on the other hand, $t = 1 + \tau(M+1) + k - \tau$ and $t > 1$, then

$$\sum_{\mu=1}^{t-1} \pi_i(a^{\mu}) = (\tau(2k-1) + k - \tau - 1)\pi_i(b^l) + (2\tau+1)\pi_i(b^P)$$

$$= (2\tau+1)\{(k-1)\pi_i(b^l) + \pi_i(b^P)\} \leq (t-1)\pi_i$$

(the last inequality follows from $(M-2)\pi_i(b^l) + 2\pi_i(b^P) < \pi_i$).

If M is odd set $k = M/2 + \frac{1}{2}$ and consider the following path:

$$d = b^P \underset{k-1}{\leftrightarrow} b^P \underset{k-2}{\leftrightarrow} b^P \underset{k}{\leftrightarrow} b^P \underset{k-3}{\leftrightarrow} b^P \underset{k+1}{\leftrightarrow}, \ldots, b^P \overset{\leftrightarrow}{1} b^P \underset{2k-3}{\leftrightarrow} b^P b^P \underset{M-1}{\leftrightarrow}$$

The same reasoning as in the case when M even establishes that the above path satisfies conditions (i)–(iii) of the Lemma when M is odd.

Case 4: $\overline{G}$ satisfies property (γ).

Let $Q = \{e(1), \ldots, e(|\overline{G}|)\}$ be the path (cycle) satisfying Lemma A2 and let $T = |\overline{G}| + M - \overline{m}$, where $\overline{m}$ is defined as in condition A5. Now define the sequence $d = (a^1, \ldots, a^T)$ as follows

$$a^t = e(t)^1 \text{ for any integer } t \leq |\overline{G}| \tag{A16}$$

$$a^{|\overline{G}|+m} = e(|\overline{G}|)^{m+1} \text{ for any integer } m \leq M - \overline{m} \tag{A17}$$

where $e(t)^i$ is the ith element of the M-string $e(t)$. It follows from the definition of $\overline{m}$—condition A5—that if ϕ is empty then $\overline{m} = M$ and thus $T = |\overline{G}|$. Also it follows from (ii) and (iii) of Lemma A2 that

$$(a^1, \ldots, a^{M-1}) = (b^P, M - \overset{\leftrightarrow}{2}, a^M) \tag{A18a}$$

$$\text{and if } \phi \text{ is not empty then } (a^{|\overline{G}|+1}, \ldots, a^{|\overline{G}|+M-\overline{m}}) = (\alpha^{\overline{m}}, \ldots, \alpha^{M-1}) \tag{A18b}$$

For any $m \leq M - \overline{m}$, define $e(|\overline{G}| + m)$ as

$$e(|\overline{G}| + m) = \left(\alpha^{\overline{m}+m-1}, \alpha^{\overline{m}+m}, \ldots, \overline{\alpha}^M, b^P, \underset{\overline{m}+m-2}{\leftrightarrow}\right) \tag{A19}$$

Clearly, it follows from Eqs. (A16), (A17), (A18a) and (A18b) that $(e(1), \ldots, e(|\overline{G}| + M - \overline{m}))$ is the SR cycle induced by d. Condition (A19) implies that

$(e(|\overline{G}|+1), \ldots, e(|\overline{G}|+M-\overline{m})) = \underline{\phi}$. Also (i) of Lemma A2 implies that $Q=(e(1), \ldots, e(|\overline{G}|)) = \overline{G}$. Since $\overline{G} \cap \underline{\phi}$ is empty, it follows that d satisfies condition (i) of Lemma A3. Moreover, since $\overline{G} \cup \underline{\phi} = G$ and $a^1 = b^P$ it follows that d satisfies condition (ii) of Lemma A3.

Finally, to show that d satisfies condition (iii) of Lemma A3, suppose that $a^t = b^P$ for some $t \leq T$. Now let $\{\bar{e}(1), \ldots, \bar{e}(t-1)\}$ be a SR *cycle* induced by $(a^1, \ldots, a^{t-1})$. Clearly,

$$\sum_{\tau=1}^{t-1} \pi_i(a^\tau) = \sum_{\tau=1}^{t-1} \overline{\pi}_i(\bar{e}(\tau)) = \sum_{\tau=1}^{t-M} \overline{\pi}_i(e(\tau)) + \sum_{\tau=t-M+1}^{t-1} \overline{\pi}_i(\bar{e}(\tau)) \qquad \text{(A20)}$$

(The last equality follows from $\bar{e}(\tau) = e(\tau)\ \forall \tau \leq t-M$). Since $(a^1, \ldots, a^{\mathrm{M}-1}) = (\mathrm{b}^{\mathrm{P}}, \underset{M-2}{\leftrightarrow})$ and $a^t = b^P$, it follows that for any integer $\tau < M$, $\bar{e}(t-M+\tau) = (a^{t-\mathrm{M}+\tau}, \ldots, a^{t-1}, \mathrm{b}^{\mathrm{P}}, \underset{\tau}{\leftrightarrow})$ and $e(t-M+\tau) = (a^{t-M+\tau}, \ldots, a^{t-1}, b^P, a^{t+1}, \ldots, a^{t+\tau-1})$.[14] Thus it follows from the definition of b^l that

$$\overline{\pi}_i(\bar{e}(t-M+\tau)) \leq \overline{\pi}_i(e(t-M+\tau)), \forall \tau < M$$

This together with $e(\tau) \varepsilon G$ for all $\tau \leq T$ imply that

$$\text{the RHS of Eq. A20} \leq \sum_{\tau=1}^{t-1} \overline{\pi}_i(e(\tau)) \leq (t-1)\pi_i$$

Thus d satisfies condition (iii) of Lemma A3. Q.E.D.

Finally I come to the proof of Lemma 1. The proof, as was mentioned before, is by induction on the number of the elements of the set B_R. Clearly, the result is true if $R=1$ (B_R contains one element). Suppose the Lemma holds for the case when $R=(P-1)$. Therefore, there exists a path $\bar{c} = (\sigma^1, \ldots, \sigma^{t(P-1)+M-1}) \in \overline{C}^M$ which satisfies Lemma 1 for the set B_{P-1}. It also follows that there exists a path $d=(a^1, \ldots, a^{|G|}) \varepsilon C^M$ satisfying conditions (i)–(iii) of Lemma A3 for the sequence α defined by

$$\alpha \equiv (\sigma^{t(P-1)+1}, \ldots, \sigma^{t(P-1)+M-1}) \qquad \text{(A21)}$$

I can define a new path c, which satisfies the conditions of Lemma 1, by amalgamating $\bar{c}$ and d as follows. Let

$$c \equiv (\bar{c}, d) = (\sigma^1, \ldots, \sigma^{t(P-1)+M-1}, a^1, \ldots, a^{|G|})$$

To simplify the notation, I will denote a^τ by $\sigma^{\tau(P-1)+M-1+\tau}$, $\forall \tau \leq |G|$. Thus let $c=(\sigma^1, \ldots, \sigma^K)$ where $K \equiv t(P-1)+M-1+|G|$. Also let $\hat{c} \equiv (e(1), \ldots, e(K))$ be the SR cycle induced by c.

---

[14] Note that if $t+\tau > T$ then $a^{t+\tau}$ is defined to be $a^{t+\tau-T}$.

Firstly, I need to show that $c \varepsilon \overline{C}^{\mathrm{M}}$. Since $c=(\bar{c},d)$, it follows from $\bar{c}$ satisfying condition (a) of Lemma 1 for the set B_{P-1} and from d satisfying condition (ii) of Lemma A3 that

$$e(\tau) \in \begin{cases} E(\pi, P-1) & \text{if } \tau < t(P-1) \\ \left\{ e \in (B_P)^M \,|\, e \notin (B_{P-1})^M \right\} & \text{if } t(P-1) \leq \tau \leq t(P-1) + |G| \end{cases} \tag{A22}$$

Since $\bar{c} \varepsilon \overline{C}^M$, $d \varepsilon \overline{C}^M$, d satisfies condition (i) of Lemma A3 for the sequence α defined by Eq. (A21) and $E(\pi, P-1) \subseteq (B_{P-1})^M$, it follows from condition A22 that $e(\tau) \neq e(\mu)$ $\forall \tau \neq \mu$, τ and $\mu \leq t(P-1) + |G|$. But this implies that $c \varepsilon \overline{C}^M$.

Secondly, note that $\bar{c}$ satisfies (a) and (b) of Lemma 1 for any $r \leq P-1$ and the path d satisfies condition (ii) of Lemma A3. Therefore c satisfies conditions (a) and (b) of Lemma 1 $\forall r \leq P$.

Finally, to show that condition (c) of Lemma 1 is satisfied, suppose that $\sigma^t = b^r$. Clearly, I only need to show that (c) of Lemma 1 holds for all $t \geq t(P-1) + M$ (this is because $\bar{c}$ satisfies condition (c) of Lemma 1 for the set B_{P-1}). Now if $r = P$, condition (c) follows immediately from $c = (\bar{c}, d)$ and from d satisfying condition (iii) of Lemma A3. If, on the other hand, $r < P$ and $t \geq t(P-1) + M$, then there exist an integer $\upsilon < t(r) + M$ such that $e(\mathrm{v}) =$ (b^{r}, $\leftrightarrow$).. (This follows from satisfying conditions (a) and (b) of Lemma 1 for the set $\overset{M-1}{B}_{P-1}$.) But

$$\sum_{\mu = t(r-1)+M}^{t-1} \pi_i(\sigma^\mu) = \sum_{\mu = t(r-1)+M}^{\upsilon-1} \pi_i(\sigma^\mu) + \sum_{\mu=\upsilon}^{t-1} \pi_i(\sigma^\mu) \tag{A23}$$

Since $\bar{c}$ satisfies (c) of Lemma 1 for the set B_{P-1}, $r < P$ and $\sigma^\upsilon = b^r$, it follows that

$$\sum_{\mu = t(r-1)+M}^{\upsilon-1} \pi_i(\sigma^\mu(f)) \leq \pi_i(\upsilon - t(r-1) - M) \tag{A24}$$

Now consider the sequence $\tilde{d} \equiv (\sigma^\upsilon, \ldots, \sigma^{t-1})$. Note that

$$\sum_{\mu=\upsilon}^{t-1} \pi_i(\sigma^\mu) = \sum_{\mu=\upsilon}^{t-1} \overline{\pi}_i(\bar{e}(\mu)) \tag{A25}$$

where $(\bar{e}(\upsilon))$, $\ldots$, $\bar{e}(t-1))$ is the SR cycle induced by $\tilde{d}$. For any μ s.t. $\upsilon \leq \mu \leq t - M$, $e(\mu) = \bar{e}(\mu)$. Since $(\sigma^\upsilon, \ldots, \sigma^{\upsilon+M-1}) = (b^r, b^l, \ldots, b^l)$ and $\sigma^t = b^r$ (by assumption), it follows from the definition of b^l that $\overline{\pi}_i(\bar{e}(\mu) \leq \overline{\pi}_i(e(\mu))$ for any μ s.t. $t - M < \mu < t$. Since $\overline{\pi}_i(e(\mu)) < \pi_i$ for every μ, the RHS of Eq. (A25) $\leq (t - \upsilon)\pi_i$. This together with Eqs. (A23) and (A24) imply that c satisfies condition (c) of Lemma 1 for the case where $\sigma^t = b^r$ and $r < P$. Q.E.D.

Appendix B. Proof of proposition 1

For any set of action profiles $(\mathrm{a}^1,\ldots,\mathrm{a}^{\mathrm{r}}) \varepsilon A^r$, let $\bar{u}_i(\mathrm{a}^1,\ldots,\mathrm{a}^{\mathrm{r}}) \equiv \sum_{\tau=1}^{r-1} \pi_i(\mathrm{a}^\tau) + u_i(\mathrm{a}^r)$. Also for any strategy $f \in F^M$ with cycle path $\Sigma(f) = (a^1,\ldots,a^R)$ define $\tilde{\pi}_i(f) = \max_{1 \le r \le \mathrm{R}} \{\bar{u}_i(\mathrm{a}^1,\ldots,\mathrm{a}^r) - \mathrm{rv_i}\}$. Now for any player i consider an optimal penal code $\bar{f} = \{\bar{s}^t(\cdot)\}_{t=1}^{\infty} \varepsilon \bar{P}_i$ such that

$$\tilde{\pi}_i(\bar{f}) \le \tilde{\pi}_i(f), \; \forall f \in \bar{P}_i \tag{B1}$$

(Since $|A|$ and M are finite such an optimal penal code exists).

For any real number $x \in R$, let $D(x)$ be the set of histories $h^t = (a^1, \ldots, a^{t-1})$ which satisfy

$$\forall \text{ integer } r \le t \text{ s.t. } \bar{u}_i\left(a^r,\ldots,a^{t-1},\bar{s}^t(h^t)\right) < v_i(t-r+1) + x \tag{B2}$$

Define the strategy $\mathrm{f}_{\mathrm{i}}^{\mathrm{x}} = \{\mathrm{s}_{\mathrm{i}}^{\mathrm{xt}}(\mathrm{h}^{\mathrm{t}})\}_{\mathrm{t}=1}^{\infty}$ for i as:

$$s_i^{xt}(h^t) \in \begin{cases} \bar{s}_i^t(h^t) & \text{if } h^t \in D(x) \\ \arg\max\limits_{a_i} \pi_i\left(a_i,\bar{s}_{-i}^t(h^t)\right) & \text{otherwise} \end{cases} \tag{B3}$$

Denote the cycle path $(f_i^x,\bar{f}_{-i})$ enters by $\Sigma(f^x,\bar{f}_{-i}) = (a^1, \ldots, a^R)$. It follows from the definition of f_i^x, that $\forall$ integer $r \le R$

$$\pi_i(a^r) = u_i(a^r) \text{ if for some } r' \le r,\, \bar{u}_i(a^{r'},\ldots,a^r) \ge v_i(r-r'+1) + x \tag{B4}$$

Lemma B1: *For any $x \le \tilde{\pi}_i(\bar{f})$, let $\Sigma(f_i^x,\bar{f}_{-i}) = (a^1,\ldots,a^R)$. Then $\exists$ integers r and r' where $r' \le r \le R$ s.t.*

$$\sum_{\tau=r'}^{r} \pi_i(a^\tau) \ge v_i(r-r'+1) + x \tag{B5}$$

Proof: It follows from Eq. (B4) that $(a^1,\ldots,a^R)$ satisfies condition B5 if $\exists$ integers r and $r' \le r \le R$ s.t.

$$\bar{u}_i(a^{r'},\ldots,a^r) \ge v_i(r-r'+1) + x \tag{B6}$$

To prove the existence of r' and r satisfying condition B6, suppose otherwise. Then it follows from the definition of f_i^x that eventually $(f_i^x,\bar{f}_{-i})$ and $\bar{f}$ will play the game in exactly the same way: $\exists$ a history h^t s.t. $\bar{\sigma}^t\ (f_i^x,\bar{f}_{-i}) = \sigma^\tau(\bar{f}|h^t)$ $\forall \tau \ge t$. Since $\bar{f}$ is a SPE, $\bar{f}|h^t$ is also a SPE. Moreover $\pi_i^\infty(\sigma(f_i^x,\bar{f}_{-i})) = \pi_i^\infty(\sigma(\bar{f}|h^t)) \ge v_i = \pi_i^\infty(\sigma(\bar{f})) \ge \pi_i^\infty(\sigma(f_i^x,\bar{f}_{-i}))$. Thus $\pi_i^\infty(f_i^x,\bar{f}_{-i}) = v_i$ and therefore $(\bar{f}|h^t) = (f_i^x,\bar{f}_{-i})$ is an optimal penal code. Hence it follows from condition B1 that $\tilde{\pi}_i(\mathrm{f}_{\mathrm{i}}^{\mathrm{x}},\bar{\mathrm{f}}_{-\mathrm{i}}) \ge \tilde{\pi}_{\mathrm{i}}(\bar{\mathrm{f}})$. Thus there exists an integer $r \le R$ s.t.

$$\bar{u}_i(a^1,\ldots,a^r) \ge rv_i + \tilde{\pi}_i(\bar{f}) \tag{B7}$$

Since $x \le \tilde{\pi}_i(\bar{f})$,, it follows that the LHS of condition B7 exceeds $rv_i + x$; but this contradicts the initial supposition. Q.E.D.

Lemma B2: *For any* $x \le \tilde{\pi}_i(\bar{f})$,

$$v_i \ge \frac{1}{M-1+|E(v_i, A(x))|}\left\{x + (M-1)\pi_i(b^l) + \sum_{e \in E(v_i, A(x))} \overline{\pi}_i(e)\right\} \tag{B8}$$

$$\text{for some } b^l \varepsilon A(x), \text{ where } A(x) = \{a | u_i(a) < v_i + x\}. \tag{B9}$$

Proof: Denote $\Sigma(f_i^x, \bar{f}_{-i})$ by $(a^1, \ldots, a^R)$. For any $r \le R$, let

$$t(r) = \max\{t \ge r | \bar{u}_i(a^r, \ldots, a^t) \ge v_i(t - r + 1) + x \text{ and } t \le R\} \tag{B10}$$

It follows from Lemma B1 that $t(r)$ is well-defined from some $r \le R$. Now define sequence $(r_1, \ldots, r_K)$ inductively as follows

$$r_1 = \min\{r \le R | t(r) \text{ exists}\}$$

$$r_k = \min\{r \le R | r > t(r_{k-1}) \text{ and } t(r) \text{ exists}\}, \forall k \text{ s.t. } 1 < k \le K$$

where K is such that for all $r > r_K$, $t(r)$ does not exist. Since $t(r)$ is well-defined for some $r \le R$ the sequence $(r_1, \ldots, r_K)$ is well defined. Moreover, it follows from the definition of r_k, conditions B4 and B10 that

$$\sum_{\tau = r_k}^{t(r_k)} \pi_i(a^\tau) = \bar{u}_i(a^{r_k}, \ldots, a^{t(r_k)}) \ge v_i(t(r_k) - r_k + 1) + x, \forall k \le K \tag{B11}$$

Now since $\bar{f}$ is a SPE, it follows that

$$v_i = \pi_i^\infty(\sigma(\bar{f})) \ge \pi_i^\infty(\sigma(f_i^x, f_{-i})) = \frac{1}{R}\sum_{\tau=1}^{R} \pi_i(a^\tau)$$

$$= \frac{1}{R}\sum_{k=1}^{K}\sum_{\tau = t(r_k)+1}^{t(r_{k+1})} \pi_i(a^\tau)$$

where $t(r_{k+1}) = R + t(r_1)$ and $(a^{R+1}, \ldots, a^{t(r_{K+1})}) = (a^1, \ldots, a^{t(r_1)})$. Thus for some $k \le K$

$$v_i \ge \frac{1}{t(r_{k+1}) - t(r_k)}\left\{\sum_{\tau = t(r_k)+1}^{t(r_{k+1})} \pi_i(a^\tau)\right\} = \frac{1}{t(r_{k+1}) - t(r_k)}$$

$$\times\left\{\sum_{\tau = t(r_k)+1}^{r_{k+1}-1} \pi_i(a^\tau) + v_i(t(r_{k+1}) - r_{k+1} + 1) + x\right\} \tag{B12}$$

(the last equality follows from Eq. (B11)). Now let $(e(1),\ldots,e(R))$ be the SR cycle induced by $(a^1,\ldots,a^R)$. Also, let $\bar{e}(\tau)=(a^{r_{k+1}-\tau},\ldots,a^{r_{k+1}-1},a^{t(r_k)+1},\ldots,a^{t(r_k)+M-\tau})$. Then $(e(t(r_k)+1),\ldots,e(r_{k+1}-M),\bar{e}(1),\ldots,\bar{e}(M-1))$ is the SR cycle induced by $(a^{t(r_k)+1},\ldots,a^{r_{k+1}-1})$. Therefore it follows from Eq. (B12) that

$$v_i\big(r_{k+1}-t(r_k)-1\big)\geq\sum_{\tau=t(r_k)+1}^{r_{k+1}-1}\pi_i(a^\tau)+x\geq\sum_{\tau=t(r_k)+1}^{r_{k+1}-M}\overline{\pi}_i(e(\tau))$$
$$+\sum_{\tau=1}^{M-1}\overline{\pi}_i(\bar{e}(\tau))+x \tag{B13}$$

Since $(e(1),\ldots,e(R))$ is the SR cycle induced by $(a^1,\ldots,a^R)$, it follows that

$$e(\tau)\neq e(\tau'),\ \forall\tau\neq\tau' \tag{B14}$$

Now let $\Gamma=\{e\,|\,e=e(\tau)$ for some τ s.t. $t(r_k)+1\leq\tau\leq r_{k+1}-M$ and $\pi_i(e)<v_i\}$. Then, it follows from Eqs. (B13) and (B14) that

$$v_i\geq\frac{1}{M-1+|\Gamma|}\left\{\sum_{e\in\Gamma}\overline{\pi}_i(e)+\sum_{\tau=1}^{M-1}\overline{\pi}_i(\bar{e}(\tau))+x\right\} \tag{B15}$$

Now it follows from the definition of r_k that $u_i(a^\tau)<v_i+x$ for all τ s.t. $t(r_k)\leq\tau\leq r_{k+1}-1$ for all k. Thus $\Gamma\subset E(v_i,A(x))$ and $\overline{\pi}_i(\bar{e}(\tau))\geq\pi_i(b^l)$ for some $b^l\in A(x)$. Therefore it follows from Eq. (B15) that

$$v_i\geq\frac{1}{M-1+|E(v_i,A(x))|}\left\{x+\sum_{e\in E(v_i,A(x))}\overline{\pi}_i(e)+(M-1)\pi_i(b^l)\right\}$$
$$\text{for some } b^l\in A(x) \tag{B16}$$

Q.E.D.

Now to complete the proof of Proposition 1, let $B=A(\tilde{\pi}_i(\bar{f}))$. Denote the elements of B by $\{b^\tau\}_{\tau=1}^R$ s.t. $u_i(b^1)\leq u_i(b^2)\leq\ldots\leq u_i(b^R)$. Then it follows from the definition of $A(x)$ that

$$A\big(u_i(b^r)-v_i\big)=B_{r-1}\equiv\{b^\tau\}_{\tau=1}^{r-1} \tag{B17}$$

Setting $x=\tilde{\pi}_i(\bar{f})$, it follows from Lemma B2 and $B=A(\tilde{\pi}_i(\bar{f}))$ that

$$v_i\geq\frac{1}{M-1+|E(v_i,B)|}\left\{\tilde{\pi}_i(\bar{f})+(M-1)\pi_i(b^l)\right.$$
$$\left.+\sum_{e\varepsilon E(v_i,B)}\tilde{\pi}_i(e)\right\} \tag{B18}$$

for some $b^l\varepsilon B$. Condition (i) of Proposition 1 follows from Eq. (B18) and the definition of $\tilde{\pi}_i(\bar{f})$. To demonstrate condition (ii) of Proposition 1, first note that $u_i(b^1)\leq u_i(a)$ for all $a\varepsilon A$. (This follows from the definition of b^1.) Since v_i can be sustained as a SPE, v_i is individually rational and thus $v_i \geq u_i(b^1)$. Now set $x = u_i(b^r) - v_i$. Then it follows from Lemma B2 and Eq. (B17) that

$$v_i \geq \frac{1}{M-1+|E(v_i,B_{r-1})|}\left\{u_i(b^r) - v_i + (M-1)\pi_i(b^l) + \sum_{e \varepsilon E(v_i,B_{r-1})} \overline{\pi}_i(e)\right\}$$

for some $b^l \varepsilon B_{r-1}$. Condition (ii) of Proposition 1 follows from rearranging the last inequality and $v_i \geq u_i(b^1)$. This completes the proof of Proposition 1.

References

Abreu, D., 1984. Infinitely repeated games with discounting: A general theory. Econometrica.

de Bruijn, N.G., 1946. A combinatorial problem. Koninklijke Nederlands Akademie van Wetenschappen, Proceedings 49 (Part 2), 758–764.

Fudenberg, D., Maskin, E., 1986. The folk theorem in repeated games with discounting or incomplete information. Econometrica 54, 533–556.

Golomb, S., 1967. Shift Register Sequences. Holden-Day.

Good, I.J., 1946. Normal recurring decimals. J. London Math. Soc. 21 (Part 3), 169–172.

Hall Jr., M., 1968. Combinatorial Mathematics.

Kalai, E., 1990. Bounded rationality and strategic complexity in repeated games. In: Chiishi, T.I., Neyman, A., Tauman, Y. (Eds.), Game Theory and Applications. Academic Press.

Pearce, 1992.

Rubinstein, A., 1979. Equilibrium in supergames with the overtaking criterion. J. Econ. Theory 21, 1–9.

Sabourian, H., 1989. Repeated games: A Survey. In: Hahn, F.H. (Ed.), The Economics of Missing Markets, Information and Games. Oxford Univ. Press.

Sabourian, H., 1989. The folk theorem of repeated games with bounded (one-period) memory. Economic Theory Discussion Paper No. 143, Cambridge.

Sabourian, H., 1989. Repeated games with M-period bounded memory (pure strategies). Economic Theory Discussion Paper No. 144, Cambridge.

Swamy, M.N.S., Thulasiraman, K., 1981. Graphs: Theory and Algorithms. Wiley, New York.