Efficient Repeated Implementation: Supplementary Material

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This Supplementary Material to Lee and Sabourian [3] (henceforth, LS) presents some formal results and proofs omitted from LS.

A Two-agent case

Proof of Theorem 3 in LS

Consider regime $\widehat{R}$ defined in Section 4.2 of LS. We prove the theorem via the following claims.

Claim A.1. *Fix any $\sigma \in \Omega^\delta(\widehat{R})$. For any $t > 1$ and $\theta(t)$, if $g^{\theta(t)} = \hat{g}$, $\pi_i^{\theta(t)} \geq v_i(f)$.*

Proof. This can be established by analogous reasoning to that behind Lemma 2 in LS. □

Claim A.2. *Fix any $\sigma \in \Omega^\delta(\widehat{R})$. Also, assume that, for each i, outcome $\tilde{a}^i \in A$ used in the construction of S^i above satisfies condition (7) in LS. Then, for any t and $\theta(t)$, if $g^{\theta(t)} = \hat{g}$ then $m_i^{\theta(t),\theta^t} = (\cdot, 0)$ and $m_j^{\theta(t),\theta^t} = (\cdot, 0)$ for any θ^t.*

Proof. Suppose not; then, for some t, $\theta(t)$ and θ^t, $g^{\theta(t)} = \hat{g}$ and the continuation regime next period at $\mathbf{h}(\theta(t), \theta^t)$ is either D^i or S^i for some i. By similar reasoning to the

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three-or-more-player case, it then follows that, for $j \neq i$,

$$\pi_j^{\theta(t),\theta^t} < v_j^j. \tag{A.1}$$

Consider two possibilities. If the continuation regime is $S^i = \Phi(\tilde{a}^i)$ then $\pi_i^{\theta(t),\theta^t} = v_i(f) = v_i(\tilde{a}^i)$ and hence (A.1) follows from (7) in LS. If the continuation regime is D^i or $S^i \neq \Phi(\tilde{a}^i)$, $d(i)$ occurs in some period. But then (A.1) follows from $v_j(\tilde{a}^i) \leq v_j^j$ and $v_j^i < v_j^j$ (where the latter inequality follows from Assumption (A)).

Then, given (A.1), agent j can profitably deviate at $(\mathbf{h}(\theta(t)), \theta^t)$ by announcing the same state as σ_j and an integer higher than i's integer choice at such a history. This is because the deviation does not alter the current outcome (given the definition of ψ of $\hat{g}$) but induces regime D^j in which, by (A.1), j obtains $v_j^j > \pi_j^{\theta(t),\theta^t}$. But this is a contradiction. □

Claim A.3. *Assume that f is efficient in the range and, for each i, outcome $\tilde{a}^i \in A$ used in the construction of S^i above satisfies condition (7) in LS. Then, for any $\sigma \in \Omega^\delta(\widehat{R})$, $\pi_i^{\theta(t)} = v_i(f)$ for any i, $t > 1$ and $\theta(t)$.*

Proof. Given Claims 1-2, and since f is efficient in the range, we can directly apply the proof of Lemma 4 in LS. □

Claim A.4. *$\Omega^\delta(\widehat{R})$ is non-empty if self-selction holds.*

Proof. Consider a symmetric Markov strategy profile in which, for any θ, each agent reports $(\theta, 0)$. Given ψ and self-selection, any unilateral deviation by i at any θ results either in no change in the current period outcome (if he does not change his announced state) or it results in current period outcome belonging to $L_i(\theta)$. Also, given the transition rules, a deviation does not improve continuation payoff at the next period either. Therefore, given self-selection, it does not pay i to deviate from his strategy. □

Finally, given Claims 3-4, the proof of Theorem 3 follows by exactly the same arguments as those behind Theorem 2 and its Corollary in LS.

Alternative condition to self-selection and condition ω (ω')

As mentioned at the end of Section 4.2 in LS, the conclusions of Theorem 3 can be obtained using an alternative condition to self-selection and condition ω (ω'), if δ is sufficiently large.

Theorem A.1. *Suppose that $I = 2$, and consider an SCF f such that there exists $\tilde{a} \in A$ such that $v_i(\tilde{a}) < v_i(f)$ for $i = 1, 2$. If f is efficient in the range, there exist a regime R and $\bar{\delta}$ such that, for any $\delta > \bar{\delta}$,(i) $\Omega^\delta(R)$ is non-empty; and (ii) for any $\sigma \in \Omega^\delta(R)$, $\pi_i^{\theta(t)}(\sigma, R) = v_i(f)$ for any i, $t \geq 2$ and $\theta(t)$. If, in addition, f is strictly efficient in the range then $a^{\theta(t),\theta^t}(\sigma, R) = f(\theta^t)$ for any $t \geq 2$, $\theta(t)$ and θ^t.*

Proof. Following Lemma 1 in LS, let S^i be the regime alternating $d(i)$ and $\phi(\tilde{a})$ from which $i = 1, 2$ can obtain payoff exactly equal to $v_i(f)$. For $j \neq i$, let $\pi_j(S^i)$ be the maximum payoff that j can obtain from regime S^i when i behaves rationally in $d(i)$. Since S^i involves $d(i)$, Assumption (A) in LS implies that $v_j^j > \pi_j(S^i)$. Then there must also exist $\epsilon > 0$ such that $v_j(\tilde{a}) < v_i(f) - \epsilon$ and $\pi_j(S^i) < v_i^i - \epsilon$. Next, define $\rho \equiv \max_{i,\theta,a,a'} [u_i(a, \theta) - u_i(a', \theta)]$ and $\bar{\delta} \equiv \frac{\rho}{\rho+\epsilon}$.

Mechanism $\tilde{g} = (M, \psi)$ is defined such that, for all i, $M_i = \Theta \times \mathbb{Z}_+$ and ψ is such that

1. if $m_i = (\theta, \cdot)$ and $m_j = (\theta, \cdot)$, $\psi(m) = f(\theta)$;
2. if $m_i = (\theta^i, z^i)$, $m_j = (\theta^j, 0)$ and $z^i \neq 0$, $\psi(m) = f(\theta^j)$;
3. for any other m, $\psi(m) = \tilde{a}$.

Let $\widetilde{R}$ denote any regime satisfying the following transition rules: $\widetilde{R}(\emptyset) = \tilde{g}$ and, for any $h = ((g^1, m^1), \ldots, (g^{t-1}, m^{t-1})) \in H^t$ such that $t > 1$ and $g^{t-1} = \tilde{g}$:

Rule 1: if $m_i^{t-1} = (\theta, 0)$ and $m_j^{t-1} = (\theta, 0)$, $\widetilde{R}(h) = \tilde{g}$;

Rule 2: if $m_i^{t-1} = (\theta^i, 0)$, $m_j^{t-1} = (\theta^j, 0)$ and $\theta^i \neq \theta^j$, $\widetilde{R}(h) = \Phi^{\tilde{a}}$;

Rule 3: if $m_i^{t-1} = (\theta^i, z^i)$, $m_j^{t-1} = (\theta^j, 0)$ and $z^i \neq 0$, $\widetilde{R}|h = S^i$;

Rule 4: if m^{t-1} is of any other type and i is lowest-indexed agent among those who announce the highest integer, $\widetilde{R}|h = D^i$.

We next prove the theorem via the following claims.

Claim A.5. *Fix any $\sigma \in \Omega^\delta(\widetilde{R})$. For any $t > 1$ and $\theta(t)$, if $g^{\theta(t)} = \tilde{g}$, $\pi_i^{\theta(t)} \geq v_i(f)$.*

Proof. Suppose not; then at some $t > 1$ and $\theta(t)$, $g^{\theta(t)} = \tilde{g}$ but $\pi_i^{\theta(t)} < v_i(f)$ for some i. Let $\theta(t) = (\theta(t-1), \theta^{t-1})$. Given the transition rules, it must be that $g^{\theta(t-1)} = \tilde{g}$ and $m_i^{\theta(t-1),\theta^{t-1}} = m_j^{\theta(t-1),\theta^{t-1}} = (\tilde{\theta}, 0)$ for some $\tilde{\theta}$.

Consider i deviating at $(\mathbf{h}(\theta(t-1)), \theta^{t-1})$ such that he reports $\tilde{\theta}$ and a positive integer. Given the output function ψ of mechanism $\tilde{g}$, the deviation does not alter the current outcome but, by Rule 3 of regime $\widetilde{R}$, can yield continuation payoff $v_i(f)$. Hence, the deviation is profitable, implying a contradiction. □

Claim A.6. *Fix any* $\delta \in (\bar{\delta}, 1)$ *and* $\sigma \in \Omega^\delta(\widetilde{R})$. *For any* t *and* $\theta(t)$, *if* $g^{\theta(t)} = \tilde{g}$, $m_i^{\theta(t),\theta^t} = m_j^{\theta(t),\theta^t} = (\theta, 0)$ *for any* θ^t.

Proof. Suppose not; then for some t, $\theta(t)$ and θ^t, $g^{\theta(t)} = \tilde{g}$ but $m^{\theta(t),\theta^t}$ is not as in the claim. There are three cases to consider.

<u>Case 1</u>: $m_i^{\theta(t),\theta^t} = (\cdot, z^i)$ and $m_j^{\theta(t),\theta^t} = (\cdot, z^j)$ with $z^i, z^j > 0$.

In this case, given ψ, $\tilde{a}$ is implemented in the current period and, by Rule 4, a dictatorship by, say, i follows forever thereafter. But then, by Assumption (A) in LS, j can profitably deviate by announcing an integer higher than z^i at such a history; the deviation does not alter the current outcome from $\tilde{a}$ but switches dictatorship to himself as of the next period.

<u>Case 2</u>: $m_i^{\theta(t),\theta^t} = (\cdot, z^i)$ and $m_j^{\theta(t),\theta^t} = (\theta^j, 0)$ with $z^i > 0$.

In this case, given ψ, $f(\theta^j)$ is implemented in the current period and, by Rule 3, continuation regime S^i follows thereafter. Consider j deviating to another strategy identical to σ_j everywhere except at $(\mathbf{h}(\theta(t)), \theta^t)$ it announces an integer higher than z^i.

Given ψ (part 3) and Rule 4, this deviation yields a continuation payoff $(1-\delta)u_j(\tilde{a}, \theta^t) + \delta v_j^j$, while the corresponding equilibrium payoff does not exceed $(1-\delta)u_j(f(\theta^j), \theta^t) + \delta \pi_j(S^i)$. But, since $v_j^j > \pi_j(S^i) + \epsilon$ and $\delta > \bar{\delta}$, the former exceeds the latter, and the deviation is profitable.

<u>Case 3</u>: $m_i^{\theta(t),\theta^t} = (\theta^i, 0)$ and $m_j^{\theta(t),\theta^t} = (\theta^j, 0)$ with $\theta^i \neq \theta^j$.

In this case, given ψ, $\tilde{a}$ is implemented in the current period and, by Rule 2, in every period thereafter. Consider any agent i deviating by announcing a positive integer at $(\mathbf{h}(\theta(t)), \theta^t)$. Given ψ (part 2) and Rule 3, such a deviation yields continuation payoff $(1-\delta)u_i(f(\theta^j), \theta^t) + \delta v_i(f)$, while the corresponding equilibrium payoff is $(1-\delta)u_i(\tilde{a}, \theta^t) + \delta v_i(\tilde{a})$. But, since $v_i(f) > v_i(\tilde{a}) + \epsilon$ and $\delta > \bar{\delta}$, the former exceeds the latter, and the deviation is profitable. □

Claim A.7. *For any* $\delta \in (\bar{\delta}, 1)$ *and* $\sigma \in \Omega^{\delta}(\widetilde{R})$, $\pi_i^{\theta(t)} = v_i(f)$ *for any* i, $t > 1$ *and* $\theta(t)$.

Proof. Given Claims A.5-A.6, and since f is efficient in the range, we can directly apply the proofs of Lemmas 3-4 in LS. □

Claim A.8. *For any* $\delta \in (\bar{\delta}, 1)$, $\Omega^{\delta}(\widetilde{R})$ *is non-empty.*

Proof. Consider a symmetric Markov strategy profile in which the true state and zero integer are always reported.

At any history, each agent i can deviate in one of the following three ways:

(i) Announce the true state but a positive integer. Given ψ (part 1) and Rule 3, such a deviation is not profitable.

(ii) Announce a false state and a positive integer. Given ψ (part 2) and Rule 3, such a deviation is not profitable.

(iii) Announce zero integer but a false state. In this case, by ψ (part 3), $\tilde{a}$ is implemented in the current period and, by Rule 2, in every period thereafter. The gain from such a deviation cannot exceed $(1-\delta)\max_{a,\theta}[u_i(\tilde{a},\theta) - u_i(a,\theta)] - \delta\epsilon < 0$, where the inequality holds since $\delta > \bar{\delta}$. Thus, the deviation is not profitable. □

B Period 1: complexity considerations

Here, we introduce players with preference for less complex strategies to the main sufficiency analysis of LS with pure strategies and show that if players have an aversion to complexity at the very margin the effcient SCF can be implemented from period 1.

Fix SCF f and consider the canonical regime with $I \geq 3$, R^*. (Corresponding results for the two-agent case can be similarly derived and, hence, omitted.) Consider *any* measure of complexity of a strategy under which taking the same action at every history with an identical state is simpler than one that takes different actions at different dates. Formally, we introduce a very weak partial order on the set of strategies that satisfies the following.[1]

[1]This partial order on the strategies is similar to the measure of complexity we use in Section 5.2 of LS on finite mechanisms. The result in this section, however, also holds if we replace this measure of complexity by any measure of complexity that stipulates that Markov strategies are less complex than non-Markov ones.

Definition B.1. *For any player i, strategy σ_i' is said to be less complex than strategy σ_i if they are identical everywhere except that there exists θ' such that σ_i' always takes the same action after observing θ' and σ_i' does not; thus,*

1. $\sigma_i'(\mathbf{h},\theta) = \sigma_i(\mathbf{h},\theta)$ *for all* $\mathbf{h}$ *and all* $\theta \neq \theta'$,
2. $\sigma_i'(\mathbf{h},\theta') = \sigma_i'(\mathbf{h}',\theta')$ *for all* $\mathbf{h},\mathbf{h}' \in \mathbf{H}^\infty$,
3. $\sigma_i(\mathbf{h},\theta') \neq \sigma_i(\mathbf{h}',\theta')$ *for some* $\mathbf{h},\mathbf{h}' \in \mathbf{H}^\infty$.[2]

Next, consider the following refinement of Nash equilibrium of regime R^*: a strategy profile $\sigma = (\sigma_1,\ldots,\sigma_I)$ constitutes a Nash equilibrium with complexity cost, NEC, of regime R if, for all i, (i) σ_i is a best response to σ_{-i}; and (ii) there exists no σ_i' such that σ_i' is a best response to σ_{-i} and σ_i' is less complex than σ_i. Then, since a NEC is also a Nash equilibrium, Lemmas 3-4 in LS hold for any NEC. In addition, we derive the following result.

Lemma B.1. *Every NEC, σ, of R^* is Markov: for all i, $\sigma_i(\mathbf{h}',\theta) = \sigma_i(\mathbf{h}'',\theta)$ for all $\mathbf{h}',\mathbf{h}'' \in \mathbf{H}^\infty$ and all θ.*

Proof. Suppose not. Then there exists some NEC, σ, of R^* such that $\sigma_i(\mathbf{h}',\theta') \neq \sigma_i(\mathbf{h}'',\theta')$ for some $i,\theta',\mathbf{h}'$ and $\mathbf{h}''$. Let $\widehat{\theta}$ be the state announced by σ_i in period 1 after observing θ'. Next, consider i deviating to another strategy σ_i' that is identical to σ_i except that at state θ', irrespective of the past history, it always announces state $\widehat{\theta}$ and and integer 1; thus, $\sigma_i'(\mathbf{h},\theta) = \sigma_i(\mathbf{h},\theta)$ for all $\mathbf{h}$ and all $\theta \neq \theta'$, and $\sigma_i'(\mathbf{h},\theta') = (\widehat{\theta},1)$ for all $\mathbf{h}$.

Clearly, σ_i' is less complex than σ_i. Furthermore, for any $\theta^1 \in \Theta$, by part (ii) of Lemma 3 in LS and the definitions of g^* and R^*, we have $a^{\theta^1}(\sigma_i',\sigma_{-i},R^*) = a^{\theta^1}(\sigma,R^*)$ and $\pi_i^{\theta^1}(\sigma_i',\sigma_{-i},R^*) = v_i(f)$. Moreover, we know from Lemma 4 in LS that $\pi_i^{\theta^1}(\sigma,R^*) = v_i(f)$. Thus, the deviation does not alter i's payoff. But, since σ_i' is less complex than σ_i, such a deviation makes i better off. This contradicts the assumption that σ is a NEC. □

This Lemma, together with Lemma 4 in LS, shows that for every NEC each player's continuation payoff at any history on the equilibrium path (including the initial history) is equal to his target payoff. Moreover, since a Markov strategy has minimal complexity (i.e. there does not exist another strategy that is less complex than the Markov strategy), it also follows that the Markov Nash equilibrium described in Lemma 5 in LS is itself a NEC. Thus, if we use NEC as the solution concept then the conclusions of Theorem 2 and its Corollary hold from period 1.

[2]We have suppressed the argument g^* in the definition of strategies here for exposition.

Theorem B.1. *If f is efficient (in the range) and satisfies conditions ω (ω'), f is payoff-repeated-implementable in Nash equilibrium with complexity cost; if, in addition, f is strictly efficient (in the range), it is repeated-implementable in Nash equilibrium with complexity cost.*

Note that the notion of NEC requires that for each player his equilibrium strategy has minimal complexity amongst all strategies that are best responses to the strategies of the other agents. As a result, NEC strategies need only to be of sufficient complexity to achieve the highest payoff on-the-equilibrium path; off-the-equilibrium payoffs do not figure in these complexity considerations. However, it may be argued that players adopt complex strategies also to deal with the off-the-equilibrium paths. In Section 5.2 of LS, as well as Section D of this Supplementary Material, we introduce an alternative equilibrium refinement based on complexity that is robust to this criticism (in order to explore what can be achieved by regimes employing only finite mechanisms). Specifically, we consider the set of subgame perfect equilibria and require players to adopt minimally complex strategies among the set of strategies that are best responses *at every history*, and not merely at the beginning of the game. We say that a strategy profile σ is a *weak* perfect equilibrium with complexity cost, or WPEC, of regime R if, for all i, (i) σ is a subgame pefect equilibrium (SPE); and (ii) there exists no σ_i' that is less complex than σ_i and best-responds to σ_{-i} at *every* (on- or off-the-equilibrium) information set.

In this equilibrium concept complexity considerations are given less priority than both on- and off-the-equilibrium payoffs. Nevertheless, the same implementation result from period 1 can also be obtained using this equilibrium notion. For this result, we have to modify the regime R^* slightly. Define $\bar{g} = (M, \psi)$ as the following mechanism: $M_i = \Theta \times \mathbb{Z}_+$ for all i and ψ is such that

1. if $m_i = (\theta, \cdot)$ for at least $I - 1$ agents then $\psi(m) = f(\theta)$;
2. otherwise, $\psi(m) = f(\theta')$ where θ' is the state announced by of the lowest-indexed agent announcing the highest integer.

Let $\bar{R}$ be any regime such that $\bar{R}(\emptyset) = \bar{g}$ and, for any $h = ((g^1, m^1), \ldots, (g^{t-1}, m^{t-1})) \in H^t$ such that $t > 1$ and $g^{t-1} = \bar{g}$, the following transition rules hold:

Rule 1: If $m_i^{t-1} = (\cdot, 0)$ for all i, $\bar{R}(h) = \bar{g}$.

Rule 2: If, for some i, $m_j^{t-1} = (\cdot, 0)$ for all $j \neq i$ and $m_i^{t-1} = (\cdot, z^i)$ with $z^i \neq 0$, $\bar{R}|h = S^i$ (Lemma 1 in LS).

Rule 3: If m^{t-1} is of any other type and i is lowest-indexed agent among those who announce the highest integer, $\bar{R}|h = D^i$.

This regime is identical to R^* except for the output function defined for the one-period mechanism when two or more agents play distinct messages; in such cases, the immediate outcome for the period is one that results from the state anounced by the agent announcing the highest integer.

Then, by the same argument as above for NEC, to obtain the result it suffices to show that any WPEC must also be Markov. To see this, assume not. Then, there exists some WPEC, σ, of R^* such that $\sigma_i(\mathbf{h}', \theta') \neq \sigma_i(\mathbf{h}'', \theta')$, for some $i, \theta', \mathbf{h}'$ and $\mathbf{h}''$.

Next, let $\overline{\theta} \in \arg\max_\theta u_i(f(\theta), \theta')$ and consider i deviating to another strategy σ_i' that is identical to σ_i except that at state θ', irrespective of the past history, it always reports state $\overline{\theta}$ and integer 1; thus, $\sigma_i'(\mathbf{h}, \theta) = \sigma_i(\mathbf{h}, \theta)$ for all $\mathbf{h}$ and all $\theta \neq \theta'$, and $\sigma_i'(\mathbf{h}, \theta') = (\overline{\theta}, 1)$ for all $\mathbf{h}$.

Clearly, σ_i' is less complex than σ_i. Furthermore, by applying the same arguments as in Lemmas 2-4 in LS to the notion of SPE, it can be shown that, at any history beyond period 1 at which $\overline{g}$ is being played, the equilibrium strategies choose integer 0 and each agent's equilibrium continuation payoff at this history is exactly the target payoff.

Thus, since σ_i' chooses 1 at any $\mathbf{h}$ if the realized state is θ', it follows that, at any such history, (i) σ_i' induces S^i in the continuation game and the target utility is achieved, and (ii) either other $I-1$ agents report the same state and the outcome in the current period is not affected, or the other players disagree on the state and $f(\overline{\theta})$ is implemented (see the modified outcome function ψ of the mechanism). Therefore, σ_i' induces a payoff no less than σ_i after any history. Since σ_i' is also less complex than σ_i we have a contradiction to σ being a WPEC.

C Mixed strategies

We next extend the main analysis of LS (Section 4.2) to incorporate mixed/behavioral strategies (also, see Section 5.1 of LS). Let $b_i : \mathbf{H}^\infty \times G \times \Theta \rightarrow \triangle\left(\cup_{g \in G} M_i^g\right)$ denote a mixed (behavioral) strategy of agent i, with b denoting a mixed strategy profile. With some abuse of notation, given R and any history $\mathbf{h}^t \in \mathbf{H}^t$, let $g^{\mathbf{h}^t}(R) \equiv (M^{\mathbf{h}^t}(R), \psi^{\mathbf{h}^t}(R))$ be the mechanism played at $\mathbf{h}^t$, $a^{\mathbf{h}^t, m^t}(R) \in A$ be the outcome implemented at $\mathbf{h}^t$ when the current message profile is m^t and $\pi_i^{\mathbf{h}^t}(b, R)$ be agent i's expected continuation payoff at $\mathbf{h}^t$ if the strategy profile b is adopted. We write $\pi_i(\sigma, R) \equiv \pi_i^{\mathbf{h}^1}(b, R)$.

Also, for any strategy profile b and regime R, let $\mathbf{H}^t(\theta(t), b, R)$ be the set of $t-1$ period histories that occur with positive probability given state realizations $\theta(t)$ and $M^{\mathbf{h}^t, \theta^t}(b, R)$ be the set of message profiles that occur with positive probability at any history $\mathbf{h}^t$ after observing θ^t. As before, the arguments in the above variables will be suppressed when the meaning is clear.

We denote by $B^\delta(R)$ denote the set of mixed strategy Nash equilibria of regime R with discount factor δ. We modify the notion of Nash repeated implementation to incorporate mixed strategies as follows.

Definition C.1. *An SCF f is payoff-repeated-implementable in mixed strategy Nash equilibrium from period τ if there exists a regime R such that (i) $B^\delta(R)$ is non-empty; and (ii) every $b \in B^\delta(R)$ is such that $\pi_i^{\mathbf{h}^t}(b, R) = v_i(f)$ for any i, $t \geq \tau$, $\theta(t)$ and $\mathbf{h}^t \in \mathbf{H}^t(\theta(t), b, R)$. An SCF f is repeated-implementable in mixed strategy Nash equilibrium from period τ if, in addition, every $b \in B^\delta(R)$ is such that $a^{\mathbf{h}^t, m^t}(R) = f(\theta^t)$ for any $t \geq \tau$, $\theta(t)$, θ^t, $\mathbf{h}^t \in \mathbf{H}^t(\theta(t), b, R)$ and $m^t \in M^{\mathbf{h}^t, \theta^t}(b, R)$.*

We now state and prove the result for the case of three or more agents. The two-agent case can be analogously dealt with and, hence, omitted to avoid repetition.

Theorem C.1. *Suppose that $I \geq 3$ and consider an SCF f satisfying condition ω. If f is efficient, it is payoff-repeated-implementable in mixed strategy Nash equilibrium from period 2; if f is strictly efficient, it is repeated-implementable in mixed strategy Nash equilibrium from period 2.*

Proof. Consider the canonical regime R^* in LS. Fix any $b \in B^\delta(R^*)$, and also fix any t, $\theta(t)$ and $\mathbf{h}^t \in \mathbf{H}^t(\theta(t), b, R^*)$ such that $g^{\mathbf{h}^t} = g^*$. Also, suppose that θ^t is observed in the current period t.

Let $r_i(m_i)$ denote player i's randomization probability of announcing message $m_i = (\theta^i, z^i)$ at this history $(\mathbf{h}^t, \theta^t)$ with $r(m) = r_1(m_1) \times \ldots \times r_I(m_I)$. Also, denote the marginals by $r_i(\theta^i) = \sum_{z^i} r_i(\theta^i, z^i)$ and $r_i(z^i) = \sum_{\theta^i} r_i(\theta^i, z^i)$.

We write agent i's continuation payoff at the given history, after observing $(\mathbf{h}^t, \theta^t)$, as

$$\pi_i^{\mathbf{h}^t,\theta^t}(b, R^*) = \sum_{m\in[\Theta\times\mathbb{Z}_+]^I} r(m)\left[(1-\delta)u_i\left(a^{\mathbf{h}^t,m}(b,R^*),\theta^t\right) + \delta\pi_i^{\mathbf{h}^t,\theta^t,m}(b,R^*)\right].$$

Then, we can also write i's continuation payoff at $\mathbf{h}^t$ prior to observing a state as

$$\pi_i^{\mathbf{h}^t}(b, R^*) = \sum_{\theta^t\in\Theta} p(\theta^t)\pi_i^{\mathbf{h}^t,\theta^t}(b, R^*).$$

We proceed by establishing the following claims. First, at the given history, we obtain a lower bound on each agent's *expected* equilibrium continuation payoff *at the next period*.

Claim C.1. $\sum_{m\in\Theta\times\mathbb{Z}_+} r(m)\pi_i^{\mathbf{h}^t,\theta^t,m} \geq v_i(f)$ *for all* i.

Proof. Suppose not; then, for some i, there exists $\epsilon > 0$ such that $\sum_m r(m)\pi_i^{\mathbf{h}^t,\theta^t,m} < v_i(f) - \epsilon$. Let $\underline{u} = \min_{i,a,\theta} u_i(a,\theta)$, and fix any $\epsilon' > 0$ such that $\epsilon'(v_i(f) - \underline{u}) < \epsilon$. Also, fix any integer $\overline{z}$ such that, given b, at $(\mathbf{h}^t, \theta^t)$ the probability that an agent other than i announces an integer greater than $\overline{z}$ is less than ϵ' (since the set of integers is infinite it is always feasible to find such an integer).

Consider agent i deviating to another strategy which is identical to the equilibrium strategy b_i except that at $(\mathbf{h}^t, \theta^t)$ it reports $\overline{z}+1$. Note from the definition of mechanism g^* and the transition rules of R^* that such a deviation at $(\mathbf{h}^t, \theta^t)$ does not alter the current period t's outcomes and expected utility while the continuation regime at the next period is S^i or D^i with probability at least $1-\epsilon'$. The latter implies that the expected continuation payoff as of the next period $t+1$ from the deviation is at least

$$(1-\epsilon')v_i(f) + \epsilon'\underline{u}. \tag{C.1}$$

Also, by assumption, the corresponding equilibrium expected continuation payoff as of $t+1$ is at most $v_i(f) - \epsilon$, which, since $\epsilon'(v_i(f) - \underline{u}) < \epsilon$, is less than (C.1). Recall that the deviation does not affect the current period t's outcomes/payoffs. Therefore, the deviation is profitable, a contradiction. □

Claim C.2. $\sum_m r(m)\pi_i^{\mathbf{h}^t,\theta^t,m} = v_i(f)$ *for all* i.

Proof. Given efficiency of f, this follows immediately from the previous claim. □

Claim C.3. $\sum_\theta r_i(\theta, 0) = 1$ *for all* i.

Proof. Suppose otherwise. Then, there exists a message profile m' which occurs with a positive probability at $(\mathbf{h}^t, \theta^t)$ such that, for some i, $m'_i = (\cdot, z^i)$ with $z^i > 0$. Since f is efficient, by similar arguments as for Claim 2 in the proof of Lemma 3 in LS, there must exist $j \neq i$ such that $\pi_j^{\mathbf{h}^t,\theta^t,m'} < v_j^j$. Then, given Claim C.2, it immediately follows that there exists $\epsilon > 0$ such that

$$v_j^j > v_j(f) + \epsilon.$$

Next, fix any $\epsilon' \in (0,1)$ such that

$$\epsilon'(v_j(f) - \underline{u}) < \epsilon r(m').$$

Also fix any integer $\overline{z} > z^i$ such that, given b, at $(\mathbf{h}^t, \theta^t)$ the probability that an agent other than j announces an integer greater than $\overline{z}$ is less than ϵ'.

Consider j deviating to another strategy which is identical to the equilibrium strategy b_j except that it reports $\overline{z}+1$ at the given history $(\mathbf{h}^t, \theta^t)$. Again, this deviation does not alter the expected outcomes in period t but, with probability $(1-\epsilon')$, the continuation regime at the next period is either S^j or D^j (Rules 2 and 3). Furthermore, since $\overline{z} > z^i$, the continuation regime is D^j with probability $\frac{r(m')}{1-\epsilon'}$. Thus, at $(\mathbf{h}^t, \theta^t)$ the expected continuation payoff at the next period $t+1$ resulting from this deviation is at least

$$\frac{r(m')}{1-\epsilon'} v_j^j + \left(1 - \epsilon' - \frac{r(m')}{1-\epsilon'}\right) v_j(f) + \epsilon' \underline{u}.$$

We know from Claim C.2 that the corresponding equilibrium expected continuation payoff at $t+1$ is $v_j(f)$. Since $v_j^j > v_j(f) + \epsilon$, $\epsilon'(v_j(f) - \underline{u}) < \epsilon r(m')$, $\epsilon' \in (0,1)$, and since the deviation does not alter the current period outcomes, the deviation is profitable, a contradiction. □

It follows from Claims C.1-C.3 that g^* must always be played on the equilibrium path. Therefore, by applying similar arguments to Lemma 2 in LS and efficiency of f, it must be that $\pi_i^{\mathbf{h}^t} = v_i(f)$ for all i, $t > 1$, $\theta(t)$ and $\mathbf{h}^t \in \mathbf{H}^t(\theta(t), b, R^*)$. The remainder of the proof follows arguments analogous to those for the corresponding results with pure strategies in Section 4.2 of LS. □

D Finite mechanisms

D.1 Three or more agents

Here, we extend the two-agent analysis on finite mechanisms (Section 5.2 of LS) to the case of $I \geq 3$.

Assumptions

As in the two-agent analysis in LS we make a minimal assumption throughout this section that each i-dictatorship $d(i)$ generates a unique payoff profile, $v^i = (v_1^i, \ldots, v_I^i)$. For example this is the case if each agent i has a unique most-preferred outcome in each state θ, i.e. $A^i(\theta)$ is a singleton set. With two agents the uniqueness of dictatorial payoffs enable us to construct for each i a history-independent and non-strategic regime S^i (by alternating the two dictatorships) that generates a unique payoff profile $w^i = (w_i^i, w_j^i)$ such that $w_i^i = v_i(f)$ and $w_j^i \leq v_j(f)$, as long as the SCF f is efficient. In LS, with almost no loss of generality, we consider the case where the latter inequality is strict.

With three or more agents, we also need to be able to construct, for each agent i, regime S^i with the above property. To do so, let $W = \{v^i\}_{i \in I} \cup \{v(a)\}_{a \in A}$ denote the set of payoff profiles from dictatorial and constant rule mechanisms and assume that, in addition to efficiency, SCF f satisfies the following.

Condition χ. For each i, there exists $w^i = (w_1^i, \ldots, w_I^i) \in co(W)$ such that $w_i^i = v_i(f)$ and $w_i^i > w_i^j$ for all $j \neq i$.[3]

By Sorin [6], any payoff profile $w \in co(W)$ could be generated by a regime that appropriately alternates some dictatorial and/or constant rule mechanisms, as long as $\delta \in \left(1 - \frac{1}{I+|A|}, 1\right)$. Assuming that δ indeed satisfies this condition (we shall assume this throughout), condition χ immediately implies that for each agent i there exists a regime S^i such that i obtains a payoff equal to the target level $v_i(f)$ but every other agent derives a payoff strictly less than his target.

While, as mentioned before, condition χ trivially holds when $I = 2$ with efficient SCF, when $I \geq 3$, one case that guarantees condition χ is the following.

[3] As discussed in LS, if the payoffs under a *restricted* dictatorship by agent i over a subset of outcomes $N \subseteq A$ are unique, for every i and N, then we could replace the requirement of $w^i \in co(W)$ in condition χ by $w^i \in co(\{v^i(N)\}_{i \in I, N \subseteq A})$.

Lemma D.1. *Suppose that there exists some* $\tilde{a}$ *such that* $v_i(f) \geq v_i(\tilde{a})$ *for each* i *and* $v_i(f) \geq v_i^j$ *for all* $i, j,\ i \neq j$. *Then, for any* i, *there exists* $w^i \in co(W)$ *such that (i)* $w_i^i = v_i(f)$ *and (ii)* $w_i^i \geq w_i^j$ *for any* $j \neq i$, *with this inequality being strict if either* $v_i(f) > v_i(\tilde{a})$ *or* $v_i(f) > v_i^j$.

Proof. For each i, by $v_i^i \geq v_i(f) \geq v_i(\tilde{a})$, there must exist $\alpha^i \in [0,1]$ such that $v_i(f) = \alpha^i v_i^i + (1-\alpha^i)v_i(\tilde{a})$. Let $w^i = \alpha^i v^i + (1-\alpha^i)v(\tilde{a})$. Clearly, w^i satisfies (i) for all i.

To show (ii), consider any $j \neq i$. Then by construction $w_i^j \leq \max\{v_i^j, v_i(\tilde{a})\}$. Since by assumption $v_i(f) \geq v_i(\tilde{a})$ and $v_i(f) \geq v_i^j$, we have $w_i^i \geq w_i^j$. Furthermore, the last inequality is strict if either $v_i(f) > v_i(\tilde{a})$ or $v_i(f) > v_i^j$. □

Given condition χ, we can show the existence of the following regimes.

Lemma D.2. *Suppose that* f *is efficient and satisfies condition* χ. *Also, fix any pair of agents,* $k, l \in I$. *Then, for any subset of agents* $C \subseteq I$ *and each date* $t = 1, 2, \ldots$, *there exist regimes* $S^C, X(t), Y$ *that respectively induce unique payoff profiles* $w^C, x(t), y \in co(W)$ *satisfying the following conditions:*[4]

$$w_k^l < y_k < x_k(t) < w_k^k \quad \text{and} \quad w_l^k < x_l(t) < y_l < w_l^l \tag{D.1}$$

$$x_k(t) \neq x_k(t') \quad \text{and} \quad x_l(t) \neq x_l(t') \quad \text{for any } t, t', t \neq t' \tag{D.2}$$

$$w_k^C < w_k^k \quad \text{if } C \neq \{k\} \quad \text{and} \quad w_l^C < w_l^l \quad \text{if } C \neq \{l\} \tag{D.3}$$

$$w_i^C \geq w_i^{C \setminus \{i\}} \quad \text{for all } i \in C. \tag{D.4}$$

Proof. To construct these regimes, let $x(t) = \lambda(t)w^k + (1-\lambda(t))w^l$ and $y = \mu w^k + (1-\mu)w^l$ for some $\mu \in (0,1)$ and a strictly monotone sequence $\{\lambda(t) : \lambda(t) \in (\mu, 1)\ \forall t\}$. Also, for any $C \subseteq I$, let $w^C = \frac{1}{|C|}\sum_{i \in C} w^i$, where w^i is given by condition χ. Since $w_i^i > w_i^j$ for all $j \neq i$, these payoffs satisfy (D.1)-(D.4). Furthermore, since for each i, $w^i \in co(W)$ can be obtained as a convex combination of dictatorial and/or constant rule mechanisms, it follows that $w^C, x(t), y \in Co(W)$ can be obtained by regimes that appropriately alternate dictatorial and/or constant rule mechanisms. □

Regime construction

We now extend the regime construction in LS for the case of $I = 2$ to our present setup. First, fix any two agents, k and l. Then, define the sequential mechanism $\hat{g}^e$ as follows:

[4] For simplicity, we write $S^{\{i\}} = S^i$ and $w^{\{i\}} = w^i$.

Stage 1 - Each agent i announces a state from Θ. If at least $I-1$ agents announce θ, $f(\theta)$ is implemented; otherwise, $f(\tilde{\theta})$ for some arbitrary but fixed $\tilde{\theta}$ is implemented.

Stage 2 - Each of agents k and l announces an integer from the set $\{0,1,2\}$; each $i \in I \backslash \{k,l\}$ announces an integer from the set $\{0,1\}$.

Notice that the outcome function in $\hat{g}^e$ is the same as that of mechanism g^* in the canonical regime R^* for the case of $I \geq 3$. This mechanism extends mechanism g^e in the finite mechanism construction with $I = 2$ by allowing only two agents to choose from $\{0,1,2\}$ while all the remaining agents choose from just $\{0,1\}$.

Next, using the constructions in Lemma D.2 above, we define new regime $\hat{R}^e$ inductively as follows: (i) mechanism $\hat{g}^e$ is implemented at $t = 1$ and (ii) if, at some date t, $\hat{g}^e$ is the mechanism played with a profile of states $\underset{\sim}{\theta} = (\theta^1, \ldots, \theta^I)$ announced in stage 1 and a profile of integers $\underset{\sim}{z} = (z^1, \ldots, z^I)$ announced in stage 2, the continuation mechanism/regime at the next period is as follows:

Rule 1: If $z^i = 0$ for all i, the mechanism next period is $\hat{g}^e$.

Rule 2: If $z^k > 0$ and $z^l = 0$ ($z^k = 0$ and $z^l > 0$), the continuation regime is S^k (S^l).

Rule 3: Suppose that $z^k, z^l > 0$. Then, we have the following:

Rule 3.1: If $z^k = z^l = 1$, the continuation regime is $X \equiv X(\tilde{t})$ for some arbitrary $\tilde{t}$, with the payoffs henceforth denoted by x.

Rule 3.2: If $z^k = z^l = 2$, the continuation regime is $X(t)$.

Rule 3.3: If $z^k \neq z^l$, the continuation regime is Y.

Rule 4: If, for some $C \subseteq I \backslash \{k,l\}$, $z^i = 1$ for all $i \in C$ and $z^i = 0$ for all $i \notin C$, the continuation regime is S^C.

This regime extends the two-agent counterpart R^e by essentially maintaining all the features for two players (k and l) and endowing the other agents with the choice of just 0 or 1. Notice from Rules 2 and 3 that if either k or l plays a non-zero integer the integer choices of other players are irrelevant to transitions.

We define histories, partial histories (within period), strategies and continuation payoffs similarly to their two-agent counterparts.

Properties of Nash equilibria

We begin by deriving the Nash equilibrium properties analogous to the corresponding results in LS.

Lemma D.3. *Consider any Nash equilibrium of regime $\hat{R}^e$. Fix any t, $\mathbf{h} \in \mathbf{H}^t$ and $d = (\theta, \underset{\sim}{\theta}) \in \Theta \times \Theta^I$ on the equilibrium path. Then, one of the following must hold at $(\mathbf{h}, d)$:*

1. Each $i \in I$ announces 0 for sure and his continuation payoff at the next period is equal to $v_i(f)$.

2. Each $i \in \{k, l\}$ announces 1 or 2 for sure, with the probability of choosing 1 equal to $\frac{x_i(t)-y_i}{x_i+x_i(t)-2y_i} \in (0,1)$. Furthermore, for all $j \in I$, the continuation payoff at the next period is less than $v_j(f)$.

Proof. At $(\mathbf{h}, d)$, the players either randomize (over the integers) or do not. We shall prove the claim by considering each case separately.

Case 1: No player randomizes.

In this case, we show that, each player must play 0 for sure. Suppose otherwise; then some i plays $z^i \neq 0$ for sure. We derive contradiction by considering the following subcases.

Subcase 1A: $z^k > 0$ and $z^l = 0$, or $z^k = 0$ and $z^l > 0$.

Consider the former case; the latter case can be handled analogously. The continuation regime at the next period is S^k (Rule 2). But then, since $y_l > w_l^k$ by (D.1), l can profitably deviate by choosing a strategy identical to the equilibrium strategy except that it announces the positive integer other than z^l at this history, which activates the continuation regime Y instead of S^k (Rule 3.3). This is a contradiction.

Subcase 1B: $z^k > 0$ and $z^l > 0$.

The continuation regime is either $X, X(t)$ or Y (Rule 3). Suppose that it is X or $X(t)$. By (D.1), we have $y_l > x_l(t')$ for all t'. But then, l can profitably deviate by choosing a strategy identical to the equilibrium strategy except that it announces the positive integer other than z^l at this history, which activates Y (Rule 3.3). This is a contradiction. Similarly, since $x_k > y_k$ by (D.1), when the continuation regime is Y, player k can profitably deviate and we obtain a similar contradiction.

Subcase 1C: For some $C \subseteq I \backslash \{k, l\}$, $z^i = 1$ for all $i \in C$ and $z^i = 0$ for all $i \notin C$.

The continuation regime is S^C (Rule 4). By (D.3), we have $w_j^j > w_j^C$ for $j \in \{k, l\}$. But then, j can profitably deviate by choosing a strategy identical to the equilibrium strategy except that it announces a positive integer at this history, which activates S^j (Rule 2).

Again by similarly applying the "odd-one-out" arguments in LS, in this case, the corresponding continuation payoffs (at the next period) equal $v(f)$.

<u>*Case 2*</u>: Some player randomizes.

For any $i \in I$, let Π_i denote i's continuation payoff at the next period if all agents announce zero. Let z^i denote the integer that i ends up choosing at this history. We proceed by establishing the following claims.

Claim 1: For each agent k or l, the continuation payoff (at the next period) from announcing 1 is greater than that from announcing 0, if there exists another player announcing a positive integer.

Proof of Claim 1. Consider k and any $\underset{\sim}{z}^{-k} \neq (0, \ldots, 0)$. The other case for l can be proved identically. There are two possibilities:

First, suppose that $z^l > 0$. In this case, if k announces zero, by Rule 2, his continuation payoff is w_k^l. If he announces 1, by Rules 3.1 and 3.3, the continuation payoff is x_k or y_k. But, by (D.1), we have $x_k > y_k > w_k^l$.

Second, suppose that $z^l = 0$. In this case, since $\underset{\sim}{z}^{-k} \neq (0, \ldots, 0)$, there must exist a non-empty set $C \subseteq I \backslash \{k, l\}$ such that $z^i = 1$ for all $i \in C$ and $z^i = 0$ for all $i \in I \backslash \{C \cup k\}$. Then if k announces 0, by Rule 4, his continuation payoff is w_k^C, whereas if he announces 1, by Rule 2, the continuation payoff is w_k^k. But, by (D.3), we have $w_k^k > w_k^C$.

Claim 2: If agent k or l announces zero with a positive probability, then every other agent must also announce zero with a positive probability.

Proof of Claim 2. Suppose not. Then, suppose that k plays 0 with a positive probability but some $i \neq k$ chooses 0 with zero probability. (The other case for l can be proved identically.) But then, by Claim 1, the latter implies that k obtains a lower continuation payoff from choosing 0 than from choosing 1. This contradicts the supposition that k chooses 0 with a positive probability.

Claim 3. Suppose that some agent $i \in \{k, l\}$ announces 0 with a positive probability. Then, $\Pi_i \geq v_i(f)$ with this inequality being strict if some other agent announces a positive integer with a positive probability.

Proof of Claim 3. For any agent $i \in \{k, l\}$, by Claim 1, playing 1 must always yield a higher continuation payoff i than playing 0, except when all other agents play 0. Since i plays 0 with a positive probability, the following must hold:

(i) If all others announce 0, i's continuation payoff when he announces 0 must be no less than that he obtains when he announces 1, i.e. $\Pi_i \geq v_i(f)$.

(ii) If some other player attaches a positive weight to a positive integer, i's continuation payoff must be greater when he chooses 0 than when he chooses 1 in the case in which all others choose 0, i.e. $\Pi_i > v_i(f)$.

Claim 4: For each agent $i \in I \backslash \{k, l\}$, the continuation payoff from announcing zero is no greater than that from announcing 1, if there exists another player announcing a positive integer.

Proof of Claim 4. For each $i \in I \backslash \{k, l\}$, the continuation payoff is independent of his choice if $z^k > 0$ or $z^l > 0$. So, suppose that $z^k = z^l = 0$. Then if i chooses 1 he obtains w_i^C, for some $C \in I \backslash \{k, l\}$ such that $i \notin C$, while he obtains $w_i^{C \cup \{i\}}$ from choosing 1. By (D.4), $w_i^C \leq w_i^{C \cup \{i\}}$. Thus, the claim follows.

Claim 5. For each agent $i \in I \backslash \{k, l\}$, $\Pi_i \geq v_i(f)$ if all players announce 0 with a positive probability.

Proof of Claim 5. Note that, if $z^j = 0$ for all $j \neq i$, i obtains Π_i from choosing 0 and obtains $v_i(f)$ from choosing 1. The claim then follows immediately from the previous claim.

Claim 6. Both k and l choose a postive integer for sure.

Proof of Claim 6. Suppose otherwise; then some $i \in \{k, l\}$ chooses 0 with a positive probability. Then, by Claim 2, every other agent must play 0 with a positive probability. By Claims 3 and 5, this implies that $\Pi_j \geq v_j(f)$ for every j. Moreover, since in this case there is randomization, some player must be choosing a positive integer with a positive probability. Then, by appealing to Claim 3 once again, we must also have that at least one of the inequalities $\Pi_k \geq v_k(f)$ or $\Pi_l \geq v_l(f)$ is strict. Since f is efficient, this is a contradiction.

Claim 7. Both k and l choose each of the integers 1 and 2 with a positive probability.

Proof of Claim 7. Suppose not; then by the previous claim one of either k or l must choose one of the positive integers for sure. But then, (D.1) implies that the other must also do the same. But, by applying (D.1) once again, this induces a contradiction (the argument is exactly the same as in Subcase 1B of Case 1 with no randomization).

Given the last two claims, simple computation verifies that both agents k and l must be playing 1 with a unique probability as in the statement. The continuation payoffs, for each $i \in I$, when k or l chooses a positive integer are $x_i, x_i(t)$ or y. Moreover, by (D.1), each of these payoffs is less than $v_i(f)$. Therefore, it follows that, in this case, the continuation payoff at the next period must be less than $v_i(f)$ for all i. □

From this Lemma, it is straightforward to establish the following (the proof is similar to the corresponding proof in LS and hence omitted).

Proposition D.1. *Fix any Nash equilibrium b of regime $\hat{R}^e$.*

1. If any player mixes over integers on the equilibrium path, then both players randomize at some partial history in stage 2 of period 1; furthermore, $\pi_i^{\mathbf{h}}(b, \hat{R}^e) \leq v_i(f)$ for all i and any on-the-equilibrium history $\mathbf{h} \in \mathbf{H}^2$ with the inequality being strict at every such history that involves randomization in stage 2 of period 1.

2. Otherwise, $\pi_i^{\mathbf{h}}(b, \hat{R}^e) = v_i(f)$ for any i, any $t > 1$ and any (on-the-equilibrium) history $\mathbf{h} \in \mathbf{H}^t$.

Refinement

We now introduce our refinement arguments. Note first that, if we apply subgame perfection, the statements of Lemma D.3 above extends to hold for any on- or *off*-the-equilibrium history after which the agents find themselves in the integer part of mechanism $\hat{g}^e$; that is, in an SPE of regime $\hat{R}^e$, at *any* $(\mathbf{h}, (\theta, \underset{\sim}{\theta}))$ they must either choose 0 for sure or mix between 1 and 2.

The definitions of complexity and WPEC can be defined analogously here to the two-agent case in LS. Note from our modified regime construction that if mixing (over integers) occurs the only relevant agents are k and l. Thus, we can apply similar WPEC arguments to the case of $I \geq 3$ as to the case of $I = 2$.

Lemma D.4. *Fix any WPEC of regime $\hat{R}^e$. Also, fix any $\mathbf{h} \in H^\infty$ and $d \in \Theta \times \Theta^I$ (on or off the equilibrium path). Then, every agent announces zero for sure at this history.*

Proof. Suppose not. Then, there exists a WPEC, b, such that, by Lemma D.3 applied to SPE, at some t, $\mathbf{h}^t \in \mathbf{H}^t$ and $d = (\theta, \underset{\sim}{\theta}) \in \Theta \times \Theta^I$, the two agents k and l play 1 or 2 for sure; each $i \in \{k, l\}$ plays 1 with probability $\frac{x_i(t) - y_i}{x_i + x_i(t) - 2y_i}$. Furthermore, by construction,

$x_k(t')$ and $x_l(t')$ are distinct for each t' and, therefore, it follows that, for some $t' \neq t$ and $\mathbf{h}^{t'} \in H^{t'}$, and for each $i \in \{k, l\}$, we have $b_i(\mathbf{h}^t, d) \neq b_i(\mathbf{h}^{t'}, d)$.

Now, consider any $i \in \{k, l\}$ deviating to another strategy b'_i that is identical to the equilibrium strategy b_i except that, for all $\mathbf{h} \in \mathbf{H}^\infty$, $b'_i(\mathbf{h}, d)$ prescribes announcing 1 for sure. Since b'_i is less complex than b_i, we obtain a contradiction by showing that $\pi_i^{\mathbf{h}}(b'_i, b_{-i}, R^e) = \pi_i^{\mathbf{h}}(b, R^e)$ for all $\mathbf{h} \in \mathbf{H}^\infty$. To do so, it suffices to fix any history $\mathbf{h}$ and consider continuation payoffs after the given partial history d. Given Lemma D.3, there are two cases to consider at $(\mathbf{h}, d)$.

First, if agents k and l mix between 1 and 2, by Lemma D.3, i is indifferent between choosing 1 and 2. Second, suppose that all agents play 0 for sure. Then, by Lemma D.3, i obtains a continuation payoff equal to $v_i(f)$ in equilibrium. Deviation also induces the same continuation payoff $v_i(f)$ as it makes i the "odd-one-out." □

Combining previous results, we obtain the following.

Proposition D.2. *1. If f is efficient, every WPEC, b, of regime $\hat{R}^e$ payoff-repeated-implements f from period 2 at every history; i.e. $\pi_i^{\mathbf{h}^t}(b, \hat{R}^e) = v_i(f)$ for all i, $t \geq 2$ and $\mathbf{h}^t \in \mathbf{H}^t$.*

2. If f is strictly efficient, every WPEC, b, of regime $\hat{R}^e$ repeated-implements f from period 2 at every history.

D.2 Alternative complexity measure

We next consider our finite mechanism analysis using another complexity measure. Let $D_\theta \equiv \Theta$ and $D_z \equiv \Theta \times \Theta^I$, respectively, refer to the set of all partial histories that the agents can face in stage 1 and in stage 2 of the extensive form mechanism. Then, the alternative complexity measure mentioned in LS can be formalized as follows.

Definition D.1. *For any i and any pair of strategies $b_i, b'_i \in B_i$, we say that b_i is more complex than b'_i if there exists $l \in \{\theta, z\}$ with the following properties:*

1. $b'_i(\mathbf{h}, d) = b_i(\mathbf{h}, d)$ for all $\mathbf{h} \in \mathbf{H}^\infty$ and all $d \notin D_l$.

2. $b'_i(\mathbf{h}, d) = b'_i(\mathbf{h}', d')$ for all $\mathbf{h}, \mathbf{h}' \in \mathbf{H}^\infty$ and all $d, d' \in D_l$.

3. $b_i(\mathbf{h}, d) \neq b_i(\mathbf{h}', d')$ for some $\mathbf{h}, \mathbf{h}' \in \mathbf{H}^\infty$ and some $d, d' \in D_l$.

With this alternative measure, our characterization of WPECs of the regimes R^e for $I = 2$ and $\hat{R}^e$ for $I \geq 3$ remain valid via identical arguments. However, these regimes may not admit an equilibrium. Consider the strategies in which the players always announce the true state and integer zero. Here, a unilateral deviation from truth-telling in stage 1 leads to either one-period outcome according to self-selection when $I = 2$ or no change in the outcome when $I \geq 3$, and hence, does not necessarily make the deviator worse-off. Thus, with Definition D.1, deviating to always announcing the same state may reduce complexity cost without affecting payoffs.

With two agents, this would not be possible if the inequalities in self-selection conditions were *strict*. Also, with more than two agents, if there existed a bad outcome (as defined in Moore and Repullo [4]) then one could deal with the problem by changing the mechanism $\hat{g}^e$ in Section D.1 so that any disagreement on the state implements the bad outcome. When strict self-selection for $I = 2$ or bad outcome for $I > 3$ do not hold, we can still handle the issue by modifying the regime construction. Below we present such a construction for the two-agent case. The three-or-more-agent construction can be obtained by similarly modifying regime $\hat{R}^e$ above and, hence, is omitted.

First, let g^e denote the same extensive form mechanism as in Section 5.2 of LS. We obtain regime $\tilde{R}^e$ by modifying regime R^e as follows. In the first period, as before, g^e is played; also, the transition rules in this period are identical to those of R^e. In any period after the first, however, the transition rules when playing g^e is identical to those of R^e (and hence of $\tilde{R}^e$ itself in $t = 1$) only if the two agents announce the same state in stage 1; otherwise, the continuation regime is X (which generate payoffs strictly dominated by $v(f)$). Formally, we have the following:

Transition rules in period 1:

Let (θ^1, θ^2) and (z^1, z^2) be the states and integers announced in period 1. The transitions rules in period 1 are as follows:

Rule 1: If $z^1 = z^2 = 0$, the mechanism next period is g^e.

Rule 2: If $z^1 > 0$ and $z^2 = 0$ ($z^1 = 0$ and $z^2 > 0$), the continuation regime is S^1 (S^2).

Rule 3: Suppose that $z^1, z^2 > 0$. Then, we have the following:

Rule 3.1: If $z^1 = z^2$, the continuation regime is $X \equiv X(\tilde{t})$ for some arbitrary $\tilde{t}$, with the payoffs henceforth denoted by x.

Rule 3.2: If $z^1 = z^2 = 2$, the continuation regime is $X(1)$.

Rule 3.3: If $z^1 \neq z^2$, the continuation regime is Y.

Transition rules in period $t \geq 2$:

Consider any date $t \geq 2$. Let $(\tilde{\theta}^1, \tilde{\theta}^2)$ and $(\tilde{z}^1, \tilde{z}^2)$ be the states and integers announced in period t. The transitions rule are as follows:

Rule A: If $\tilde{\theta}^1 \neq \tilde{\theta}^2$, the continuation regime is X.

Rule B: If $\tilde{\theta}^1 = \tilde{\theta}^2$ and $\tilde{z}^1 = \tilde{z}^2 = 0$, the mechanism next period is g^e.

Rule C: If $\tilde{\theta}^1 = \tilde{\theta}^2$, $\tilde{z}^1 > 0$ and $\tilde{z}^2 = 0$ ($\tilde{z}^1 = 0$ and $\tilde{z}^2 > 0$), the continuation regime is S^1 (S^2).

Rule D: Suppose that $\tilde{\theta}^1 = \tilde{\theta}^2$ and $\tilde{z}^1, \tilde{z}^2 > 0$. Then, we have the following:

Rule D1: If $\tilde{z}^1 = \tilde{z}^2$, the continuation regime is X.

Rule D2: If $\tilde{z}^1 = \tilde{z}^2 = 2$, the continuation regime is $X(t)$.

Rule D3: If $\tilde{z}^1 \neq \tilde{z}^2$, the continuation regime is Y.

The Nash equilibria of this regime feature the same properties as those reported in Lemma 6 and Proposition 1 for regime R^e in Section 5.2 of LS.

Next, let us consider WPECs of $\tilde{R}^e$ with the new complexity measure of Definition D.1. Recall first that this regime is identical to regime R^e everywhere except for Rule A, which applies only to the state part of the mechanism from period 2 if there is a disagreement. This implies that, Lemma 7 in LS, which is concerned only with the integer part of R^e, must hold for any history along which players agree on the state announcements.

To complete the characterization of WPECs in this regime, we next show that indeed the players must always report the same state after the first period. This will then imply that every WPEC here will also payoff-repeated-implement the SCF from period 2 (analogously to Proposition 2 in LS).

Lemma D.5. *Fix any WPEC of regime $\tilde{R}^e$ under complexity measure of Definition D.1. Fix any $t \geq 2$ and $\mathbf{h}^t \in \mathbf{H}^t$ at which mechanism g^e is played. Then, the agents report the same state for sure in stage 1 of g^e after $(\mathbf{h}^t, \theta)$, for any θ.*

Proof. Let $r(\theta, \underset{\sim}{\theta})$ denote the probability with which partial history $(\theta, \underset{\sim}{\theta})$ occurs at $\mathbf{h}^t$ under the given equilibrium, and let $a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}}$ represent the corresponding outcome. Also, let $\underset{\sim}{\Theta}' = \{\ (\theta^1, \theta^2) \in \Theta^2 \mid \theta^1 = \theta^2\}$ denote the set of state profiles in which the players agree and $\underset{\sim}{\Theta}'' = \Theta^2 \backslash \underset{\sim}{\Theta}'$ denote the set of state profiles in which they disagree.

Given that at $\mathbf{h}^t$ mechanism g^e is played, in the previous period $t-1$ of history $\mathbf{h}^t$, the same mechanism must have been in force, and the players must have agreed on the state in stage 1 and integer 0 in stage 2. Therefore, by applying analogous arguments to those in Section 4.2 of LS, we must have $\pi_i^{\mathbf{h}^t} = v_i(f)$ for all i.

Similarly, for any partial history $(\theta, \underset{\sim}{\theta})$ with $\underset{\sim}{\theta} \in \underset{\sim}{\Theta}'$, and for integer profile $\underset{\sim}{z} = (0, 0)$, we also have $\pi_i^{\mathbf{h}^t,\theta,\underset{\sim}{\theta},\underset{\sim}{z}} = v_i(f)$ for all i. Also, when there is agreement on the state in stage 1, by applying the arguments of Lemma 7 in LS, the agents report zero for sure in stage 2. It then follows that the continuation payoffs after any $d = (\theta, \underset{\sim}{\theta})$ with $\underset{\sim}{\theta} \in \underset{\sim}{\Theta}'$ are $v(f)$.

Note also that when there is disagreement in stage 1 at $\mathbf{h}^t$, by Rule A, the continuation regime at $t+1$ is X and the continuation payoff is x_i for each i.

It follows from above that we can write each i's continuation payoff at $\mathbf{h}^t$ as

$$\begin{aligned}
\pi_i^{\mathbf{h}^t} &= \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}'} r(\theta,\underset{\sim}{\theta}) \left[(1-\delta)u_i(a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}},\theta) + \delta v_i(f)\right] + \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}''} r(\theta,\underset{\sim}{\theta}) \left[(1-\delta)u_i(a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}},\theta) + \delta x_i\right] \\
&= (1-\delta) \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\Theta^2} r(\theta,\underset{\sim}{\theta}) u_i(a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}},\theta) + \delta \left[v_i(f) \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}'} r(\theta,\underset{\sim}{\theta}) + x_i \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}''} r(\theta,\underset{\sim}{\theta}) \right].
\end{aligned}$$

Since $\pi_i^{\mathbf{h}^t} = v_i(f)$ and $x_i < v_i(f)$ for all i, if $\sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}''} r(\theta,\underset{\sim}{\theta}) \neq 0$ then it must be that $\sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\Theta^2} r(\theta,\underset{\sim}{\theta})\ u_i(a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}},\theta) > v_i(f)$ for all i. But this is not feasible with f being efficient. It therefore follows that $\sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}''} r(\theta,\underset{\sim}{\theta}) = 0$. □

Next, we show the existence of an WPEC.

Lemma D.6. *Regime $\tilde{R}^e$ admits a WPEC under complexity measure of Definition D.1.*

Proof. Consider a strategy profile b in which, regardless of past history, each agent always announces the true state in stage 1 and reports integer 0 in stage 2 irrespective of partial history within the period.

Clearly, this strategy profile is such that no player can economize on its complexity regarding the integer part of the regime. Thus, to show that the above strategies constitute a WPEC, it suffices to consider that some i deviates to a less complex strategy

b_i' which always announces the same state and integer 0, regardless of the past history; thus $b_i'(\mathbf{h}, \theta) = b_i'(\mathbf{h}', \theta')$ for all $(\mathbf{h}, \theta), (\mathbf{h}', \theta') \in \mathbf{H}^\infty \times \Theta$ and $b_i'(\mathbf{h}, \theta, \underset{\sim}{\theta}) = b_i'(\mathbf{h}', \theta', \underset{\sim}{\theta}')$ for all $(\mathbf{h}, \theta, \underset{\sim}{\theta}), (\mathbf{h}', \theta', \underset{\sim}{\theta}') \in \mathbf{H}^\infty \times \Theta \times \Theta^2$.

By self-selection, the deviation does not improve the one-period payoff at any history in periods 1 and 2, and also Rule A implies that the continuation payoffs from this deviation at different histories at period 3 fall below $v_i(f)$. Since the equilibrium continuation payoff equals $v_i(f)$ at every history, the deviation is not worthwhile. □

D.3 Alternative equilibrium notion

In all of our finite mechanism analysis thus far, we have chosen to refine Nash equilibrium by applying complexity considerations directly to the set of SPEs. Notice, however, from the above arguments that in any Nash equilibrium mixing over integers can in fact occur only period 1; after such randomization, the game effectively shuts down. Our complexity arguments are based on economizing on the response to a particular partial history over different periods and, thus, the role of complexity cost as a refinement is to economize on unnecessarily complex behavior *off the equilibrium*.

Off-the-equilibrium could be thought of as arising out of the possibility of trembles. Therefore, an alternative way of thinking about the issue of credibility of strategies and complexity considerations is to introduce two kinds of perturbations into the basic model and looking at the limiting Nash equilibrium behavior as these perturbations become arbitrarily small (e.g. Chatterjee and Sabourian [1], Sabourian [5] and Gale and Sabourian [2]). One perturbation allows for a small but positive cost of choosing a more complex strategy; another perturbation represents a small but positive and independent probability of making an error (off-the-equilibrium-path move). Since our results hold with the WPEC concept that only requires minimal complexity amongst the set of best responses at every information set, they also hold for such limiting equilibria, *independently* of the order of the limiting arguments.[5] In the paper, we have opted to present our results in terms of WPEC for expositional reasons.

[5] In terms of the limiting arguments, the standard equilibrium definition with complexity that makes only on-the-equilibrium comparisons corresponds to a limiting equilibrium that first lets the complexity cost go to zero and then considers the probability of error (e.g. Chatterjee and Sabourian [1]).

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