Complexity and Competition[1]

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Abstract

Extensive-form market games typically have a large number of non-competitive equilibria. In this paper, we argue that the complexity of non-competitive behavior provides a justification for competitive equilibrium in the sense that, if rational agents have an aversion to complexity (at the margin), then maximizing behavior will result in simple behavioral rules and hence in a competitive outcome. For this purpose, we use a class of extensive-form dynamic matching and bargaining games with a finite number of agents. In particular, we consider markets with heterogeneous buyers and sellers and deterministic, exogenous, sequential matching rules, although the results can be extended to other matching processes. If the complexity costs of implementing strategies enter players' preferences lexicographically with the standard payoff, then every equilibrium strategy profile induces a competitive outcome.

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1 Introduction

Perfect competition is an abstract ideal and yet, from the beginning of the neoclassical period, it has been the central paradigm of economics. Competition is clearly imperfect in many markets. Moreover, in theory, perfect competition requires the assumption that individual traders have no market power, an assumption that can only be justified in general if the number of traders is infinite. So the empirical usefulness of perfect competition relies on showing that it is a good approximation to markets with a finite number of traders.

In general, it has not been easy to model perfectly competitive markets with a finite number of agents, despite some notable attempts. This has been particularly the case in dynamic settings (see Gale (2000)). Extensive-form market games — in contrast to the elegant and simple model of perfect competition — tend to be complex and intractable because, among other things, they have large numbers of equilibria, sustained by threats and counter-threats.

In this paper, we present an alternative theoretical argument for the relevance of perfect competition in finite markets. Our argument is based on the complexity of non-competitive behavior. One of the striking features of perfect competition is that the rules of behavior are very simple. As Hayek (1945) noted, in a competitive market, economic agents only need to know their own endowments, preferences and technologies, and the vector of prices at which trade takes place. Then economic agents, either maximizing utility subject to a budget constraint or maximizing profits subject to a technological constraint, will make Pareto-efficient choices in equilibrium. We suggest that the converse might also be true: *if rational agents have, at least at the margin, an aversion to complex behavior, then their maximizing behavior will result in simple behavioral rules and a perfectly competitive equilibrium outcome.*

We consider a simple model of the market for a single indivisible good. The market is made up of a finite set of buyers and sellers who exchange the good for money. Each buyer wants at most one unit of the good and each seller has one unit of the good for sale. We allow for heterogeneity in the valuations of both the buyers and the sellers. Trade is the result of pairwise matching and bargaining between buyers and sellers. More precisely, we consider the following process. At each date $t = 1, 2, ...$, one buyer and one seller are matched. One member of the pair is the *proposer*, who makes a price offer. The other is the *responder*, who accepts or rejects the proposer's price offer. If the proposal is accepted, the good is traded at the agreed price and both agents leave the market. If the proposal is rejected, there is no trade and all agents begin the next period with the same endowments.

In this paper we also assume that the matching is exogenous and deterministic. That is, the identities of the proposer and the responder at each date are an exogenous and deterministic function of the set of agents remaining in the market and the date. We consider it the natural place to start for two reasons. First, our basic approach is to show that minimally complex strategies imply competitive behavior, but this is only true, given our extremely weak assumptions about complexity costs, if the environment itself is not too complex.[1] Making stronger assumptions about complexity costs would allow us to deal with a richer set of matching procedures, but would raise questions about the empirical validity of our complexity measure. On balance, we felt the notion of complexity adopted in the present approach is the most defensible one. A second, more practical, consideration is that the deterministic matching process makes the analysis tractable.

The first result we establish is that the market game described above has a continuum of non-competitive subgame perfect equilibria. This is not an entirely new result. In a seminal paper, Rubinstein and Wolinsky (1990), henceforth RW, analyze a *homogeneous* market, that is, one in which all buyers are identical and all sellers are identical. This market has a unique competitive price but, as RW show, the model possesses a continuum of non-competitive sequential equilibrium outcomes, a result reminiscent of the Folk Theorem for repeated games.[2]

RW also consider conditions under which sequential equilibrium outcomes are competitive. For example, they show that a sequential equilibrium is competitive if the equilibrium strategies are Markov or, in their terminology, anonymous. This suggests that perfect competition may be the outcome of non-cooperative behavior if agents are required to use simple strategies.

Following this suggestion, Sabourian (2004), henceforth S, assumes there

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[1] Extensions to alternative matching technologies are discussed in Section 4.

[2] An important feature of RW is that it analyzes a market with a finite number of agents and with no restriction on the set of strategies. The preceding literature (Rubinstein and Wolinsky (1985), Gale (1986a,b,c, 1987), Binmore and Herrera (1988a,b), McLennan and Sonnenschein (1991)) either assumes a non-atomic continuum of agents, each of whom has a negligible effect on equilibrium, or restricts the set of strategies (to stationary ones). See also the monographs by Osborne and Rubinstein (1990) and Gale (2000).

is a 'small' cost associated with choosing a more complex strategy. In equilibrium, additional complexity is justified only if it is necessary to implement a best response. S shows that the equilibria of RW's model satisfy this refinement if and only if the equilibrium outcome is competitive, in the sense that all trade occurs at the competitive price. Moreover, S also establishes that such a refinement implies that the equilibrium strategies are Markov. The selection result is obtained with and without discounting, with exogenous and endogenous matching, and with one or more sellers.[3]

Unfortunately, the homogeneous markets considered by RW and S are very special: in non-homogeneous markets, things are more complicated, both analytically and conceptually. In fact, as we discuss in Section 2, heterogeneous markets require a substantially different theory, not a simple extension.

In this paper, we follow S and refine the set of equilibria by introducing complexity costs lexicographically with the standard payoff into the players' preference ordering.[4] Complexity in S is represented by a partial ordering of the set of individual strategies.[5] Here, we use a similar approach. Very informally, if two strategies are otherwise identical except that, for some nonempty set of histories, where the agent is always proposer or always responder and there is a fixed set of remaining agents, one strategy does different things whereas the second does the same thing, then the first strategy is said to be more complex than the second. This definition of complexity implies a very weak partial ordering of strategies.

We define a *Nash equilibrium with complexity costs* (NEC) to be a strategy profile f such that each agent k's strategy f_k is a best response and there does not exist a best response f'_k that is less complex than f_k. A *perfect equilibrium with complexity costs* (PEC) is a (subgame) perfect equilibrium[6] that is also a NEC.

The main result of the paper is that a PEC is always competitive in

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[3] Some of the results in S depend on the precise definition of complexity used and on whether complexity costs are positive or vanishingly small.

[4] Rubinstein (1986) and Abreu and Rubinstein (1988) were the first to introduce complexity of strategies lexicographically into dynamic games. However, in these papers, players are modelled as finite-state automata involved in a two-player repeated game and complexity is measured by the number of states of the automaton.

[5] Chatterjee and Sabourian (2000) use a similar notion of complexity costs to justify Markov (stationary) equilibria in n-player alternating-offer bargaining games.

[6] Henceforth, the term perfect equilibrium refers to *subgame perfect equilibrium.*

every subgame. It shows that complexity costs are sufficient to characterize competitive behavior in a heterogeneous market, thus supporting the view that competitive equilibrium may arise in a finite market where complex behavior is costly. As a corollary, we also establish that every PEC is Markov in the sense of being history-independent.

The rest of the paper is organized as follows. In Section 2 we discuss the differences between homogeneous and heterogeneous markets. The main results of the paper are in Section 3 where we deal with the exogenous, deterministic, sequential matching model. First we describe the model. Next, we show that, as in RW's random matching model, there is a continuum of non-competitive perfect equilibria. Then we introduce the concept of complexity and show that any PEC induces a competitive outcome. Section 4 contains further discussion of our assumptions and results. The appendix contains the proof of the characterization result.

2 Heterogeneous markets

As we pointed out earlier, the markets analyzed by RW and by S are *homogeneous*, that is, comprising B identical buyers and S identical sellers. Without loss of generality, RW and S focus on the case in which $B > S$ (the case in which $B < S$ is symmetric) and the valuations of buyers and sellers are normalised to 1 and 0, respectively. The unique competitive price is 1 but, as we have noted, there is a continuum of non-competitive sequential equilibrium outcomes.

A *heterogeneous* market, by contrast, allows for a much richer set of equilibrium outcomes. We define a heterogeneous market as follows. As in the homogeneous case, there is a single indivisible good that is exchanged for money and each agent wants to trade at most one unit of the good. We denote the set of buyers and sellers by I and J respectively. Without loss of generality, we can assume that there are equal numbers of buyers and sellers.[7] Buyers are indexed by $i = 1, ..., n$ and sellers are indexed by $j = 1, ..., n$. Buyer i's valuation of the good is denoted by $v_i \geq 0$ and seller j's valuation is denoted by $w_j \geq 0$. We assume, for simplicity, that buyers' and sellers' valuations are generic. Then buyers and sellers can be ordered

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[7] Sellers with extremely high valuations and buyers with extremely low valuations cannot trade in any case.

so that $v_1 > v_2 > ... > v_n$ and $w_1 < w_2 < ... < w_n$ and $v_i \neq w_j$ for all i and j.

These valuations define the demand and supply curves that determine the competitive, market-clearing price(s) in the usual way.

The *marginal traders* $i = j = m$ are defined by the conditions

$$v_m > w_m, \ v_{m+1} < w_{m+1},$$

and the interval of *perfectly competitive prices* for this market is defined by the conditions

$$\max\{w_m, v_{m+1}\} \leq p \leq \min\{w_{m+1}, v_m\}.$$

Also, an agent k is called *inframarginal* (*extramarginal*) if $k \leq m$ ($k > m$).

The first point to note is that the arguments used in S to show that the unique PEC outcome is competitive do not extend to heterogeneous markets. To see this, first consider a homogeneous market. For the case of more than one seller the selection result is established by induction on the number of sellers. In the case of *a single seller* the main steps of the arguments in S are the following. First, in any perfect equilibrium, because there are more buyers than sellers, competition ensures there exists a buyer with a zero continuation payoff at some history. Second, in any PEC each agent either always rejects a price offer $p = 1$ or always accepts $p = 1$, irrespective of the past; otherwise he can economise on complexity[8]. This implies that in any *non-competitive* PEC there cannot be an agreement at $p = 1$ between a buyer and the seller at any history.[9] But then it follows that in any *non-competitive* PEC, after every history, the seller's continuation payoff is less than 1 and thus *all* continuation payoffs of *all* buyers are positive. Since, by the first step above, the continuation payoff of some buyer is zero at some history, it must be the case that every PEC is competitive.

Now consider a heterogeneous market. We can see immediately why the analysis of S will not suffice. In the homogeneous market, except for the

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[8] For example, if this were not the case for a buyer then, since accepting $p = 1$ results in a zero payoff, he could economize on complexity without forgoing any payoff by adopting another strategy that is the same as the equilibrium strategy except that it always rejects an offer of 1.

[9] For example, if this were not the case, and a buyer accepts $p = 1$ after some history, then by the previous argument he must accept $p = 1$ after every history. But this guarantees the seller an equilibrium payoff of 1 and hence contradicts the assumption that the outcome is non-competitive.

special case $B = S$, the competitive equilibrium price is either 0 or 1 and all of the surplus goes to one side of the market. In S this property of the competitive equilibrium is used extensively to obtain the selection result. In a heterogeneous market, by contrast, there will typically be agents receiving positive payoffs on both sides of the market in a competitive equilibrium. Therefore, one cannot rule out non-competitive outcomes simply by focusing on extreme outcomes in which one party derives no surplus from trade.

Moreover, there are several additional differences between homogeneous and heterogeneous markets that require substantive changes in the theory.

(i) *Efficient trade*: In a heterogeneous market, trade between an inframarginal agent and an extramarginal agent is always inefficient, but can be individually rational. In a homogeneous market, by contrast, individually rational trade is by definition efficient.

(ii) *The uniqueness of equilibrium:* As mentioned above, in the generic homogeneous case, the competitive price is uniquely determined. In the generic heterogeneous case, the set of competitive prices is a non-degenerate interval. The uniqueness of the equilibrium price makes the homogeneous case easier to analyze because, after one pair of agents has traded, the competitive price remains the same, so a proof by induction on the number of agents left in the market (as in S) can take for granted that the price is uniform if it is competitive. In the heterogeneous case, no such presumption can be made. After one trade, the assumption that the remaining trades take place at a competitive price does not guarantee that all take place at the same price. As a result, an induction argument that refers to “the” competitive price may become problematic.

(ii) *Invariance of the competitive interval*: As we mentioned above, in the homogeneous market, the set of competitive prices remains constant, independently of the set of agents remaining in the market. In the heterogeneous market, this need not be so. If the marginal traders trade first or if an inframarginal agent trades with an extramarginal agent then the competitive interval may change. In some cases, the new competitive interval may not even intersect the old one. The fact that the competitive interval of prices may change as the result of trade exacerbates the problems associated with using an induction hypothesis that refers to “the” competitive price (future prices may be conditioned on past trades even if prices are restricted to be competitive ones).

One way of summarizing these differences is to say that heterogeneous markets can be non-competitive in many more ways than homogeneous markets.[10] For all these reasons, the analysis of heterogeneous markets turns out to be more complicated and more subtle than the analysis of the homogeneous market.

3 Exogenous sequential matching

3.1 The model

The data for the heterogenous market described in the last section is given by a triple (K, v, w), where $K = I \cup J$ is the set of buyers and sellers, and $v = (v_1, ..., v_n)$ and $w = (w_1, ..., w_n)$ denote the valuations of the buyers and sellers, respectively, for a single unit of the good. Given the market data (K, v, w), a dynamic matching and bargaining game is defined by the following rules:

- Trade takes place at a sequence of dates $t = 1, 2, ...$ At each date t, a pair of agents consisting of one buyer and one seller is chosen from the set $N \subseteq K$ of agents remaining in the market. One member of the pair is the proposer $\pi_t(N) \in N$; the other is the responder $\rho_t(N) \in N$.

- The agent who is chosen as the proposer offers a price p. The responder can accept (A) or reject (R) this price. If the proposal is accepted, the good is traded at the agreed price and both agents leave the market. If the proposal is rejected, there is no trade and all agents begin the next period with the same endowments.

There is no discounting of utilities. If buyer i and seller j exchange a unit of the good at a price p, buyer i's payoff is $v_i - p$ and seller j's payoff is $p - w_j$.

Notice that exactly one pair is formed at each date. If the proposer $\pi_t(N)$ is a buyer then the responder $\rho_t(N)$ is a seller and vice versa.

A set $N \subseteq K$ is called *balanced* if it contains equal numbers of buyers and sellers. Let $\mathcal{N}$ denote the set of non-empty balanced sets. Then the game form is defined by a sequence of *matching functions* $(\pi, \rho) = \{(\pi_t, \rho_t)\}_{t=1}^{\infty}$

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[10] In contrast with the homogeneous market studied by RW, even the Markov property is not sufficient for perfect competition in heterogeneous markets with *random matching*. See Gale and Sabourian (2003a).

such that for any date t and any $N \in \mathcal{N}$, $(\pi_t(N), \rho_t(N)) \in N \times N$ and $[\pi_t(N) \in I] \Longleftrightarrow [\rho_t(N) \in J]$.

To obtain the characterization result of this section, we make the following assumption, which ensures the pattern of matches is sufficiently rich.

Assumption 1 For any $N \in \mathcal{N}$, for any feasible matches $(k, \ell) \in N^2$ and $(k', \ell') \in N^2$ such that $(k, \ell) \neq (k', \ell')$, and for any date T, there exists a $t > T$ such that

$$(\pi_t(N), \rho_t(N)) = (k, \ell) \text{ and } \left(\pi_{t+1}(N), \rho_{t+1}(N)\right) = (k', \ell').$$

The dynamic matching and bargaining game is one of complete and perfect information and is described fully by the data (K, u, v, π, ρ).

The outcome of play at any date is described by an ordered four-tuple $(k, \ell, p, r) \in K \times K \times \mathbf{R}_+ \times \{A, R\}$, where k is the proposer, ℓ the responder, p the price offer and r the response. The history of the game up to the beginning of date t consists of a sequence $h_t = ((k^1, \ell^1, p^1, r^1), ..., (k^{t-1}, \ell^{t-1}, p^{t-1}, r^{t-1}))$. Let H^t denote the set of histories at date t and let $H = \cup_{t=1}^{\infty} H^t$. The set of histories at date 1 is denoted by H^1 and consists of trivial (null) history. Note that the history $h \in H^t$ uniquely defines the set of agents $N(h)$ remaining in the game at date t. Thus, the history also uniquely defines the identity of the proposer and responder at date t. For any history $h \in H^t$, we denote the proposer by $\pi(h) \equiv \pi_t(N(h))$ and the responder by $\rho(h) \equiv \rho_t(N(h))$.

For any agent k, let $H_k^p = \pi^{-1}(k)$ denote the set of histories at which k is the proposer and let $H_k^r = \rho^{-1}(k)$ denote the set of histories at which k is the responder. At any history $h \in H_k^p$, agent k knows the history h, the set of remaining agents $N(h)$ and the identities of the proposer $\pi(h) = k$ and the responder $\rho(h)$ before he makes his offer. For any history $h \in H_k^r$, agent k knows the history h, the identities of the proposer $\pi(h)$ and the responder $\rho(h) = k$ and the price offer p before he makes his response r. A strategy for agent k is a function f_k defined on $H_k \equiv H_k^p \cup (H_k^r \times \mathbf{R}_+)$ such that

$$\begin{array}{ll} f_k(h) \in \mathbf{R}_+, & \forall h \in H_k^p \\ f_k(h, p) \in \{A, R\}, & \forall (h, p) \in H_k^r \times \mathbf{R}_+. \end{array}$$

Let F_k denote the strategy set for agent k and let $F = \times_{k \in K} F_k$ denote the set of strategy profiles.

Given any strategy profile $f = \{f_k\}_{k \in K}$, the outcome is uniquely determined. Let $U_k(f)$ denote the payoff to agent k from this outcome.

The market game is defined by the set of players K, the set of strategy profiles F, and the payoff function $U = (U_k)_{k \in K}$.

3.2 A continuum of non-competitive perfect equilibria

To show that the problem of characterizing the competitive equilibria of the market is non-trivial, we first show that the model defined above has a *continuum of non-competitive perfect equilibria* and that some of these equilibria are inefficient. We illustrate the point with an example of a market with two buyers and two sellers. The results extend immediately to arbitrary numbers of buyers and sellers.[11]

Proposition 1 *Suppose that* $N = 2$*,* $m = 1$ *and*

$$w_1 < v_2 < w_2 < v_1.$$

Then, for any $p_h \in (w_2, v_1)$ *and any* $p_\ell \in (w_1, v_2)$*,*

(i) there exists a perfect equilibrium such that w_2 *and* v_1 *trade at* p_h *and* w_1 *and* v_2 *trade at* p_ℓ*;*

(ii) there exists a perfect equilibrium such that v_1 *and* w_1 *trade at* p_b *where* $b = h$ *or* ℓ*.*

Notice that the perfect equilibrium described in part (i) is inefficient and the equilibria described in parts (i) and (ii) are not competitive (the competitive equilibria of this example involve v_1 and w_1 trading at a price $p \in [v_2, w_2]$). For a complete proof of this result, the reader is referred to Gale and Sabourian (2003b). Here we merely sketch the construction of an equilibrium for each pair of parameters (p_h, p_ℓ).

The equilibrium strategy profile can be described by a collection of three states $\{s, s_h, s_\ell\}$ and the transition rules between them. We denote the transition function by $\mu : \{s, s_\ell, s_h\} \times \Sigma \to \{s, s_\ell, s_h\}$, where Σ is the set of outcomes in a given period. In each of the three states and for any set of remaining agents, unless explicitly mentioned, the players behave as follows.

State s: In this state (v_1, w_1) do not agree when there are four agents (in particular, v_1 offers p_ℓ and accepts p if and only if $p \leq p_\ell$ and w_1 offers p_h and accepts p if and only if $p \geq p_h$); (v_1, w_1) agree on p_ℓ when there are two agents; (v_1, w_2) agree on p_h; and (v_2, w_1) agree on p_ℓ.

State s_ℓ: In this state (v_1, w_1) agree on p_ℓ; (v_1, w_2) do not agree; (v_2, w_1) do not agree (in particular, v_2 offers p_ℓ and accepts p if and only if $p \leq p_\ell$ and w_1 offers p_h and accepts p if and only if $p \geq p_h$).

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[11] The construction has a family resemblence to the RW construction for the random matching model. Equivalent results can also be established if the players discount the future by sufficiently small amounts.

State s_h: In this state (v_1, w_1) agree on p_h; (v_1, w_2) do not agree (in particular, v_1 offers p_ℓ and accepts p if and only if $p \leq p_\ell$ and w_2 offers p_h and accepts p if and only if $p \geq p_h$); (v_2, w_1) do not agree.

Finally, in any state where the agents (v_2, w_2) are matched they do not agree.

For any outome $\sigma \in \Sigma$ the transition function μ satisfies the following

$$\mu(s, \sigma) = \begin{cases} s_h & \text{if either } [\sigma = (v_1, w_1, p, R) \text{ and } p \neq p_\ell] \text{ or } \sigma = (v_2, w_2, p, A) \\ s_\ell & \text{if } \sigma = (w_1, v_1, p, R) \text{ and } p \neq p_h \\ s & \text{otherwise} \end{cases}$$

$$\mu(s_\ell, \sigma) = \begin{cases} s_h & \text{if } \sigma = (v_2, w_1, p, R) \text{ and } p \neq p_\ell \\ s & \text{if } \sigma = (w_1, v_2, p, R) \text{ and } p \neq p_h \\ s_\ell & \text{otherwise} \end{cases}$$

$$\mu(s_h, \sigma) = \begin{cases} s & \text{if } \sigma = (v_1, w_2, p, R) \text{ and } p \neq p_\ell \\ s_\ell & \text{if } \sigma = (w_2, v_1, p, R) \text{ and } p \neq p_h \\ s_h & \text{otherwise} \end{cases}$$

Note that if the current state is s then the above strategy profile results in (v_1, w_2) agreeing to p_h and (v_2, w_1) agreeing to p_ℓ. Similarly, if the current state is s_b (for $b = h$ or ℓ) then the above strategy profile results in (v_1, w_1) agreeing on p_b. Therefore, amongst the three states, s is best for the extramarginal agents, s_ℓ and s_h are best for v_1 and w_1, respectively.

To verify that the profile defined above is a perfect equilibrium, one would need to show that in each of the three states the strategy attributed to each player is optimal, given the strategies of the others. The argument turns on first showing that whenever the agent who is chosen as the proposer deviates from the equilibrium proposal by making an improving offer, the proposal is rejected and the proposer receives the same continuation payoff as he would have obtained had he not deviated. Secondly, one would need to show that in any state the responder rejects any such deviation by the proposer because rejection results in a transition that induces a payoff for the responder no less than that he would obtain from accepting the deviation proposal.[12]

Finally, note that appropriate choice of the initial state establishes the results in part (i) and (ii) of the Proposition.

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[12] The basic argument is the same as that in Rubinstein's alternating bargaining model with more than two players or with discrete offers: no player makes an improving offer because the responder, who correctly perceives that he will do better by waiting, rejects the offer (see Osborne and Rubinstein, 1990).

3.3 Complexity

Before introducing the notion of complexity, we need some further notation.

For any two players $k, \ell \in K$, let $\langle k, \ell \rangle$ denote the match between k and ℓ in which k is the proposer and ℓ the responder. Let $H\langle k, \ell \rangle = \{ h \in H | \, \pi(h) = k \text{ and } \rho(h) = \ell \}$ denote the set of histories resulting in the match $\langle k, \ell \rangle$. For any balanced set of agents $N \in \mathcal{N}$, let $H(N) = \{ h \in H | \, N(h) = N \}$. Finally, for any $N \in \mathcal{N}$ and any match $\langle k, \ell \rangle$, let $H(N, \langle k, \ell \rangle) \equiv H(N) \cap H\langle k, \ell \rangle$ be the set of histories at which N is the set of remaining agents, k is the proposer and ℓ is the responder.

Definition 2 *For any agent k, a strategy f_k is called a Markov strategy* [13] *if, for all ℓ and for all $N \in \mathcal{N}$,*

$$\begin{array}{ll} f_k(h) = f_k(h'), & \forall h, h^{'} \in H(N, \langle k, \ell \rangle) \\ f_k(h, p) = f_k(h', p), & \forall h, h^{'} \in H(N, \langle \ell, k \rangle) \text{ and } \forall p \in \mathbf{R}_+. \end{array} \tag{1}$$

As we mentioned before, we measure complexity by a very weak partial order on the set of strategies. Roughly, one strategy is considered more complex than another if the two strategies are identical *except that*, for a given set of remaining agents, in a given match either as a proposer or as a responder to some price offer p, the first strategy is a constant function whereas the second strategy is not.

Definition 3 *(i) For any player k, a strategy f_k is called a simple strategy at N when k is the proposer in a match with player ℓ if*

$$f_k(h) = f_k(h'), \forall h, h^{'} \in H(N, \langle k, \ell \rangle).$$

(ii) For any player k, a strategy f_k is simple at N when k is the responder to a price p in a match with player ℓ if

$$f_k(h, p) = f_k(h', p), \ \forall h, h^{'} \in H(N, \langle \ell, k \rangle).$$

(iii) For any player k, a strategy f_k is simple at N if and only if, for any $\ell \in N$ and $p \in \mathbf{R}_+$, f_k is a simple strategy at N both when k is the proposer and when k is the responder to p, in a match with ℓ.

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[13] Note that Markov strategies, according to this definition, are history-independent. Since the matching technology is not stationary, this definition is stronger that the usual definition of Markov strategies, which would allow strategies to depend on all payoff relevant variables, including the date t.

Note that a strategy f_k is Markov if and only if, for any $N \in \mathcal{N}$, f_k is a simple strategy at N.

Definition 4 *For any agent k, a strategy f'_k is more complex than f_k, denoted by $f^{'}_k \succ f_k$, if one of the following two conditions is satisfied:*

1. *there exist a balanced set N and a player $\ell \in N$ such that f_k and f'_k are otherwise identical except that f_k is simple at N when k is the proposer in a match with ℓ and f'_k is not; formally there exist a balanced set $N \in \mathcal{N}$ and a player $\ell \in N$ such that*

$$\begin{array}{ll} f_k(h) = f'_k(h), & \forall h \in H^p_k \backslash H(N, \langle k, \ell\rangle), \\ f_k(h,p) = f'_k(h,p), & \forall h \in H^r_k \text{ and } \forall p \in \mathbf{R}_+, \\ f_k(h) = f_k(h') & \forall h, h^{'} \in H(N, \langle k, \ell\rangle), \\ f'_k(h) \neq f'_k(h') & \text{for some } h \text{ and } h^{'} \in H(N, \langle k, \ell\rangle); \end{array}$$

2. *there exist a balanced set N, a player $\ell \in N$ and a price p such that f_k and f'_k are otherwise identical except that f_k is simple at N when k is the responder to p in a match with ℓ and f_k is not; formally there exist a balanced set $N \in \mathcal{N}$, a player $\ell \in N$ and a price $p \in \mathbf{R}_+$ such that*

$$\begin{array}{ll} f^{'}_k(h) = f_k(h), & \forall h \in H^p_k, \\ f^{'}_k(h,p') = f_k(h,p'), & \text{if either } h \in H^r_k \backslash H(N, \langle \ell, k\rangle) \text{ or } p' \neq p, \\ f_k(h,p) = f_k(h',p), & \forall h, h^{'} \in H(N, \langle \ell, k\rangle), \\ f'_k(h,p) \neq f'_k(h',p), & \text{for some } h \text{ and } h' \in H(N, \langle \ell, k\rangle). \end{array}$$

This complexity criterion ranks two strategies only if, in some situation, one of them is simple and the other is not.

3.4 Nash equilibrium and complexity

We are now ready to define the Nash equilibrium of the game with complexity cost.

Definition 5 *A strategy profile $f \in F$ constitutes a Nash equilibrium with lexicographic complexity cost (denoted by NEC) if, for each player k, the following two conditions hold*

$$\begin{array}{l} (i)\ U_k(f_k, f_{-k}) \geq U_k(f'_k, f_{-k}),\ \forall f'_k \in F_k; \\ (ii)\ \forall f'_k \in F_k,\ f_k \succ f'_k \text{ implies } U_k(f_k, f_{-k}) > U_k(f'_k, f_{-k}). \end{array}$$

Note that this notion of equilibrium does not assume that players choose Markov strategies. Given the equilibrium behaviour of his opponents, a best response may require a player to engage in non-Markov behavior. NEC simply requires that additional complexity be justified by a higher payoff.

In the rest of this section, we explore the restrictions on the strategies that are implied by NEC. These restrictions are used repeatedly in the proof of the main result, which is formally stated in the next subsection. The propositions that follow illustrate well how the the principle of lexicographically minimizing complexity costs leads to simple behavior. Note that we do not appeal to subgame perfection to establish these results: these are all properties of *Nash equilibrium* with complexity costs.

The first proposition shows that, if an agent accepts an offer from another agent along the equilibrium path, then he always accepts the offer, irrespective of the previous history, whenever the two agents are matched in the same way with the same remaining set of agents.

Proposition 6 *Let f be a NEC and let $E \subset H$ denote the set of finite histories that occur along the equilibrium path of f. Suppose that, for some finite history $h_0 \in E \cap H(N, \langle k, \ell \rangle)$,*

$$f_k(h_0) = p_0, f_\ell(h_0, p_0) = A.$$

Then

$$f_\ell(h, p_0) = A, \forall h \in H(N, \langle k, \ell \rangle). \tag{2}$$

Proof. Suppose not; then $f_\ell(h_1, p_0) = R$ for some $h_1 \in H(N, \langle k, \ell \rangle)$. Define a new strategy f'_ℓ as follows. For any $h \in H^p_\ell$ put $f'_\ell(h) = f_\ell(h)$ and for any $(h, p) \in H^r_\ell \times \mathbf{R}_+$ let

$$f'_\ell(h, p) = \begin{cases} A & \forall h \in H(N, \langle k, \ell \rangle) \text{ and } p = p_0, \\ f_\ell(h, p) & \text{otherwise.} \end{cases}$$

By inspection we see that $f_\ell \succ f'_\ell$ because $f_\ell(h_0, p_0) \neq f_\ell(h_1, p_0)$. Further, f'_ℓ must give the same payoff as f_ℓ. To see this, note that if ℓ chooses f'_ℓ then after any history $h \in H(N, \langle k, \ell \rangle)$, either agent k proposes p_0 and ℓ accepts, so the payoff to agent ℓ is the same as from the equilibrium strategy, or agent k does not offer p_0, in which case the change in the strategy is not observed and the play of the game is unaffected by the deviation. In either case, the payoff to agent ℓ from f'_ℓ is the same as from f_ℓ, contradicting the definition of NEC. ■

The second result shows that in any NEC each player's response to another player's offer is simple (either always accept the offer or always reject).

Proposition 7 *Let f be a NEC. Then for any $N \in \mathcal{N}$, any players $k, \ell \in N$ and any price offer p_0 strategy f_ℓ is simple at N when ℓ is the responder to p_0 in a match with k.*

Proof. Suppose not; then for some $N \in \mathcal{N}$, some match $\langle k, \ell \rangle$ and some price offer p_0, there exist histories $h', h'' \in H(N, \langle k, \ell \rangle)$ such that

$$f_\ell(h', p_0) = R \text{ and } f_\ell(h'', p_0) = A,$$

It then follows from Proposition 6 that

$$[f_k(h) = p_0] \Longrightarrow [f_\ell(h, p_0) = R], \forall h \in E \cap H(N, \langle k, \ell \rangle), \tag{3}$$

where $E \subset H$ denotes the set of histories that occur along the equilibrium path of f. Now define a new strategy f'_ℓ as follows. For any $h \in H^p_\ell$ put $f'_\ell(h) = f_\ell(h)$ and for any $(h, p) \in H^r_\ell \times \mathbf{R}_+$ let

$$f'_\ell(h, p) = \begin{cases} R & \text{if } h \in H(N, \langle k, \ell \rangle) \text{ and } p = p_0 \\ f_\ell(h, p) & \text{otherwise.} \end{cases}$$

Clearly, $f_\ell \succ f'_\ell$. Further, f'_ℓ must give the same payoff as f_ℓ because, by (3), they do not differ on the equilibrium path. But since f is a NEC this is a contradiction. ■

The next result shows that if at some history two agents trade at a price that results in a payoff no less than the proposer's equilibrium payoff, then they make the same trade whenever they are matched in the same way with the same remaining set of agents.

Proposition 8 *Let f be a NEC. Consider any NEC profile f, any player k and any price p_0 such that*

$$\begin{array}{ll} v_k - p_0 \geq U_k(f) & \textit{if } k \textit{ is a buyer} \\ p_0 - w_k \geq U_k(f) & \textit{if } k \textit{ is a seller.} \end{array}$$

Suppose that, for some N and for some finite history $h_0 \in H(N, \langle k, \ell \rangle)$,

$$f_k(h_0) = p_0 \textit{ and } f_\ell(h_0, p_0) = A. \tag{4}$$

Then we have

$$f_k(h) = p_0, \forall h \in H(N, \langle k, \ell \rangle), \tag{5}$$

$$f_\ell(h, p_0) = A, \forall h \in H(N, \langle k, \ell \rangle). \tag{6}$$

Proof. Condition (6) follows immediately from condition (4) and Proposition 7. Now to show condition (5) suppose otherwise. Define a new strategy f'_k as follows. Put $f'_k(h,p) = f_k(h,p)$ for any $(h,p) \in H^r_k \times \mathbf{R}_+$ and let

$$f'_k(h) = \begin{cases} p_0 & \forall h \in H\left(N, \langle k, \ell \rangle\right) \\ f_k(h) & \text{otherwise,} \end{cases}$$

for any $h \in H^p_k$. Then $f_k \succ f'_k$ by inspection. Moreover, the choice of (f'_k, f_{-k}) either induces a history $h \in H\left(N, \langle k, \ell \rangle\right)$, in which case, by the definition of f'_k and condition (6), k and ℓ trade at the price p_0 and k obtains a payoff no less than $U_k(f)$, or no history $h \in H\left(N, \langle k, \ell \rangle\right)$ occurs, in which case the payoff of agent k is unaffected by the deviation to f'_k. Therefore, f'_k induces at least the same payoff as f_k contradicting the definition of NEC. ■

The fourth result shows that if, along the equilibrium path, k does not make an offer to ℓ, then *k's* behavior is simple as a proposer in a match with ℓ, assuming the set of remaining agents is the same in each case.

Proposition 9 *Let f be a NEC and let E denote the set of equilibrium histories induced by f. Then for all $N \in \mathcal{N}$ and any $k, \ell \in N$ if $E \cap H\left(N, \langle k, \ell \rangle\right) = \emptyset$ then f_k is simple at N when k is the proposer in a match with ℓ.*

Proof. Suppose not; then there exists histories $h', h'' \in H\left(N, \langle k, \ell \rangle\right)$ such that $f_k(h') \neq f_k(h'')$. Define a new strategy f'_k as follows. For any $(h,p) \in H^r_k \times \mathbf{R}_+$ put $f'_k(h,p) = f_k(h,p)$, for any $h \in H^p_k$ put

$$f'_k(h) = \begin{cases} f_k(h') & \text{if } h \in H\left(N, \langle k, \ell \rangle\right) \\ f_k(h) & \text{otherwise.} \end{cases}$$

Note that by inspection $f_k \succ f'_k$. Further, f'_k must give the same payoff as f_k because by assumption they do not differ on the equilibrium path. This contradicts the definition of a NEC. ■

3.5 Perfect equilibrium and competitive equilibrium

Now we are ready to define perfect equilibria with complexity costs introduced lexicographically and state the central result.

Definition 10 *A strategy profile f is called a* perfect equilibrium with complexity costs *(PEC) if it is both a perfect equilibrium and a NEC.*

The next theorem is the main result of the paper. It establishes that for any $N \in \mathcal{N}$, every PEC strategy profile induces a competitive outcome (for the set N) in all subgames defined by the set of histories $H(N)$. Formally, for any $N \in \mathcal{N}$, let $\mathcal{I}(N)$, $\mathcal{X}(N)$ and $C(N)$ denote respectively the set of inframarginal players, the set of extramarginal players and the competitive price interval when N is the set of (remaining) agents in the market. Also, denote the set of subgames defined by the set of histories $H(N)$ by $\Gamma(N)$.

Theorem 11 *Consider any PEC profile f and any $N \in \mathcal{N}$. Then the strategy profile f is such that either $C(N)$ is empty and there is no trade in any subgame $\gamma \in \Gamma(N)$ or there exists a competitive price $\bar{p} \in C(N)$ such that in any subgame $\gamma \in \Gamma(N)$ all inframarginal agent $k \in \mathcal{I}(N)$ trades at $\bar{p}$ and all extramarginal agent $\ell \in \mathcal{X}(N)$ does not trade.*

Corollary 12 *Any PEC profile f is Markov.*

We appeal to the three properties of NEC described in Propositions 7 to 9 to prove Theorem 11. The formal proof of Theorem 11 (and its Corollary), described in the Appendix, is by induction on the set of all subgames ordered by the number of remaining agents in the subgames. In particular, the Induction Step involves showing that for any integer r if the result holds for any pool of agents consisting of $2(r-1)$ agents then it also holds for any pool of agents consisting of $2r$ agents. To establish this Induction Step we asume that the result holds for subgames with $2(r-1)$ agents. We then show, using the restrictions implied by Propositions 7-9, that the strategy profile is simple for any set of remaining agents with cardinality $2r$. Finally, we use the simplicity of the profile to show that for any set of remaining agents $N_r \in \mathcal{N}$ consisting of $2r$ agents and any subgame $\gamma_r \in \Gamma(N_r)$, if the pair of agents i_r and j_r trade first in this subgame at the price q and the remaining agents trade subsequently at some $p_{r-1} \in C(N \backslash \{i_r, j_r\})$ (this follows from the induction assumption for $r-1$) then $q = p_{r-1} \in C(N_r)$.

To obtain the above characterisation results, Assumption 1 (or a similar condition) is needed. This is most easily seen by considering a counter-example. More precisely, consider a market where $n = 3$ and

$$w_2 < v_3 < w_3 < v_2. \tag{7}$$

That is, v_1, v_2, w_1, and w_2 are inframarginal and v_3 and w_3 are extramarginal. Also, any $p \in [v_3, w_3]$ is a competitive outcome.

Notice that if the pair v_i and w_3 with $i < 3$ leave the market any price level $q \in [w_2, v_3]$ is a competitive price and that if the pair v_3 and w_j with $j < 3$ leave the market any price level $r \in [w_3, v_2]$ is a competitive price.

If there is no agreement, suppose that the sequence of matches consists of cycles of the following of finite sequence of matches:

$$\langle v_3, w_2\rangle\langle v_1, w_2\rangle\langle v_2, w_3\rangle\langle v_2, w_2\rangle\langle v_3, w_1\rangle\langle v_2, w_1\rangle\langle v_1, w_3\rangle\langle v_1, w_1\rangle\langle v_3, w_3\rangle \tag{8}$$

followed by the same sequence with the roles reversed

$$\langle w_2, v_3\rangle\langle w_2, v_1\rangle\langle w_3, v_2\rangle\langle w_2, v_2\rangle\langle w_1, v_3\rangle\langle w_1, v_2\rangle\langle w_3, v_1\rangle\langle w_1, v_1\rangle\langle w_3, v_3\rangle. \tag{9}$$

This deterministic matching process ensures that every possible pair $\langle \ell, k\rangle$ occurs infinitely often, but it does not satisfy Assumption 1 because there are some possible sequences $\{\langle \ell, k\rangle, \langle \ell', k'\rangle\}$ that do not occur even once. With this matching process a PEC is not necessarily competitive as the following result shows.

Proposition 13 *Suppose a market with $n = 3$ satisfies (7), (8) and (9). Fix any $q \in (w_2, v_3)$ and $r \in (w_3, v_2)$. Then there exists a perfect equilibrium in Markov strategies (hence a PEC) in which v_3 and w_2 trade at the price q and, in the continuation game that follows, all trade occurs at the competitive equilibrium price r.*

The description of the required equilibrium is given in the appendix. In this equilibrium, v_3 and w_2 anticipate that if they do not trade at a non-competitive price, the next pair (v_1, w_2) will trade in a way that changes the interval of competitive prices and makes both v_3 and w_2 no better off. So it is rational for v_3 and w_2 to trade at q. Once they have traded, the competitive interval changes and in the subgame that follows the remaining agents to trade at the price r. Why is this not an perfect equilibrium under Assumption 1? As we have seen in Section 3.4, behavior on the equilibrium path has strong implications in subgames that do not occur on the equilibrium path. In particular, if trade occurs at the price q on the equilibrium path then it should occur in any subgame with the same set of remaining players when the pair $\langle v_3, w_2\rangle$ is matched. However, under Assumption 1 in some of these subgames the prescribed behavior will not be optimal, contradicting the definition of perfect equilibrium.

4 Concluding remarks

The main result of this paper — that every PEC of the market game induces a competitive outcome and hence is Markov — can be extended to other deterministic matching models. One possible extension is to the case of semi-endogenous sequential matching, in which the proposer is chosen exogenously and then chooses the responding partner endogenously. The same general approach works with this variation of the model: Gale and Sabourian (2003b) show that in a model with an endogenous choice of responders, any PEC induces a competitive outcome.

A more drastic change would be to allow the choice of proposers and responders to be endogenous and to allow simultaneous matching and bargaining to take place. Then each agent can make a proposal to any other agent at any date. In a related paper, Gale and Sabourian (2003c) show that complexity costs can be used to select the competitive equilibrium, in a model where agents are allowed to make offers simultaneously in continuous time. Simultaneous offers and responses raise new difficulties, however. A major part of the analysis is the development of a plausible protocol to deal with inconsistent offers and responses (e.g., what happens when an agent simultaneously accepts an offer and has an offer accepted).

The fact that the characterization result holds for three deterministic matching models — the exogenous model used in this paper, the semi-endogenous model of Gale and Sabourian (2003b) and the endogenous model of Gale and Sabourian (2003c) — is evidence of the robustness of the relationship between complexity costs and competitive behavior in *deterministic* matching and bargaining games.

The assumption that matching is deterministic rather than random may also be restrictive. Random matching is convenient because it allows for a stationary model, which is impossible with exogenous deterministic matching. This is the only reason we can see for introducing uncertainty into an otherwise non-stochastic setting. If the matching procedure were completely endogenous, as in Gale and Sabourian (2003c), it would be deterministic in any pure-strategy equilibrium. Random matching is problematic because, as Gale and Sabourian (2003a) show, a SPE in Markov strategies is not necessarily perfectly competitive in the random matching model. Since Markov strategies are minimally complex, according to the definition used in this paper, this notion of complexity is not sufficient to select a competitive outcome. The problem with random matching is simply that there are more information sets on the equilibrium path, so a stronger notion of complexity costs (represented by a more complete partial ordering) is required to deliver the perfectly competitive outcome. Gale and Sabourian (2003a) show that a stronger notion of complexity, together with the assumption of Markov Nash equilibrium with perfect responses, does indeed characterize perfect competition.

In Section 3, a PEC (NEC) was defined to be a profile of perfect (Nash) equilibrium strategies $f = (f_k, f_{-k})$ such that for each player k strategy f_k has minimal complexity amongst all strategies for k that are best responses to f_{-k}. [14] Although the assumptions underlying our definition of complexity are mild, this is not the only possible approach. Our approach puts more weight on complexity costs than on the off-the-equilibrium-path moves: *in considering complexity*, players ignore any consideration of payoffs off-the-equilibrium path and the trade-off is between the equilibrium payoffs of the two strategies and the complexity of the two. Therefore, although complexity costs are negligible, they take priority over optimal behavior after deviations.

We can think of the concept of PEC as the limit as two kinds of perturbations become vanishingly small. One perturbation is to impose a small but positive cost of choosing a more complex strategy. Another perturbation is to introduce a small but positive probability of making an off-the-equilibrium-path move. To obtain PEC, we first let the probability of making an off-the-equilibrium-path move go to zero and then let the cost of choosing a more complex strategy go to zero. (See Chatterjee and Sabourian, 2000). This is reflected in the fact that we require agents to choose a minimally complex strategy within the set of best responses.

Alternatively, it may be that, in some situations, complexity costs are less significant than the possibility of a deviation from the equilibrium path. In terms of limiting arguments, we would represent this by letting the cost of choosing a more complex strategy go to zero first and then letting the

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[14] We could also consider Nash equilibria with positive complexity costs and thus allow tradeoff between complexity and equilibrium payoffs. Since such preference ordering induces at least as much economy in complexity as the lexicographic criterion the properties of NEC profiles described in Propositions 7-9 also hold trivially if for Nash equilibria with positive complexity costs. Therefore, the selection results of this paper also hold for any perfect equilibrium that is also Nash with positive complexity costs. (It may be that perfect equilibrium is not an appropriate concept with a positive complexity costs; we discuss below the case in which PEC is the limit of a model with trembles and positive complexity costs.)

probability of making an off-the-equilibrium-path move go to zero. In the extreme case, in which complexity costs are less significant than the probability of *every* off-the-equilibrium-path move, this would require agents to choose minimally complex strategies among the set of strategies that are best responses on and off the equilibrium path (see Kalai and Neme (1992)).[15]

For the case of a market with a *homogeneous* set of buyers and sellers, S shows that the competitive selection result is independent of the relative importance of complexity costs and off-the-equilibrium payoffs (trembles). In the case of heterogeneous markets, however, the proofs of main results do depend on the relative importance of complexity costs and off-the-equilibrium payoffs (trembles). In our working paper (Gale and Sabourian 2004a), we construct an example of a non-competitive perfect equilibrium strategy profile $f = (f_k, f_{-k})$ such that f_k is least complex amongst all strategies that are best responses at *every* information set.

One conclusion to draw from the above discussion is that a competitive outcome is more likely when complexity costs are more significant than the pertubations that induce off-the-equilibrium-path behaviour. However, this conclusion — that the characterization may depend on the order of limits — is at least partly a reflection of the fact that we are working with a very weak notion of complexity (the partial ordering $\succ$) to begin with. Because we have little data about complexity costs, we want to make the weakest possible assumptions about the complexity ordering $\succ$. This may be the main reason that there are difficulties in weakening the concept of PEC in this particular direction.

Finally, it is important to note that the market considered here (and in the rest of the literature) is clearly very special: only one good is being traded (for money) and the agents are allowed to trade one unit one time. Allowing more than one unit introduces a real possibility of monopoly power (for example a seller may withhold supply while still participating in trade). Clearly, it will be important to extend our analysis of complexity to these richer markets. Our conjecture is that the results obtained here do extend to models where multiple units are traded. Note that the restrictions (in terms of simple behaviour) implied by Propositions 7-9 on the set of NECs also hold in a multi-unit market game where each proposal consists of a quantity

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[15] The above two alternatives are rather extreme. For example, one could allow the order in which complexity costs and trembles enter the limiting arguments to be such that complexity costs are more important than some (off-the-equilibrium paths) trembles and less important than some others.

and a price offer. The induction argument will be quite different from the one used in this paper, however.

If the conjecture is not valid, it may still be possible to use the properties established in Propositions 7-9 to prove that an NEC of any finite economy has Markov strategies. This would allow us to use the approach of Gale (2000) to show that our characterization can be obtained asymptotically for large finite games.[16] For example, even when agents have a small amount of market power, they may not exploit it because of the complexity costs.[17]

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[16] In the context of random matching model, Gale (2000) shows that the Markov assumption, together with some other technical assumptions, are sufficient to justify a competitive outcome in a large and finite exchange economy.

[17] We would like to thank a referee and the Associate Editor for raising the issue of complexity and market power.

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Appendix: Proof of Theorem 11 and Corollary 12

Before stating the proof of Theorem 11 we need to introduce one further piece of notation and establish three lemmas. We define the set of histories $h \in H(N)$ such that k and ℓ are, respectively, the proposer and the responder at h and k' and ℓ' are respectively the proposer and the responder the next period if k and ℓ do not reach an agreement at h by

$$H(N, \langle k, \ell \rangle, \langle k', \ell' \rangle) = \left\{ h \in H(N) \,\middle|\, \begin{array}{l} \text{if } h \in H^t \text{ then } (\pi_t(N), \rho_t(N)) = (k, \ell) \\ \text{and } (\pi_{t+1}(N), \rho_{t+1}(N)) = (k', \ell') \end{array} \right\}$$

Lemma 14 *Let f be a PEC. Then for any k, f_k is Markov if $U_k(f) = 0$.*

Proof. Suppose not; then it follows from Proposition 7 that for some $N \in \mathcal{N}$ and $\ell \in N$, $f_k(h_0) \neq f_k(h_1)$ for some $h_0, h_1 \in H(N, \langle k, \ell \rangle)$. Next consider another strategy f'_k that is otherwise identical to f_k except that $f'_k(h) = f_k(h_0)$ $\forall h \in H(N, \langle k, \ell \rangle)$. Clearly, $f_k \succ f'_k$. Moreover since f is a perfect equilibrium it must be that k's continuation payoff is non-negative in any subgame. Therefore, by the definition of f'_k and Proposition 7 (each player's response is simple), f'_k guarantees at least the equilibrium payoff of zero. But this contradicts f being a NEC. ■

Lemma 15 *Consider any PEC f. Suppose that for some $N \in \mathcal{N}$ there is no trade in some subgame $\gamma' \in \Gamma(N)$. Then in every subgame $\gamma \in \Gamma(N)$ there is no trade and the outcome is competitive ($N = \mathcal{X}(N)$ and $C(N)$ is empty).*

Proof. We establish this result in several steps.

Step 1: There cannot be an agreement at any $h \in H(N) \cap E$, where E is the equilibrium history that f induces. Otherwise, there exists players $k, l \in N$ and a history $h' \in H(N, \langle k, \ell \rangle) \cap E$ such that k and l trade at h'. By Proposition 8 this implies that players k and l always trade at any $h \in E \cap H(N, \langle k, \ell \rangle)$. But this, together with Assumption 1, contradicts the hypothesis that there is no trade in the subgame $\gamma' \in \Gamma(N)$.

Step 2: There is no trade in any subgame $\gamma \in \Gamma(N)$. Otherwise, there exists players $k', l' \in N$ and a history $h' \in H(N, \langle k', \ell' \rangle)$ such that k' and l' trade at h'. Now by Step 1 either $E \cap H(N)$ is empty or $U_k(f) = 0$ for all $k \in N$. But then, by Propositions 7 and 9, and Lemma 14, it must be that f_k is simple at N for each $k \in N$. This implies that players k' and l' always trade at any history $h \in H(N, \langle k', \ell' \rangle)$. But this, together with Assumption 1, contradict the hypothesis that there is no trade in the subgame $\gamma' \in \Gamma(N)$.

Step 3: Every $k \in N$ is extramarginal and $C(N)$ is empty. Otherwise, for some i and $j \in N$ we have $v_i > w_j$. But by Step 2, the continuation payoff of each player $k \in N$ is zero at every history $h \in H(N)$. This implies that j accepts any offer $p' > w_j$ at any $h' \in H(N, \langle i, j \rangle)$. Thus, buyer i can obtain a positive payoff by offering a price $p' \in (w_j, v_j)$; but this contradicts the conclusion that i's continuation payoff at any $h \in H(N)$ is zero. ■

Lemma 16 *Consider any PEC strategy profile f. Suppose that for some $N \in \mathcal{N}$*

1. *there exists a subgame $\gamma' \in \Gamma(N)$ such that there is one and only one trade in the subgame γ';.*
2. *f_k is simple at N when k is the proposer in a match with ℓ for any pair of buyers and sellers $(k, \ell) \in N \times N$.*

Then there exists a competitive price $p_1 \in C(N)$ such that in any subgame $\gamma \in \Gamma(N)$ the outcome is competitive and any trade takes place at p_1.

Proof. Let i_1 and j_1 denote respectively the (only) buyer and the (only) seller that trade in the subgame γ'. Denote all other agents in N by

$$N' = N \backslash \{i_1, j_1\}.$$

Also denote the history, the match and the price at which i_1 and j_1 trade in this subgame by $h_1, \mid i_1, j_1 \mid$ and p_1, respectively. Thus if players choose f in the subgame γ' it results in a history $h_1 \in H(N_1)$ such that the following conditions hold

$$\mid i_1, j_1 \mid = \langle \pi(h_1), \rho(h_1) \rangle$$

$$f_{\pi(h_1)}(h_1) = p_1, f_{\rho(h_1)}(h_1, p_1) = A$$

where $\pi(h_1)$ is the proposer and $\rho(h_1)$ is the responder in the match $\mid i_1, j_1 \mid$. But then since by assumption $f_{\pi(h_1)}$ is simple at N when $\pi(h_1)$ is the proposer in a match with $\rho(h_1)$ it follows from Proposition 7 that

$$i_1 \text{ and } j_1 \text{ reach an agreement at any } h \in H(N_1, \mid i_1, j_1 \mid) \qquad (10)$$

Now we show that i_1 and j_1 are the only inframarginal agents in N (for any $i, j \in N'$, $v_i < w_j$), $p_1 \in C(N)$ and in any subgame $\gamma \in \Gamma(N)$, i_1 and

j_1 trade at the price p_1 and this is the only trade that occurs. This involves establishing the following steps.

Step 1: There are no individually rational trades in the set N' and for any $k \in N'$ the continuation payoff of k is zero at any $h \in H(N')$. Since after i_1 and j_1 trade at h_1 no other agent belonging to the set N' trade it follows from Lemma 15 that in any subgame $\gamma \in \Gamma(N')$ the outcome is competitive and there is no trade. This implies that there are no individually rational trades in the set N' and the continuation payoff of any agent $k \in N'$ is zero at every $h \in H(N')$.

Step 2. Any $i \in N'$ will accept any offer $p < v_i$ at any history $h \in H(N, \langle j_1, i\rangle)$. Consider any history $h' \in H(N, \langle j_1, i\rangle, \mid i_1, j_1 \mid)$ (by Assumption 1 such a history exists). If at h' no agreement is reached in the match $\langle j_1, i\rangle$, then, by (10), i_1 and j_1 reach an agreement in the next match $\mid i_1, j_1 \mid$ and thus i receives a zero payoff in the continuation game by Step 1. Thus, by (subgame) perfection, i should accept any offer $p < v_i$ in the match $\langle j_1, i\rangle$ at h'. But then from Proposition 7 buyer i will accept $p < v_i$ at any history $h \in H(N, \langle j_1, i\rangle)$.

Step 3. Any $j \in N'$ will accept any offer $p > w_j$ at any history $h \in H(N, \langle i_1, j\rangle)$. The proof is the same as that of Step 2.

Step 4. Both i_1 and j_1 are inframarginal agents in the set N. The proof is by contradiction. Suppose that, contrary to what we want to prove, $j_1 \in \mathcal{X}(N)$ (and $i_1 \in \mathcal{I}(N)$). Then there must be a seller $j \neq j_1$ such that $j \in \mathcal{I}(N)$. By Step 1 and individual rationality we have $v_i < w_j < w_{j_1} \leq p_1$ for any $i \in N'$. This means that in any subgame $\gamma \in \Gamma(N)$, i_1 can refuse to trade with any seller and ensure that he reaches a history $h \in H(N, \langle i_1, j\rangle)$. At this point, he offers $p \in (w_j, p_1)$ and, by Step 3, j accepts. Since at $h_1 \in H(N)$, i_1 trades at p_1 this is a contradiction.

The assumption that $i_1 \in \mathcal{X}(N)$ leads to a similar contradiction. This completes the proof that i_1 and j_1 are inframarginal.

Step 5. Buyer i_1 will not accept a price $p > p_1$ at any history $h \in H(N, \langle j, i_1\rangle)$ for any seller $j \in N$. Consider any history $h' \in H(N, \langle j, i_1\rangle, \mid i_1, j_1 \mid)$. Suppose that j offers a price $p > p_1$ at h'. Buyer i_1 must reject the offer because by (10), at the next date, he will trade at the price p_1. Then by Proposition 7 he makes the same response at every history $h \in H(N, \langle j, i_1\rangle)$.

Step 6. Seller j_1 will not accept a price $p < p_1$ at any history $h \in H(N, \langle i, j_1\rangle)$ for any buyer $i \in N$. The proof is the same as Step 5.

Step 7. $\mathbf{p}_1 \in \mathbf{C}(\mathbf{N})$. We need to show that

$$\max\{v_{i'}, w_{j_1}\} \leq p_1 \leq \min\{v_{i_1}, w_{j'}\},$$

where i' is the highest-valuation extramarginal buyer and j' be the lowest valuation. First we show that $\max\{v_{i'}, w_{j_1}\} \leq p_1$. Individual rationality requires $w_{j_1} \leq p_1$. If $v_{i'} \leq w_{j_1}$ then clearly $\max\{v_{i'}, w_{j_1}\} \leq p_1$.

Now suppose $v_{i'} > w_{j_1}$ and consider any history $h \in H(N, \langle j, i_1\rangle, \langle j_1, i'\rangle)$ for any seller $j \in N$. Then i_1 knows, by Step 2, that if he does not trade in the first match $\langle j, i_1\rangle$, j_1 can achieve at least $v_{i'} - w_{j_1}$ in the continuation game. Thus the continuation payoff to i_1 if there is no agreement at h is bounded above by $\text{Max}\{v_{i_1} - v_{i'}, v_{i_1} - w_{j'}\} = v_{i_1} - v_{i'}$. Therefore, i_1 must accept any offer $p < v_{i'}$ at h in the match $\langle j, i_1\rangle$. Step 5 then implies that $p_1 \geq v_{i'}$.

A similar argument shows that $p_1 \leq \min\{v_{i_1}, w_{j'}\}$.

Step 8. Suppose that $v_{i'} < p_1 < w_{j'}$, where i' and j' are the extramarginal agents identified in Step 7. Then there cannot be trade between any pair of agents other than i_1 and j_1 in any subgame $h \in H(N)$. Suppose not; then by Step 1 there exists either an extramarginal seller j such that j and i_1 trade at some $h \in H(N)$ or an extramarginal buyer i such that i and j_1 trade at some $h \in H(N)$. By Assumption 1, as long as the set of remaining agents is N, the match $\mid i_1, j_1 \mid$ will eventually occur and by (10) i_1 and j_1 trade at the price p_1. Consider the sequence of matches that starts at h and ends at the first match $\mid i_1, j_1 \mid$, assuming there has been no intervening trade. Denote the last match before $\mid i_1, j_1 \mid$ in this sequence that includes either i_1 or j_1 by $\langle k, \ell\rangle$. By refusing to trade, the inframarginal agent i_1 or j_1 in the match $\langle k, \ell\rangle$ can ensure that he will get p_1 when the match $\mid i_1, j_1 \mid$ next occurs, so no trade will occur at a price the inframarginal agent regards as worse than p_1. Therefore, as shown in Step 7, $p_1 \in C(N)$ and by assumption $v_{i'} < p_1 < w_{j'}$ it follows that no trade can occur between an inframarginal agent and an extramarginal agent at $\langle k, \ell\rangle$. By induction, one can show that there cannot be a trade in any match after h until the first match $\mid i_1, j_1 \mid$. But this is a contradiction.

Step 9. Suppose that $p_1 \in \{v_{i'}, w_{j'}\}$, where i' and j' are the extramarginal agents identified in Step 7. Then there cannot be trade between any pair of agents other than i_1 and j_1 at any history $h \in H(N)$. Suppose not. Consider the case $p_1 = v_{i'}$ (the arguments for the other case is identical). Then i' and j_1 must trade at some history $h' \in H(N)$. But as long as the set of remaining agents is N, by Step 6, j_1 will never accept $p < p_1 = v_{i'}$ and by Step 2 i' will accept any price $p < v_{i'}$; thus at h', i' and j_1 trade at the price p_1. But then since f_k is simple at N when k is the proposer in a match with ℓ for all $k, l \in N$, it follows from Proposition 7 that i' and j_1 always trade at p_1 in $H(N)$ whenever they have the same roles. Since $p_1 = v_{i'} < \min\{v_{i_1}, w_{j'}\}$ this implies that i_1 must accept any price $p_1 < p < \min\{v_{i_1}, w_{j'}\}$ from j_1 (consider the match $\langle j_1, i_1\rangle$ followed by $\langle i', j_1\rangle$ or $\langle j_1, i'\rangle$, whichever is appropriate), contradicting Step 5.

Step 10. In any subgame $\gamma \in \Gamma(N)$, i_1 and j_1 trade at the price p_1 and this is the only trade that occurs. From Steps 5 and 6 it is clear that i_1 and j_1 cannot trade at any price other than p_1 and we have seen that they will eventually trade at the price p_1 if no other trade intervenes. Steps 8 and 9 show that no trade can occur between any pair of agents other than i_1 and j_1. ■

The proof of Theorem 11 is by induction on the set of all subgames ordered by the number of remaining agents in the subgames. More precisely, let f be a fixed but arbitrary PEC and let E be the history that occurs along the equilibrium path of f (for the rest of the induction proof f and E are fixed). For any integer $r \geq 1$ let

$$\mathcal{N}_r = \{N \in \mathcal{N} \,|\text{the number of agents in } N \text{ is } 2r\}$$

be the collection of balanced sets with exactly $2r$ agents. The proof is by backward induction on the number of remaining agents, beginning with $r = 1$. Thus we first show that for any set of agents $N_1 \in \mathcal{N}_1$ the outcome is competitive and any trade occurs at the same competitive price in any subgame $\gamma \in \Gamma(N_1)$. Next we fix r and show that if for any set of remaining agents $N_{r-1} \in \mathcal{N}_{r-1}$ the outcome is competitive and any trade occurs at the same competitive price in every subgame $\gamma \in \Gamma(N_{r-1})$ then the same holds for any $N_r \in \mathcal{N}_r$.

Starting the induction
We begin by considering the set of subgames in which the remaining agents consist of two traders $i \in I$ and $j \in J$. First we establish the following result.

Lemma 17 *Consider any $N_1 \in \mathcal{N}_1$ consisting of two agents $i \in I$ and $j \in J$. Then for any $k, l \in N_1$ and $k \neq l$ strategy f_k is simple at N_1 when k is the proposer in a match with player ℓ; thus we have*

$$f_k(h) = f_k(h') \;\forall h, h' \in H(N_1, \langle k, \ell\rangle). \tag{11}$$

Proof. By Propositions 9 and Lemma 14 this is clearly the case if $H(N_1\}) \cap E = \varnothing$ or if $U_k(f) = 0$. Suppose now that $H(N_1) \cap E \neq \varnothing$ and $U_k(f) > 0$; then there exists $h_0 \in H(N_1) \cap E$ at which k and ℓ trade at some price p_0. Now there are two cases to consider.

Case A: k is the proposer at history h_0. Then, by Proposition 8, condition (11) holds.

Case B: l is the proposer at history h_0. To show that condition (11) holds suppose otherwise. Then define a new strategy f'_k that is otherwise identical to f_k except that $f'_k(h) = p_0$ for any $h \in H(N_1, \langle k, l\rangle)$. Then $f_k \succ f'_k$ by inspection. Moreover, since l and k reach an agreement at $h_0 \in H(N_1) \cap E$, by Proposition 8, the match $\langle l, k\rangle$ always induces an agreement at p_0 at any $h \in H(N_1, \langle l, k\rangle)$. But this, together with Assumption 1, imply that the choice of (f'_k, f_{-k}) induces an agreement at p_0 at some history $h \in H(N_1)$ either when k is the proposer or when l is the proposer. Therefore, f'_k induces at least the same payoff as f_k. This contradicts f being a NEC. ■

Next we show that for any $N_1 \in \mathcal{N}_1$ the outcome is competitive and any trade is at the same competitive price in any subgame $\gamma \in \Gamma(N_1)$.

Lemma 18 *Consider any $N_1 \in \mathcal{N}_1$. Then either $C(N_1)$ is empty and in every $\gamma \in \Gamma(N_1)$ there is no trade or there exists a price $p_0 \in C(N_1)$ such that in every subgame $\gamma \in \Gamma(N_1)$ each agent $k \in N_1$ trades at p_0.*

Proof. Since in any subgame $\gamma \in \Gamma(N_1)$ there is at most one trade the result follows immediately from Lemmas 15, 16 and 17. ■

The induction step

We take as our induction hypothesis the claim that for $(r-1)$ and for any set of agents $N_{r-1} \in \mathcal{N}_{r-1}$ the outcome is competitive and the same in every subgame $\gamma \in \Gamma(N_{r-1})$ — more specifically either there is no trade in all such subgames or there exists a price $p_{r-1} \in C(N_{r-1})$ such that, in any subgame $\gamma \in \Gamma(N_{r-1})$, every $k \in \mathcal{I}(N_{r-1})$ trades at the price p_{r-1} and every $\ell \in \mathcal{X}(N_{r-1})$ does not trade. Then we establish that the same is true for r.

Lemma 19 *Consider any integer $r > 1$. Suppose that the induction hypothesis is true for $r-1$. Then for any $N \in \mathcal{N}_r$, any $k, l \in N$ and $k \neq \ell$ strategy f_k is simple at N when k is the proposer in a match with player ℓ.*

Proof. If $H(N) \cap E = \varnothing$ then the result follows immediately from Proposition 9. Therefore suppose that $H(N) \cap E$ is not empty. Then if there is no trade at any history $h \in H(N) \cap E$ the equilibrium payoff $U_k(f)$ of each player $k \in N$ must equal zero and thus the result follows immediately from Lemma 14. Thus suppose that there is trade at some history belonging to the set $H(N) \cap E$.

Next enumerate the set of histories $H(N) \cap E$ in order of appearance on the equilibrium path by $h^0, h^1, ..., h^Q$. Denote the periods at which $h^0, h^1, ..., h^Q$ occur on the equilibrium path (the length of each such history) by $v, v+1, ..., v+Q$ respectively. Also, for any $q \leq Q$ denote the proposer, the responder and the price offer at h^q by k^q, l^q and p^q, respectively. Since there is trade at some history belonging to the set $H(N) \cap E$ it follows that the price offer p^q is rejected at h^q for any $q < Q$ and the offer p^Q at h^Q is accepted. By the induction hypothesis for $r-1$, once k^Q and l^Q reach an agreement at h^Q the continuation equilibrium outcome is competitive for the remaining players. The rest of the proof involves the following steps.

Step 1: f_k is simple at N when k is the proposer in a match with ℓ if the match $\langle k, l \rangle$ that does not occur at any $h \in H(N) \cap E$. This follows immediately from Proposition 9.

Step 2: The match $\langle k^Q, l^Q \rangle$ always induces an agreement at p^Q at any $h \in H(N)$ (thus f_{k^Q} is simple at N when k^Q is the proposer in a match with player ℓ^Q). Since $\langle k^Q, l^Q \rangle$ trade at p^Q at history $h^Q \in E$ this step follows immediately from Proposition 8.

Step 3: For any $q < Q$ there is no an agreement in the match $\langle k^q, l^q \rangle$ at any history $h \in H(N, \langle k^q, l^q \rangle) \cap H^{v+q}$ (note that H^t refers to histories of length t). First consider the match $\langle k^q, l^q \rangle$ for any $q < Q$. Then since $\langle k^q, l^q \rangle$ results in a rejection at $h^q \in H(N, \langle k^q, l^q \rangle)$ it follows from Proposition 8 that $\forall h_0 \in H(N, \langle k^q, l^q \rangle)$

$$\text{if } f_{k^q}(h_0) = p_0 \text{ and } f_{\ell^q}(h_0, p_0) = A \text{ then } \begin{cases} p_0 > v_{k^q} - U_{k^q}(f) & \text{if } k^q \in I \\ p_0 < w_{k^q} + U_{k^q}(f) & \text{if } k^q \in J. \end{cases} \tag{12}$$

Next consider the match $\langle k^{Q-1}, l^{Q-1} \rangle$. Since $H(N \backslash \langle k^Q, l^Q \rangle) \cap E$ is nonempty (because $\langle k^Q, l^Q \rangle$ trade on the equilibrium path at $h^Q \cap H(N) \cap E$) and, by induction hypothesis for $r-1$, in any subgame at which $N \backslash \langle k^Q, l^Q \rangle$ is the set of remaining agents the strategy profile f always induces the same competitive outcome, it must be that each $k \in N \backslash \langle k^Q, l^Q \rangle$ receives his equilibrium payoff in any subgame $\gamma \in \Gamma(N \backslash \langle k^Q, l^Q \rangle)$. But this, together with Step 2 and Proposition 7, imply that if at any $h \in H(N, \langle k^{Q-1}, l^{Q-1} \rangle) \cap H^{v+Q-1}$ the match $\langle k^{Q-1}, l^{Q-1} \rangle$ does not result in an agreement then $\langle k^Q, l^Q \rangle$ trade the next period and any remaining agent receives his equilibrium payoff. By perfection and (12), this implies that f is that at any $h \in H(N, \langle k^{Q-1}, l^{Q-1}\rangle) \cap H^{v+Q-1}$ the match $\langle k^{Q-1}, l^{Q-1}\rangle$ does not reach an agreement.

Next by induction one can show that for any $q < Q - 1$ the match $\langle k^q, l^q\rangle$ does not reach an agreement at any $h \in H(N, \langle k^q, l^q\rangle) \cap H^{v+q}$. Assume that the statement holds for any q' such that $q < q' < Q$ then we can show by a similar argument as above that the statement is true for q. Since each $k \in N \backslash \langle k^Q, l^Q\rangle$ receives his equilibrium payoff in any subgame $\gamma \in \Gamma(N \backslash \langle k^Q, l^Q\rangle)$, the match $\langle k^Q, l^Q\rangle$ always reaches an agreement at any $h \in H(N, \langle k^Q, l^Q\rangle)$ and the match $\langle k^{q'}, l^{q'}\rangle$ does not result in an agreement at any $h \in H(N, \langle k^{q'}, l^{q'}\rangle) \cap H^{v+q'}$ for any $Q > q' > q$ then it follows that at any $h \in H(N, \langle k^q, l^q\rangle) \cap H^{v+q}$ player k^q can guarantee himself his equilibrium payoff of $U_{k^q}(f)$ by not reaching an agreement with l^q at h. But this together with (12) imply that at any $h \in H(N, \langle k^q, l^q\rangle) \cap H^{v+q}$ the match $\langle k^q, l^q\rangle$ does not result in an agreement.

Step 4: f_{k^q} is simple at N when k^q is the proposer in a match with player l^q for all $q < Q$. Suppose not; then there exists $q < Q$, h and $h' \in H(N, \langle k^q, l^q\rangle)$ such that $f_{k^q}(h) \neq f_{k^q}(h')$. Next define a new strategy f'_{k^q} as follows. Put $f'_{k^q}(h, p) = f_{k^q}(h, p)$ for any $(h, p) \in H^r_{k^q} \times \mathbf{R}_+$ and let

$$f'_{k^q}(h) = \begin{cases} p^q & \forall h \in H\left(N, \langle k^q, l^q\rangle\right) \\ f_k(h) & \text{otherwise;} \end{cases}$$

for any $h \in H^p_{k^q}$. Then $f_{k^q} \succ f'_{k^q}$ by inspection. Moreover, since for any q' such that $q < q' < Q$ the match $\langle k^{q'}, l^{q'}\rangle$ does not result in an agreement at any $h_0 \in H(N, \langle k^{q'}, l^{q'}\rangle) \cap H^{v+q'}$ (Step 3), the match $\langle k^Q, l^Q\rangle$ always induces an agreement at any $h \in H(N, \langle k^Q, l^Q\rangle)$ (Step2), it follows that the choice of (f'_{k^q}, f_{-k^q}) induces an outcome path from period v consisting of a series of disagreements in the matches $\langle k^{q'}, l^{q'}\rangle$ for $q < Q$, followed by an agreement in the match $\langle k^Q, l^Q\rangle$ and the same competitive outcome for the set $N \backslash \langle k^Q, l^Q\rangle$ of remaining agents as when (f_{k^q}, f_{-k^q}) is chosen. Therefore, f'_{k^q} induces at least the same payoff as f_k contradicting the definition of NEC. ■

Lemma 20 *If the induction hypothesis is true for* $r - 1$ *then it is true for* r*, for any* $r > 1$.

Proof. Fix any $N_r \in \mathcal{N}_r$. Let m_r denote the number of inframarginal agents on each side of the market when the remaining set of agents is N_r and (with some abuse of notation) label the buyers and sellers in the set N_r so that $v_1 > ... > v_{m_r} > ... > v_r$ and $w_1 < ... < w_{m_r} < ... < w_r$.

Now, by Lemmas 15, 16 and 19, if there exists a subgame $\gamma \in \Gamma(N_r)$ such that there is at most one trade then the outcome is competitive in every $\gamma \in \Gamma(N_r)$ and any trade is at the same competitive price belonging to the set $C(N_r)$ in all such subgames. Therefore, for the rest of the proof we consider only the case in which in every subgame $\gamma \in \Gamma(N_r)$ there is more than one trade.

Fix any $\gamma_r \in \Gamma(N_r)$. Suppose buyer i_r and seller j_r are the first pair of agents that trade in the subgame γ_r. Denote the history, the match and the price at which i_r and j_r trade in this subgame by h_r, $\mid i_r, j_r \mid = \langle \pi(h_r), \rho(h_r) \rangle$ and q respectively. Let us also denote respectively the set of remaining agents and the history immediately after i_r and j_r trade and leave the market in this subgame by $N_{r-1} \equiv N_r \backslash \{i_r, j_r\}$ and $h_{r-1} \equiv (h_r, (\pi(h_r),\ \rho(h_r), q, A))$. Also, let γ_{r-1} be the subgame defined by the history h_{r-1}.

Clearly, since $h_r \in H(N_r)$ we have $\gamma_{r-1} \in \Gamma(N_{r-1})$ and $N_{r-1} \in \mathcal{N}_{r-1}$. Thus by the hypothesis of the induction argument for $r-1$ there exists a price p_{r-1} belonging to the competitive interval $C(N_{r-1})$ such that, in any subgame $\gamma \in \Gamma(N_{r-1})$, every $k \in \mathcal{I}(N_{r-1})$ trades at the price p_{r-1} and every $\ell \in \mathcal{X}(N_{r-1})$ does not trade. We now establish the result by showing that $q = p_{r-1}$, $q \in C(N_r)$ and the outcome is competitive for the market with N_r and all trades occur at q at every subgames $\gamma \in \Gamma(N_r)$. The proof follows in a number of steps.

Step 1: The match $\mid i_r, j_r \mid$ always results in an agreement at q at every $h \in H(N_r)$.

Since the match $\mid i_r, j_r \mid$ induces an agreement at $h_r \in H(N_r)$ this step follows immediately from Proposition 7 and Lemma 19, together with the assumption that the induction hypothesis holds for $r-1$.

Step2: $q \geq p_{r-1}$.

Suppose not; then $p_{r-1} > q$. Consider any $i \in N_{r-1} \cap I$ that trades in the subgame γ_{r-1}. By Assumption 1 there exists a history $h_0 \in H(N_r, \langle j_r, i \rangle, \mid i_r, j_r \mid)$. First we claim that i and j_r will trade at a price $p_0 \geq p_{r-1}$ at h_0. By perfection, i must accept any price $p < p_{r-1}$ at h_0 (otherwise, by Step 1 i_r and j_r will trade the next period and i will trade at p_{r-1}). But then there must exist some price p_0 such that j_r offers p_0 and i accepts at h_0 in the match $\langle j_r, i \rangle$. Otherwise, i_r and j_r will trade at the next date and j_r will receive $q - w_{j_r} < p - w_{j_r}$ for any $p \in (q, p_{r-1})$. Moreover, note that $p_0 \geq p_{r-1}$ because otherwise j_r can increase his payoff by offering $p \in (q, p_{r-1})$.

Second, by Proposition 7 and Lemma 19 and the assumption that the induction hypothesis holds at $r-1$, it follows from j_r and i trading at p_0 at h_0 that the match $\langle j_r, i\rangle$ always results in trade at p_0 at any $h \in H(N_r, \langle j_r, i\rangle)$.

By Assumption 1 there also exists a history $h' \in H(N_r, \mid i_r, j_r \mid, \langle j_r, i\rangle)$. Then since $p_0 \geq p_{r-1} > q$ and $\langle j_r, i\rangle$ always results in trade at the price p_0, perfection requires j_r not to trade at q with i_r in the first match at h'. But this contradicts Step 1.

Step 3: $q \leq p_{r-1}$.

The proof is symmetric to the proof of Step 2.

Step 4: If $j_r \leq m_r$ then

$$\begin{array}{ll} p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\} & \text{if } j_r < m_r \\ p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r-1}\} & \text{if } j_r = m_r. \end{array} \tag{13}$$

The proof of this step follows from $p_{r-1} \in C(N_{r-1})$.

First suppose $j_r < m_r$. Then in the subgame γ_{r-1} there are three possible cases.

- Case A: The set of inframarginal traders $\mathcal{I}(N_{r-1})$ consists of $2m_r$ agents. Then the competitive price $p_{r-1} \in C(N_{r-1})$ must exceed w_{m_r+1}. But $w_{m_r+1} > w_{m_r}$ and $w_{m_r+1} > v_{m_r+1}$. This implies that $p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

- Case B: The set of inframarginal traders $\mathcal{I}(N_{r-1})$ consists of $2(m_r-1)$ agents and $i_r \leq m_r$. Then the competitive price p_{r-1} must be no less than $\max\{v_{m_r+1}, w_{m_r}\}$.

- Case C: The set $\mathcal{I}(N_{r-1})$ consists of $2(m_r-1)$ agents and $i_r > m_r$. Then the competitive price p_{r-1} must be no less than $\max\{v_{m_r}, w_{m_r}\} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

Next suppose $j_r = m_r$. Then in the subgame γ_{r-1} there are two cases to be considered.

- Case A: The set $\mathcal{I}(N_{r-1})$ consists of $2(m_r-1)$ agents. This occurs if $i_r \leq m_r$ or if $i_r > m_r$ and $w_{m_r+1} > v_{m_r}$. Then the competitive price p_{r-1} must be no less than $\max\{v_{m_r+1}, w_{m_r-1}\}$.

- Case B: The set $\mathcal{I}(N_{r-1})$ consists of $2m_r$ agents. This occurs if $i_r > m_r$ and $w_{m_r+1} < v_{m_r}$. Then the competitive price p_{r-1}must be at least $w_{m_r+1} > \max\{v_{m_r+1}, w_{m_r-1}\}$.

Step 5: If $i_r \leq m_r$ then

$$\begin{array}{ll} p_{r-1} \leq \min\{v_{m_r}, w_{m_r+1}\} & \text{if } i < m_r \\ p_{r-1} \leq \min\{v_{m_r-1}, w_{m_r+1}\} & \text{if } i = m_r. \end{array}$$

This follows by the same reasoning as in Step 4.

Step 6: Suppose $j_r \leq m_r$ then $q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

By Steps 2-4, $q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$ for $j_r < m_r$. Now consider $j_r = m_r$. Since $j_r = m_r$ trades at q individual rationality requires

$$q \geq w_{m_r}.$$

Also, by Steps 2-4 we have

$$q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r-1}\}.$$

The last two conditions imply that $q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

Step 7: Suppose $i_r \leq m_r$ then $q = p_{r-1} \leq \min\{v_{m_r}, w_{m_r+1}\}$.

The proof is symmetrical to the proof of Step 6.

Step 8: Suppose $j_r \leq m_r$ then $q = p_{r-1} \leq \min\{v_{m_r}, w_{m_r+1}\}$.

If not, then $q > \min\{v_{m_r}, w_{m_r+1}\} > v_{m_r+1}$. Therefore, since i_r trades at q individual rationality implies that $i_r \leq m_r$. But then by Step 7 we have $p_{r-1} = q \leq \min\{v_{m_r}, w_{m_r+1}\}$.

Step 9: Suppose $i_r \leq m_r$ then $q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

The argument is symmetrical to that in Step 8.

Step 10: $\max\{v_{m_r+1}, w_{m_r}\} \leq q = p_{r-1} \leq \max\{v_{m_r}, w_{m_r+1}\}$.

By individual rationality of the trade between i_r and j_r it must be that either $i_r \leq m_r$ or $j_r \leq m_r$. But then this step follows from the previous four Steps.

Step 11: i_r and $j_r \in \mathcal{I}(N_r)$.

Suppose i_r is not inframarginal; then $i_r > m_r$ and

$$p_{r-1} \geq \begin{cases} w_{m_r+1} & \text{if } \mathcal{I}(N_{r-1}) \text{ consists of } 2m_r \text{ traders} \\ v_{m_r} & \text{if } \mathcal{I}(N_{r-1}) \text{ consists of } 2(m_r-1) \text{ traders.} \end{cases}$$

Since both w_{m_r+1} and v_{m_r} exceed v_{m_r+1} it follows that $q = p_{r-1} > v_{m_r+1}$. But since $i_r > m_r$ this contradicts the individual rationality condition $v_{i_r} \geq q$ for i_r. Therefore, i_r must be an inframarginal buyer.

The arguments to show that j_r is an inframarginal seller is analogous.

Step 12: The subgame γ_r induces a competitive outcome and every inframarginal trader $k \in \mathcal{I}(N_r)$ trades at the price $q \in C(N_r)$.

This follows from the induction hypothesis and Steps 10 and 11.

Step 13: Any subgame $\gamma \in \Gamma(N_r)$ induces a competitive outcome and all trade is at the same competitive price $q \in C(N_r)$.

Since γ_r was fixed to be any arbitrary subgame belonging to the set $\Gamma(N_r)$ it follows from Step 12 that any $\gamma \in \Gamma(N_r)$ induces a competitive outcome. To complete this step we need to show that all trades occur at the same competitive price q at every subgame $\gamma \in \Gamma(N_r)$.

Consider any history $h \in H(N_r)$ at which an inframarginal buyer $i \in \mathcal{I}(N_r) \cap I$ trades with an inframarginal seller $j \in \mathcal{I}(N_r) \cap J$ at a competitive price $p \in C(N_r)$. We next show that $p = q$.

Denote the match at h by $\mid i,j \mid = \langle \pi(h), \rho(h) \rangle$. Next note that since the match $\mid i,j \mid$ induces an agreement at $h \in H(N_r)$ at p it follows immediately from Proposition 7 and Lemma 19, together with the assumption that the induction hypothesis holds at $r-1$, that

$$\mid i,j \mid \text{ results in a trade at } p \text{ at any } h' \in H(N_r, \mid i,j \mid). \tag{14}$$

By Assumption 1 there also exists a history $h'' \in H(N_r, \mid i,j \mid, \mid i_r, j_r \mid)$. If i and j do not trade at h'' then, by Step 1, i_r and j_r will trade at the next period and $N_{r-1}(= N_r \backslash \{i_r, j_r\})$ will be the set of remaining players. Since the outcome is competitive and every inframarginal player trades at the price p_{r-1} in any subgame $\gamma \in \Gamma(N_{r-1})$, and $p_{r-1} = q$ (Steps 2 and 3), we have that if i and j do not trade at h'' then each will eventually trade at q.

Now, by (14), the match $\mid i,j \mid$ results in trade at the price p at history h''. But then perfection requires $p = q$. Otherwise either i or j could make himself better off by not agreeing to trade at h'' and then eventually trading at q. $\blacksquare$

Now Theorem 11 follows by induction from Lemmas 18 and 20.

To demonstrate Corollary 12, by Proposition 7, it is sufficient to show that for any $N \in \mathcal{N}$ and any $k, \ell \in N$ and $k \neq \ell$ strategy f_k is simple at N when k is the proposer in a match with player ℓ. But this follows immediately from Lemmas 17 and 19, and Theorem 11.

Proof of Proposition 13

Fix any $q \in (w_2, v_3)$ and $r \in (w_3, v_2)$. Next consider the the following Markov strategies. If no agent has left the market then the strategies satisfy the following.

Any match between v_i and w_j for $i = j < 3$ involves an offer of r followed by a rejection (thus the strategy is such that the proposer always makes the offer r the responder accepts an offer if and only if the offer is less than r).

Any match between v_i and w_j for $i \neq j < 3$ involves an offer of q followed by a rejection.

Any match between v_3 and w_i for $i < 3$ involves an offer of q followed by an acceptance. Any match between v_i and w_3 for $i < 3$ involves an offer of r followed by an acceptance.

Any match between v_3 and w_3 results in no agreement.

If v_3 and w_i for $i < 3$ leave the market, the strategies result in a competitive trade at the price of r. Also, if v_i and w_3 for $i < 3$ leave the market the strategies result in a competitive trade at the price of q. If a pair of inframarginal types leave the market, then the strategies result in a competitive trade at the price of $p \in (v_3, w_3)$.

Clearly, the above strategies are Markov. Moreover, no player can make himself better off. To show this, first note that the strategies are trivially optimal in any subgame with two remaining agents. Next assume that no agent has left the market and consider each of the following cases.

When v_i and w_j for $i = j < 3$ are matched they cannot make themselve better off by deviating because if the match does not result in an agreement v_3 is going to trade with one of the inframarginal sellers and thus the continuation payoffs of v_i and w_j will be $v_i - r$ and $r - w_j$ respectively.

When v_i and w_j for $i \neq j < 3$ are matched they cannot make themselve better off by deviating because if the match does not result in an agreement w_3 is going to trade with one of the inframarginal sellers and thus the continuation payoffs of v_i and w_j will be $v_i - q$ and $q - w_j$ respectively.

When v_3 and w_j for $j < 3$ are matched they cannot make themselve better off by deviating because if v_3 and w_j do not agree to trade at q, w_3 is going to trade with one of the inframarginal buyers and thus the continuation payoff of v_3 and w_j will be $v_3 - q$ and $q - w_j$, repectively.

When v_i and w_3 for $i < 3$ are matched they cannot make themselve better off by deviating because if v_i and w_3 do not agree to trade at r, v_3 is going to trade with one of the inframarginal sellers and thus the continuation payoff of v_i and w_3 will be $v_i - r$ and $w_j - r$.

This completes the proof.