Herd Behavior in Efficient Financial Markets

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Abstract

Rational herd behavior and informationally efficient security prices have long been considered to be mutually exclusive but for exceptional cases. In this paper we describe conditions on the underlying information structure that are necessary and sufficient for informational herding. Employing a standard sequential security trading model, we argue that people may be subject to herding if and only if there is sufficient amount of noise and, loosely, their information leads them to believe that extreme outcomes are more likely than moderate ones. We then show that herding has a significant effect on prices: prices can move substantially during herding and they become more volatile than if there were no herding. Furthermore, herding can be persistent and can affect the process of learning. We also outline conditions for contrarian and for no-herding/no-contrarian behavior and thus provide a complete characterization of trading behavior. Our analysis suggests that herding (and contrarian behavior) may be more pervasive than was originally thought. Hence, the paper provides a new perspective on herding in financial markets with efficient prices.

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1 Introduction

When markets crash in the absence of significant shifts in economic fundamentals, an explanation frequently heard is that investors behaved like a herd that stampeded without cause. Mass-uniform behavior, however, need not be triggered by 'animal spirits' — it can be fully rational. So called 'rational herding' can occur in situations with information externalities, when agents' private information is swamped by the information derived from observing others' actions. Such 'herders' deliberately act against their private information (but technically they do not discard it) and follow the crowd.

At first sight, this concept seems tailor-made to explain financial market frenzies, crashes and panics. However, when prices are assumed to be informationally efficient, reflecting all public information, it is not clear that herd-behavior can occur at all. For example, suppose a crowd of people buys frantically and consider the case of an investor with unfavorable private information. Such an investor will update his information, and, indeed many buys will increase his expectation. At the same time, prices also adjust upward. Then it is not clear that at this stage such an investor, with an unfavorable-signal, buys — to him the security may still be overvalued.

So to observe herding, private expectations and prices must diverge substantially — once unfavorable expectations must rise faster or favorable expectations must drop faster than prices. In this paper we characterize conditions for this to happen. We further show that with herding prices can move substantially and that they become more volatile than if there were no herding. Herding is also persistent and it can hamper the process of learning (the true state). In addition to herding, we also characterize conditions under which contrarian behavior (moving against the crowd) arises. Finally, by contraposition we obtain the conditions under which neither herding nor contrarian behavior is possible. We thus provide a complete characterization of trading behavior.

We follow the microstructure literature and employ a stylized specialist sequential security trading model à la Glosten and Milgrom (1985), in which we require there to be at least three possible security-values (states) and three signals. We need at least three states because with only two liquidation values, there can be no herding (see Avery and Zemsky 1998).[1] We assume that the signals obey the *Monotone Likelihood Ratio Property* (MLRP). This assumption is the standard signal monotonicity requirement in the literature, found for instance in rational expectations models or auctions. It is a convenient tool, as, for instance, investors' expectations are ordered so that higher signals imply higher

[1]Two recent experimental studies, Drehmann, Oechssler, and Roider (2005) and Cipriani and Guarino (2005), confirm this theoretical result employing a two-states–two-signals world; the first author is currently preparing a project that tests the model of this paper experimentally.

expected liquidation values. We make this very strong, though standard, restriction on the information structure because our objective is to show the possibility of herding (and its consequences) even with a very restrictive condition on the signal distribution;[2] this makes it harder to establish herding and strengthens the result.

In our model asymptotically the true state is revealed and prices converge to the true value. In this paper, however, we shall not be concerned with such long-run results. As Chamley (2004) has pointed out: "Convergence results are often overstated. It is certainly more relevant to study how people may be wrong over an extended length of time and how a sudden price change may occur." This is precisely what we analyze here.

We follow Avery and Zemsky (1998) and define herding as any history-induced switch of opinion (behavior) by a type of agent in the direction of the crowd. We show in Section 3 that herding can occur if and only if (i) for at least one signal the conditional signal distribution is "U-shaped" in liquidation values (see the next paragraph for an informal definition) and (ii) there is a sufficient amount of noise, i.e. the proportion of informed investors is not too large.

The intuition behind these requirements is as follows. First, the probability of informed trading cannot be too large because otherwise the bid-ask spread would be too large to induce 'appropriate' trading. Second, a signal has a U-shaped conditional distribution in values if both extreme values generate this signal with large probability. When forming his posterior, the recipient of such a signal will shift weight away from the center to the extremes. Thus, the signal recipient discounts the possibility of the intermediate value and updates the probabilities of *either* extreme value faster than an agent who receives only the (noisy) public information. So even if this investor's prior belief is pessimistic, after observing a large number of buys (favourable news), he updates his belief and puts more weight on the highest value than the general public. Such traders are more volatile in their decisions, switching from selling to buying and back. Putting it differently the recipients of U-shaped signals are "extreme" (they believe more in extreme than in moderate values).

We show next (Section 4) that prices can move substantially during herding and that investors continue to herd as long as trades are 'in the direction of the crowd'. The range of herding-prices can even comprise almost the entire range of feasible prices. This contrasts, for instance, Avery and Zemsky (1998), who show that during herding prices hardly move at all.

We then investigate the impact of herding on price volatility and potential mispricings. To tackle this issue (in Section 5) we compare price movements in our set-up in which traders learn from each other and herding is possible with those in a hypothetical economy

[2]For instance, the MLRP implies signals obey first-order stochastic dominance.

in which agents rely only on their private information and social learning (and thus herding) does not occur (e.g. agents are naïve and ignore the public information). We show that prices respond more to individual trades when herding occurs relative to a situation in which agents ignore public information and herding does not occur. This is a surprising result because casual intuition suggests that during herding little information is revealed. This intuition, however, is incorrect: herding is informationally significant due to the U-shape of the herders' signal distributions. Therefore, short-run price movements are more volatile with herding compared with the case in which social learning does not occur.[3]

To round up the analysis, we provide numerical simulations of price-paths to highlight some additional implications of herd behavior. First, as some types of traders may change their trading modes (e.g. during herding), prices become history-dependent. More specifically, as the entry order of traders is permutated, prices with the same population of traders can be strikingly different. Second, herding results in price paths that are very sensitive to changes in some key parameters. In particular, as we noted before, a necessary condition for herding is that the proportion of informed agents is below some critical level. Comparing two situations, one with the proportion of informed agents just below the critical level to trigger herding and one with it just above to prevent herding, prices deviate substantially in the two cases. Also, herding slows down the convergence to the true value if the herd moves away from that true value, but it accelerates convergence if the herd moves into the right direction. The differences in speeds of convergence speak to the prevalence of herding.

Next, we investigate the conditions so that a trader, after observing a trading history, changes his action to act *against* the crowd. Such a trader would be considered to engage in "contrarian" behavior. Necessary and sufficient conditions for such behavior are a sufficiently large amount of noise and the existence of signals with "hill-shaped" conditional signal distributions (extreme values generate this signal with smaller probability relative to middle values).

This observation then completes our understanding of the different kinds of behavior triggered by conditional signal distributions. With a U-shaped conditional signal distribution the signal is more likely to occur under extreme than middle value realizations. This makes the recipient's expectation more volatile and prone to flip-flopping and following the majority (herding). A hill-shaped signal distribution achieves the opposite: the posterior distribution of the recipient of such a signal shifts weight to the center and this makes the

[3]The increase in price-volatility associated with herding is only relative to a hypothetical scenario. Even when herding is possible, in the long-run volatility settles down and prices react less to individual trades. Overall, it is well known that the variance of Martingale price-processes such as ours is bounded by model primitives.

recipient's expectation more stable. Thus, he may act against the general movement of prices (contrarian). Lastly, we show that the signal recipient always does the same thing and remains on one side of the market irrespective of public information if and only if the conditional signal distribution is monotonic in values.[4]

It is important to understand the difference between cases in which the asset has two possible liquidation values with that in which the asset has three (or possibly more) values. In a two-state world, an informative signal is always either good or bad news. Further, different good-news signals can be compared with respect to their signal quality, the chance of being correct. With three (or more) values, this intuition no longer applies. U-shaped or hill-shaped signals have no such interpretations. Instead, three (or more) liquidation values allow a much richer set of signals which in turn allow more interesting outcomes.

To see the intuition behind the richness of our three-value, three-signal formulation, consider the example of a company which is rumored to be taken over. The takeover can be good or bad for the company, or there may be a third middle state in which the takeover does not occur at all; for simplicity assume all three outcomes are equally likely. Our three-signal structure always contains a favorable signal, so that the recipient of this signal puts most weight on the takeover being good; likewise there is always an unfavorable signal indicating the opposite. These two signals are extreme. There is also a third signal that, loosely, is somewhere between the good and the bad signal. Agents who receive one of the two extreme signals are settled in their ways and will not be herding or following contrarian behavior.

However, the third signal, even though it is somewhere in between the two extreme signals, is consistent with different stories. It may indicate that a takeover is extremely unlikely; in which case the recipient of such a 'middle' signal becomes a contrarian by shifting probability weight to the center (no takeover): for example, one can imagine that he may, for instance, engage in short-selling if prices move up too much. Alternatively, the third signal may indicate that a takeover is extremely likely but that it is not clear whether it is a good or bad one. Such a piece of information corresponds to a U-shaped distribution for this signal. Traders with such a signal may 'go with the flow' and engage in herding behavior. Finally, another possibility is that the 'middle' signal is simply a weaker version of one of the extreme signals (it would be monotonic in values); recipients of such a signal would always remain on one side of the market.[5]

[4]Note that the discussion above shows that the MLRP, which is in itself a very strong monotonicity assumption, is not strong enough to preclude herding or contrarian behavior — one must impose the additional restriction that the conditional signal distributions are all monotonic in values.

[5]Our arguments are in no way restricted to situations with three signals — in Section 8 we argue that herd behavior can also arise in models with a continuum of signals; likewise, we also provide a three-state–four-signal example that allows both buy- and sell-herding in the same model.

Herding in our model does not imply that all traders act alike, and thus the herd behavior that we document does not match the intuition suggested in flashy newspaper headlines. Rather, it signifies a substantial shift in "sentiment", which involves an accelerating rate of same-direction trades (e.g. 'buys beget more buys'). In fact, in our model, the MLRP implies that agents with extreme signals always take the same action and do not herd or behave as contrarians. Therefore, the amount of herding in our model is restricted by the likelihood of middle-signal types, and the MLRP further places tight bounds on this likelihood. When signals do not satisfy the MLRP, however, it seems even more likely that there is a large fraction of somewhat informed yet "extreme" investors who fit into the herding-prone signal-category. (Similarly, in reality there are probably also stubborn investors that are prone to contrarian behavior; Drehman, Oechsler and Roider (2005) document such behavior in their experimental study.) This is, in fact what Avery and Zemsky establish in their important paper on financial market herding.

Employing the same Glosten and Milgrom (1985) specialist sequential security trading model, Avery and Zemsky (1988), henceforth called AZ, argue that herd behavior with informationally efficient asset prices is not possible unless signals are "non-monotonic" and risk is "multi-dimensional". (They introduce a non-standard notion of monotonic signals that almost by definition excludes herding possibility; see Section 9 below.) AZ then show that for a specific, non-monotone information structure with two-dimensional risk there can be herding. However, there is very little price movement during such a herd phase. To show that extreme price movements (bubbles) with herding are possible they introduce further information asymmetries and thus more risk dimensions; in particular, in addition to the above information structure, they assume that traders have different abilities to interpret the signals and that this is private information. However, even with these further informational asymmetries, the likelihood of large price movements in their set-up during a herd phase is extremely small (of the order of $10^{-6}\times$the probability of a particular sequence of trades, see Chamley 2004).

The profession, for instance Brunnermeier (2001), Bikhchandani and Sunil (2000), Chamley (2004) have derived three messages from AZ's paper. First, with 'monotonic' signals, herding is impossible. Second, for herding one needs 'multidimensionality' of risk. Third, herding does not involve violent price movements except in the most unlikely environments. Therefore, since in AZ the information structure needed to induce herding is very special and large price movements cannot easily be attributed to herd-type behavior, it has been concluded that rational herding models are not so relevant to understanding the functioning of efficient financial markets.[6]

[6]Lee (1998) also uses a financial market model with moving prices. His herding, or rather 'information avalanches' result is, however, based on frictions induced by transaction costs. (He also allows loss making

We do not contest AZ's very insightful results, nor do we argue that their information structures do not have intuitive appeal. But the results of our paper demonstrate that the conclusions derived from AZ and the profession's perception need to be corrected. First, for financial market herding one needs neither non-monotonic signals nor multidimensionality of risk. Even with signals satisfying the MLRP there may be a great deal more rational informational herding than is currently expected in the literature. Second, extreme price movements with herding are possible under not so unlikely situations. And third, price volatility may even be exacerbated by herding. It is noteworthy that agents who herd in AZ's model with non-monotonic information structure have signals that have similarly extreme structures (U-shaped) as those identified in this paper as responsible for herding.

In the next section we outline the basic setup, the assumptions on signal distributions, and the definition of herding and contrarian behavior. Our main results follow in Sections 3, 4, and 5. In Section 3 we discuss which assumptions on MLRP signal structures ensure that herding occurs with positive probability. In Sections 4 and 5 we then show that herding can persist and explain why herding-prices are more extreme. In Section 6 we simulate the speed of convergence and the impact of herding on the trading and price history. Section 7 outlines contrarian behavior, Section 8 argues that the lessons gleaned from the three signal case extend to there being more than three signals. We briefly discuss Avery and Zemsky's results and the relation to our findings in Section 9. All proofs are in the appendix.

2 The Model

2.1 The Basic Setup

Security: There is a single risky asset with a liquidation value V from a set of three potential values $\mathbb{V} = \{V_1, V_2, V_3\}$ with $V_1 < V_2 < V_3$. The prior distribution over $\mathbb{V}$ is denoted by $\mathsf{Pr}(\cdot)$. To simplify the computation we assume that $\{V_1, V_2, V_3\} = \{0, \mathcal{V}, 2\mathcal{V}\}$, $\mathcal{V} > 0$ and that the prior distribution is symmetric around V_2; thus $\mathsf{Pr}(V_1) = \mathsf{Pr}(V_3)$.[7]

Traders: There is a pool of traders consisting of two kinds of agents: *Noise Traders* and *Informed Agents.* At each discrete date t one trader arrives at the market in an exogenous and random sequence. Each trader can only trade once at the point in time at which he arrives. We assume that at each date the entering trader is an informed agent

market-makers.) In Chari and Kehoe (2004), herding occurs with respect to real investment decisions that are external to the financial market, but there is no herding with respect to the underlying market trading in which prices are set efficiently. Dasgupta and Prat (2005) develop a sequential trading model in which traders' reputation concerns lead to a breakdown of learning. For a recent, comprehensive survey of the herding literature see Hirshleifer and Teoh (2003).

[7]The results of this paper remain valid without these assumptions on symmetry.

with probability $\mu > 0$ and a noise trader with probability $1 - \mu > 0$.

The informed agents are risk neutral and rational. Each receives a private, conditionally i.i.d. signal $S \in \{S_1, S_2, S_3\}$ about V. We assume that the signals are ordered such that $S_1 < S_2 < S_3$.

Noise traders have no information and trade randomly. These traders are not necessarily irrational, but they trade for reasons not included in this model, such as liquidity.[8]

Market Maker: Trade in the market is organised by a market maker who has no private information. He is subject to competition and thus makes zero-expected profits.[9] In every period t, prior to the arrival of a trader, he posts a bid-price p_t^B at which he is willing to buy the security and an ask-price p_t^A at which he is willing to sell the security. Consequently he sets prices in the interval $[V_1, V_3]$.[10]

Traders' Actions: Each trader can buy or sell *one* unit of the security at prices posted by the market maker, or he can be inactive. So the set of possible actions for any trader is $\mathbb{A} := \{\text{buy}, \text{hold}, \text{sell}\}$. We denote the action taken in period t by the trader that arrives at that date by $a_t \in \mathbb{A}$.

We assume that noise traders trade with equal probability. Therefore, in any period, a noise-trader buy, hold or sale occurs with probability $\gamma = (1 - \mu)/3$ each.

Information: The structure of the model is common knowledge among all market participants. The identity of a trader and his signal are private information, but everyone can observe past trades and transaction prices. The history (public information) at any date $t > 1$, the sequence of the traders' past actions together with the realised past transaction prices, is denoted by $H_t = ((a_1, \mathsf{p}_1), \ldots, (a_{t-1}, \mathsf{p}_{t-1}))$ for $t > 1$, where a_τ and p_τ are traders' actions and realised transaction prices at any date $\tau < t$ respectively. Also, H_1 refers to the initial history before trade occurs.

The Informed Trader's Optimal Choice: An informed trader enters the market in period t, receives his signal S_t and observes history H_t. We assume, for simplicity, the tie-breaking rule that, in the case of indifference, agents always prefer not to trade. Therefore, an informed trader's optimal action is (i) to *buy* if he values the security no less than the ask-price: $\mathsf{E}[V|H_t, S_t] > \mathsf{p}_t^A$, (ii) to *sell* if he thinks the security is worth no more than the bid price: $\mathsf{p}_t^B < \mathsf{E}[V|H_t, S_t]$, and (iii) to *hold* in all other cases.

The Market Maker's Price-Setting: To ensure that the market maker receives zero expected profits, bid and ask prices have to be such that at any date t and any publicly

[8]As is common in the literature on micro-structure with asymmetric information, we assume that noise traders have a positive weight ($\mu < 1$) to prevent "no-trade" outcomes à la Milgrom-Stokey (1982).

[9]Alternatively, we could also assume a model with many identical market makers setting prices as in Bertrand competition.

[10]The market maker in our model resembles a market 'specialist'.

available information H_t,

$$\mathsf{p}_t^A = \mathsf{E}[V|a_t = \text{buy at } \mathsf{p}_t^A, H_t] \text{ and } \mathsf{p}_t^B = \mathsf{E}[V|a_t = \text{sell at } \mathsf{p}_t^B, H_t]$$

Informed agents are better informed than the market maker. Consequently, if the market maker always sets prices equal to public expectation, $\mathsf{E}[V|H_t]$, he makes an expected loss on trades with informed agents. However, if he sets an ask-price and a bid-price respectively above and below the public expectation, he gains on noise traders, as their trades have no information value. Thus, in equilibrium the market maker makes profit on trades with noise traders to compensate for losses against informed agents. This implies that at any date there is a spread between the bid and ask price; in particular at any date t and for any public information H_t we have $\mathsf{p}_t^A > \mathsf{E}[V|H_t] > \mathsf{p}_t^B$. Moreover, the spread $\mathsf{p}_t^A - \mathsf{p}_t^B$ increases in μ, the probability that a trader is informed.

Equilibrium concept: Since the game played by the informed agents is one of incomplete information the appropriate equilibrium concept is the Perfect Bayesian equilibrium.

Long-run behavior of the model: Price formation in our model is standard. Therefore, by standard arguments we have that transaction prices form a Martingale process, and beliefs and prices converge to the truth (see Glosten and Milgrom (1985)). However, as we mentioned in the introduction, here we are interested in short-run behavior and fluctuations.

2.2 Properties of the Signal Distribution

We assume, as is standard in models that use informative signals, that signals are strictly monotonic in the sense of the *monotone likelihood ratio property* (MLRP) (introduced by Karlin and Rubin (1956) and Milgrom (1981)). This means that for any signals $S_l, S_h \in \mathbb{S}$ and any values $V_l, V_h \in \mathbb{V}$ such that $S_l < S_h$ and $V_l < V_h$ we have

$$\frac{\mathsf{Pr}(S_h|V_h)}{\mathsf{Pr}(S_l|V_h)} > \frac{\mathsf{Pr}(S_h|V_l)}{\mathsf{Pr}(S_l|V_l)}.$$

This assumption is a strong condition and thus, establishing the possibility of herding with MLRP signals is a very strong result.[11]

As we mentioned in the introduction, it turns out that the possibility of herding for any informed agent with signal S depends critically on the shape of the conditional signal distribution that S has; we will henceforth refer to the conditional signal distribution as *csd*. We will also employ the following terminology to describe six different types of csds

[11]Note that the MLRP is trivially satisfied if there are only two signals, two values and if signals are conditionally independent.

that any signal S may have:

$$\begin{aligned}
\text{increasing} &\Leftrightarrow \mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2) < \mathsf{Pr}(S|V_3)\\
\text{decreasing} &\Leftrightarrow \mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_2) > \mathsf{Pr}(S|V_3)\\
\text{U-shaped} &\Leftrightarrow \mathsf{Pr}(S|V_i) > \mathsf{Pr}(S|V_2) \text{ for } i = 1,3\\
\text{Hill-shaped} &\Leftrightarrow \mathsf{Pr}(S|V_i) < \mathsf{Pr}(S|V_2) \text{ for } i = 1,3\\
\text{Negatively biased} &\Leftrightarrow \mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_3)\\
\text{Positively biased} &\Leftrightarrow \mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_3)
\end{aligned}$$

We shall call a signal csd-monotonic if its csd is either increasing or decreasing. The MLRP allows us to establish the following set of results.

Proposition 1
(a) *Conditional expectations are monotonic in signals. Formally, for any $S_l, S_h \in \mathbb{S}$, if $S_l < S_h$ then $\mathsf{E}[V|S_l, H_t] < \mathsf{E}[V|S_h, H_t]$ for any date t and any history H_t.*
(b) *The csd for S_1 is decreasing and the csd for S_3 is increasing.*

Proposition 2
In any equilibrium the following holds at any history:
(a) *Informed traders with signal S_1 (S_3) always sell (buy).*
(b) *The probability of a buy (sale) increases (decreases) in V by a positive amount independent of the past. Formally, there exists $\epsilon > 0$ such that for every H_t and for any V_h and V_l with $V_h > V_l$, the following two conditions hold*

$$\mathsf{Pr}(\text{buy}|V_h, H_t) - \mathsf{Pr}(\text{buy}|V_l, H_t) > \epsilon \quad \textit{and} \quad \mathsf{Pr}(\text{sale}|V_l, H_t) - \mathsf{Pr}(\text{sale}|V_h, H_t) > \epsilon.$$

Proposition 1 (a) implies that informed agents' conditional expectations are ordered after any history of trade. Since the MLRP implies First Order Stochastic Dominance, it follows that conditional expectations are ordered ex-ante before any trade. Our result is simply an extension of this observation to expectations after any history.

Proposition 1 (b) says that the conditional probability of the lowest (highest) signal decreases (increases) in the true liquidation value. However, for the middle signal, no such general rule applies! Csds for S_2 can be decreasing, increasing, or they can be hill-shaped or U-shaped with a negative or a positive bias. To see this consider Table 1 which contains six numerical examples of MLRP information structures. Each information structure is described by a 3×3 matrix; for each such matrix the MLRP is equivalent to all minors of order 2 being positive. This property holds for all matrices. The different information structures in the table exhibit all the six csd described above for the middle signal S_2.

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{2}{9}$
S_2	$\frac{6}{18}$	$\frac{4}{18}$	$\frac{3}{18}$
S_3	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{11}{18}$

decreasing for S_2

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{1}{9}$
S_2	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{9}{27}$
S_3	$\frac{7}{27}$	$\frac{1}{3}$	$\frac{5}{9}$

increasing for S_2

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{5}{6}$	$\frac{1}{3}$	0
S_2	$\frac{11}{120}$	$\frac{40}{120}$	$\frac{20}{120}$
S_3	$\frac{3}{40}$	$\frac{1}{3}$	$\frac{5}{6}$

hill-shape and positive bias for S_2

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{3}{40}$
S_2	$\frac{20}{120}$	$\frac{40}{120}$	$\frac{11}{120}$
S_3	0	$\frac{1}{3}$	$\frac{5}{6}$

hill-shape and negative bias for S_2

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{31}{100}$	$\frac{1}{5}$	$\frac{1}{100}$
S_2	$\frac{59}{100}$	$\frac{50}{100}$	$\frac{60}{100}$
S_3	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{39}{100}$

U-shape and positive bias for S_2

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{50}$
S_2	$\frac{60}{100}$	$\frac{50}{100}$	$\frac{59}{100}$
S_3	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{39}{100}$

U-shape and negative bias for S_2

Table 1: **Six Examples of MLRP Signal distributions** For very matrix each entry represents the probability of the row-signal given the true liquidation value given by the column. Therefore, for each matrix the sum of the entries in each column add up to 1. In all the above examples, the signal distributions for S_1 and S_3 are csd-monotonic whereas each matrix exhibits a different kind of signal distribution for S_2.

Proposition 2 (a) establishes that the lowest and highest signal types always take the same action. Therefore, the only agents that might change their behavior depending on the history of past actions are agents with signal S_2.

Proposition 2 (b) implies that for any two states V_l and V_h, with $V_h > V_l$, a buy (sale) increases (decreases) the likelihood of V_h relative to that of V_l.

2.3 Definitions of Herding and Contrarian Behavior

We adopt the same definition of herding as in AZ.

Definition 1 (Herding) *A trader with signal S engages in herd-buying in period t after history H_t if and only if* (H1) $\mathsf{E}[V|S] < \mathsf{p}_1^B$, (H2) $\mathsf{E}[V|S, H_t] > \mathsf{p}_t^A$, (H3) $\mathsf{E}[V|H_t] > \mathsf{E}[V]$.

If at history H_t conditions (H1)-(H3) *are satisfied for some signal type, then we say that history H_t involves buy-herding.*

Herd-selling is defined analogously.

(H1) requires the S type agent to (strictly) prefer to sell ex-ante, before observing the action of others; (H2) requires the S type to (strictly) prefer to buy, after observing the history H_t; and (H3) requires the public expectation to 'move in the direction' of the herd.

Note that a 'history with buy-herding' only implies that there could be types that buy-herd — it does not mean that the actual trades are by herders.

According to the above definition, agents with a particular signal engage in herding if, as a result of observing the behavior of others, they take a different action from the one that they would take initially. Thus, herding in our set-up (as well as in AZ) represents any history-induced switch of opinion *in the direction of the crowd*.[12] Conditions (H1) and (H2) capture the sense of changing from selling to buying after observing the actions of others; (H3) ensures that the switch occurs following the crowd (see the discussion below).[13]

However, similar types of agents (with the same signal) may change their action from selling to buying (or vice versa) as the history unfolds without engaging in herd-behavior. For example, it may be that a type changes, for some trading history, from selling to buying because the market price (public expectations) has fallen. In the literature, such a change of opinion against the crowd is typically referred to as contrarian behavior.[14] Formally, we define contarian-behavior as follows.

Definition 2 (Contrarian) *A trader with signal S engages in contrarian-buying in period t after history H_t if and only if* (C1) $\mathsf{E}[V|S] < \mathsf{p}_1^B$, (C2) $\mathsf{E}[V|S, H_t] > \mathsf{p}_t^A$, (C3) $\mathsf{E}[V|H_t] < \mathsf{E}[V]$. *Contrarian-selling is defined analogously.*

The first two conditions in the definition of contrarian behavior are the same as those for herd behavior: thus for contrarian-buying (C1) requires the S type to (strictly) prefer to sell ex-ante, before observing the action of others and (C2) requires the S type to (strictly) prefer to buy, after observing the history.

The key difference between herding and contrarianism lies in the difference between Conditions (H3) and (C3): the former ensures that the change of action from selling to buying is not due to a decline in the price (public expectation) but instead is *with* the general movement of the crowd. The latter condition, (C3), requires the public expectation to have dropped so that after this history a trader who buys acts *against* the general movement of prices.

[12]In the literature, there are other definitions of herding (and informational cascades). For instance, Smith and Sørensen (2000) and also Cipriani and Guarino (2003) define herding as ‘action convergence’ — agents of the same ‘type’ take the same action. They describe an informational cascade as a situation where an agent takes the same decision irrespective of his private signal. Herding in our set-up refers to the actions of a particular signal type, not to all informed agents collectively. In our model the market-maker’s zero-profit condition alone precludes action-convergence of all informed traders — it is not possible that all informed agents trade on the same side of the market (see Proposition 2 (a)). In Cipriani and Guarino (2003) action convergence refers to specific types with type-characteristics other than just signals — even in their model different types take different actions. In any case, the definition of herding which we (and also AZ) employ is in spirit of herd-mentality: people switch actions to follow the crowd.

[13]In our analysis we look only at the most extreme case when a player switches from selling to buying (or the reverse). One could argue, however, that a switch from holding to buying/selling also constitutes herding. This would require less restrictions and may include further cases of information structures that would trigger herding. To ensure consistency with the literature, we focus on the extreme cases.

[14]Avery and Zemsky use this term too, but their definition of ‘contrarian’ includes a feature closely related to their definition of monotonicity; see Section 9.

3 Herding with MLRP Signal Structures

Proposition 2 (a) and Proposition 1 (b) imply respectively the following: First, the only possible herding candidate is an informed agent with middle signal S_2 (S_1 traders always sell and S_3 traders always buy), and second, the only type of informed agent that could have a U-shaped or hill-shaped csd has signal S_2 (S_1 and S_3 are csd-monotonic).

The above two observations allow us to establish that herding is possible. More specifically, we show below that, with MLRP signals, necessary and sufficient conditions for herding are:

- U-shaped csd for signal S_2;
- 'enough' noise traders.

In addition, we show that, depending on the relative values of $\mathsf{Pr}(S_2|V_1)$ and $\mathsf{Pr}(S_2|V_3)$ (negative bias or positive bias), either buy-herding is possible or sell-herding but not both.

To illustrate the basic ideas for this characterization result consider the case of buy-herding. As we mentioned before, by Proposition 2(a), S_2 types are the only possible candidates for buy-herding. For this type to buy-herd he must be willing to sell initially at H_1 and to buy at some later history in the direction of the crowd (i.e. after a sufficiently large number of buys).

An S_2 type drawn at H_1 sells if and only if the following two conditions hold:

(i) Signal S_2 has a negatively biased csd: $\mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_3)$.

(ii) There are enough noise traders.

The intuition for this is as follows. First, for S_2 to sell initially he has to have a negative opinion prior to trading; but then since the prior is symmetric this is equivalent to a negative bias in his csd. Second, the ask price initially at H_1 is a decreasing function of the weight of the noise traders, the bid-price is increasing. Thus, if there were not enough noise traders then at H_1 the spread would be too large and S_2 types would not sell.

For S_2 types to buy after observing a series of buys we need that the S_2-informed agent updates his expectation faster upwards than the market maker raises the ask-price. This turns out to be equivalent to the following two conditions:

(iii) $\mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_2)$.

(iv) There are enough noise traders.

The intuition for this is as follows. By Proposition 2 (b), after a history with a sufficiently large number of buys, state V_1 can be ignored (the conditional probability of V_1 is small relative to those of V_2 and V_3). Then (iii) implies that S_2 types attach more weight to V_3 than to V_2. On the other hand, the market maker, in setting his ask price (the expected value conditional on a buy), has to allow for the possibility of noise traders. With a sufficiently large probability of noise trading, there is not much informational content to a buy for the market maker. As a result, at such a history the expected value of the asset for the S_2 types exceeds the ask-price.

Before turning to the formal statement of the results, there are several points to note. First, to obtain herding the weight of noise traders must be sufficiently large for two different reasons: (a) to induce a S_2 type to sell initially, and (b) to ensure that, later after a sufficient number of buys, the ask price set by the market maker is below the expectation of the S_2 types, and hence informed agents with signal S_2 change their behavior and buy. Therefore, for buy-herding the proportion of informed agents μ has to be smaller than two different bounds. We shall refer to the bound on the size of μ to generate the initial sale μ_b^{in} and the one to induce a change of behavior later on by μ_b^{ch}. Below we shall define these bounds formally.

Second, (i) and (iii) are equivalent to saying that the csd of S_2 is U-shaped and has negative bias.

Third, analogous reasoning explains that sell-herding is equivalent to S_2 types having a U-shaped, positively biased csd and the proportion of noise trading to be sufficiently large. The difference here is that for sell-herding S_2 types have to buy initially; this implies, given the symmetric prior, that S_2 types must have a positively biased csd.

Fourth, since S_2 types are the only herding candidates and their csd either have a negative or a positive bias but not both, it follows from the above discussion, depending on the sign of the bias (relative values of $\mathsf{Pr}(S_2|V_1)$ and $\mathsf{Pr}(S_2|V_3)$), either buy-herding is possible or sell-herding but not both.

Next, we present a formal set of necessary and sufficient conditions for herding and then characterise these conditions in terms of the shape of the csd for S_2. Define

$$\kappa_b := \frac{\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2)}{\rho_{23}^{23}}, \quad \kappa_s := \frac{\mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_2)}{\rho_{12}^{12}}$$

$$\theta_b := \frac{\mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_3)}{\mathsf{Pr}(V_2)(\rho_{12}^{12} + \rho_{12}^{23}) + (1 - \mathsf{Pr}(V_2))\rho_{12}^{13}}, \quad \theta_s := \frac{\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2)}{\mathsf{Pr}(V_2)(\rho_{23}^{12} + \rho_{23}^{23}) + (1 - \mathsf{Pr}(V_2))\rho_{23}^{13}}$$

where for any i, j, k, l,

$$\rho_{ij}^{kl} := \mathsf{Pr}(S_i|V_k)\mathsf{Pr}(S_j|V_l) - \mathsf{Pr}(S_j|V_k)\mathsf{Pr}(S_i|V_l).$$

Denote the two upper bounds for the proportion of informed agents for buy-herding by

$$\mu_b^{in} = \theta_b/(\theta_b + 3) \quad \text{and} \quad \mu_b^{ch} = \kappa_b/(\kappa_b + 3).$$

Similarly, for sell-herding define the two upper bounds by

$$\mu_s^{in} = \theta_s/(\theta_s + 3) \quad \text{and} \quad \mu_s^{ch} = \kappa_s/(\kappa_s + 3).$$

Proposition 3 (Herding with MLRP Signals)

Suppose that the information structure satisfies the MLRP; then the following hold.

(a) *Informed agents with signal* S_2 *buy-herd with positive probability if and only if* $0 < \mu < \mu_b$ *where*

$$\mu_b = \min\{\mu_b^{in}, \mu_b^{ch}\}.$$

(b) *Informed agents signal* S_2 *sell-herd with positive probability if and only if* $0 < \mu < \mu_s$ *where*

$$\mu_s = \min\{\mu_s^{in}, \mu_s^{ch}\}.$$

Consider the case of buy-herding in the above proposition. Part (a) of the proposition says that two conditions $0 < \mu < \mu_b^{in}$ and $0 < \mu < \mu_b^{ch}$ are necessary and sufficient conditions for buy-herding. This is because it turns out that the former condition is equivalent to S_2 types wanting to sell initially ($\mathsf{E}[V|S_2, H_1] - \mathsf{p}_1^B < 0$) and the latter is equivalent is to S_2 types wanting to buy after some history H_t ($\mathsf{E}[V|S_2, H_t] - \mathsf{p}_t^A > 0$), in the direction of the crowd.

More specifically, in the proof of the above proposition we first show that after any history of length H_t, $\mathsf{E}[V|S_2, H_t] - \mathsf{p}_t^A$ has the same sign as

$$[\beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2)] + \frac{q_1^t}{q_3^t}[\beta_1^t \mathsf{Pr}(S_2|V_2) - \beta_2^t \mathsf{Pr}(S_2|V_1)] + \frac{2q_1^t}{q_2^t}[\beta_1^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_1)] \quad (1)$$

and $\mathsf{E}[V|S_2, H_t] - \mathsf{p}_t^B$ has the same sign as

$$\frac{q_3^t}{q_1^t}[\sigma_2^t \mathsf{Pr}(S_2|V_3) - \sigma_3^t \mathsf{Pr}(S_2|V_2)] + [\sigma_1^t \mathsf{Pr}(S_2|V_2) - \sigma_2^t \mathsf{Pr}(S_2|V_1)] + \frac{2q_3^t}{q_2^t}[\sigma_1^t \mathsf{Pr}(S_2|V_3) - \sigma_3^t \mathsf{Pr}(S_2|V_1)], \quad (2)$$

where $\beta_i^t = \mathsf{Pr}(\text{buy}|H_t, V_i)$, $\sigma_i^t = \mathsf{Pr}(\text{sale}|H_t, V_i)$ and $q_i^t = \mathsf{Pr}(V_i|H_t)$ denote, respectively, the conditional probability of a buy in state V_i, the conditional probability of a sale in state V_i and the posterior probability of state V_i at date t.

The above implies that for buy-herding by S_2 types the following two conditions hold:

(I) the expression in (2) is negative at the initial date $t = 1$;

(II) the expression in (1) is positive at some finite history H_t.

Simple manipulation shows that $0 < \mu < \mu_b^{in}$ is equivalent to (I). Condition $0 < \mu < \mu_b^{ch}$ is equivalent to (II). This is because by simple manipulations one can show that $0 < \mu < \mu_b^{ch}$ is necessary and sufficient for the first term in (1) to be positive. Moreover, if there is sufficient evidence in favour of high values (i.e. sufficiently many more 'buys' than 'sales') then, by part (b) of Proposition 2, the posterior probability q_1^t will be sufficiently small relative to q_2^t and q_3^t and hence the last two terms in (1) vanish. Therefore, the expression in (1) is indeed positive and S_2 types change their behavior at such a history if and only if $0 < \mu < \mu_b^{ch}$.

Analogous reasoning as in the previous two paragraghs applies to sell-herding and the conditions in part (b) of the above proposition.

Clearly, to ensure that the characterisation in Propostion 3 is not vacuous μ_b and μ_s have to be positive. However, notice that by the MLRP we have

$$\kappa_b > 0 \quad \Leftrightarrow \quad \mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2) > 0$$
$$\kappa_s > 0 \quad \Leftrightarrow \quad \mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_2) > 0$$
$$\theta_b > 0 \quad \Leftrightarrow \quad \mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_3) > 0$$
$$\theta_s > 0 \quad \Leftrightarrow \quad \mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_1) > 0$$

Therefore, $\mu_b > 0$ if and only if $\mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_2)$. Also $\mu_s > 0$ if and only if $\mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_2)$. Thus we have the following corollary to Proposition 3.

Corollary 1 (Necessary and Sufficient Conditions for Herding Revisited)
Suppose that the information structure satisfies the MLRP; then the following hold.
1. BUY-HERDING
(*a*) *A negatively biased and U-shaped csd for signal S_2 is necessary for buy-herding.*
(*b*) *Suppose that signal S_2's csd is negatively biased and U-shaped. Then there exists $\mu_b > 0$ such that there is a positive probability of buy-herding if and only if $0 < \mu < \mu_b$. Furthermore, μ_b is uniquely defined by $\mu_b = \min\{\mu_b^{in}, \mu_b^{ch}\}$.*
2. SELL-HERDING
(*a*) *A positively biased and U-shaped csd for signal S_2 is necessary for sell-herding.*
(*b*) *Suppose that S_2's csd is positively biased and U-shaped. Then there exists $\mu_s > 0$ such that there is a positive probability of sell-herding if and only if $0 < \mu < \mu_s$. Furthermore, μ_s is uniquely defined by $\mu_s = \min\{\mu_s^{in}, \mu_s^{ch}\}$.*
3. HERDING
(*a*) *A U-shaped csd for signal S_2 is necessary for herding.*
(*b*) *Suppose that S_2's csd is U-shaped. Then there exists $\overline{\mu} > 0$ such that there is a positive probability of herding if and only if $0 < \mu < \overline{\mu}$. Furthermore, $\overline{\mu}$ is uniquely defined by $\overline{\mu} = \max\{\mu_b, \mu_s\}$.*

Finally, note that by the above corollary if the csd for S_2 is hill-shaped or monotonic, herding cannot occur. However, a hill-shaped or monotonic csd cannot be interpreted as yielding global signal-monotonicity (e.g. in the sense of the MLRP); these properties say nothing about the relation between signals.

4 Large Price Movements during Herding

In contrast to the literature, in our setting prices may move significantly during herding.[15] To see the intuition for this possibility, suppose that buy-herding occurs at some date t and consider the effect of more buys after t (analogous reasoning holds for sell-herding followed by more sales).

Clearly, further buys increase the market maker's expectation, and hence prices, because, by Proposition 2 (b), the probability of a buy increases in the liquidation value V. Moreover, prices can approach the maximum value V_3 if the number of buys becomes arbitrarily large (the probability of an infinite number of consecutive buys is, of course, zero because of the existence of noise traders).

The important thing to note is that buy-herding will not stop if buying persists and there are no sales. Further herd-buys simply increase the herding agents' expectations by more than that of the marker maker and thus the herd is not broken. The intuition for this is as follows. As we discussed in the last section, the expectation of S_2 types exceeds the ask price at any date t if and only if the herding expression (1) is positive. Moreover, the possibility of buy-herding ($0 < \mu < \mu_b$) implies that the second and the third terms in (1) are negative and the first term is positive. Thus, when buy-herding starts, the second and the third terms in (1) are small relative to the first term, and hence the expression in (1) is positive. Since signals satisfy the MLRP, additional buys further decrease both q_1^t/q_2^t and q_1^t/q_3^t (Proposition 2 (b)), making the second and the third terms in (1) even smaller. Thus, S_2 types will continue to buy and herding persists. Of course, if there are many sales, the second and third terms in (1) increase, and eventually the S_2 type's expectation can again drop below the ask-price. Yet the herd is robust — the more herd-buys there are, the more sales it takes to break the herd. This is in contrast to standard herding models (e.g. Banerjee (1992)) in which herding is very fragile: a *single* action against the herd results in a collapse of the herd.

We further show that in our set-up there exists a set of priors on $\mathbb{V}$ such that herding can start when prices are close to the middle value, V_2. Indeed, the minimum number of same-direction trades necessary to induce herding turns out to be independent of the exact prior, and thus if $\mathsf{Pr}(V_2)$ is sufficiently close to one, then the transaction price will start near V_2 and will remain so even by the time herding is triggered.

The left panel in Figure 1 plots simulated transaction prices that illustrate the above

[15] Namely, in AZ with Event Uncertainty information structure (described formally in Section 9 below) price movements during herding are strictly limited. This is because during herding *all* informed agents that trade take the same action; thus trades in the direction of the herd do not convey information about the high value and hence the expectations of such agents do not move so that a very small price movement stops herding (see Section 9 below for a further discussion of this point).

points: Herding starts for prices near V_2, and during herding prices rise substantially.

The following result states that prices *can* move during herding. The extent to which herding causes prices to move 'more' (than a benchmark) is discussed in Section 5.

Proposition 4 (Persistence of Herding and the Range of Herd-Prices)
1. Herding towards Extreme Prices
(*a*) *Suppose that there is there is buy-herding after history $H_{t'}$. Then for any $\epsilon > 0$, there exists history $H_{t'+\tau}$ following $H_{t'}$ such that (i) there is buy-herding at every H_t such that $t' \leq t \leq t' + \tau$ and (ii) the transaction price $\mathsf{p}_{t'+\tau}$ exceeds $V_3 - \epsilon$.*
(*b*) *Suppose that there is sell-herding after history $H_{t'}$. Then for any $\epsilon > 0$, there exists history $H_{t'+\tau}$ following $H_{t'}$ such that (i) there is sell-herding at every H_t such that $t' \leq t \leq t' + \tau$ and (ii) the transaction-price $\mathsf{p}_{t'+\tau}$ is below $V_1 + \epsilon$.*
2. Herding at the Middle Value
(*a*) *Suppose that S_2 has a U-shaped csd with a negative bias. Then there exists $\mu_* > 0$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mathsf{Pr}(V_2) > 1 - \delta$ and if $0 < \mu < \mu_*$ then herd-buying can start for $\mathsf{p}^* \in (V_2, V_2 + \epsilon)$.*
(*b*) *Suppose that S_2 has a U-shaped csd with a positive bias. Then there exists $\mu_* > 0$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mathsf{Pr}(V_2) > 1 - \delta$ and if $0 < \mu < \mu_*$ then herd-selling can start for $\mathsf{p}^* \in (V_2, V_2 - \epsilon)$.*

5 Price Movements With and Without Social Learning/Herding

In this section we present our main result on the impact of herding on price volatility. To measure it, we need a benchmark; as such we compare price movements when agents can follow the crowd and herd with price movements that may arise when agents ignore public information. In particular, we address the following two questions. First, will buys move prices less with herding than when herding/social learning is not allowed? And second, will sales move prices more with herding than when no herding/social learning is allowed? In what follows we focus on price-impacts for buy-herding; sale-herding effects are analogous.[16]

To answer these questions we compare bid- and ask-prices in a buy-herding situation, henceforth referred to as the *rational* case, with prices in a hypothetical economy, called *naïve*, that is otherwise identical to the rational world except that

[16] Our result does not address the total variance of prices for it is well know (since Glosten and Milgrom (1985), and as was also argued in Avery and Zemsky (1998)) that the absolute variance of price paths is bounded. Moreover, the bound is a function of primitives that have no relation to signals. We are instead concerned with the incremental price variability that herding causes relative to a no-learning benchmark.

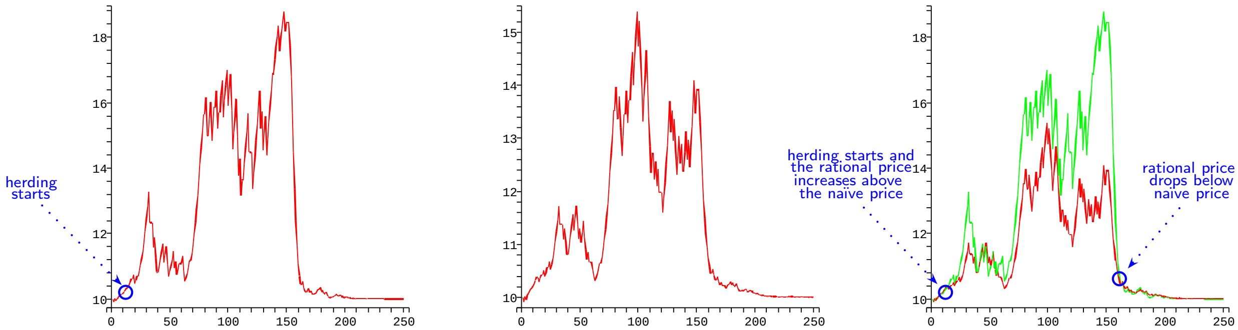


Figure 1: **Simulated Transaction Prices.** The left panel displays a simulation of transaction prices when traders behave rationally (and thus may herd). As can be seen, herding starts for middle prices (i.e. prices close to V_2), and prices during herding can move up substantially ($V_3 = 20,\ V_2 = 10$). The middle panel plots transaction prices for the same sequence of traders, but for "naïve" agents, i.e. the S_2 types merely follow their prior expectation and ignore all information in the trading history. The right panel combines both scenarios. Details of the data are available from the authors upon request. The underlying signal distribution is listed in Appendix B where we use $\mu = \mu_b - 0.001$.

- at each date informed agents with signal S_2 are naïve and unable (unwilling) to interpret the public information; they therefore buy if their expected value conditional on their private information $\mathsf{E}[V|S_2]$ exceeds the ask price, sell if their expected value conditional on their private information is less than the bid price and hold otherwise;
- the market maker sets prices as before taking into account that the strategies of the informed agents with signal S_2 are indeed naïve.

One justification for such naïve behavior (by S_2 types) is simply that they do not observe the public history of past actions and prices. Alternatively, one can think of the naïve traders as automata that always buy or sell depending on their signals.[17]

Casual intuition suggests that once buy-herding is possible, buys in a rational world should move prices less and sales more than in the naïve world. This is, loosely, because (*i*) a 'buy' carries less information when agents are rational and buy-herd (both S_2 and S_3 buy) than when they do not (only S_3 buys); and (*ii*) a 'sale' is a stronger negative signal in the rational buy-herd case (only S_1 sells) than in the naïve (S_1 and maybe S_2 sell) case.

While the intuition for sales is accurate, the intuition for buys is misleading. When buy-herding starts prices move stronger in *both* directions. Again, the reason lies in the

[17]The naïve world construction is simply a benchmark to compare the effect of social learning. In the construction of this naïve economy we assume that only S_2 types behave naïvely and other informed agents behave as in the rational world (S_1 and S_3 types always sell and buy, respectively). This assumption is made to ensure that the only difference between the two economies is due to the behavior of the herding types, and to simplify the analysis. Otherwise, we need to allow for S_1 and S_3 types changing behavior in the hyothetical world (in this case the naïve world may not even have an equilibrium in pure strategies).

U-shape of S_2's csd. At any history H_t at which buy-herding starts, a buy in the naïve world reveals that the buyer is either a S_3 type or a noise trader whereas in the rational world a buy reveals that the buyer is a S_2 type, a S_3 type or a noise trader. Thus, prices in the two worlds differ because of the information inferred from signal S_2. But when there is buy-herding, signal S_2 induces a large weight on V_3 and little on other values. This is because first, S_2 has a U-shaped csd and therefore, given S_2, the weight on V_2 is small relative to those of V_1 *and* V_3; and second, there must have been a sufficiently large number of buys (and thus a large number of S_3 signals) before any buy-herding history and therefore the weight on V_1 at such a history must be small.

Before stating the formal result on the comparison between price movements in the two worlds of naïve and rational traders, we need to introduce some futher notations and definitions. First, let $\mathsf{E}_n[V|H_t]$, $\mathsf{p}^A_{t,n}$, $\mathsf{p}^B_{t,n}$, $\beta_{i,n}$ and $\sigma_{i,n}$ be respectively the public (market) expectation, the ask-price, the bid-price, the probability of a buy in state i and the probability of a sale in state i in the naïve world.

Next suppose that $0 < \mu < \mu_b$ (so that buy-herding in the rational world is possible) and consider any history $H_t = (a_1 \ldots, a_{r+b+s})$ of outcomes, with $r > 1, b \geq 0$ and $s \geq 0$, that satisfies the following three conditions:

- (N1) for any truncation $H_\tau = (a_1, \ldots, a_\tau)$ a buy-herd in the rational world is possible if and only if $\tau \geq r$,
- (N2) the path $(a_{r+1}, \ldots, a_{r+b+s})$ consists of b buys and s sells,
- (N3) the posteriors of the market maker at H_r, $\mathsf{Pr}(V_i|H_r)$, are identical for the rational and the naïve case.

Finally, define $\mu_{hb} = \kappa_{hb}/(3 + \kappa_{hb})$ where $\kappa_{hb} = (\mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_3))/\rho^{13}_{12}$.

Proposition 5 (The Impact of Herding on Prices)
Consider any history $H_t = (a_1, \ldots, a_{r+b+s})$ that satisfies (N1)–(N3)*.*
(a) *Suppose that $b > 0$ and $s = 0$; then $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] > 0$.*
(b) *Suppose that $b = 0$ and $s > 0$. Then there exists $\bar{s} > 0$ such that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] < 0$ for any $s \leq \bar{s}$. Moreover, if $\mu_{hb} < \mu < \mu_b$ and $\mathsf{E}[V|H_t, S_2] > \mathsf{E}[V|H_t]$, then $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] < 0$ for all s.*
(c) *For any s there exists $\overline{b}$ such that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] > 0$, for all $b > \overline{b}$.*

Part (a) of the above proposition shows that if once herding starts there are only buys then the public expectation at H_t (which here coincides with last period's ask-price) is higher in the rational world than in the naïve economy. In part (b) if once herding starts there are only sales and the number $\bar{s}$ does not exceed some upper bound then the public expectation (which here coincides with last-period's bid price) will be smaller in the rational world than in the naïve economy; together with (a) this implies that the bid-ask-spread is wider in the rational case and thus prices move more than in the naïve world. Moreover, when the weight of the informed agents is not too small (μ exceeds μ_{hb}),[18] the public expectation in the rational world is and remains below that in the naïve case while people keep *selling*, as long as the rational S_2 type remains in the buy-herding mode. Part (c) shows that if the number of buys b is sufficiently large relative to the number of sales then prices increase more in the rational case than in the naïve case.

Using a series of simulated transaction prices, Figure 1 illustrates the proposition: The left panel displays rational prices, the middle panel displays naïve prices, and the right panel plots both simultaneously.

6 Further Features of Herding

Simple History Dependence. The order of trades and traders does not affect the price path as long as the model primitives do not allow any type of trader to change behavior. Clearly, herding or contrarian behavior involve such a change of behavior; changes from buying to holding or selling to holding also qualify as a change of behavior.

Without changes in behavior, it suffices to study the order imbalance (number of buys minus number of sales) to determine prices, but with changes, the order of arrival matters a great deal. Consider the following numerical example[19] of an MLRP signal structure with U-shaped and negatively biased csd for S_2

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{40}{49}$	$\frac{4}{49}$	0
S_2	$\frac{9}{49}$	$\frac{9}{490}$	$\frac{243}{12250}$
S_3	0	$\frac{9}{10}$	$\frac{12007}{12250}$

$\mu = \frac{1209}{1600}$,
$\mathbb{V} = (0, 10, 20)$, and
$\mathsf{Pr}(V) = (1/6, 2/3, 1/6)$.

For illustrative purposes, assume that the first fifteen traders are all informed and each signal S_i, $i = 1, 2, 3$, is received by five of the first fifteen traders. Next, we compare the price paths for different arrival orders of these traders.

SERIES 1: The arrival order is $5{\times}S_1$–$5\ {\times}S_2$–$5{\times}S_3$ (meaning the first five receive S_1, the next five S_2 and the last five S_3). The S_1 types, who move first, all sell and thus the price drops. The S_2 types also sell and the S_3 types buy. Computations show that after

[18]Condition $\mu_{hb} < \mu_b$ is generally feasible; for instance for the distribution described in Appendix B, condition (42), $\mu_{hb} \approx .258$, $\mu_b \approx .324$.

[19]We chose the numbers so that there can be herding after a small number of trades.

these 15 trades the public expectation will drop from 10 to .15.

SERIES 2: $5\times S_1$–$5\times S_3$–$5\times S_2$. Here the outcome is the same as in the previous series with S_1 traders selling, S_3 types' buying and finally the S_2 types selling. The public expectation also drops from 10 to .15.

SERIES 3: $5\times S_3$–$5\times S_2$–$5\times S_1$. The S_3 traders move first and buy. The S_2 types will now behave differently from the previous two series and will be buy-herding. The public expectation now rises to about 13.5. Finally, the five S_1 type sell, and then the public expectation drops to 10.31.

The difference between the outcome for Series 3 with those of Series 1 and 2 illustrates how the arrival order of traders matters: since there are S_2 types who trade, this type's change in trading-mode (from selling to buying) strongly affects the price-path.

Note, however, that even if there are no S_2-types directly involved in trading, the market maker has to consider the possibility that this type trades and thus has to account for this type's change of trading mode. To illustrate this, we next compare the outcome when the same number of buys and sales occurs, but in different orders.

SERIES 4: 20 BUYS FOLLOWED BY 20 SALES. After 20 buys, the public expectation is 15.36, after 20 subsequent sales it is 3.12.

SERIES 4: 20 SALES FOLLOWED BY 20 BUYS. After 20 sales, the public expectation is 1.16×10^{-13}, after 20 subsequent buys it is 10.0064.

In summary, the S_2-type can change trading modes in response to observing the order flow; thus the order flow affects prices and the frequency of different types of future trades. Although there is convergence in the long run, in the short run the fluctuations may be influenced by the precise order of trades.

Price Sensitivity. To further elaborate on the price sensitivity induced by herding, consider the following simulations of our model which uses the specification outlined in Appendix B, expression (43). The simulated prices paths are plotted in Figure 2.

In the left panel, there are two relevant price paths: the first (in gray) is for a setting with $\mu = \mu_b - \epsilon$, $\epsilon = 1/10,000$; in other words, there is just enough noise so that herding is possible. The second price path (in red) is for $\mu = \mu_b$ so that there cannot be herding.[20] The entry series for the graph is as follows: first, there is a long series of S_3 types, who all buy; this is followed by a group of S_2 types and eventually by some S_1 types. The point when S_2 types start entering is clearly marked; the S_1 types enter at the point when both curves peak. The point at which herding starts is marked too.

[20]The third price path (in blue) is for the case of naïve agents as described in the preceding section. For the naïve case the differences in prices for the two levels of μ are negligible.

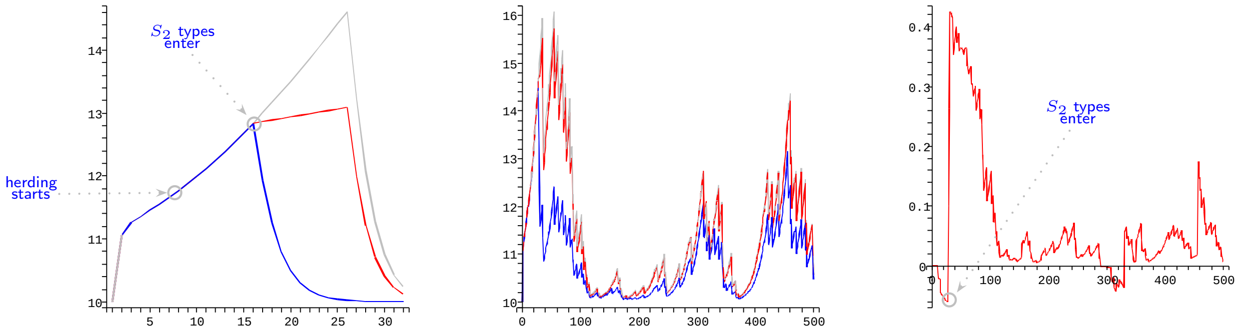


Figure 2: **Illustrations of the Sensitivity in Prices Paths with and without Herding.**

The series is constructed so that there are S_3 types who enter during herding. When the S_2 types enter, in the herding case, they buy, in the no-herding case, they hold. Even with holds, however, prices increase (this is due to the U-shaped csd).[21]

In the middle panel we plot prices for the same specifications, this time for a random sequence of traders; both series have the same sequence of traders but due to herding their actions may differ.[22] In the right panel we plot the difference of the two rational price-series from the middle panel. The series with herding-prices has more noise (because $\mu < \mu_b$). Thus initially, the price for the no-herding series is above the price of the herd series. Once herding starts (here after 8 trades), and once an S_2 type enters, this relation flips; this illustrates that due to herding prices move stronger in the direction of the herd than in the no-herding case.

Does Herding Hamper Learning? The common perception of herding is that it slows down learning. With rational agents and informationally efficient prices, this is not so obvious. With U-shaped signal distributions, the S_2-herding-type occurs with high probability in both the highest and the lowest state — so their herding may speed up learning.

To explore this more generally, we use Monte Carlo simulations and compare the two scenarios outlined when discussing price sensitivity. That is, for the first series, there is just enough noise so that buy-herding can be triggered, $\mu = \mu_b - \epsilon$, $\epsilon \approx 1/10,000$. In the second series, herding cannot occur, because there is too much informed trading, $\mu = \mu_b^{ch} \equiv \mu_b$. We will refer to prices in the first setting as herding-prices, irrespective of whether or not herding actually occurred; we refer to prices in the second setting as no-herding prices.

[21] The same simulation for the naïve case of the proceeding section results in S_2 types selling and prices falling for both levels of μ.

[22] There is also a series for the naïve case which, not surprisingly, is entirely below both rational series. Again, the naïve price series for $\mu = \mu_b$ and $\mu = \mu_b - \epsilon$ are almost identical.

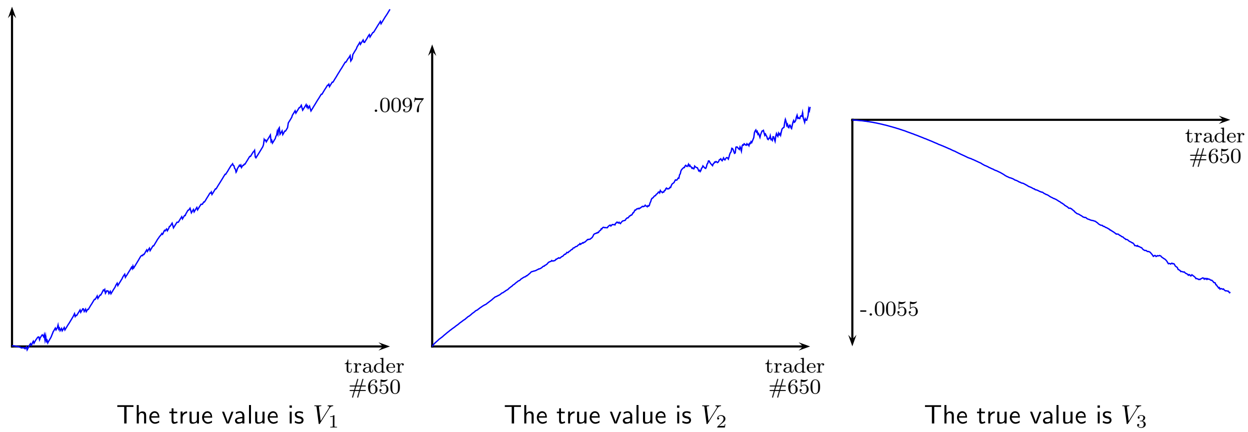


Figure 3: **The Difference in Speeds of Convergence.** Each graph plots the difference of the negative of the average log-distance of the transaction prices of herding and no-herding case. An up-sloping line thus indicates that for any t herding-prices are further from the true value than no-herding prices. All graphs are scaled to fit the page. The underlying signal distribution is listed in Appendix B.

Comparing the speeds of convergence for our two sets of simulations we note the following two observations:

1. if the true value is V_1 or V_2, then herding-prices converge slower;
2. if the true value is V_3, then convergence with herding is faster.

These observations are based on the following: For the simulations we again used the specification of the parameters given by (43) in Appendix B. Fixing the true liquidation values, we then drew 650 traders at random (noise and informed) assuming that $\mu_b \approx .766$. Since the proportion of the informed agents μ is large — approximately three quarters for both simulations — the 650 trades are almost always sufficient to obtain convergence to the true value. Next, we computed the time series of the transaction prices for both the herding and the no-herding case, and then recorded for each t and for both cases the absolute distance of the transaction price from the true value (which we know). We then repeated this procedure a large number of times, and calculated for each t and for each case the average distance from the true value. Since prices converge to the true value, these average distances decline in t. In the simulations, this distance declines approximately exponentially to zero. Thus the slope of the logarithm of the average distance measures the speed of convergence.

As the final step, we subtract at each t the log-averages for the no-herding from the herding series. A positive number indicates that the herding series is slower, i.e. that the average herding price is further away from the true value. Figure 3 plots these differences and the graphs are striking; they confirm our two observations mentioned above.[23]

To see the intuition for the these observations compare the effects of buy-herding on the herding and no-herding prices. First, when buy-herding occurs, S_2 types buy in the herding case and thus there are more buys with herding than in the no-herding case. Second, in the case of a buy, prices in the herding case tend to be higher than in the no-herding case. Since the no-herding prices here are the same as the ones that arise in the 'naïve' economy of the previous section (only S_3 types buy in both cases), this second effect follows from the same reasoning used in the previous section to explain why, in the case of a buy, prices in the rational world, when herding starts, exceed those in the naïve hypothetical economy (see Proposition 5(a)). Third, when there is a sale, prices in the herding and no-herding cases are almost identical and unaffected by buy-herding. This is because in both cases only S_1 types sell: in the herding case this is so by definition and in the no-herding case, the S_2 type's expectation is almost equal to the ask-price (expression (1) is almost zero) and thus larger than the bid-price.[24]

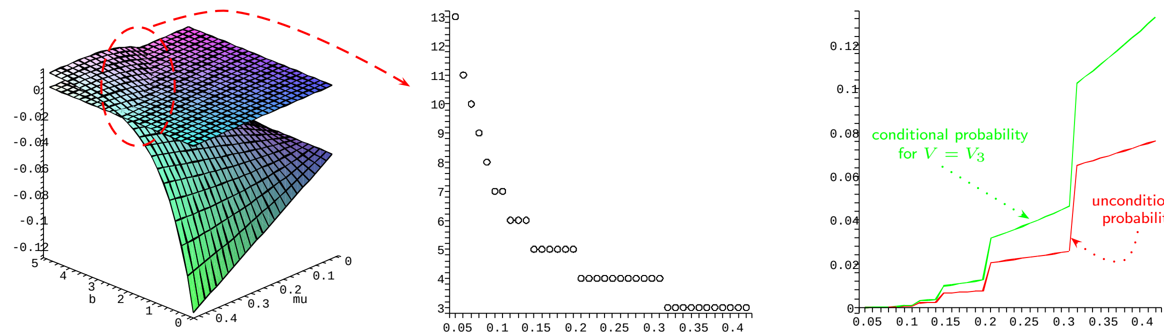
Now it follows from the above that if the true value is V_1 or V_2, herding prices converge slower: during herding, herd-buys move prices *away* from the true value by a larger magnitude and there are more such buys than in the no-herding case (sales have a similar effect in both cases). If, however, the true value is V_3 then once herding starts, prices in the herding-case move up more strongly because of the first two effects and thus they move faster *towards* the true value. This leads to a higher speed of convergence in the herding case. Figure 3 documents these three cases.

The Probability of the Fastest Herd. The shortest sequence of trades that leads to buy-herding is one with only buys; this is the 'fastest' herd. We now want get a sense of how likely this sequence is. Keeping the csd and the prior distribution fixed but varying the proportion of informed trading, we compute first how many buys are needed for buy-herding to begin, and then we determine how likely this sequence of buys is. The same type of analysis clearly applies to sell-herding.

As was explained before, S_2 types buy at any history H_t if the expression in (1) is positive. As the amount of informed trading increases from 0 to μ_b, there are then two

[23]We have also made a formal analysis by regressing the log-distance on time and, using the Chow test, checking if one slope is steeper than the other. The results were highly significant.

[24]The herding and no-herding price paths may also differ even if no buy-herding occurs (if S_2 types behave the same way in the two cases) because the proportions of informed trading μ are different for the two cases. In particular, when S_2 types do not buy-herd, since μ is smaller in the herding case, each price-movement in the herding-price series is smaller than than in the no-herding case, and as a result speed of convergence is slower in the former series. However, since for the simulations the difference between the values of μ is small ($\epsilon = 1/10,000$), the consequence of this effect is small relative to the first two effects mentioned above.



Equation (1) | Minimum b given μ | Probability of b given μ

Figure 4: **Trades needed for Herding the Probabilities for these trades.** The left panel plots the value of expression (1) as a function of μ, with $\mu \in (0, \mu_b)$, and of no-herd buys b. Whenever the bend curve crosses the 0-surface from below, herding is triggered. The middle panel computes the minimum integer number of no-herd buys that would trigger herding as a function of noise level μ. The right panel computes two probabilities: the first is the probability of having exactly the threshold number of buys at the beginning of trade (the thresholds are taken from the middle panel) conditional on the true state being V_3. The second probability is the unconditional likelihood of this threshold number. The plots in the right panel are functions of the μ. The signal distribution that underlies these plots is listed in Appendix B.

opposing effects. First, as noise decreases, the positive term in expression (1) (the first term) becomes smaller. This implies that for any history, the difference between the market maker's and the S_2 type's expectation becomes smaller; thus to get buy-herding one needs more buys. Second, as noise decreases, the informational content of past behavior (public information) improves and this makes herding more likely. Formally, the second and third terms in (1), the negative terms, decline as μ increases. This is because for any $i = 2, 3$, $\frac{\beta_1}{\beta_i} = \frac{\mu \mathsf{Pr}(S_3|V_1)+\gamma}{\mu \mathsf{Pr}(S_3|V_i)+\gamma}$, $\frac{\partial(\beta_1/\beta_i)}{\partial\mu} = (\mathsf{Pr}(S_3|V_1) - \mathsf{Pr}(S_3|V_i))/{\beta_i}^2$ and thus, since S_1's csd is decreasing, $\frac{\partial(\beta_1/\beta i)}{\partial\mu} < 0$.

While we do not have an analytical result on the net effect of increasing μ from 0 to μ_b, in all numerical examples that we computed the second effect dominates. Thus as noise trading declines (μ increases to μ_b) it takes *fewer* buys to trigger buy-herding. Figure 4 plots the minimum number of such consecutive time-zero buys needed to trigger buy-herding for our simulations. As the amount of noise decreases, ex ante it gets more likely that these consecutive buy-trades occur. (Figure 4's right panel illustrates these probabilities.)

7 Contrarians

As we mentioned before, contrarianism, as is the case with herding, represents changes of behavior resulting from observing the action of others, except that the change is against the crowd.

As with herding, by Proposition 2(a), S_2 is the only possible candidate for contrarian behavior. Now, however, the necessary and sufficient conditions for contrarian behavior turn out to be first, a hill-shaped csd for signal S_2 and second, that there are enough noise traders.

The second condition on the sufficient number of noise traders exists for the same reasons as in the case of herding (although the upper bound on the size of the informed is different from that for herding; see below). Also, as in the case of herding, depending on the relative values of $\Pr(S_2|V_1)$ and $\Pr(S_2|V_3)$, either buy-contrarianism or sell-contrarianism is possible, but not both. In contrast to herding, however, the critical condition for contrarian behavior is a *hill-shaped* csd for S_2. The intuition for this is that an informed agent with a hill-shaped S_2 signal increases his posterior on V_2 — he is more certain that the true value is V_2. Thus, if prices fall (rise) he would buy (sell) as the asset seems relatively undervalued (overvalued) to him. Then if such a type has a negative (positive) bias, he sells (buys) initially but becomes a buy-contrarian (sell-contrarian) after a sufficiently large number of sales (buys).

Before stating the formal result for contrarian behavior, we define the upper bound on the size of the informed for contrarian behavior as follows. Let

$$\kappa_b^{con} := \frac{\Pr(S_2|V_2) - \Pr(S_2|V_1)}{\rho_{23}^{12}}, \quad \kappa_s^{con} := \frac{\Pr(S_2|V_2) - \Pr(S_2|V_3)}{\rho_{12}^{23}}.$$

Then define

$$\mu_b^{con} = \kappa_b^{con}/(\kappa_b^{con} + 3), \quad \text{and} \quad \mu_s^{con} = \kappa_s^{con}/(\kappa_s^{con} + 3).$$

Proposition 6 (Contrarian with MLRP Signals)
Suppose that the information structure satisfies the MLRP; then the following hold.
(a) *Informed agents with signal S_2 become buy-contrarians with positive probability if and only if $0 < \mu < \mu_b^c$ where*

$$\mu_b^c = \min\{\mu_b^{in}, \mu_b^{con}\}.$$

(b) *Informed agents with signal S_2 become sell-contrarians with positive probability if and only if $0 < \mu < \mu_s^c$ where*

$$\mu_s^c = \min\{\mu_s^{in}, \mu_s^{con}\}.$$

The intuition for the above is similar to that for Proposition (3). To see this consider the case of becoming a buy-contrarian and part (a) of the above proposition (the arguments for becoming a sell-contrarian are analogous). For exactly the same reason as in the case of buy-herding, $0 < \mu < \mu_b^{in}$ is equivalent to S_2 types wanting to sell initially. Also, the condition $0 < \mu < \mu_b^{con}$ turns out to be equivalent to S_2 types wanting to buy after some history H_t against the crowd. This is because, as we mentioned before, type S_2 buys after any history H_t if and only if the expression in (1) is positive. By multiplying the expression in (1) by q_3^t/q_1^t, this is equivalent to the following expression being positive

$$\frac{q_3^t}{q_1^t}[\beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2)] + [\beta_1^t \mathsf{Pr}(S_2|V_2) - \beta_2^t \mathsf{Pr}(S_2|V_1)] + \frac{2q_3^t}{q_2^t}[\beta_1^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_1)].$$

Requiring $0 < \mu < \mu_b^{con}$ is equivalent to the the second term in the above expression being positive. If there is sufficient evidence in favour of low values (i.e. sufficiently many more sales than buys) the first and last terms in the above expression become arbitrarily small. It then follows that $0 < \mu < \mu_b^{con}$ is equivalent to S_2 types wanting to buy at some history H_t after prices have fallen (sufficiently many sales).

The formal proof of Proposition 6 is similar to that of Proposition 3 and is omitted.

As with the case of herding, to ensure that the characterisation in Propostion 6 is not vacuous μ_b^c and μ_s^c have to be positive. Simple computation demonstrates that $\mu_b^c > 0$ if and only if $\mathsf{Pr}(S_2|V_2) > \mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_3)$. Also $\mu_s^c > 0$ if and only if $\mathsf{Pr}(S_2|V_2) > \mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1)$. Thus we have the following corollary to the above result.

Corollary 2 (Necessary and Sufficient Conditions for Contrarianism Revisited) *Suppose that the information structure satisfies the MLRP; then the following hold.*

1. Buy-contrarian

(*a*) *A negatively biased and hill-shaped csd for S_2 is necessary for buy-contrarianism.*

(*b*) *Suppose that S_2's csd is negatively biased and hill-shaped. Then there exists $\mu_b^c > 0$ such that there is a positive probability of buy-contrarianism if and only if $0 < \mu < \mu_b^c$. Furthermore, μ_b^c is uniquely defined by $\mu_b^c = \min\{\mu_b^{in}, \mu_b^{con}\}$.*

2. Sell-contrarian

(*a*) *A positively biased and hill-shaped csd S_2 is necessary for sell-contrarianism.*

(*b*) *Suppose that S_2's signal distribution is positively biased and hill-shaped. Then there exists $\mu_s^c > 0$ such that there is a positive probability of sell-contrarianism if and only if $0 < \mu < \mu_s^c$. Furthermore, μ_s^c is uniquely defined by $\mu_s^c = \min\{\mu_s^{in}, \mu_s^{con}\}$.*

3. Contrarianism

(*a*) *A hill-shaped csd for S_2 is necessary for contrarian behavior.*

(*b*) *Suppose that S_2's csd is hill-shaped. Then there exists $\overline{\mu}^c > 0$ such that there is a positive probability of contrarian behavior if and only if $0 < \mu < \overline{\mu}^c$. Furthermore, $\overline{\mu}^c$ is uniquely defined by $\overline{\mu}^c = \max\{\mu_b^c, \mu_s^c\}$.*

8 More Than Three Signals

Buy- and Sell-Herding in the Same Framework. In our model, with the MLRP, informed agents with the highest and the lowest signals always take the same action (by Proposition 2(a)) and thus the herding candidate must be a middle signal-type. If there is only one middle type, as is the case with three signals, there can be only buy- or sell-herding but not both. This is merely the result of a setup with three signals. For example, with *four* MLRP signals, there are two middle types. If one type's signal distribution is negatively biased and U-shaped, and another type's is positively biased and U-shaped, then there can be both types of herding in the same model. We provide an example for such an MLRP csd in Appendix B.

A Continuum of Signals. Our herding result continues to hold if there is a continuum of signals, because even in this case, the csd for some signals can be U-shaped in V. Let the signal space $\mathbb{S}$ be an interval on the real line and suppose that for every V_i, $i = 1, 2, 3$, there exists a continuous density function $f(\cdot|V_i) : \mathbb{S} \to \mathbb{R}_+$.

Analogous to the discrete case, the information structure is said to satisfy the continuous version of the MLRP if for any $S_l, S_h \in \mathbb{S}$ and any values $V_l, V_h \in \mathbb{V}$ such that $S_l < S_h$ and $V_l < V_h$, f satisfies $f(S_h|V_h)/f(S_l|V_h) > f(S_h|V_l)/f(S_l|V_l)$. Likewise, all restrictions on noise can now be reformulated in terms of signal densities. For instance, constant κ_b, which is used to determine μ_b^{ch} (the level of noise allowing a change from selling to buying), will now be

$$\kappa_b := \frac{f(S^*|V_3) - f(S^*|V_2)}{f(S^*|V_3)(1 - F(S^*|V_2)) - f(S^*|V_2)(1 - F(S^*|V_3))},$$

where S^* is the signal type that is prone to buy-herding.

It can be then be shown, as in the case with 3 signals, that type S^* buy-herds with positive probability if and only if $0 < \mu < \mu_b$. Moreover, $\mu_b > 0$ is equivalent to $f(S^*|V_1) > f(S^*|V_3) > f(S^*|V_2)$, which is the continuous-signal analogue of a U-shaped, negatively biased csd. In Appendix B, we provide an example of continuous MLRP signal distributions such that for a continuum of signals the csd is U-shaped and negatively biased, i.e. there exists $\underline{S}$ and $\overline{S} \in \mathbb{S}$, such that $f(S^*|V_1) > f(S^*|V_3) > f(S^*|V_2)$ for all $S^* \in [\underline{S}, \overline{S}]$.

9 Discussion of Avery and Zemsky's Event Uncertainty Herding and their Definition of Signal Monotonicity

AZ analyse the same micro-structure model as ours and reach a very different set of conclusions. In particular, they argue that herd behavior with informationally efficient asset prices is not possible unless signals are "non-monotonic" and risk is "multi-dimensional". In this section we explain why our conclusions differ from theirs.

AZ reach their conclusions by (i) showing that herding is not possible when the information structure satisfies their definition of monotonicity and (ii) demonstrating the possibility of herding using a special non-monotone information structure, that has "two-dimensional" risk. AZ argue that it is this information structure's inherent non-monotonicity that triggers herding.

More specifically, AZ show that herding cannot occur if all signals satisfy the following non-standard signal monotonicity condition.

Definition 3 (AZ Monotonicity)
A signal S is monotonic if there exists a function $w(S)$ such that for all histories H_t, $\mathsf{E}[V|S,H_t]$ is always weakly between $\mathsf{E}[V|H_t]$ and $w(S)$.

There are several points to note concerning AZ's definition of monotonic signals. First, this condition precludes herding almost by definition (for example, if S types sell initially then $w(S) \leq \mathsf{E}[V|H_t,S] \leq \mathsf{E}[V|H_t]$ for any H_t, and buy-herding is not possible). Second, AZ's defnition of monotonicity does not imply nor is implied by the standard MLRP definition of monotonicity. It is easy to construct an example to show that AZ's definition does not imply MLRP; to show that the converse also does not hold it suffices to note that the former does not allow herding whereas, as we have shown, the latter does if the middle signal has a U-shaped csd.[25] Third, AZ's definition is not a condition on the primitives, i.e. on the signal distribution, but a requirement on endogenuous variables that must hold for all trading histories. This makes it difficult to determine ex ante whether or not a given signal distribution is monotonic. The following proposition however helps to clarify this point by providing a condition on primitives that implies to AZ's monotonicity.

Proposition 7 (Relation of the Monotonicity Concepts)
If S is csd-monotonic then S satisfies Definition 3.

Having shown that AZ's differing conclusion from ours on the possibility of herding with a monotonic information structure is simply to do with the non-standard nature of the definition of monotonicity that they adopt, we shall next discuss their example of herding and their conclusions on the role of multi-dimensionality of risks in inducing herding.

Using the same three-states–three-signals world as we, AZ provide a specific information structure that allows herding; their example uses *Event Uncertainty*, a concept first

[25] We have another numerical example of an MLRP csd in which there is no herding and yet AZ's definition of monotonicity is violated. In this example, the csd allows contrarian behavior.

employed by Easley and O'Hara (1987). The idea behind Event Uncertainty is that ex ante, it is possible that the asset's value has not moved at all. They further attach the following information structure to Event Uncertainty (called "IS1" in AZ). First, informed agents know *if* something has happened, i.e. if there was an event which moved the fundamental value of the asset. Second, they receive noisy information about how this event has influenced the liquidation value. Formally, AZ assume

$$\begin{aligned} \mathsf{Pr}(S_1|V_1) &= \mathsf{Pr}(S_3|V_3) = q, \;\; \mathsf{Pr}(S_3|V_1) = \mathsf{Pr}(S_1|V_3) = 1-q, \\ \mathsf{Pr}(S_2|V_3) &= \mathsf{Pr}(S_2|V_1) = 0, \;\; \mathsf{Pr}(S_2|V_2) = 1, \quad \text{for some } q > 1/2. \end{aligned} \tag{3}$$

AZ show that with the above information structure S_1 types can buy-herd and S_3 types can sell-herd. When this happens, S_2 types are not present and thus one can say that all rational agents that trade act alike.[26] The proof of AZ's herding result is simple and compelling. For example, for buy-herding, suppose first that there is a series of holds for the first n periods, which leaves the informed agent's beliefs unchanged, but which causes the market maker's conditional distribution to place a large weight on the middle value V_2. Then assume that there are m periods of buys which increase the (trading) informed agents' expected value. Now if both m and n/m are sufficiently large then the market maker's conditional distribution after this history places a large weight on the middle value V_2 (this follows from n/m being large) whereas the informed agent's conditional distribution after this history places a large weight on V_3 (this follows from m being large). Thus after such a history the S_1 types' expectations rise above the ask-price. Now if S_1 types are assumed to sell initially then it follows that they will change their action in the direction of the crowd and herd after this history.

There are three points to note about this signal structure. First, it is not monotonic, in the sense of the MLRP, because

$$\mathsf{Pr}(S_3|V_2)\mathsf{Pr}(S_2|V_1) = 0 \cdot 0 = 0 < 1-q = 1 \cdot (1-q) = \mathsf{Pr}(S_2|V_2)\mathsf{Pr}(S_3|V_1).$$

Second, knowing that some liquidation value has not occurred is an additional piece of information that causes informed agents' partitions of the set of liquidation values to be finer

[26] It is important to note, however, that AZ's herding also does not constitute an informational cascade: 'informed' traders who receive the 'no event'-signal (the middle signal) may trade in the opposite direction of S_1 and S_3 types. The S_2 types know that the true value is the intermediate value and thus whenever the bid is higher than V_2 they would sell and if the ask is lower than V_2 they would buy. When prices are such that, say, the S_1 types would buy-herd, then the bid must be above $1/2$, and so at such prices the S_2 types would sell. Consequently, public information continues to accumulate even during herding, so this is not a cascade (AZ themselves show that informational cascades do not occur with informationally efficient prices).

than the market maker's. AZ interpret this as a different 'dimension' of risk and attribute their herding result to this property. However, the explanation for herding outlined in the previous paragraph also works for small perturbations of AZ's signal structure, i.e. herding is also possible if the information structure in AZ is perturbed so that $\Pr(S_i|V_2) = \epsilon$, $i = 1, 3$, for some small and positive ϵ. Consequently, even if agents with signal S_1 and S_3 believe that the "no-information" event V_2 has happened with sufficiently small but positive probability there can be herding. Since such a perturbed information structure would no longer be multidimensional it follows that multidimensionality is not necessary for herd behavior. Our analysis in the previous sections shows that with the MLRP it is not the additional 'dimension' but rather the general shape of the conditional signal distribution that determines if herding is possible. In fact, in AZ the candidate herding types (those that receive signals S_1 and S_3) also have U-shaped csds with herd-buyer S_1 types having a negative bias and herd-seller S_3 types having a positive bias.

Finally, as we mentioned before, AZ's herding outcome is somewhat unappealing as an explanation of price volatility because price movements during herding are strictly limited: For informed agents, trades do not convey information, thus their expectations do not move. To break buy-herding (sell-herding), it suffices that prices rise above (fall below) the (constant) expectation of S_1 (S_3) types, and this is generally a very small movement.[27] In our setting with MLRP and a U-shaped signal type, on the other hand, prices may move significantly during herding: if buying persists and there are no sales, buy-herding will not stop while at the same time the market maker keeps raising the price.

A Omitted Proofs

Proof of Proposition 1

(a) By standard results on MLRP and stochastic dominance it must be that $\mathsf{E}[V|S_l] < \mathsf{E}[V|S_H]$. By a similar reasoning, at any history H_t, $\mathsf{E}[V|S_l, H_t] < \mathsf{E}[V|S_H, H_t]$ if the following MLRP condition holds at H_t: for any $S_l < S_h$ and any $V_l < V_h$

$$\frac{\Pr(S_h|V_h, H_t)}{\Pr(S_l|V_h, H_t)} > \frac{\Pr(S_h|V_l, H_t)}{\Pr(S_l|V_l, H_t)}. \tag{4}$$

To show this note first that $\Pr(V|H_t, S) = \Pr(V|S)\Pr(H_t|V) / \sum_{V' \in \mathbb{V}} \Pr(V'|S)\Pr(H_t|V')$. Then we have by the following manipulations that the MLRP condition $\frac{\Pr(S_h|V_h)}{\Pr(S_l|V_h)} > \frac{\Pr(S_h|V_l)}{\Pr(S_l|V_l)}$

[27] In fact, in AZ the required price movement during any herding vanishes in the limit as $\mu \to 0$ and $q \to 1/2$ (as the informativeness of the signals of the informed agents disappears); see Proposition 8 in AZ.

implies the MLRP condition (4) at any H_t:

$$\begin{aligned}
\mathsf{Pr}(S_l|V_l)\mathsf{Pr}(S_h|V_h) &> \mathsf{Pr}(S_l|V_h)\mathsf{Pr}(S_h|V_l) \\
\Leftrightarrow \mathsf{Pr}(V_l|S_l)\mathsf{Pr}(V_h|S_h) &> \mathsf{Pr}(V_h|S_l)\mathsf{Pr}(V_l|S_h) \\
\Leftrightarrow \frac{\mathsf{Pr}(V_l|S_l)\mathsf{Pr}(H_t|V_l)}{\sum_{\mathbb{V}}\mathsf{Pr}(V|S_l)\mathsf{Pr}(H_t|V)}\frac{\mathsf{Pr}(V_h|S_h)\mathsf{Pr}(H_t|V_h)}{\sum_{\mathbb{V}}\mathsf{Pr}(V|S_h)\mathsf{Pr}(H_t|V)} &> \frac{\mathsf{Pr}(V_h|S_l)\mathsf{Pr}(H_t|V_h)}{\sum_{\mathbb{V}}\mathsf{Pr}(V|S_l)\mathsf{Pr}(H_t|V)}\frac{\mathsf{Pr}(V_l|S_h)\mathsf{Pr}(H_t|V_l)}{\sum_{\mathbb{V}}\mathsf{Pr}(V|S_h)\mathsf{Pr}(H_t|V)} \\
\Leftrightarrow \mathsf{Pr}(V_l|H_t,S_l)\mathsf{Pr}(V_h|H_t,S_h) &> \mathsf{Pr}(V_h|H_t,S_l)\mathsf{Pr}(V_l|H_t,S_h).
\end{aligned}$$

(b) First we show that $\mathsf{Pr}(S_1|V_1) > \mathsf{Pr}(S_1|V_3)$. Suppose otherwise; thus $\mathsf{Pr}(S_1|V_1) \leq \mathsf{Pr}(S_1|V_3)$. Then the two MLRP conditions $\mathsf{Pr}(S_1|V_1)\mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1)\mathsf{Pr}(S_1|V_3)$ and $\mathsf{Pr}(S_1|V_1)\mathsf{Pr}(S_3|V_3) > \mathsf{Pr}(S_3|V_1)\mathsf{Pr}(S_1|V_3)$ imply respectively that $\mathsf{Pr}(S_2|V_1) < \mathsf{Pr}(S_2|V_3)$ and $\mathsf{Pr}(S_3|V_1) < \mathsf{Pr}(S_3|V_3)$. Hence, since $\mathsf{Pr}(S_1|V_1) \leq \mathsf{Pr}(S_1|V_3)$ we have $\sum_{i=1}^{3}\mathsf{Pr}(S_i|V_3) > \sum_{i=1}^{3}\mathsf{Pr}(S_i|V_1)$. But this contradicts $\sum_{i=1}^{3}\mathsf{Pr}(S_i|V_j) = 1$ for every j.

The same argument can be applied to show that $\mathsf{Pr}(S_1|V_1) > \mathsf{Pr}(S_1|V_2)$ and $\mathsf{Pr}(S_1|V_2) > \mathsf{Pr}(S_1|V_3)$, and also in the reverse direction for $\mathsf{Pr}(S_3|V_1) < \mathsf{Pr}(S_3|V_2) < \mathsf{Pr}(S_3|V_3)$.

Proof of Proposition 2

(a) Suppose contrary to the claim, an informed trader with signal S_1 buys at some history H_t. Then by Proposition 1 (a) every informed trader buys at H_t. This implies that at history H_t, $\mathsf{p}_t^A = \mathsf{p}_t^B = \mathsf{E}[V|H_t]$. But then, since by Proposition 1 (a) $\mathsf{E}[V|H_t] > \mathsf{E}[V|S_1, H_t]$, we have $\mathsf{p}_t^B > \mathsf{E}[V|S_1, H_t]$. Hence, an informed trader with signal S_1 sells at H_t. This is a contradiction.

The proof of the claim that informed traders with signal S_3 always buy is analogous.

(b) We shall prove the result in the case of buys. The same reasoning applies to sales.

Consider any arbitrary history H_t and any $V_h > V_l$. Then, by part (a) of this Proposition, at H_t there are two possibilities: either S_3 types buy and S_1 and S_2 types sell or S_2 and S_3 types buy and S_1 types sell. In the former case

$$\mathsf{Pr}(\mathsf{buy}|H_t, V_h) - \mathsf{Pr}(\mathsf{buy}|H_t, V_l) = \mu\left(\mathsf{Pr}(S_3|V_h) - \mathsf{Pr}(S_3|V_l)\right).$$

In the latter case

$$\begin{aligned}
\mathsf{Pr}(\mathsf{buy}|H_t, V_h) - \mathsf{Pr}(\mathsf{buy}|H_t, V_l) &= \mu\left(\mathsf{Pr}(S_3|V_h) + \mathsf{Pr}(S_2|V_h) - \mathsf{Pr}(S_3|V_l) - \mathsf{Pr}(S_2|V_l)\right) \\
&= \mu\left(1 - \mathsf{Pr}(S_1|V_h) - (1 - \mathsf{Pr}(S_1|V_l))\right) \\
&= \mu\left(\mathsf{Pr}(S_1|V_l) - \mathsf{Pr}(S_1|V_h)\right).
\end{aligned}$$

The result follows by setting $\epsilon = \min\{\mu(\mathsf{Pr}(S_3|V_h) - \mathsf{Pr}(S_3|V_l)), \mu(\mathsf{Pr}(S_1|V_l) - \mathsf{Pr}(S_1|V_h))\}$ and by noting that, by Proposition 1 (b), ϵ must be positive.

Proof of Proposition 3

In this proof, to simplify the exposition, we shall at times denote q_i^t, β_i^t and σ_i^t by q_i, β_i and σ_i when the meaning is clear. Also, we use the symbol $\propto$ to denote that two expressions have the same sign; thus for any real numbers x and y, $x \propto y$ stands for x and y having the same sign. We shall next prove the result in 7 steps.

Step 1: *For any H_t, we have*

$$\begin{aligned} \mathsf{E}[V|S_i,H_t]-\mathsf{p}_t^A \quad \propto \quad & [\beta_2^t \mathsf{Pr}(S_i|V_3)-\beta_3^t \mathsf{Pr}(S_i|V_2)] \\ & +\tfrac{q_1^t}{q_3^t}[\beta_1^t \mathsf{Pr}(S_i|V_2)-\beta_2^t \mathsf{Pr}(S_i|V_1)] \\ & +\tfrac{2q_1^t}{q_2^t}[\beta_1^t \mathsf{Pr}(S_i|V_3)-\beta_3^t \mathsf{Pr}(S_i|V_1)]. \end{aligned} \tag{5}$$

Note that

$$\mathsf{E}[V|S_i,H_t]-\mathsf{p}_t^A=\mathcal{V}q_2\left(\frac{\mathsf{Pr}(S_i|V_2)}{\mathsf{Pr}(S_i)}-\frac{\beta_2}{\mathsf{Pr}(\mathsf{buy}|H_t)}\right)+2\mathcal{V}q_3\left(\frac{\mathsf{Pr}(S_i|V_3)}{\mathsf{Pr}(S_i)}-\frac{\beta_3}{\mathsf{Pr}(\mathsf{buy}|H_t)}\right).$$

The RHS of the above has the same sign as

$$\begin{aligned} & \mathsf{Pr}(S_i|V_2)\sum_j \beta_j q_j-\beta_2\sum_j \mathsf{Pr}(S_i|V_j)q_j+2\,\frac{q_3}{q_2}\left(\mathsf{Pr}(S_i|V_3)\sum_j \beta_j q_j-\beta_3\sum_j \mathsf{Pr}(S_i|V_j)q_j\right) \\ = \quad & q_1\left(\beta_1 \mathsf{Pr}(S_i|V_2)-\beta_2 \mathsf{Pr}(S_i|V_1)\right)+q_3\left(\beta_3 \mathsf{Pr}(S_i|V_2)-\beta_2 \mathsf{Pr}(S_i|V_3)\right) \\ & +2\,\frac{q_3}{q_2}\left(q_1\ \left(\beta_1 \mathsf{Pr}(S_i|V_3)-\beta_3 \mathsf{Pr}(S_i|V_1)\right)+q_2\left(\beta_2 \mathsf{Pr}(S_i|V_3)-\beta_3 \mathsf{Pr}(S_i|V_2)\right)\right) \\ = \quad & q_3[\beta_2 \mathsf{Pr}(S_i|V_3)-\beta_3 \mathsf{Pr}(S_i|V_2)]+q_1\left(\beta_1 \mathsf{Pr}(S_i|V_2)-\beta_2 \mathsf{Pr}(S_i|V_1)\right) \\ & +2\frac{q_3}{q_2}\ \left(\beta_1 \mathsf{Pr}(S_i|V_3)-\beta_3 \mathsf{Pr}(S_i|V_1)\right). \end{aligned}$$

This implies (5).

Step 2: *For any H_t, we have*

$$\begin{aligned} \mathsf{E}[V|S_i,H_t]-\mathsf{p}_t^B \quad \propto \quad & [\sigma_2^t \mathsf{Pr}(S_i|V_3)-\sigma_3^t \mathsf{Pr}(S_i|V_2)] \\ & +\tfrac{q_1^t}{q_3^t}[\sigma_1^t \mathsf{Pr}(S_i|V_2)-\sigma_2^t \mathsf{Pr}(S_i|V_1)] \\ & +2\ \tfrac{q_1^t}{q_2^t}[\sigma_1^t \mathsf{Pr}(S_i|V_3)-\sigma_3^t \mathsf{Pr}(S_i|V_1)]. \end{aligned}$$

This follows by analogous arguments as in Step 1.

Step 3: $\mathsf{E}[V|S_2] - \mathsf{p}_1^B < 0$ *if and only if* $\mu < \mu_b^{in}$.
Since $\mathsf{Pr}(V_1) = \mathsf{Pr}(V_3)$ it follows from Step 2 that $\mathsf{E}[V|S_i] - \mathsf{p}_1^B < 0$ is equivalent to

$$\begin{aligned} &[\sigma_2^1 \mathsf{Pr}(S_i|V_3) - \sigma_3^1 \mathsf{Pr}(S_i|V_2)] + [\sigma_1^1 \mathsf{Pr}(S_i|V_2) - \sigma_2^1 \mathsf{Pr}(S_i|V_1)] \\ &\qquad + 2\, \tfrac{\mathsf{Pr}(V_1)}{\mathsf{Pr}(V_2)} [\sigma_1^1 \mathsf{Pr}(S_i|V_3) - \sigma_3^1 \mathsf{Pr}(S_i|V_1)] < 0. \end{aligned} \tag{6}$$

Since S_3 types always buy and S_1 types always sell we have that

$$\sigma_i^1 = \begin{cases} \gamma + \mu \mathsf{Pr}(S_1|V_i) & \text{if } S_2 \text{ types do not sell at } \mathsf{p}_1^B \\ \gamma + \mu(\mathsf{Pr}(S_1|V_i) + \mathsf{Pr}(S_2|V_i)) & \text{if } S_2 \text{types sell at } \mathsf{p}_1^B. \end{cases}$$

Substituting for $\sigma_i^1, i = 1,2,3$, in (6) and simplifying we have

$$\mathsf{E}[V|S_2] - \mathsf{p}_1^B < 0 \;\Leftrightarrow\; \frac{\mu}{\gamma} < \theta_b.$$

(The last expression holds irrespective of whether or not S_2 types sell at p_1^B.) This implies that $\mathsf{E}[V|S_i] - \mathsf{p}_1^B < 0$ if and only if $\mu < \frac{\theta_b}{3+\theta_b} = \mu_b^{in}$.

Step 4: *For any* $\eta > 0$ *there exists a history* H_t *consisting of only buys such that* $\frac{q_1^t}{q_2^t} < \eta$ *and* $\frac{q_1^t}{q_3^t} < \eta$.
Since by Proposition 2 (b) there exists $\epsilon > 0$ such that $\mathsf{Pr}(\mathsf{buy}|V_h, H_t) > \mathsf{Pr}(\mathsf{buy}|V_l, H_t) + \epsilon$ for any history H_t and $V_h > V_l$, it follows that for sufficiently large t any history H_t consisting only of buys is such that $\frac{q_1^t}{q_2^t} < \eta$ and $\frac{q_1^t}{q_3^t} < \eta$.

Step 5: *For any date* t *we have* $\beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2) = \rho_{23}^{23}\left(\frac{k_b(1-\mu)}{3} - \mu\right)$.
Since S_1 types always sell and S_3 types always buy we have that

$$\beta_i^t = \begin{cases} \gamma + \mu \mathsf{Pr}(S_3|V_i) & \text{if } S_2 \text{ types do not buy at } \mathsf{p}_t^A \\ \gamma + \mu\left(\mathsf{Pr}(S_3|V_i) + \mathsf{Pr}(S_2|V_i)\right) & \text{if } S_2 \text{ types buy at } \mathsf{p}_t^A. \end{cases}$$

This implies, irrespective of whether S_2 types buy or not at p_t^A, that

$$\begin{aligned} &\{\beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2)\} \\ = \;& \gamma\left(\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2)\right) + \mu\left(\mathsf{Pr}(S_3|V_2)\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_3|V_3)\mathsf{Pr}(S_2|V_2)\right) \\ = \;& \frac{(1-\mu)\left(\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2)\right)}{3} - \mu\rho_{23}^{23} = \rho_{23}^{23}\left(\frac{k_b(1-\mu)}{3} - \mu\right). \end{aligned}$$

Step 6: $\mathsf{E}[V|S_2, H_t] - \mathsf{p}_t^A > 0$ *and* $\mathsf{E}[V|H_t] > \mathsf{E}[V]$ *for some history* H_t *if* $\mu < \mu_b^{ch}$.
Suppose that $\mu + \eta < \mu_b^{ch}$ for some $\eta > 0$. By Step 4, there exists a history H_t of only buys

such that first, $q_1^t/q_3^t < 1$ and second, that the sum of the second and the third term of (1) are less than $\eta\rho_{23}^{23}(1+\frac{k_b}{3})$:

$$\frac{q_1^t}{q_3^t}[\beta_1^t\mathsf{Pr}(S_2|V_2)-\beta_2^t\mathsf{Pr}(S_2|V_1)]+\frac{2q_1^t}{q_2^t}[\beta_1^t\mathsf{Pr}(S_2|V_3)-\beta_3^t\mathsf{Pr}(S_2|V_1)]<\eta\rho_{23}^{23}(1+\frac{k_b}{3}). \tag{7}$$

Since $\mu+\eta<\mu_b^{ch}=\frac{k_b}{3+k_b}$ it follows that $\eta(1+\frac{k_b}{3})<\frac{k_b(1-\mu)}{3}-\mu$. But then, by Step 5, the first term of (1), $[\beta_2^t\mathsf{Pr}(S_2|V_3)-\beta_3^t\mathsf{Pr}(S_2|V_2)]$, exceeds $\eta\rho_{23}^{23}(1+\frac{k_b}{3})$. This, together with (7), establishes that $\mathsf{E}[V|S_2,H_t]-\mathsf{p}_t^A>0$.

To show that $\mathsf{E}[V|H_t]>\mathsf{E}[V]$, note that

$$\mathsf{E}[V|H_t]-\mathsf{E}[V]=\mathcal{V}\{(1-q_1^t-q_3^t)+2q_3^t\}-\mathcal{V}=\mathcal{V}(q_3^t-q_1^t). \tag{8}$$

This, together with $\frac{q_1^t}{q_3^t}<1$, imply that $\mathsf{E}[V|H_t]>\mathsf{E}[V]$.

Step 7: *If* $\mathsf{E}[V|S_2]-\mathsf{p}_1^B<0$, $\mathsf{E}[V|S_2,H_t]-\mathsf{p}_t^A>0$ *and* $\mathsf{E}[V|H_t]>\mathsf{E}[V]$ *for some history* H_t *then* $\mu<\frac{k_b}{3+k_b}$.
By Step 1 and $\mathsf{E}[V|S_2,H_t]-\mathsf{p}_t^A>0$ we have

$$\begin{aligned}&[\beta_2\mathsf{Pr}(S_2|V_3)-\beta_3\mathsf{Pr}(S_2|V_2)]+\tfrac{q_1}{q_3}[\beta_1\mathsf{Pr}(S_2|V_2)-\beta_2\mathsf{Pr}(S_2|V_1)]\\&\qquad+\tfrac{2q_1}{q_2}[\beta_1\mathsf{Pr}(S_2|V_3)-\beta_3\mathsf{Pr}(S_2|V_1)]>0.\end{aligned} \tag{9}$$

Now since $\mathsf{E}[V|S_2]-\mathsf{p}_1^B<0$, by Step 3, we have $0<\mu<\frac{\theta_b}{3+\theta_b}$. Thus, $\theta_b>0$ and hence

$$\mathsf{Pr}(S_2|V_3)-\mathsf{Pr}(S_2|V_1)<0. \tag{10}$$

But then, by condition (9), we have

$$\beta_2\mathsf{Pr}(S_2|V_3)-\beta_3\mathsf{Pr}(S_2|V_2)+\frac{q_1^t}{q_3^t}[\beta_1\mathsf{Pr}(S_2|V_2)-\beta_2\mathsf{Pr}(S_2|V_1)]>0. \tag{11}$$

This, together with (10) and $0<\beta_1<\beta_3$ (Proposition 2 (b)), imply that

$$\beta_2\mathsf{Pr}(S_2|V_1)-\beta_1\mathsf{Pr}(S_2|V_2)+\frac{q_1^t}{q_3^t}[\beta_1\mathsf{Pr}(S_2|V_2)-\beta_2\mathsf{Pr}(S_2|V_1)]>0.$$

But then

$$(\beta_2\mathsf{Pr}(S_2|V_1)-\beta_1\mathsf{Pr}(S_2|V_2))\left(1-\frac{q_1^t}{q_3^t}\right)>0. \tag{12}$$

Also, since $\mathsf{E}[V|H_t] > \mathsf{E}[V]$ it follows from (8) that $q_3^t - q_1^t > 0$. This together with (12) imply that $\beta_2 \mathsf{Pr}(S_2|V_1) - \beta_1 \mathsf{Pr}(S_2|V_2) > 0$. Thus, by (11), we have

$$\beta_2 \mathsf{Pr}(S_2|V_3) - \beta_3 \mathsf{Pr}(S_2|V_2) > 0. \tag{13}$$

Thus by Step 5, we have $k_b(1-\mu) - 3\mu > 0$; hence, $\mu < \frac{k_b}{3+k_b}$. This completes the proof of this step.

Claims (a) and (b) in the Proposition follow immediately from Steps 3, 6 and 7.

Proof of Proposition 4

Proof of 1(a): Since there is buy herding at $H_{t'}$ it follows from Step 1 in the proof of Proposition 3 that

$$\begin{aligned} \beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2) + \tfrac{q_1^t}{q_3^t} \left[\beta_1^t \mathsf{Pr}(S_2|V_2) - \beta_2^t \mathsf{Pr}(S_2|V_1)\right] \\ + \tfrac{2q_1^t}{q_2^t} \left[\beta_1^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_1)\right] > 0. \end{aligned} \tag{14}$$

for $t = t'$. Moreover, buy-herding persists if (14) holds for any $t > t'$.

Also, since buy-herding occurs at $H_{t'}$, by Corollary 1 it must be that S_2 has a U-shaped csd with a negative bias. But then the second and the third terms in (14) are negative and, since (14) holds for $t = t'$, the first term in (14) is positive for $t = t'$. This implies that at any date $t > t'$ inequality (14) will continue to hold, as long as $\frac{q_1^t}{q_3^t}$ and $\frac{q_1^t}{q_2^t}$ are non-increasing in t. This is indeed the case if the unfolding history involves only buys because, by Proposition 2 (b), $\beta_3^t > \beta_2^t > \beta_1^t$ at any history. Thus for continuing buys, herding persists beyond period t'.

We now show that for continuing buys, beyond period t', the prices (during this buy-herding phase) will approach V_3 . To see this observe that

$$\mathsf{E}[V|H_t] = \sum_i V_i q_i^t = q_3^t \left(V_2 \frac{q_2^t}{q_3^t} + V_3 \right).$$

But since, by Proposition 2 (b), there exists $\eta > 0$ such that $\beta_3^t > \beta_2^t + \eta$, it follows that $\frac{q_2^t}{q_3^t}$ is arbitrarily small at any history H_t that includes a sufficiently large number of buys. Consequently, for every $\epsilon > 0$, there exists a finite history H_t consisting of $H_{t'}$ followed by sufficiently many (herd-)buys such that $\mathsf{E}[V|H_t] > V_3 - \epsilon$.

Proof of 1(b): The proof for this part is analogous to that for part 1(a).

Proof of 2(a): Before proving the claim, note that since S_2 has a U-shaped csd with a negative bias we have (i) μ_b^{ch} is positive and independent of the value of q_2^1 and (ii) θ_b, and hence μ_b^{in}, are always positive and bounded away form zero for any value of q_2^1. Also, since $\mu_b = \min\{\mu_b^{ch}, \mu_b^{in}\}$ it then follows that there exists $\mu_* > 0$ such that $\mu_b \geq \mu_*$ for any value of $q_2^1 \in [0,1]$.

Next, fix any $\mu < \mu_*$. Then, by Step 5 in the proof of the previous Proposition and $\mu_* \leq \mu_b$, there exists $\eta > 0$ such that we have

$$[\beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2)] > \eta, \text{ for every } t. \tag{15}$$

Also, fix any $b > 1$ such that

$$\left(\frac{\mu \mathsf{Pr}(S_3|V_1) + \gamma}{\mu \mathsf{Pr}(S_3|V_3) + \gamma}\right)^{b-1} [\beta_2^b \mathsf{Pr}(S_2|V_1) - \beta_1^b \mathsf{Pr}(S_2|V_2)] < \eta/2. \tag{16}$$

(Since $\mathsf{Pr}(S_3|V_1) < \mathsf{Pr}(S_3|V_3)$ such a b exists.) Let H_b be the history consisting only of $b-1$ no-herd buys, and denote the posterior probabilities for V_i at H_b by q_i^b. Then, since $\frac{q_1^b}{q_3^b} = \left(\frac{\mu \mathsf{Pr}(S_3|V_1)+\gamma}{\mu \mathsf{Pr}(S_3|V_3)+\gamma}\right)^{b-1}$, it follows from (15) and (16) that

$$[\beta_2^b \mathsf{Pr}(S_2|V_3) - \beta_3^b \mathsf{Pr}(S_2|V_2)] + \frac{q_1^b}{q_3^b}[\beta_1^b \mathsf{Pr}(S_2|V_2) - \beta_2^b \mathsf{Pr}(S_2|V_1)] > \eta/2. \tag{17}$$

Next, fix any $\epsilon > 0$. Note that $\frac{q_i^b}{q_2^b} = \frac{q_i^1}{q_2^1}\left(\frac{\mu \mathsf{Pr}(S_3|V_1)+\gamma}{\mu \mathsf{Pr}(S_3|V_2)+\gamma}\right)^{b-1}$ for $i = 1, 3$. Then there must exist $\delta > 0$ such that if $q_2^1 > 1 - \delta$ the following two conditions hold:

$$\mathsf{p}_b^A = \mathsf{E}[V|H_b, \text{buy}] = q_2^b(V_2 + \frac{q_3^b}{q_2^b} V_3) \in (V_2, V_2 + \epsilon) \tag{18}$$

and

$$2\, \frac{q_1^b}{q_2^b}[\beta_3 \mathsf{Pr}(S_2|V_1) - \beta_1 \mathsf{Pr}(S_2|V_3)] < \eta/2. \tag{19}$$

Hence, if $q_2^1 > 1 - \delta$ it follows from (17) and (19) that

$$[\beta_2^b \mathsf{Pr}(S_2|V_3) - \beta_3^b \mathsf{Pr}(S_2|V_2)] + \frac{q_1^b}{q_3^b}[\beta_1^b \mathsf{Pr}(S_2|V_2) - \beta_2^b \mathsf{Pr}(S_2|V_1)] + 2\, \frac{q_1^b}{q_2^b}[\beta_1 \mathsf{Pr}(S_2|V_3) - \beta_3 \mathsf{Pr}(S_2|V_1)] > 0.$$

But this, together with (18), establish that if $q_2^1 > 1 - \delta$ then at H_b there is buy-herding and the ask price belongs to the interval $(V_2, V_2 + \epsilon)$.

Proof of 2(b): The proof for this part is analogous to that for part 2 (a).

Proof of Proposition 5

Proof of (a): First note that

$$\begin{aligned}
&\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] = \mathcal{V}\{(q_2^t - q_{2,n}^t) + 2(q_3^t - q_{3,n}^t)\} \\
= \;& \mathcal{V}\left\{ q_2^r \left(\frac{\beta_2^b \sigma_2^s}{\sum_i q_i^r \beta_i^b \sigma_i^s} - \frac{\beta_{2,n}^b \sigma_{2,n}^s}{\sum_i q_i^r \beta_{i,n}^b \sigma_{i,n}^s} \right) + 2q_3^r \left(\frac{\beta_3^b \sigma_3^s}{\sum_i q_i^r \beta_i^b \sigma_i^s} - \frac{\beta_{3,n}^b \sigma_{3,n}^s}{\sum_i q_i^r \beta_{i,n}^b \sigma_{i,n}^s} \right) \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] \;\; \propto \;\; & q_2^r q_1^r [(\beta_2 \beta_{1,n})^b (\sigma_2 \sigma_{1,n})^s - (\beta_{2,n}\beta_1)^b (\sigma_{2,n}\sigma_1)^s] \\
& + 2q_3^r q_1^r [(\beta_3 \beta_{1,n})^b (\sigma_3 \sigma_{1,n})^s - (\beta_{3,n}\beta_1)^b (\sigma_{3,n}\sigma_1)^s] \\
& + q_3^r q_2^r [(\beta_3 \beta_{2,n})^b (\sigma_3 \sigma_{2,n})^s - (\beta_{3,n}\beta_2)^b (\sigma_{3,n}\sigma_2)^s].
\end{aligned}$$

Dividing the RHS of the above by $q_3^r q_2^r$ and rearranging, we have

$$\begin{aligned}
\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] \;\; \propto \;\; & [(\beta_3 \beta_{2,n})^b (\sigma_3 \sigma_{2n})^s - (\beta_{3,n}\beta_2)^b (\sigma_{3,n}\sigma_2)^s] \\
& + \frac{q_1^r}{q_3^r} [(\beta_2 \beta_{1,n})^b (\sigma_2 \sigma_{1,n})^s - (\beta_{2,n}\beta_1)^b (\sigma_{2,n}\sigma_1)^s] \\
& + \frac{2q_1^r}{q_2^r} [(\beta_3 \beta_{1,n})^b (\sigma_3 \sigma_{1,n})^s - (\beta_{3,n}\beta_1)^b (\sigma_{3,n}\sigma_1)^s].
\end{aligned} \tag{20}$$

Manipulating the RHS of (20), we have that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t]$ has the same sign as

$$\begin{aligned}
& [(\beta_3 \beta_{2,n})^b - (\beta_{3,n}\beta_2)^b](\sigma_3\sigma_{2,n})^s + [(\sigma_3\sigma_{2,n})^s - (\sigma_{3,n}\sigma_2)^s](\beta_{3,n}\beta_2)^b \\
& + \frac{q_1^r}{q_3^r} \left\{ [(\beta_2 \beta_{1,n})^b - (\beta_{2,n}\beta_1)^b](\sigma_2\sigma_{1,n})^s + [(\sigma_2\sigma_{1,n})^s - (\sigma_{2,n}\sigma_1)^s](\beta_{2,n}\beta_1)^b \right\} \\
& + \frac{2q_1^r}{q_2^r} \left\{ [(\beta_3 \beta_{1,n})^b - (\beta_{3,n}\beta_1)^b](\sigma_3\sigma_{1,n})^s + [(\sigma_3\sigma_{1,n})^s - (\sigma_{3,n}\sigma_1)^s](\beta_{3,n}\beta_1)^b \right\}.
\end{aligned} \tag{21}$$

Next note that since $\beta_3 > \beta_2 > \beta_1$ and $\beta_{3,n} > \beta_{2,n} > \beta_{1,n}$ it follows that

$$\sum_{\tau=0}^{b-1} (\beta_3 \beta_{2,n})^{b-1-\tau} (\beta_{3,n}\beta_2)^{\tau} \;\; > \;\; \sum_{\tau=0}^{b-1} (\beta_2 \beta_{1,n})^{b-1-\tau} (\beta_{2,n}\beta_1)^{\tau}, \tag{22}$$

$$\sum_{\tau=0}^{b-1} (\beta_3 \beta_{2,n})^{b-1-\tau} (\beta_{3,n}\beta_2)^{\tau} \;\; > \;\; \sum_{\tau=0}^{b-1} (\beta_3 \beta_{1,n})^{b-1-\tau} (\beta_3\beta_1)^{\tau}. \tag{23}$$

Since there is herding at r, by Step 1 of the proof of Proposition 3, we have

$$[\beta_2 \mathsf{Pr}(S_2|V_3) - \beta_3 \mathsf{Pr}(S_2|V_2)] + \tfrac{q_1^r}{q_3^t}[\beta_1 \mathsf{Pr}(S_2|V_2) - \beta_2 \mathsf{Pr}(S_2|V_1)] \\ +2\ \tfrac{q_1^r}{q_2^r}[\beta_1 \mathsf{Pr}(S_2|V_3) - \beta_3 \mathsf{Pr}(S_2|V_1)] > 0. \tag{24}$$

Also, by Corollary 1, buy-herding at r implies that the csd for S_2 is U-shaped and negatively biased. Thus, we have

$$\begin{aligned} \beta_3\beta_{2,n} - \beta_{3,n}\beta_2 &= \mu[\beta_2 \mathsf{Pr}(S_2|V_3) - \beta_3 \mathsf{Pr}(S_2|V_2)] > 0, \\ \beta_2\beta_{1,n} - \beta_{2,n}\beta_1 &= \mu[\beta_1 \mathsf{Pr}(S_2|V_2) - \beta_2 \mathsf{Pr}(S_2|V_1)] < 0, \\ \beta_3\beta_{1,n} - \beta_{3,n}\beta_1 &= \mu[\beta_1 \mathsf{Pr}(S_2|V_3) - \beta_3 \mathsf{Pr}(S_2|V_1)] < 0. \end{aligned} \tag{25}$$

Therefore, it follows from (24) that

$$[\beta_3\beta_{2,n} - \beta_{3,n}\beta_2] + \frac{q_1^r}{q_3^t}[\beta_2\beta_{1,n} - \beta_{2,n}\beta_1] + \frac{2q_1^r}{q_2^r}[\beta_3\beta_{1,n} - \beta_{3,n}\beta_1] > 0. \tag{26}$$

Now when $s = 0$, by expanding the expression in (21), it must be that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t]$ has the same sign as

$$\begin{aligned} &[(\beta_3\beta_{2,n}) - (\beta_{3,n}\beta_2)] \sum\nolimits_{\tau=0}^{b-1} (\beta_3\beta_{2,n})^{b-1-\tau}(\beta_{3,n}\beta_2)^\tau \\ &+\frac{q_1^r}{q_3^r}\left\{(\beta_2\beta_{1,n} - \beta_{2,n}\beta_1)\sum\nolimits_{\tau=0}^{b-1}(\beta_2\beta_{1,n})^{b-1-\tau}(\beta_{2,n}\beta_1)^\tau\right\} \\ &+2\frac{q_1^r}{q_2^r}\left\{(\beta_3\beta_{1,n} - \beta_{3,n}\beta_1)\sum\nolimits_{\tau=0}^{b-1}(\beta_3\beta_{1,n})^{b-1-\tau}(\beta_{3,n}\beta_1)^\tau\right\}. \end{aligned}$$

But then, by (22), (23) and (26), $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] > 0$ for $s = 0$. This completes the proof of part (a) of the Proposition.

Proof of (b): We prove the first part of (b) by showing that if $b = 0$ and $s = 1$ then $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] < 0$.

First, since $\sigma_1 > \sigma_2 > \sigma_3$ and $\sigma_{1,n} > \sigma_{2,n} > \sigma_{3,n}$ it follows that

$$\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau < \sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,n})^{s-1-\tau}(\sigma_{2,n}\sigma_1)^\tau, \tag{27}$$

$$\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau < \sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,n})^{s-1-\tau}(\sigma_{3,n}\sigma_1)^\tau. \tag{28}$$

Next, note that $\mathsf{E}[V|S_2, H_r] > \mathsf{p}_r^B$ because there is herding at r. By Step 2 in the proof of

Proposition 3, this implies

$$\begin{aligned}[\sigma_2 \mathsf{Pr}(S_2|V_3) - \sigma_3 \mathsf{Pr}(S_2|V_2)] + \tfrac{q_1^r}{q_3^r}[\sigma_1 \mathsf{Pr}(S_2|V_2) - \sigma_2 \mathsf{Pr}(S_2|V_1)] \\ +2\ \tfrac{q_1^r}{q_2^r}[\sigma_1 \mathsf{Pr}(S_2|V_3) - \sigma_3 \mathsf{Pr}(S_2|V_1)] > 0.\end{aligned} \tag{29}$$

Simple computation shows that

$$\begin{aligned}\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2 &= -\mu[\sigma_2 \mathsf{Pr}(S_2|V_3) - \sigma_3 \mathsf{Pr}(S_2|V_2)], \\ \sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1 &= -\mu[\sigma_1 \mathsf{Pr}(S_2|V_3) - \sigma_3 \mathsf{Pr}(S_2|V_1)], \\ \sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1 &= -\mu[\sigma_1 \mathsf{Pr}(S_2|V_2) - \sigma_2 \mathsf{Pr}(S_2|V_1)].\end{aligned} \tag{30}$$

Therefore, (29) is equivalent to

$$[\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2] + \frac{q_1^r}{q_3^r}[\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1] + \frac{2q_1^r}{q_2^r}[\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1] < 0. \tag{31}$$

But the LHS of (31) is the same as the RHS of (20) with $b = 0$ and $s = 1$. Thus, $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] < 0$ when $b = 0$ and $s = 1$. This completes the proof of the first part of (b).

We will now turn to the proof of the second part of (b). First, note that, by (30), we have

$$\begin{aligned}\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2 &= -\mu^2\rho_{12}^{23} + \mu\gamma(\mathsf{Pr}(S_2|V_2) - \mathsf{Pr}(S_2|V_3)) < 0 \\ \sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1 &> \sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1\end{aligned} \tag{32}$$

Also, since we require $\mathsf{E}[V|S_2, H_1] < \mathsf{p}_1^b$, we must have

$$[\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2] + \frac{q_1^1}{q_3^1}[\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1] + \frac{2q_1^1}{q_2^1}[\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1] > 0.$$

But then it follows from (32) that

$$\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1 > 0. \tag{33}$$

Finally, by simple manipulations, we have that

$$\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1 < 0 \text{ if } \mu > \mu_{hb}. \tag{34}$$

Now since $\mathsf{p}_t^b - \mathsf{E}[V|S_2, H_t] < 0$, it must be that

$$[\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2] + \frac{q_1^r}{q_3^r}\left(\frac{\sigma_1}{\sigma_3}\right)^s [\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1] + \frac{2q_1^r}{q_2^r}\left(\frac{\sigma_1}{\sigma_2}\right)^s [\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1] < 0. \quad (35)$$

Also, since $E[V|H_t] - E_n[V|H_t]$ has the same sign as the expression in (21), by simple expansion of the expression in (21) we have that if $b = 0$ then

$$\begin{aligned} E[V|H_t] - E_n[V|H_t] \;\propto\; & [(\sigma_3\sigma_{2,n}) - (\sigma_{3,n}\sigma_2)] \sum_{\tau=0}^{s-1} (\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau \\ & + \frac{q_1^r}{q_3^r}\left\{(\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1)\sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,n})^{s-1-\tau}(\sigma_{2,n}\sigma_1)^\tau\right\} \\ & + 2\frac{q_1^r}{q_2^r}\left\{(\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1)\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,n})^{s-1-\tau}(\sigma_3\sigma_1)^\tau\right\}. \end{aligned}$$

Rearranging the RHS of the above expression we have that if $b = 0$ then

$$\begin{aligned} E[V|H_t] - E_n[V|H_t] \;\propto\; & [\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2] \\ & + \frac{q_1^r}{q_3^r}\frac{\sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,n})^{s-1-\tau}(\sigma_{2,n}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau}[\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1] \qquad (36) \\ & + \frac{2q_1^r}{q_2^r}\frac{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,n})^{s-1-\tau}(\sigma_3\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau}[\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1]. \end{aligned}$$

Now simple manipulations show that

$$\left(\frac{\sigma_1}{\sigma_3}\right)^s > \frac{\sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,n})^{s-1-\tau}(\sigma_{2,n}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau} \Leftrightarrow \sum_{\tau=0}^{s-1}{\sigma_2}^{s-1-\tau}{\sigma_{2,n}}^\tau(\sigma_1\sigma_3)^\tau\left((\sigma_1\sigma_{3,n})^{s-1-\tau} - (\sigma_3\sigma_{1,n})^{s-1-\tau}\right) > 0.$$

But this, together with (34), imply that

$$\left(\frac{\sigma_1}{\sigma_3}\right)^s > \frac{\sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,n})^{s-1-\tau}(\sigma_{2,n}\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau} \quad \text{if } \mu > \mu_{hb}. \qquad (37)$$

Likewise, simple manipulations show that

$$\left(\frac{\sigma_1}{\sigma_2}\right)^s < \frac{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,n})^{s-1-\tau}(\sigma_3\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau} \Leftrightarrow \sum_{\tau=0}^{s-1}{\sigma_3}^{s-1-\tau}{\sigma_{3,n}}^\tau(\sigma_1\sigma_2)^\tau\left((\sigma_2\sigma_{1,n})^{s-1-\tau} - (\sigma_1\sigma_{2,n})^{s-1-\tau}\right) > 0.$$

But this together with (33), imply that

$$\left(\frac{\sigma_1}{\sigma_2}\right)^s < \frac{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,n})^{s-1-\tau}(\sigma_3\sigma_1)^\tau}{\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau}. \qquad (38)$$

Then it follows from condition (35), together with (33), (34), (37) and (38), that the expression in the RHS of (36) is negative if $\mu > \mu_{hb}$. Then so is the LHS in (36) and this completes the proof of the second part of (b).

Proof of (c): First, note that the RHS of (20) can be written as

$$(\beta_3\beta_{2,n})^b\left[(\sigma_3\sigma_{2n})^s - \frac{(\beta_{3,n}\beta_2)^b}{(\beta_3\beta_{2,n})^b}(\sigma_{3,n}\sigma_2)^s\right] + \frac{q_1^r}{q_3^r}(\beta_{2,n}\beta_1)^b\left[\frac{(\beta_2\beta_{1,n})^b}{(\beta_{2,n}\beta_1)^b}(\sigma_2\sigma_{1,n})^s - (\sigma_{2,n}\sigma_1)^s\right] + \frac{2q_1^r}{q_2^r}(\beta_{3,n}\beta_1)^b\left[\frac{(\beta_3\beta_{1,n})^b}{(\beta_{3,n}\beta_1)^b}(\sigma_3\sigma_{1,n})^s - (\sigma_{3,n}\sigma_1)^s\right]. \tag{39}$$

(Simply factor out the largest part of the RHS of (20) that involves βs.) Next, we divide the entire expression above by $(\beta_3\beta_{2,n})^b$:

$$\left[(\sigma_3\sigma_{2n})^s - \frac{(\beta_{3,n}\beta_2)^b}{(\beta_3\beta_{2,n})^b}(\sigma_{3,n}\sigma_2)^s\right] + \frac{q_1^r}{q_3^r}\frac{(\beta_{2,n}\beta_1)^b}{(\beta_3\beta_{2,n})^b}\left[\frac{(\beta_2\beta_{1,n})^b}{(\beta_{2,n}\beta_1)^b}(\sigma_2\sigma_{1,n})^s - (\sigma_{2,n}\sigma_1)^s\right] + \frac{2q_1^r}{q_2^r}\frac{(\beta_{3,n}\beta_1)^b}{(\beta_3\beta_{2,n})^b}\left[\frac{(\beta_3\beta_{1,n})^b}{(\beta_{3,n}\beta_1)^b}(\sigma_3\sigma_{1,n})^s - (\sigma_{3,n}\sigma_1)^s\right]. \tag{40}$$

Now let $b \to \infty$. Then since by (25) $\beta_3\beta_{2,n} > \beta_{3,n}\beta_2$ we have that the first term in (40) converges to $(\sigma_3\sigma_{2n})^s$ as $b \to \infty$. Next, since $\beta_3 > \beta_2 > \beta_1$ it follows that $\beta_{2,n}\beta_1 < \beta_{2,n}\beta_3$ and $\beta_{3,n}\beta_2 > \beta_{3,n}\beta_1$. The former, together with (25), imply that the second term in (40) vanishes as $b \to \infty$. The latter, together with (25), imply that $\beta_3\beta_{2,n} > \beta_{3,n}\beta_1$; therefore, using (25) again, the last term in (40) also vanishes. Consequently, as $b \to \infty$ the expression in (40) converges to $(\sigma_3\sigma_{2n})^s$. Since $(\sigma_3\sigma_{2n})^s > 0$ and $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t]$ has the same sign as the expression in (40), the claim in part (c) is established.

Proof of Proposition 7

By a similar set of calculations as those used to derive expression (1) (which we did for S_2, but the same manipulations yield the expression for S_i), it can be shown that for any H_t

$$\mathsf{E}[V|S_i, H_t] - \mathsf{E}[V|H_t] \propto \mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_2) + \frac{q_1^t}{q_3^t}[\mathsf{Pr}(S_i|V_2) - \mathsf{Pr}(S_i|V_1)] + \frac{2q_1^t}{q_2^t}[\mathsf{Pr}(S_i|V_3) - \mathsf{Pr}(S_i|V_1)]. \tag{41}$$

Now, if S_2's distribution is csd-monotonic in V, then it follows that

$$\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2),\ \mathsf{Pr}(S_2|V_2) - \mathsf{Pr}(S_2|V_1),\ \text{and } \mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_1)$$

have the same sign. But then, by examining the RHS of (41) with $S_i = S_2$, it follows that $\mathsf{E}[V|S_2, H_t] - \mathsf{E}[V|H_t]$ will be either always positive (if S_2's csd is increasing) or always negative (if S_2's csd is decreasing).

B Distributions Used for Numerical Computations

Figures 1 and 4 employ

$$\begin{aligned} &\mu_b^{ch} = \kappa_b/(3+\kappa_b) = 0.9496 \\ &\mu_b^{in} = \theta_b/(3+\theta_b) = 0.4294 \equiv \mu_b \\ &\mathbb{V} = (0, 10, 20), \\ &\mathsf{Pr}(V) = (1/100, 98/100, 1/100), \text{ and} \end{aligned}$$

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{601}{1000}$	$\frac{27}{100}$	0
S_2	$\frac{399}{1000}$	$\frac{18}{100}$	$\frac{245}{1000}$
S_3	0	$\frac{55}{100}$	$\frac{755}{1000}$

(42)

Figure 2 employs

$$\begin{aligned} &\mu_b^{ch} = \kappa_b/(3+\kappa_b) = 0.7656 \equiv \mu_b \\ &\mu_b^{in} = \theta_b/(3+\theta_b) = 0.9215 \\ &\mathbb{V} = (0, 10, 20), \\ &\mathsf{Pr}(V) = (1/10, 4/5, 1/10), \text{ and} \end{aligned}$$

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{40049}{49000}$	$\frac{4}{49}$	0
S_2	$\frac{8951}{49000}$	$\frac{9}{490}$	$\frac{243}{12250}$
S_3	0	$\frac{9}{10}$	$\frac{12007}{12250}$

(43)

A Four-Signal MLRP distribution that has both signals with negatively biased and U-shaped and positively-biased and U-shaped csds (so that there can be buy-and sell-herding in the same model) is given by

Define
$a = .3$,
$b = .25$,
$x = a/(2(1-a)) \approx .214$.

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$1-a-b$	$1-x/2$	0
S_2	a	x	b
S_3	b	x	a
S_4	0	$1-x/2$	$1-a-b$

(44)

A set of continuous csds that allows U-shaped or hill-shaped csds are given by the following densities for each signal $S \in [1, e] = \mathbb{S}$:

$$f(S|V_1) = \frac{1}{S}, \; f(S|V_2) = \frac{e-1}{c} + \frac{2(S-1)}{e-1}\left(\frac{1}{e-1} - \frac{e-1}{c}\right), \quad \text{and} \quad f(S|V_3) = \ln(S),$$

where e is Euler's number. These densities satisfy the MLRP for $c \in (2.38, 5.48)$. For each $c \in (2.38, 3.83)$, there is an interval in $\mathbb{S}$ so that over this interval the csd is hill-shaped. Also, for any $c \in (3.83, 5.48)$, there is an interval in $\mathbb{S}$ so that over this interval the csd is U-shaped.

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