Sequential Elimination vs. Instantaneous Voting[*]

Parimal Kanti Bag[†] Hamid Sabourian[‡] Eyal Winter[§]

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Abstract

A class of sequential elimination voting eliminating candidates one-at-a-time based on repeated ballots is superior to well-known single-round and semi-sequential elimination voting rules: if voters are strategic the former will induce the Condorcet winner in unique equilibrium, whereas the latter may fail to select it. In addition, when there is no Condorcet winner the outcome of sequential elimination voting always belongs to the 'top cycle'. The proposed sequential family includes an appropriate adaptation of almost all standard single-round voting procedures. The importance of one-by-one elimination and repeated ballots for Condorcet consistency are further emphasized by its failure for voting rules such as *plurality runoff* rule, *exhaustive ballot* method, and a variant of *instant runoff* voting. ***JEL*** Classification Numbers: P16, D71, C72. **Key Words:** Sequential elimination voting, Condorcet winner, top cycle, weakest link voting, scoring rule, exhaustive ballot, instant runoff voting, Markov equilibrium, complexity aversion.

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[†] Department of Economics, University of Surrey, Guildford, Surrey, GU2 7XH, U.K.; E-mail: p.bag@surrey.ac.uk

[‡] Faculty of Economics, University of Cambridge, Sidgwick Avenue, Cambridge CB39 DD, U.K.; E-mail: Hamid.Sabourian@econ.cam.ac.uk

[§] Economics Department, The Hebrew University of Jerusalem, Jerusalem 91905, Israel; E-mail: mseyal@mscc.huji.ac.il

1 Introduction

Any assessment of a voting rule is likely to be based on the extent it aggregates individual preferences. It is reasonable to suggest that any voting rule must be ratified by at least a majority of the members of a society if it is going to function as a democratic way of reaching collective decisions. Condorcet (1785) had first epitomized an extreme version of this view by suggesting that whenever there is a candidate (or alternative) always preferred by a majority of voters against any other candidate on pairwise comparisons, such a candidate, later to become known as the *Condorcet winner* (henceforth, CW) must be elected by the chosen voting rule. This requirement, called *Condorcet consistency*, is "widely regarded as a compelling democratic principle" (Moulin, 1988; sect. 9.4).

In this paper, we will argue that a class of voting procedures based on repeated ballots and sequential elimination is superior to ones in which the winner is determined in a single-round voting or procedures that allow more than one candidate to be eliminated in some voting round. We show that when voters behave strategically,[1] sequential, one-by-one elimination procedures often aggregate voter preferences much better.[2] Roughly, in sequential elimination (with repeated ballots) the voting outcome is determined gradually allowing voters more influence on the outcome and preventing them from getting locked in a "bad" equilibrium – a non-Condorcet outcome (when a CW exists[3]) – because of miscoordination. Even when the CW does not exist, sequential elimination procedures have a nice feature: the voting outcome always belongs to the 'top cycle,' thus dominating, on majority comparison, any other candidate either directly or indirectly. In contrast, miscoordination tends to be pervasive in single-round and various semi-sequential voting and thus may fail to possess the above desirable properties.

The idea of sequential elimination voting is best conveyed by the following repeated application of the one-person-one-vote principle. Voting takes place in

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[1] Strategic voting, also known as *sophisticated voting*, was popularized by Farquharson (1969).

[2] Sequential elimination voting in this paper differs from *sequential voting* that mostly concerns with the important issue of information aggregation, as in Dekel and Piccione (2000), Strumpf (2002), Battaglini (2005) etc. Our main concern is with *preference aggregation*, and our voters cast their votes simultaneously and repeatedly in successive rounds.

[3] For single-peaked preferences CW always exists. Moreover, for small number of alternatives a CW exists with a high probability (under random preference) – more than 90% for 3 alternatives and around 70% for 7 alternatives (Fishburn, 1973).

rounds with all the voters simultaneously casting their votes in each successive round. In any round the candidate receiving the smallest number of votes is eliminated, with any tie involving the smallest number of votes broken by a deterministic tie-breaking rule. This process continues until all but one of the candidates have been eliminated. We call this the *weakest link voting.* At present the contests for the leadership of the Conservative Party in Great Britain roughly follow the weakest link procedure.[4] Even the recent contest in 2005 to select the host city for the 2012 olympic games had the characteristics of weakest link voting (London emerged the winner after Moscow, New York, Madrid and Paris were eliminated in that order in successive votes held over four rounds).[5] One interpretation of the weakest link voting is that it is a natural sequential extension of the plurality voting principle, with elimination of only the worst plurality loser in each round.[6] Similar one-by-one sequential elimination method can be adopted to extend any familiar single-round voting rule to its appropriate sequential equivalent.

Our results are as follows. When there is a CW and voters are strategic, the unique equilibrium of a broadly defined sequential elimination voting game will select the CW (Theorems 1–3; Propositions 1, 2). Furthermore, if there is no CW, we show that the elected candidate in the appropriately defined sequential voting game must be in the top cycle (Theorem 4). The sufficient condition on the sequential voting game that ensure these results requires that any (group of) majority voters have some minimal collective influence on vote proceedings: by coordinating their votes in any round a majority can always ensure that any particular candidate who survived up to that round is not eliminated in that round; further, such vote coordinations by the majority must be "stable" in the sense that should the majority fail to choose some appropriate coordination of votes that may lead to the particular

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[4] The Conservative Party leaders in 2001 and 2005 were chosen in a two-stage voting, where the first stage basically followed the weakest link rule: the party's parliamentary members voted in successive rounds to reduce a small number of candidates, respectively five and four, to only two candidates by eliminating a candidate in each round with the least number of votes, and eventually the party members at large voted to elect the final winner from the two remaining candidates; see an explanation of the election procedure by Julian Glover in The Guardian, July 10, 2001 at $http://politics.guardian.co.uk/Print/0,3858,4196604,00.html$.

[5] See $http://news.bbc.co.uk/sport1/hi/front\_page/4655555.stm$.

[6] The weakest link voting is similar to *sequential runoff election* where alternatives are eliminated one-at-a-time but based on the voters submitting a full strict-order ranking of the remaining alternatives, eliminating in each round the alternative with the least number of first place votes.

candidate's elimination, there will be at least one member of the majority group who will have an incentive – if his aim were to protect that candidate – to further deviate by changing his vote. We call these twin requirements, the ***majority non-elimination property***.

We show that the majority non-elimination property will be satisfied by sequential versions of most familiar single-round voting procedures (the unique exception is the sequential analogue of negative voting). To understand how majority influence works, consider for instance sequential scoring rules (which eliminate, at any round, only one candidate with the lowest total score). Clearly, for any candidate and any majority, placing the candidate at the top by every member of a majority is stable; furthermore the candidate will have a total score that is strictly higher than the average score of the remaining candidates, even if every voter outside the majority places that candidate at the bottom, if the following property holds: the scores in any round for various ranks be such that the average of the two scores corresponding to the top and the bottom ranks weakly exceeds the average score for all the intermediate ranks combined (this property clearly holds for the weakest link and the sequential analogue of Borda). Thus, if this property holds the majority is able to protect the candidate from being eliminated and hence satisfies the majority non-elimination property.

In contrast, we identify a large class of instantaneous (single-round) voting rules that fail to be *Condorcet consistent*, henceforth *CC* (Theorem 5 and Proposition 3), or specific voting rules within this class may even fail to include a member in the top cycle when there is no *CW*.[7] This class includes all *scoring rules* (for example, the popular voting rules such as plurality rule, Borda rule and negative voting; see Moulin, 1988), plus approval voting, two alternative versions of *instant runoff voting*,[8] and some less well-known voting rules such as *Copeland* and *Simpson*. (Our

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[7] Roughly, *instantaneous voting* means voting takes place only once but the winner may be selected in one or more rounds of eliminations.

[8] The standard version of instant runoff voting, also known as *alternative vote method*, *full preferential voting* and *single transferrable voting* etc. (see Wikipedia, the free encyclopedia, at http://en.wikipedia.org/wiki/Instant-runoff_voting), requires voters to submit a full ranking of candidates in a single ballot. If no candidate wins a majority of the top rank, the candidate receiving the smallest number of top-rank votes gets eliminated and a fresh count is taken with the rankings rearranged. The vote counting (using transferrable votes) continues until some candidate secures a majority of the top rank. A second variant of the voting rule does not use the *majority top-rank trigger* but instead eliminates candidates sequentially, one-by-one: first eliminate the

special mention of the last two rules is motivated by an interesting observation in Moulin (1988, ch. 9) that these rules are somewhat unique in that they exhibit Condorcet consistency under *sincere voting* – a feature not shared by most standard voting rules.)

We also highlight two specific features of the sequential voting method that are important for Condorcet consistency: *one-by-one elimination* and *repeated ballots*. For the first, we provide two counter examples: (*i*) a *plurality runoff* rule eliminating all but two candidates in the first round using plurality rule and then choosing the winner in a second ballot from the remaining two candidates using majority rule, is shown to fail Condorcet consistency;[9] (*ii*) an *exhaustive ballot* method, which is same as the weakest link voting except for a *majority vote trigger* (that is, if at any round a candidate receives majority votes then that candidate immediately becomes the winner), also fails Condorcet consistency.[10] These examples are summarized in Proposition 4. For the second, in Proposition 5 we consider a one-shot version of the weakest link and show that it is not *CC*. Here repeated ballots, in contrast to one-shot games, allow backward induction type reasoning important for Condorcet consistency.[11]

Both the positive and negative results on Condorcet consistency in this paper should be viewed as taking an important issue significantly further that has only been intermittently studied in the voting literature. One strand of the literature analyzing the issue of Condorcet consistency under sophisticated voting focus on binary voting and its variants (McKelvey and Niemi, 1978; Banks, 1985; Dutta and Pattanaik, 1985; Dutta and Sen, 1993; Dutta, Jackson and Le Breton, 2002; Bernheim and Nataraj, 2004; and Bernheim et. al., 2006).[12] An alternative focus

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candidate with the least number of top-rank votes, then do vote transfers and eliminate the candidate who on recounting has the least number of top-rank votes, and so on (with possible ties broken by a deterministic tie-breaking rule).

[9] Plurality runoff rule falls between the two extremes – weakest link and plurality voting. It is also known under alternative names such as two-ballot, double ballot, second ballot, majority runoff, and two round system.

[10] This voting rule is used to select the host city for the olympic games; see http://www.gamesbids.com/english/archives/past.shtml.

[11] Similarly, the instant runoff voting without the majority top-rank trigger is also not *CC*.

[12] In binary voting voters vote in each round over only two choices and a choice may include one or more than one candidate. The voting proceeds by elimination of candidates through the voting rounds using majority rule. With voter choice in each round restricted to only two candidates is the well-known *sequential binary voting*. The last two papers consider a variant of

on the same issue makes the assumption of sincere voting (see ch. 9 of Moulin, 1988 for example). However, very little is available by way of a general characterization of *CC* voting rules under strategic voting.[13] Our first task therefore has been to bring together various voting rules under a unified analysis and closely scrutinize the Condorcet consistency question. But more importantly, we like to argue that sequential, one-by-one elimination voting methods in general deserve much greater attention than have been awarded so far.[14] In particular, in practice some of these methods may be adopted in important committee/electoral decisions where only a small number of alternatives are considered from which a single alternative is to be elected. With a small number of alternatives the costs of multi-round elections are unlikely to be of serious concern. Furthermore, while some important voting methods used in practice, such as plurality runoff rule and exhaustive ballots, do coarsely follow the principle of sequential elimination (as in the weakest link rule), these rules fail to *fully* implement one-by-one eliminations. In normative terms, our recommendation is that if possible such rules should be modified by carrying out one-by-one eliminations. In general, while the sequential voting family that we propose in this paper is large, individual voting rules can be simple/intuitive and easy to administer so that they are adaptable to a broad range of applications. However, it is worth noting that though all voting rules satisfying our suggested procedures are *CC* they do not necessarily satisfy other desirable objectives such as *Pareto efficiency* or *neutrality* (with respect to agendas),[15] so any decision regarding the choice of a voting mechanism might also consider some of these other objectives.

The rest of the paper is organized as follows. In the next section we discuss an example, followed by formal descriptions of voting rules and related equilibrium solution concepts in section 3. Section 4 contains results on sequential elimination voting. In section 5, we analyze single-round and some other popular voting mech-

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sequential binary voting that involve dynamic legislative decision making with real-time agenda-setting and/or discounting.

[13] Some papers (Bernheim and Nataraj 2004 and Benheim et. al. 2006) rely on a behavioral assumption – as-if pivotal voting by the voters – to cut through the usual strategic coordination problem of voting games.

[14] Sequential binary voting is an exception, which is well-studied. It is well known that such voting rule is *CC*. Our analysis here shows that the two critical properties of the sequential binary voting that ensure Condorcet consistency are the one-by-one elimination and repeated ballots.

[15] For instance, weakest link voting is neutral but not necessarily Pareto efficient (with more than three candidates) while sequential binary voting is Pareto efficient but certainly not neutral.

anisms. Section 6 concludes. The proofs not included in the main body of the paper appear in the Appendix A–C and the *Supplemental material.*

2 Preliminary observations on the weakest link

Much of our insight about sequential elimination voting versus one-shot voting can be gained by studying the *weakest link voting* and comparing against its one-shot counterpart. This we do with the help of an example.

Example. Consider three voters, and four candidates: $\{w, x, y, z\}$. The voters' strict preferences over candidates (in a descending order) are as follows:

$$\begin{aligned} 1 &: y, x, z, w \\ 2 &: z, x, y, w \\ 3 &: w, x, z, y. \end{aligned}$$

Note that candidate x is the CW.

Consider first the case of plurality voting with a fixed deterministic tie-breaking rule. Then any candidate x, y and z can be the winner in some (weakly) undominated Nash equilibrium. To see this, note that voting for any candidate other than the one lowest in one's ranking is a weakly undominated strategy. Thus, for example, one undominated Nash equilibrium is voters 1 and 2 choose y and voter 3 chooses z; this results in y being elected. Similarly, one can construct equilibria that result in x or z being elected (w can also be a winner if we assume further that the tie-breaking rule eliminates x first). Thus, plurality voting is not CC because there is a coordination problem (in terms of selecting the CW) among the voters that cannot be resolved in equilibrium.[16]

Consider next the weakest link voting. Before characterizing the strategic equilibrium outcome, suppose first that all three voters vote *sincerely* so that at each voting stage each votes for his most preferred candidate among the surviving ones. Clearly, x will be eliminated at the first stage (regardless of the tie-breaking rule) and thus cannot be the winner.

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[16] This is hardly surprising – Dhillon and Lockwood (2004) have shown this applying iterative deletion of weakly dominated strategies.

In contrast, assume now that the three voters behave strategically.[17] Without defining formally a solution concept at this stage, we will argue informally why strategic behavior must result in x winning. We will invoke a backward induction argument: Consider first the last stage in which only two candidates remain. If x is one of these candidates then clearly x will win at this stage because two out of three voters prefer x to the other candidate (and for them voting for x weakly dominates voting for the other candidate). Similar argument implies that if the two candidates at the last stage are (z, w), then z wins. If they are (z, y) then z wins as well and for (y, w) candidate y emerges the winner. Consider now the second-to-last stage of voting: If the three surviving candidates at this stage are (y, z, w), then z must emerge the winner. This is because given the possible continuations (as specified above), if z survives this stage he will become the eventual winner and otherwise y will be the winner. So, voting for z is the only (weakly) undominated choice for voters 2 and 3. Hence z will not be eliminated at this stage and will emerge the winner. A similar argument implies that if x is one of the remaining candidates at the second-to-last stage then he must emerge the eventual winner. We now consider the first stage of the voting. Given the possible continuations as described above, if x is eliminated at the first stage then z will emerge the winner, whereas x will emerge the winner if any of the other three candidates is eliminated. Hence, the only weakly undominated choice for voters 1 and 3 is to vote for x. Thus x will survive and will emerge the winner.

In contrast to plurality voting, in the weakest link by working backwards in each stage the voters are able to coordinate their votes and ensure that the CW is never eliminated. The voters' ability to coordinate in the weakest link derives from both the extensive form, sequential structure of the game as well as the power of the equilibrium refinement (based on backward inductions) associated with the dynamic structure of the game.[18]

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[17] Strategic/non-sincere voting is quite common: in the 2005 election to host the 2012 olympic games and also for the leadership contests for the Conservative party in 2001 and 2005, there were instances of vote dropping (i.e. number of votes in favor of particular candidates in later rounds, with fewer candidates remaining, were less than in earlier rounds).

[18] To see this, further consider a one-shot game in which voters submit at the outset their entire weakest link strategies (that maps from the set of available candidates in any elimination round to a single candidate for non-elimination in that round) and then the winner is determined according to the weakest link rule. In this one-shot weakest link voting, backward inductions type reasoning cannot be applied (is not an appropriate solution concept) and as a result the CW may fail to be

Before leaving this section, we like to mention that this example is rather special because all the candidates can be majority-ranked (therefore any subset of candidates has a CW). Nevertheless, it highlights the advantage of a voting procedure in which voters are called to step in repeatedly in submitting their preferences over ones that involve a single stage. As we will show later, this advantage goes beyond the weakest link rule. Indeed we will introduce a large class of sequential elimination procedures which guarantee the selection of the CW in strategic voting and will also argue that almost all well-known single-round voting procedures fail to have this property. $||$

3 The Voting Rules and Equilibrium Solutions

Voting Games

First we describe the class of voting games considered in this paper. This class is quite general.

The set of candidates is denoted as $\mathcal{K}$ with cardinality k, and the voter set is denoted as $\mathcal{N}$ with cardinality n, both k and n at least three. Throughout we assume n to be an odd number, but this can be relaxed (see footnote 22). Also for simplicity of exposition, $\mathcal{K} \cap \mathcal{N} = \emptyset$. Each voter $i \in \mathcal{N}$ has a strict, ordinal preference ordering over the candidates given by $\succ_i$.

The class of voting games we consider are as follows. Each voting rule consists of the voters/players voting in at most J rounds/stages, $J < k$. At each stage the voters simultaneously vote (i.e., take an action) and at least one candidate is removed. At the end of a maximum of J rounds voting one candidate survives who is the winner. If C is the set of candidates left at any stage $j \leq J$ with $|C| \geq 2$ ($|.|$ denoting cardinality) then a choice for voter i at that stage consists of choosing an element from an arbitrary choice set $A_i(C, j)$. Moreover, if each i chooses $a_i \in A_i(C, j)$ at this stage then we shall denote the set of eliminated candidate(s) by $e(a^j, C) \in C$ where $a^j = (a_1, ..., a_n)$ is the profile of votes at stage j. So if the voting finishes in some $\mathcal{J} \leq J$ rounds and voters choose the sequence of votes $\{a^j\}_{j=1}^{\mathcal{J}}$, then the winning candidate is $w \notin \cup_{j=1}^{\mathcal{J}} e(a^j, C)$.

For any $j \leq J$ let $h^j = (a^1, ..., a^{j-1})$ be a complete history (description) of the

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elected – see Proposition 5 for a replicated version of the above example (with two voters of each type).

actual voting decisions up to stage j. Define $\mathcal{H}^j$ to be the set of histories at round j and $\mathcal{H} = \bigcup_j \mathcal{H}^j$ be the set of all histories, with the convention that $\mathcal{H}^0$ refers to the initial null history. Also, let $C(h)$ be the set of remaining candidates at $h \in \mathcal{H}$.

Now a (pure) strategy for voter i is a function $s_i : \mathcal{H} \to \bigcup_{j,C} A_i(C, j)$ such that $s_i(h) \in A_i(C(h), j)$ if $h \in \mathcal{H}^j$. Also, denote the set of (pure) strategies of voter i by S_i and let $S = \times_i S_i$.

The above set of games clearly includes the weakest link voting, and more generally any sequential (elimination) voting, and any single-round voting such as plurality rule, approval voting, Borda voting and negative voting. In the case of the weakest link, the number of voting rounds J is $k-1$, the set of choices $A_i(C, j) = C$ and at each stage one candidate is eliminated so that $|e(a^j, C)| = 1$.

In general, *sequential elimination voting* rules are such that at each stage only one candidate is eliminated.

In the case of single-round voting, $J = 1$, all voters submit their strategies at the first stage and all the candidates except one are eliminated simultaneously.

Also, included in our voting games will be three other categories: one with $J > 1$ but if (and only if) at any round a candidate gets majority votes he is immediately declared the winner ending any further ballot (exhaustive ballot, for example); a second one that eliminates candidates in one or more attempts following a single ballot (so that $J = 1$, as in the case of instant runoff voting); finally, voting involving repeated ballots ($1 < J < k-1$) and more than one candidate being eliminated in some round (such as plurality runoff voting with $J = 2$).

The equilibrium

Since the voting games we consider may have a dynamic structure, we require our equilibrium concept to be subgame-perfect. In addition, as is common in the literature on voting, we need to eliminate choices that are weakly dominated, otherwise there are a large number of trivial equilibria in which each voter's choice is immaterial. Therefore, an equilibrium in our set-up is a strategy profile for the voters that is a subgame perfect equilibrium and is such that at each stage the votes of each player is not weakly dominated given the equilibrium continuation strategies of others in future stages.

In other words, any equilibrium strategy profile $s^* \in S$ in a voting game must have the following properties. In any final stage subgame (i.e., at stage J), s^* must be a weakly undominated Nash equilibrium in the subgame. In any subgame starting with stage $J-1$, the voters' strategies must be an undominated Nash equilibrium in the subgame given that the voters play the game according to s^* in the continuation game (thus the *permissible strategies* of the other voters with respect to which the weak-domination check is carried out are consistent with the equilibrium strategy in the next stage). This backward elimination procedure continues all the way to stage 1.

Formally, for any history $h \in \mathcal{H}$, let $\Gamma(h)$ be the subgame at h and $w(s,h)$ be the candidate elected in the subgame $\Gamma(h)$ if the voters follow strategy profile s in this subgame. Also, for any strategy profile $s \in S$ and any history $h \in \mathcal{H}$, define the set of strategies for all players other than i that are consistent with s in every subgame after h by

$$\widetilde{S}_{-i}(h,s) = \{s'_{-i} \in S_{-i} \mid s'_{-i}(h,h') = s_{-i}(h,h') \text{ for all non-empty } h' \text{ s.t. } (h,h') \in \mathcal{H}\}.$$

Definition 1. *A strategy profile s^* is an equilibrium if for any history $h \in \mathcal{H}$ it satisfies the following properties in the subagme $\Gamma(h)$:*

$$\left.\begin{array}{rl} \textit{(Nash)} \quad \textit{For any } i, \quad w(s^*,h) & \succeq_i \quad w(s_i, s^*_{-i}, h) \quad \forall s_i \in S_i, \\ & \textit{where } \succeq_i \textit{ means either } \succ_i \textit{ or } =; \\ \textit{(Weak non-domination)} \quad \textit{For any } i, & \nexists s_i \in S_i \textit{ s.t.} \\ w(s_i, s_{-i}, h) \succeq_i w(s^*_i, s_{-i}, h) & \forall s_{-i} \in \widetilde{S}_{-i}(h,s^*) \\ w(s_i, s_{-i}, h) \succ_i w(s^*_i, s_{-i}, h) & \textit{for some } s_{-i} \in \widetilde{S}_{-i}(h,s^*). \end{array}\right\} \quad (1)$$

Notice that for any $s \in S$, at any $h \in \mathcal{H}^j$ we can define a one-shot reduced form voting game $\widehat{\Gamma}(h,s)$ in which voter i's strategy set is $A_i(C(h),j)$ and, given any profile $a^j \in A(C(h),j)$ $(= \prod_i A_i(C(h),j))$ of votes, the outcome of the game is given by $w(s,(h,a^j))$ elected. Clearly, our definition of equilibrium strategy in Definition 1 is equivalent to showing that the choices that the equilibrium strategies prescribe at any history h constitute an undominated Nash equilibrium of the one-shot reduced voting game at h. Thus, s^* *is an equilibrium if and only if* $s^*(h)$ *is an undominated Nash equilibrium of* $\widehat{\Gamma}(h,s^*)$, for all h.

Remark 1. *Our equilibrium concept is effectively a backward elimination procedure. However, note that it differs from the more familiar procedure of iterative elimination of (weakly) dominated strategies; while in the latter approach the weak-domination check is carried out in relation to the entire game, ours is only along* *the subgames.*[19,20] *Iterative elimination on its own is unlikely to solve the miscoordination problems that result in undesirable outcomes; in fact, it is well known in other voting contexts that iterative elimination can have very little elimination power.*[21]

Remark 2. *Note also that any trembling hand perfect equilibrium in extensive form satisfies our definition of equilibrium. This is because any trembling hand perfect equilibrium in extensive form is a subgame perfect equilibrium and excludes weakly dominated choices at different information sets. We could have alternatively started with trembling hand perfect equilibrium in extensive form as our equilibrium concept (see also our remark at the end of subsection 4.1). However, for ease of exposition we adopt the above definition of equilibrium.*

Remark 3. *In the case of single-round voting the standard equilibrium concept is* undominated Nash. *Note that our twin requirements of subgame perfection and non-domination boil down to this standard equilibrium definition for single-round voting rules. Thus, the comparisons to be made in section 5 between sequential elimination voting and single-round voting are based on the same benchmark solution concept.*

Next we define Markov equilibrium.

Definition 2. *An equilibrium s^* is said to be Markov if for any i and any j,*

$$s_i^*(h) = s_i^*(h') \quad \forall h, h' \in \mathcal{H}^j \text{ such that } C(h) = C(h').$$

Markov equilibrium strategies are such that at any stage onwards the strategies depend only on the candidates who have survived up to that stage and *not* on the specific history leading up to it. In Appendix B, we justify the use of the Markov assumption for our sequential elimination voting.

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[19] Moulin (1979) formally analyzed the iterative elimination procedure to generalize the concept of sophisticated voting and applied it to a significant class of voting – voting by veto, kingmaker and voting by binary choices.

[20] In our setup the two definitions may differ because at each stage our voters vote simultaneously (the game is not one of perfect information) over more than two alternatives.

[21] For example, for plurality rule Dhillon and Lockwood (2004) show that *anything* other than one's lowest-ranked candidate will survive iterative eliminations of weakly dominated strategies.

4 Sequential (elimination) voting

4.1 Condorcet consistency of the weakest link

Much of our insight about sequential elimination voting can be gained by studying the *weakest link voting*, so we start with this particular voting rule and then broaden our analysis to a very general class of sequential elimination voting.

First, some notations. Given the voters' strict preference ordering over candidates, a binary comparison operator T defines a candidate x to be *majority preferred* over another candidate y, written as xTy, if the number of voters preferring x over y exceeds the number of voters preferring y over x.[22]

Next, the CW, if it exists, is defined as a candidate $z \in \mathcal{K}$ such that zTz', for all $z' \in \mathcal{K}$. Similarly, for any set of remaining candidates $C \subseteq \mathcal{K}$ the CW with respect to C, if it exists, is a candidate $z \in C$ such that zTz' for all $z' \in C$.

We say that an equilibrium s^* of a voting rule is CC at every subgame if for every $h \in \mathcal{H}$ such that the set of remaining candidates $C(h)$ has a CW winner $z(h)$, the equilibrium strategy induces the CW with respect to $C(h)$ in the subgame defined by h (i.e. $w(s^*, h) = z(h)$ if $z(h)$ is defined for h).

Our first result is an equilibrium characterization of the weakest link game:

Theorem 1. *Any Markov equilibrium of the weakest link voting is* CC *at every subgame.*

Proof. We demonstrate this by (backward) inductions on the number of remaining candidates in any subgame.

First, consider any subgame at stage $k-1$ with only two candidates, z and z'. Because sincere voting is the only Nash equilibrium that is also undominated in this final stage subgame, the CW must be the winner.

Now suppose the following induction hypothesis is true:

For every history $h \in \mathcal{H}$ such that the set of remaining candidates $C(h)$ consists of j candidates, the following holds: *if $C(h)$ has a CW, z, then z will become the ultimate winner in the subgame defined by h* (i.e., $w(s^*, h) = z$).

We then prove that the same holds at any history/subgame with $j+1$ remaining candidates.

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[22] To relax the assumption of *odd* number of voters, extend the definition of majority preference, whenever there is a tie, by applying a tie-breaker.

Suppose not; then there exists a subgame defined by some history h such that the set of the remaining candidates $C(h)$ has $j + 1$ candidates, $C(h)$ has a CW, z, and some other candidate $z' \neq z$ becomes the ultimate winner in this subgame. Now since z is the CW with respect to $C(h)$, it follows by the induction hypothesis that z is eliminated immediately at h at stage $k - j$ (since at h there are $j + 1$ candidates, the subgame defined by h begins in round $k - j$). Otherwise, since z is also the CW with respect to the set of candidates in the next round, by the hypothesis he will become the ultimate winner.

Next, consider those voters who prefer z over z' and their immediate vote at h in stage $k - j$. By definition of z, these voters will form a majority. We claim that for any such voter i, voting for z weakly dominates voting for any other candidate z'' at this stage, given the equilibrium continuation strategies in the future stages. To show this, first notice that if voter i chooses z'' there are two possible outcomes depending on the choices of others at this stage: either (i) z survives at this stage and, by the induction hypothesis, all the subsequent stages and becomes the ultimate winner; or (ii) z is eliminated and, by the Markov property of the equilibrium strategies, z' becomes the ultimate winner. Now if (i) is the case then if i switches his vote from z'' to z the outcome will be the same with z surviving all stages and becoming the winner. If (ii) is the case then if i switches his vote from z'' to z, either z is eliminated and the outcome will be the same with z' becoming the ultimate winner or z survives this stage, and by the induction hypothesis, all the subsequent stages and becomes the ultimate winner. Finally, note that there is a vote profile for all voters other than i (for example, $(n-1)/2$ of them vote for z'' and the remaining $(n-1)/2$ voters vote for z) such that if voter i votes for z'' then z would be eliminated and z' goes on to win whereas if he votes for z then z is not eliminated and z wins. Since voter i prefers z to z', the choice of z thus weakly dominates z'' for i. This implies a majority of voters would vote for z, contradicting the supposition that z is eliminated at this stage.

Since we already proved our hypothesis for subgames with two candidates, it follows by the induction step above that if there is a CW for the set C, he will be elected in any subgame with C. **Q.E.D.**

The above result is a characterization result for Markov equilibria of the weakest link voting when the set of (remaining) candidates has a CW. However, in order to ensure that the result is not vacuous one has to show that the weakest link game has a (Markov) equilibrium. This is particularly important because even if a set of candidates has a CW, there could be subgames off-the-equilibrium path without a CW among the remaining candidates and it is by no means clear that there is an equilibrium in such subgames. Thus, Theorem 1 should be viewed in combination with Theorem 2 below.[23]

Theorem 2. *Assume* $n \geq 2k - 1$*. Then in the weakest link game there exists a Markov equilibrium.*

The proof of this result is rather technical and can be found in Appendix A.

There are several further points to note concerning the characterization result in Theorem 1. First, notice that the arguments in the proof of this result does not make any reference to the tie-breaking rule; thus the weakest link voting is CC for any arbitrary deterministic tie-breaking rule. Also, if the preferences of the voters can be represented using expected utility framework then by an analogous argument one can show that Theorem 1 holds for random tie-breaking rules.

Second, as we mentioned before, the weakest link is the sequential analogue of (one-shot) plurality voting. The latter will be later shown to fail Condorcet consistency (Theorem 5; Proposition 3). Thus, Theorem 1 illustrates the distinct advantage of the sequential elimination procedure over a single-round elimination. Later, based on Theorem 3, similar parallels can be made between other well-known single-round voting rules and their sequential counterparts.

Third, limiting the result to equilibria that are Markov could be considered a limitation of Theorem 1. However, there are two points that we like to make with respect to the Markov restriction. First, a weaker version of the Markov property would suffice for the proof of Theorem 1. All we require to obtain the result is that the equilibrium strategies do not depend on the history through the specific configuration of votes that lead to the particular candidates' eliminations. However, the strategies can still depend on the order in which the candidates are eliminated. In fact, if we assume that the votes are not revealed between stages but only the

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[23] Since we wrote an earlier version of this paper (available under a different title: Bag, Sabourian and Winter, 2002), we recently came across Peress (2004) who also examines the issue of Condorcet consistency using the weakest link (that he calls multistage runoff) but under a very restrictive assumption that every subset of candidates has a CW (all candidates can be majority ranked). In particular, he does not need to consider the possibility that off-equilibrium subgames may not have a CW. This makes the required analysis in Peress (2004) much simpler. Also, his equilibrium concept seems to have similarity with ours but is not clearly defined.

identity of the eliminated candidate at each stage is announced, then we do not need the Markov property. Second, it could be shown that if, in choosing the strategies, players have, at least at the margin (lexicographically), a preference for simplicity (aversion to complexity) then all equilibria are Markov.[24] Basic reason is that in our sequential voting games, for any equilibrium strategy profile every set of remaining candidates occur on the equilibrium path at most once. If any player i's strategy is non-Markov, then i makes a different choice at two different subgames with the same set of remaining candidates C; but then since C occurs at most once on the equilibrium path, player i could economize on complexity by always making the same choice at every subgame with C without sacrificing payoffs. In Appendix B (Theorem 6) we provide a formal justification for this claim for the general sequential elimination voting game.

Finally, as discussed after the equilibrium definition, since every trembling hand perfect equilibrium in extensive form satisfies our equilibrium concept, it follows that every Markov trembling hand perfect equilibrium in extensive form of the weakest link voting is CC at every subgame. $||$

4.2 Sequential elimination with majority property

Next we generalize the Condorcet consistency result in Theorem 1 to a rather general *sequential* process of elimination where in each round only one candidate is eliminated. In these games, as we mentioned before, players vote in $k-1$ rounds, the set of votes for voter i at round $j < k$ when C is the set of remaining candidates is $A_i(C, j)$, and one candidate $e(a^j, C)$ is eliminated at each round j.

An important aspect of this procedure would be the decisive role that any group of majority voters can play: at any round a majority of voters can ensure that any particular candidate is *not* eliminated. We now specify this important property for the set of sequential (elimination) voting games as follows.

Majority non-elimination* (MNE) *property: *For any stage $j < k$, any set of remaining candidates C, any $c \in C$, and any set of majority voters $\phi \subseteq \mathcal{N}$, there exists a set of strategy profiles $\mathcal{D}^c_\phi(C, j) \subseteq \Pi_{i\in\phi} A_i(C, j)$ for the majority ϕ such that the following two conditions hold:*

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[24] Properties of Markov equilibrium in general dynamic games have been studied by Chatterjee and Sabourian (2000), Sabourian (2004), and Gale and Sabourian (2005).

[i] **(Majority protection)** *If all members of ϕ choose some profile $a_\phi \in \mathcal{D}^c_\phi(C, j)$ then c is not eliminated, i.e.,*

$$e(a_\phi, a_{-\phi}, C) \neq c, \; \forall a_{-\phi} \in \Pi_{\ell \notin \phi} A_\ell(C, j).$$

[ii] **(Protection stability)** *For any profile $a_\phi \notin \mathcal{D}^c_\phi(C, j)$ such that $e(a_\phi, a_{-\phi}, C) = c$ for some $a_{-\phi} \in \Pi_{\ell \notin \phi} A(C, j)$, there exists some member of the majority $i \in \phi$ and an action $a^c_i \in A_i(C, j)$ such that*

$$\forall a'_{-i} \in A_{-i}(C, j) \; \text{ if } \; e(a_i, a'_{-i}, C) \neq c \; \text{ then } \; e(a^c_i, a'_{-i}, C) \neq c \tag{2}$$

$$\textit{and} \;\; \exists a'_{-i} \in A_{-i}(C, j) \; \text{ s.t. } \; e(a_i, a'_{-i}, C) = c \; \text{ and } \; e(a^c_i, a'_{-i}, C) \neq c. \tag{3}$$

That is, a_i is "inferior" to a^c_i in protecting c.

All sequential voting rules satisfying these two non-elimination conditions constitute the family $\mathcal{F}$. ||

Note that $\{\mathcal{D}^c_\phi(C, j)\}$ are sets of actions/votes for non-elimination of any candidate c. For instance, if each stage of the sequential voting involves voters ranking the candidates, one can think of $\{\mathcal{D}^c_\phi(C, j)\}$ as all actions by the majority that place c at the top of their ranking; then the two conditions in the **MNE**-property require that [i] if a majority of voters place c at the top then c cannot be eliminated, and [ii] if a majority fails to place c at the top and c is eliminated then there is some voter from that majority who will have an action that is (weakly) better than his particular action in the 'failed majority action profile' in protecting c. Later we will verify that ***sequential extensions*** of approval voting and a class of ***scoring voting rules*** (that includes plurality and Borda rules as special cases) plus Copeland and Simpson rules (see Moulin, 1988, ch. 9 for the last three voting rules) fall under the family $\mathcal{F}$. Further, it can be checked that the important class of ***sequential binary voting*** comes under $\mathcal{F}$.[25]

Theorem 3. *All Markov equilibria of any sequential voting rule in the family $\mathcal{F}$ are* CC *at every subgame.*

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[25] Note that our sequential elimination voting is quite general in the sense that voters can submit a (weak or strict) ranking, or the preference submission may even be more abstract than a simple ranking of candidates.

Proof. Go back to the proof of Theorem 1. It is not difficult to see that the arguments there will apply equally for the entire family $\mathcal{F}$. In particular, fix any voting game belonging to the set $\mathcal{F}$ and any Markov equilibrium. Assume the hypothesis for the case of j candidates: if a set of candidates C has j candidates and a CW then he will be the ultimate winner in any subgame with the set of remaining candidates C. As in Theorem 1 we can then show that the same holds for any set of $j+1$ candidates with subgame starting at stage $k-j$. Otherwise, there exists a set $\tilde{C}$ with $j+1$ candidates at some subgame at stage $k-j$ such that $\tilde{C}$ has a CW, z, and $z' \neq z$ is elected in this subgame. Then z is eliminated immediately in this subgame at stage $k-j$ (otherwise, by the induction hypothesis, z will become the ultimate winner). Moreover, since z is the CW with respect to $\tilde{C}$, there exists a set of majority voters $\phi \subseteq \mathcal{N}$ who prefer z over z'. Now by the **MNE**-property, there exists sets of action profiles $\mathcal{D}^z_\phi(\tilde{C}, k-j) \subseteq \Pi_{i\in\phi} A_i(\tilde{C}, k-j)$ satisfying conditions *[i]* and *[ii]* (of the non-elimination property).

Next suppose the voters choose a profile $a \in A(\tilde{C}, k-j)$ at stage $k-j$ in this subgame. Then since at this stage z is eliminated it follows from condition *[i]* of the **MNE**-property that $a_\phi \notin \mathcal{D}^z_\phi(\tilde{C}, k-j)$. Then since $e(a, \tilde{C}) = z$ by condition *[ii]* of the **MNE**-property, it follows that there exists a voter $i \in \phi$ and an action $a^z_i \in A_i(\tilde{C}, k-j)$ such that conditions (2) and (3) hold for the case that $c = z$.

We now claim that the vote a^z_i weakly dominates a_i at this stage $k-j$ given the equilibrium continuation strategies in the future stages. To show this first note that if voter i chooses a_i there are two possible outcomes depending on the choices of others at this stage: either ***[1]*** z survives at this stage and, by the induction hypothesis, all the subsequent stages and becomes the ultimate winner; or ***[2]*** z is eliminated and, by the Markov property of the equilibrium strategies, z' becomes the ultimate winner.

Now if *[1]* is the case then if the voter switches his vote from a_i to a^z_i, the outcome will be the same with z surviving all stages and becoming the winner; this is because by (2) in condition *[ii]* of the **MNE**-property $e(a_i, a_{-i}, \tilde{C}) \neq z \Rightarrow e(a^z_i, a_{-i}, \tilde{C}) \neq z$, for any $a_{-i} \in A_{-i}(\tilde{C}, k-j)$.

If *[2]* is the case then if the voter switches his vote from a_i to a^z_i, either z is eliminated and the outcome is the same with z' becoming the ultimate winner or z survives this stage, and by the induction hypothesis, all the subsequent stages and becomes the ultimate winner. Furthermore, the switch will ensure the latter (z surviving and becoming the ultimate winner) in some situation because, by (3) in condition *[ii]* of the **MNE**-property, there is some $a_{-i} \in A_{-i}(\tilde{C}, k-j)$ such that $e(a_i, a_{-i}, \tilde{C}) = z$ (so that z' would have been the eventual winner) and yet $e(a_i^z, a_{-i}, \tilde{C}) \neq z$ (so that z is the winner).

Since voter i prefers z to z', the choice a_i^z weakly dominates a_i. But this is a contradiction; hence the induction hypothesis holds when there are $j+1$ candidates.

By a similar argument as above, it is easy to check that the hypothesis is also true for $j = 2$. Therefore, by induction it follows that at any subgame if the set of remaining candidates has a *CW*, then he will be elected in the subgame. **Q.E.D.**

Two brief remarks at this stage. First, our Condorcet consistency result, and more generally the top-cycle result to be stated in Theorem 4, do not require strategies to be Nash as part of the equilibrium definition; we impose the Nash requirement mainly to make the equilibrium definition consistent with the non-sequential voting games of section 5 and a related negative result in Theorem 5. Second, the Markov assumption for Theorem 2, like Theorem 1, is justifiable as we have argued before (see Theorem 6 in Appendix B). Also, as in Theorem 1, any Markov trembling hand perfect equilibrium in extensive form is also *CC*.

The scope of $\mathcal{F}$. To fully appreciate Theorem 3, it is important that we further elaborate the scope of the voting family $\mathcal{F}$. First consider *scoring rules*.

Definition 3. (***Scoring voting rules*** [***Moulin, 1988, ch.9***]) *Fix a nondecreasing sequence of real numbers $\varsigma_1 \leq \varsigma_2 \leq \ldots \leq \varsigma_k$ with $\varsigma_1 < \varsigma_k$. Voters rank the candidates, thus giving ς_1 score to the one ranked last, ς_2 to the one ranked next to last, and so on. A candidate with a maximal total score is elected.*

Thus, there are k ranks and the rankings are not necessarily strict.

Definition 4. (***Sequential scoring rule***) *A sequential scoring rule is the sequential, one-by-one elimination analogue of scoring rules:*

- *At any stage and for any set of remaining $J \leq k$ candidates, fix a nondecreasing sequence of real numbers $\varsigma_1 \leq \varsigma_2 \leq \cdots \leq \varsigma_J$ with $\varsigma_1 < \varsigma_J$.*
- *At the particular stage, voters rank the candidates according to the above sequence, and the candidate receiving the lowest total score is eliminated.*

Proposition 1. *Any sequential scoring rule belongs to the family $\mathcal{F}$, if at each stage the scores associated with different ranks are such that*

$$\frac{1}{2}(\varsigma_1 + \varsigma_J) \geq \frac{1}{(J-2)} \sum_{j=2}^{J-1} \varsigma_j. \tag{4}$$

Condition (4) implies that if any majority voters place a candidate c at the top and the remaining voters place c at the bottom then the resulting total score of c can never be the lowest (exceeds the average score of the other candidates). Therefore, this condition ensures that c is not eliminated, irrespective of what others do, and thus the set of actions by a majority that place a candidate at the top satisfy majority protection and hence the **MNE**-property (protection stability is also satisfied by any strategy that does not place c at the top because it is then always possible to protect c better by improving its ranking). In fact, the **MNE**-property cannot be guaranteed with sequential scoring if condition (4) were not to hold.

Both plurality and Borda rules satisfy (4), thus the corresponding sequential extensions – the weakest link and sequential Borda rules – satisfy the **MNE**-property. However, the negative voting with $\varsigma_1 = 0$ and $\varsigma_j = 1$ for all $j > 1$ would fail (4). Moreover, one can show that its sequential extension – *sequential veto* rule (in each stage each voter vetoes one candidate and the one receiving the maximum number of vetoes is eliminated) – fails the **MNE**-property. This is because a majority of voters may not always be able to guarantee non-elimination of a candidate c by giving it the maximum point, 1: The only way to ensure non-elimination of c is for the majority to coordinate to veto some other candidate(s) other than c; but this may violate *protection stability* condition because strategies that do not coordinate on vetoing some other candidate(s) need not be inferior in protecting the particular candidate c.

Three other one-shot rules (not part of scoring rules) – approval voting, Copeland rule and Simpson rule – have similar sequential extensions with the Condorcet consistency property, as summarized below.[26]

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[26] In approval voting, a voter is allowed to approve or disapprove any number of candidates (point 1 to indicate approval of a candidate and point 0 to denote disapproval) except that the voter cannot approve all or disapprove all the candidates. The candidate with maximal votes wins (see Brams and Fishburn, 1978, and Myerson, 2002).

Copeland and Simpson rules (Moulin 1988, ch. 9) are based on voters submitting only strict

Proposition 2. *The sequential extension of approval, Copeland and Simpson voting rules belongs to the family $\mathcal{F}$.*

The proof of Proposition 1 appears in Appendix C. Proposition 2 proof is very similar and omitted (see also footnote 39).

Arbitrary voter preferences including no *CW*

So far our analysis is based on the assumption that a CW exists. The structure of equilibrium in the absence of a CW should be of interest. The next result applies to the sequential family $\mathcal{F}$, with or without a CW.

Before stating our result let us define for any set of candidates $C \subseteq \mathcal{K}$ the set of *top cycle* with respect to C by

$$\begin{aligned} \mathcal{TC}(C) &= \{x \in C : \forall y \in C, y \neq x, \text{ either } xTy \\ \text{or there exist } x_1, x_2, \ldots, x_\tau &\in C \text{ candidates such that } xTx_1T\ldots Tx_\tau Ty\} \end{aligned}$$

where, as before, T is the binary operator representing majority preference. We also refer to $\mathcal{TC}(\mathcal{K})$ simply by the top cycle.

Theorem 4. *In all Markov equilibria of any sequential voting rule in the family $\mathcal{F}$, candidate w is the winner in any subgame with remaining candidates C only if $w \in \mathcal{TC}(C)$.*

Theorem 4 is clearly a generalization of Theorem 3. We choose this exposition for ease of presentation.

For binary voting trees (see footnote 12), McKelvey and Niemi (1978) had shown a similar result. McKelvey and Niemi's equilibrium, that they call multistage sophisticated solution, is similar in spirit to Farquharson's (1969) sophisticated solution: presenting the voting game as a tree of binary choices and treating each

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order rankings (so that $J = k$). For Copeland rule, candidate a, compared with another candidate b, is assigned a score $+1$ if a majority prefers a to b, -1 if a majority prefers b to a, and 0 if it is a tie. Summing up the scores over all $b, b \neq a$, yields the Copeland score of a. A candidate with the highest such score, called a *Copeland winner*, is elected. For Simpson rule, for candidate a denote by $N(a, b)$ the number of voters preferring a to another candidate b. The Simpson score of a is the minimum of $N(a, b)$ over all $b, b \neq a$. A candidate with the highest such score, called a *Simpson winner*, is elected.

Sequential extensions of the above voting rules would eliminate, at any round, the candidate that receives the lowest score, applying a tie-breaker wherever necessary.

decision node with its specific binary choices as a constituent game, McKelvey and Niemi solve recursively the various constituent stage games backwards using elimination of weakly dominated strategies. Our sequential, one-by-one elimination voting family is inherently different from the class of binary voting games (with the exception of sequential binary voting) studied by McKelvey and Niemi. In particular, in our framework with more than two remaining candidates voters may have more than two choices. Furthermore, binary voting procedures may involve multiple candidates being eliminated in a single stage and it can even happen that the ultimate winner is determined in the first stage.[27]

5 Single-round and some other voting mechanisms

In this section we look at voting rules that differ from the sequential family $\mathcal{F}$ in two important respects: either (1) the elimination of candidates is *not* one-by-one, or (2) the elimination which may even be sequential is through a *single ballot*, or both. This complementary class[28] includes all single-round voting, a plurality runoff rule, the exhaustive ballot method, instant runoff voting, etc. We shall examine the Condorcet consistency property (or the lack of it) of this complementary class.

First we define a class of single-round voting rules. For any set of candidates $\mathcal{K}$ with cardinality k (as defined in section 3), the set of strategies for a voter is to rank the k candidates in J different categories for some J such that $1 < J \leq k$ subject to some bounds on the number of candidates in each category. Denote the minimum and the maximum number of candidates in each category $j \leq J$ by $m(j)$ and $M(j)$, respectively. Let Λ be the set of all such J rankings over $\mathcal{K}$. Thus, the strategy for voter i, denoted by $R_i \in \Lambda$, is a profile $(X_1, ..., X_J)$ with J components such that it partitions the set $\mathcal{K}$ into J non-empty cells $X_1, ..., X_J$ and $m(j) \leq |X_j| \leq M(j)$. Since it is a partition it must be the case that $\sum_j m(j) \leq k$. From R_i we can also specify for each $x, y \in \mathcal{K}$ whether x is ranked strictly above y, denoted by $x \, P_i \, y$, or y is ranked strictly above x, denoted by $y \, P_i \, x$, or x is ranked the same as y (in

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[27] McKelvey and Niemi do not require the Markov assumption because of the binary nature of choices at every decision node. The equilibrium in any continuation game of their binary voting following elimination of the CW is essentially unique. For more than two choices possible (as in our case), the uniqueness can be guaranteed only by assuming the Markov property.

[28] It is also worth noting that, this complementary class includes any sequential elimination voting that fails the **MNE**-property.

the same category), denoted by $x \, I_i \, y$. For any set of n voters, the one-shot voting game also specifies the winning candidate as a function of the submitted strategies of the n voters given by an outcome function $f^n : \Lambda^n \rightarrow \mathcal{K}$. A voting rule with k candidates is then defined by the number of categories, the bounds on the size of each category and the outcome function.We refer to such a one-shot voting rule by $v(k) = (J, \{m(j)\}_{j \leq J}, \{M(j)\}_{j \leq J}, \{f^n\}_{n \in \mathbb{N}})$, where $\mathbb{N}$ is the set of odd numbers (as elsewhere, this restriction is made for simplicity).

Rankings Λ can accommodate strict order submissions as in the case of Borda, Copeland and Simpson rules, or standard voting rules that ask for submission of candidates of a particular rank such as plurality rule, negative voting etc. (Note that the rankings induced by the strategies are strict/unique if and only if $J = k$). Further, it can accommodate *approval voting*, which asks voters to partition candidates into 1's and 0's. Thus, our one-shot voting game is the most comprehensive (one-shot) generalization of Moulin's (1988) **scoring rules**.[29]

We shall see that, for any fixed number of candidates k, all single-round voting games satisfying two intuitive properties, called scale invariance and responsiveness, are not CC in strategic voting. This will be a strong assertion because the lack of Condorcet consistency is demonstrated for any fixed k. (The number of voters can of course vary). The meaning of scale invariance is rather straightforward.

Definition 5. *A voting rule $v(k)$ with k candidates is scale invariant if replicating the set of voters with their submitted strategies by any multiple will not alter the winner.*

Responsiveness is about voter pivotalness. Roughly it requires that for each voter and any pair of candidates, there is a scenario at which the voter is pivotal in determining the winner between the two candidates. Before defining responsiveness, we need to define sincere behavior and Condorcet consistency (in sincere voting) in the above class of voting games.

We say that a strategy $R_i = (X_1, ..., X_J) \in \Lambda$ submitted by voter i is *sincere* if $X_1 = \{c^1, ..., c^{m(1)}\}$, $X_2 = \{c^{m(1)+1}, ..., c^{m(1)+m(2)}\}$,...,$X_J = \{c^{\sum_{j<J} m(j)+1}, ..., c^k\}$, when the true preference ranking of voter i is $c^1 \succ_i \ldots \succ_i c^k$.[30]

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[29] Copeland and Simpson rules and approval voting are *not* part of scoring rules.

[30] Notice that this definition of sincere behavior is same as the standard definition when $J = k$ (strict submissions are allowed); thus our definition is a generalization of the standard definition to deal with cases in which the ranking is not strict.

Finally, for any k, a voting rule $v(k)$ is said to be *CC* under sincere voting if for any number of voters and any preference profile over k candidates that admits a *CW*, the voting rule $v(k)$ selects the *CW* whenever the voters' strategies are sincere.

Definition 6. *A voting rule $v(k)$ with k candidates is* <u>*responsive*</u> *if it satisfies the following three conditions for each voter i:*

1. For any pair of candidates x and y and any two strategies R_i and R_i' such that $x \, P_i \, y$ and $y \, P_i' \, x$ there exists a profile of strategies R_{-i} by the remaining voters such that (R_i, R_{-i}) elects x as the winner, and (R_i', R_{-i}) elects y as the winner.

2. For any ranking $j \leq J$ and any two strategies $R_i = (X_1, ..., X_J)$ and $R_i' = (X_1', \ldots, X_J')$ such that $X_\ell' = X_\ell$ for all $\ell < j$, $X_j \subset X_j'$ and $y \in X_j' \backslash X_j$ for some candidate y, there exists a profile of strategies R_{-i} by the remaining voters such that (R_i, R_{-i}) elects some $x \in X_j$ as the winner, and (R_i', R_{-i}) elects y as the winner.

3. If the voting rule is *CC* under sincere voting, then (i) the submissions are strict ($J = k$);[31] and (ii) for any three candidates $X = \{x, y, z\}$, there exists a candidate z in X such that the following holds: for any pair of strategies $R_i = (X_1, X_2, X_3, ..., X_k)$ and $R_i' = (X_2, X_1, X_3, ..., X_k)$ such that $(X_1, X_2, X_3) = (x, y, z)$, there exists a profile of strategies R_{-i} by the remaining voters such that (R_i, R_{-i}) elects x as the winner, and (R_i', R_{-i}) elects z as the winner.

Conditions 2 and 3 in the above definition hold trivially for many standard one-shot voting games. In particular, all *scoring rules* trivially satisfy condition 2 of Definition 6. This is because in any such voting game, the number of candidates in each ranking j of the player is always the same ($m(j) = M(j)$), whereas condition 2 applies only to voting rules in which this is not the case for some ranking j (more precisely, note that in condition 2 above $\mid X_j \mid < \mid X_j' \mid$). Also, if a voting rule is *not CC* under sincere voting then condition 3 of Definition 6 holds vacuously. Therefore, since scoring rules as well as *approval voting* (not part of scoring rules) and instant runoffs are not *CC* under sincere voting for any number of candidates $k \geq 3$ (we demonstrate this in the proof of Proposition 3), it follows that they satisfy condition 3 of Definition 6 trivially.

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[31] We assume in this case $J = k$ because all voting rules that are *CC* with respect to sincere voting are based on strict rankings. This assumption is not necessary for our results but it simplifies the proofs substantially.

Theorem 5. *For any single-round voting rule $v(k)$ with k candidates, if $v(k)$ satisfies responsiveness and scale invariance then $v(k)$ is* <u>*not*</u> *CC.*

Proposition 3. *Suppose $n \geq k-1$. Then standard one-shot voting rules, in particular, all scoring rules (including plurality rule, negative voting, Borda rule), approval voting, the two variants of Instant runoff voting (with and without the majority top-rank trigger[32]), Copeland rule and Simpson rule will all satisfy responsiveness and scale invariance conditions of Theorem 5. Hence none of these voting rules will be CC.*

The proof of Proposition 3 appears in Appendix C.

While failure of Condorcet consistency for specific one-shot voting rule(s) is perhaps not that surprising,[33] to our knowledge there is no general result verifying that Condorcet consistency may fail for most prominent voting rules. (Note that Theorem 5 is applicable more generally to any one-shot voting rule, not just the ones stated in Proposition 3.) On the contrary, significant positive results in the implementation literature would have led one to believe otherwise. However, the requirement that voting mechanisms be relatively simple somewhat constrains the mechanisms' scope in achieving desirable objectives. In that respect, failure of Condorcet consistency of one-shot voting rules should also be worth noting. Next we turn to formally establish Theorem 5.

Proof of Theorem 5: Fix any voting rule satisfying the two assumptions of responsiveness and invariance. We need to show that for some preference profile admitting a *CW*, there exists a Nash equilibrium with weakly undominated strategies such that a non-Condorcet winner is elected. We will prove the assertion for the case of $\mathcal{K} = \{c^1, c^2, ..., c^k\}$, the set of candidates.

We shall first show that for a voter *sincere* submission of his ranking is never a weakly dominated strategy. Without any loss of generality assume that voter i has the preference relation $c^1 \succ_i c^2 \succ_i ... \succ_i c^k$. Suppose $R_i = (X_1, ..., X_J)$ is sincere and suppose it is dominated by $R'_i = (X'_1, ..., X'_J)$. Then we obtain a contradiction by showing by induction that $X'_j = X_j$ for all $j \leq J$. To show this it is sufficient to show that for any $j \leq J$ if either $j = 1$ or $j > 1$ and $X'_\ell = X_\ell$ for all $\ell < j$ then $X'_j = X_j$. We show this in two steps.

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[32] See footnote 8.

[33] See Dhillon and Lockwood (2004) in the case of plurality rule, and De Sinopoli et. al. (2006) for approval voting.

Step 1: If $x \in X_j$ then $x \in X'_j$. Suppose not, then, since $|X_j| = m(j)$ and $|X'_j| \geq m(j)$, there exists $y \in X'_j$ such that $y \notin X_j$. Now since either $j = 1$ or $X'_\ell = X_\ell$ for all $\ell < j$, $x \in X_j$, $y \notin X_j$, $x \notin X'_j$, and $y \in X'_j$ imply that $x\, P_i y$, and $y\, P'_i x$. But then by condition 1 in Definition 6 there exists R_{-i} such that R_i results in x winning, and R'_i results in y winning, thus contradicting that R_i is dominated by R'_i (note that since R_i is sincere xP_iy implies that i prefers x to y).

Step 2: If $y \in X'_j$ then $y \in X_j$. Suppose not; then since either $j = 1$ or $X'_\ell = X_\ell$ for all $\ell < j$, it must be that $y \in X_\ell$ for some $\ell > j$. But then for every $x \in X_j$, $x\, P_i\ y$ and thus, since R_i is sincere, it must be that i prefers x to y. But then since by the previous step $X_j \subseteq X'_j$, by condition 2 in Definition 6, there exists R_{-i} such that R_i results in some $x \in X_j$ winning, and R'_i results in y winning, thus contradicting that R_i is dominated by R'_i.

By induction we thus have $R_i = R'_i$, contradicting that R_i is dominated by R'_i.

Now consider two separate cases.

Case A: The voting rule is not CC with respect to sincere voting.

Consider a set of voters, a given preference profile $(\succ_1, ..., \succ_n)$, and the sincere strategy profile $R_\mathcal{N}$ for which Condorcet consistency is violated in sincere voting. By the arguments above each voter i submitting R_i is not using a dominated strategy. Consider now a sufficiently large replica of the voting game with every voter with preference ordering $\succ_i$ submitting R_i (so that the *scale invariance* of Definition 6 applies) and such that unilateral deviation does not alter the outcome. Then the corresponding strategy combination is a Nash equilibrium with undominated strategies yielding a candidate which is not a *CW*.

Case B: The voting rule is CC in sincere voting.

Consider the first three candidates c^1, c^2 and c^3. Without any loss of generality assume that c^3 is the candidate among the first three candidates that satisfies the property in condition 3 in Definition 6 (i.e. c^3 is in the role of candidate z in condition 3). Next, let $\kappa = \max\{\kappa' \mid 3\kappa' \leq n\}$, where n is the number of voters. Suppose that the true preference profile of the voters is such that the set of voters can be partitioned into three sets $\mathcal{S}^1, \mathcal{S}^2$ and $\mathcal{S}^3$ as follows: The set $\mathcal{S}^1$ consists of $n - 2\kappa$ voters and each $i \in \mathcal{S}^1$ has preferences given by $c^1 \succ_i c^2 \succ_i c^3 \succ_i ... \succ_i c^k$; the set $\mathcal{S}^2$ consists of κ voters and each $i \in \mathcal{S}^2$ has preferences given by $c^2 \succ_i c^1 \succ_i c^3 \succ_i c^4 \succ_i ... \succ_i c^k$; the set $\mathcal{S}^3$ consists of κ voters and each $i \in \mathcal{S}^3$ has preferences given by $c^3 \succ_i c^1 \succ_i c^2 \succ_i c^4 \succ_i ... \succ_i c^k$. Then note that c^1 is the *CW* and c^2 is the *CW* among all candidates other than c^1. Also, denote the set of voters that prefer c^2 to c^3 by $\mathcal{S} = \mathcal{S}^1 \cup \mathcal{S}^2$; clearly, $\mathcal{S}$ forms a majority.

Now since the voting rule is CC in sincere voting, by condition 3 in Definition 6, $J = k$ (all rankings are strict). Next consider for any $i \in \mathcal{S}$ the strategy $R_i = (c^2, c^1, c^3, ..., c^k)$. First we show that for any $i \in \mathcal{S}$, R_i is not weakly dominated.

Suppose not; then for some $i \in \mathcal{S}$, R_i is weakly dominated by another strategy $R'_i = (X'_1, ..., X'_k)$. Now since R_i is sincere if $i \in \mathcal{S}^2$ and voting sincerely is not weakly dominated (see above), it follows that $i \in \mathcal{S}^1$ and $c^1 \succ_i c^2 \succ_i c^3 \succ_i ... \succ_i c^k$. Using this, we next establish in several steps that $X'_\tau = c^\tau$ for all $\tau \leq k$.

Step 1: We claim that $X'_1 \neq c^\tau$ for any $\tau > 2$. Suppose not; then by condition 1 in Definition 6 there exists R_{-i} such that R_i results in c^2 winning, and R'_i results in c^τ for some $\tau > 2$ winning, thus contradicting that R_i is dominated by R'_i.

Step 2: We claim that $X'_1 = c^1$. Suppose not; then by the previous step $X'_1 = c^2$. But then since $R_i = (c^2, c^1, c^3, ..., c^k)$ and $c^1 \succ_i c^3 \succ_i ... \succ_i c^k$ by induction it follows that $R'_i = R_i$ (use the same induction reasoning as that in the proof of voting sincerely is not dominated). But this is a contradiction.

Step 3: We claim that for $X'_j = c^j$ for all $j \leq J$. Since by the previous step the claim is true for $j = 1$, by induction, it suffices to show that for any $j \leq J$, if $X'_{j'} = c^{j'}$ for all $j' < j$, then $X'_j = c^j$. To show this suppose contrary to the claim that $X'_{j'} = c^{j'}$ for all $j' < j$ and $X'_j \neq c^j$. Then $X'_j = c^\tau$ for some $\tau > j$. This implies, by condition 1 in Definition 6, that there exists R_{-i} such that R_i results in either c^1 (if $j = 2$) or c^j (if $j > 2$) winning, and R'_i results in c^τ. Since i prefers both c^1 and c^j to c^τ ($\tau > j$), this contradicts R_i being dominated by R'_i.

Now since $R_i = (c^2, c^1, c^3, ..., c^k)$ and $R'_i = (c^1, c^2, c^3, ..., c^k)$, by condition 3 in Definition 6, there exists a strategy profile R_{-i} such that (R'_i, R_{-i}) elects c^3 whereas (R_i, R_{-i}) elects c^2. Since c^3 is worse than c^2 in i's true ranking, R'_i cannot weakly dominate R_i. But this is a contradiction. Hence R_i is not weakly dominated.

Now consider any strategy combination $R_{\mathcal{N}}$ in which every $i \in \mathcal{S}$ submits the strategy R_i, and the rest of the voters submit any undominated strategies (e.g. they vote sincerely by submitting $(c^3, c^1, c^2, c^4, ..., c^k)$). We next show that such a profile results in c^2 being elected. Consider any preference profile $\succ' = (\succ'_1, ..., \succ'_n)$ such that $c^2 \succ'_i c^1 \succ'_i c^3 \succ'_i ... \succ'_i c^k$ for every $i \in \mathcal{S}$ and $R_{i'}$ is sincere with respect to $\succ'_{i'}$ for every $i' \in \mathcal{N} \backslash \mathcal{S}$. Clearly, $R_{\mathcal{N}}$ is sincere with respect to $\succ'$. Moreover, since $\succ'$ is such that c^2 is the most preferred for every $i \in \mathcal{S}$ and the set $\mathcal{S}$ constitutes a majority, it follows that c^2 is the CW with respect to $\succ'$. Hence, since the voting rule is, by assumption, CC in sincere voting and $R_{\mathcal{N}}$ is sincere with respect to $\succ'$, it follows that c^2 must be elected when the voters submit $R_{\mathcal{N}}$.

Now assume that $n > 5$ and $R_{\mathcal{N}}$ is chosen. Then no individual voter can affect the outcome because for any single deviation there are at least $n - 2\kappa + \kappa - 1 = n - k - 1 \geq 2\kappa - 1$ voters (the numbers of $\mathcal{S}_1$ and $\mathcal{S}_2$ minus 1) who put c^2 first. Since $2\kappa - 1$ forms a majority if $n > 5$ and the voting rule is *CC* with respect to sincere voting, it follows that c^2 is still elected if any single voter deviates. Thus the strategy profile $R_{\mathcal{N}}$ is a Nash equilibrium with undominated strategies, yielding the candidate c^2. But c^1 is the *CW* with respect to the true preferences. **Q.E.D.**

Remark. The use of 'Replica invariance' in the above proof is mainly to bring under a single theorem all the standard voting rules. However, it is not that difficult to construct counter-examples specific to each of the voting rules in Proposition 3 that involve only a small number of voters.

So far in this section we have considered only single-round voting rules that rank candidates. Next we consider voting rules that do not belong to either the above class of single-round voting or the sequential (elimination) family of section 4. Obviously one can think of many voting rules that come under a third complementary group. We are not going to make any general observation here. Instead, we present two voting rules in Proposition 4 and a third voting rule in Proposition 5 (see sections 1 and 2 for more complete descriptions of these rules) to indicate why both *one-by-one elimination* and *repeated ballots* are potentially important for Condorcet consistency: the *plurality runoff rule* and the *exhaustive ballot method*, sharing features of weakest link voting, both use multiple ballots but fail *one-by-one elimination*; a *one-shot version of the weakest link voting* eliminate candidates one-by-one but fail *repeated ballots* (and likewise for the instant runoff voting without the majority top-rank trigger – see Proposition 3). Of these voting rules, plurality runoff and exhaustive ballot are used in various political appointments. (Proofs of Propositions 4 and 5 appear in the *Supplemental material.*)

Proposition 4. *Both the plurality runoff rule and the exhaustive ballot method are not CC.*

An intuition on why elimination of more than one candidate in some round may lead to a non-Condorcet outcome would be instructive. The basic idea is that with one-by-one elimination, when the *CW* is eliminated in some voting round the (off-equilibrium) outcome is unique in the induction argument. When more than one candidate are eliminated, following the CW's elimination the outcome is not necessarily unique – it depends on who else is being eliminated along with the CW; as a result, in this case, the voters may not vote for the CW in order to influence the final outcome in the case when the CW is eliminated.

Proposition 5. *The one-shot weakest link voting (with voters submitting their entire weakest link strategies once-for-all in a single round followed by one-by-one eliminations) is not CC.*

In Theorem 5 and Propositions 3-5 we have shown that most standard voting rules are not CC due to miscoordination of voter strategies. The problem can be worse when there is no CW, as these voting rules, in contrast to the class of sequential elimination in the previous section (see Theorem 4), may not even select a member of the top cycle. We shall next provide an intuition for such possibilities by providing examples of winning candidate outside the top cycle set in the context of plurality rule and plurality runoff voting (plurality and plurality runoff are respectively typical examples of one-shot voting and multi-round voting without one-by-one elimination).

Consider the case of five voters and six alternatives with the following preferences: type 1: a, b, c, d, e, f (two voters); type 2: b, c, a, e, d, f (two voters) ; type 3: c, a, b, d, e, f (one voter). Assume further that the tie-breaker is such that e is eliminated last and d second last. Clearly, d is outside the top cycle. In the case of plurality rule, voting for d by each voter is an equilibrium outcome (that is, Nash and undominated) because d is not lowest in any one's ranking (Dhillon and Lockwood, 2004). In the case of the plurality runoff it can be checked that the following strategies will be an equilibrium: in the second stage voters vote sincerely; in the first stage two type 1 voters vote for d, two type 2 voters vote for e, and the type 3 voter votes for d. Thus, in both voting rules, the alternative d will be the winner in an equilibrium.

6 Conclusion and further remarks

A common concern in the voting literature is that often votes are wasted in the sense that a sizeable proportion of the electorate vote for a candidate who later on are found out to be far below the top few candidates. It is also widely acknowledged that one-shot voting rules are more likely to suffer from this deficiency – there is no second or fall back option for those who back a loser.

Repeated ballots seem to be a reasonable answer to this drawback of one-shot rules. In this paper we show that, in fact, repeated ballots with one-by-one elimination are an integral part of any voting mechanism when some popular candidate is to be elected. For a suitable choice of a popular candidate, our focus in this paper has been on CW when such an alternative exists. When there is no CW, our sequential voting still has the desirable property that the winner belongs to the top cycle set (a familiar bound for voting equilibria in other contexts; see sect. 3 of Dutta, Jackson and Le Breton, 2002). Despite this positive aspect, it is still the case that when there is no CW our proposed voting rules may allow for a broad range of outcomes (the top cycle set may be too large). In particular, given its diverse domain, it is to be expected that the voting rules from the sequential family will exhibit differing characteristics in respect of other social choice criteria like Pareto efficiency, 'neutrality' (voting outcomes be independent of the identity of candidates), etc.[34] A voting rule may satisfy one criterion and not the other. For instance, sequential binary voting (where votes are taken over pairs of candidates arranged in a specific order), a popular member of the voting family $\mathcal{F}$, is known to be Pareto efficient whereas weakest link voting may fail Pareto efficiency, as we show in an example with four candidates in the *Supplemental material.*[35,36] On the other hand, sequential binary voting is *not* neutral because the equilibrium outcome depends on the order of agendas when there is no CW (anyone with a power to choose agendas can significantly influence the outcome), whereas the weakest link voting (as well as sequential Borda, approval voting and other examples of the voting family $\mathcal{F}$ considered in this paper), are neutral up to the tie-breaking rule; in fact if we allow expected utility (i.e., cardinal preferences unique up to affine transformation) and random tie-breaking rule (with equal probabilities of eliminations for the tied candidates), these voting rules are fully neutral.

Our broad purpose here, however, is not to argue for or against specific sequen-

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[34] The issue of Pareto efficiency has been explored in Dutta and Pattanaik (1985) and Moulin (1980; 1988).

[35] Obviously, for such a result to occur it must be that the voter preferences do not admit a CW; CW, when it exists, *is* Pareto efficient.

[36] For three candidates *all* sequential elimination rules are Pareto efficient, given that top cycle coincides with the uncovered set (see Moulin, 1988). Note also that in one-shot voting games, Pareto efficiency is not even guaranteed for three candidates case.

tial voting rules. Instead, the important point that we like to emphasize in this paper is that Condorcet consistency, and more generally top cycle, is a general property of a broad class of sequential elimination voting, which is not shared by one-shot or any of the familiar semi-sequential voting rules.

Finally, our interest in *CC voting rules* is also borne out of the fact that voting mechanisms seem to be the most natural and decentralized way of reaching collective decisions. Substantial research in the important literature on implementation theory investigate questions of how to achieve desirable social objectives including Condorcet outcomes and often employ abstract/general mechanisms. Our focus on voting mechanisms, rather than general mechanisms, thus differ from this tradition.

Appendix A

Existence of a Markov equilibrium for the weakest link voting when $n \geq 2k - 1$.

To prove existence we need to show that there exists a Markov strategy profile s^* such that at each stage it is Nash and undominated assuming that all players play according to s^* in any later stages. This is done by defining s^* inductively in subgames with a given number of candidates as the inductive variable, as follows.

First, let $k(h)$ denote the number of candidates at h. Then at any h with $k(h) = 2$, assume that voter i chooses *sincerely.* Clearly such a strategy profile is an undominated Nash equilibrium in this last stage and is independent of h.

Induction hypothesis. *Now suppose for all* h' *such that* $k(h') \leq J - 1$*,* $s^*(h')$ *is defined, and is undominated Nash and Markov*[37] *from here onwards.*

We need to define a profile of choices for all voters $s^*(h)$, for all h such that $k(h) = J$, such that $s^*(h)$ is an undominated Nash equilibrium and Markov from here onwards, assuming that all follow $s^*(h')$ at all later stages h' s.t. $k(h') \leq J-1$.

Fix any h s.t. $k(h) = J$. Let $C = \{c^1, ..., c^J\}$ be the set of candidates at h. Without any loss of generality assume that $c^{j'}$ is higher in the tie-breaking rule than c^j (i.e., if at all, c^j is eliminated before eliminating $c^{j'}$) if and only if $j' < j$.

Also let $\sigma(c)$ be the winner if c is eliminated at the start of play of the subgame $\Gamma(h)$. Notice that $\sigma(c)$ is unique because by the induction hypothesis $s^*(h)$, when there are $J - 1$ candidates, is independent of the past history.

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[37] That is, the strategies depend only on the candidates around and not on the precise history leading up to it.

Next define, for any $i \in \mathcal{N}$,

$$M_i = \begin{cases} \Theta_i & \text{if } \exists c \text{ and } c' \in C \text{ s.t. } \sigma(c) \neq \sigma(c'); \\ \emptyset & \text{otherwise.} \end{cases}$$

where $\Theta_i = \{c \in C \mid \nexists c' \in C \text{ s.t. } \sigma(c') \succ_i \sigma(c)\}$ consists of voter i's best elimination candidate(s) in this round of play. Note that M_i is empty-valued when the subgame is degenerate (the identity of the eventual winner is independent of who is eliminated at this round). Finally, let $M_i^c = C \backslash M_i$. Clearly, $M_i^c \neq \emptyset$.

Lemma 1. *In the subgame $\widehat{\Gamma}(h)$, any $c \in M_i^c$ is not weakly dominated for voter i.*

Proof of Lemma 1. Suppose $M_i \neq \emptyset$ (if $M_i = \emptyset$, Lemma 1 holds trivially). Fix $c \in M_i^c$ and any $c' \in C,\ c' \neq c$. We want to argue that switching his vote from c to c' would be worse for voter i for at least one profile of other voters' votes.

If the tie-breaker places *some* $\hat{c} \in M_i$ ahead of c and $\hat{c} \neq c'$, let the distribution of votes of all the voters other than i be as follows:

$$\vartheta(\hat{c}) = \vartheta(c) = 0 \text{ and } \vartheta(\tilde{c}) > 0,\ \forall \tilde{c} \neq \hat{c}, c,$$

where $\vartheta(\cdot)$ denotes the number of votes in favor of a candidate by all voters other than i. Now if i votes for c then the distribution of votes as above leads to the elimination of $\hat{c}$. However, if i switches to c' while the rest stay with their votes as above, candidates $\hat{c}$ and c will be tied with minimal votes and by the tie-breaker c will be eliminated, which is worse for voter i. If $\hat{c} = c'$, the argument holds with even greater force as c would be eliminated (as i switches to c') without having to invoke the tie-breaker.

If the tie-breaker is such that c is placed ahead of *all* $\hat{c} \in M_i$, let the distribution of votes of all the voters other than i be as follows:

$$\vartheta(c) = 0,\ \vartheta(\hat{c}) = 1\ \ \forall\ \hat{c} \in M_i, \text{ and } \vartheta(\tilde{c}) \geq 2\ \ \forall \tilde{c} \neq c,\ \tilde{c} \in M_i^c.$$

Now if i votes for c then this leads to the elimination of some $\hat{c}$. However, if i switches to c' while the rest stay with their votes as above, c will be unique with minimal votes and therefore be eliminated, which is worse for voter i. This completes the proof of Lemma 1. ||

Next for any $r = 1, ..., J$ we define the following property.

Definition 7. *Any $r \in \{2, ..., J\}$ satisfies property α if there exists a set of voters $\Omega = (u_1, v_1, u_2, v_2, ..., u_{r-1}, v_{r-1})$ consisting of $2(r-1)$ different voters such that*

$$c^j \in M_i^c \text{ for } i = u_j, v_j \text{ for all } j < r. \tag{5}$$

Lemma 2. *Suppose that the following two conditions hold for some $1 \leq r < J$: (i) either $r = 1$ or r satisfies property α; and (ii) $r + 1$ does not satisfy property α. Then there exist a choice profile $s^*(h)$ that is Nash, is not weakly dominated, and is Markov.*

Proof of Lemma 2. Given that r satisfies (i) and (ii) above, there exists a set of voters Ω (that is empty if $r = 1$) consisting of $2(r-1)$ different voters $(u_1, v_1, u_2, v_2, ..., u_{r-1}, v_{r-1}) \subset \mathcal{N}$ such that

$$c^j \in M_i^c \text{ for } i = u_j, v_j \text{ for all } j < r \tag{6}$$

and there exists a set of voters $V \subset \mathcal{N} \backslash \Omega$ such that

$$\mid V \mid = n - 2(r-1) - 1 \tag{7}$$

and

$$c^r \in M_v \text{ for any } v \in V. \tag{8}$$

Let

$$C^r = \{c \in C \,| \sigma(c) = \sigma(c^r)\} \quad \text{and} \quad \overline{C}^r = \{C \backslash C^r\} \cap \{c^{r+1}, ..., c^J\}.$$

Since the preferences of each voter is strict, it follows that

$$\overline{C}^r \subset M_v^c \text{ for any } v \in V. \tag{9}$$

Also since $\mid V \mid = n - 2(r-1) - 1$, $\mid \overline{C}^r \mid \leq J - r$ and by assumption $n \geq 2k - 1 \geq 2J - 1$ and $r < J$, it follows that the number of voters in V is at least twice the number of candidates in $\overline{C}^r$. But this implies that there exists a choice profile $\{s_v^*(h)\}_{v \in V}$ such that

$$s_v^*(h) \in \overline{C}^r \text{ for each } v \in V, \tag{10}$$

$$|\{v \in V \,| s_v^*(h) = c\}| \geq 2 \text{ for each } c \in \overline{C}^r. \tag{11}$$

(The second condition says that each candidate $c \in \overline{C}^r$ receive at least two votes). Next set the choice $s_i^*(h)$ of each $i \in \Omega$ to be such that

$$s_i^*(h) = c^j \text{ for } i = u_j, v_j. \tag{12}$$

Finally, denote the remaining voter $\mathcal{N}\backslash\{V \cup \Omega\}$ by x and set the choice of voter x to be such that

$$\begin{array}{ll} s_x^*(h) \in M_x^c\backslash c^r & \text{if } \; M_x^c\backslash c^r \text{ is not empty;} \\ s_x^*(h) = c^r & \text{otherwise.} \end{array} \tag{13}$$

Now by Lemma 1 and conditions (6), (9), (10), (12) and (13), the choice $s_\ell^*(h)$ is undominated in this round for any voter $\ell \in \mathcal{N}$ and is Markov.[38] Next we show that $s^*(h) = \{s_\ell^*(h)\}_{\ell\in\mathcal{N}}$ is Nash. There are two possible cases.

Case A. $M_x^c \neq c^r$**.** First, note that by (11) and (12), in this round each candidate $c \in \overline{C}^r \cup \{c^1, ..., c^{r-1}\}$ receives at least two votes, c^r receives zero vote (follows from (13), given the fact that $M_x^c \neq \emptyset$ and $M_x^c \neq c^r$), and any other $c' \in C^r \cap \{c^{r+1}, \ldots, c^J\}$ receives at most one vote. This means that some candidate $c \in C^r$ is eliminated and $\sigma(c^r)$ will be the final winner. Moreover, since c^r receives zero vote, it must be that the eliminated candidate $c^e \in C^r$ receives zero vote and $e \geq r$.

Since, by (8), $\sigma(c^r)$ is a best outcome for each $v \in V$, it follows that $s_v^*(h)$ is a best choice for any $v \in V$. Moreover, no voter $i \in \Omega$ can change the final outcome $\sigma(c^r)$ by changing its action because the choice $s_i^*(h) \in \{c^1, ..., c^{r-1}\}$ receives at least two votes, the eliminated candidate c^e has zero vote and $e \geq r$. Finally, voter x cannot change the final outcome $\sigma(c^r)$ by changing its action because either the choice $s_x^*(h) \in \{c^1, ..., c^{r-1}\} \cup \overline{C}^r$, in which case $s_x^*(h)$ receives at least three votes and as before c^e has zero vote, or $s_x^*(h) \in C^r \cap \{c^{r+1}, \ldots, c^J\}$ in which case $s_x^*(h)$ receives one vote and any deviation results in some candidate in the set C^r to be eliminated.

Case B. $M_x^c = c^r$**.** Then for each $c' \neq c^r$, $c' \in M_x$. Therefore

$$\forall c', c'' \neq c^r, \;\; \sigma(c') = \sigma(c''). \tag{14}$$

This implies that $\overline{C}^r = \{c^{r+1}, ..., c^J\}$. But then, by (11) and (12), in this round each candidate $c \neq c^r$ receives at least two votes, c^r receives one vote (the vote of x), c^r is eliminated and $\sigma(c^r)$ will be the final winner. As in the previous case, since this is a best outcome for each $v \in V$ it follows that $s_v^*(h)$ is a best choice for any $v \in V$. Next note that for each voter $i = u_j, v_j$ for $j < r$ we have $s_i^*(h) = c^j \in M_i^c$ and thus $c^r \in M_i$. Therefore, eliminating c^r is also the best outcome for any $i \in \Omega$.

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[38] If $M_x^c\backslash c^r$ is an empty set then c^r must be an element of M_x^c because $M_x^c \neq \emptyset$. Thus, $s_x^*(h)$ is undominated.

Finally, note that voter x cannot change the final outcome $\sigma(c^r)$ by changing its action because every $c \neq c^r$ receives two votes. $||$

Lemma 3. *Suppose that J satisfies property α. Then there exist a choice profile $s^*(h)$ that is Nash, is not weakly dominated, and is Markov.*

Proof of Lemma 3. Given that J satisfies property α, there exists a set of voters $\Omega = (u_1, v_1, u_2, v_2, ..., u_{J-1}, v_{J-1})$ consisting of $2(J-1)$ different voters such that

$$c^j \in M_i^c \ \text{ for } \ i = u_j, v_j \text{ for all } \ j < J. \tag{15}$$

Set the choice profile $\{s_i^*(h)\}_{i\in\Omega}$ to be such that

$$s_i^*(h) = c^j \text{ if } i = u_j, v_j. \tag{16}$$

Also partition the remaining voters as follows:

$$\begin{aligned} \Gamma^J &= \left\{v \in \mathcal{N}\backslash\Omega | M_v^c = c^J\right\} \\ \overline{\Gamma}^J &= \left\{v \in \mathcal{N}\backslash\Omega | M_v^c \neq c^J\right\}. \end{aligned}$$

Let the choice profile $\{s_v^*(h)\}_{v\in\mathcal{N}\backslash\Omega}$ be such that

(i)
$$s_v^*(h) \in \begin{cases} c^J & \text{if } \ v \in \Gamma^J \\ M_v^c \backslash c^J & \text{if } \ v \in \overline{\Gamma}^J; \end{cases}$$

(ii) if Γ^J is non-empty,

$$|n(c) - n(c')| \leq 1 \ \ \forall c, c' \neq c^J \ \text{ s.t. } \ \sigma(c) = \sigma(c'), \tag{17}$$

where

$$n(c) = |\{v \in \mathcal{N} \,| s_v^*(h) = c\}| \ \text{ for any } \ c.$$

(Note that $M_v^c \backslash c^J \neq \emptyset$ for $v \in \overline{\Gamma}^J$.) Notice that if Γ^J is non-empty, (17) is possible for the following reasons. First, since Γ^j is non-empty,

$$\forall c', c'' \neq c^J, \ \ \sigma(c') = \sigma(c'') \neq \sigma(c^J). \tag{18}$$

Next note that each c^j, $j < J$ receives two votes from the set of voters Ω. The only other voters that vote for the candidates c^j, $j < J$ are from the set $\overline{\Gamma}^J$. Because Γ^J is non-empty it follows from (18) that for each $v \in \Gamma^J$, $M_v = \{c^1, \ldots, c^{J-1}\}$; therefore votes by the members of $\overline{\Gamma}^J$ can be arranged so that (17) is satisfied: the first member of $\overline{\Gamma}^J$ votes for c^1, the second for c^2 etc. until the $(J-1)$st member votes for c^{J-1}, the J-th member for c^1, $(J+1)$st for c^2 etc.

By Lemma 1, $s^*(h)$ is not weakly dominated. Also by definition, $s^*(h)$ is Markov. Next we show that it is a Nash equilibrium.

Case A. Γ^J is empty. Then every $c \neq c^J$ receives at least two votes, c^J receives no votes and is eliminated. This together with c^J having the lowest rank in the tie-breaking rule imply that no player can change the final outcome by changing their choices and thus $s^*(h)$ constitutes a Nash equilibrium.

Case B. Γ^J is non-empty. By (18), since for each $v \notin \Gamma^J$ there exists a $c \neq c^J$ such that $c \in M_v^c$, it follows that

$$\forall v \notin \Gamma^J, \;\; c^J \in M_v. \tag{19}$$

Now there are two possibilities.

Subcase 1: Candidate c^J is eliminated. Then, by (19), this is the best outcome for any $v \notin \Gamma^J$ and therefore, each such v is choosing his optimal action. Moreover, each $v \in \Gamma^J$ cannot change the outcome by deviating from $s_v^*(h)$ because $s_v^*(h) = c^J$ and c^J is the candidate that is eliminated.

Subcase 2: Some $c \neq c^J$ is eliminated. Then, by the tie-breaking rule

$$n(c) < n(c^J). \tag{20}$$

Next note that by (18) and the definition of Γ^J, this is the best outcome for any $v \in \Gamma^J$ and therefore, each such v is choosing his optimal action. Next we show that no voter $v \notin \Gamma^J$ can change the outcome by deviating. Suppose not; then some voter $v \notin \Gamma^J$ can deviate from $s_v^*(h) = c^j (\neq c)$ for some $j < J$ and change the final outcome $\sigma(c)$ by voting for another candidate. Since the outcome is changed, by (18), it must be that c^J is eliminated. This implies that

$$n(c^j) - 1 \geq n(c^J).$$

But this together with (20) imply that

$$n(c^j) > n(c) + 1$$

But this contradicts condition (17). Therefore no $v \notin \Gamma^J$ can change the final outcome by deviating. $||$

The last two lemmas together establish that there exists a choice profile $s^*(h)$ that is Nash, not weakly dominated, and is Markov. **Q.E.D.**

Appendix B

Justifying the use of Markov strategies.

Recall, S_i is the strategy set of voter i with $s_i : \mathcal{H} \rightarrow \cup_{C,j} A_i(C,j)$ s.t. $s_i(h) \in A_i(C,j)\ \ \forall h \in \mathcal{H}_C^j$, where $\mathcal{H}_C^j = \mathcal{H}_C \cap \mathcal{H}^j$. Also, let $S = \Pi_i S_i$.

Definition 8. *A strategy $s_i \in S_i$ is* <u>*more complex*</u> *than another strategy $s_i' \in S_i$ if $\exists C$ and j s.t.*

$$\begin{array}{rl} (i) & s_i(h) = s_i'(h) \ \ \forall h \notin \mathcal{H}_C^j; \\ (ii) & s_i'(h) = s_i'(h') \ \ \forall h, h' \in \mathcal{H}_C^j; \\ (iii) & s_i(h) \neq s_i(h') \ \ \textit{for some}\ h, h' \in \mathcal{H}_C^j. \end{array}$$

The above ordering of complexity is only a partial ordering. Nevertheless, it will prove a powerful one for our purpose. Based on this ordering, let us refine our earlier definition of *equilibrium* as follows.

Definition 9. *A equilibrium strategy profile $s^* \in S$ will be called a* <u>*simple equilibrium*</u> *if for any $i \in \mathcal{N}$*

$$\nexists s_i \in S_i \ \textit{s.t.}\ w(s_i, s_{-i}^*) = w(s_i^*, s_{-i}^*) \ \ \textit{and}\ \ s_i^* \ \ \textit{is more complex than}\ \ s_i, \tag{21}$$

where $w(s)$ is the winner if profile s is adopted.

Note that while the definition of *simple* equilibrium allows history-dependent (i.e., non-Markov) strategies, the condition in (21) reflects the implicit assumption that the voters are averse to complexity *unless* it helps to change the final outcome. Thus, simplicity of the simple equilibrium is a very weak, and in our view plausible, requirement for any descriptive analysis. We can therefore use the simplicity criterion for equilibrium selection.

Theorem 6. *Any simple equilibrium is also a Markov equilibrium.*

Proof. Suppose $s^* \in S$ is a simple equilibrium but not a Markov equilibrium. Then there exists some i, C, j and $h, h' \in \mathcal{H}_C^j$ s.t. $s_i^*(h) \neq s_i^*(h')$. Clearly, if $\mathcal{H}_C^j \cap E \neq \emptyset$ where E is the equilibrium path corresponding to the simple equilibrium s^*, then $\mathcal{H}_C^j \cap E$ is *unique*; that is, C happens on the equilibrium path at stage j at most once. Now consider another strategy $s_i \in S_i$ s.t.

$$\begin{array}{rcl} s_i(h) & = & s_i^*(h) \ \ \forall h \notin \mathcal{H}_C^j; \\ \forall h \in \mathcal{H}_C, \quad s_i(h) & = & \left\{ \begin{array}{ll} s_i^*(\mathcal{H}_C^j \cap E) & \text{if } \mathcal{H}_C^j \cap E \neq \emptyset, \\ a_i \in C & \text{if } \mathcal{H}_C^j \cap E = \emptyset, \end{array} \right. \end{array}$$

where a_i denotes any arbitrary element of C (note that s_i is well-defined because $\mathcal{H}_C^j \cap E$ is unique when defined).

It is easy to see that s_i is simpler than s_i^*. Moreover, because s_i differs from s_i^* only possibly for histories in $\mathcal{H}_C^j$ that are off-the-equilibrium path, (s_i, s_{-i}^*) will result in the same winner as the equilibrium s^*, so that $w_i(s_i, s_{-i}^*) = w_i(s_i^*, s_{-i}^*)$. Hence, s^* cannot be a simple equilibrium – a contradiction. **Q.E.D.**

Appendix C

Proof of Proposition 1: Fix a stage with the set of remaining candidates C having the cardinality J. Also, fix a candidate $c \in C$ and a majority ϕ.

For any voter i, let $\mathcal{D}_i^c(C, j)$ be the set of all strategies that place c at the top (with no other restriction on the positions of other candidates).[39] Also, let $\mathcal{D}_\phi^c(C, j) = \Pi_{i \in \phi} \mathcal{D}_i^c(C, j)$.

First we verify condition *[i]*. Fix any $a \in A(\mathcal{C}, j)$ such that $a_\phi \in \mathcal{D}_\phi^c(C, j)$. We need to show that $e(a, C) \neq c$.

For any $x \in C$ and any $a' \in A(\mathcal{C}, j)$, denote the total score of candidate x at this stage when action profile a' is chosen by $TS(x, a')$.

Next, define θ_{top} to be the total score of a candidate if he receives the highest score, ς_J, from a majority of $(n+1)/2$ voters and gets the lowest score, ς_1, from the remaining $n - (n+1)/2$ voters:

$$\theta_{\text{top}} = \frac{(n+1)}{2} \varsigma_J + [n - \frac{(n+1)}{2}] \varsigma_1.$$

Since a is such that the majority ϕ place c at the top, it follows that $TS(c, a) \geq \theta_{\text{top}}$. Therefore, the average score that the other candidates receive when a is chosen cannot exceed

$$\theta_{\min} = \frac{n[\varsigma_J + \ldots + \varsigma_1] - \theta_{\text{top}}}{J - 1}.$$

But then there must exist a candidate $d \in C$ such that $TS(d, a) \leq \theta_{\min}$. Now to complete verification of condition *[i]*, it suffices to show that $\theta_{\text{top}} - \theta_{\min} > 0$. Note

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[39] Proposition 2 proof, omitted, will follow a similar argument as in the rest of Proposition 1 proof. For sequential extension of approval voting, $\mathcal{D}_i^c(C, j)$ will consist of the unique strategy of voter i approving only candidate c and disapproving all the remaining candidates. For sequential extensions of Copeland and Simpson rules – given that these rules are based on strict order submissions – $\mathcal{D}_i^c(C, j)$ will place only candidate c at the top.

that

$$\begin{aligned}(J-1)(\theta_{\text{top}}-\theta_{\text{min}}) &= J\cdot\theta_{\text{top}} - n\sum_{\ell=1}^{J}\varsigma_\ell \\ = \quad & \frac{(J-2)n+J}{2}\varsigma_J + \frac{(J-2)n-J}{2}\varsigma_1 - n\sum_{\ell=2}^{J-1}\varsigma_\ell.\end{aligned}$$

$$\text{Therefore,}\quad \theta_{\text{top}}-\theta_{\text{min}} > 0 \Leftrightarrow \frac{1}{2}(\varsigma_J+\varsigma_1) + \frac{J}{2n(J-2)}(\varsigma_J-\varsigma_1) > \frac{1}{(J-2)}\sum_{\ell=2}^{J-1}\varsigma_\ell.$$

But since, by assumption, $\frac{1}{2}(\varsigma_J+\varsigma_1) \geq \frac{1}{(J-2)}\sum_{\ell=2}^{J-1}\varsigma_\ell$ and $\varsigma_J > \varsigma_1$, condition *[i]* must hold. $||$

Next, we verify condition *[ii]*. Fix $a \in A(C,j)$ such that $a_\phi \notin \mathcal{D}^c_\phi(C,j)$ and $e(a,C)=c$. For any i, let m^i be a candidate to whom i attaches the highest score ς_J: $a_i(m^i)=\varsigma_J$.

Without loss of generality denote the set of voters in the ϕ-majority by $\{1,2,\ldots,|\phi|\}$. Next, consider the sequence of vote profiles, $a^{(0)},a^{(1)},\ldots,a^{(|\phi|)}$, defined as follows: $a^{(0)}=a$ and for any i and ℓ such that $1\leq i,\ell\leq|\phi|$,

$$a^{(i)}_\ell(x) = \begin{cases} \varsigma_J & \text{if } x=c \text{ and } \ell\leq i \\ a_\ell(c) & \text{if } x=m^\ell \text{ and } \ell\leq i \\ a_\ell(x) & \text{otherwise.}\end{cases}$$

Note that $a^{(|\phi|)}$ is such that $a^{(|\phi|)}_i(c)=\varsigma_J$ for all $i\in\phi$. Therefore, $a^{(|\phi|)}_\phi \in \mathcal{D}^c_\phi(C,j)$ and hence, by condition *[i]*, $e(a^{(|\phi|)},C)\neq c$. Moreover, by assumption $e(a^{(0)},C)=c$. Therefore, there exists some i, $1\leq i\leq|\phi|$, such that

$$e(a^{(i-1)},C)=c \text{ and } e(a^{(i)},C)\neq c.$$

Furthermore, by the definition of the sequence $a^{(0)},a^{(1)},\ldots,a^{(|\phi|)}$ we have that $a^{(i-1)}_i=a_i$ and $a^{(i-1)}_{-i}=a^{(i)}_{-i}$. Therefore, we have $e(a_i,a^{(i-1)}_{-i},C)=c$ and $e(a^{(i)}_i,a^{(i-1)}_{-i},C)\neq c$ verifying (3) in condition *[ii]*.

To verify (2), for $a_i(=a^{(i-1)}_i)$ and $a^{(i)}_i$ note that $a^{(i)}_i(c)=\varsigma_J$, $a^{(i)}_i(m^i)=a_i(c)$ and $a^{(i)}_i(x)=a_i(x)$ for all $x\neq c$. Thus, for any $a_{-i}\in A_{-i}(C,j)$ we have $TS(c,a^{(i)}_i,a_{-i})\geq TS(c,a_i,a_{-i})$, $TS(m^i,a^{(i)}_i,a_{-i})\leq TS(m^i,a_i,a_{-i})$ and $TS(x,a^{(i)}_i,a_{-i})=TS(c,a_i,a_{-i})$ for all $x\neq c$. But this implies that if $e(a_i,a_{-i},C)\neq c$ then $e(a^{(i)}_i,a_{-i},C)\neq c$ for all $a_{-i}\in A_{-i}(C,j)$, hence verifying (2). **Q.E.D.**

Proof of Theorem 4: We use induction on the number of remaining candidates.

Suppose C consists of two candidates. Then the result is true by Theorem 3. Now assume the following hypothesis: *Theorem 4 is true for any subgame with j candidates.*

We want to show that the result is also true for any subgame with $j+1$ remaining candidates. Suppose not. Then there is a subgame Γ at stage $k-j$ with remaining candidates C of cardinality $j+1$ such that w is the ultimate winner and $w \notin \mathcal{TC}(C)$. This implies there exists some $y \in C$ such that

$$y \; T \; w \;\; \underline{\text{and}} \text{ it is } \textit{not} \text{ the case that } \; w \; T \; x_1 \; T \; x_2 \; T \; \ldots T \; x_\ell \; T \; y, \tag{22}$$

for some $x_1, x_2, ..., x_\ell \in C$.

Next we establish two intermediate claims.

Claim 1: y must be the first eliminated candidate in the subgame Γ.

If not, let $y' \neq y$ be the candidate eliminated at this stage. Then in this subgame the remaining candidate set is $C \setminus y'$ and w wins, which implies by hypothesis $w \in \mathcal{TC}(C \setminus y')$. But then there will be a (direct or an indirect) chain such that $w \; T \; x_1 \; T \; x_2 \; T \; \ldots T \; x_\ell \; T \; y$, contradicting (22). $\quad ||$

Claim 2: y is the ultimate winner in any subgame at stage $k-j$ with remaining candidates C if y is not the first eliminated candidate in this subgame.

Let $a \neq y$ be the candidate that is eliminated first. Denote the winner after (a is eliminated) by $\hat{w}$.

First note that, $y \in C \setminus a$ implies that $w \notin \mathcal{TC}(C \setminus a)$, and hence by hypothesis $\hat{w} \neq w$.

Now suppose Claim 2 is false; then $\hat{w} \neq y$. Since $\hat{w} \in \mathcal{TC}(C \setminus a)$, it must then be that

$$\hat{w} \; T \; \ldots T \; y. \tag{23}$$

Also, since by Claim 1 and hypothesis $w \in \mathcal{TC}(C \setminus y)$, and $\hat{w} \neq w$ (as established above), it must be that

$$w \; T \; \ldots T \; \hat{w}. \tag{24}$$

Now (23) and (24) together imply

$$w \; T \; \ldots T \; \hat{w} \; T \; \ldots T \; y,$$

but this contradicts (22). So Claim 2 must be true. $||$

The rest of the proof is the same as in the case of having a CW, as follows. In the subgame with remaining candidates C, consider any voter i such that $y \succ_i w$; there will be a majority of such voters because $y\,T\,w$. Denote these majority voters by ϕ. By condition *[i]* of the **MNE**-property, there exist vote profiles $a_\phi \in \mathcal{D}^y_\phi(C, k-j)$ such that $e(a_\phi, a_{-\phi}, C) \neq y$, $\forall a_{-\phi} \in \Pi_{\ell \notin \phi} A_\ell(C, k-j)$. Then since by Claim 1 y must be the first eliminated candidate in the subgame Γ, it must be that the majority ϕ chose some vote profile $\tilde{a}_\phi \notin \mathcal{D}^y_\phi(C, k-j)$. But then by condition *[ii]* of the **MNE**-property, there is some voter $i \in \phi$ whose vote choice $\tilde{a}_i$ (corresponding to the profile $\tilde{a}_\phi$) is "inferior" to some other vote choice a^y_i (as defined in condition *[ii]* of the **MNE**-property in section 4.2) in protecting y. But then we have a contradiction because then for such i voting for a^y_i weakly dominates voting for $\tilde{a}_i$: either y is eliminated in which case, by claim 1, w wins; or there is at least one vote profile a'_{-i} by the remaining voters such that $e(a^y_i, a_{-i}, C) \neq y$ (by (3) in condition *[ii]*) and, by claim 2, y will be the ultimate winner (the details of this argument is similar to that in Theorem 3). Thus, $w \notin \mathcal{TC}(C)$ cannot be the winner. **Q.E.D.**

Proof of Proposition 3: We prove that the voting rules considered satisfy the three conditions in Definition 6 separately.

Condition 1

To show this, fix any x and y and any two strategies $R_i = (X_1, ..., X_J)$ and $R'_i = (X'_1, ..., X'_J)$ such that $x \in X_\tau$, $y \in X_{\tau'}$ and $\tau < \tau'$, so that $x\,P_i\,y$. Suppose also $x \in X'_\nu$, $y \in X'_{\nu'}$ and $\nu' < \nu$, so that $y\,P'_i\,x$. Also, let $m = (n-1)/2$ and consider the set of voters other than i. Enumerate this set (we are assuming an odd number of voters) and denote the enumeration by $\{\alpha_1, ..., \alpha_{2m}\}$, with a typical voter denoted as α_ℓ. Also enumerate the candidates other than x, y as $\{c^1, c^2, \ldots, c^{k-2}\}$.

Next, for any voter α_ℓ consider any strategy $R_{\alpha_\ell} = (\hat{X}_1, ..., \hat{X}_J)$ satisfying[40]

$$x \in \begin{cases} \hat{X}_1 & \text{if either } \ell \leq m \quad \text{or} \quad M(1) > 1 \\ \hat{X}_2 & \text{if } \ell > m \quad \text{and} \quad M(1) = 1 \end{cases}$$

$$y \in \begin{cases} \hat{X}_1 & \text{if either } \ell > m \quad \text{or} \quad M(1) > 1 \\ \hat{X}_2 & \text{if } \ell \leq m \quad \text{and} \quad M(1) = 1 \end{cases}$$

$$\text{and} \qquad c^r \in \hat{X}_J \qquad \text{for voter } \alpha_r, \quad 1 \leq r \leq k-2.$$

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[40] One should index the cells to reflect individualistic voting, but we keep to minimal notations.

Thus, each of the candidates other than x and y is placed in at least one voter's lowest-ranked cell. This is possible because there are $k-2$ such candidates and $k-2 \leq 2m$ (by assumption $k-1 \leq n$).

Next, consider the different voting rules under consideration.

Scoring rules and Approval Voting: Denote the score attached to the j-th cell in either of the two voting rules by ς_{J-j+1}. Also, denote respectively the total score that any candidate c receives for strategy profile (R_i, R_{-i}) and (R_i', R_{-i}) by $TS(c, R_i, R_{-i})$ and $TS(c, R_i', R_{-i})$. Then it follows from the definition of R_{-i} above that:

$$\left.\begin{aligned} TS(x, R_i, R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\tau+1} \\ TS(y, R_i, R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\tau'+1} \\ TS(c^r, R_i, R_{-i}) &\leq \varsigma_J + (2m-1)\gamma + \varsigma_1, \qquad 1 \leq r \leq k-2, \end{aligned}\right\} \tag{25}$$

$$\text{where} \qquad \gamma = \begin{cases} \varsigma_J & \text{if } M(1) > 1 \\ \varsigma_{J-1} & \text{if } M(1) = 1. \end{cases}$$

Therefore, it follows from (25) and $\varsigma_{J-\tau+1} > \varsigma_{J-\tau'+1} \geq \varsigma_1$ that $TS(x, R_i, R_{-i}) - TS(y, R_i, R_{-i}) > 0$, and $TS(x, R_i, R_{-i}) - TS(c^r, R_i, R_{-i}) > 0$. Therefore, (R_i, R_{-i}) results in x being elected.

Also, it follows from the definition of R_{-i} above that:

$$\begin{aligned} TS(x, R_i', R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\nu+1} \\ TS(y, R_i', R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\nu'+1} \\ TS(c^r, R_i', R_{-i}) &\leq \varsigma_J + (2m-1)\gamma + \varsigma_1, \qquad 1 \leq r \leq k-2. \end{aligned}$$

But this together with $\varsigma_{J-\nu'+1} > \varsigma_{J-\nu+1} \geq \varsigma_1$ imply that $TS(y, R_i', R_{-i}) - TS(x, R_i', R_{-i}) \geq \varsigma_{J-\nu'+1} - \varsigma_{J-\nu+1} > 0$, and $TS(y, R_i', R_{-i}) - TS(c^r, R_i', R_{-i}) \geq \varsigma_{J-\nu'+1} - \varsigma_1 > 0$. Thus, (R_i', R_{-i}) results in y being elected.

Instant runoff voting (with and without the majority top-rank trigger). For both variants of instant runoff voting, the strategy profile (R_i, R_{-i}) described above results in candidate x having the highest number of votes at each round and therefore in x being elected, and the strategy profile (R_i', R_{-i}) described above results in candidate y having the highest number of votes at each round and therefore in y being elected (this follows from the same reasoning as in the previous case with scoring rules and approval voting).

Copeland rule. To calculate Copeland scores for (R_i, R_{-i}) submissions, let us do binary comparisons: comparing x against any other candidate yields x each time a score of $+1$, thus the Copeland score of x is $k-1$. Since $k-1$ is the maximum possible Copeland score, it follows that (R_i, R_{-i}) results in x being the winner. By the same reasoning, the Copeland score of y when (R_i', R_{-i}) is chosen is $k-1$; therefore in this case the Copeland winner is y. Thus, condition 1 is satisfied.

Simpson rule. The strategy profile (R_i, R_{-i}) described above results in candidate x having the highest Simpson score and therefore being elected. This is because the Simpson score of x in this case is $m+1$ ($N(x,a) = 2m$ for all $a \neq x, y$ and $N(x,y) = m+1$), whereas the Simpson score of y is m and that of any other candidate $a \neq x, y$ is no greater than 1. By the same reasoning, it follows that the strategy profile (R_i', R_{-i}) described above results in candidate y having the highest Simpson score and therefore being elected. ||

Condition 2

Note that since in the case of scoring rules, instant runoff voting (with and without the majority top-rank trigger), and Copeland and Simpson rules each player's strategies are such that the number of candidates in each rank is fixed, it follows that these voting rules satisfy condition 2 in Definition 6 vacuously. Therefore, to complete the proof we only need to show that approval voting satisfies condition 2.

To show this, fix any two strategies $R_i = (X_1, X_2)$ and $R_i' = (X_1', X_2')$ such that $X_1 \subset X_1'$ and $y \in X_1' \backslash X_1$. Also, fix any $x \in X_1$. Now there are two cases to consider.

First, assume that the tie-breaker places y ahead of x. Then consider any R_{-i} such that m voters submit $(x, \mathcal{K} \backslash \{x\})$ and other m voters submit $(y, \mathcal{K} \backslash \{y\})$. This means x is elected when (R_i, R_{-i}) is chosen. However, since $x, y \in X_1'$, if i were to submit R_i' so that the tie-breaker is invoked, y will win.

Next, assume that the tie-breaker places x ahead of y. Let R_{-i} be such that among the remaining $n-1$ voters, $m-1$ voters submit $(x, \mathcal{K} \setminus \{y\})$, another m voters submit $(y, \mathcal{K} \setminus \{y\})$, one voter submits $(\{x, y\}, \mathcal{K} \setminus \{x, y\})$. This means x is elected when (R_i, R_{-i}) is chosen, after invoking the tie-breaker. However, since $x, y \in X_1'$, if i were to submit R_i' candidate y will win. ||

Condition 3

Scoring rules, approval voting and instant runoff voting (with and without the majority top-rank trigger). For these voting rules condition 3 holds vacuously because these voting rules are not CC with respect to sincere voting. To see this, consider each of the voting rules under consideration.

For scoring rules, the assertion follows from a three candidates, seventeen voters example due to Fishburn (1973) with a CW that fails to be elected under sincere voting (see also Theorem 9.1 in Moulin, 1988). To show that the same holds for arbitrary number of candidates k, consider Fishburn's example and add $k-3$ more candidates below the three candidates for all voters.

For approval voting and instant runoff voting (with and without the majority top-rank trigger), see our example in section 2 and add $k-4$ more candidates below the four candidates in the example for all voters.

Copeland rule and Simpson rule. Consider any three candidates $X = \{x, y, z\}$. Suppose that the tie-breaker places z above x and y. Fix any two strategies $R_i = (x, y, z, X_4, \ldots, X_k)$ and $R'_i = (y, x, z, X_4, \ldots, X_k)$, for any $(X_4, \ldots, X_k)$. Next, specify R_{-i} as follows: $(n-1)/2$ voters submit $(x, z\,, y, X_4, \ldots, X_k)$ and $(n-1)/2$ other voters submit $(z, y, x,\ X_4, \ldots, X_k)$.

First let us calculate the Copeland rule. For R_i submission by i, comparing x against any other candidate yields x each time a score of $+1$, so candidate x's Copeland score $CSc(x) = k-1$, and thus x is the Copeland winner. On the other hand if i submits R'_i instead, the Copeland scores are calculated as follows. Candidate x: comparison x, y yields x the score -1 and comparison of x against any other candidate yields each time x the score $+1$, so $CSc(x) = k-3$. Candidate y: comparison y, z yields y the score -1 and comparison of y against any other candidate yields each time y the score $+1$, so $CSc(y) = k-3$. Candidate z: comparison z, x yields z the score -1 and comparison of z against any other candidate yields each time z the score $+1$, so $CSc(z) = k-3$. Since z is ahead of x and y in the tie-breaker, it follows that if R'_i is chosen z will be the Copeland winner (for any other candidate w, $CSc(w) \leq k-7$).

Next, consider the Simpson rule. For R_i submission by i, the Simpson scores are $SSc(x) = (n-1)/2 + 1,\ SSc(y) = 1,\ SSc(z) = (n-1)/2$ and $SSc(w) = 0$ for any other w; thus the Simpson winner is x. On the other hand, for R'_i submission the Simpson scores are $SSc(x) = (n-1)/2,\ SSc(y) = 1,\ SSc(z) = (n-1)/2$ and $SSc(w) = 0$ for any other w. With a tie-breaker placing z ahead of x, the Simpson winner is x. ||

Our required verifications for the specific one-shot voting rules are now complete. Thus, by Theorem 5, none of the voting rules considered in Proposition 3 are CC under strategic voting. **Q.E.D.**

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Supplemental Material (Sequential Elimination vs. Instantaneous Voting)

Proof of Proposition 4:

(i) [**Plurality runoff**] Suppose there are three voters and four candidates, w, x, y, z. Fix a tie-breaking rule y, z, w, x. The voters' ranking over candidates are as follows:

$$\begin{aligned} 1: &\quad x, w, y, z \\ 2: &\quad x, y, z, w \\ 3: &\quad z, w, y, x. \end{aligned}$$

The CW is x. Also, $y\,T\,z\,T\,w\,T\,y$.

Under plurality-runoff rule, an equilibrium strategy profile is

$$1:\ y\,; \qquad 2:\ z\,; \qquad 3:\ \ w$$

in stage 1, followed by sincere voting in stage 2. In stage 1, x and w are eliminated, so that y is picked as the ultimate winner.

Given that sincere voting in stage 2 constitute a Nash equilibrium that is also weakly undominated, we only need to check that the proposed stage 1 strategies will be Nash equilibrium and weakly undominated. Clearly, the strategies are best responses to each other and therefore Nash in stage 1. So we will only verify that for any voter no other strategy weakly dominates his proposed equilibrium strategy.

The votes by voters 1 and 2 are the unique best responses, thus also weakly undominated. So let us consider voter 3's strategy. Let voters 1 and 2 choose in stage 1 respectively x and z. If voter 3 chooses w the outcome is z; on the other hand, if voter 3 chooses x or z the outcome is x, and if he chooses y the outcome is y, and both are worse compared to z.

Thus, plurality-runoff rule is not CC.[41]

(ii) [**Exhaustive ballot**] There are three types of voters, with three voters of each

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[41] Note that given a counter example for a particular tie-breaking rule, one can construct similar counter examples for other tie-breakers by simply permuting voter preferences in the same way the alternatives are permuted to obtain the new tie-breaking rule.

type. There are three alternatives with the following preferences:

type 1 : x, y, z
type 2 : y, z, x
type 3 : $x, z, y.$

The CW is x.

Consider the following strategy profile: In round 1, type 1 voters vote for x and types 2 and 3 vote for z. In round 2, if reached, each type vote for the alternative (from the remaining two) which he prefers most.

The above strategy induces z as the winner. We claim that this strategy will be an equilibrium. That round 2 voting satisfies the equilibrium conditions is trivial. So consider round 1 voting. First, note that no player can gain by deviating unilaterally in this round (this is because each type has three voters). It thus remains to argue that no weakly dominated strategies are used in round 1. As type 1 voters vote for their top-ranked candidate, clearly the strategy is undominated. So we need to argue that voter types 2 and 3 are not using weakly dominated strategies in round 1. First consider type 3 voters. Let the strategy combination in round 1 be as follows: all type 1 vote for y, all type 2 vote for z, two type 3 vote for z and one type 3 votes for x. This leads to z being elected. If on the other hand one of the two type 3 voters who voted for z now switches to either y or x, then x will be eliminated and y is the ultimate outcome, which is worse for a type 3 voter. Consider now type 2 voters. Let the strategy combination in round 1 be follows: all type 1 vote for x, all type 3 vote for z, one of type 2 votes for x and the other two vote for z. For this profile the outcome is z. If, however, one of the voters who earlier voted for z now switches to either y or x, then y will be eliminated and x is the ultimate outcome, which is worse for a type 2 voter. **Q.E.D.**

Proof of Proposition 5: Consider the one-shot version of the weakest link game. Suppose there are four candidates, w, x, y, z. Fix a tie-breaking rule y, z, w, x. Below we specify voter preferences for which CW will not be elected. For any other tie-breaking rule, a counter example can be constructed by permuting voter preferences appropriately (see footnote 41).

Consider three types of voters with two voters of each type with the following preferences (this is a replicated version of the example in section 2):

$$\begin{array}{lcl} \text{type } 1 & : & y,\ x,\ z,\ w \\ \text{type } 2 & : & z,\ x,\ y,\ w \\ \text{type } 3 & : & w,\ x,\ z,\ y. \end{array}$$

The CW is x.

For only two candidates sincere submission (i.e., voting for one's favorite candidate) is clearly the only undominated strategy for any voter. We now specify the strategies for each type of voter for three and four candidates:

$$\begin{array}{lllll} s_1(w,y,z)=y, & \mathbf{s_1(x,y,z)=x}, & s_1(w,x,z)=x, & s_1(w,x,y)=y, & s_1(w,x,y,z)=y \\ s_2(w,y,z)=z, & s_2(x,y,z)=z, & s_2(w,x,z)=z, & s_2(w,x,y)=x, & s_2(w,x,y,z)=z \\ s_3(w,y,z)=w, & s_3(x,y,z)=x, & \mathbf{s_3(w,x,z)=x}, & \mathbf{s_3(w,x,y)=x}, & s_3(w,x,y,z)=w. \end{array}$$

We verify that the strategies constitute a Nash equilibrium. For candidates $\{w,y,z\}$, the proposed strategies lead to z as the ultimate winner (given that for any two candidates under consideration the voters would vote sincerely). For a type 1 voter deviation to w or z, and for a type 3 voter deviation to y or z, still result in z as the winner; for a type 2 voter clearly deviation cannot be optimal.

For candidates $\{x,y,z\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to y or z, for a type 2 voter deviation to y or x, and for a type 3 voter deviation to y or z – all leave the voting outcome unchanged (i.e., x is the winner).

For candidates $\{w,x,z\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to w or z, and for a type 3 voter deviation to y or w, still result in x as the winner; for a type 2 voter deviation to w or x also result in x as the winner (in the first deviation, w gets eliminated using the tie-breaker).

For candidates $\{w,x,y\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to w or x, and for a type 3 voter deviation to w or y, leave the winner x unchanged; and for a type 2 voter clearly deviation cannot be optimal.

For candidates $\{w,x,y,z\}$, the proposed strategies lead to x's elimination in the first round and ultimately lead to z being the ultimate winner. Given that x is placed below other alternatives in the tie-breaker and there are two voters of each type with the same types choosing the same strategy in the proposed equilibrium, no individual voter can prevent x's elimination by altering his vote. Thus, deviation by any of the voters is never optimal.

We next argue why the proposed strategies are undominated as well. We will make our assertions for only the three vote choices indicated in bold; for the remainder, to verify that the choices are undominated is easy because in each case the voter votes for his top-ranked candidate.

First, consider a type 1 voter's decision $s_1(x, y, z) = x$. To show that for this voter voting for y cannot (weakly) dominate, suppose all other five voters vote for z, and for only two candidates remaining all vote sincerely. Then the type 1 voter by voting for y will induce z as the winner, whereas if he votes for x the winner is x; and he prefers x over y. To show that voting for z cannot dominate either, suppose the other type 1 voter votes for x and the remaining four voters all vote for y. Then if the type 1 voter votes for z the winner is z, whereas if he votes for x the winner is x; and he prefers x over z.

Next consider a type 3 voter's decision $s_3(w, x, z) = x$. To show that voting for w cannot dominate, suppose the other type 3 voter votes for x and the remaining four voters all vote for z. Then if the type 3 voter votes for w, the tie-breaker would eliminate x and sincere voting in the next round of elimination would elect z; on the other hand, if the type 3 voter voted for x then in the next round of elimination (with x and z as the candidates) sincere voting would elect x, which the type 3 voter prefers over z. To show that voting for z cannot dominate either, suppose the other type 3 voter votes for x, one from the remaining four votes for w and three others vote for z. Then if the type 3 voter votes for z, the tie-breaker eliminates x and sincere voting (with two candidates remaining) elects z as the winner. On the other hand, if the type 3 voter votes for x, alternative w will be eliminated and sincere voting with two candidates remaining would elect x, which the type 3 voter prefers.

Finally, by an argument similar to the one just given, one can show that a type 3 voter's decision $s_3(w, x, y) = x$ is undominated.

Thus, we have constructed a Nash equilibrium with undominated strategies in which the *CW*, x, is eliminated and z gets elected. **Q.E.D.**

An Example of a winner in the weakest link voting that is Pareto dominated

Consider 4 candidates, 9 voters of 3 different types with the following preferences:

- type $I(\times 3)$: z_1, z_2, z_4, z_3
- type $II(\times 3)$: z_3, z_1, z_2, z_4
- type $III(\times 3)$: z_4, z_3, z_1, z_2.

First, in order to describe the equilibrium strategy profile in this example, we need to define some notation similar to some of those in the the existence proof of Theorem 2 in Appendix A: In each subgame with C as the set of remaining candidates denote the winner in this subgame if candidate $c \in C$ is first eliminated by $\sigma(c)$ and let $M_i^c = \{c \in C \mid \sigma(c') \succ_i \sigma(c) \text{ for some } c' \in C\ \}$, for each type $i = I, II, III$ (the notation is slightly different from that in Appendix A). By the proof of Lemma 1, at this stage voting for any $c \in M_i^c$ is not weakly dominated for any type i.

Next, we describe the following equilibrium strategies:

1. C consists of two candidates: sincere voting.
2. $C = \{z_1, z_2, z_3\}$. Here z_3 is the CW and therefore the winner.
3. $C = \{z_1, z_2, z_4\}$. Here z_1 is the CW and therefore the winner.
4. $C = \{z_1, z_3, z_4\}$. Here As choose z_1, Bs choose z_1 and Cs choose z_4 and z_3 is eliminated and z_1 ends up winning.

   Note that since no agent is pivotal the above choices are best responses. Moreover, in this case $z_1 \in M_I^c = \{z_1, z_2\}$, $z_1 \in M_{II}^c = \{z_1, z_3\}$ and $z_4 \in M_{III}^c = \{z_3, z_4\}$; therefore these choices are not weakly dominated.
5. $C = \{z_2, z_3, z_4\}$. Here As choose z_2, Bs choose z_2 and Cs choose z_4 and z_3 is eliminated and z_2 ends up winning.

   Note that since no agent is pivotal the above choices are best responses. Moreover, in this case $z_2 \in M_I^c = \{z_4, z_2\}$, $z_2 \in M_{II}^c = \{z_2, z_3\}$ and $z_4 \in M_{III}^c = \{z_3, z_4\}$; therefore these choices are not weakly dominated.
6. $C = \{z_1, z_2, z_3, z_4\}$. Here type Is choose z_4, type IIs choose z_3 and type IIIs choose z_2 and z_1 is eliminated and z_2 ends up winning.

Note that since no agent is pivotal, the above choices are best responses. Moreover, in this case $z_4 \in M_I^c = \{z_1, z_4\}$, $z_3 \in M_{II}^c = \{z_1, z_2, z_3\}$ and $z_2 \in M_{III}^c = \{z_1, z_2, z_3\}$; therefore these choices are not weakly dominated.

The winner is z_2 and is Pareto dominated by z_1. ||