Complexity and Competition, Part I: Sequential Matching

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Abstract

Extensive-form market games often have a large number of non-competitive equilibria. This paper uses the complexity of non-competitive behaviour to provide a new justification for competitive equilibrium in the context of extensive-form market games with a finite number of agents. The paper demonstrates that if rational agents have (at least at the margin) an aversion for complex behaviours then their maximizing behavior will result in simple behavioral rules and thereby in a perfectly competitive outcome. In particular, we consider sequential market games with heterogenous sets of buyers and sellers and show that if the complexity costs of implementing strategies enter players' preferences, together with the standard payoff in the game, then every equilibrium strategy profile induces a competitive outcome. This is done for sequential deterministic matching/bargaining models in which at any date either the identities of the matched players are determined exogenously or one player is exogenously selected to choose his partner and make a price proposal.

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1 Introduction

Perfect competition is an abstract ideal and yet, from the beginning of the neoclassical period, it has been the central paradigm of economics. The main reason may be its attractive welfare properties. Thanks to the fundamental theorems of welfare economics, the theory of competitive equilibrium provides an important rationale for market capitalism. Another reason may be its analytical elegance and tractability. Notwithstanding these attractive features, the empirical relevance of perfect competition depends on showing that it is a good approximation to actual markets. Competition is clearly imperfect in many markets. Moreover, in theory, perfect competition requires the assumption that individual traders have no market power, an assumption that can only be justified in general if the number of traders is infinite.

There have been attempts to show that competition is a good description of finite markets. On the empirical side, there is evidence from experimental markets such as double oral auctions, as well as from financial markets with a small number of dealers or a specialist market maker. On the theoretical side, we have examples such as Bertrand competition and the durable goods monopoly model. However, in general it has not been easy to model perfectly competitive markets with a finite number of agents. This has been particularly the case in dynamic settings (see Gale (2000)). Extensive-form market games, in contrast to the elegant and simple model of perfect competition, tend to be complex and intractable because, among other things, they have large numbers of equilibria, sustained by threats and counter-threats.

In this paper, we present an alternative theoretical argument for the relevance of perfect competition in finite markets. Our argument is based on the complexity of non-competitive behavior. One of the striking features of perfect competition is that the rules of behavior are very simple. As Hayek (1945) noted, in a competitive market, economic agents only need to know their own endowments, preferences and technologies and the vector of prices at which trade takes place. Then economic agents maximizing utility subject to a budget constraint or maximizing profits subject to a technological constraint will make Pareto-efficient choices in equilibrium. We suggest that the converse might also be true: *if rational agents have, at least at the margin, an aversion to complex behaviour, then their maximizing behavior will result* *in simple behavioral rules and thereby in a perfectly competitive equilibrium.*

We start with a simple model of the market for a single indivisible good. The market is made up of a *finite* and *heterogeneous* set of buyers and of sellers who exchange the good for money. Each buyer wants at most one unit of the good and each seller has one unit of the good for sale. Trade is the result of pairwise matching and bargaining between buyers and sellers. More precisely, we consider the following process. There is a countable number of dates indexed $t = 1, 2, ...$ At each date t, one buyer and one seller are matched. One member of the pair is the *proposer*, who makes a price offer. The other is the *responder*, who accepts or rejects the proposer's price offer. If the proposal is accepted, the good is traded at the agreed price and both agents leave the market. If the proposal is rejected, there is no trade and all agents begin the next period.

The first version of the model we consider is characterized by *exogenous and deterministic* matching. That is, the identities of the proposer and the responder at each date are an exogenous and deterministic function of the set of agents remaining in the market and the date. We consider it the natural place to start for two reasons. First, our basic approach is to show that minimally complex strategies imply competitive behavior, but this is only true, given our extremely weak assumptions about complexity costs, if the environment itself is not too complex. For example, as we have shown elsewhere (Gale and Sabourian (2003a)), if the matching process is random, the notion of complexity costs used in this paper may not be sufficient to limit the equilibrium outcomes. Making stronger assumptions about complexity costs would allow us to deal with a richer set of matching procedures, but would raise questions about the empirical validity of our complexity measure. On balance, we felt the notion of complexity adopted in the present approach is the most defensible one. A second, more practical, consideration is that the simplicity of this matching process is extremely helpful in making the analysis tractable.

The first result we establish is that the market game described above has a continuum of non-competitive subgame perfect equilibria. This is not a new result. In a seminal paper, Rubinstein and Wolinsky (1990), henceforth RW, analyze a *homogeneous* market, that is, one consisting of identical buyers and identical sellers. This market has a unique competitive price but, as RW show, their model possesses a continuum of non-competitive perfect equilibrium outcomes,[1] a result reminiscent of the Folk Theorem for repeated games.[2]

RW also consider conditions under which perfect-equilibrium outcomes are competitive. For example, it is shown that a perfect equilibrium is competitive if the equilibrium strategies are Markov (stationary) or, in their terminology, anonymous. This result suggests that perfect competition may be the outcome of non-cooperative behavior if agents are required to use simple strategies.

Following this suggestion, Sabourian (2001), henceforth S, assumes there is a 'small' cost associated with choosing a more complex strategy. More precisely, agents minimize complexity costs lexicographically: they first choose a best response and then choose the least complex strategy within the set of best responses. In equilibrium, additional complexity is justified only if it is necessary to implement a best response. S shows that the perfect equilibria of RW's model satisfy this refinement if and only if the equilibrium outcome is competitive, in the sense that all trade occurs at the competitive price. Moreover, S also establishes that such a refinement implies that the equilibrium strategies are Markov. The selection result is obtained with and without discounting, with exogenous and endogenous matching, and with one or more sellers.[3]

Unfortunately, the homogeneous markets considered by RW and S are very special. In non-homogeneous markets, things are more complicated, both analytically and conceptually. In fact, as we discuss in Section 2, heterogeneous markets require a substantially different theory, not a simple ex-

[1]We use the term perfect equilibrium to embrace both subgame perfect equilibrium and sequential or perfect Bayesian equilibrium, as appropriate. In a market where a single pair of agents is matched at any time, the matching game has perfect information and it is sufficient to use the concept of subgame perfect equilibrium. Sequential or perfect Bayesian equilibrium is needed when simultaneous matching and bargaining are allowed. In the following discussion, where the technical differences are not important, we use the term perfect equilibrium to cover both cases.

[2]An important feature of RW is that it analyzes a market with a finite number of agents and with no restriction on the set of strategies. The preceding literature (Rubinstein and Wolinsky (1985), Gale (1986a,b,c, 1987), Binmore and Herrera (1988a,b), McLennan and Sonnenschein (1991)) either assumes a non-atomic continuum of agents, each of whom has a negligible effect on equilibrium or restrict the set of strategies (to stationary ones). See also the texts by Osborne and Rubinstein (1990) and Gale (2000).

[3]Some of his results depend on the precise definition of complexity used in S and on whether or not complexity costs are positive.

tension.

In this paper, we follow S and refine the set of equilibria by introducing complexity costs lexicographically with the standard payoff into the players' preference ordering.[4] Complexity is represented by a partial ordering of the set of individual strategies. Very informally, if two strategies are otherwise identical except that in some instance the first strategy uses more information than that available in the current period of bargaining and the second uses only the information available in the current period, then the first strategy is said to be more complex than the second.[5] Here, we use a similar approach. More formally, for any agent k a strategy f'_k is more complex than f_k, if there exists a non-empty set of histories $\bar{H}$ such that k is always proposer or always responder, the set of remaining agents is N, f'_k and f_k agree outside of $\bar{H}$, f_k specifies a single action on $\bar{H}$ and f'_k specifies more than one action on $\bar{H}$. This definition is a very weak measure of complexity based on complexity of behavioral rules within a period.[6]

We define a *Nash equilibrium with complexity costs* (NEC) to be a strategy profile f such that each agent k's strategy f_k is a best response and there does not exist a best response f'_k that is less complex than f_k. A *perfect equilibrium with complexity costs* (PEC) is a perfect equilibrium that is also a NEC.

The main result of the paper is that a PEC is always competitive. It shows that complexity costs are sufficient to characterize competitive behavior in a heterogeneous market, thus supporting the view that competitive equilibrium may arise in a finite market where complex behavior is costly. The result is obtained by showing that in any PEC each agent's behaviour is sufficiently simple so that a competitive outcome is the only possibility. As a corollary of our main selection result, we also demonstrate that any PEC profile must be 'Markov' or to be more precise history-independent.

Having characterized the set of PEC profiles for the simple exogenous se-

[4] Rubinstein (1986) and Abreu and Rubinstein (1988) were the first to introduce complexity of strategies lexicographically into dynamic games. However, in these papers, players are modelled as finite-state automata involved in a two-player repeated game and complexity is measured by the number of states of the automaton.

[5] Chatterjee and Sabourian (2000) use a similar notion of complexity costs to justify Markov/stationary equilibria in n-player alternating bargaining games.

[6] Note that this definition of complexity is with reference to a given set of remaining agents and thus it has a Markov flavour (for any history $h \in \bar{H}$ the set agents remaining in the market is the same). Later in the paper we discuss an alternative notion of complexity that is independent of the set of remaining agents.

quential deterministic matching model, one must next ask how far the competitive characterization of PECs extends to other (deterministic) matching processes. One easy extension is to the case of semi-endogenous sequential matching in which the choice of partners is endogenous. In this model the identity of the proposer at any date is exogenous and is given by a deterministic function of the set of remaining players; and at each date the proposer chooses the identity of his partner/responder. For this variation, the same approach will work. We show that in such a model, with an endogenous choice of responders, any PEC induces a competitive outcome.

A more drastic change would be to allow the choice of proposers and responders to be endogenous and simultaneous matching and bargaining to be observed. In this model each agent can make a proposal to any agent at any date. We consider the role of complexity in such a model in an accompanying paper (Gale and Sabourian (2003,b)). Such a model raises new difficulties because it is now possible that simultaneous and inconsistent moves are made (for example, in some period a player could simultaneously make an offer that is accepted and accept another offer). In order to make the trading process well defined, we develop an elaborate and appealing protocol to decide what happens when inconsistent behavior takes place. Given the protocol adopted we show that the selection results of this paper extend to the case of totally endogenous matching.

Since the selection result holds for all the above three matching models we can conclude that complexity considerations inducing a competitive outcome seem to be a robust result in deterministic matching and bargaining market games.

The rest of the paper is organized as follows. In the next section we discuss the differences between homogeneous and heterogeneous markets. The main results of the paper are in Sections 3 and 4. In Section 3 we deal with the exogenous sequential matching model. First we describe the model. Next, we show that, as in RW's random matching model, there is a continuum of non-competitive perfect equilibria. Then we introduce the concept of complexity and show that any PEC induces a competitive outcome. In Section 4, we establish an identical set of results for a sequential semi-endogenous matching model in which the choice of partner is endogenous. We discuss the issues that arise concerning the relative importance of complexity and off-the-equilibrium behavior in Section 5. In Section 6, we explain the difference between our definition of complexity, which refers to a given set of remaining agents, and a global notion of complexity, which is defined independently of the set of remaining agents. This later definition of complexity is rather extreme. In Section 6, we explain that such a global notion of complexity may be a too strong a concept to use in games with heterogenous agents. The final section contains further discussion of our assumptions and results. The Appendix contains the proof of the selection result for the semi-endogenous matching model of Section 4.

2 Heterogeneous markets

As we pointed out earlier, the markets analyzed by RW and by S are *homogeneous*, that is, comprising B identical buyers and S identical sellers. Without loss of generality, RW and S focus on the case in which $B > S$ (the other generic case of $B < S$ is similar) and the valuations of buyers and sellers are normalised to 1 and 0, respectively. The unique competitive price is 1 but, as we have noted, there is a continuum of non-competitive perfect equilibrium outcomes.

A *heterogeneous* market, by contrast, allows for a much richer set of equilibrium outcomes. We define a heterogeneous market as follows. As in the homogeneous case, there is a single indivisible good that is exchanged for money and each agent wants to trade at most one unit of the good. We denote the set of buyers and sellers by I and J respectively. Without loss of generality, we can assume that there are equal numbers of buyers and sellers.[7] Buyers are indexed by $i = 1, ..., n$ and sellers are indexed by $j = 1, ..., n$. Buyer i's valuation of the good is denoted by $v_i \geq 0$ and seller j's valuation is denoted by $w_j \geq 0$. We assume for simplicity that buyers and sellers can be ordered so that $v_1 > v_2 > ... > v_n$ and $w_1 < w_2 < ... < w_n$ and that $v_i \neq w_j$ for all i and j.

These valuations define the demand and supply curves that determine the competitive, market-clearing price(s) in the usual way.

Insert a demand supply figure

The marginal traders $i = j = m$ are defined by the conditions

$$v_m > w_m, \; v_{m+1} < w_{m+1},$$

[7]Sellers with extremely high valuations and buyers with extremely low valuations cannot trade in any case.

and the interval of perfectly competitive prices for this market is defined by the conditions

$$\max\{w_m, v_{m+1}\} \leq p \leq \min\{w_{m+1}, v_m\}.$$

To see why the homogeneous case is special, consider S's result showing that the unique PEC outcome is one in which the good is sold for at the competitive price $p = 1$. The intuition for S's result *in the case of a single seller* is as follows. In any non-competitive PEC there cannot be an agreement at a price of 1 between a buyer and a seller at any history; otherwise some player can economize on complexity. (For example, consider what happens if, at some history, a buyer is offered the price $p = 1$ and he accepts; then by a complexity argument the buyer should accept $p = 1$ whenever it is offered. Accepting $p = 1$ whenever it is offered guarantees the seller an equilibrium payoff of one and thus the competitive outcome). This implies that for any non-competitive PEC *all* continuation payoffs of all buyers are positive; but this cannot be so in any perfect equilibrium in which there are more buyers than sellers. (By competition, in any perfect equilibrium with $B > S$, there must be a buyer with a zero continuation payoff at some history). The result for $S > 1$ is established by induction on the number of sellers.

Now consider a heterogeneous market. We can see immediately why the analysis of S will not suffice.

- *Division of surplus*: In the homogeneous market, except for the special case $B = S$, the competitive equilibrium price is either 0 or 1 and all of the surplus goes to one side of the market. In S this property of the competitive equilibrium is used extensively to obtain the selection result. (See the discussion above on the informal intuition for S's selection result). In a heterogeneous market, by contrast, there will typically be agents receiving positive payoffs on both sides of the market in a competitive equilibrium. Therefore, one cannot rule out non-competitive outcomes simply by focusing on extreme outcomes in which one party derives no surplus from trade.

Moreover, there are several additional differences between homogeneous and heterogeneous markets that require substantive changes in the theory.

- *Efficient trade*: In a heterogeneous market, trade between an inframarginal seller $j \leq m$ and an extramarginal buyer $i > m$ is always inefficient, but can be individually rational if $v_i > w_j$. Likewise, trade between an inframarginal buyer $i \leq m$ and an extramarginal seller $j > m$ is always inefficient, but can be individually rational if $v_i > w_j$. In a homogeneous market, by contrast, individually rational trade is by definition efficient.

- *The uniqueness of equilibrium:* As mentioned above, in the generic homogeneous case, the competitive price is uniquely determined and equal to zero or one. In the generic heterogeneous case, the set of competitive prices is a non-degenerate interval. The uniqueness of the equilibrium price makes the homogeneous case easier to analyze because after one pair of agents has traded, the competitive price remains the same, so a proof by induction as in S (on the number of agents left in the market) can take for granted that the price is uniform if it is competitive. In the heterogeneous case, no such presumption can be made. After one trade, the assumption that the remaining trades take place at a competitive price does not guarantee that all take place at the same price. As a result, an induction argument (on the number of agents left in the market) that refers to "the" competitive price may become problematic.

- *Invariance of the competitive interval*: As we mentioned above, in the homogeneous market, the set of competitive prices remains constant, independently of the set of agents remaining in the market. For example, if $B > S$ then no matter how many pairs of agents have traded, the number of remaining buyers is greater than the number of remaining sellers and the competitive price remains equal to 1. In the heterogeneous market, this need not be so. For example, if the competitive interval is $[w_m, v_m]$ and the marginal buyer and seller trade first, the competitive interval becomes

$$[\max\{v_{m+1}, w_{m-1}\}, \min\{v_{m-1}, w_{m+1}\}],$$

which is strictly larger given the assumption that $v_{m-1} > v_m > v_{m+1}$ and $w_{m-1} < w_m < w_{m+1}$. Similarly, if a buyer $i \leq m$ trades with a seller $j > m$, or a buyer $i > m$ trades with a seller $j \leq m$, then the competitive interval changes. In some cases, the new competitive interval may not even intersect the old one. The fact that the competitive interval of prices may change as the result of trade exacerbates the problems associated with using an induction hypothesis that refers to "the" competitive price (future prices may be conditioned on past trades even if prices are restricted to be competitive ones).

One way of summarizing these differences is to say that, in a heterogeneous market, there are many more types of non-competitive behavior.[8] A refinement that characterizes competitive behavior in the homogeneous case may be consistent with non-competitive behavior in the heterogeneous case; alternatively, a refinement that is consistent with equilibrium in the homogeneous case may lead to non-existence in the heterogeneous case. For all these reasons, the analysis of heterogeneous markets turns out to be more complicated and more subtle than the analysis of the homogeneous market.

3 Exogenous Sequential matching

3.1 The model

The data for the heterogenous market described in the last section is given by a triple (K, v, w) where $K = I \cup J$ is the set of buyers and sellers, and $v = (v_1, ..., v_n)$ and $w = (w_1, ..., w_n)$ denote the valuations of the buyers and sellers, respectively, for a single unit of the good. For any such market data (K, u, v) a deterministic exogenous sequential dynamic matching and bargaining game is defined by the following rules:

- Trade takes place at a sequence of dates $t = 1, 2, ...$ At each date t, a pair of agents consisting of one buyer and one seller is chosen from the set $N \subseteq K$ of agents remaining in the market. One member of the pair is the proposer $\pi_t(N) \in N$; the other is the responder $\rho_t(N) \in N$.
- The agent who is chosen as the proposer offers a price p. The responder can accept (A) or reject (R) this price. If the proposal is accepted, the good is traded at the agreed price and both agents leave the market. If the proposal is rejected, there is no trade and all agents begin the next period with the same endowments.

[8] In contrast with the homogeneous market studied by RW, even the Markov property is not sufficient for perfect competition in heterogeneous markets with *random matching*. Gale and Sabourian (2003a) present robust examples with heterogenous sets of buyers and of sellers in which there exists a continuum of non-competitive Markov perfect equilibria.

- There is no discounting of utilities. If buyer i and seller j exchange a unit of the good at a price p, buyer i's payoff is $v_i - p$ and seller j's payoff is $p - w_j$.

Note that only one pair is formed at each date. Furthermore, if a player is matched at date t he must be either a proposer or a responder. If the proposer $\pi_t(N)$ is a buyer then the responder $\rho_t(N)$ is a seller and vice versa.

A set $N \subseteq K$ is called *balanced* if it contains equal numbers of buyers and sellers. Let $\mathcal{N}$ denote the set of non-empty balanced sets. Then the game form is defined by a sequence of *matching functions* $(\pi, \rho) = \{(\pi_t, \rho_t)\}_{t=1}^{\infty}$ such that for any date t and any $N \in \mathcal{N}$,

$$(\pi_t(N), \rho_t(N)) \in N \times N,$$

$$[\pi_t(N) \in I] \Longleftrightarrow [\rho_t(N) \in J] .$$

To obtain the selection result of this section we shall also make the following assumption which ensures the richness of the matching.

Assumption 1 For any $N \in \mathcal{N}$, for any feasible matches $(k, \ell) \in N^2$ and $(k', \ell') \in N^2$ such that $(k, \ell) \neq (k', \ell')$, and for any date T, there exists a $t > T$ such that

$$(\pi_t(N), \rho_t(N)) = (k, \ell) \text{ and } \left(\pi_{t+1}(N), \rho_{t+1}(N)\right) = (k', \ell').$$

The dynamic matching and bargaining game is one of complete and perfect information and is described fully by the data (K, u, v, π, ρ).

The outcome of the game at any date is described by an ordered four-tuple $(k, \ell, p, r) \in K \times K \times \mathbf{R}_+ \times \{A, R\}$, where k is the proposer, ℓ the responder, p the price offer and r the response. The history of the game up to the beginning of date t consists of a sequence $h_t = ((k^1, \ell^1, p^1, r^1), ..., (k^{t-1}, \ell^{t-1}, p^{t-1}, r^{t-1}))$. Let H^t denote the set of histories at date t and let $H = \cup_{t=1}^{\infty} H^t$. The trivial (null) history at date 1 is denoted by H^1. Note that the history $h \in H^t$ uniquely defines the set of agents $N(h)$ remaining in the game at date t. Thus, the history also uniquely defines the identity of the proposer and responder at date t. For any history $h \in H^t$, we denote the proposer by $\pi(h) \equiv \pi_t(N(h))$ and the responder by $\rho(h) \equiv \rho_t(N(h))$.

For any agent k, let $H_k^p = \pi^{-1}(k)$ denote the set of histories at which k is the proposer and let $H_k^r = \rho^{-1}(k)$ denote the set of histories at which k is the responder. At any history $h \in H_k^p$, agent k knows the history h, the set of remaining agents $N(h)$ and the identities of the proposer $\pi(h) = k$ and the responder $\rho(h)$ before he makes his offer. For any history $h \in H_k^r$, agent k knows the history h, the identities of the proposer $\pi(h)$ and the responder $\rho(h) = k$ and the price offer p before he makes his response r. A strategy for agent k is a function f_k defined on $H_k \equiv H_k^p \cup (H_k^r \times \mathbf{R}_+)$ such that

$$\begin{array}{ll} f_k(h) \in \mathbf{R}_+, & \forall h \in H_k^p \\ f_k(h,p) \in \{A, R\}, & \forall (h,p) \in H_k^r \times \mathbf{R}_+. \end{array}$$

Let F_k denote the strategy set for agent k and let $F = \times_{k \in K} F_k$ denote the set of strategy profiles.

Given any strategy profile $f = \{f_k\}_{k \in K}$, the outcome is uniquely determined. In fact, the outcome of the offers and the responses $\{(k^t, \ell^t, p^t, r^t)\}_{t=1}^{\infty}$ is defined recursively by putting

$$(k^1, \ell^1, p^1, r^1) = \left(\pi(\emptyset), \rho(\emptyset), f_{\pi(\varnothing)}(\varnothing), f_{\rho(\varnothing)}(\varnothing, f_{\pi(\varnothing)}(\varnothing))\right)$$

and

$$(k^t, \ell^t, p^t, r^t) = \left(\pi(h), \rho(h) f_{\pi(h)}(h), f_{\rho(h)}(h, f_{\pi(h)}(h))\right),$$

where $h = ((k^1, \ell^1, p^1, r^1), ..., (k^{t-1}, \ell^{t-1}, p^{t-1}, r^{t-1}))$, for $t = 2, 3, ...$ Let $U_k(f)$ denote the payoff to agent k from this outcome.

The market game $\Gamma = (K, F, U)$ is defined by the set of players K, the set of strategy profiles F, and the payoff function $U = (U_k)_{k \in K}$.

3.2 A continuum of non-competitive perfect equilibria

In this section we show that there is a continuum of non-competitive perfect equilibria[9] in the above matching model. Moreover, we show that some of these non-competitive equilibria are inefficient.

To illustrate the ideas, it suffices to consider the simple case of two buyers and two sellers and one inframarginal buyer and one inframarginal seller. It will be evident from the proof below that the construction of non-competitive

[9] Henceforth, since the games we consider are one of perfect and complete information, the term perfect equilibrium refers to *subgame perfect equilibrium* (see footnote 1).

outcomes below has a similar flavor to that of RW for the random matching model and that the results extend immediately to the case of arbitrary numbers of buyers and sellers.

Proposition 1 *Suppose that $N = 2$, $m = 1$ and*

$$w_1 < v_2 < w_2 < v_1.$$

Consider any deterministic exogenous sequential matching technology that satisfies Assumption 1. Then for any $p_h \in [w_2, v_1]$ and any $p_\ell \in [w_1, v_2]$

(i) there exists a perfect equilibrium such that w_2 and v_1 trade at p_h and w_1 and v_2 trade at p_ℓ;

(ii) there exists a perfect equilibrium such that v_1 and w_1 trade at p_b where $b = h$ or ℓ.

Before stating the proof of the above proposition notice, that the perfect equilibrium described in part (i) is inefficient and the equilibria described in parts (i) and (ii) are not competitive (the competitive equilibria of this example involve v_1 and w_1 trading at a uniform price $p \in [v_2, w_2]$).

Proof. The equilibrium strategy profile will be described by a collection of three states $\{s, s_h, s_\ell\}$ and rules of transition between them. We shall denote the transition function by $\mu : \{s, s_\ell, s_h\} \times \Sigma \to \{s, s_\ell, s_h\}$, where Σ is the set of outcomes in a given period. In each of the three states, irrespective of the number of players left, the players behave as follows.

State s : In this state each pair $(v_i, w_j) \neq (v_2, w_2)$ does the following if matched, irrespective of who is the proposer.

- (v_1, w_1) do not agree; in particular v_1 offers p_ℓ and accepts a price p if and only if $p \leq p_\ell$ and w_1 offers p_h and accepts a price p if and only if $p \geq p_h$.

- (v_1, w_2) agree on p_h; in particular in this match both offer p_h as the proposer, v_1 accepts p if and only if $p \leq p_h$ and w_2 accepts p if and only if $p \geq p_h$.

- (v_2, w_1) agree on p_ℓ; in particular in this match both offer p_ℓ as the proposer, v_2 accepts p if and only if $p \leq p_\ell$ and w_2 accepts p if and only if $p \geq p_\ell$.

State s_ℓ : In this state each pair $(v_i, w_j) \neq (v_2, w_2)$ does the following if matched, irrespective of who is the proposer.

- (v_1, w_1) agree on p_ℓ; in particular in this match both offer p_ℓ as the proposer, v_1 accepts p if and only if $p \leq p_\ell$ and w_1 accepts p if and only if $p \geq p_\ell$.

- (v_1, w_2) do not agree; in particular v_1 offers p_ℓ and accepts p if and only if $p \leq p_\ell$ and w_2 offers p_h and accepts p if and only if $p \geq w_2$.

- (v_2, w_1) do not agree; in particular v_2 offers p_ℓ and accepts p if and only if $p \leq p_\ell$ and w_1 offers p_h and accepts p if and only if $p \geq p_h$.

State s_h : In this state each pair $(v_i, w_j) \neq (v_2, w_2)$ does the following if matched, irrespective of who is the proposer.

- (v_1, w_1) agree on p_h in particular in this match both offer p_h as the proposer, v_1 accepts p if and only if $p \leq p_h$ and w_1 accepts p if and only if $p \geq p_h$.

- (v_1, w_2) do not agree; in particular v_1 offers p_ℓ and accepts p if and only if $p \leq p_\ell$ and w_2 offers p_h and accepts p if and only if $p \geq p_h$.

- (v_2, w_1) do not agree in particular v_2 offers p_ℓ and accepts p if and only if $p \leq v_2$ and w_1 offers p_h and accepts p if and only if $p \geq p_h$.

Finally, in any state where the agents (v_2, w_2) are matched they do not agree; in particular v_2 offers w_1 and accepts p if and only if $p \leq v_2$ and w_2 offers v_1 and accepts p if and only if $p \geq w_2$.

For any $e \in E$ the transition function μ satisfies the following

$$\mu(s, e) = \begin{cases} s_h & \text{if } e = (v_1, w_1, p, R) \text{ and } p \neq p_\ell \\ s_\ell & \text{if } e = (w_1, v_1, p, R) \text{ and } p \neq p_h \\ s & \text{otherwise} \end{cases}$$

$$\mu(s_\ell, e) = \begin{cases} s_h & \text{if } e = (v_2, w_1, p, R) \text{ and } p \neq p_\ell \\ s & \text{if } e = (w_1, v_2, p, R) \text{ and } p \neq p_h \\ s_\ell & \text{otherwise} \end{cases}$$

$$\mu(s_h, e) = \begin{cases} s & \text{if } e = (v_1, w_2, p, R) \text{ and } p \neq p_\ell \\ s_\ell & \text{if } e = (w_2, v_1, p, R) \text{ and } p \neq p_h \\ s_h & \text{otherwise} \end{cases}$$

Notice that if the current state is s then the above strategy profile results in (v_1, w_2) agreeing to p_h and (v_2, w_1) agreeing to p_ℓ. Similarly, if the current state is s_b (for $b = h$ or ℓ) then the above strategy profile results in (v_1, w_1) agreeing on p_b. Thus the continuation payoff of each player in each state is as follows:

	s	s_ℓ	s_h
v_1	$v_1 - p_h$	$v_1 - p_\ell$	$v_1 - p_h$
v_2	$v_2 - p_\ell$	0	0
w_1	$p_\ell - w_1$	$p_\ell - w_1$	$p_h - w_1$
w_2	$p_h - w_2$	0	0

(1)

To verify that the above profile is a perfect equilibrium we need to show that in each of the three states the strategy attributed to each player is optimal given the strategies of the others. To show this, note the following points. First, in any state and in any match, any deviation offer by a proposer that is better for the proposer than his continuation payoff in that state results in a rejection by the responder and the same continuation payoff as the proposer would have obtained had he not deviated. For example, if v_1 deviates from his equilibrium proposal in state s and proposes a price $p < p_h$ and $p \neq p_\ell$ then the responder rejects, the state in the next period will be either s or s_h and v_1 would obtain a payoff of $v_1 - p_h$. But this is just v_1's continuation payoff in state s if he does not deviate.

Secondly, in any state the responder rejects any deviation by the proposer that is better for the proposer than his continuation payoff because rejection results in a transition that induces a payoff for the responder no less than that he would obtain from accepting the deviation proposal. For example, in state s or s_ℓ buyer v_1's strategy prescribes accepting an offer p from w_1 if and only if $p \leq p_\ell$. Such a response is optimal for v_1 in states s or s_ℓ because (i) if $p = p_h$ and the current state is s then rejection by v_1 induces the same state s the next period and v_1 will obtain a continuation payoff of $v_1 - p_h$ and (ii) if $p \neq p_h$ or the current state is s_ℓ then rejection by v_1 induces state s_ℓ the next period and v_1 will receive a continuation payoff $v_1 - p_\ell$. Similarly, in state s_h buyer v_1's strategy prescribes accepting an offer p from w_1 if and only if $p \geq p_h$. Such a response is optimal for v_1 in state s_h because if v_1 rejects the offer p by w_1 the players remain in state s_h and v_1 receives a continuation payoff $v_1 - p_h$. By inspection it can be verified that the responses of all players in all states are optimal.

Finally, note that appropriate choice of the initial state establishes the results. Thus if the initial state is s then the above describes the required perfect equilibrium strategy profile in part (i) of the Proposition and if the initial state is s_b ($b = \ell$ or h) then the above describes the required perfect equilibrium strategy profile in part (ii) of the Proposition. ■

3.3 Complexity and Equilibrium Characterisation

3.3.1 Complexity

Before introducing the notion of complexity, we need some further notation.

For any two players $k, \ell \in K$ let $\langle k, \ell \rangle$ denote the match between k and ℓ in which k is the proposer and ℓ the responder. Let $H\langle k, \ell \rangle = \{ h \in H | \, \pi(h) = k \text{ and } \rho(h) = \ell \}$ denote the set of histories resulting in the match $\langle k, \ell \rangle$. For any balanced set of agents $N \in \mathcal{N}$, let $H(N) = \{ h \in H | \, N(h) = N \}$. Finally, for any $N \in \mathcal{N}$ and any match $\langle k, \ell \rangle$, let $H(N, \langle k, \ell \rangle) \equiv H(N) \cap H\langle k, \ell \rangle$ be the set of histories at which N is the set of remaining agents, k is the proposer and ℓ is the responder.

Definition 2 *For any agent k, a strategy f_k is Markov*[10] *if for all ℓ and for all $N \in \mathcal{N}$*

$$\begin{array}{ll} f_k(h) = f_k(h'), & \forall h, h' \in H(N, \langle k, \ell \rangle) \\ f_k(h, p) = f_k(h', p), & \forall h, h' \in H(N, \langle \ell, k \rangle) \text{ and } \forall p \in \mathbf{R}_+. \end{array} \tag{2}$$

Complexity of a strategy can be measured in many different ways. As we mentioned before, in this paper we use a very plausible measure of the complexity that consists of a partial order on the set of strategies that very roughly satisfies the following property: if two strategies are otherwise identical except that in a given match either as a proposer or as a responder to some price offer p, the first strategy always behaves the same way (always makes the same proposal as a proposer or always accepts or always rejects as a responder) whereas the second strategy does not, then the second strategy should be considered as being more complex than the first.

[10]Note that Markov strategies here refer to strategies that are history-independent. Since the matching technology may not be staionary the above definition of Markov strategies does not strictly correspond to the standard Markov definition that strategies depend only on payoff relevant variables.

Definition 3 *(i) For any player k, a strategy f_k is simple at N when k is the proposer in a match with player ℓ if*

$$f_k(h) = f_k(h'), \forall h, h^{'} \in H(N, \langle k, \ell \rangle).$$

(ii) For any player k, a strategy f_k is simple at N when k is the responder to a price p in a match with player ℓ if

$$f_k(h,p) = f_k(h',p),\ \forall h, h^{'} \in H(N, \langle \ell, k \rangle).$$

(iii) For any player k, a strategy f_k is simple at N if and only if for any $\ell \in N$ and $p \in \mathbf{R}_+$ strategy f_k is simple at N both when k is the proposer and when k is the responder to p in a match with ℓ.

Note that a strategy f_k is Markov if and only if for any $N \in \mathcal{N}$ strategy f_k is simple at N.

Definition 4 *For any agent k, a strategy f'_k is more complex than f_k, denoted by $f^{'}_k \succ f_k$, if one of the following two conditions is satisfied:*

1. *there exists a balanced set N and a player $\ell \in N$ such that f_k and f'_k are otherwise identical except that f_k is simple at N when k is the proposer in a match with ℓ and f'_k is not; formally $\exists N \in \mathcal{N}$ and $\ell \in N$ such that*

$$\begin{array}{ll} f_k(h) = f'_k(h), & \forall h \in H^p_k \backslash H(N, \langle k, \ell \rangle), \\ f_k(h,p) = f'_k(h,p), & \forall h \in H^r_k \text{ and } \forall p \in \mathbf{R}_+, \\ f_k(h) = f_k(h') & \forall h, h^{'} \in H(N, \langle k, \ell \rangle), \\ f'_k(h) \neq f'_k(h') & \text{for some } h \text{ and } h^{'} \in H(N, \langle k, \ell \rangle); \end{array}$$

2. *there exists a balanced set N, a player $\ell \in N$ and a price p such that f_k and f'_k are otherwise identical except that f_k is simple at N when k is the responder to p in a match with ℓ and f_k is not; formally $\exists N \in \mathcal{N}, \ell \in N$ and $p \in \mathbf{R}_+$ such that*

$$\begin{array}{ll} f^{'}_k(h) = f_k(h), & \forall h \in H^p_k, \\ f^{'}_k(h,p') = f_k(h,p'), & \text{if either } h \in H^r_k \backslash H(N, \langle \ell, k \rangle) \text{ or } p' \neq p, \\ f_k(h,p) = f_k(h',p), & \forall h, h^{'} \in H(N, \langle \ell, k \rangle), \\ f'_k(h,p) \neq f'_k(h',p), & \text{for some } h \text{ and } h' \in H(N, \langle \ell, k \rangle). \end{array}$$

If f_k is not more complex than f'_k we write $f'_k \succeq f_k$.

This complexity criterion ranks any two strategies only if, in some situation, one of them is simple and the other is not. Effectively, our definition of complexity imposes a partial order that reflects the degree of stationarity of the strategies. Notice that, although Markov strategies are in some sense minimally complex, this does not imply that players necessarily choose Markov strategies in equilibrium. A player prefers a non-Markov strategy to a Markov strategy if the former guarantees a higher payoff than the latter. In other words, our notion of equilibrium requires that additional complexity be justified by a higher payoff. Given the equilibrium behaviour of his opponents, a best response may require a player to engage in non-Markov behavior.

3.3.2 Nash equilibrium and complexity

We now define *Nash equilibrium* of the game with complexity cost.

Definition 5 *A strategy profile $f \in F$ constitutes a Nash equilibrium with lexicographic complexity cost (denoted by NEC) if, for each player k, the following two conditions hold*

$$\begin{array}{ll} U_k(f_k, f_{-k}) \geq U_k(f'_k, f_{-k}), & \forall f'_k \in F_k; \\ \left[U_k(f_k, f_{-k}) = U_k(f'_k, f_{-k})\right] \Longrightarrow \left[f'_k \succeq^r f_k\right], & \forall f'_k \in F_k. \end{array}$$

From this definition we can immediately infer some useful restrictions on the strategies that form a NEC. The first result shows that if an agent accepts an offer from another agent along the equilibrium path, then he always accepts the offer, irrespective of the previous history, whenever the two agents are matched in the same way with the same remaining set of agents. More precisely, let f be a NEC and let $E \subset H$ denote the set of finite histories that occur along the equilibrium path of f. Suppose the match $\langle k, \ell \rangle$ at some history $h_0 \in E$ results in an agreement at a price p_0 when the remaining set of agents is $N = N(h_0)$. Then agent ℓ accepts p_0 whenever the match $\langle k, \ell \rangle$ occurs and the set of remaining agents is N.

Proposition 6 *Let f be a NEC and let $E \subset H$ denote the set of finite histories that occur along the equilibrium path of f. Suppose that, for some finite history $h_0 \in E \cap H(N, \langle k, \ell \rangle)$,*

$$f_k(h_0) = p_0, f_\ell(h_0, p_0) = A.$$

Then we have

$$f_\ell(h, p_0) = A, \forall h \in H\left(N, \langle k, \ell \rangle\right). \tag{3}$$

Proof. The proof is by contradiction. Suppose first that (3) does not hold; then $f_\ell(h_1, p_0) = R$ for some $h_1 \in H\left(N, \langle k, \ell \rangle\right)$. Define a new strategy f'_ℓ as follows. For any $h \in H^p_\ell$ put $f'_\ell(h) = f_\ell(h)$ and for any $(h, p) \in H^r_\ell \times \mathbf{R}_+$ let

$$f'_\ell(h, p) = \begin{cases} A & \forall h \in H\left(N, \langle k, \ell \rangle\right) \text{ and } p = p_0 \\ f_\ell(h, p) & \text{otherwise} \end{cases}$$

By inspection we see that $f_\ell \succ f'_\ell$ because $f_\ell(h_0, p_0) \neq f_\ell(h_1, p_0)$. Further, f'_ℓ must give the same payoff as f_ℓ. To see this, note that if ℓ chooses f'_ℓ then after any history $h \in H\left(N, \langle k, \ell \rangle\right)$, either agent k proposes p_0 and ℓ accepts, so the payoff to agent ℓ is the same as from the equilibrium strategy, or agent k does not offer p_0, in which case the change in the strategy is not observed and the play of the game is unaffected by the deviation. In either case, the strategy f'_ℓ is less complex than f_ℓ and the payoff to agent ℓ from f'_ℓ is the same as from f_ℓ, contradicting the definition of NEC. Thus, $f_\ell(h, p_0) = A$ for any $h \in H\left(N, \langle k, \ell \rangle\right)$. ■

The second result establishes that in any NEC each player's response to another player's offer is simple. Thus for any match $\langle k, \ell \rangle$ and any price p_0, either ℓ always accepts or ℓ always rejects the offer p_0 by k.

Proposition 7 *Let f be a NEC. Then for any $N \in \mathcal{N}$, any players $k, \ell \in N$ and any price offer p_0 strategy f_ℓ is simple at N when ℓ is the responder to p_0 in a match with k.*

Proof. Suppose not; then for some $N \in \mathcal{N}$, some match $\langle k, \ell \rangle$ and some price offer p_0, there exist histories $h', h'' \in H\left(N, \langle k, \ell \rangle\right)$ such that

$$f_\ell(h', p_0) = R \text{ and } f_\ell(h'', p_0) = A.$$

It then follows from Proposition 6 that

$$\left[f_k\left(h\right) = p_0\right] \Longrightarrow \left[f_\ell\left(h, p_0\right) = R\right], \forall h \in E \cap H\left(N, \langle k, \ell \rangle\right). \tag{4}$$

where $E \subset H$ denotes the set of histories that occur along the equilibrium path of f. Now define a new strategy f'_ℓ as follows. For any $h \in H^p_\ell$ put $f'_\ell(h) = f_\ell(h)$ and for any $(h, p) \in H^r_\ell \times \mathbf{R}_+$ let

$$f'_\ell(h, p) = \begin{cases} R & \text{if } h \in H\left(N, \langle k, \ell \rangle\right) \text{ and } p = p_0 \\ f_\ell(h, p) & \text{otherwise.} \end{cases}$$

Clearly, $f_\ell \succ^r f'_\ell$. Further, f'_ℓ must give the same payoff as f_ℓ because, by (4), they do not differ on the equilibrium path. This contradicts the definition of a NEC. ■

The next result shows that if at some history two agents trade at a price that results in a payoff no less than the proposer's equilibrium payoff, then they make the same trade whenever they are matched in the same way with the same remaining set of agents.

Proposition 8 *Let f be a NEC. Consider any player k and any price p_0 such that trading at p_0 for player k results in a payoff no less than k's equilibrium payoff $U_k(f)$; thus*

$$\begin{array}{ll} v_k - p_0 \geq U_k(f) & \text{if } k \text{ is a buyer} \\ p_0 - w_k \geq U_k(f) & \text{if } k \text{ is a seller.} \end{array}$$

Suppose that, for some N and for some finite history $h_0 \in H(N, \langle k, \ell \rangle)$,

$$f_k(h_0) = p_0 \text{ and } f_\ell(h_0, p_0) = A. \tag{5}$$

Then we have

$$f_k(h) = p_0, \forall h \in H(N, \langle k, \ell \rangle), \tag{6}$$

$$f_\ell(h, p_0) = A, \forall h \in H(N, \langle k, \ell \rangle). \tag{7}$$

Proof. Condition (7) follows immediately from condition (5) and Proposition 7.

Now to show condition (6) suppose otherwise. Then $f_k(h_1) \neq p_0$, for some $h_1 \in H(N, \langle k, \ell \rangle)$. Define a new strategy f'_k as follows. Put $f'_k(h,p) = f_k(h,p)$ for any $(h,p) \in H_k^r \times \mathbf{R}_+$ and let

$$f'_k(h) = \begin{cases} p_0 & \forall h \in H(N, \langle k, \ell \rangle) \\ f_k(h) & \text{otherwise,} \end{cases}$$

for any $h \in H_k^p$. Then $f_k \succ f'_k$ by inspection. Moreover, the choice of (f'_k, f_{-k}) either induces a history $h \in H(N, \langle k, \ell \rangle)$ in which case, by the definition of f'_k and condition (7), k and ℓ trade at the price p_0 and k obtains a payoff no less than $U_k(f)$ or no history $h \in H(N, \langle k, \ell \rangle)$ occurs in which case the payoff of agent k is unaffected by the deviation to f'_k. Therefore, f'_k induces at least the same payoff as f_k contradicting the definition of NEC. ■

The fourth result shows trivially that if, along the equilibrium path, k does not make an offer to ℓ, then k's behavior is simple as a proposer in a match with ℓ, assuming the set of remaining agents is the same in each case.

Proposition 9 *Let f be a NEC and let $E \subset H$ denote the set of finite histories that occur along the equilibrium path of f. Then for all $N \in \mathcal{N}$ and any $k, \ell \in N$ if $E \cap H(N, \langle k, \ell \rangle) = \emptyset$ then f_k is simple at N when k is the proposer in a match with ℓ.*

Proof. Suppose not; then there exists histories $h', h'' \in H(N, \langle k, \ell \rangle)$ such that $f_k(h') \neq f_k(h'')$. Define a new strategy f'_k as follows. For any $(h, p) \in H^r_k \times \mathbf{R}_+$ put $f'_k(h, p) = f_k(h, p)$, for any $h \in H^p_k$ put

$$f'_k(h) = \begin{cases} f_k(h') & \text{if } h \in H(N, \langle k, \ell \rangle) \\ f_k(h) & \text{otherwise.} \end{cases}$$

Note that by inspection $f_k \succ f'_k$. Further, f'_k must give the same payoff as f_k because by assumption they do not differ on the equilibrium path. This contradicts the definition of a NEC. ■

3.3.3 Perfect equilibrium, complexity and competitive equilibrium

Next we consider perfect equilibria and complexity costs introduced lexicographically.

Definition 10 *A strategy profile f is called a* perfect equilibrium with complexity costs *(PEC) if it is both a perfect equilibrium and a NEC.*

In this subsection we characterize the set of PECs of the above game.

First note that trivially for any PEC profile f if any player k receives a zero payoff then k must have a Markov strategy. Otherwise, it follows from Proposition 7 that for some $N \in \mathcal{N}$ and $\ell \in N$, $f_k(h_0) \neq f_k(h_1)$ for some $h_0, h_1 \in H(N, \langle k, \ell \rangle)$. But then consider another strategy f'_k that is otherwise identical to f_k except that $f'_k(h) = f_k(h_0)$ for any $h \in H(N, \langle k, \ell \rangle)$. Clearly, $f_k \succ f'_k$. Moreover since f is a perfect equilibrium it must be that k's continuation payoff is non-negative in any subgame. Therefore, by the definition of f'_k and Proposition 7 (each player's response is simple), f'_k guarantees at least the equilibrium payoff of zero. Thus we have the following.

Proposition 11 *Let f be a PEC. Then for any k, f_k is Markov if $U_k(f) = 0$.*

Next, for any $N \in \mathcal{N}$, let $C(N)$, $\mathcal{I}(N)$ and $\mathcal{X}(N)$ denote respectively the competitive price interval, the set of inframarginal players and the set of extramarginal players when N is the set of (remaining) agents in the market. We also denote the set of subgames defined by the set of histories $H(N)$ by $\Gamma(N)$.

The next theorem is the main results of this Section.

Theorem 12 *Suppose the matching technology satisfies Assumption 1. Consider any PEC profile f and any $N \in \mathcal{N}$. Then the strategy profile f induces a competitive outcome (for the set N) in all subgames $\Gamma(N)$ and any trade occurs at the same competitive price in all such subgames. Therefore, strategy profile f is such that*

1. *either $C(N)$ is empty and there is no trade in any subgame $\gamma \in \Gamma(N)$;*
2. *or there exists a competitive price $\bar{p} \in C(N)$ such that in any subgame $\gamma \in \Gamma(N)$ all inframarginal agents $\mathcal{I}(N)$ trade at $\bar{p}$ and all extramarginal agents $\mathcal{X}(N)$ do not trade.*

Corollary 13 *Suppose the matching technology satisfies Assumption 1. Then any PEC profile f is Markov.*

Remark 1 *To obtain the above characterisation results, Assumption 1 (or a similar condition) is needed. We have examples that do not satisfy the assumption and have non-competitive PECs.*

Proof of Theorem 12 and Corollary 13

Before stating the proof of Theorem 12 we need to introduce one further piece of notation and establish two lemmas. We define the set of histories $h \in H(N)$ such that k and ℓ are, respectively, the proposer and the responder at h and k' and ℓ' are respectively the proposer and the responder the next period if k and ℓ do not reach an agreement at h by

$$H(N, \langle k, \ell \rangle, \langle k', \ell' \rangle) = \left\{ h \in H(N) \,\middle|\, \begin{array}{l} \text{if } h \in H^t \text{ then } (\pi_t(N), \rho_t(N)) = (k, \ell) \\ \text{and } (\pi_{t+1}(N), \rho_{t+1}(N)) = (k', \ell') \end{array} \right\}$$

Lemma 14 *Consider any fixed but arbitrary PEC f. Suppose that for some $N \in \mathcal{N}$ there is no trade in some subgame $\gamma' \in \Gamma(N)$. Then in every subgame $\gamma \in \Gamma(N)$ there is no trade and the outcome is competitive (all agents belonging to N are extramarginal and $C(N)$ is empty).*

Proof. We establish this result in several steps.

Step 1: There cannot be an agreement at any $h \in H(N) \cap E$. Otherwise, there exists players $k, l \in N$ and a history $h' \in H(N, \langle k, \ell \rangle) \cap E$ such that k and l trade at h'; by Proposition 8 this implies that players k and l always trade at any $h \in E \cap H(N, \langle k, \ell \rangle)$. But this, together with Assumption 1, contradicts the hypothesis that there is no trade in the subgame $\gamma' \in \Gamma(N)$.

Step 2: There is no trade in any subgame $\gamma \in \Gamma(N)$. Otherwise, there exists players $k', l' \in N$ and a history $h' \in H(N, \langle k', \ell' \rangle)$ such that k' and l' trade at h'. Now by Step 1 either $E \cap H(N)$ is empty or $U_k(f) = 0$ for all $k \in N$. But then, by Propositions 7, 9 and 11, it must be that f_k is simple at N for each $k \in N$. This implies that players k' and l' always trade at *any* history $h \in H(N, \langle k', \ell' \rangle)$. But this, together with Assumption 1, contradict the hypothesis that there is no trade in the subgame $\gamma' \in \Gamma(N)$.

Step 3: every $k \in N$ is extramarginal and $C(N)$ is empty. Otherwise, for some i and $j \in N$ we have $v_i > w_j$. But by Step 2, the continuation payoff of each player $k \in N$ is zero at every history $h \in H(N)$. This implies that j accepts any offer $p' > w_j$ at any $h' \in H(N, \langle i, j \rangle)$. Thus, buyer i can obtain a positive payoff by offering a price $p' \in (w_j, v_j)$; but this contradicts the conclusion that i's continuation payoff at any $h \in H(N)$ is zero. ■

Lemma 15 *Consider any fixed but arbitrary PEC strategy profile f. Suppose that for some $N \in \mathcal{N}$*

1. *there exists a subgame $\gamma' \in \Gamma(N)$ such that there is one and only one trade in the subgame γ';.*
2. *f_k is simple at N when k is the proposer in a match with ℓ for any pair of buyers and sellers $k, \ell \in N$.*

Then there exists a competitive price $p_1 \in C(N)$ such that in any subgame $\gamma \in \Gamma(N)$ the outcome is competitive and any trade takes place at p_1.

Proof. Let i_1 and j_1 denote respectively the (only) buyer and the (only) seller that trade in the subgame γ'. Also denote the history, the match and the price at which i_1 and j_1 trade in this subgame by $h_1, \mid i_1, j_1 \mid$ and p_1, respectively. Thus if players choose f in the subgame γ' it results in a history $h_1 \in H(N_1)$ such that the following conditions hold

$$\mid i_1, j_1 \mid = \langle \pi(h_1), \rho(h_1) \rangle$$

$$f_{\pi(h_1)}(h_1) = p_1, f_{\rho(h_1)}(h_1, p_1) = A$$

where $\pi(h_1)$ is the proposer and $\rho(h_1)$ is the responder in the match $\mid i_1, j_1 \mid$. But then since by assumption $f_{\pi(h_1)}$ is simple at N when $\pi(h_1)$ is the proposer in a match with $\rho(h_1)$ it follows from Proposition 7 that

$$i_1 \text{ and } j_1 \text{ reach an agreement at any } h \in H(N_1, \mid i_1, j_1 \mid) \tag{8}$$

Now we show that i_1 and j_1 are the only inframarginal agents in N (for any $i, j \in N \backslash \{i_1, j_1\}$, $v_i < w_j$), $p_1 \in C(N)$ and in any subgame $\gamma \in \Gamma(N)$, i_1 and j_1 trade at the price p_1 and this is the only trade that occurs. This involves establishing the following steps.

Step 1: There are no individually rational trades in the set $N' = N\backslash\{i_1, j_1\}$ and for any $k \in N'$ the continuation payoff of k is zero at any $h \in H(N')$. Since after i_1 and j_1 trade at h_1 no other agent belonging to the set N' trades, it follows from Lemma 14 that in any subgame $\gamma \in \Gamma(N')$ the outcome is competitive and there is no trade. This implies that there are no individually rational trades in the set N' and the continuation payoff of any agent $k \in N'$ is zero at every $h \in H(N')$.

Step 2. Any $i \in N' = N\backslash\{i_1, j_1\}$ will accept any offer $p < v_i$ at any history $h \in H(N, \langle j_1, i\rangle)$. Consider any history $h' \in H(N, \langle j_1, i\rangle, \mid i_1, j_1 \mid)$ (by Assumption 1 such a history exists). If at h' no agreement is reached in the match $\langle j_1, i\rangle$, then, by (8), i_1 and j_1 reach an agreement in the next match $\mid i_1, j_1 \mid$ and thus i receives a zero payoff in the continuation game by Step 1. Thus, by (subgame) perfection, i should accept any offer $p < v_i$ in the match $\langle j_1, i\rangle$ at h'. But then from Proposition 7 buyer i will accept $p < v_i$ at any history $h \in H(N, \langle j_1, i\rangle)$.

Step 3. Any $j \in N' = N\backslash\{i_1, j_1\}$ will accept any offer $p > w_j$ at any history $h \in H(N, \langle i_1, j\rangle)$. The proof is the same as that of Step 2.

Step 4. Both i_1 and j_1 are inframarginal agents in the set N. The proof is by contradiction. Suppose that, contrary to what we want to prove, j_1 is extramarginal (and i_1 is inframarginal). Then there must be an inframarginal seller $j \neq j_1$. By Step 1 and individual rationality we have $v_i < w_j < w_{j_1} \leq p_1$ for any $i \in N'$. This means that in any subgame $\gamma \in \Gamma(N)$, i_1 can refuse to trade with any seller and ensure that he reaches a history $h \in H(N, \langle i_1, j\rangle)$. At this point, he offers $p \in (w_j, p_1)$ and by Step 3 j accepts. Since at $h_1 \in H(N)$, i_1 trades at p_1 this is a contradiction.

The assumption that i_1 is extramarginal leads to a similar contradiction. This completes the proof that i_1 and j_1 are inframarginal.

Step 5. Buyer i_1 will not accept a price $p > p_1$ at any history $h \in H(N, \langle j, i_1 \rangle)$ for any seller $j \in N$. Consider any history $h' \in H(N, \langle j, i_1 \rangle, \mid i_1, j_1 \mid)$. Suppose that j offers a price $p > p_1$ at h'. Buyer i_1 must reject the offer because by (8), at the next date, he will trade at the price p_1. Then by Proposition 7 he makes the same response at every history $h \in H(N, \langle j, i_1 \rangle)$.

Step 6. Seller j_1 will not accept a price $p < p_1$ at any history $h \in H(N, \langle i, j_1 \rangle)$ for any buyer $i \in N$. The proof is the same as Step 5.

Step 7. The price p_1 belongs to the competitive interval of N; thus

$$\max \{v_{i'}, w_{j_1}\} \leq p_1 \leq \min \{v_{i_1}, w_{j'}\}$$

where i' is the highest-valuation extramarginal buyer and j' is the lowest valuation extramarginal seller. First we show that $\max \{v_{i'}, w_{j_1}\} \leq p_1$. Individual rationality requires $w_{j_1} \leq p_1$. If $v_{i'} \leq w_{j_1}$ then clearly $\max \{v_{i'}, w_{j_1}\} \leq p_1$.

Now suppose $v_{i'} > w_{j_1}$ and consider any history $h \in H(N, \langle j, i_1 \rangle, \langle j_1, i' \rangle)$ for any seller $j \in N$. Then i_1 knows, by Step 2, that if he does not trade in the first match $\langle j, i_1 \rangle$, j_1 can achieve at least $v_{i'} - w_{j_1}$ in the continuation game. Thus the continuation payoff to i_1 if there is no agreement at h is bounded above by $\text{Max}\{v_{i_1} - v_{i'}, v_{i_1} - w_{j'}\} = v_{i_1} - v_{i'}$. Therefore, i_1 must accept any offer $p < v_{i'}$ at h in the match $\langle j, i_1 \rangle$. Step 5 then implies that $p_1 \geq v_{i'}$.

A similar argument shows that $p_1 \leq \min \{v_{i_1}, w_{j'}\}$.

Step 8. Suppose that $v_{i'} < p_1 < w_{j'}$, where i' and j' are the extramarginal agents identified in Steps 7. Then there cannot be trade between any pair of agents other than i_1 and j_1 at any history $h \in H(N)$. Suppose not; then by Step 1 there exists either an extramarginal seller j such that j and i_1 trade at some $h \in H(N)$ or an extramarginal buyer i such that i and j_1 trade at some $h \in H(N)$. By Assumption 1, as long as the set of remaining agents is N, the match $\mid i_1, j_1 \mid$ will eventually occur and by (8) i_1 and j_1 trade at the price p_1. Consider the sequence of matches that starts at h and ends at the first match $\mid i_1, j_1 \mid$, assuming there has been no intervening trade. Consider the last match $\langle k, \ell \rangle$ including either i_1 or j_1 between h and the match $\mid i_1, j_1 \mid$. By refusing to trade, the inframarginal agent in the match $\langle k, \ell \rangle$ can ensure that he will get p_1 when the match $\mid i_1, j_1 \mid$ next occurs, so no trade will occur at a price the inframarginal agent regards as worse than p_1. Therefore, since by Step 7 p_1

belongs to the competitive interval for N and by assumption $v_{i'} < p_1 < w_{j'}$ it follows that no trade can occur between an inframarginal agent and an extramarginal agent at $\langle k, \ell \rangle$ and if $\langle k, \ell \rangle$ consists of two inframarginal agent they can only trade at the price p_1. By induction, one can show that only i_1 and j_1 trade after h and that trade occurs at the price p_1.

Step 9. Suppose that $p_1 \in \{v_{i'}, w_{j'}\}$, where i' and j' are the extramarginal agents identified in Step 7. Then there cannot be trade between any pair of agents other than i_1 and j_1 at any history $h \in H(N)$. Suppose not. Consider the case $p_1 = v_{i'}$ (the arguments for the other case is identical). Then i' and j_1 must trade at some history $h' \in H(N)$. But as long as the set of remaining agents is N, by Step 6 j_1 will never accept $p < p_1 = v_{i'}$ and by Step 2 i' will accept any price $p < v_{i'}$; thus at h', i' and j_1 trade at the price p_1. But then since f_k is simple at N when k is the proposer in a match with ℓ for all $k, l \in N$, it follows from Proposition 7 that i' and j_1 always trade at p_1 in $H(N)$ whenever they have the same roles. Since $p_1 = v_{i'} < \min\{v_{i_1}, w_{j'}\}$ this implies that i_1 must accept any price $p_1 < p < \min\{v_{i_1}, w_{j'}\}$ from j_1 (consider the match $\langle j_1, i_1 \rangle$ followed by $\langle i', j_1 \rangle$ or $\langle j_1, i' \rangle$, whichever is appropriate), contradicting Step 5.

Step 10. In any subgame $\gamma \in \Gamma(N)$, i_1 and j_1 trade at the price p_1 and this is the only trade that occurs. From Steps 5 and 6 it is clear that i_1 and j_1 cannot trade at any price other than p_1 and we have seen that they will eventually trade at the price p_1 if no other trade intervenes. Steps 8 and 9 show that no trade can occur between any pair of agents other than i_1 and j_1. ■

The proof of Theorem 12 is by induction on the set of all subgames ordered by the number of remaining agents in the subgames. More precisely, let f be a fixed but arbitrary PEC (f is fixed for the rest of the proof). We want to show for any N and any $\gamma \in \Gamma(N)$ the outcome is competitive and trade occurs at the same competitive price.

For any integer $r \geq 1$ let

$$\mathcal{N}_r = \{N \in \mathcal{N} \,|\text{the number of agents in } N \text{ is } 2r\}$$

be the collection of balanced sets with exactly $2r$ agents. The proof is by backward induction on the number of remaining agents, beginning with $r = 1$. Thus we first show that for any set of agents $N_1 \in \mathcal{N}_1$ the outcome is competitive and any trade occurs at the same competitive price in any subgame $\gamma \in \Gamma(N_1)$. Next we show that if for any set of remaining agents

$N_{r-1} \in \mathcal{N}_{r-1}$ the outcome is competitive and any trade occurs at the same competitive price in every subgame $\gamma \in \Gamma(N_{r-1})$ then the same holds for any $N_r \in \mathcal{N}_r$.

Starting the induction
We begin by considering the set of subgames in which the remaining agents consist of two traders $i \in I$ and $j \in J$. First we establish the following result.

Lemma 16 *Consider any two traders $i \in I$ and $j \in J$. Then for any $k, \ell \in \{i, j\}$ and $k \neq \ell$ strategy f_k is simple at $\{i, j\}$ when k is the proposer in a match with player ℓ; thus we have*

$$f_k(h) = f_k(h') \; \forall h, h' \in H(\{i, j\}, \langle k, \ell \rangle) \tag{9}$$

Proof. By Propositions 9 and 11 this is clearly the case if $H(\{i, j\}) \cap E = \varnothing$ or if $U_k(f) = 0$. Suppose now that $H(\{i, j\}) \cap E \neq \varnothing$ and $U_k(f) > 0$; then there exists $h_0 \in H(\{i, j\}) \cap E$ at which k and ℓ trade at some price p_0. Now there are two cases to consider.

Case A: k is the proposer at history h_0. Then, by Proposition 8, condition (9) holds.

Case B: l is the proposer at history h_0. To show that condition (9) holds suppose otherwise. Then define a new strategy f'_k that is otherwise identical to f_k except that $f'_k(h) = p_0$ for any $h \in H(\{i, j\}, \langle k, l \rangle)$. Then $f_k \succ f'_k$ by inspection. Moreover, since l and k reach an agreement at $h_0 \in H(\{i, j\}) \cap E$, by Proposition 8, the match $\langle l, k \rangle$ always induces an agreement at p_0 at any $h \in H(\{i, j\}, \langle l, k \rangle)$. But this, together with Assumption 1, imply that the choice of (f'_k, f_{-k}) induces an agreement at p_0 at some history $h \in H\{i, j\}$ either when k is the proposer or when l is the proposer. Therefore, f'_k induces at least the same payoff as f_k and thus f cannot be a NEC. But this is a contradiction. ■

Lemma 17 *Consider any two traders $i \in I$ and $j \in J$. Then the outcome is competitive and any trade is at the same competitive price in any subgame $\gamma \in \Gamma(\{i, j\})$; thus either $C(\{i, j\})$ is empty and in every $\gamma \in \Gamma(\{i, j\})$ there is no trade or there exists a price $p_0 \in C(\{i, j\})$ such that in every subgame $\gamma \in \Gamma(\{i, j\})$ agents i and j trade at p_0.*

Proof. Since in any subgame $\gamma \in \Gamma(\{i, j\})$ there is at most one trade the result follows immediately from Lemmas 14, 15 and 16. ■

The induction step

We take as our induction hypothesis the claim that for $(r-1)$ and any set of agents $N_{r-1} \in \mathcal{N}_{r-1}$ the outcome is competitive and the same in every subgame $\gamma \in \Gamma(N_{r-1})$; more specifically either there is no trade in all such subgames or there exists a price $p_{r-1} \in C(N_{r-1})$ such that, in every subgame $\gamma \in \Gamma(N_{r-1})$, the inframarginal agents in N_{r-1} trade at the price p_{r-1} and extramarginal agents in N_{r-1} do not trade. Then we establish that the same is true for r.

Lemma 18 *Consider any integer $r > 1$. Suppose that the induction hypothesis is true for $r-1$. Then for any $N \in \mathcal{N}_r$, any $k, l \in N$ and $k \neq \ell$ strategy f_k is simple at N when k is the proposer in a match with player ℓ.*

Proof. If $H(N) \cap E = \varnothing$ then the result follows immediately from Proposition 9. Therefore suppose that $H(N) \cap E$ is not empty. Then if there is no trade at any history $h \in H(N) \cap E$ the equilibrium payoff $U_k(f)$ of each player $k \in N$ must equal zero and thus the result follows immediately from Proposition 11. Thus suppose that there is trade at some history belonging to the set $H(N) \cap E$.

Next enumerate the set of histories $H(N) \cap E$ in order of appearance on the equilibrium path by $h^0, h^1, ..., h^Q$. Denote the periods at which $h^0, h^1, ..., h^Q$ occur on the equilibrium path (the length of each such history) by $v, v+1, ..., v+Q$ respectively. Also, for any $q \leq Q$ denote the proposer, the responder and the price offer at h^q by k^q, l^q and p^q, respectively. Since there is trade at some history belonging to the set $H(N) \cap E$ it follows that the price offer p^q is rejected at h^q for any $q < Q$ and the offer p^Q at h^Q is accepted. By the induction hypothesis for $r-1$, once k^Q and l^Q reach an agreement at h^Q the continuation equilibrium outcome is competitive for the remaining players. The rest of the proof involves the following steps.

Step 1: f_k is simple at N when k is the proposer in a match with ℓ if the match $\langle k, l \rangle$ that does not occur at any $h \in H(N) \cap E$. This follows immediately from Proposition 9.

Step 2: The match $\langle k^Q, l^Q \rangle$ always induces an agreement at p^Q at any $h \in H(N)$ (thus f_{k^Q} is simple at N when k^Q is the proposer in a match with player ℓ^Q). Since $\langle k^Q, l^Q \rangle$ trade at p^Q at history $h^Q \in E$ this step follows immediately from Proposition 8.

Step 3: For any $q < Q$ strategy profile f does not induce an agreement in the match $\langle k^q, l^q \rangle$ at any history $h \in H(N, \langle k^q, l^q \rangle) \cap H^{v+q}$

(note that H^t refers to histories of length t). First consider the match $\langle k^q, l^q\rangle$ for any $q < Q$. Then since $\langle k^q, l^q\rangle$ results in a rejection at $h^q \in H(N, \langle k^q, l^q\rangle)$ it follows from Proposition 8 that, for any $h_0 \in H(N, \langle k^q, l^q\rangle)$,

$$\begin{array}{ll} \text{if } f_{k^q}(h_0) = p_0 \text{ and } f_{\ell^q}(h_0, p_0) = A \text{ then} & \\ p_0 > v_{k^q} - U_{k^q}(f) \quad \text{if } k^q \in I & \\ p_0 < w_{k^q} + U_{k^q}(f) \quad \text{if } k^q \in J & \end{array} \tag{10}$$

Next consider the match $\langle k^{Q-1}, l^{Q-1}\rangle$. Since $H(N\backslash\langle k^Q, l^Q\rangle) \cap E$ is non-empty (because $\langle k^Q, l^Q\rangle$ trade on the equilibrium path at $h^Q \cap H(N) \cap E$) and, by induction hypothesis for $r-1$, in any subgame at which $N\backslash\langle k^Q, l^Q\rangle$ is the set of remaining agents the strategy profile f always induces the same competitive outcome, it must be that each $k \in N\backslash\langle k^Q, l^Q\rangle$ receives his equilibrium payoff in any subgame $\gamma \in \Gamma(N\backslash\langle k^Q, l^Q\rangle)$. But this, together with Step 2 and Proposition 7, imply that if at any $h \in H(N, \langle k^{Q-1}, l^{Q-1}\rangle) \cap H^{v+Q-1}$ the match $\langle k^{Q-1}, l^{Q-1}\rangle$ does not result in an agreement then $\langle k^Q, l^Q\rangle$ trade the next period and any remaining agent receives his equilibrium payoff. By perfection and (10), this implies that f is that at any $h \in H(N, \langle k^{Q-1}, l^{Q-1}\rangle) \cap H^{v+Q-1}$ the match $\langle k^{Q-1}, l^{Q-1}\rangle$ does not reach an agreement.

Next by induction one can show that for any $q < Q-1$ the match $\langle k^q, l^q\rangle$ does not reach an agreement at any $h \in H(N, \langle k^q, l^q\rangle) \cap H^{v+q}$. Assume that the statement holds for any q' such that $q < q' < Q$ then we can show by a similar argument as above that the statement is true for q. Since each $k \in N\backslash\langle k^Q, l^Q\rangle$ receive his equilibrium payoff in any subgame $\gamma \in \Gamma(N\backslash\langle k^Q, l^Q\rangle)$, the match $\langle k^Q, l^Q\rangle$ always reaches an agreement at any $h \in H(N, \langle k^Q, l^Q\rangle)$ and the match $\langle k^{q'}, l^{q'}\rangle$ does not result in an agreement at any $h \in H(N, \langle k^{q'}, l^{q'}\rangle) \cap H^{v+q'}$ for any $Q > q' > q$ then it follows that at any $h \in H(N, \langle k^q, l^q\rangle) \cap H^{v+q}$ player k^q can guarantee himself his equilibrium payoff of $U_{k^q}(f)$ by not reaching an agreement with l^q at h. But this together with (10) imply that at any $h \in H(N, \langle k^q, l^q\rangle) \cap H^{v+q}$ the match $\langle k^q, l^q\rangle$ does not result in an agreement.

Step 4: f_{k^q} is simple at N when k^q is the proposer in a match with player l^q for all $q < Q$. Suppose not; then there exists $q < Q$, h and $h' \in H(N, \langle k^q, l^q\rangle)$ such that $f_{k^q}(h) \neq f_{k^q}(h')$. Next define a new strategy f'_{k^q} as follows. Put $f'_{k^q}(h,p) = f_{k^q}(h,p)$ for any $(h,p) \in H^r_{k^q} \times \mathbf{R}_+$ and let

$$f'_{k^q}(h) = \begin{cases} p^q & \forall h \in H\left(N, \langle k^q, l^q\rangle\right) \\ f_k(h) & \text{otherwise;} \end{cases}$$

for any $h \in H^p_{k^q}$. Then $f_{k^q} \succ f'_{k^q}$ by inspection. Moreover, since for any q' such that $q < q' < Q$ the match $\langle k^{q'}, l^{q'} \rangle$ does not result in an agreement at any $h_0 \in H(N, \langle k^{q'}, l^{q'} \rangle) \cap H^{v+q'}$ (Step 3), the match $\langle k^Q, l^Q \rangle$ always induces an agreement at any $h \in H(N, \langle k^Q, l^Q \rangle)$ (Step2), it follows that the choice of (f'_{k^q}, f_{-k^q}) induces an outcome path from period v consisting of a series of disagreements in the matches $\langle k^{q'}, l^{q'} \rangle$ for $q < Q$, followed by an agreement in the match $\langle k^Q, l^Q \rangle$ and the same competitive outcome for the set of $N \backslash \langle k^Q, l^Q \rangle$ of remaining agents as when (f_{k^q}, f_{-k^q}) is chosen. Therefore, f'_{k^q} induces at least the same payoff as f_k contradicting the definition of NEC. ■

Lemma 19 *If the induction hypothesis is true for $r - 1$ then it is true for r, for any $r > 1$.*

Proof. Fix any $N_r \in \mathcal{N}_r$. Let m_r denote the number of inframarginal agents on each side of the market when the remaining set of agents is N_r and (with some abuse of notation) label the buyers and sellers in the set N_r by $i = 1, ..., m_r, ..., r$ and $j = 1, ..., m_r, ..., r$, respectively.

Now, by Lemmas 14, 15 and 18, if there exists a subgame $\gamma \in \Gamma(N_r)$ such that there is at most one trade then the outcome is competitive in every $\gamma \in \Gamma(N_r)$ and any trade is at the same competitive price belonging to the set $C(N_r)$ in all such subgames. Therefore, for the rest of the proof we consider only the case in which in every subgame $\gamma \in \Gamma(N_r)$ there is more than one trade.

Fix any $\gamma_r \in \Gamma(N_r)$. Suppose buyer i_r and seller j_r are the first pair of agents that trade in the subgame γ_r. Denote the history, the match and the price at which i_r and j_r trade in this subgame by h_r, $\mid i_r, j_r \mid = \langle \pi(h_r), \rho(h_r) \rangle$ and q respectively. Let us also denote respectively the set of remaining agents and the history immediately after i_r and j_r trade and leave the market in this subgame by $N_{r-1} \equiv N_r \backslash \{i_r, j_r\}$ and $h_{r-1} \equiv (h_r, (\pi(h_r),\ \rho(h_r), q, A))$. Also, let γ_{r-1} be the subgame defined by the history h_{r-1}

Clearly, since $h_r \in H(N_r)$ we have $\gamma_{r-1} \in \Gamma(N_{r-1})$ and $N_{r-1} \in \mathcal{N}_{r-1}$. Thus by the hypothesis of the induction argument for $r - 1$ there exists a price p_{r-1} belonging to the competitive interval $C(N_{r-1})$ such that, in *every* subgame $\gamma \in \Gamma(N_{r-1})$, the inframarginal agents in N_{r-1} trade at the price p_{r-1} and extramarginal agents do not trade. We now establish the result by showing that $q = p_{r-1}$, $q \in C(N_r)$ and the outcome is competitive for the market with N_r and all trades occur at q at every subgames $\gamma \in \Gamma(N_r)$. The proof follows in a number of steps.

Step 1: The match $\mid i_r, j_r \mid$ always results in an agreement at q at every $h \in H(N_r)$.

Since the match $\mid i_r, j_r \mid$ induces an agreement at $h_r \in H(N_r)$ this step follows immediately from Lemmas 7 and 18, together with the assumption that the induction hypothesis holds for $r-1$.

Step2: $q \geq p_{r-1}$.

Suppose not; then $p_{r-1} > q$. Consider any $i \in N_{r-1}$ that trades in the subgame γ_{r-1}. By Assumption 1 there exists a history $h_0 \in H(N_r, \langle j_r, i\rangle, \mid i_r, j_r \mid)$. First we claim that i and j_r will trade at a price $p_0 \geq p_{r-1}$ at h_0. By perfection, i must accept any price $p < p_{r-1}$ at h_0 (otherwise, by Step 1 i_r and j_r will trade the next period and i will trade at p_{r-1}). But then there must exist some price p_0 such that j_r offers p_0 and i accepts at h_0 in the match $\langle j_r, i\rangle$. Otherwise, i_r and j_r will trade at the next date and j_r will receive $q - w_{j_r} < p - w_{j_r}$ for any $q < p < p_{r-1}$. Moreover, note that $p_0 \geq p_{r-1}$ because otherwise j_r can increase his payoff by offering $p_0 < p < p_{r-1}$.

Second, by Lemmas 7 and 18 and the assumption that the induction hypothesis holds at $r-1$, it follows from j_r and i trading at p_0 at h_0 that the match $\langle j_r, i\rangle$ always results in trade at p_0 at any $h \in H(N_r, \langle j_r, i\rangle)$.

By Assumption 1 there also exists a history $h' \in H(N_r, \mid i_r, j_r \mid, \langle j_r, i\rangle)$. Then since $p_0 \geq p_{r-1} > q$ and $\langle j_r, i\rangle$ always results in trade at the price p_0, perfection requires j_r not to trade at q with i_r in the first match at h'. This is because if j_r is the responder in the match $\mid i_r, j_r \mid$ then he could make more than $q - w_{j_r}$ by rejecting q from i_r and if j_r is the proposer in the match $\mid i_r, j_r \mid$ then he could obtain $p_0 - w_{j_r} > q - w_{j_r}$ by offering p_0 to i_r at h' (either the offer is accepted or j_r trades with i at p_0 the next period). But this contradicts Step 1.

Step 3: $q \leq p_{r-1}$.

The proof is symmetric to the proof of Step 2 and we shall state it for completeness. Suppose not; then $p_{r-1} < q$. Consider any $j \in N_{r-1}$ that trades in the subgame γ_{r-1}. By Assumption 1 there exists a history $h_0 \in H(N_r, \langle i_r, j\rangle, \mid i_r, j_r \mid)$. First we claim that i_r and j will trade at a price $p_0 \leq p_{r-1}$ at h_0. By perfection, j must accept any price $p > p_{r-1}$ at h (otherwise, i_r and j_r will trade the next period and j will trade at p_{r-1}). But then there must exist some price p_0 such that i_r offers p_0 and j accepts at in the match $\langle i_r, j\rangle$. Otherwise, i_r and j_r will trade at the next date and i_r will receive $v_{i_r} - q < p - w_{j_r}$ for any $q > p > p_{r-1}$. Moreover, note that $p_0 \leq p_{r-1}$ because otherwise i_r can increase his payoff by offering $p_0 > p > p_{r-1}$.

Second, by Lemmas 7 and 18 and the assumption that the induction hypothesis holds at $r-1$, it follows that the match $\langle i_r, j\rangle$ always results in trade at p_0 at any $h \in H(N_r, \langle i_r, j\rangle)$.

By Assumption 2 there also exists a history $h' \in H(N_r, \mid i_r, j_r \mid, \langle i_r, j\rangle)$. Then since $p_0 \leq p_{r-1} < q$ and $\langle i_r, j\rangle$ always results in trade at the price p_0, subgame perfection requires i_r not to trade at q with j_r in the first match at h'. But this contradicts Step 1.

Step 4: If $j_r \leq m_r$ then

$$\begin{array}{ll} p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\} & \text{if } j_r < m_r \\ p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r-1}\} & \text{if } j_r = m_r. \end{array} \tag{11}$$

The proof of this step follows from $p_{r-1} \in C(N_{r-1})$.

First suppose $j_r < m_r$. Then in the subgame γ_{r-1} there are three possible cases.

- Case A: The set of inframarginal traders $\mathcal{I}(N_{r-1})$ consists of $2m_r$ agents. Then any the competitive price $p_{r-1} \in C(N_{r-1})$ must exceed w_{m_r+1}. But $w_{m_r+1} > w_{m_r}$ and $w_{m_r+1} > v_{m_r+1}$. This implies that $p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

- Case B: The set of inframarginal traders $\mathcal{I}(N_{r-1})$ consists of $2(m_r-1)$ agents, $j_r < m_r$ and $i_r \leq m_r$. Then the competitive price p_{r-1} must be no less than $\max\{v_{m_r+1}, w_{m_r}\}$.

- Case C: The set of inframarginal traders $\mathcal{I}(N_{r-1})$ consists of $2(m_r-1)$ agents, $j_r < m_r$ and $i_r > m_r$. Then the competitive price p_{r-1} must be no less than $\max\{v_{m_r}, w_{m_r}\} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

Next suppose $j_r = m_r$. Then in the subgame γ_{r-1} there are two cases to be considered.

- Case A: The set of inframarginal traders $\mathcal{I}(N_{r-1})$ consists of $2(m_r-1)$ agents. This occurs if $i_r \leq m_r$ or if $i_r > m_r$ and $w_{m_r+1} > v_{m_r}$. Then the competitive price p_{r-1} must be no less than $\max\{v_{m_r+1}, w_{m_r-1}\}$.

- Case B: The set of inframarginal traders $\mathcal{I}(N_{r-1})$ consists of $2m_r$ agents. This occurs if $i_r > m_r$ and $w_{m_r+1} < v_{m_r}$. Then the competitive price p_{r-1}must be at least $w_{m_r+1} > \max\{v_{m_r+1}, w_{m_r-1}\}$.

Step 5: If $i_r \leq m_r$ then

$$\begin{array}{ll} p_{r-1} \leq \min\{v_{m_r}, w_{m_r+1}\} & \text{if } i < m_r \\ p_{r-1} \leq \min\{v_{m_r-1}, w_{m_r+1}\} & \text{if } i = m_r. \end{array}$$

This follows by the same reasoning as in Step 4.

Step 6: Suppose $j_r \leq m_r$ then $q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

By Steps 2-4, $q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$ for $j_r < m_r$. Now consider $j_r = m_r$. Since $j_r = m_r$ trades at q individual rationality requires

$$q \geq w_{m_r}.$$

Also, by Steps 2-4 we have

$$q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r-1}\}.$$

The last two conditions imply that

$$q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}.$$

Step 7: Suppose $i_r \leq m_r$ then $q = p_{r-1} \leq \min\{v_{m_r}, w_{m_r+1}\}$.

The proof is symmetrical to the proof of Step 6.

Step 8: Suppose $j_r \leq m_r$ then $q = p_{r-1} \leq \min\{v_{m_r}, w_{m_r+1}\}$.

If not, then $q > \min\{v_{m_r}, w_{m_r+1}\} > v_{m_r+1}$. Therefore, since i_r trades at q individual rationality implies that $i_r \leq m_r$. But then by Step 7 we have $p_{r-1} = q \leq \min\{v_{m_r}, w_{m_r+1}\}$.

Step 9: Suppose $i_r \leq m_r$ then $q = p_{r-1} \geq \max\{v_{m_r+1}, w_{m_r}\}$.

The argument is symmetrical to that in Step 8.

Step 10: $\max\{v_{m_r+1}, w_{m_r}\} \leq q = p_{r-1} \leq \max\{v_{m_r}, w_{m_r+1}\}$.

By individual rationality of the trade between i_r and j it must be that either $i_r \leq m_r$ or $j_r \leq m_r$. But then this step follows from the previous four Steps.

Step 11: $i_r \in \mathcal{I}(N_r)$ and $j_r \in \mathcal{I}(N_r)$.

Suppose i_r is not inframarginal; then $i_r > m_r$ and

$$p_{r-1} \geq \begin{cases} w_{m_r+1} & \text{if the inframarginal agents } \mathcal{I}(N_{r+1}) \text{ consists of } 2m_r \text{ traders} \\ v_{m_r} & \text{if the inframarginal agents } \mathcal{I}(N_{r+1}) \text{ consists of } 2(m_r-1) \text{ traders.} \end{cases}$$

Since both w_{m_r+1} and v_{m_r} exceed v_{m_r+1} it follows that $q = p_{r-1} > v_{m_r+1}$. But since $i_r > m_r$ this contradicts the individual rationality condition $v_{i_r} \geq q$ for i_r. Therefore, i_r must be an inframarginal buyer.

The arguments to show that j_r is an inframarginal seller is analogous.

Step 12: The subgame γ_r induces a competitive outcome and every inframarginal trader in $\mathcal{I}(N_r)$ trades at the price $q \in C(N_r)$.

This follows from the induction hypothesis and Steps 10 and 11.

Step 13: Any subgame $\gamma \in \Gamma(N_r)$ induces a competitive outcome and all trade is at the same competitive price $q \in C(N_r)$.

Since γ_r was fixed to be any arbitrary subgame belonging to the set $\Gamma(N_r)$ it follows from Step 12 that any $\gamma \in \Gamma(N_r)$ induces a competitive outcome. To complete this step we need to show that all trades occur at the same competitive price q at every subgame $\gamma \in \Gamma(N_r)$.

Consider any history $h \in H(N_r)$ at which an inframarginal buyer $i \in \mathcal{I}(N_r) \cap I$ trades with an inframarginal seller $j \in \mathcal{I}(N_r) \cap J$ at a competitive price $p \in C(N_r)$. We next show that $p = q$.

Denote the match at h by $\mid i, j \mid = \langle \pi(h), \rho(h) \rangle$. Next note that since the match $\mid i, j \mid$ induces an agreement at $h \in H(N_r)$ at p it follows immediately from Lemmas 7 and 18, together with the assumption that the induction hypothesis holds at $r-1$, that

$$\mid i, j \mid \text{ results in a trade at } p \text{ at any } h' \in H(N_r, \mid i, j \mid) \qquad (12)$$

By Assumption 1 there also exists a history $h'' \in H(N_r, \mid i, j \mid, \mid i_r, j_r \mid)$. If i and j do not trade at h'' then, by Step 1, i_r and j_r will trade at the next period and $N_r \backslash \{i_r, j_r\}$ will be the set of remaining players. Since the outcome is competitive and every inframarginal player trades at the price p_{r-1} in any subgame $\gamma \in \Gamma(N_r \backslash \{i_r, j_r\})$, and $p_{r-1} = q$ (Steps 2 and 3), we have that if i and j do not trade at h'' then each will eventually trade at q.

Now, by (12), the match $\mid i, j \mid$ results in trade at the price p at history h'. But then perfection requires $p = q$. Otherwise either i or j could make himself better off by not agreeing to trade at h'' and then eventually trading at q. ■

Now Theorem 12 follows by induction from Lemmas 17 and 19.

To demonstrate Corollary 13, by Proposition 7, it is sufficient to show that for any $N \in \mathcal{N}$ and any $k, \ell \in N$ and $k \neq \ell$ strategy f_k is simple at N when k is the proposer in a match with player ℓ. But this follows immediately from Lemmas 16 and 18, and Theorem 12.

4 Sequential matching with an endogenous choice of partners

In this section, we consider a model in which at each period one player is chosen exogenously to (i) select his partner and (ii) to make a proposal to his chosen partner. We show that our selection results extend naturally to this semi-endogenous model of matching.

Given the market data (K, v, w) the matching and bargaining game is now defined by the following rules:

- At each date t, a single agent is selected as the proposer from the set $N \subset K$ of agents remaining in the market. We denote the selected player by $\pi_t(N) \in N$.
- The agent $\pi_t(N)$ who is selected at t chooses a partner and a price offer $(\ell, p) \in (N, \mathbf{R}_+)$ such that $\ell \in I$ if and only if $\pi_t(N) \in J$.[11] The partner (responder) can accept or reject this price. If the proposal is accepted, the good is traded at the agreed price and both agents leave the market. If the proposal is rejected, there is no trade and all agents begin the next period with the same endowments.

Note that, as before, only one pair is formed at each date. The following assumption is analogous to Assumption 1.

Assumption 1′: For any balanced set N, and for any $k, k' \in N$, and any date T there exists a $t > T$ such that

$$\pi_t(N) = k \text{ and } \pi_{t+1}(N) = k'.$$

As in the previous section H^t denotes the set of histories at date t. Let $H = \cup_{t=1}^{\infty} H^t$. Also, as before, for any history $h \in H^t$, we denote respectively the set of agents remaining and the proposer at h by $N(h)$ and $\pi(h) = \pi_t(N(h))$. (The history uniquely defines the identity of the proposer). Also, let $H_k^p = \pi^{-1}(k)$ denote the set of histories at which k is the selected player (proposer). Denote the set of histories at which ℓ could be a responder by $H_k^r = \{h \in H \backslash H_k^p \,|\, k \in N(h)\}$. We also denote the set of histories at which

[11] Here we are assuming that the player who selects his partner also makes the price proposal; this is not essential for our result.

k is the proposer and N is the remaining set of agents by $H(N, \langle k\rangle) = H(N) \cap H_k^p$.

A strategy for agent k is now a function f_k defined on $H_k^p \cup (H_k^r \times \mathbf{R}_+)$ such that

$$\begin{array}{ll} f_k(h) \in M(k) \times \mathbf{R}_+, & \forall h \in H_k^p \\ f_k(h,p) \in \{A, R\}, & \forall h \in H_k^r \text{ and } \forall p \in \mathbf{R}_+. \end{array}$$

where

$$M(k) = \begin{cases} I & \text{if } k \in J \\ J & \text{if } k \in I \end{cases}$$

The rest of the notation in this section is the same as that in the exogenous sequential matching model of the previous section.

The definitions of a Markov strategy, simple behavior and complexity are the same as in the previous section except that here at the beginning of each period a player chooses both a partner and a price.

Definition 20 *(i) For any player k, a strategy f_k is simple at N when k is the proposer if*

$$f_k(h) = f_k(h'), \forall h, h' \in H(N, \langle k\rangle),$$

(ii) For any player k, a strategy f_k is simple at N when k is the responder to a price p in a match with player ℓ if

$$f_k(h,p) = f_k(h',p), \ \forall h, h' \in H(N, \langle \ell\rangle),$$

(iii) For any player k, a strategy f_k is simple at N if and only if for any $\ell \in N$ and $p \in \mathbf{R}_+$ strategy f_k is simple at N both when k is the proposer and when k is the responder to p in a match with ℓ.

Definition 21 *For any agent k, a strategy f_k' is more complex than f_k, denoted by $f_k' \succ f_k$, if one of the following conditions is satisfied:*

1. *there exists a balanced set N such that f_k and f_k' are otherwise identical except that f_k is simple as a proposer at N and f_k' is not; formally $\exists$ $N \in \mathcal{N}$ such that*

$$\begin{array}{ll} f_k'(h) = f_k(h), & \forall h \in H_k^p \backslash H(N, \langle k\rangle), \\ f_k'(h,p) = f_k(h,p), & \forall (h,p) \in H_k^r \times \mathbf{R}_+, \\ f_k(h) = f_k(h'), & \forall h, h' \in H(N, \langle k\rangle), \\ f_k'(h) \neq f_k'(h'), & \textit{for some } h, h' \in H(N, \langle k\rangle); \end{array}$$

2. *there exists a balanced set N, a player ℓ and a price p such that such that f_k and f_k' are otherwise identical except that f_k is simple at N when k is the responder to p in a match with ℓ and f_k' is not; formally there exists $N \in \mathcal{N}, \ell \in N$, and a price p' such that*

$$\begin{array}{ll} f_k^{'}(h) = f_k(h), & \forall h \in H_k^p, \\ f_k^{'}(h, p') = f_k(h, p'), & \textit{if either } h \in H_k^r \backslash H(N, \langle \ell \rangle)) \textit{ or } p' \neq p, \\ f_k(h, p) = f_k(h', p), & \forall h, h^{'} \in H(N, \langle \ell \rangle), \\ f_k^{'}(h, p) \neq f_k^{'}(h', p), & \textit{for some } h, h' \in H(N, \langle \ell \rangle); \end{array}$$

The following result is the analogue of Theorem 12.

Theorem 22 *Suppose the matching technology satisfies Assumption $1'$. Consider any PEC profile f and any $N \in \mathcal{N}$. Then the strategy profile f induces a competitive outcome (for the set N) in all subgames $\Gamma(N)$ and any trade occurs at the same competitive price in all such subgames. Therefore, either $C(N)$ is empty and there is no trade in any subgame $\gamma \in \Gamma(N)$ or there exists a competitive price $\bar{p} \in C(N)$ such that in any subgame $\gamma \in \Gamma(N)$ all inframarginal agents $\mathcal{I}(N)$ trade at $\bar{p}$ and all extramarginal agents $\mathcal{X}(N)$ do not trade.*

Corollary 23 *Suppose the matching technology satisfies Assumption 1'. Then any PEC profile f is Markov.*

The proofs of Theorem 22 and Corollary 23 are similar to those of Theorem 12 and Corollary 13. We sketch the proofs of these result in the Appendix.

5 Complexity and off-the-equilibrium payoff

In Section 3 a PEC (NEC) was defined to be a profile of perfect (Nash) equilibrium strategies $f = (f_k, f_{-k})$ such that for each player k strategy f_k has minimal complexity amongst all strategies for k that are best responses to f_{-k}. Thus, in the definitions above, complexity costs enter lexicographically into each player's preference ordering over the set of strategies.[12]

[12] We could also, as in S, treat complexity as a (small) positive fixed cost of choosing a more complex strategy and define a Nash (perfect) equilibrium with a fixed positive complexity costs. The selection result of this paper also holds for positive complexity costs.

Although the assumptions underlying our definition of complexity are mild, this is not the only possible approach. Our approach puts more weight on complexity costs than on the off-the-equilibrium-path moves: *in considering complexity*, players ignore any consideration of payoffs off-the-equilibrium path and the trade-off is between the equilibrium payoffs of the two strategies and the complexity of the two. The trade-off between complexity and off-the-equilibrium-path payoffs does not arise. Therefore, although complexity costs are negligible, they take priority over optimal behavior after deviations.

We can think of the concept of PEC as the limit as two kinds of perturbations become vanishingly small. One perturbation is to impose a small but positive cost of choosing a more complex strategy. Another perturbation is to introduce a small but positive probability of making an off-the-equilibrium-path move. To obtain PEC, we first let the probability of making an off-the-equilibrium-path move go to zero and then let the cost of choosing a more complex strategy go to zero. (See Chatterjee and Sabourian, 2000). This is reflected in the fact that we require agents to choose a minimally complex strategy within the set of best responses.

Alternatively, it may be that in some situations complexity is a less significant criterion than the off-the-equilibrium payoffs. In terms of the above limiting arguments this would involve letting the cost of choosing a more complex strategy go to zero and then letting the probability of making an off-the-equilibrium-path move go to zero. In the extreme case, in which complexity costs are less significant than the probability of *every* off-the-equilibrium-path move in the limiting arguments, this would require agents to choose minimally complex strategies among the set of strategies that are best responses on and off the equilibrium path (see Kalai and Neme (1992)). Unfortunately, our results for this alternative extreme concept are negative.[13]

Formally, as our alternative to PEC, we define an equilibrium that gives priority to *every* off-the-equilibrium-path move over complexity costs as follows.

Definition 24 *A strategy profile $f = (f_k, f_{-k})$ is a weak PEC strategy profile, denoted by WPEC, if it is both a perfect equilibrium and is such that for*

[13] The above two alternatives are rather extreme. For example, one could allow the order in which complexity costs and trembles enter the limiting arguments to be such that complexity costs are more important than some (off-the-equilibrium paths) trembles and less important than some others.

any k and for any information set $\theta \in H_k^p \cup (H_k^r \times \mathbf{R}_+)$,

$$U_k(f_k, f_{-k} \mid \theta) = U_k(f'_k, f_{-k} \mid \theta) \text{ implies } f'_k \succeq f_k,$$

where for any profile $f'' \in F$, $U_k(f'' \mid \theta)$ refers to continuation payoff of k at information set θ.

Complexity costs, in the above definition, impose a restriction only among strategies that are best responses at *every* information set. Clearly, any PEC strategy profile is a WPEC.

For the case of a market with a *homogeneous* set of buyers and sellers, S shows that any WPEC also induces a competitive outcome. Thus, in the case of a homogeneous market the competitive selection result is independent of the relative importance of complexity costs and off-the-equilibrium payoffs (trembles). In the case of heterogeneous markets, however, the proofs of main results do not extend to the weaker concept of WPEC. For example, in the proof of Proposition 6 it was shown that for any PEC if player ℓ accepts an offer p from k on the equilibrium path then ℓ always accepts an offer p from k at all histories. Otherwise ℓ could economise on complexity and obtain the same equilibrium payoff as before by choosing an alternative strategy that is otherwise identical to the equilibrium strategy except that it always accepts an offer p by k at all histories. Clearly, this argument does not necessarily hold in the case of WPEC because, off-the-equilibrium path, ℓ may prefer to reject p because he anticipates a higher payoff than that associated with trading at p.

In fact we next show that there may exist a continuum of non-competitive WPEC in the model with exogenous sequential matching. Moreover, we show that some of these non-competitive equilibria are inefficient.

To illustrate the ideas, it suffices to consider the simple case of two buyers and two sellers and one inframarginal buyer and one inframarginal seller of Section 2.2.

Proposition 25 *Suppose that $N = 2$, $m = 1$ and*

$$w_1 < v_2 < w_2 < v_1.$$

Consider any deterministic matching technology as described in Section 2 that satisfies Assumption 1. Then for any $p_h \in [w_2, v_1]$ and any $p_\ell \in [w_1, v_2]$

(i) there exists a WPEC such that w_2 and v_1 trade at p_h and w_1 and v_2 trade at p_ℓ;

(ii) there exists a WPEC such that v_1 and w_1 trade at p_b where $b = h$ or ℓ.

Proof. The equilibrium strategy profile is identical to that in the proof of Proposition 1 except that the players always make the same proposal in each role. Thus the strategy profile is described as a collection of three states $\{s, s_h, s_\ell\}$ and rules of transition between them. The transition function $\mu : \{s, s_\ell, s_h\} \times \Sigma \to \{s, s_\ell, s_h\}$ is the same as that in Proposition 1. In each of the three states the players behave as follows.

State s : In this state, each pair $(v_i, w_j) \neq (v_2, w_2)$ does the following if matched.

- (v_1, w_1) do not agree: v_1 offers p_ℓ and accepts p iff $p \leq p_\ell$ and w_1 offers p_h and accepts p iff $p \geq p_h$.
- (v_1, w_2) agree on p_h iff w_2 is the proposer: v_1 offers p_ℓ and accepts p iff $p \leq p_h$ and w_2 offers p_h as the proposer and accepts p iff $p \geq p_h$.
- (v_2, w_1) agree on p_ℓ iff v_2 is the proposer: v_2 offers p_ℓ as the proposer and accepts p iff $p \leq p_\ell$ and w_2 offers p_h and accepts p iff $p \geq p_\ell$.

State s_ℓ : In this state each pair $(v_i, w_j) \neq (v_2, w_2)$ does the following if matched.

- (v_1, w_1) agree on p_ℓ iff v_1 is the proposer: v_1 offers p_ℓ and accepts p iff $p \leq p_\ell$ and w_1 offers p_h and accepts p iff $p \geq p_\ell$.
- (v_1, w_2) do not agree: v_1 offers p_ℓ and accepts p iff $p \leq p_\ell$ and w_2 offers p_h and accepts p iff $p > w_2$.
- (v_2, w_1) do not agree: v_2 offers p_ℓ and accepts p iff $p \leq p_\ell$ and w_1 offers p_h and accepts p iff $p \geq p_h$.

State s_h : In this state, each pair $(v_i, w_j) \neq (v_2, w_2)$ does the following.

- (v_1, w_1) agree on p_h iff w_1 is the proposer: v_1 offers p_ℓ and accepts p iff $p \leq p_h$ and w_1 offers p_h and accepts p iff $p \geq p_h$.
- (v_1, w_2) do not agree: v_1 offers p_ℓ and accepts p iff $p \leq p_\ell$ and w_2 offers p_h and accepts p iff $p \geq p_h$.

- (v_2, w_1) do not agree: v_2 offers p_ℓ and accepts p iff $p < v_2$ and w_1 offers p_h and accepts p iff $p \geq p_h$.

Finally, in any state, (v_2, w_2) do not agree: v_2 offers w_1 and accepts p iff $p \leq v_2$ and w_1 offers v_1 and accepts p iff $p \geq w_2$.

Note that the above strategy profile is identical to the construction in the proof of Proposition 1 except that in the above construction each player in each match always makes the same proposal irrespective of the state whereas this is not the case in the construction in the proof of Proposition 1 (the responses of each player in each state and in each match are the same in both constructions).

Notice also that, as in the Proof of Proposition 1, if the current state is s then the above strategy profile results in $\langle w_2, v_1 \rangle$ agreeing on p_h and $\langle v_2, w_1 \rangle$ on p_h. Similarly, for any $b = \ell, h$ if the current state is s_b then the above strategy profiles results in $\langle v_1, w_1 \rangle$ agreeing on p_b. Also the continuation payoff of the players in each state is the same as in (1) in the proof of Proposition 1.

As in the proof of Proposition 1, it can be verified by inspection that the profile above is a perfect equilibrium.

To verify that the above profile is a WPEC we also need to check that it is not possible to economise on complexity without sacrificing payoffs on or off the equilibrium path. First notice that the above profile is such that for any pair of agents $\langle k, \ell \rangle$ it is not possible to economise on complexity of k's behavior as a proposer because k always makes the same proposal to ℓ independent of the state (past history). Next we need to check for the complexity of response behavior of the each agent to a given proposal. For this purpose consider the response behaviour in each of the following matches.

- $\langle v_1, w_1 \rangle$: In this case, in all the three states, w_1 accepts any $p \geq p_h$ and rejects any $p < p_\ell$. Therefore, there is no possibility of economizing on complexity in response to such price offers. Next consider any price offer $p \in [p_\ell, p_h)$. The above strategy for w_1 rejects the offer in state s_h and accepts the offer in state s_ℓ. But in either of the two states s_h and s_ℓ a change to the response of w_1 to p results in a loss of payoffs in that state. Therefore, it is not possible to economise on complexity without sacrificing payoffs in some state.

- $\langle w_1, v_1 \rangle$: In this case, in all the three states, v_1 accepts any $p \leq p_\ell$ and rejects any $p > p_h$. Therefore, there is no possibility of economizing on complexity in response to such price offers. Next consider any price offer $p \in (p_\ell, p_h]$. The above strategy for v_1 rejects the offer in state s_ℓ and accepts the offer in state s_h. But in either of the two states s_ℓ and s_h a change to the response of v_1 to p results in a loss of payoffs in that state. Therefore, it is not possible to economise on complexity without sacrificing payoffs in some state.

- $\langle v_1, w_2 \rangle$: In this case, in all the three states, w_2 accepts any $p \geq p_h$ and rejects any $p \leq w_2$. Therefore, there is no possibility of economizing on complexity in response to such price offers. Next consider any price offer $p \in (w_2, p_h)$. The above strategy for w_2 rejects the offer in state s and accepts the offer in state s_ℓ. But in either of the two states s and s_ℓ a change to the response of w_2 to p results in a loss of payoffs in that state. Therefore, it is not possible to economise on complexity without sacrificing payoffs in some state.

- $\langle w_2, v_1 \rangle$: In this case, in all the three states, v_1 accepts any $p \leq p_\ell$ and rejects any $p > p_h$. Therefore, there is no possibility of economizing on complexity in response to such price offers. Next, consider any price $p \in (p_\ell, p_h]$. The above strategy for v_1 rejects the offer in state s_ℓ and accepts the offer in state s. But in either of the two states s_ℓ and s a change to the response of v_1 to p results in a loss of payoffs in that state. Therefore, it is not possible to economise on complexity without sacrificing payoffs in some state.

- $\langle w_1, v_2 \rangle$: The argument in this case is analogous to the case of $\langle v_1, w_2 \rangle$.

- $\langle v_2, w_1 \rangle$: The argument in this case is analogous to the case of $\langle w_2, v_1 \rangle$.

Finally, note that appropriate choice of the initial state establishes the results. Thus, if the initial state is s then the above describes the required subgame perfect equilibrium strategy profile in part (i) of the Proposition and if the initial state is s_b ($b = \ell$ or h) then we have described the required subgame perfect equilibrium strategy profile in part (ii) of the Proposition. ■

We conclude from the selection results for PEC (Theorem 12 and its Corollary) and for WPEC (Proposition 25) that a competitive outcome is more likely when complexity costs are more significant than the pertubations that induce off-the-equilibrium-path behaviour.

6 Global complexity

The matching model considered in this paper is by definition non-stationary. As agents leave the market the set of remaining agents changes. The notion of complexity contained in Definitions 4 and 21 respects this aspect of the game. In other words, our notion of complexity has a 'local' character — it consists of a partial order on the set of strategies with reference to a given set N of remaining agents. Gale and Sabourian (2003a), on the other hand, use a 'global' concept of complexity that does not refer to the set of remaining agents. They show that the combination of perfect equilibrium and lexicographic global complexity costs (applied to Markov strategies) may be too strong in the sense that a solution may not exist; however, GS also show that lexicographic global complexity costs can provide a justification for the competitive equilibrium in the random matching models if a equilibrium concept weaker than perfect equilibrium is used. Here we can also establish that in the sequential matching models considered above the combination of perfect equilibrium and lexicographic global complexity costs may be too strong in the sense that a solution may not exist. In the rest of this section we will only consider the exogenous sequential model of Section 3 and demonstrate this result for this case. The discussion below applies equally well to the semi-endogenous matching model of Section 4; however to save space we shall restrict the discussion below to the case of exogenous sequential matching.

Definition 26 *(i) For any player k, a strategy f_k is globally simple when k is the proposer in a match with player ℓ if*

$$f_k(h) = f_k(h'), \forall h, h^{'} \in H\langle k, \ell\rangle.$$

(ii) For any player k, a strategy f_k is globally simple when k is the responder to price p in a match with player ℓ if

$$f_k(h, p) = f_k(h', p),\ \forall h, h^{'} \in H\langle \ell, k\rangle,$$

Definition 27 *(Global complexity)For any agent k, a strategy f'_k is more globally complex than f_k, if one of the following conditions is satisfied:*

1. *there exists a player $\ell \in K$ such that f_k and f'_k are otherwise identical except that strategy f_k is globally simple when k is the proposer in a match with ℓ and strategy f'_k is not;*

2. *there exists a player* $\ell \in K$ *and a price* p *such that such that* f_k *and* f'_k *are otherwise identical except that strategy* f_k *is globally simple when* k *is the responder to* p *in a match with* ℓ *and strategy* f'_k *is not.*

We refer to NEC (PEC) strategy profiles with the above notion of complexity as global NEC (global PEC).

The definition of global complexity is in some sense independent of the set of remaining agents after different subgames. We have adopted Definitions 4 and 21 because they have the same flavor as the structure of the game. Moreover, it turns out that global complexity together with perfection is simply too strong. We show below that the set of global PECs may be empty for markets that satisfy the following property.

Property α : A market (K, v, w) satisfies property α if there exist a balanced set $N \in \mathcal{N}$ such that $C(K) \cap C(N)$ is empty.

First note that any global PEC satisfies the conditions specified in Propositions 6, 7 8, 9 and 11 in Section 3. The proof of this claim for any global PEC follows exactly the same reasonings as in the proofs of Propositions 6, 7 8, 9 and 11. In fact, for the case of global PECs, the conclusions specified in Propositions 6, 7 8, 9 and 11 can be strengthened to show that for any match $\langle k, \ell \rangle$ they hold for any history $h \in H\langle k, \ell \rangle$ and not just for any $h \in H(N, \langle k, \ell \rangle))$. In particular, for the case of global NEC, the conclusions of Proposition 8 can be strengthened to show that if at some history two agents trade at a price that results in a payoff no less than the proposer's equilibrium payoff, then they make the same trade whenever they are matched in the same way.

Proposition 28 *Let* f *be a global NEC. Consider any player* k *and any price* p_0 *such that trading at* p_0 *for player* k *results in a payoff no less than* k*'s equilibrium payoff* $U_k(f)$*. Suppose that, for some balanced set* N *and for some finite history* $h_0 \in H(\langle k, \ell \rangle)$*,*

$$f_k(h_0) = p_0 \text{ and } f_\ell(h_0, p_0) = A. \tag{13}$$

Then we have

$$f_k(h) = p_0, \forall h \in H(\langle k, \ell \rangle), \tag{14}$$

$$f_\ell(h, p_0) = A, \forall h \in H(\langle k, \ell \rangle). \tag{15}$$

The proof of the above proposition is almost identical to that of Proposition 8; simply replace the set of histories $H(N, \langle k, \ell \rangle)$ in the later by the set of histories $H(\langle k, \ell \rangle)$.

Next we establish the selection result for the concept of global PEC.

Proposition 29 *Suppose the matching technology satisfies Assumption 1. Any global PEC profile f induces a competitive outcome.*

Sketch of the proof: The proof of this result follows the same reasoning as in the proof of Theorem 12 and its Corollary. In Section 3, Theorem 12 was established by showing that any perfect equilibrium that satisfies the conclusions of Propositions 6, 7 8, 9 and 11 induces a competitive outcome at any $h \in \bar{H}(N)$, for all balanced N. Since any global PEC profile also satisfies these four Propositions it follows that f satisfies the conclusions of Theorem 12 and its corollary. To save space we shall omit the details. ■

Now we show that the market game may not have a global PEC for economies that satisfy property α. To illustrate the ideas, it suffices to consider the simple case of five buyers and five sellers with three inframarginal buyers and three inframarginal sellers. It will be evident from the discussion below that this kind of non-existence result extends immediately to the case with an arbitrary number of buyers and sellers satisfying property α.

Proposition 30 *Suppose $n = 5$, $m = 3$, $w_4 > w_1$ and $v_5 > w_3$. Then the set of global PEC is empty.*

Proof. Suppose not; consider any PEC f. By Proposition 29 on the equilibrium path w_1 trades at some competitive price $\bar{p} \in [v_4, v_3]$ with some inframarginal buyer v_i $(i \leq 3)$ in some match $\mid w_1, v_i \mid$ (thus $\mid w_1, v_i \mid$ is either $\langle w_1, v_i \rangle$ or $\langle v_i, w_1 \rangle$). Then by Proposition 28 we have that in any match $\mid w_1, v_i \mid$, seller w_1 always trades at $\bar{p}$ with buyer v_i at any history $h' \in H(\mid w_1, v_i \mid)$.

Next consider a history h such that the set of remaining players is $N \equiv \{v_1, v_2, v_3, w_1, w_4, w_5\}$. In this subgame v_1 and w_1 are the inframarginal agents and by assumption the only individually rational trade is between $v_{i''}$ $(i'' = 1, 2, 3)$ and w_1. Now let $v_{i'}$ be the buyer such that $v_{i'} \neq v_i$ and $i \leq 2$. Consider any $h \in H(N, \langle w_1, v_{i'} \rangle, \mid w_1, v_i \mid)$. Since $\mid w_1, v_i \mid$ always reach an agreement at $\bar{p} \leq v_3$ it follows that at h in the match $\langle w_1, v_{i'} \rangle$ buyer $v_{i'}$ always accepts any price $p < v_{i'}$. Therefore, at h seller w_1 must offer a price $p = v_{i'}$ and it will be accepted by $v_{i'}$. Since $\bar{p} \leq v_3 < v_2 \leq v_{i'}$ it then follows from Proposition 28 that the match $\langle w_1, v_{i'} \rangle$ always results in an agreement at $v_{i'}$ at any $h' \in H\left(\langle w_1, v_{i'} \rangle\right)$. But this implies that the continuation payoff of w_1 at any $h'' \in H(N, \mid w_1, v_i \mid, \langle w_1, v_{i'} \rangle)$ is at least $v_{i'} - w_1$ (he could refuse to trade until he is matched with $v_{i'}$ as the proposer). Since $\bar{p} \leq v_3 < v_2 \leq v_{i'}$ this contradicts the assumption that the match $\mid w_1, v_i \mid$ always reaches an agreement at $\bar{p}$ at any history $h' \in H(\mid w_1, v_i \mid)$. ■

7 Concluding Remarks

The main result of this paper (Theorem 12) provides sufficient conditions for every PEC of the market game to be perfectly competitive. This result can be extended to endogenous matching as shown in Section 4 and in Gale and Sabourian (2003b). We have also pointed out several natural seeming extensions that "don't work". What these positive and negative results illustrate is a tension between different notions of complexity and different notions of equilibrium, rather than the fragility of the results. For example, in Section 5, we show that the concept of WPEC is too weak to guarantee perfectly competitive behavior. There are two limits at work here, both motivated by the equilibrium concept we need to work with. We let complexity costs converge to zero in order to ensure that agents choose best responses in equilibrium. We rationalize subgame perfection by introducing small perturbations of the equilibrium strategies, and letting them converge to zero. Depending on whether we let complexity costs or perturbations converge to zero faster, we obtain WPEC or PEC. We rationalize our preferred notion of equilibrium PEC by assuming that complexity costs matter more than perturbations. The fact that we are on a knife-edge, that the order of limits matters, is really a reflection of the fact that we are working with a very weak notion of complexity (the partial ordering $\succ$) to begin with. Because we have little data about complexity costs, we want to make the weakest possible assumptions about the complexity ordering $\succ$. It is for this reason, rather than any lack of robustness, that we cannot weaken the concept of PEC (in this particular direction).

Similarly, we have shown in Section 6 that strengthening the concept of complexity to global complexity may lead to non-existence. This is because we have applied the very strong concept of subgame perfect equilibrium (SPE). In Gale and Sabourian (2003a) we use a weaker notion of equilibrium, which we call Nash equilibrium with perfect responses, that is consistent with the global concept of complexity discussed in Section 6. Roughly speaking, Nash equilibrium with perfect responses requires rationality on the equilibrium path and at information sets that can be reached by one deviation off the equilibrium path. Imposing stricter rationality requirements off the equilibrium path necessarily makes equilibrium strategies more complex. We would argue that if complexity matters more than perturbations, it does not make sense to impose very strong requirements for rationality off the equilibrium path.

These two examples may suggest that PEC can be neither strengthened nor weakened if we want to maintain the conclusion of Theorem 12, but that is not the conclusion we would draw. In both the cases mentioned, the problem arises because the concept of SPE already makes assumptions about the complexity of strategies that are being used by players and one cannot adjust one's view on complexity without making corresponding changes in the equilibrium concept.

Another important restriction we have noted is to deterministic rather than random matching. Random matching is convenient because it allows for a stationary model, which is impossible with exogenous deterministic matching. There is no other reason for introducing uncertainty into an otherwise non-stochastic setting. If the matching procedure were completely endogenous, as in Gale and Sabourian (2003b), it would automatically be deterministic. In any case, in Gale and Sabourian (2003a) we show that a SPE in Markov strategies is not necessarily perfectly competitive for the random matching model. This implies that with random matching the complexity definition used in this paper is not sufficient to select a competitive outcome. In addition, another problem with random matching is simply that there are more information sets on the equilibrium path and so a stronger notion of complexity costs (more complete partial ordering) is required to deliver the perfectly competitive outcome. We show in Gale and Sabourian (2003a) that something like global complexity with Markov Nash equilibrium with perfect responses is sufficient to characterize perfect competition.

8 Appendix: Proof of Theorem 18

As in the proof of Theorem 1, we first establish analogous results to Propositions 6, 7 8, 9 and 11 for the case of the sequential matching with endogenous choice of partners.

Proposition 31 *Let f be a NEC and let $E \subset H$ denote the set of finite histories that occur along the equilibrium path of f. Suppose that, for some finite history $h_0 \in H(N, \langle k \rangle) \cap E$,*

$$f_k(h_0) = (\ell, p_0), f_\ell(h_0, p_0) = A.$$

Then we have

$$f_\ell(h, p_0) = A, \forall h \in H(N, \langle k \rangle)$$

Proposition 32 *Let f be a NEC. Then for any $N \in \mathcal{N}$, any players $k,\ell \in N$ and any price offer p_0 strategy f_ℓ is simple at N when ℓ is the responder to p_0 in a match with k.*

Proposition 33 *Let f be a NEC. Consider any player k and any price p_0 such that trading at p_0 for player k results in a payoff no less than k's equilibrium payoff $U_k(f)$. Suppose that, for some finite history $h_0 \in H\left(N, \langle k \rangle\right)$,*

$$f_k(h_0) = (\ell, p_0) \text{ and } f_\ell(h_0, p_0) = A.$$

Then

$$\begin{aligned} f_k(h) &= (\ell, p_0), \forall h \in H\left(N, \langle k \rangle\right), \\ f_\ell(h, p_0) &= A, \forall h \in H\left(N, \langle k \rangle\right). \end{aligned} \tag{16}$$

Proposition 34 *Let f be a NEC and let $E \subset H$ denote the set of finite histories that occur along the equilibrium path of f. Then for all N and any $k, \ell \in N$ if $E \cap H\left(N, \langle k \rangle\right) = \emptyset$ then f_k is simple at N when k is the proposer.*

Proposition 35 *Let f be a PEC. Then f_k is Markov if $U_k(f) = 0$.*

The proofs of the last five Propositions are almost identical to those of Propositions 6, 7 8, 9 and 11 for the exogenous matching model. To avoid repetition we shall not provide proofs for the last five Propositions. Simply replace the set of histories $H\left(N, \langle k, \ell \rangle\right)$ and the price proposals p_0 and p in the proofs of Propositions 6, 7 8, 9 and 11 by the set of histories $H\left(N, \langle k \rangle\right)$ and by the proposals (ℓ, p_0) and (ℓ, p), respectively.

Next we define the set of histories $h \in H(N)$ such that k is the proposer at h and k' is the proposer if k does not reach an agreement at h by

$$H(N, \langle k \rangle, \langle k' \rangle) = \left\{ h \in H(N) \middle| \text{ if } h \in H^t \text{ then } \pi_t(N) = k \text{ and } \pi_{t+1}(N) = k' \right\}.$$

As in Section 3, we need to establish two further results before proving Theorem 22.

Lemma 36 *Consider any fixed but arbitrary PEC f. Suppose that for some $N \in \mathcal{N}$ there is no trade in some subgame $\gamma' \in \Gamma(N)$. Then in every subgame $\gamma \in \Gamma(N)$ there is no trade and the outcome is competitive.*

Proof. The proof involves the same three Steps as those in the proof of Lemma 14.

Step 1: There cannot be an agreement at any $h \in H(N) \cap E$. Otherwise, there exists players $k, l \in N$ and a history $h' \in E \cap H(N, \langle k \rangle)$ such that k and l trade at h'; by Proposition 33 this implies that players k and l always trade at any $h \in E \cap H(N, \langle k \rangle)$. But this, together with Assumption 1', contradicts the hypothesis that there is no trade in $\gamma' \in \Gamma(N)$.

Step 2: There is no trade in any subgame $\gamma \in \Gamma(N)$. Otherwise, there exists players $k', l' \in N$ and a history $h' \in H(N, \langle k' \rangle)$ such that k' and l' trade at h'. Now by Step 1 either $E \cap H(N)$ is empty or $U_k(f) = 0$ for all $k \in N$. But then, by Propositions 32, 34 and 35, f_k is simple at N for each $k \in N$. This implies that players k' and l' always trade at *any* $h \in H(N, \langle k' \rangle)$. But this, together with Assumption 1', contradict the hypothesis that there is no trade in the subgame $\gamma' \in \Gamma(N)$.

Step 3: every $k \in N$ is extramarginal and $C(N)$ is empty. Otherwise, for some i and $j \in N$ we have $v_i > w_j$. By Step 2, the continuation payoff of each $k \in N$ is zero at every $h \in H(N)$. This implies that j accepts any offer $p' > w_j$ at any $h' \in H(N, \langle i \rangle)$. Thus, buyer i can obtain a positive payoff by offering a price $p' \in (w_j, v_j)$; but this is a contradiction ■

Lemma 37 *Consider any fixed but arbitrary PEC strategy profile f. Suppose that for some $N \in \mathcal{N}$*

1. *there exists a subgame $\gamma' \in \Gamma(N)$ such that there is one and only one trade in the subgame γ';.*
2. *f_k is simple at N when k is the proposer for any $k \in N$.*

Then there exists a competitive price $p_1 \in C(N)$ such that in any subgame $\gamma \in \Gamma(N)$ the outcome is competitive and any trade takes place at p_1.

Proof. Let i_1 and j_1 denote respectively the (only) buyer and the (only) seller that trade in the subgame γ'. Also denote the history,the proposer, the responder and the price at which i_1 and j_1 trade in this subgame by

$h_1, \pi(h_1), \rho(h_1)$ and p_1. Thus if players choose f in the subgame γ' it results in a history $h_1 \in H(N)$ such that

$$f_{\pi(h_1)}(h_1) = (\rho(h_1), p_1), f_{\rho(h_1)}(h_1, p_1) = A$$

But then since by assumption $f_{\pi(h_1)}$ is simple at N when $\pi(h_1)$ is the proposer it follows from Proposition 7 that

$$f_{\pi(h_1)}(h_1) = (\rho(h_1), p_1), f_{\rho(h_1)}(h_1, p_1) = A, \forall h \in H(N, \langle \pi(h_1) \rangle). \tag{17}$$

Thus, i_1 and j_1 always reach an agreement at p_1 at any $h \in H(N, \langle \pi(h_1) \rangle)$.

Now we show that i_1 and j_1 are the only inframarginal agents in N, $p_1 \in C(N)$ and in any subgame $\gamma \in \Gamma(N)$, i_1 and j_1 trade at the price p_1 and this is the only trade that occurs. This involves establishing the following steps.

Step 1: There are no individually rational trades in the set $N' = N \backslash \{i_1, j_1\}$ and $\forall k \in N'$ the continuation payoff of k is zero at any $h \in H(N')$. The proof is almost identical to that of Step 1 in the proof of Lemma 15; simply replace Lemma 14 in the latter by Lemma 36.

Step 2. Any $i \in N' = N \backslash \{i_1, j_1\}$ will accept any offer $p < v_i$ at any history $h \in H(N, \langle j_1 \rangle)$. Consider any history $h' \in H(N, \langle j_1 \rangle, \langle \pi(h_1) \rangle)$ (by Assumption $1'$ such a history exists). If at h' seller j_1 chooses (i, p) and i rejects, then by (17) i_1 and j_1 reach an agreement in the next period and thus i receives a zero payoff in the continuation game by Step 1. Thus, by perfection, i should accept any offer $p < v_i$ by j_1 at h'. But then from Proposition 32 i will accept $p < v_i$ at any history $h \in H(N, \langle j_1 \rangle)$.

Step 3. Any $j \in N' = N \backslash \{i_1, j_1\}$ will accept any offer $p > w_j$ at any history $h \in H(N, \langle i_1 \rangle)$. The proof is the same as that of Step 2.

Step 4. Both i_1 and j_1 are inframarginal agents in the set N. The proof is almost identical to that of Step 4 in the proof of Lemma 15; simply replace '$H(N, \langle i_1, j \rangle)$' and the offer '$p$' in the latter by '$H(N, \langle i \rangle)$' and '$(j, p)$', respectively.

Step 5. Buyer i_1 will not accept a price $p > p_1$ at any history $h \in H(N, \langle j \rangle)$ for any seller $j \in N$. Consider any history $h' \in H(N, \langle j \rangle, \langle \pi(h_1) \rangle)$. Suppose that j offers i_1 a price $p > p_1$ at h'. Buyer i_1 must reject the offer because by (17), at the next date, he will trade at the price p_1. Then by Proposition 32 he makes the same response at every history $h \in H(N, \langle j \rangle)$.

Step 6. Seller j_1 will not accept a price $p < p_1$ at any history $h \in H(N, \langle i \rangle)$ for any buyer $i \in N$. The proof is the same as in Step 5.

Step 7. $p_1 \in C(N)$; thus

$$\max\{v_{i'}, w_{j_1}\} \leq p_1 \leq \min\{v_{i_1}, w_{j'}\}$$

where i' is the highest-valuation extramarginal buyer and j' is the lowest valuation extramarginal seller. First we show that $\max\{v_{i'}, w_{j_1}\} \leq p_1$. Individual rationality requires $w_{j_1} \leq p_1$. If $v_{i'} \leq w_{j_1}$ then clearly $\max\{v_{i'}, w_{j_1}\} \leq p_1$.

Now suppose $v_{i'} > w_{j_1}$ and consider any history $h \in H(N, \langle j \rangle, \langle j_1 \rangle)$ for any seller $j \in N$. If at h seller j proposes a price to i_1 then i_1 knows, by Step 2, that if he does not accept, j_1 can achieve at least $v_{i'} - w_{j_1}$ in the continuation game. Thus the continuation payoff to i_1 if there is no agreement at h is bounded above by $\text{Max}\{v_{i_1} - v_{i'}, v_{i_1} - w_{j'}\} = v_{i_1} - v_{i'}$. Therefore, i_1 must accept any offer $p < v_{i'}$ at h by j. Step 5 then implies that $p_1 \geq v_{i'}$.

A similar argument shows that $p_1 \leq \min\{v_{i_1}, w_{j'}\}$.

Step 8. Suppose that $v_{i'} < p_1 < w_{j'}$, where i' and j' are the extramarginal agents identified in Steps 7. Then there cannot be trade between any pair of agents other than i_1 and j_1 at any history $h \in H(N)$. The proof is almost identical to that of Step 8 in the proof of Lemma 15; simply replace 'Assumption 1', 'the match $\mid i_1, j_1 \mid$' and '(8)' in Step 8 of Lemma 15 by 'Assumption 1′', 'the proposer being $\pi(h_1)$' and '(17)', respectively.

Step 9. Suppose that $p_1 \in \{v_{i'}, w_{j'}\}$, where i' and j' are the extramarginal agents identified in Step 8. Then there cannot be trade between any pair of agents other than i_1 and j_1 at any history $h \in H(N)$. Suppose not. Consider the case $p_1 = v_{i'}$ (the arguments for the other case is identical). By the same reasoning as in the proof of Step 9 in the proof of Lemma 15 we have that i' and j_1 must trade at some history $h' \in H(N)$ at the price p_1. Since f_k is simple at N when k is the proposer, it follows from Proposition 32 that i' and j_1 always trade at p_1 at any $h \in H(N, \langle \rho(h') \rangle)$. Since $p_1 = v_{i'} < \min\{v_{i_1}, w_{j'}\}$ this implies that i_1 must accept any price $p_1 < p < \min\{v_{i_1}, w_{j'}\}$ from j_1 (suppose j_1 selects (i_1, p) at some history $h \in H(N_1, \langle j_1 \rangle, \langle \rho(h') \rangle))$, contradicting Step 5.

Step 10. In any subgame $\gamma \in \Gamma(N)$, i_1 and j_1 trade at the price p_1 and this is the only trade that occurs. The proof is identical to that of Step 10 in the proof of Lemma 15. ■

As in the previous section, the proof of Theorem 22, is by induction on the set all subgames ordered by the number of remaining agents in the subgames. For the rest of the proof fix an arbitrary PEC profile f.

Starting the induction

As in Section 3 we begin by considering the set of subgames in which the remaining agents consists of two traders $i \in I$ and $j \in J$. First we establish the following result.

Lemma 38 *Consider any two traders $i \in I$ and $j \in J$. Then $\forall k \in \{i, j\}$*

$$f_k(h) = f_k(h') \; \forall h, h' \in H(\{i, j\}, \langle k \rangle) \tag{18}$$

Proof. By Propositions 34 and 35 this is clearly the case if $H(\{i, j\}) \cap E = \varnothing$ or if $U_k(f) = 0$. Suppose now that $H(\{i, j\}) \cap E \neq \varnothing$ and $U_k(f) > 0$; then there exists $h_0 \in H(\{i, j\}) \cap E$ at which k trades with another agent $l \in \{i, j\}$ at some price p_0.

Now there are two cases to consider.

Case A: k is the proposer at history h_0. Then, by Proposition 33, condition (18) holds.

Case B: l is the proposer at history h_0. To show that condition (18) holds suppose otherwise. Then define a new strategy f'_k that is otherwise identical to f_k except that $f'_k(h) = (l, p_0) \; \forall h \in H(\{i, j\}, \langle k \rangle)$. Then $f_k \succ f'_k$ by inspection. Moreover, since l and k reach an agreement at $h_0 \in H(\{i, j\}, \langle l \rangle)) \cap E$, by Proposition 33, l always reaches an agreement with k at p_0 at any $h \in H(\{i, j\}, \langle l \rangle)$. But this, together with Assumption 1', imply that the choice of (f'_k, f_{-k}) induces an agreement at p_0 at some history $h \in H\{i, j\}$. Therefore, f'_k induces at least the same payoff as f_k. But this is a contradiction. ■

Lemma 39 *Consider any two traders $i \in I$ and $j \in J$. Then the outcome is competitive and any trade is at the same competitive price in any subgame $\gamma \in \Gamma(\{i, j\})$.*

Proof. Since in any subgame $\gamma \in \Gamma(\{i, j\})$ there is at most one trade the result follows immediately from Lemmas 36, 37 and 38. ■

The induction step

As in Section 3, we take as our induction hypothesis the claim that for any $r > 1$ and any history $N_{r-1} \in \mathcal{N}_{r-1}$ the outcome is competitive after any subgame $\gamma \in \Gamma(N_{r-1})$; in particular if the competitive interval $C(N_{r-1})$ is not empty then there exists a price $p_{r-1} \in C(N_{r-1})$ such that, in any subgame $\gamma \in \Gamma(N_{r-1})$, the inframarginal agents in N_{r-1} trade at the price p_{r-1} and extramarginal agents in N_{r-1} do not trade. Then we establish that the same is true for r.

Lemma 40 *Fix any $r > 1$. Suppose that the induction hypothesis is true for $r-1$. Then for any $N \in \mathcal{N}_r$, any $k, l \in N$ and $k \neq \ell$ strategy f_k is simple at N when k is the proposer.*

Proof. If $H(N) \cap E = \varnothing$ then the result follows immediately from Proposition 34. Therefore suppose that $H(N) \cap E$ is not empty. Then if there is no trade at any history $h \in H(N) \cap E$ the equilibrium payoff $U_k(f)$ of each player $k \in N$ must equal zero and thus the result follows immediately from Proposition 35. Thus suppose that there is trade at some history belonging to the set $H(N) \cap E$.

Next, as in the proof of Lemma 18 enumerate the set of histories $H(N) \cap E$ in order of appearance on the equilibrium path by $h^0, h^1, ..., h^Q$. Denote the periods at which $h^0, h^1, ..., h^Q$ occur on the equilibrium path by $v, v+1, ..., v+Q$, respectively. Also, for any $q \leq Q$ denote the proposer, the responder and the price offer at h^q by k^q, l^q and p^q, respectively. Note that the price offer p^q is rejected at h^q for any $q < Q$, the offer p^Q at h^Q is accepted and, by the induction hypothesis for $r-1$, the continuation equilibrium outcome after period $v+Q$ is competitive for the remaining players. The rest of the proof involves the following steps.

Step 1: f_k is simple at N when k is the proposer if k is not the proposer at any $h \in H(N) \cap E$. This follows immediately from Proposition 34.

Step 2: f_{k^Q} is simple at N when k^Q is the proposer. Since k^Q and l^Q trade at p^Q at history $h^Q \in E$ this step follows immediately from Proposition 33.

Step 3: For any $q < Q$ the strategy profile f does not induce an agreement at any history $h \in H(N, \langle k^q \rangle) \cap H^{v+q}$ (note that H^t refers to histories of length t). For any $q < Q$, since there is no agreement at $h^q \in H(N, \langle k^q \rangle)$ between k^q and l^q it follows from Proposition 33 that $\forall h_0 \in H(N, \langle k^q \rangle)$

$$\begin{array}{ll} \text{if } f_{k^q}(h_0) = (l^q, p_0) \text{ and } f_{\ell^q}(h_0, p_0) = A \text{ then} & \\ p_0 > v_{k^q} - U_{k^q}(f) \quad \text{if } k^q \in I & \\ p_0 < w_{k^q} + U_{k^q}(f) \quad \text{if } k^q \in J & \end{array} \tag{19}$$

Next consider agents k^{Q-1} and l^{Q-1}. Since $H(N\backslash\langle k^Q, l^Q\rangle) \cap E$ is nonempty and, by induction hypothesis for $r-1$, in any subgame at which $N\backslash\langle k^Q, l^Q\rangle$ is the set of remaining agents f always induces the same competitive outcome, it must be that each $k \in N\backslash\langle k^Q, l^Q\rangle$ receives his equilibrium payoff in any subgame $\gamma \in \Gamma(N\backslash\langle k^Q, l^Q\rangle)$. But this, together with Step 2 and Proposition 32, imply that if at any $h \in H(N, \langle k^{Q-1}\rangle) \cap H^{v+Q-1}$ there is no agreement then k^Q and l^Q will trade the next period and any remaining agent receives his equilibrium payoff. Since f is a perfect equilibrium and condition (19) holds, this implies that at any $h \in H(N, \langle k^{Q-1}\rangle) \cap H^{v+Q-1}$ there is no an agreement.

Next by induction one can show that for any $q < Q-1$ there is no agreement at any $h \in H(N, \langle k^q\rangle) \cap H^{v+q}$. Assume that the statement holds for any q' such that $q < q' < Q$ then we can show by a similar argument as above that the statement is true for q. Since each $k \in N\backslash\langle k^Q, l^Q\rangle$ receive his equilibrium payoff in any subgame $\gamma \in \Gamma(N\backslash\langle k^Q, l^Q\rangle)$, the players k^Q and l^Q always trade at any $h \in H(N, \langle k^Q\rangle)$ and there is no agreement at any $h \in H(N, \langle k^{q'}\rangle) \cap H^{v+q'}$ for any $Q > q' > q$ then it follows that at any $h \in H(N, \langle k^q\rangle) \cap H^{v+q}$ player k^q can guarantee himself his equilibrium payoff of $U_{k^q}(f)$ by not reaching an agreement at h. But this together with (19) imply that at any $h \in H(N, \langle k^q\rangle) \cap H^{v+q}$ there is no agreement.

Step 4: f_{k^q} is simple at N when k^q is the proposer for all $q < Q$. Suppose not; then there exists $q < Q$, h and $h' \in H(N, \langle k^q\rangle)$ such that $f_{k^q}(h) \neq f_{k^q}(h')$. Next define a new strategy f'_{k^q} as follows. Put $f'_{k^q}(h,p) = f_{k^q}(h,p)$ for any $(h,p) \in H^r_{k^q} \times \mathbf{R}_+$ and let

$$f'_{k^q}(h) = \begin{cases} (l^q, p^q) & \forall h \in H\left(N, \langle k^q\rangle\right) \\ f_k(h) & \text{otherwise;} \end{cases}$$

for any $h \in H^p_{k^q}$. Then $f_{k^q} \succ f'_{k^q}$ by inspection. Moreover, since for any q' such that $q < q' < Q$ there is no agreement at any $h_0 \in H(N, \langle k^{q'}\rangle) \cap H^{v+q'}$ (Step 3), agents k^Q and l^Q always induces an agreement at any $h \in H(N, \langle k^Q\rangle)$ (Step2), it follows that (f'_{k^q}, f_{-k^q}) induces an outcome path from period v consisting of a series of disagreements between $k^{q'}$ and $l^{q'}$ for $q < Q$, followed by an agreement between k^Q and l^Q and the same competitive outcome for the set of $N\backslash\langle k^Q, l^Q\rangle$ of remaining agents as when (f_{k^q}, f_{-k^q}) is chosen. Therefore, f'_{k^q} induces at least the same payoff as f_k contradicting the definition of NEC. ■

Lemma 41 *If the induction hypothesis is true for $r-1$ then it is true for r, for any $r>1$.*

Proof. The proof of this lemma is similar to that of Lemma 19. Fix any $N_r \in \mathcal{N}_r$. Let m_r denote the number of inframarginal agents on each side of the market when the remaining set of agents is N_r and (with some abuse of notation) label the buyers and sellers in the set N_r by $i = 1, ..., m_r, ..., r$ and $j = 1, ..., m_r, ..., r$, respectively.

Now, by Lemmas 36, 37 and 40, if there exists $\gamma \in \Gamma(N_r)$ such that there is at most one trade then the outcome is competitive in every $\gamma \in \Gamma(N_r)$ and any trade is at the same competitive price belonging to the set $C(N_r)$ in all such subgames. Therefore, for the rest of the proof we consider only the case in which in every $\gamma \in \Gamma(N_r)$ there is more than one trade.

Fix any $\gamma_r \in \Gamma(N_r)$. Suppose buyer i_r and seller j_r are the first pair of agents that trade in the subgame γ_r. Denote the history, the proposer, the responder and the price at which i_r and j_r trade in this subgame by h_r, $\pi(h_r)$, $\rho(h_r)$ and q respectively. Also denote respectively the set of remaining agents and the history immediately after i_r and j_r trade and leave the market in this subgame by $N_{r-1} \equiv N_r \backslash \{i_r, j_r\}$ and $h_{r-1} \equiv (h_r, (\pi(h_r),\ \rho(h_r), q, A))$. Also, let γ_{r-1} be the subgame defined by the history h_{r-1}

Clearly, since $h_r \in H(N_r)$ we have $\gamma_{r-1} \in \Gamma(N_{r-1})$ and $N_{r-1} \in \mathcal{N}_{r-1}$. Thus by the hypothesis of the induction argument for $r-1$ there exists a price p_{r-1} belonging to the competitive interval $C(N_{r-1})$ such that, in *every* subgame $\gamma \in \Gamma(N_{r-1})$, the inframarginal agents in N_{r-1} trade at the price p_{r-1} and extramarginal agents do not trade. We now establish the result by showing that $q = p_{r-1}$, $q \in C(N_r)$ and the outcome is competitive for the market with N_r and all trades occur at q at every subgames $\gamma \in \Gamma(N_r)$.

Step 1: i_r and j_r always trade at the price q at every $h \in H(N_r, \langle \pi(h_r \rangle)$.

Since i_r and j_r induces an agreement at $h_r \in H(N_r)$ this step follows immediately from Lemmas 32 and 40, together with the assumption that the induction hypothesis holds for $r-1$.

Step2: $q \geq p_{r-1}$.

Suppose not; then $p_{r-1} > q$. Let

$$I' = \left\{ i \in N_{r-1} \cap I \,\middle|\, i \text{ trades in the subgame } \gamma_{r-1} \right\}.$$

By Assumption 1$'$ there exists a history $h \in H(N_r, \langle j_r \rangle, \langle \pi(h_r \rangle)$. First we claim that j_r will trade with some i at a price $p_0 \geq p_{r-1}$ at h. By perfection, any $i' \in I'$ must accept any price $p < p_{r-1}$ by j_R at h (otherwise, i_r and j_r will trade the next period and i' will trade at p_{r-1}). But then there must exists a buyer i and some price p_0 such that at h seller j_r offers i a price p_0 and i accepts. Otherwise, i_r and j_r will trade at the next date and j_r will receive $q - w_{j_r} < p - w_{j_r}$ for any $q < p < p_{r-1}$. Moreover, note that $p_0 \geq p_{r-1}$ because otherwise j_r can increase his payoff by offering $p_0 < p < p_{r-1}$ to i.

Second since $p_0 \geq p_{r-1} > q$ it then follows from Propositions 33 and 32 that at any $h'' \in H(N_r, \langle j_r \rangle)$ seller j_r always chooses i and they always trade at p_0 .

By Assumption 1 there also exists a history $h' \in H(N_r, \langle \pi(h_r\rangle, \langle j_r \rangle)$. Then since $p_0 \geq p_{r-1} > q$ and at any $h'' \in H(N_r, \langle j_r \rangle)$ seller j_r always chooses i and they always trade at p_0, perfection requires j_r not to trade at q with i_r at h'. But this contradicts Step 1.

Step 3: $q \leq p_{r-1}$.

The proof is symmetric with the proof of Step 2 and to save space we shall not state the argument.

The rest of the proof of Lemma 41 follows exactly the same reasoning as that of Lemma 19; thus it involves establishing Steps 4-13 in the proof of Lemma 19 for the above sequential mapping with an endogenous choice of partners. Since the proofs of these steps are the same as those of Steps 4-14 in the proof of Lemma 19 we shall not repeat the arguments. ■

Now Theorem 22 follows by induction from Lemmas 39 and 41. Also, Corollary 23 follows immediately from Proposition 32, Lemmas 38 and 40, and Theorem 22.

References

Abreu D. and A. Rubinstein (1988) "The Structure of Nash Equilibria in Repeated Games with Finite Automata," *Econometrica*, 56**,** 1259-1282.

Binmore, K. and M. Herrero (1988a) "Matching and Bargaining in Dynamic Markets," *Review of Economic Studies* **55**, 17-31.

— (1988b) "Security Equilibrium," *Review of Economic Studies* **55**, 33-48.

Chatterjee K. and H. Sabourian (2000) "Multiperson Bargaining and Strategic Complexity," *Econometrica* **68**, 1491-1509.

Gale D. (1986a) "A Simple Characterization of Bargaining Equilibrium in a Large Market Without the Assumption of Dispersed Characteristics," University of Pennsylvania Center for Analytic Research in Economics and the Social Sciences (CARESS) Working Paper: 86-05, pages 19.

— (1986b) "Bargaining and Competition Part I: Characterization," *Econometrica* **54**, pages 785-806.

— (1986c) "Bargaining and Competition Part II: Existence," *Econometrica* **54**, pages 807-818.

— (1987) "Limit Theorems for Markets with Sequential Bargaining," *Journal of Economic Theory* **43** 20-54.

— (2000) *Foundations of General Equilibrium: Dynamic Matching and Bargaining Games.* Cambridge: Cambridge University Press

Gale D. and H. Sabourian (2003a) "Markov equilibria in Dynamic Matching and Bargaining Games" Cambridge University and NYU, unpublished.

— and — (2003b) "Competition and Complexity, Part II: Simultaneous Endogenous Matching," in preparation.

Hayek, F. (1945) "The Use of Knowledge in Society," *American Economic Review* **35**, 519-530.

Kalai E. and A. Neme (1992) 'The strength of a little perfection' *International Journal of game theory* **20**, 335-355.

McLennan A. and H. Sonnenschein (1991) "Sequential Bargaining as a Noncooperative Foundation for Walrasian Equilibrium," *Econometrica* **59**, 1395-1424.

Osborne M. and A. Rubinstein (1990) *Bargaining and Markets* San Diego; London; Sydney and Toronto: Harcourt Brace Jovanovich, Academic Press.

Rubinstein A. (1986) "Finite Automata Play the Repeated Prisoner's Dilemma" *Journal of Economic Theory* 38, 83-96.

Rubinstein A. and A. Wolinsky (1985) "Equilibrium in a Market with Sequential Bargaining," *Econometrica* **53**, 1133-50.

Rubinstein A. and A. Wolinsky (1990) "Decentralized Trading, Strategic Behaviour and the Walrasian Outcome," *Review of Economic Studies* **57**, 63-78.

Sabourian H. (2001) "Bargaining and Markets: Complexity and the Walrasian Outcome," Cambridge University, forthcoming *Journal of Economic Theory*.