Competitive Equilibrium and Noncooperative Game Theory : Noise and Bounded Rationality*†

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Abstract

In this paper I survey some of the issues that arise when a non-cooperative game-theoretic foundation is provided for the competitive behaviour in dynamic settings. It is often claimed that a market with a 'large' number of 'insignificant' agents result in a competitive outcome. One way of formalizing this in a dynamic game-theoretic context is to assume that there is a continuum of anonymous agents. Here, I consider the equilibria of two types of dynamic games - repeated games, and bargaining and matching models - with a large but finite number of players. Equilibria in these models can be shown to be radically different from those found in models with a continuum of anonymous players. The reason for this discontinuity at infinity in dynamic games with a finite number of players is that players can choose history-dependent strategies. The possibility of conditioning behaviour on histories allows one to construct a large number of equilibria in these settings. In this paper, I discuss, in the context of repeated games and bargaining and matching models, some attempts at explaining away the above discontinuity. These attempts show that noise and/or some elements of 'bounded rationality' can provide some game-theoretic foundation for competitive behaviour in dynamic models with a finite number of agents.

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†The content of this lecture has been greatly influenced by the forthcoming book of Douglas Gale, which addresses the issue of dynamic games and competitive behaviour in more detail.

1. Introduction

In this paper I survey some of the issues that arise when a non-cooperative game-theoretic foundation is provided for the competitive behaviour in dynamic settings. In a competitive market agents take prices as given. Usually, this is justified by saying that there is a 'large' number of 'insignificant' agents. One way of formalizing this in a dynamic game-theoretic context is to assume that there is a continuum of anonymous agents. Such models induce the competitive outcome(s) under some regularity conditions. Here, I discuss the problems that arise in providing a competitive limit theorem. In particular, I consider the equilibria of two types of dynamic games - repeated games, and bargaining and matching models - with a large but finite number of players. Equilibria in these models can be shown to be radically different from those found in models with a continuum of anonymous players. While this discontinuity at infinity may seem narrow, the issue is important because it is critical in terms of providing a game-theoretic foundation to the competitive behaviour in a dynamic-setting. Clearly, economies with a continuum of anonymous agents are of limited value if their equilibria are radically different from those found in economies with a large but finite number of agents. The reason for this discontinuity at infinity in dynamic games is the following. With a continuum of anonymous agents, a single agent's actions cannot influence the information the others receive during the game, whereas with a finite number of players, at each information set players can condition their behaviour on ('payoff irrelevant') past history, and in particular on past plays of other players. This possibility of choosing history-dependent strategies allows one to construct equilibria with the property that at each stage every player has to consider the possible response of others to different moves. By selecting appropriate response rules to past behaviour of others, a large number of equilibrium outcome paths can be constructed.[1] In such equilibria a single agent has a large effect. As a result, even in a frictionless market with a large but finite number of agents, these equilibria induce non-competitive outcomes.

However, notice that in dynamic games it is assumed that a great deal of precise information is observed and remembered at no costs, and that the players do not make any mistakes. If there are observational errors, or if players make mistakes or if there are some costs in remembering past information then the equilibria of dynamic games may look very different. In this paper, I discuss, in the context of repeated games and in the context of dynamic matching and bargaining models, how the above discontinuities disappear if there is noise in the model and /or if players, in taking decisions, care about the complexity of their rules of behaviour ('bounded rationality').

[1] For example, in the case of the repeated games, the Folk Theorem holds for games with a finite number of (anonymous) players but not for the case of a continuum of anonymous players (see below).

Noise and/or some elements of 'bounded rationality' can provide some game-theoretic foundation for the competitive behaviour and can ensure that in equilibrium players choose history-independent (sometimes referred to as stationary or Markov) strategies in dynamic models with a finite number of agents.

2. Non-cooperative repeated game

2.1. Oligopoly model

The earliest and best known example is that of the Cournot model of oligopoly. This is a static one-shot game with K firms choosing quantities simultaneously. An inverse demand function, representing the behaviour of the consumers, determines the market clearing price as a function of aggregate output. When K is finite, the Nash equilibrium of this game results in a price above that predicted by the price-taking competitive model. However, one might conjecture that as K becomes unboundedly large any Nash equilibrium of this model converges to a perfectly competitive equilibrium. The reasoning behind this conjecture is as follows. With a large number of firms, each small firm could benefit from producing an extra unit of output if the market price is above the competitive one because the effect of an extra unit of output on the price is of a lower order of magnitude than the revenue from selling an extra unit. Roberts (1980) and others showed that this conjecture is correct under some strong regularity conditions.[2] He also provided a counter-example to this limit result by showing that there may be a discontinuity in the inverse demand function . Such a discontinuity arises from inverting a selection from the demand correspondence of the consumers. The strong regularity conditions ensure that this does not arise. Without such regularity conditions, a small increase in the aggregate output may result in a sharp (discontinuous) fall in the market price. As a result, at such a discontinuity point, a small increase in output may result a loss of revenue on inframarginal units that is of a higher order of magnitude than the revenue from the additional output. Thus, at such discontinuity points, each firm has a non-negligible effect on the price, no matter how many firms and consumers are in the market.

The discontinuity in the inverse demand function pointed out by Roberts has been studied in a more general setting by Green (1984). In particular, Green looks at the limits of Nash equilibria in static games as the number of players increases without any bound. He shows that to achieve a limit result characterising the limits of such equilibria, some continuity restrictions on the payoffs may be necessary. All these may appear to be purely technical and in static games we may assume them away by only considering games in which the payoff functions are 'well-behaved'. However, when one looks at dynamic games such discontinuities can arise naturally and no reasonable assumptions on the underlying parameter can remove the discontinuities that arise

[2] See also Mas-Colell (1983) for a survey on the static Nash equilibrium foundation of the competitive equilibrium.

when a ('small') player changes its action by a 'small' amount. This is because in these games in equilibrium players can choose complex history-dependent strategies. As a result, there is no guarantee that in equilibrium a change in a single player's strategy will not result in a large response by others.

Applying the Folk Theorem of the repeated game to the repeated Cournot oligopoly model, it is trivial to show that the monopoly outcome can be sustained as an equilibrium of the repeated Cournot model with an arbitrary number of firms if the firms are sufficiently patient and there is perfect monitoring (all firms observe the past output actions of every other firm).[3] Clearly, one could argue that perfect monitoring may be very costly in a large game. However, Green (1980) shows that the above result holds even if the game is anonymous. In particular, he demonstrated the result for the case in which the firms observe only some sufficient statistics (the past prices of the good) reflecting the past aggregate actions of all the firms. The basic idea of Green's result was to construct the following trigger strategy profile. This profile requires each firm to produce $1/K$th of the monopoly output (K refers to the number of the firms) at the initial period and at every other period if the past prices were the monopoly ones. If non-monopoly prices have been observed play Cournot- Nash output forever. The Cournot-Nash behaviour is the punishment that deters deviation from the monopoly behaviour. Clearly, this strategy profile constitutes a subgame perfect equilibrium if the players are sufficiently patient. Notice also that as K increases it is easier to support the monopoly outcome: as the number of the firms increases, the Cournot- Nash equilibria converge to the competitive outcome (given Robert's regularity condition) and therefore the threat of punishing a deviator by reverting to the Cournot-Nash equilibrium becomes more severe.

The message of Green's result is that even when K is large a single player can be 'informationally significant' because any change in his behaviour results in a non-monopoly price and this will trigger a punishment phase (of all players playing the Cournot Nash equilibrium of the one-shot game). Putting it differently, in any finite game there is information about the past behaviour of the firms that past market prices reveal. In particular, past prices can reveal if any player has deviated from the monopoly output in the past, although the identity of the deviator is not revealed.

One way of reducing the informativeness of past prices is to allow prices to vary because of, say, random demand in the industry. If the scale of each firm's output is small relative to the size of the market, any stochastic demand disturbance should make the strategies involving punishment threats more difficult to enforce. Effectively, one is hoping that the introduction of exogenous noise will allow agents to become 'informationally insignificant' as the number of firms increases without any bounds. In the rest of this section, I shall make this idea precise in the context of an abstract repeated game.

2.2. The abstract game with random outcome (imperfect monitoring)

[3]With perfect monitoring this result holds even for the case with a continuum of firms.

2.2.1. Stage game

Consider a stage game $H_K = (\mathcal{K}, \overline{A}, A, X, F, \pi)$ where

$\mathcal{K} = \{1, ..., K\}$ is a finite set of players;

$\overline{A}$ is the set of actions available to each player k;

$A = \overline{A}^K$ is the set of action profiles;

X is set of publicly observed outcomes (signals);

$F : A \to \Delta(X)$ maps action profiles into the set of outcome distributions (for any set Q, $\Delta(Q)$ refers to the set of (Borel) probability measures on Q)

$\pi_k : \overline{A} \times X \to \mathcal{R}$ denotes player k's payoff function.

Thus in this set-up player k's payoff depends on the action taken by k and the realization of the outcome. Also, I assume that both $\overline{A}$ and X are subsets of Euclidean spaces (in fact, complete separable metric spaces will suffice) and that all mappings from measurable spaces into these spaces will be assumed to be Borel measurable.

Now for any action profile $a \in A$, define the expected payoff to k in the one shot game H_K by

$$\Pi_k(a) = E_{F(a)}\pi_k(a_k, x) = \int_x \pi_k(a_k, x)dF(a)(x).$$

Definition 1. *An ϵ-Nash equilibrium of H_K is a profile $a^* = (a_1^*, ..., a_K^*) \in A$ such that for any player k*

$$\Pi_k(a_k, a_{-k}^*) - \Pi_k(a^*) < \epsilon \text{ for all } a_k$$

2.2.2. Repeated game with random outcome

H_K^∞ refers to the infinitely repeated game H_K with discount factor $\delta < 1$. Players are assumed to observe the outcomes of past play (and not the past actions); thus strategy for player k can be written as a sequence of functions $s_k = \{s_k^t(.)\}_{t=0}^\infty$ where $s_k^t : X^t \to \overline{A}$ and X^t is the t-fold Cartesian product of X. Let s be a profile of strategies and S the set of the strategy profiles in H_K^∞.

We shall also assume that the game is anonymous. Formally, this anonymity is defined as follows. Let $m : A \to \Delta(\overline{A})$ map profiles of actions into the distribution of actions; thus for any $a \in A$ and any $B \in \overline{A}$

$$m(a)(B) = 1/K \,|\{k \in \mathcal{K}\,|a_k \in B\}|$$

where $|Q|$ refers to the cardinality of the set Q. Now H_K^∞ is anonymous if for any action profile $a \in A$ the outcome function $F(a)$ depends on the action profile a only through the distribution of actions $m(a)$. Formally, there exists a function $G : \Delta(\overline{A}) \to \Delta(X)$ such that $F(a) = G(m(a))$ for all $a \in A$.

Green demonstrates that the equilibria of the repeated game coincides with that of the one shot game under the above anonymity condition for a model with a continuum of players. To obtain an equivalent result for a model with a 'large' but finite number of players, one needs to restrict the amount of information the publicly observed outcomes (signals) of the game carry even further than that implied by anonymity. Effectively, it is necessary to make the agents' actions 'informationally' insignificant as the number of players is increased. Clearly, the mapping G from the the set of distribution over actions to the set of outcome distribution needs to have some continuity property. As a first attempt one could assume G is continuous when both $\Delta(\overline{A})$ and $\Delta(X)$ are endowed with weak topology. It turns out that this is not sufficient to obtain the result on the equivalence of the equilibria of H_K and H_K^∞ as the number of players increases (see Green 1980). Sabourian (1990) shows that the result holds for large but finite number of players if we assume that G is continuous when $\Delta(\overline{A})$ is endowed with the weak topology and $\Delta(X)$ is endowed with the total variation norm (TVN) topology.[4] (See the appendix for a basic exposition of these topologies.) Continuity of G with respect to the TVN topology (for $\Delta(X)$)) turns out to be precisely the condition needed to ensure that each player becomes 'informationally insignificant' as the number of players increases without bounds (see below).

Theorem 2.1. *Consider a family of one-shot games $\{H_K = (K, \overline{A}, X, \pi_k, F, G\}_{K\in\mathcal{N}}$ with their associate repeated games $\{H_K^\infty\}_{K\in\mathcal{N}}$ ($\mathcal{N}$ refers to the set of natural numbers). Assume that the payoffs of $\{H_K\}_{K\in\mathcal{N}}$ are uniformly bounded and G is continuous when $\Delta(\overline{A})$ is endowed with the weak topology and $\Delta(X)$ is endowed with the TVN topology. Then for any $\epsilon > 0$ there exists a $\overline{K}$ such that for all $K > \overline{K}$, for any Nash equilibrium strategy profile f of H_K^∞ and after any history h of the game H_K^∞ that occurs with a positive probability (given f), the action profile perscribed by f after h is an ϵ-Nash equilibrium of the one shot game H_K.*

Now I would like to provide some intuition for the above result and provide some explanation for the need for the continuity of G with respect to this stronger (TVN) topology.

Fix any repeated game H_k^∞. For any strategy profile $s = (s_k, s_{-k})$ in this game let $E\pi_k^\infty(s)$ be the repeated game payoff of player k. Using techniques that are by now standard in the literature on repeated games one can factorize the long-run payoff $E\pi_k^\infty(s)$ into the first period payoff and the continuation payoff. Formally, for each k there exists a bounded and measurable (continuation payoff) function $V_k : S \times X \to \mathcal{R}$ such that[5]

$$E\pi_k^\infty(s) = \Pi_k(s^0) + \delta \int_x V_k(s, x) dF(s^0)(x)$$

where s^0 is the action profile prescribed by s at the initial period. Given the

[4]Green (1980) demonstrated this result by restricting players to trigger-type strategies. Sabourian (1990) generalises it to all strategies.

[5]Measurability of V_k is with respect to its second argument.

anonymity condition, the above can be written as

$$E\pi_k^{\infty}(s) = \Pi_k(s^0) + \delta \int_x V_k(s,x)dG(m(s^0))(x)$$

Thus, a player takes into account the future reaction of others to his own action to the extent that a change in his action affects the second term in the RHS of the above expression. Thus, to establish the equivalence of the equilibria of H_K and H_K^{∞} as the number of players increases, one needs to show that the effect of a player's action on the second term in the above expression goes to zero as the number of players increases. Formally, one needs to show that for large K and for any Nash equilibrium $s \epsilon S$

$$\left| \int_x V_k(s,x)dG(m(s^0))(x) - \int_x V_k(s,x)dG(m(a_k, s^0_{-k}))(x) \right| \simeq 0 \text{ for all } a_k \in \overline{A} \quad (2.1)$$

Clearly, for large K the action of any single player has a small effect on the distribution of actions; thus $m(s^0)$ is arbitrarily close to $m(a_k, s^0_{-k})$ for large K. This together with the continuity of G imply that for large K the distributions $G(m(s^0))$ and $G(m(a_k, s^0_{-k}))$ are close with respect to the TVN topology. Now, if a probability space is endowed with TVN topology then two probability measures are 'close' if the expected values of *any bounded* measurable function are close with respect to these two measures (and the closeness of the expected values hold uniformly for any set of uniformly bounded functions).[6] Since $V_k(s,x)$ is a bounded measurable (measurability is with respect to x) function, the expression in (2.1) follows from $G(m(s^0))$ being close to $G(m(a_k, s^0_{-k}))$ with respect to the TVN topology.

Finally, notice that if $\Delta(X)$ was endowed with the weak topology the expression in (2.1) would not necessarily be true. This is because weak topology is equivalent to claiming that two probability measures are close if the expected values of any bounded *continuous* function with respect to these two measures are close. (See the appendix for a formal statement of this result.) The continuation payoff function $V_k(s,x)$ is not necessarily continuous as function of x in repeated games. Therefore, weak topology does not limit the amount of information that the outcomes of the game carries.

In repeated games, the only assumptions one can impose on $V_k(s,x)$ are (uniform) boundedness and measureability. But closeness of two measures with respect to the TVN topology is equivalent to saying that expected values of any bounded and measureable function with respect to the two measures are close (with some uniformity). Thus continuity of G with respect to the TVN topology is exact the condition needed to ensure that in a game with a finite but large number of players, a change in the action of a single agent has a small influence on the continuation payoff of that agent.

Before concluding this section, I like to briefly discuss how strong the assumption of the continuity of G with respect to TVN topology is. If the range of G contains

[6] See the appendix for a formal statement of this result.

degenerate distributions (no noise) then G may not be continuous with respect to this strong topology. For example, suppose the sequence of distributions of actions $\{m_n\}$ with $m_n \in \Delta(\overline{A})$ is such that m_n converges weakly to $m \in \Delta(\overline{A})$,

$$G(m_n) = \{\text{the degenerate probability distribution that attaches probablility 1 to 1/n and zero elsewhere}\}$$

and

$$G(m) = \{\text{the degenerate probability distribution that attaches probablility 1 to 0 and zero elsewhere}\}$$

Then $G(.)$ is not continuous with respect to TVN topology (see the Appendix).

On the other hand if the range of G (and thus F) is any of the usual parametrized families of distribution (for example normal) and does not contain any degenerate distribution, then (weak) continuity of the parameters of these distributions as a function of the distribution of actions ensure that G is continuous with respect to TVN topology.[7]

Finally, I like to mention a new paper by Al-Najjar and Smorodinsky (1999) on this subject that has to come to my attention since I wrote the first draft of this paper. This paper extends the result of Sabourian (1990) in Theorem 2.1 to repeated games with non-anonymous signal functions by imposing a stronger set of conditions on the underlying game (in particular, it assumes that the stage payoff functions are continuous). This result is established by using the notion of 'influence' (pivotal) that measures the impact of a change of a single player's action on the expected value of the collective outcome. Building on their earlier work (Al-Najjar and Smorodinsky (1998)), they show that for any given level of influence ϵ, the number of players who have more than ϵ influence on the collective outcome in any period is bounded by number $\gamma(\epsilon)$. This number, in their set-up, turns out to be independent of the number of players and it holds uniformly over all strategy profiles. This establishes that as the number of players increases to infinity the proportion of agents that have a positive influence on the continuation game becomes arbitrarily small. Thus, with a large number of players almost all players play the repeated game as if it is a one-shot game.

3. Dynamic matching and bargaining with a finite number of players

The theory of bargaining has been extensively studied. Dynamic matching models with explicit bargaining and a continuum of agents have been used to provide

[7] In the context of finite (three period) extensive form game, Levine and Pesendorfer (1995) establish a similar result as in Sabourian (1990) by assuming directly that the range of G are distributions that have continuously diffentiable density functions and by assuming that the level of noise disappears as the number of players increases, but not too rapidly.

a game-theoretic foundation to the competitive equilibrium (see Rubinstein and Osborne (1990) and the forthcoming book by Gale). Rubinstein and Wolinsky (1990) - henceforth referred to as RW - consider a simple dynamic matching and bargaining model with a finite number of players. The predictions of this model turn out to be radically different from that of the competitive market even for the case in which there are no transaction costs or frictions. In this section, I shall first discuss RW's results and then discuss some recent work by Gale (2000) and Sabourian (1999) which show that the predictions of RW's model are consistent with the competitive outcome if the agents are assumed to be 'boundedly rational' in specific ways.

3.1. RW's model with random matching

RW consider a market with B identical buyers and S identical sellers. Each seller has a single unit of an indivisible commodity. Each buyer would like to buy precisely one unit of the commodity. The utility that each buyer derives from consuming one unit of the good is one and the disutility of parting with a unit of the good (the reservation price) for each seller is zero. Assume, throughout, that

$$B > S.$$

Thus the competitive solution in this model is such that the sellers receive the entire surplus and all available units are sold at the price 1.

The play of the dynamic matching and bargaining game takes place in discrete time. Let $\delta \in [0, 1]$ be the common discount factor for each player. Thus if at any period $t = 0, 1, 2, ...$ a buyer and a seller agree to the sale of a single unit of the good at a price p then the seller and the buyer receive a payoff of $\delta^t p$ and $\delta^t(1-p)$ respectively.

At each period the remaining agents in the market are matched in pairs of one seller and one buyer. The matching process is random and all possible seller-buyer matches are equally likely. When two agents meet, each has an equal probablity (probability 1/2) of being chosen as a proposer. Once an offer is made by the proposer, the other agent accepts (A) or rejects (R) the offer. If the proposal results in an acceptance, the agreement is implemented and the parties leave the market. If the offer is rejected the match is disolved and the parties return to the market to join the next matching paeriod. In any period, unmatched buyers are required to remain inactive until the next matching stage.

At each period t each agent is assumed to know everything that has happened in the market up to the end of period $t-1$, including all the past outcomes in matches in which he was not a party to. In addition, at each period t, each player knows the identity of his match at t, the selection of the proposer, and, in the case of a responder, the proposal. However, at any period t, when players select their actions they do not know the identity of the other matches and what actions are being simultaneously chosen by other agents.

Notice that if $S > 1$ then there is more than one match in each period. Thus, the game is one of imperfect information when there is more than one seller. Therefore, the appropriate equilibrium concept, in this case, is sequential equilibrium (perfect Bayesian equilibrium). With $S = 1$, it is sufficient to use subgame perfect equilibrium as the solution concept.

3.2. Equilibrium characterisation for the case of $\delta = 1$.

The case for the competitive outcome(s) is often made for economies in which there are no transaction costs and/or frictions. In RW's model the only possible transaction cost (friction) is due to the assumption of discounting. For the case of no discounting, RW's provide the following characterisation result.

Theorem 3.1. *If $\delta = 1$ then for every price $\overline{p}$ between 0 and 1 and for every one to one function β from the set of sellers to the set of buyers there exists a sequential equilibrium in which each seller s sells one unit of the good to buyer $\beta(s)$ for a price $\overline{p}$.*

Since in a competitive equilibrium all goods are sold at the unique price of 1 (because $B > S$), the above result demonstrate that, even when there are no transaction costs, a continuum of non-competitive prices can be sustained as a sequential equilibria in the above dynamic matching and bargaining game with a finite number of agents.

The intuition behind the proof for the case of $S = 1$ is the following. There is a distinguished buyer $\overline{b}$ who has the 'right' to buy the single seller's good at $\overline{p}$ ($\overline{b}$ depends on the past history of play). The equilibrium strategies are such that whenever the seller meets the distinguished buyer, which ever is chosen as the proposer offers a price $\overline{p}$ and the responder accepts. Whenever the seller meets a buyer $b \neq \overline{b}$, the seller as a proposer offers the good at a price $p = 1$ and the buyer $b \neq \overline{b}$ offers to pay a price $p = 0$. In both cases the responders reject the offers. The outcome of these strategies is that the seller sells the good to buyer $\overline{b}$ at $\overline{p}$.

To show that it does not pay players to deviate from the above, the strategies further specify the following responses to any deviations. If the seller proposes a price different from the equilibrium price ($\overline{p}$ to $\overline{b}$ and 1 to $b \neq \overline{b}$) to any buyer $\widetilde{b}$ then this buyer rejects and he has the 'right' to buy the good at a price $\widetilde{p} = 0$. Thus, the continuation strategy is the same as above with the price $\widetilde{p}$ in place of $\overline{p}$ and the buyer $\widetilde{b}$ in place of $\overline{b}$.

If one of the buyers deviates from their equilibrium strategies then the seller rejects and another buyer $\widehat{b}$ gets the right to buy the good at a price $\widehat{p} = 1$. The continuation strategy is the same as before with the price $\widehat{p}$ in place of $\overline{p}$ and the buyer $\widehat{b}$ in place of $\overline{b}$.

Further deviations can be treated in exactly the same way.

It is easy to check that it does not pay any player to deviate from the above strategy after any history. Clearly, any initial deviator is no better off from deviating given the punishments. Also after any deviation any responder is at least as well off rejecting the proposed deviation and following the punishments than from accepting the proposed deviation.

Notice that the strategies are quite complicated and the behaviour of each agent at any period depends on the history play up to that period - there are potentially an indefinite number of potential deviations and for each deviation the above strategy profile specifies a tailor-made response in order to deter the deviation. Thus the agents need a large amount of information to implement the above strategy profile.[8] At the other extreme, we can assume that at any period the agents only have access to the history of play in that period and can not condition their behaviour on the previous history of plays. Thus for the purpose of comparison, one can consider history-independent (stationary or Markov) strategies. RW show that the only stationary equilibrium outcome is the competitive one. In fact, their result is slightly stronger.

Theorem 3.2. *If at any time each player's information consists only of the set of players that are present in the market at time t and the time itself then the unique sequential equilibrium price is the competitive price of 1.*

The above informational restriction prevents agents from punishing a deviator since the deviator is not remembered. For example, in the proof of Theorem 3.1, any deviation by the seller was rejected by the responder because the rejection led to a reward for the buyer. In Theorem 3.2 with stationary strategies the buyer could not be rewarded because the deviation of the seller could not be observed.

In the literature on dynamic games, it is often the case that only stationary/Markov equilibria are considered. There are very few formal attempts at justifying such equilibria (Piccione and Rubinstein (1992), Maskin and Tirole (1998) and Chatterjee and Sabourian (1999, 2000) are some exceptions). By appealing to some elements of 'bounded rationality', Gale (2000) and Sabourian (2000) show that the competitive price is the unique outcome in RW's model and thereby provide some justification for stationary/Markov equilibria in these class of games. However, these two papers formalise bounded rationality in different ways; here, I will briefly sketch their arguments. Before addressing the works of Gale (2000) and Sabourian (2000), I will briefly discuss the no discounting assumption and alternative matching technologies in RW's model.

3.3. Discounting and matching

With discounting and random matching, it is not possible to establish the existence a of continuum of sequential equilibrium prices as in Theorem 3.1. RW establish this Theorem by constructing strategies that induce special 'relationships' between

[8] RW also obtain the same result for the case in which each player observes only his own past history.

buyers and sellers after every history. As was described in the previous subsection, these relationship involve a buyer obtaining the right to buy a good provided by a specific seller at a particular price. With no discounting, the threat of forming new relationship deter deviations from the equilibrium strategies in RW's set-up. With discounting, the threat of such new relationships may no longer be credible. This is because, with random matching, after any history the expected time that elapses before the designated members of this new relationship meet each other may be very long (for example, this will be the case if S and B are large). This is not important when there is no discounting. However, with discounting, new relationship that take (on average) a long time to occur do not have sufficient deterrence value. As a result, with discounting such strategies may not constitute a credible equilibrium. In fact, RW establish the following result for the one seller case.

Theorem 3.3. *(See RW) Suppose that* $S = 1$ *and* $\delta \in (0, 1)$. *Then the subgame perfect equilibrium outcome is unique and the agreement is reached in the first period. Moreover, the unique equilibrium price converges to the competitive price of 1 as* $B \to \infty$ *or as* $\delta \to 1$.[9]

The convergence of the equilibrium prices to the competitive price of 1 as $\delta \to 1$ seems to throw some doubt on the multiplicity result in Theorem 3.1. However, RW argue that with random matching discounting has an unrealistic feature. If players discount the future then holding a special relationship becomes costly. In particular, a pair of agents face a cost of maintaining their relationship even after ther are matched. But why should staying with one's current partner be costly? Thus, they consider a new model of matching in which a matched pair of agents do not have to separate at the end of a bargaining session. With this new matching model with an endogenous choice of partner they establish the continuum of (non-competitive) equilibria even for the case in which the players discount the future.

Theorem 3.4. *(See RW) Suppose that there is one seller* s *and in each period* s *chooses the buyer with which to bargain with. Then for each buyer* b *and any price* $\frac{1}{1+\delta} \leq p \leq 1$ *there exists a subgame perfect equilibrium outcome such that (i)* s *always chooses* b *in the first period and (ii) either* s *is selected as the proposer, in which case they agree on the price* p, *or* b *is selected as the proposer, in which case they agree on the price* $\frac{\delta p}{2-\delta}$.

Thus, even with discounting, it is possible to demonstrate the existence of a large number of non-competitive sequential equilibrium outcomes, irrespective of the number of buyers.[10]

[9] No equivalent result is known for the case of more than one seller.

[10] Theorem 3.4 can be extended to the case in which there is more than one seller.

3.4. Some attempts at providing a justification for the competitive outcome in RW's model

First some notation. Let

- e outcome of all actions (plays) in all matches in a given period
- $h^t = (e^1, ..., e^{t-1})$ history of outcome of plays in all matchs up to period t
- H^t the set all possible histories of plays (outcomes in all matchs) up to period t
- $H^\infty = \cup_{t=1}^\infty H^t$ the set of all finite histories of plays
- d_i information that any player i receives in any period of the bargaining; thus either d_i says that i has been selected to make a proposal to some player j or d_i consists of a price offer by some player j to i
- D_i the set of all d_is
- f_i strategy of player i in the bargaining game; thus $f_i : H \times D_i \to A \cup R \cup [0, 1]$
- F_i the set strategies for i
- $\langle f_i \mid h \rangle$ strategy induced by $f_i \in F_i$ after history $h \in H^\infty$; thus for any h and $h' \in H^\infty$
$$\langle f_i \mid h \rangle (h') = f_i(h, h')$$
- $\pi_i(f_i, f_{-i})$ the expected payoff to player i if the strategy profile (f_i, f_{-i}) is chosen.

3.4.1. Gale (2000)

For any strategy profile f, let $I(f) \equiv \{f' \in F \,| f' = \langle f \mid h \rangle$ for some $h\}$ be the set of all strategies induced by f after some h. Effectively $I(f)$ is the set of rules (strategies) within the strategy profile f. At the beginning of the game the players follow the strategy profile $f \in I(f)$; if the outcome e^1 occurs in the first period then the players follow the rule (strategy) $\langle f \mid e^1 \rangle \in I(f)$ in the second period and so on. Thus, if the players choose a profile f then the players effectively make transitions between rules depending on the outcome in any period. For any outcome of plays e in a period, this can be formally described by a transition function

$$\Psi(f', e) = \langle f' \mid e \rangle \text{ for all } f' \in I(f).$$

Now Gale (2000) introduces a small amount of randomness into the transitions between rules. In particular, he fixes the strategy profile f and some $\epsilon > 0$; then he assumes that if the players are following a rule $f' \in I(f)$ and an outcome e happens then they follow the rule (strategy) $\langle f' \mid e \rangle$ with probability $(1-\epsilon)$ and with probability ϵ they uniformly follow all the rules in $I(f)$. Thus, for strategy profile f such that $I(f)$ is finite, the transition between members of $I(f)$, in Gale's set-up, is given by:

$$\Psi(f', e) = \begin{cases} \langle f' \mid e \rangle & \text{with probability } 1 - \epsilon + \epsilon/N \\ f'' \neq \langle f' \mid e \rangle & \text{with probability } \epsilon/N \end{cases}$$

where N refers to the cardinality of the set $I(f)$.

The above randomness ensures that there is always a small probability that in any continuation game all the rules within the strategy profile is chosen with a positive probability. Now, Gale's notion of an equilibrium consists of a profile of strategies $f = (f_i, f_{-i})$ that is a subgame perfect equilibrium in the game with the above ϵ transition trembles. Clearly, his equilibrium notion is the same as the standard subgame perfect equilibrium when $\epsilon = 0$. However, with $\epsilon > 0$, Gale shows that the unique equilibrium outcome is the competitive one in the case in which there is a single seller s. I shall now provide a sketch of Gale's arguments for the case of an equilibrium profile f for which $I(f)$ is finite and the seller always makes the offer.

Denote the elements of $I(f)$ by $f^1, f^2, ..., f^N$. Let $v_i(f^n)$ be the expected payoff of i in the game with transition error ϵ if the players choose f^n with probability $(1-\epsilon)$ and with probability ϵ they uniformly follow all the rules in $I(f)$. . Thus

$$v_i(f^n) = (1-\epsilon)\pi_i(f^n) + \epsilon/N \mid \sum_{f' \in I(f)} \pi_i(f')$$

Also, let

$$1 - z = \min_n v_s(f^n) \tag{3.1}$$

Then,

$$\sum_{b \in \mathcal{B}} v_b(f^n) \leq z \text{ for all } n$$

where $\mathcal{B}$ is the set of buyers. Thus for each n there exists at most one buyer $b(n)$ such that $v_{b(n)}(f^n) > z/2$. But this implies that there exists a buyer b such that $v_b(f^n) \leq z/2$ for at least a fraction $(N-1)/N$ of the rules (otherwise, at some n more than one buyer have a continuation payoff of more than $z/2$). But then for any n

$$v_b(f^n) \leq (1-\epsilon)z + \epsilon(\frac{1}{N}z + (\frac{N-1}{N})z/2)$$

$$= z(1 - \epsilon\frac{N-1}{2N})$$

Thus buyer b always accepts any offer $p < 1-\ z(1-\epsilon\frac{N-1}{2N})$. But then z must be equal to 0. Otherwise, s can always guarantee himself more than $1-z$ by making a price offer p such that $1 - z < p < 1 - z(1 - \epsilon\frac{N-1}{2N})$. But this contradicts (3.1)

3.4.2. Sabourian (1999)

It is often argued that if players prefer simple strategies to more complicated ones then in equilibrium stationary strategies would be chosen.[11] Sabourian (1999) attempts to formalise this intuition, in the context of RW's model, by introducing complexity costs lexicographically with the standard payoff into the players' preference ordering as in Rubinstein (1986), Abreu and Rubinstein (1988), Piccione and Rubinstein (1993) and others.

The earlier papers of Rubinstein and others addressed the issue of equilibrium selection in a two-player repeated games by modelling players as finite-state automata. Complexity costs in these papers was measured by the number of states of a machine. Here, since the game played in each period is itself an extensive form, there is not a unique way of specifying an automaton in the above matching and bargaining game. Sabourian (1999) formulates a particular specification of an automaton. With this specification, the minimum number of machine states needed to implement a particular strategy $f_i \in F_i$ turns out to be the cardinality of the set of induced strategy

$$I_i(f_i) = \{f_i' \,|\, f_i' = \langle f_i \mid h \rangle \text{ for some } h\}\,. \tag{3.2}$$

But this measure of complexity does not measure fully the complexity of behaviour in each period. In the context of the random matching model, (Sabourian) introduce a different definition of complexity to measure the complexity of behaviour within a period. In particular, I assume the following complexity criterion. *A* strategy f_i is more complex than another machine f_i', denoted by $f_i \succ f_i'$, if the strategies f_i and f_i' are otherwise identical except that given some partial history (information) $d \in D_i$ in a single period, f_i' changes its action less often in response d than f_i (thus f_i' conditions less on the previous history of the game prior to d than f_i).[12]

This definition of complexity is a very weak concept and is sufficient to ensure that in equilibrium the competitive outcome is selected in the random matching model.

The basic equilibrium notion used in Sabourian (2000) is *Nash Equilibrium* of the game with complexity cost introduced lexicographically. Formally, a Nash equilibrium with complexity cost (denoted by NEC) is a strategy profile $f = (f_1, .., f_n)$ that satisfies the following two conditions:

$$\pi_i(f_i,\, f_{\sim i}) \geq \pi_i(f_i', f_{\sim i}),\ \forall f_i',$$

$$\nexists f_i' \text{ such that} \qquad \pi_i(f_i,\, f_{\sim i}) = \pi_i(f_i',\, f_{\sim i}) \qquad \text{and} \qquad f_i \succ f_i'.$$

[11] For example, Gul (1989) mentions simplicity as a reason for selecting stationary equilibria in the non-cooperative coalitional bargaining. (See also Osborne and Rubinstein (1994) chapter ? for an argument against this view.)

[12] Chatterjee and Sabourian (1999,2000) use a similar notion of complexity to justify stationary equilibria in n-person alternating bargaining games.

Clearly, the concept of NEC does not put any restriction on the behaviour of the agents 'off-the-equilibrium' path. One way of ensuring credibility is to restrict attention to NEC profiles that are perfect Bayesian equilibrium (sequential equilibrium) of the underlying game. Sabourian (1999) does precisely this and refers to such a strategy profile as perfect Bayesian equilibrium with complexity costs(PBEC).[13]

The main selection result of Sabourian (1999) corresponds to this notion of equilibrium. In particular, for the random matching model of RW, I show that any PBEC-strategy profile f is stationary and induces the unique competitive price of 1, if set $I(f)$ is finite (finite memory).[14]

Here, to provide some intuition for the role of complexity I shall explain why the strategies that are used in the proof of Theorem 3.1 can not constitute a PBEC. First, assume otherwise. Next, note that these strategies are non-stationary. In particular, the set of buyers $B' \equiv \{b \in B \,|b \neq \beta(s) \text{ for all } s \in S\}$ - this is the set of all those buyers who do not have any rights to any good - also follow non-stationary (complex) strategies. Third, note that any $b \in B'$ has a payoff of zero in the equilibrium constructed in the proof of Theorem 3.1. Since any $b \in B'$ can always obtain at least a payoff of zero by following a simpler strategy that always makes the same offer and accepts all offers, it follows that any such b can economize on complexity without sacrificing any payoff. But this is a contradiction.

The proof of the selection result for the case of a single seller in Sabourian (1999) is similar to above but applied to the continuation payoff. The selection result for an arbitrary number of sellers is demonstrated by applying an induction argument to the set of sellers.

Sabourian (1999) also considers RW's voluntary matching model. With voluntary matching, at the beginning of each period, each seller chooses his bargaining partner during that period. This introduces an additional element of complexity. In Sabourian (1999), I also show that the above selection result for the random matching model extends to models with voluntary matching if the notion of complexity is strengthened to include both the complexity of behaviour in a given period, as above, and the 'counting states' notion of complexity (the cardinality of the set $I_i(.)$ defined in(3.2)). Effectively, counting the number of states measures the complexity of conditioning the choice of one's partner at the beginning of each period on past history.

Appendix

Weak convergence and total variation norm topology

For any measurable space (X, Σ), where X is an arbitrary set and Σ is the $\sigma-$algebra on X, I define the set of probability measures on (X, Σ) by $\Delta(X, \Sigma)$.

[13] Alternatively, as in Chatterjee and Sabourian (1999, 2000), one could ensure credibility by introducing noise into the system and consider extensive form trembling hand equilibrium with complexity costs.

[14] If complexity costs enter the preference ordering of the agents as an arbitrary fixed positive cost (rather than lexicographically) then one does not need to assume that $I(f)$ is finite.

Definition 2 (Weak convergence). *A sequence of probability measures* $\{m_n\}_{n=1}^{\infty}$ *in the set* $\Delta(X, \Sigma)$ *converges weakly to a probability measure* $m \in \Delta(X, \Sigma)$ *if and only if for all bounded continious real-valued function* $f : X \to \mathcal{R}$ *the following holds*

$$\lim_{n\to\infty} \int f \; dm_n = \int f \; dm$$

Next, I define the total variation norm.

Definition 3. *Consider any measurable set* (X, Σ)*. The total variation norm of any probability measure* $m \in \Delta(X, \Sigma)$ *is defined by*

$$|| \; m \; ||= \sup \sum_{i=1}^{n} | \; m(B_i) \; |$$

where the supremum is over all finite partitions of X *into disjoint meaurable sets* $B_1, ..., B_n$ *and for any arbitrary real number* r, $| \; r \; |$ *refers to the magnitude of* r.

The above total variation norm defines a metric on the space $\Delta(X, \Sigma)$. This metric defines the total variation topology.

Definition 4 (Total variation norm topology). *A sequence of probability measures* $\{m_n\}_{n=1}^{\infty}$ *in the set* $\Delta(X, \Sigma)$ *converges in the total variation norm to the probability measure* $m \in \Delta(X, \Sigma)$ *if and only if* $\lim_{n\to\infty} || \; m_n - m \; ||= 0$.

The following result (see Lucas and Stokey (1989) section 11.3) summarizes the properties of the total variation norm topology.

Definition 5. *For any sequence of probability measures* $\{m\}_{n=1}^{\infty}$ *in the set* $\Delta(X, \Sigma)$ *and for any probability measure* $m \in \Delta(X, \Sigma)$*, the following conditions are equivalent:*
(i) $\lim_{n\to\infty} || \; m_n - m \; ||= 0$,
(ii) $m_n(B)$ *converges uniformly to* $m(B)$ *as* $n \to \infty$ *for any meausrable set* $B \in \Sigma$,
(iii) for any real-valued bounded measurable function f *defined on* (X, Σ)

$$\lim_{n\to\infty} \int f dm_n = \int f dm$$

and this convergence is uniform for all f *such that* $\sup_{x\in X} | \; f(x) \; |\leq 1$.

To illustrate how strong the TVN topology is relative to weak convergence consider the following deterministic example (also found in Lucas and Stokey (1989)).

Example 1. *Let* $X = [0, 1]$ *and for any integer* n *let* $m_n = \{$*the degenerate probability distribution that attaches probablility* 1 *to* $1/n$ *and zero else*$\}$*. Clearly, the sequence* $\{m_n\}$ *is equivalent to the sequence*

$$1, \frac{1}{2}, \frac{1}{3}, \ldots$$

and this sequence converges to zero. Now let $m = \{$*the degenerate probability distribution that attaches probablility* 1 *to* 0 *and zero else*$\}$*.*

Does m_n *converges to* m *either with respect to the TVN topolgy or with respect to weak topology? First, consider the real valued function* $f : X \to \mathcal{R}$ *given by*

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $\int f dm_n = 0$ *for all* n *and* $\int f dm = 1$*. Therefore, by* (iii) *of Definition 5,* m_n *does not converge to* m *with respect to TVN topolgy.*

Next note that for any bounded real valued function $f : X \to \mathcal{R}$ *we have* $\int f dm_n = f(1/n)$ *for all* n *and* $\int f dm = f(0)$*. Therefore, if* f *is continuous then* $\lim_n \int f dm_n = \int f dm$*. Thus* m_n *converges weakly to* m*.*

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