Lecture Notes in Mathematics

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Contents

Preliminary reading

The mathematics part of Paper 3 assumes that students are familiar with the material set out below (which is basically the content of the Core Mathematics modules C1 – C4 of a standard A-Level Mathematics course):

> Sets, notation and basic facts; Definition of integers, rational and real numbers; Indices; Pairs of simultaneous linear equations; Quadratic equations; Graphs of linear and quadratic equations, and simple coordinate geometry; Definition of function, domain, range and inverse function, composition of functions; An intuitive notion of limits and continuity; Differentiation and its geometric interpretation, rate of change; Natural logarithm and exponential function; Differentiation of natural log and exponential functions; Product, quotient and chain rules for differentiation; Differentiation of polynomial functions; Multi-variable functions and partial derivatives; Integrals as antiderivatives, definite integrals; Sequences, series, sum of geometric progression; Unconstrained optimisation of a function of one variable; integration of $1/x$ and exponential function; Integration by substitution and by parts; Vectors (addition, subtraction and scalar product).

You don't have to revise all of the above, but a careful revision of chapters 1, 2, 3 of Sydsaeter & Hammond's *Essential Mathematics for Economics Analysis* (8th edition) is essential.

1 Real numbers and real tuples

We denote by $\mathbb{R}$ the set of real numbers. We typically visualise real numbers as points on a line which we will refer to as the **real line**. The **distance** between the points x and y on the real line will be $|x - y|$.

1.1 The least upper bound property

A subset S of $\mathbb{R}$ is called **bounded above** if there exists a number B greater than or equal to all elements of S. $\mathbb{R}$ satisfies the so-called **least upper bound property** which states that:

> If a subset S of $\mathbb{R}$ is bounded above, then it has an upper bound which is smaller than all other bounds, that is, there exists b such that
>
> - $x \leq b$ for all $x \in S$, and
> - if $x \leq B$ for all $x \in S$, then $b \leq B$
>
> This bound b is called the **least upper bound (lub)** of S. Another name for lub is **supremum**.

If the supremum of a set belongs to the set itself, then it is necessarily the maximum (greatest) element of that set. Note that a set being bounded above does not imply that the set necessarily has a maximum. For example the open interval $(0, 1)$ in $\mathbb{R}$ is bounded above, but does not have a maximum element. 1 is its supremum.

Analogously, we can define the notion of being bounded below and the related **greatest lower bound** property. The greatest lower bound of a set is also called its **infimum**.

Finally, a set S in $\mathbb{R}$ is called **bounded** if there exists B such that $|x| \leq B$ for all $x \in S$.

1.2 The set of ordered real pairs: $\mathbb{R}^2$

We denote an **ordered pair** of real numbers by (x_1, x_2), where $x_1 \in \mathbb{R}$ is called the **first entry**, and $x_2 \in \mathbb{R}$ is called the **second entry**. The set of all ordered pairs of real numbers is nothing but the *cartesian product* of $\mathbb{R}$ with itself. That is, $\mathbb{R} \times \mathbb{R}$, also denoted by $\mathbb{R}^2$:

$$\mathbb{R}^2 \quad = \quad \{(x_1, x_2) \ \mid \ x_1 \in \mathbb{R}, \ x_2 \in \mathbb{R}\}$$

The term 'ordered' is to remind us that $(3,5)$ is not the same as $(5,3)$.

Similar to our visualisation of $\mathbb{R}$ as a line, we will visualise $\mathbb{R}^2$ as a plane, and on this plane we will often draw two lines explicitly:

- $\{(x_1, 0) \mid x_1 \in \mathbb{R}\}$ called the horizontal axis
- $\{(0, x_2) \mid x_2 \in \mathbb{R}\}$ called the vertical axis

The point $(0,0)$, also denoted $\mathbf{0}$ or simply 0, is called the **origin**.

We can appeal to geometry to extend our notion of distance from the real line (a visualisation of the set $\mathbb{R}$) to the real plane (a visualisation of the set $\mathbb{R}^2$). And then take that geometric notion of distance between two points $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ (provided by Pythagoras) and use that very notion as the distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^2$.

$$\text{dist}((x_1, x_2), (y_1, y_2)) \quad = \quad \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)^{1/2}$$

This also suggests a generalisation of the notion of absolute value that applies to $\mathbb{R}^2$. Given two points $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ in $\mathbb{R}^2$, let's define $\boldsymbol{x} - \boldsymbol{y}$ as $(x_1 - y_2, x_2 - y_2)$. If we define the **norm** of a point $\boldsymbol{a} = (a_1, a_2)$ in $\mathbb{R}^2$ as $\sqrt{a_1^2 + a_2^2}$ and denote it by $\|\boldsymbol{a}\|$, then

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) \quad = \quad \|\boldsymbol{x} - \boldsymbol{y}\| \quad = \quad \|(x_1 - y_1, x_2 - y_2)\| \quad = \quad \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)^{1/2}$$

1.3 The set of ordered n-tuples

If we can talk about pairs, why not also define n-tuples of real numbers? That is, ordered strings of n reals:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for all } i = 1, \ldots, n\}$$

We will also refer to such n-tuples as **points** in the **n-dimensional real space**. We define the following algebraic operations for points in $\mathbb{R}^n$. For every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$:

- $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
- $\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$

The **norm** of a point $\boldsymbol{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ is defined as

$$\|\boldsymbol{x}\| \quad = \quad \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$$

and the **distance** between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$ is given by

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) \quad = \quad \|\boldsymbol{x} - \boldsymbol{y}\| \quad = \quad \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

Note that the norm of a point is its distance from the origin.

A convention is to treat the cartesian product $\mathbb{R}^m \times \mathbb{R}^n$ as $\mathbb{R}^{m+n}$. This is quite intuitive as the first set is a set of ordered m-tuples, and the second set is a set of ordered n-tuples. An ordered pair of an m-tuple and an n-tuple can be seen as an $m+n$-tuple:

$$\begin{aligned} ((x_1, \ldots, x_m), (\tilde{x}_1, \ldots, \tilde{x}_n)) \quad &= \quad (x_1, \ldots, x_m, \tilde{x}_1, \ldots, \tilde{x}_n) \\ &= \quad (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \qquad \text{where } x_{m+i} = \tilde{x}_i \text{ for all } i = 1, \ldots, n \end{aligned}$$

1.4 Open, closed, and compact subsets of $\mathbb{R}^n$

The **ε-neighbourhood** of a point a in $\mathbb{R}$ (also called the **open ball with centre a and radius ε**) is the set of all real numbers which are less than ε away from a. In other words, it is the open interval $(a - \varepsilon, a + \varepsilon)$.

This definition of a *neighbourhood* can be generalised to $\mathbb{R}^n$ word-by-word:

The ε-**neighbourhood** of a point $\boldsymbol{a}$ in $\mathbb{R}^n$ (also called the **open ball with centre** $\boldsymbol{a}$ **and radius** ε) is the set of all points in $\mathbb{R}^n$ which are less than ε away from $\boldsymbol{a}$. In other words,

$$N(\boldsymbol{a}, \varepsilon) \quad = \quad \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}\| < \varepsilon\}$$

We refer to any such set as an **open ball around** a.

A point s is called an **interior** point of a set $S \subseteq \mathbb{R}^n$ if there exists an open ball around s all of which is contained in S.

The set of all interior points of S is called the **interior of** S, and is denoted by $\text{int}(S)$.

For example the interior of the closed interval $[0, 1]$ is the open interval $(0, 1)$.

A subset S of $\mathbb{R}^n$ is **open** if every point in S is an interior point of S.

In other words, S is open if and only if for every $\boldsymbol{x} \in S$, there exists $r > 0$ such that $N(\boldsymbol{x}, r) \subseteq S$.

Note that a set is open if and only if it is equal to its interior.

For example, open intervals in $\mathbb{R}$, and more generally open balls in $\mathbb{R}^n$ are open sets. The empty set is also open since the condition of being open is automatically satisfied (since there is no point in the empty set, the condition becomes vacuous). The set $\mathbb{R}^n$ is open too. The union of open sets is open.

A subset S of $\mathbb{R}^n$ is called **closed** if its complement (i.e., $\mathbb{R}^n \backslash S$) is open.

Most calculus textbooks define closed sets via limits. The above definition is, in some sense, cleaner, but for our purposes the limit based definition is sufficient. So we will revisit closed sets with another definition later.

It is worth noting that a set which is not open is not necessarily closed. For example the interval $[0, 1)$ in $\mathbb{R}$ is neither open nor closed. It is not open because, no neighbourhood of 0 is contained in this set. It is not closed, because no neighbourhood of 1 is contained in its complement (which renders the complement "not open").

A set consisting of a single point is closed. The empty set is also closed. The empty set and $\mathbb{R}^n$ are the only sets in $\mathbb{R}^n$ which are both open and closed.

A subset S of $\mathbb{R}^n$ is said to be **bounded** if there exists B such that $\|x\| \leq B$ for all x in S. If a set S in $\mathbb{R}^n$ is both closed and bounded, then it is called **compact**.

1.5 Convex sets in $\mathbb{R}^n$

A subset K of $\mathbb{R}^n$ is called **convex** if for every $\boldsymbol{x}$ and $\boldsymbol{y}$ in K, and every $\lambda \in [0, 1]$, we have $\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in K$.

Moreover, we call K **strictly convex** if for every $\boldsymbol{x}$ and $\boldsymbol{y}$ in K, and every $\lambda \in (0, 1)$, the point $\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}$ is an interior point of K.

2 Functions

Functions summarise/capture/report relationships between two or more "variables". Writing down a function which specifies/describes a relationship between one variable and another doe not imply or assume causality, does not necessarily provide some structural explanation. A function simply keeps track of the relationship.

2.1 Basic definitions

When we speak of a function, we have two sets and a "rule/mapping" in mind. That is, a function is not only the "rule" that describes what is mapped to what (i.e. the relationship), but also the specification of the domain and the codomain.

$$\begin{array}{rccc} f & : & A & \longrightarrow & B \\ & & a & \longmapsto & b \end{array}$$

This notation means: for every element a of A, there is an element in B which is called the **image** of a under f. We denote by $f(a)$ the image of a under f. If $b = f(a)$, one can also say "f sends a to b" or "f maps a to b", etc.

The set A is called the **domain** of the function f. And the set B is called the **codomain**.

We can also talk about the **image of a set S under f**, which is nothing but the collection of the images of the elements in S. I.e., if S is a subset of the domain, then

$$f(S) \quad = \quad \{f(x) |\ x \in S\}.$$

Given a function $f : A \to B$, the image of the domain is called the **range** of f.

A function $f : A \to B$ is called **one-to-one** (or **1-1**, or **injective**) if for any $a, a' \in A$

$$f(a) = f(a') \quad \Longrightarrow \quad a = a'$$

or equivalently

$$a \neq a' \quad \Longrightarrow \quad f(a) \neq f(a')$$

That is, two distinct elements of the domain are never mapped to the same image.

A function $f : A \to B$ is called **onto** (or **surjective**) if for each $b \in B$, there is $a \in A$ such that $f(a) = b$.

The **inverse image** of b under f is the set of elements in A which are mapped to b:

$$f^{-1}(b) = \{a \mid f(a) = b\}$$

If $f : A \to B$ is 1-1 and onto, we can talk about its **inverse function**, denoted f^{-1}, which is the unique function from B to A pinned down by

$$f(f^{-1}(b)) = b \quad \text{for all } b \in B \quad \text{and} \quad f^{-1}(f(a)) = a \quad \text{for all } a \in A.$$

We say $f : A \to B$ and $f^{-1} : B \to A$ are the inverses of each other.

Simple algebra of functions. If a set B admits basic algebraic operations such as addition and multiplication (e.g., if B is the set of real numbers), we can extend such rules to the set of functions from A to B. In other words, given functions f and g from A to B,

- The sum of two functions f and g is another function from A to B, denoted $f+g$, and defined by $(f + g)(a) = f(a) + g(a)$ for all $a \in A$.

  Likewise,

- The product of two functions f and g is another function from A to B, denoted fg, and defined by $(fg)(a) = f(a)g(a)$ for all $a \in A$.

- The product of a function f with a scalar b in B is another function from A to B, denoted bf, and defined by $(bf)(a) = bf(a)$ for all $b \in B$ and all $a \in A$.

Composition of functions. Given functions $f : A \to B$ and $g : B \to C$ we define the function $g \circ f : A \to C$ by setting

$$(g \circ f)(a) \quad = \quad g(f(a)) \qquad \text{for all } a \in A$$

For example, given a one-to-one and onto function $f : A \to B$, composing f with its inverse f^{-1} would yield the **identity function**:

$$f \circ f^{-1} \quad = \quad \mathrm{id}_B : B \to B \qquad \text{where } \mathrm{id}_B(b) = b \quad \text{for all } b \in B$$

and

$$f^{-1} \circ f \quad = \quad \mathrm{id}_A : A \to A \qquad \text{where } \mathrm{id}_A(a) = a \quad \text{for all } a \in A$$

Note that unless $A = B$, the functions $f \circ f^{-1}$ and $f^{-1} \circ f$ are not the same functions. They are both identity functions, but they do not have the same domain.

2.2 Sequences

A **sequence** with values in set A is nothing but a function $f : \mathbb{N} \to A$, where $\mathbb{N}$ stands for the set of natural numbers (non-negative integers). The custom is to use a different notation though. Instead of using parentheses, we will use subindices: e.g., instead of writing $f(n)$, we will write a_n, and the sequence will be denoted (a_n). We will refer to a_n as the n-th term of the sequence. (There is nothing special about the letter "a" here. It just seems like a reasonable notation given that the sequence takes values in a set denoted by the letter "A".)

We will be dealing mainly with sequences with values in real numbers (or later in the course with values in $\mathbb{R}^m$ for $m > 1$).

If (n_k) is an increasing sequence of positive integers, then a_{n_k} is called a **subsequence** of (a_n). For example, if we set $b_n = a_{2n}$, then (b_n) would be a subsequence of (a_n).

A real-valued sequence is **non-increasing** if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$. Likewise, we say a sequence is **non-decreasing** if $a_{n+1} \geq a_n$ for all n. Of course, a sequence can be neither. If a sequence is non-increasing or non-decreasing, then it is called **monotone**.

A real-valued sequence (a_n) is said to be **bounded** if there exists B such that $|a_n| \leq B$ for all n. Analogously, a sequence $(\boldsymbol{a}_n)$ with terms in $\mathbb{R}^n$ is called bounded if there exists B such that $\|\boldsymbol{a}_n\| \leq B$ for all n.

2.3 $\mathbb{R} \to \mathbb{R}$ functions

When we refer to the class of $\mathbb{R} \to \mathbb{R}$ functions, we actually have a larger class of functions in mind, namely **real-valued functions of one real variable**. The analysis we are developing will typically concern functions whose domain might be smaller than $\mathbb{R}$, for example, functions the like $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $f(x) = 1/x$.

Given a function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, we will say

- f is **non-decreasing** on D if for every $a, b \in D$, we have $a > b$ implies $f(a) \geq f(b)$
- f is **increasing** on D if for every $a, b \in D$, we have $a > b$ implies $f(a) > f(b)$
- f is **non-increasing** on D if for every $a, b \in D$, we have $a > b$ implies $f(a) \leq f(b)$
- f is **decreasing** on D if for every $a, b \in D$, we have $a > b$ implies $f(a) < f(b)$
- If f satisfies any of the above, it is called **monotonic**

We say $f : D \to \mathbb{R}$ is **bounded above** on $E \subseteq D$ if there exists a number $\bar{b}$ such that $f(x) \leq \bar{b}$ for all $x \in E$. Likewise, we say f is **bounded below** on $E \subseteq D$ if there exists a number $\underline{b}$ such that $f(x) \geq \underline{b}$ for all $x \in E$. If f is both bounded above and below, it is simply called **bounded**. For example, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x^2$ is bounded above on $\mathbb{R}$, is bounded below on every finite interval $I \subset \mathbb{R}$, but is not bounded below on $\mathbb{R}$ or on $\mathbb{R}_-$ or on $\mathbb{N}$.

We say f is **single-peaked** on D if there exists p such that f is non-decreasing on the left hand side of p, and non-increasing on the right hand side of p. That is, for every $a, b \in D$, if $a < b \leq p$ then $f(a) \leq f(b)$; and if $p \leq a < b$, then $f(a) \geq f(b)$.

2.4 Graphs of $\mathbb{R} \to \mathbb{R}$ functions

Plotting the graph of a function from $\mathbb{R}$ to $\mathbb{R}$ amounts to creating a visualisation, a picture of the following subset of $\mathbb{R}^2$

$$\text{Graph}(f) = \{(x, f(x)) \mid x \in \mathbb{R}\}$$

on the real plane.

Obviously, if the domain of f is $D \subset \mathbb{R}$, then its graph is the picture of

$$\text{Graph}(f) = \{(x, f(x)) \mid x \in D\}$$

on the real plane.

A function which looks like

$$f(x) = ax + b \qquad \text{where } a \neq 0$$

is also called a linear function, because if we plot its graph, that is, if we draw the picture of the following set in $\mathbb{R}^2$

$$\text{Graph}(f) = \{(x, ax + b) \mid x \in \mathbb{R}\}$$

in the real plane, we get a line that crosses the vertical axis at $(0, b)$ and the horizontal axis at $(-b/a, 0)$.

A quadratic function looks like

$$g(x) = mx^2 + nx + p \qquad \text{where } m \neq 0$$

and the graph of a quadratic function is called a **parabola**.

Note on graphs of inverse functions: If a function f maps a to b, and if f has an inverse function f^{-1}, then by definition f^{-1} maps b to a. Bringing this fact into the graphs, we can see that the graph of f^{-1} can be obtained by reflecting the graph of f about the 45^o line that goes through the origin. In other words, in the x-y coordinate plane, the graph of f is the mirror image of the graph of f^{-1} if we think of the $x = y$ line as the mirror.

What's the point of being able to plot graphs by hand? Especially when a cheap calculator can plot them in an instant... we are less interested in the end product (a graph), than the process (developing a feel for a function). For example, when sketching the graph of a function, we examine several properties of the function: where it is increasing or decreasing, its critical points, its behaviour for arbitrarily large values of x (i.e., asymptotic behaviour as $x \to \pm\infty$), the points around which its value grows arbitrarily (vertical asymptotes), and so on. These qualitative features are what we care about, and we want the graph to illustrate these features (as opposed to the graph being drawn to prefect scale).

In addition to the graphs of $\mathbb{R} \to \mathbb{R}$ functions, we will also be interested in the graphs of sets (e.g., the set of points which satisfy a given set of equations of inequalities.) Take, for example, the set C defined as the solutions $(x, y) \in \mathbb{R}^2$ to the equation $x^2 + y^2 = 4$. The graph of C is the circle with centre at $(0, 0)$ and radius equal to 2, and for brevity we usually refer to C as a circle. Graphs of "constraint sets" defined by equalities or inequalities (such as budget sets and indifference curves) can often help us understand economic problems.

2.5 Polynomials and polynomial functions

An nth degree polynomial with real coefficients is the following object

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that $a_n \neq 0$. These numbers $a_n, \ldots, a_0$ are called the **coefficients** of the polynomial. a_n is called the **leading coefficient** of P.

Each polynomial $P = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ corresponds to a polynomial function $P(\cdot) : \mathbb{R} \to \mathbb{R}$ in an obvious manner which maps

$$x \quad \longmapsto \quad P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

A number r is called a **root** of a polynomial P if $P(r) = 0$.

If $P(r) = 0$, then the polynomial P is divisible by $(x - r)$, meaning it can be written as

$$P(x) = (x - r)Q(x)$$

where Q is an $(n-1)$st degree polynomial.

An nth degree polynomial P has at most n roots. If P has n distinct real roots $r_1, \ldots, r_n$, and if its leading coefficient is a_n, then

$$P(x) \quad = \quad a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

The expression on the right hand side above is called the **factorisation** of polynomial P. Each $x - r_i$ is called a factor of P.

The nth binomial $(x+1)^n$ is

$$x^n + \binom{n}{n-1} x^{n-1} + \binom{n}{n-2} x^{n-2} + \cdots + x^2 \binom{n}{2} + x \binom{n}{1} + 1$$

where

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1}$$

We read the above expression as "n choose k". Using the **factorial** notation

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

we can rewrite "n choose k" as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

3 Limits

A-level mathematics might not discuss limits explicitly, but it does use the intuition behind limits in its interpretation and (admittedly vague) definition of differentiation. Now, before we get into the notion of a limit of an $\mathbb{R} \to \mathbb{R}$ function, there is in fact an even simpler context in which we can formally introduce limits. When we talked about the idea that irrational numbers can be approximated by rational numbers, we gave a constructive procedure to "get closer and closer" to a given irrational number using only rational numbers. That actually was an example of a sequence with a limit, and if we really wanted to construct irrational numbers formally, we could do so with the aid of limits. But all that rests on a well-defined notion of a limit of a sequence.

3.1 Limits of $\mathbb{R}$-valued and $\mathbb{R}^n$-valued sequences

A real valued sequence (a_k) is said to **converge** to L if for each $\varepsilon > 0$,

$$\text{there exists } K \text{ such that} \quad k > K \implies |a_k - L| < \varepsilon.$$

That is, however small the given ε is, eventually all terms of the sequence are within the ε-neighbourhood of L. Pay attention to the wording: after which term the sequence is trapped in to the ε-neighbourhood of L can (and will typically) depend on ε. The smaller the ε, the bigger K might need to be. But, the key is, for every $\varepsilon > 0$, we can indeed find a K that works.[1]

For example, suppose $a_0 = 1$ and $a_k = 1/\sqrt{k}$ for $k > 0$. It is not hard to guess that $a_k \to 0$. Let's prove this. Given any ε, can we find a corresponding K such that all terms of the sequence after the K-th term are in the ε-neighbourhood of 0? Well, what does it mean for a_k to be in the ε-neighbourhood of 0? It means

$$|a_k| < \varepsilon, \quad \text{that is,} \quad \frac{1}{\sqrt{k}} < \varepsilon \quad \text{which is equivalent to} \quad k > \frac{1}{\varepsilon^2}$$

If we choose K to be an integer larger than $\frac{1}{\varepsilon^2}$, then any k larger than K is automatically larger than $\frac{1}{\varepsilon^2}$, and hence $|a_k| < \varepsilon$. Done!

We say L is the limit of sequence (a_k) as k approaches to infinity. There are two common notations to express this, and you will see both, sometimes in the same text:

$$\lim_{k\to\infty} a_k = L \qquad \text{and} \qquad a_k \to L \qquad \text{both mean the same thing.}$$

If a sequence does not converge to any L in $\mathbb{R}$, we say that the sequence **diverges**. But it is useful to distinguish two types of divergence:

Diverging to ∞ or to $-\infty$. We say that (a_k) diverges to **infinity** if

$$\text{for every bound } M > 0 \text{ there exists an index } K > 0, \text{ such that } k > K \implies a_k > M.$$

Similarly, we say that (a_k) diverges to **minus infinity** if

$$\text{for every bound } M < 0 \text{ there exists an index } K > 0, \text{ such that } k > K \implies a_k < M.$$

Diverging without a clear trend. If a sequence diverges, but not to ∞ nor $-\infty$, then we have even less to say about its "long-run trend". In order to illustrated this sort of divergence, consider for instance the sequence defined by $a_k = (-1)^k$ which alternates between 1 and -1.

[1] While ε-δ arguments are not examinable, it is crucial to have at least an intuitive/geometric understanding of limits of sequences to be able to guess/calculate limits for the types of example we frequently encounter in our simple economic models. In what follows, formal ε-δ arguments are included when such arguments are fairly simple and instructive for those who are keen to understand the ideas behind the related statements.

For $\mathbb{R}^n$-valued sequences

The same definition, phrased via neighbourhoods, applies: we say an $\mathbb{R}^n$-valued sequence $(\boldsymbol{a}^k)$ converges to $\boldsymbol{a}$ if

$$\forall \varepsilon \quad \exists K \quad \text{such that} \quad k > K \implies \|\boldsymbol{a}^k - \boldsymbol{a}\| < \varepsilon$$

Note that we used a superscript as opposed to a subscript when referring to the kth element of the sequence. This is simply to avoid a messy notation if we want to write each element (a point in $\mathbb{R}^n$) explicitly as in $\boldsymbol{a}^k = (a_1^k, a_2^k, \ldots, a_n^k)$.

A set $S \subseteq \mathbb{R}^n$ is **closed** if and only if every convergent sequence with elements in S converges to a point in S.

The above statement can be taken as the definition of set in $\mathbb{R}^n$ being closed. Or alternatively, one can take the definition as being the complement of an open set, and show that it is equivalent to the above definition.

3.2 Limit of an $\mathbb{R} \to \mathbb{R}$ function

If we evaluate a function in smaller and smaller neighbourhoods of a point a, is it the case that the values are trapped in smaller and smaller neighbourhoods of a particular number? The existence of a limit of a function is concerned with this question.

We say the limit of a single-real-variable, real-valued function f as x approaches to a is L if given any neighbourhood of L, there exists a corresponding neighbourhood of a such that all $x \neq a$ in the latter neighbourhood are mapped to the given neighbourhood of L.

In other words the **limit of a function f as x approaches to a is L** if the following statement holds:

Given whatever $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

When this is the case, we write $\lim_{x \to a} f(x) = L$.

Note that the existence of this limit as x approaches to L makes no reference to the value of the function at a. It only requires those points "close to a" to be mapped "close to L". In fact, it is possible that f is not even defined at the point a.

Sometimes we focus on the behaviour of the function only on one side of the point of interest. We might ask what happens to $f(x)$ as x gets closer and closer to a, but always from the right hand side. This focus motivates the weaker concept of one-sided limits.

We say the **right limit** of f as x approaches to a is L if

$$\forall \varepsilon \quad \exists \delta \quad \text{such that} \quad 0 < x - a < \delta \implies |f(x) - L| < \varepsilon$$

The notation for L being the right limit of f as x approaches to a is $\lim_{x \to a^+} f(x) = L$.

Analogously, the **left limit** of f as x approaches to a is L if

$$\forall \varepsilon \quad \exists \delta \quad \text{such that} \quad 0 < a - x < \delta \implies |f(x) - L| < \varepsilon$$

The notation for L being the left limit of f as x approaches to a is $\lim_{x \to a^-} f(x) = L$.

For an example of all these concepts, consider the function

$$f : \mathbb{R}_{<0} \cup \mathbb{R}_{>0} \to \mathbb{R} \quad \text{where} \quad f(x) = \frac{|x|}{x}$$

which does not have a limit as $x \to 0$, but its right limit at 0 is equal to 1, and left limit at 0 is equal to -1.

We say a function f is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$.

Moreover, we say f is continuous over a set A if it is continuous at every point $a \in A$.

Note that the definition of a limit phrased in the language of neighbourhoods can be automatically extended to functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. We say the limit of f as $\boldsymbol{x}$ approaches to $\boldsymbol{a}$ is equal to $\boldsymbol{L}$ if given any ε-neighbourhood of $\boldsymbol{L}$, there exists a δ such that all $\boldsymbol{x} \neq \boldsymbol{a}$ in the δ-neighbourhood of $\boldsymbol{a}$ are mapped into the ε-neighbourhood of $\boldsymbol{L}$:

$$\forall\varepsilon \quad \exists\delta \quad \text{such that} \quad 0 < \|\boldsymbol{x} - \boldsymbol{a}\| < \delta \quad \Longrightarrow \quad \|f(\boldsymbol{x}) - \boldsymbol{L}\| < \varepsilon$$

The concept of limit is critical in formalising a key notion, used extensively in economics: that is, the notion of *marginal*. The language of the marginal requires us to look at what happens "at the margin" of a point of interest. For example, how does a consumer's taste change when she is "very close" to the point of spending all of her income? Note, however, that the adjectives "close" or "small" have no descriptive meaning on their own. They can only make sense in relative terms. *What do we mean, then, when we say "at the margin" or "for small changes"?*

3.3 Some key results about $\mathbb{R}$-valued continuous functions

Back in subsection 2.3 we defined what it means for a real-valued function to be bounded.

Extreme value theorem. A continuous function f defined on a closed interval $[a, b]$ is bounded. Moreover, it takes its minimum and maximum values over the interval.

In order to appreciate the above statement, note that it does not hold for an open interval. Consider (and draw the graphs for) the functions $f : (0, 1) \to \mathbb{R}$ with $f(x) = 1/x$; and $g : (0, 1) \to \mathbb{R}$ with $g(x) = x$. Now evaluate how the conclusion of the theorem fails for these two functions.

Intermediate Value Theorem. If f is continuous on $[a, b]$, then f takes all the values between $f(a)$ and $f(b)$.

For this theorem, the interval being closed is not of importance. One implication is that if f is defined over an interval I, and its maximum and minimum values are given by M and m, respectively, then f takes all the values between m and M. In other words, for every $\mu \in [m, M]$, there must be $c \in I$ such that $f(c) = \mu$.

3.4 Limits of a few familiar functions

Let's begin with *constant functions*. If a function takes the same value at every element of its domain, then it is called a **constant function**. When talking about functions from $\mathbb{R}$ to $\mathbb{R}$, a function which takes the value 5 at every point is also denoted by 5 even though this notation occasionally leads to confusion, because 5 happens to be the symbol for a number as well. Having said that this double usage of the same symbol both for a function and a number is more often convenient than confusing. For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 0$, we will simply refer to this function as 0. Zero everywhere! It is almost immediate to verify that a constant function c has the limit c at every point. (*Can you check this?*)

Here is a more complicated function, the so-called **identity function** from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x$. Once again, we might intuit that

$$\lim_{x\to a} f(x) \quad = \quad \lim_{x\to a} x \quad = \quad a \; = \; f(a)$$

and thus conclude that the identity function on $\mathbb{R}$ is continuous everywhere! (*Once again, you might want to verify the statement formally if you want to practise the use of the definition.*)

3.5 Algebra of limits

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then

- $\lim_{x\to a}(f+g)(x) = L + M$.
- $\lim_{x\to a}(fg)(x) = LM$

If we also have $M \neq 0$, then

- $\lim_{x\to a}(f/g)(x) = L/M$

Combining these with just the knowledge of the limit of the identity function and constant functions, we can argue:

- Polynomial functions are continuous at every point of $\mathbb{R}$.
- Rational functions are continuous at every point where they are defined.

3.6 Limits of composition of functions

Limits are all about getting close! They talk about tendencies. Say g tends to L as its argument tends to a. And say f tends to M as its argument tends to L.

Well, then what would you expect from $f \circ g$ as its argument tends to a:

$$f(\underbrace{g(\overbrace{\square}^{\text{goes to } a})}_{\text{goes to } L}) \longrightarrow M$$

In other words:

$$\lim_{x\to a} g(x) = L \quad \text{and} \quad \lim_{x\to L} f(x) = M \quad \Longrightarrow \quad \lim_{x\to a} f(g(x)) = M$$

Proof. Given $\varepsilon > 0$, we'd like to show that there exists a δ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |f(g(x)) - M| < \varepsilon$$

First, we know that for the same ε given above, there exists a $\Delta > 0$ such that

$$0 < |x - L| < \Delta \quad \Longrightarrow \quad |f(x) - M| < \varepsilon$$

So what we need from δ is to make sure that $|g(x) - L| < \Delta$ for $0 < |x - a| < \delta$. But since $g(x) \to L$ as $x \to a$, it is indeed possible to find such δ. And therefore

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |g(x) - L| < \Delta \quad \Longrightarrow \quad |f(g(x)) - M| < \varepsilon$$

□

3.7 A quick detour around ∞

We have already formalised what it means for the argument of a function to go to ∞. However, we haven't given a clear definition of what it means to say a function is approaching to ∞ as its argument approaches to this or that.[2]

What is ∞? Is it a number? Does it really exist? No, and yes, respectively.

∞ is not a real number. That is, it is not a member of $\mathbb{R}$. Can we just add ∞ to this set and talk about a new and larger set of numbers? If we really want, of course we can, but we need to be careful, because ∞ does not behave like other numbers in algebraic operations. And, therefore,

[2]Likewise, we have formalised what it means for the terms of a sequence to go to ∞, but we haven't given a clear definition of what it means to say a sequence approaching to ∞

we can't really treat it as if it is a number as far as algebra goes in the way we have been used to. And so much of our analysis relies on our ability to carry out algebra as we know it so far.

Since we can't treat ∞ as we treat real numbers, let's just not force "numberhood" on ∞ and carry on. One way to think about ∞ is to observe that it summarises a **particular behaviour of a set of numbers**. As such ∞ is a property which really describes something about a set. What kind of set? What kind of behaviour? For example, if we say a sequence goes (diverges) to ∞, what we mean is that given whatever bound, the terms of the sequence will eventually exceed that bound. To make it more precise, given any $B > 0$, there exists k such that $a_n > B$ for all $n \geq k$. All terms of the sequence will be larger than that bound B, once we look at the kth term and beyond. What is k? That will depend on the sequence and the bound B. For example we might need a bigger k as B gets bigger. The key is that, for every given B, there exists such a k.

Adapting this formalism to limits of functions, we say f **goes (diverges) to infinity as** x **approaches to** a, and write

$$\lim_{x \to a} f(x) = \infty$$

if

$$\text{for every given } B, \text{ there exists } \delta > 0 \text{ such that } 0 < |x - a| < \delta \Longrightarrow f(x) > B.$$

In other words, if you look at points close enough to a, the values of the function at those points will be guaranteed to exceed M. How about the point a? Well, remember again that the limit of f as x goes to a is a concept unrelated to what happens at the point a. Note that the requirement $f(x) > M$ is for those points which satisfy $0 < |x - a| < \delta$.

While we are at it, let's also make clear what it means to say "as the variable (or the argument) of the function goes to infinity, such and such happens". We say $f(x)$ **approaches to** L **as** x **approaches to** ∞, and write

$$\lim_{x \to \infty} f(x) = L$$

if

$$\text{given any } \varepsilon > 0, \text{ there exists } B \text{ such that } x > B \Longrightarrow |f(x) - L| < \varepsilon.$$

Note. We never say $\frac{1}{0}$ is equal to ∞. Instead we would say "$\frac{1}{0}$ is not defined". Likewise, we would not (at least formally) write things like $\frac{1}{\infty} = 0$. We would say: as x goes to ∞, the expression $\frac{1}{x}$ goes to 0.

3.8 Limits of rational functions $P(x)/Q(x)$ as $x \to \infty$

Say $P(x)$ and $Q(x)$ are polynomial functions of degree m and n, respectively, so they have the form:

$$P(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 \quad \text{and} \quad Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

where a_m and b_n are non-zero.

The limit of $P(x)/Q(x)$ as $x \to \infty$ is

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)} = \begin{cases} a_m/b_n & \text{if } m = n \\ 0 & \text{if } m < n \\ \infty & \text{if } m > n \text{ and } a_m/b_n > 0 \\ -\infty & \text{if } m > n \text{ and } a_m/b_n < 0 \end{cases}$$

By the way, you may treat all of these results as exercises and try to verify them.

The algebra of continuity

Following the so-called algebra of limits, it is immediate to conclude that continuity is a property preserved by addition, multiplication, division, powers, inverses, and composition. To be more precise, if f and g are functions continuous at x, then so is $f \pm g$, fg and f^g. If, moreover, $g(x) \neq 0$, then f/g too is continuous at x. If f has an inverse over a neighbourhood of x, then the inverse of f is continuous at $f^{-1}(x)$. Finally, if f is continuous at x and g is continuous at $f(x)$, then $g \circ f$ is continuous at x.

4 Differentiation

An intuitive notion of differentiation (of $\mathbb{R} \to \mathbb{R}$ functions) as a measure of *rate of change*, its connection with the graph of the function.

4.1 The derivative of an $\mathbb{R} \to \mathbb{R}$ function

The derivative of a function f with respect to its argument (variable) is a measure of the **rate of change** of $f(x)$ in relation to changes in x. For example, if the argument of the function increases from a to $a+h$, then the change in the value of the function is $f(a+h)-f(a)$. Looking at the rate of change is to have a measure of this change $f(a+h)-f(a)$ with respect (in proportion) to the change h in the argument of the function. That is, the derivative of the function aims to formalise the following ratio

$$\frac{f(a+h)-f(a)}{h}$$

In order to capture what the rate of change is exactly at the point a, we look at the above ratio for smaller and smaller changes denoted by h. And if by looking at smaller and smaller h, a unique number emerges, that is, if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{exists}$$

then we say this limit is the **derivative of f at** a and denote it with $f'(a)$.

Note that we can also write this limit as

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Differentiability implies continuity. That is, if f is differentiable at a, it must be continuous at a. Why is that? Well just have look at the definition. The denominator of the fraction goes to 0 as $x \to a$. If the fraction has any chance of having a limit, the numerator must also be going to 0. That is we must have $f(x) \to f(a)$ as $x \to a$, which is nothing but the definition of f being continuous at a.

Linear approximations. Here's another definition (or important interpretation) of derivatives. The function f having a derivative at the point a can also be interpreted as f having a "reasonable linear approximation" around the point a. What does a linear approximation mean, and what does it mean for it to be reasonable? We would like a linear function, that is, something of the form $Mx+N$ which is

"sufficiently close" to $f(x)$ when x is "close" to a

But what does that really mean? Surely, we should demand something better than "$f(x)-(Mx+N)$ goes to 0 as $x-a$ goes to 0", because even the constant function with $M=0$ and $N=f(a)$ would satisfy that since f is continuous.

So we need the **error term** of our approximation

$$f(x)-(Mx+N)$$

not only to go to zero, but also to go to zero at a speed an order of magnitude faster than $h = f(x)-f(a)$ does.

More specifically, we want

$$\frac{\text{the error term}}{h} \to 0 \quad \text{as } h \to 0$$

And this quality of approximation is possible if f is differentiable at a. The function f has a linear approximation around a, that is

$$f(a+h) \approx f(a) + f'(a)h$$

in the sense that

$$f(a+h) = f(a) + f'(a)h + \text{Error}(h) \quad \text{where} \quad \lim_{h\to 0} \frac{\text{Error}(h)}{h} = 0$$

The derivative of f: If we have the above limit for every x in a domain D of the function f, we say f is differentiable over the domain D. We write f' for the function which associates $f'(x)$ to x, and call this function f' the derivative of f.

Leibniz notation for derivatives: Derivative of f with respect to x can also be denoted as

$$\frac{df}{dx}$$

Sometimes, we'd like to keep track of the name of the variable x separately from the particular points at which we are analysing the function. This notation is useful:

$$\left.\frac{df}{dx}\right|_{x=a}$$

to stand for the derivative of f with respect to x evaluated at the point a. In other words: $f'(a)$. The advantage of the Leibniz notation is in reminding us what the differentiation variable is when lots of symbols are floating around; a reminder especially handy when we are dealing with functions of multiple variables.

4.2 The derivatives of two simple functions

The simplest of all $\mathbb{R} \to \mathbb{R}$ functions is a constant function. Is this function differentiable anywhere? Well, let's remember the definition of what it means to be differentiable at a point a, and rephrase that last question. Does the following limit exist:

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

The fact that f is a constant function implies, in particular, that $f(a+h) = f(a)$ whatever h is. Thus the above expression is nothing but

$$\lim_{h\to 0} \frac{0}{h} \quad = \quad \lim_{h\to 0} 0 \quad = \quad 0$$

So, yes, constant functions are differentiable everywhere, and their derivative is 0 everywhere. Makes sense: constant means to change. Hence zero rate of change!

Our next function of interest is the identity function on $\mathbb{R}$, that is $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$. Is it differentiable anywhere? Just apply the definition, and ask if the following exists:

$$\lim_{h\to 0} \frac{a+h-a}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1$$

which obviously exists and is equal to 1 at every a. Thus, $f(x) = x$ is a differentiable function with the derivative being 1 everywhere.

4.3 Basic algebra of derivatives

We have already established that the derivative of a constant function is 0, and the derivative of x is 1.

Given f and g differentiable at a, we have the following convenient results:

- $(f+g)'(a) = f'(a) + g'(a)$

- if c is a constant, then $(cf)'(a) = cf'(a)$
- $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
- $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$ as long as $g(a) \neq 0$.

Now, use these rules with the knowledge of the derivatives of constants and the identity function, and you can calculate the derivative of all polynomials (and rational functions)!

In particular

- $(x^n)' = nx^{n-1}$ for integers $n \geq 0$
- $\left(\frac{1}{x}\right)' = (x^{-1})' = -\frac{1}{x^2}$
- $\left(\frac{1}{x^n}\right)' = (x^{-n})' = -n\frac{1}{x^{n+1}}$ for integer $n < 0$

Actually, it looks like we can summarise all of the above in one expression

$$(x^n)' = nx^{n-1} \qquad \text{for all integers } n$$

Note: We must acknowledge that we have been a bit lazy in our notation above. In particular, when we were talking about the function $f(x) = x^n$, we just wrote x^n. In doing so, we are keeping in mind that x stands for a variable. These kinds of shortcuts in notation are convenient, but sometimes potentially confusing.

4.4 Higher order derivatives

If f is differentiable everywhere in a domain, then we can talk about its derivative f' as another function on this domain. Perhaps f' is also differentiable. If so, the derivative of f' is called the second derivative of f. In general, if exists, we can talk about the nth derivative of f and denote it by $f^{(n)}$, so

$$f^{(n)}(x) = (f^{(n-1)}(x))'$$

i.e., the nth derivative of f is the derivative of the $(n-1)$st derivative of f.

Note, for example, that the kth derivative of the function $f(x) = x^n$ with $n > 0$ exists:

$$\text{If} \quad f(x) = x^n, \quad \text{then} \quad f^{(k)}(x) = \begin{cases} n(n-1)\cdots(n-k+1)x^{n-k} & \text{if } 0 < k \leq n \\ 0 & \text{if } 0 < n < k \end{cases}$$

4.5 Composing functions, and the chain rule

For functions $f : A \to B$ and $g : B \to C$, we can talk about the composed function $g \circ f : A \to C$ defined as

$$(g \circ f)(x) = g(f(x))$$

The Chain Rule. If f is differentiable at x, and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

The idea behind the chain rule. How does this come about? Since this is a rule we will be using over and over and over, it is worth developing a feel for why it indeed holds. It is actually fairly intuitive.

For small δ, we have

$$f(x+\delta) - f(x) \approx f'(x)\delta \qquad \text{and therefore} \qquad f(x+\delta) \approx f(x) + f'(x)\delta \qquad (\star)$$

Likewise, for small Δ we have

$$g(y+\Delta) - g(y) \approx g'(y)\Delta \qquad \text{and therefore} \qquad g(y+\Delta) \approx g(y) + g'(y)\Delta \qquad (\star\star)$$

First, g is continuous, so when δ is small, we can appeal to $(\star)$ to infer

$$g(f(x+\delta)) \approx g(f(x) + f'(x)\delta)$$

If δ is small, we can treat $f'(x)\delta$ as the small Δ in $(\star\star)$, and set $y = f(x)$, so $(\star\star)$ becomes

$$g(f(x) + f'(x)\delta) \approx g(f(x)) + g'(f(x))f'(x)\delta$$

Combining the last two approximate equations:

$$g(f(x+\delta)) \approx g(f(x)) + g'(f(x))f'(x)\delta$$

Rearranging this yields

$$g(f(x+\delta)) - g(f(x)) \approx g'(f(x))f'(x)\delta$$

and therefore we must have $(g \circ f)'(x) = g'(f(x))f'(x)$. $\diamond$

4.6 The first order condition for optimisation

Recall the extreme value theorem which states that if f is a continuous function on $[a, b]$, then f attains its maximum and minimum values over this interval. That is, there exist points c and d in $[a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

Now suppose f is also differentiable on (a, b). If those points c and d are not the end points, that is, if they are in (a, b), let's look at the rate of change of f around c and d.

To begin with, look at c first. The fact that f is minimised at c when x varies over $[a, b]$ implies that

$$f(c+h) - f(c) \geq 0 \qquad \text{for all } h \neq 0$$

Dividing this expression by $h \neq 0$:

$$(\star) \qquad \frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{for } h > 0$$

$$(\star\star) \qquad \frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{for } h < 0$$

The fact that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists means both $(\star)$ and $(\star\star)$ approach to that same limit as h approaches to 0. (In the case of $(\star)$, we are talking about positive values of h approaching to 0. In the case of $(\star\star)$, we are talking about negative values of h approaching to 0.)

But $(\star)$ always stays ≥ 0 because the numerator is always non-negative, and the denominator is always positive. Therefore it cannot approach to something negative. Likewise, $(\star\star)$ always stays ≤ 0 because the numerator is always non-positive, and the denominator is always positive. Thus it cannot approach to something positive. So, given that both approach to the same limit (which is $f'(c)$), that limit has to be 0.

We can tell a similar story for the point $d \in (a, b)$ where the function is maximised as x varies over $[a, b]$. The only difference with the above analysis would be that $(\star)$ would now be ≤ 0, whereas $(\star\star)$ would be ≥ 0. But the same logic will apply: if these expressions converge to the same number, i.e., $f'(d)$, that number has to be 0.

In fact we can reach this conclusion at any point $e \in (a, b)$ as long as e minimises or maximises a differentiable function in a neighbourhood around e. In other words, e does not have to be the point at which f is minimised or maximised for the above analysis to hold. It is sufficient for e to be

an **interior local minimum** or an **interior local maximum**. How so? Well, suppose f takes its minimum value over $(e-\delta, e+\delta)$ at the point e. Then we can carry out all of the above reasoning, because as we look at smaller and smaller h, the points $e-h$ and $e+h$ are eventually in the δ-neighbourhood of e. Thus we have the conclusion which is known as the **first order condition** for a function's minima and maxima.[3]

Theorem. Suppose a function is differentiable on some open interval containing the point d. If d is a local minimum or a local maximum, then $f'(d) = 0$

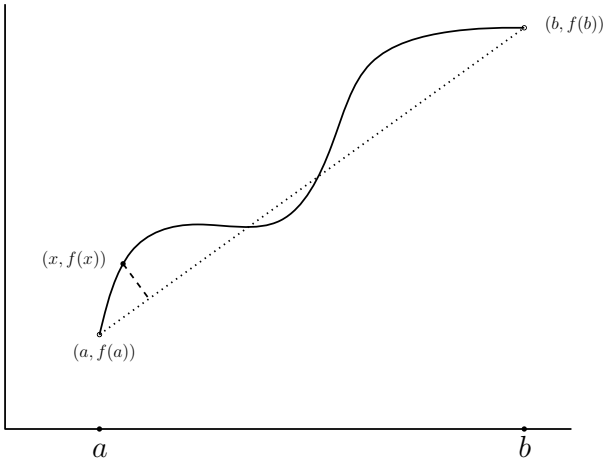
Rolle's Theorem. If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

This can be seen as an application of the EVT and the FOC. If f is constant over $[a, b]$, then its derivative is zero for all $x \in (a, b)$. If it is not constant, then there exist points in (a, b), where f takes greater or smaller values than $f(a) = f(b)$. Say it takes higher values. Since f is continuous, it attains its maximum value over this domain at some point, and that point cannot be a or b since f takes higher values somewhere between a and b. Say $c \neq a, b$ is where f takes its maximum value. The FOC implies that $f'(c) = 0$ □

The mean value theorem

Suppose we drove from Cambridge to London. The journey was exactly 60 miles long and it took us exactly 1 hour to complete this journey. The speedometer must have shown 60 mph at some point of the journey. That's what the mean value theorem (MVT) says. Denoting an hour with the unit interval $[0, 1]$, let $x \in [0, 1]$ stand for time since departure, and let $f(x)$ stand for the distance travelled in the first x hours of the journey. So we have $f(0) = 0$ and $f(1) = 60$. The speed of the car at time x is the rate of change of the distance travelled with respect to time. That is $f'(x)$. The average speed over the whole journey is $(f(1) - f(0))/1 = 60$. To paraphrase the conclusion of the MVT, the average (i.e., mean) speed of the car is attained at some point during the journey: there exists c such that $f'(c) = 60$.

Mean Value Theorem. Let f be continuous over $[a, b]$, and differentiable over (a, b). There exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



The theorem suggests a point where the slope of the tangent to the graph is the same as the slope of the dotted line that connects $(a, f(a))$ and $(b, f(b))$. It's not hard to convince yourself that the points on the graph whose distance from that dotted line is maximised (or even locally

[3] A function f is said to have a **local maximum** at e if there exists a non-empty neighbourhood of e in which f attains its maximum value at e. That is, if there exists $\delta > 0$ such that $f(e) \geq f(x)$ for all $x \in (e-\delta, e+\delta)$.

maximised) will be the points where the tangent will be parallel to the dotted line. (The dashed line segment marked on the graph represents the distance from the point $(x, f(x))$ on the graph to the dotted line. This is clearly a point which does not maximise the distance to the dotted line. The tangent at this point being steeper than the dotted line suggests that moving northeast further along the curve will increase the distance to the dotted line.)

Exercise: If the derivative of a function f is 0 at every point of an interval $[a, b]$, then f must be constant over that interval.

4.7 The monotonicity of a function and the sign of its derivative

The derivative, remember, captures the rate of change of f with respect to its argument. So, it should not be surprising that it carries some information regarding whether f is increasing or decreasing. The following conclusions are fairly intuitive (can be easily verified using the mean value theorem):

- If $f' > 0$ on an interval (a, b), then f is increasing on (a, b).
- If $f' < 0$ on an interval (a, b), then f is decreasing on (a, b).
- If f is non-decreasing on an interval, then $f' \geq 0$ on that interval.
- If f is non-increasing on an interval, then $f' \leq 0$ on that interval.
- f is constant on (a, b) if and only if $f' = 0$ on (a, b).

4.8 L'hôpital's rule and variations to compute limits

Given two functions f and g such that

- $\lim_{x\to a} f(x) = L$, where $L \in \mathbb{R}$ or $L = \pm\infty$
- $\lim_{x\to a} g(x) = M$, where $M \in \mathbb{R}$ or $M = \pm\infty$

what can we say about $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ if $L = M = 0$? Or if $L = \pm\infty$ and $M = \pm\infty$.

L'Hôpital's rule. Suppose two functions f and g are differentiable on an open interval I except possibly at a contained in I. Suppose also that $g'(x) \neq 0$ for all $x \in I$ with $x \neq a$. If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ or $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$, and if $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

In the same fashion, here's a list of strategies to deal with limits which resemble one of

$$\infty^0 \quad , \quad \infty - \infty \quad , \quad 0^0 \quad , \quad 1^\infty$$

- For fg where $f \to \infty$ and $g \to 0$, look at $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$ instead.
- For f^g where $f \to \infty$ and $g \to 0$, look at $\ln(f^g) = g \ln f$ (can be viewed as $\frac{g}{1/\ln f}$ or $\frac{\ln f}{1/g}$).
- For $f - g$ where $f \to \infty$ and $g \to \infty$, first try to simplify and see if you can arrive at an expression of the form $u(x)/v(x)$. Having exhausted options, have a look at $\exp(f - g) = \exp f / \exp g$.
- For f^g where $f \to 0$ and $g \to 0$, look at $\ln(f^g) = g \ln f$ (can be viewed as $\frac{g}{1/\ln f}$ or $\frac{\ln f}{1/g}$).

- For f^g where $f \to 1$ and $g \to \infty$, look at $\ln(f^g) = g \ln f$ (can be viewed as $\frac{g}{1/\ln f}$ or $\frac{\ln f}{1/g}$).

In other words, try to convert the limit question to one that looks like $0/0$ or ∞/∞ to be able to apply L'hôpital's rule.

4.9 Concavity and convexity of real valued functions

An $\mathbb{R} \to \mathbb{R}$ function f is said to be **concave** over an interval I if for every $a, b \in I$ and any $\lambda \in (0, 1)$:

$$f(\lambda a + (1 - \lambda)b) \quad \geq \quad \lambda f(a) + (1 - \lambda) f(b).$$

Intuitively, concavity of f on interval I is equivalent to the following geometric property of its graph: whichever two points you pick on the graph of $f : I \to \mathbb{R}$, and connect them with a line segment, the graph of the function will lie *above* that line segment.

A function $f(x)$ is said to be **convex** over an interval I if for every $a, b \in I$ and any $\lambda \in (0, 1)$:

$$f(\lambda a + (1 - \lambda)b) \quad \leq \quad \lambda f(a) + (1 - \lambda) f(b).$$

Intuitively, the graph of the function between $x = a$ and $x = b$ lies *below* the line segment connecting $(a, f(a))$ and $(b, f(b))$.

It is not hard to verify that f is concave if and only if $-f$ is convex. (Treat this as an exercise in algebra to practice your skills in writing the conditions for concavity and convexity.) Hence we can formulate our statements for concave functions, and appropriate versions for convex functions should follow by changing the signs in the right places, replacing maximums with minimums, and so on.

Theorem. Let I be an open interval and $f : I \to \mathbb{R}$ be twice differentiable such that f'' is continuous. f is concave on I if and only if $f''(x) \leq 0$ for all $x \in I$.

Moreover, if $f'' < 0$ on an interval, then f is strictly concave over that interval. If $f'' > 0$ on an interval, then f is strictly convex over that interval.

4.10 Classifying stationary points of $\mathbb{R} \to \mathbb{R}$ functions

Stationary points of a single-variable real-valued differentiable function are those points at which the derivative is equal to zero. Each such point corresponds to one of the following:

- A local minimum (At this point $f'' \geq 0$.)
- A local maximum (At this point $f'' \leq 0$.)
- An inflection point, if it is neither a local min or a local max.

Knowing the second derivative at a stationary point can help identify the nature of that stationary point. Suppose $f'(a) = 0$.

- If $f''(a) > 0$, then a is a local minimum.
- If $f''(a) < 0$, then a is a local maximum.
- If $f''(a) = 0$, then no conclusion can be drawn on the basis of f'' alone.

These conditions are also referred to as the second-order conditions (SOC) for a single-variable function.

Summary: studying an $\mathbb{R} \to \mathbb{R}$ function

If we are given a functional expression which describes $f(x)$ in terms x, but if we are not told what the domain of the function is, then we can say the function is not properly described. However, you may frequently see such incomplete descriptions, and in those circumstances, it makes sense that the function f we are talked about has the largest possible domain. For example, if we are told to think about the function $f(x) = 1/(x^2 - 4)$, we will usually assume that the domain is $\mathbb{R}\backslash\{-2, 2\}$, that is, the set of all real numbers except -2 and 2. (Because the expression $1/(x^2 - 4)$ doesn't make sense for $x = \pm 2$.)

The graph of a function is often useful to get a sense for the behaviour of the function. But how do we know where to start if we want to draw its graph? We don't know, because there isn't really a correct answer to the question of where we start. However, we have a few ideas which might be good starting points. Let's try to answer these specific questions:

1. What is the domain of the function? (If we are not told what exactly the domain is, then we wonder what the largest domain would be.)

2. Can we figure out if the graph intersects the vertical axis? Note that this would be the point $(0, f(0))$. So, if f is not defined at 0, then no such point will exist.

3. How about intersections with the horizontal axis? The points of the graph which happen to be on the horizontal axis are, by definition, those points x for which $f(x) = 0$. Such points do not have to exist. (Take for example the constant function which takes the value 7 everywhere.) There might be finitely many such points (e.g., $f(x) = x^2 - 4$) or infinitely many (e.g., the constant function 0).

   Sometimes, it is not obvious where we have $f(x) = 0$, or if f ever takes the value 0. The intermediate value theorem might help. It basically says if f is positive at a, and negative at b, and if f is continuous on $[a, b]$, then it takes the value 0 somewhere in (a, b). So the graph must hit the horizontal axis somewhere between a and b.

4. Are there points a in $\mathbb{R}$ such that as x approaches a, the value $f(x)$ approaches to ∞ or to $-\infty$? If so, we say the graph has a **vertical asymptote** at a. Put differently, the vertical line described by the equation $x = a$ is called a vertical asymptote of (the graph of) the function.

5. If the domain of the function is not bounded above then we can ask whether the function approaches to a limit as $x \to \infty$. If for example $f(x) \to L$ as $x \to \infty$, then we refer to the line described by the equation $y = L$ as a **horizontal asymptote** (on the positive side). Likewise, if the domain of the function is not bounded below, and if $f(x) \to M$ as $x \to -\infty$, then we call the line described by the equation $y = M$ as its horizontal asymptote on the negative side. The asymptotic behaviour of the functions (i.e., what happens to $f(x)$ as $x \to \pm\infty$) could be different from a horizontal line, but might still be easy to picture. For example, it could be that as x grows larger and larger, $f(x)$ gets closer and closer to the graph of another function g we are familiar with. In other words, we might have $f(x) - g(x) \to 0$ as $x \to \infty$, where we know the graph/behaviour of g well. That helps us get a sense of f as $x \to \infty$.

6. Is the function increasing or decreasing anywhere? Simply knowing the sign of f' can say loads. In particular, identify over which intervals the derivative is negative, and over which intervals the derivative is positive. Remember and use the fact that f is strictly increasing on an interval if $f' > 0$ on that interval. Likewise, we know that f is strictly decreasing on an interval if $f' < 0$ on that interval.

7. Can we say something regarding the curvature of the graph? The second derivative is useful in answering this question, because we also know that f is strictly convex on an interval if $f'' > 0$ on that interval. Likewise, we know that f is strictly concave on an interval if $f'' < 0$ on that interval.

5 Integration

We first provide a quick review of integration which involves a mechanical description of integration (of $\mathbb{R} \to \mathbb{R}$ functions) as the "inverse operator" of differentiation, and the geometric interpretation of definite integrals as areas associated with the graphs of $\mathbb{R} \to \mathbb{R}$ functions. Then we will discuss the foundations of integration to develop a more coherent view.

5.1 A quick review

If the derivative of F is f, then we say an **indefinite integral** (also called an **antiderivative**) of f is F. Note that if F is an indefinite integral of f, so is $F + c$ for every constant c, because $F' = (F + c)'$ (because the derivative of any constant function is 0).

If F is an indefinite integral of f, then the **definite integral of f from a to b** is denoted

$$\int_a^b f(x)dx \quad \text{and is equal to} \quad F(b) - F(a).$$

Note that in the last expression it does not matter if we replace F with G where $G = F + c$ and c is a constant. Thus, "the definite integral" is a uniquely determined number, as opposed to indefinite integrals which are functions determined up to a constant.

We can think of *indefinite integration* as an operator which takes as its input an integrable function f, and returns as its output a class of differentiable functions whose derivative is f

$$f \xmapsto{\text{integrated}} \{F \quad \text{such that } F'(x) = f(x)\}.$$

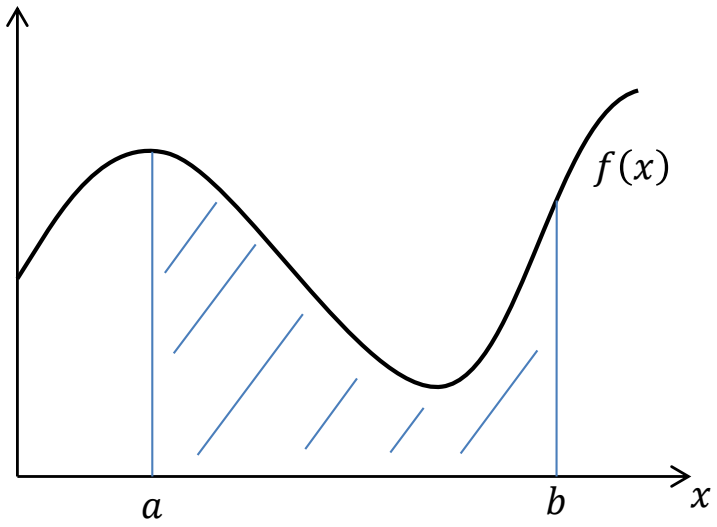


Figure 1: The area of the shaded region is $\int_a^b f(x)\,dx$.

If f is non-negative on $[a, b]$, then $\int_a^b f(x)\,dx$ will capture the area between the graph of f, the horizontal axis, and the vertical lines at $x = a$ and $x = b$. More generally, if f is at times positive, and at times negative over the interval $[a, b]$, then $\int_a^b f(x)\,dx$ will capture the sum of areas between the graph of f and the horizontal axis, with a minus sign in front of the areas that lie below the horizontal axis.

Basic rules

Being the inverse operation of differentiation, it is not surprising that integration will readily inherit the basic algebraic properties of differentiation: namely, addition, subtraction and scalar multiplication: given integrable functions f and g,

$$\int \alpha f + \beta g = \alpha \int f + \beta \int g$$

where α and β are constants (i.e., scalars).

Again, the anti-derivative nature of integration implies that

- $\int x^n dx = \frac{1}{n+1}x^{n+1}$ for $n \neq -1$
- $\int x^{-1} dx = \ln x$
- $\int \exp x dx = \exp x$

Remember that the above expressions are about indefinite integrals, and therefore both sides of the equalities are functions, and the equalities are "correct up to a constant".

5.2 Foundations of Riemann Integration

Let f be a continuous function on interval $[a, b]$. We'd like to partition this interval, beginning with halving it, then halving both half intervals, and then halving all four quarter intervals, and so on. So, if we denote by ℓ the length of the original interval, i.e., setting $\ell = b - a$, first we create two intervals of length $\ell/2$ each. Then four intervals of length $\ell/4$ each, and then eight intervals of length $\ell/8$ each, etc. After n iterations, we have divided the interval into 2^n equal sub-intervals of length $\ell/2^n$ each.

Now, for this partition, let m_i be the minimum value of f over ith subinterval, and M_i be the maximum value of f over that same subinterval. So, we have $m_i \leq M_i$ for every subinterval $i = 1, \ldots, 2^n$.

Next, define the nth lower sum s_n and nth higher sum S_n as follows

$$s_n = \frac{1}{2^n}\sum_{i=1}^{2^n} m_i \qquad S_n = \frac{1}{2^n}\sum_{i=1}^{2^n} M_i$$

Clearly $s_n \leq S_n$. (Geometrically speaking, if f is nonnegative over $[a, b]$, then s_n is meant to approximate, from the inside, the area between the graph of f and the x-axis, whereas S_n is meant to approximate, from the outside.)

One more critical observation.[4] If we make the partition finer, that is, if we divide the subintervals into half one more time, the lower sum gets higher, and the upper sum gets lower. That is

$$s_n \leq s_{n+1} \leq S_{n+1} \leq S_n.$$

Note that s_n is a non-decreasing sequence which is bounded above. Hence it is convergent. Say it converges to s. Likewise, S_n is a non-increasing sequence bounded below, and hence is convergent as well, say, with limit S. The above observations tell us that $s \leq S$.

If $s = S$, then we say that the **integral of f from a to b** is this number s, and we write

$$\int_a^b f(t)dt = s$$

Remark. These numbers s and S are not equal for all functions. Thus, there are non-Riemann-integrable functions. For example think about the limits of lower and higher sums for the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

over the interval $[0, 1]$. What do you get for s? What do you get for S?

Remark. We have discussed one particular way subdividing the interval $[a, b]$, and then computing the associated lower and upper sums. As the subintervals get smaller and smaller, if the lower sums

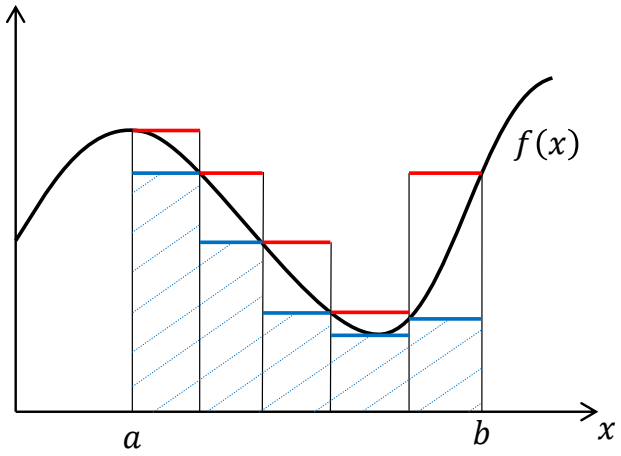
[4] This is something you should verify. To get an idea, simply compare, for an arbitrary function g, the following two quantities: $\min_{x\in[0,2]} g(x)$ versus $\frac{1}{2}\min_{x\in[0,1]} g(x) + \frac{1}{2}\min_{x\in[1,2]} g(x)$.

and upper sums converge to the same limit, we defined that limit as the integral. But in fact, we could have carried out this exercise with partitions of arbitrary subdivisions as long as the length of the longest subinterval at each iteration goes to zero as we keep subdividing.

Notational convention. For practical purposes, it will be useful to define what we mean by an integral from b to a when f is a function on $[a, b]$. And we will define it as

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Remark. The interpretation of integral as the area between the graph of the function and the horizontal axis makes sense if the function is nonnegative as can be seen in the figure. The calculation of lower and upper sums involves multiplying values of the function with the lengths of subintervals. When those values are positive, the products can be seen as the areas if rectangles whose heights are given by the values of the function. If those values, however, are negative, then the products will be negative, and we can think of rectangles below the horizontal axis having negative areas.



5.3 A few basic properties of integrals

1. If $a < b < c$, and if f is continuous on $[a, c]$, then
$$\int_a^b f(x)dx + \int_b^c f(x)dx \quad = \quad \int_a^c f(x)dx$$

2. If f and g are continuous on $[a, b]$, and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f(x)dx \quad \leq \quad \int_a^b g(x)dx$$

3. If f and g are continuous on $[a, b]$, and if α, β are real numbers, then
$$\int_a^b (\alpha f(x) + \beta g(x))dx \quad = \quad \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

4. **[Integral Mean Value Theorem]** Let f be continuous over $[a, b]$. There exists $c \in [a, b]$ such that
$$(b - a)f(c) = \int_a^b f(x)dx.$$

   Proof. Since f is continuous over $[a, b]$, it takes its minimum and maximum values over this interval. Call these values m and M, respectively so $m \leq f(x) \leq M$, for every $x \in [a, b]$. Therefore,
$$m(b - a) \quad \leq \quad \int_a^b f(x)dx \quad \leq \quad M(b - a).$$

Hence

$$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$$

On the other hand, by the Intermediate Value Theorem, any value between m and M is attained by f on $[a,b]$. Hence there exists $c \in [a,b]$, such that $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$. □

5.4 Integral as antiderivative

Now that we have defined definite integrals of a function f, we can look at an associated function F defined as

$$F(x) = \int_a^x f(t)dt$$

(provided that f is integrable of course).

Fundamental Theorem of Calculus. Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and $F : [a,b] \to \mathbb{R}$ be defined by $F(x) = \int_a^x f(t)dt$. Then

(i) $F'(x) = f(x)$ for all $x \in (a,b)$.

(ii) $\int_c^d f(t)dt = F(d) - F(c)$ for every c and d in $[a,b]$.

Proof. (Part (i)): By definition, $F'(x)$ is:

$$\begin{aligned}\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} &= \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}\\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}\\ &= \lim_{h\to 0} f(c_h) \qquad \text{for some } c_h \in [x, x+h] \text{ by IMVT.}\end{aligned}$$

Since $c_h \in [x, x+h]$, and f is continuous, we know that $f(c_h) \to f(x)$ as $h \to 0$.

(Part (ii)): We have by definition $F(d) = \int_a^d f(t)dt$ and $F(c) = \int_a^c f(t)dt$. Writing

$$\int_a^d f(t)dt = \int_a^c f(t)dt + \int_c^d f(t)dt$$

and rearranging, we get

$$\int_c^d f(t)dt = \int_a^d f(t)dt - \int_a^c f(t)dt = F(d) - F(c)$$

□

Thus, the Fundamental Theorem of Calculus (FTC) lets us think of integrals as sort of "antiderivatives". This antiderivative of f is *unique up to a constant*. It is customary to call this **the indefinite integral** of f, though it is important to keep in mind that it is really *an* antiderivative, and not *the*. After all, we can define another function $G(x)$ as

$$G(x) = \int_b^x f(t)dt$$

where b is some constant. FTC applies to G as well. So $G' = f$ and $\int_c^d f(t)dt = G(d) - G(c)$. Note that G differs from F. More specifically, at every point x, the difference between these two functions is

$$F(x) - G(x) = \int_a^x f(t)dt - \int_b^x f(t)dt = \int_a^b f(t)dt$$

Note that the last expression is independent of x. I.e., it is a constant, and this is the constant by which F and G differ.

5.5 Integration by parts

The so-called "integration by parts" is often expressed concisely as

$$\int udv = uv - \int vdu$$

In this notation, both u and v stand for functions of a variable, say x. Moreover

$$dv \quad \text{stands for} \quad v'(x)dx$$

and

$$du \quad \text{stands for} \quad u'(x)dx$$

So, what's really meant by the above formula is

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

In order to see where this comes from, simply rearrange it to obtain

$$\begin{aligned} \int u(x)v'(x)dx + \int v(x)u'(x)dx &= u(x)v(x) \\ \int (u(x)v'(x) + v(x)u'(x))dx &= u(x)v(x) \\ \int (u(x)v(x))'dx &= u(x)v(x) \end{aligned}$$

which is nothing but the fundamental theorem of calculus applied to the function uv.

5.6 Change of variables

Recall the chain rule: if u is defined around t and differentiable at t, and if f is defined around $u(t)$, and differentiable at $u(t)$, then $f \circ u$ is differentiable at t with the derivative

$$(f \circ u)'(t) = f'(u(t))u'(t)$$

By the fundamental theorem of calculus, the integral of the left hand side is $f(u(t)) + C$, so we can write

$$f(u(t)) + C = \int f'(u(t))u'(t)dt$$

But, let us express the right hand side a bit differently. In particular, let's "forget about t" for a moment, and adopt the notation

$$du = u'(t)dt$$

so that the integral can be rewritten as

$$\int f'(u)du.$$

This last expression is integration with respect to u. By the fundamental theorem of calculus, it is equal to

$$f(u).$$

But recall that $u = u(t)$, so indeed the answer is $f(u(t))$.

A long exercise on differentiation and integration

In our lectures, we have already introduced the natural logarithm function as a differentiable bijection from $\mathbb{R}_{>0}$ to $\mathbb{R}$ which takes the value 0 at 1, and has the derivative $1/x$ for every $x > 0$. For completeness, we show in this section that such a function indeed exists.

The natural logarithm function

The **natural logarithm** is a function $\ln x : \mathbb{R}_{>0} \to \mathbb{R}$ specified as

$$\ln x = \int_1^x \frac{1}{t} dt$$

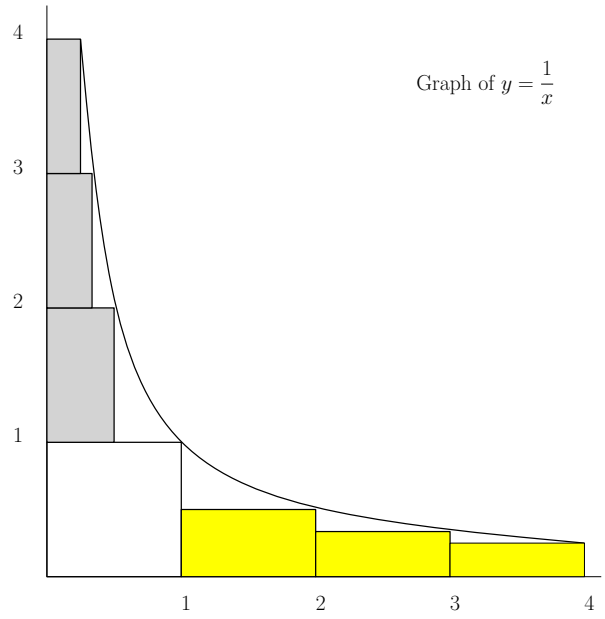
Note that

- $\ln 1 = 0$
- $\ln x$ is strictly increasing (and therefore one-to-one)
- If $x \in (0, 1)$, then $\ln x < 0$
- By the fundamental theorem of calculus:

$$(\ln x)' = \frac{1}{x}$$

Question. Is $\ln : \mathbb{R}_{>0} \to \mathbb{R}$ bounded above or below? Since it is an increasing function, if it is bounded above then its least upper bound would be $\lim_{x\to\infty} \ln(x)$. Likewise, if it is bounded below, its greatest lower bound would be $\lim_{x\to 0} \ln(x)$.

We will use the following figure to adress the above question:



First let's observe that the boxes that lie on the x-axis and below the curve $y = 1/x$ in the following figure all have the same width 1, and decreasing heights given by

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \text{and so on}$$

Our next observation is that adding up the areas of n boxes from left to right would amount to

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

and this area must be smaller than the area underneath the curve

$$\int_1^n \frac{1}{t} dt$$

Thus we can conclude that

$$\ln n \quad > \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

Now, if we can show that the limit

$$\lim_{N\to\infty} \sum_{i=2}^{N} \frac{1}{i} \quad = \quad \lim_{N\to\infty} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right)$$

diverges, then we can conclude that $\ln n \to \infty$ as $n \to \infty$.

Now observe that

$$\begin{aligned} & \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} + \cdots + \frac{1}{N} \\ > \quad & \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16} + \cdots \end{aligned}$$

because comparing the two sums term-by-term, the bottom sum has equal or smaller terms at every term.

And finally, observe that as N grows, the bottom sum

$$\frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}} + \underbrace{\frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16}}_{\frac{1}{2}} + \cdots$$

will involve infinitely many groups each of which adds up to $1/2$. Thus, as N grows, this bottom sum diverges to ∞, therefore the top sum diverges, and therefore $\ln n$ diverges to ∞.

How about $\lim_{x\to 0^+} \ln x$? For this one, note that

$$\lim_{x\to 0^+} \ln x = -(\text{Area underneath the curve from } x = 0 \text{ to } x = 1)$$

But that area clearly is bigger than the sum of the boxes stacked along the vertical axis. Moreover, those boxes have the same area as the boxes lying along the horizontal boxes, and therefore their total area, too, diverges to infinity. That means $\lim_{x\to 0^+} \ln x$ diverges to $-\infty$.

Thus can conclude that the range of the natural logarithm functions is the whole real line.

Other logarithm functions

Given $a > 0$, we will define **logarithm with base** a as

$$\log_a x = \frac{\ln x}{\ln a}$$

The exponential function

Since the natural logarithm

$$\ln : \mathbb{R}_{>0} \to \mathbb{R}$$

is one-to-one and onto, it admits an inverse function from $\mathbb{R}$ to $\mathbb{R}_{>0}$ which we will call the **exponential function**.

The fact that exp is the inverse of ln can be used in a fairly straightforward fashion to obtain the following properties:

- $\exp x > 0$ for all x
- $\lim_{x\to-\infty} \exp x = 0$ and $\lim_{x\to\infty} \exp x = \infty$
- $\exp 0 = 1$
- $\exp(x+y) = \exp x \exp y$
- $(\exp x)^y = \exp(xy)$

Note that $\exp x$ is the same as $(\exp 1)^x$. So, if we decide to denote $\exp 1$ with e, then the exponential function can also be denoted as e^x.

Raising a number to an arbitrary power x

Now, given a real number a, let us define the function a^x. We sometimes talk about raising a to a power, but if that power x is not a rational number, it is not as easy to make sense of this with the interpretation of multiplying a with itself x times. But the following definition does make sense

$$a^x = \exp(x \ln a) = e^{x \ln a}$$

since we know what $\ln a$ means (the integral of $1/x$ from 1 to a), and we know what the exponential of $x \ln a$ means (the inverse image of $x \ln a$ under the natural logarithm function).

Deriving the logarithm laws and the power laws

Suppose we define the **natural logarithm** function $\ln : \mathbb{R}_{>0} \to \mathbb{R}$ as

$$\ln x = \int_1^x \frac{1}{t} dt$$

and the **exponential function** $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ as the inverse of ln.

Exercises

1. Show that $e < 3$.
2. Show that $(\exp x)' = \exp x$

   Hint: Differentiate the function $\ln \circ \exp$ *via the chain rule.*
3. Show that $\exp(x+y) = (\exp x)(\exp y)$

   Hint: Differentiate $\exp(x+y)/\exp x$ *with respect to* x.
4. Show that $\ln(xy) = \ln x + \ln y$

   Hint: Differentiate $\ln(xy) - \ln x$ *with respect to* x.
5. Remember that a^x is defined as $\exp(x \ln a)$. Show that a^x satisfies the power rules:
   (a) $a^{x+y} = a^x a^y$
   (b) $(a^x)^y = a^{xy}$
   (c) $a^x b^x = (ab)^x$
6. Remember that $\log_a x$ is defined as $\dfrac{\ln x}{\ln a}$. Show that $\log_a x$ is the inverse of a^x, and that it satisfies the logarithm rules:
   (a) $\log_a(xy) = \log_a x + \log_a y$
   (b) $\log_a x^y = y \log_a x$
7. Evaluate the derivative and the integral of a^x.

6 Multi-variable functions

We have already introduced $\mathbb{R}^n$, a distance notion in this set, and various algebraic operations in Section 1.3. Before we take a closer look at functions from $\mathbb{R}^n$ to $\mathbb{R}$, we revise the key definitions.

6.1 n-dimensional Euclidean space: $\mathbb{R}^n$

In the context of consumer choice, we often talk about bundles of two goods which we represent with pairs of real numbers. In general, we can talk about bundles of n goods, or in other contexts we might be interested in n variables all of which have an effect on a quantity of our interest. Therefore, we need a domain consisting of ordered n-tuples denoted by $(x_1, x_2, \ldots, x_n)$. In this set, we also have the algebraic operations of **addition** and **scalar multiplication**.

Addition. Given $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$,

$$\boldsymbol{x} + \boldsymbol{y} \quad = \quad (x_1 + y_1, \ldots, x_n + y_n)$$

Scalar multiplication. Given $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $c \in \mathbb{R}$,

$$c\boldsymbol{x} \quad = \quad \boldsymbol{x}c \quad = \quad (cx_1, \ldots, cx_n)$$

In this last definition, the number c is referred to as a scalar, because unlike an n-dimensional point, c is a number in $\mathbb{R}$, and multiplying a point $\boldsymbol{x}$ with that number c is essentially scaling $\boldsymbol{x}$ by c.

With the above operations, elements in $\mathbb{R}^n$ are also called n-dimensional vectors. x_i is called the ith entry of $\boldsymbol{x}$.

The origin. The point $\mathbf{0} = (0, 0, \ldots, 0)$ in $\mathbb{R}^n$, also called the **zero vector**, is the standard reference point of the n-dimensional real space, and as such deserves the special name: **the origin**. For brevity, we will usually denote this point as 0 with the understanding that it really corresponds to $(0, \ldots, 0) \in \mathbb{R}^n$.

You might also hear people referring to $\mathbb{R}^n$ as the **n-dimensional real Euclidean space**, because $\mathbb{R}^n$ admits a notion of distance inspired by good old Euclidean geometry. Take, for instance, $\mathbb{R}^2$ which is easy to visualise as the Euclidean plane of classical geometry. If $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ are two points in $\mathbb{R}^2$, we have

$$\text{The distance between } (a_1, a_2) \text{ and } (b_1, b_2) \text{ as } \quad \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \quad .$$

Distance in $\mathbb{R}^n$. We simply adapt the above geometric distance notion to the world of n-dimensional points and define

$$\text{the distance between two points } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ in } \mathbb{R}^n \text{ as } \quad \sqrt{(a_1 - b_1)^2 + \cdots + (a_i - b_i)^2 + \cdots + (a_n - b_n)^2} \, .$$

Note that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are not the same points, the distance between them will be positive.

The norm of a point (vector) in $\mathbb{R}^n$. We refer to the distance between $\boldsymbol{x}$ and the origin as the **norm** of $\boldsymbol{x}$, and denote it by $\|\boldsymbol{x}\|$.

$$\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

The r-neighbourhood of a point in $\mathbb{R}^n$. Remember that the r-neighbourhood of a point a in $\mathbb{R}$ is the open interval $(a - r, a + r)$. Described in the terminology of distance, this is the set of all numbers whose distance from a is less than r. This latter distance formulation can be used in the

same way to define the notion of the r**-neighbourhood** of a point $\boldsymbol{a}$ in $\mathbb{R}^n$. That is, the set of all points in $\mathbb{R}^n$ whose distance from $\boldsymbol{a}$ is less than r. Denoting it by $N(\boldsymbol{a}, r)$, we can also write it as

$$\begin{aligned} N(\boldsymbol{a}, r) &= \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}\| < r\} \\ &= \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < r\right\} \\ &= \{\boldsymbol{x} \in \mathbb{R}^n \mid (x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 < r^2\} \end{aligned}$$

Some people call this set **the open r-ball centred at $\boldsymbol{a}$**. The qualifier *open* refers to the fact that the surface of the ball (i.e., the set of points which are exactly r away from $\boldsymbol{a}$) is not included in the set. So, the r-ball centred at $a \in \mathbb{R}$ is the open interval $(a - r, a + r)$. The r-ball centred at $\boldsymbol{a} \in \mathbb{R}^2$ is the open disk in the plane with centre $\boldsymbol{a}$ and radius r.

6.2 Limits and Continuity of $\mathbb{R}^n \to \mathbb{R}$ functions

(This subsection is a formal discussion, and is not examinable.)

We can simply take the definition of a limit of an $\mathbb{R} \to \mathbb{R}$ function and adapt it to multi-variable functions. Let's do that first in words, then in a slightly more formal notation.

We say f approaches to L as its argument approaches to $\boldsymbol{a}$ if for every given neighbourhood of L, there exists a neighbourhood of $\boldsymbol{a}$ such that all points in that neighbourhood $\boldsymbol{a}$ (possibly except $\boldsymbol{a}$ itself) have their images in the given neighbourhood of L.

Wait, is this a definition for single-variable or multi-variable functions?

It applies to both single-variable and multi-variable functions! It is exactly the same definition when expressed in the language of neighbourhoods.

Let's write it again, this time in more formal language. We write

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = L$$

if the following condition holds:

$$\text{given any } \varepsilon > 0, \text{ there exists a } \delta > 0 \text{ such that } \underbrace{0 < \|\boldsymbol{x} - \boldsymbol{a}\| < \delta}_{\substack{\text{for every } \boldsymbol{x} \neq \boldsymbol{a} \text{ in the} \\ \delta\text{-neighbourhood of } \boldsymbol{a} \text{ in } \mathbb{R}^n}} \implies \underbrace{|f(\boldsymbol{x}) - L| < \varepsilon}_{\substack{f(\boldsymbol{x}) \text{ must be in the} \\ \varepsilon\text{-neighbourhood of } L \text{ in } \mathbb{R}}}$$

The notion of continuity is the same as before. We say

$$f \text{ is } \textbf{continuous} \text{ at } \boldsymbol{a} \text{ if } \lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a}).$$

Alternatively, we can phrase the definition more directly, and say f is **continuous** at $\boldsymbol{a}$ if

$$\text{given any } \varepsilon > 0, \text{ there exists a } \delta > 0 \text{ such that } \underbrace{\|\boldsymbol{x} - \boldsymbol{a}\| < \delta}_{\substack{\text{for every } \boldsymbol{x} \text{ in the} \\ \delta\text{-neighbourhood of } \boldsymbol{a} \text{ in } \mathbb{R}^n}} \implies \underbrace{|f(\boldsymbol{x}) - L| < \varepsilon}_{\substack{f(\boldsymbol{x}) \text{ must be in the} \\ \varepsilon\text{-neighbourhood of } L \text{ in } \mathbb{R}}}$$

We might also be tempted to adapt the definition of differentiability to multi-variable functions since the derivative is defined via a limit. That is we can ask:

$$\text{Does } \lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - f(\boldsymbol{a})}{\boldsymbol{x} - \boldsymbol{a}} \text{ exist?}$$

One problem with the above question is that while $f(\boldsymbol{x}) - f(\boldsymbol{a})$ is a real number, $\boldsymbol{x} - \boldsymbol{a}$ is a point in $\mathbb{R}^n$, and it doesn't make sense to divide an element of $\mathbb{R}$ with an element of $\mathbb{R}^n$.

Instead of looking for a way of getting around this, we will now turn to a more manageable approach as far as differentiating multi-variable functions go.

6.3 Partial derivatives

Let us first begin with an example. Take a two-variable function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2y^3$.

A key observation is that we can view and treat this object much more narrowly, by fixing one of its variables, and letting only the other variable to actually vary. For example, suppose y is fixed at $y = 7$. Then we can define a new function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = f(x, 7)$$

Now that we have this $\mathbb{R} \to \mathbb{R}$ function, we can appeal to our earlier understanding of differentiating it. In particular we can talk about the derivative of g with respect to x

$$\frac{dg}{dx} = g'(x)$$

The above derivative will also be called the "partial derivative of f with respect to x, evaluated at $y = 7$". Now, of course, we can repeat the above exercise for every y. The key is to treat y as fixed, and study how $f(x, y)$ varies as x varies. As such we get the **partial derivative of f with respect to x** given by

$$\frac{\partial f}{\partial x} = h'(x) \qquad \text{where } h(x) = f(x, y)$$

We used the notation of $h : \mathbb{R} \to \mathbb{R}$ above to emphasise that when we are differentiating with respect to x, we treat y as a constant.

Going back to the example $f(x, y) = x^2y^3$, we must have

$$\frac{\partial f}{\partial x} = 2xy^3 \qquad \frac{\partial f}{\partial y} = 3x^2y^2$$

Defining partial derivatives in general

Given a multi-variable function $f(x_1, \ldots, x_n)$, viewing it as a function of x_1 alone means analysing the behaviour of the single-variable function given by

$$x_1 \quad \longmapsto \quad f(x_1, x_2, \ldots, x_n)$$

The derivative of this function at x_1 is called the **partial derivative of f with respect to its first variable at** $(x_1, x_2, \ldots, x_n)$.

Note that, for every specification of $x_2, \ldots, x_n$, we'd get another function. For example

$$x_1 \quad \longmapsto \quad f(x_1, a_2, \ldots, a_n)$$

is yet another single-variable function, and the derivative of this function would then be the **partial derivative of f with respect to its first variable at** $(x_1, a_2, \ldots, a_n)$.

In this fashion, we can define the **partial derivative of f with respect to its first variable** as the following function from $\mathbb{R}^n$ to $\mathbb{R}$:

$$\frac{\partial f}{\partial x_1} \;:\; (x_1, \ldots, x_n) \longmapsto \lim_{h \to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}$$

A careful notation for the evaluation of this function at a specific point $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is

$$\left.\frac{\partial f}{\partial x_1}\right|_{\boldsymbol{x}=\boldsymbol{a}}$$

Likewise we define the **partial derivative of f with respect to its kth variable x_k**:

$$\frac{\partial f}{\partial x_k} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_k + h, \ldots, x_n) - f(\boldsymbol{x})}{h}$$

The briefest and (usually) the preferred notation for partial derivatives.

When it is convenient to refer to the variables of a multi-variable function as first variable, second variable, third variable, and so on, we will denote

$$f_k \quad = \quad \text{the partial derivative of } f \text{ with respect to its } k\text{th variable}$$

Higher order partial derivatives

If $f : \mathbb{R}^n \to \mathbb{R}$ admits partial derivatives, then f_i is also an n-variable function with values in $\mathbb{R}$. If f_i admits partial derivatives, then we denote these partial derivatives as

$$f_{ij} \quad = \quad (f_i)_j \quad = \quad \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) \quad = \quad \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Pay attention to the order of writing for i and j !

In the same vein, we can define third and higher order partial derivatives if they exist. For example

$$f_{ijk} \quad = \quad \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right) \quad = \quad \frac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}$$

Young's theorem. If a function $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second partial derivatives at $\boldsymbol{a}$, then

$$f_{ij}(\boldsymbol{a}) = f_{ji}(\boldsymbol{a})$$

This property is sometimes quoted as *mixed partials commuting*. We will mostly deal with multi-variable functions whose second partial derivatives are continuous, and therefore it won't matter in which order we take partial derivatives.

6.4 The chain rule with multi-variable functions

Remember that the chain rule is about relating the derivative of composition of functions to the derivatives of the functions composed. For example, when we have two functions f and g from $\mathbb{R}$ to $\mathbb{R}$, say differentiable everywhere for ease of exposition, the composition $f \circ g$ is differentiable with the derivative

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Now, we'd like to generalise this result to be able to relate the derivatives of the three functions

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad x : \mathbb{R} \to \mathbb{R} \qquad y : \mathbb{R} \to \mathbb{R}$$

to the derivative of the function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$t \quad \longmapsto \quad f(x(t), y(t)) \tag{♡}$$

We were being very careful in not referring to $f(x(t), y(t))$ as a function because without any clarification as to what the variable is, it is not obvious how this expression describes a function. In contrast, expressing it as in (♡) above makes it clear that we are talking about a single-variable function.

The Chain Rule in two dimensions. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives with respect to both its variables. If x and y are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, then

$$\frac{d}{dt} f(x(t), y(t)) \quad = \quad \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t)$$

The notation on the right hand side is prone to confusion, because when we write $\frac{\partial f}{\partial x}$ we are not really referring to the function $x : \mathbb{R} \to \mathbb{R}$. Rather, we are talking about partially differentiating $f : \mathbb{R}^2 \to \mathbb{R}$ with respect to its first variable. The symbol x in ∂x and the symbol in $x(t)$ are not referring to the same thing. The one in ∂x is referring to the first variable of f, whereas $x(t)$ is a function from $\mathbb{R} \to \mathbb{R}$. It just happens that we are plugging this function into the first variable of f. The expression of the left hand side is really the derivative (with respect to t) of the function defined by (♡).

Sometimes, you see the chain rule expressed in the more concise Leibniz notation:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

When we read this expression, we implicitly know that we are dealing with the function $t \mapsto f(x(t), y(t))$, and that $\frac{\partial f}{dx}$ refers to the partial derivative of f with respect to its first variable, and $\frac{\partial f}{\partial y}$ refers to the partial derivative of f with respect to its second variable.

Yet another alternative notation for partial derivatives is to write f_i for the partial derivative of f with respect to its ith variable when we denote the ith variable by x_i. So, the chain rule can also be expressed as

$$\frac{d}{dt}f(x_1(t), x_2(t)) = f_1(x_1(t), x_2(t))x_1'(t) + f_2(x_1(t), x_2(t))x_2'(t)$$

Or we can stick to referring to the variables of f as x and y, and denote the corresponding partial derivatives by f_x and f_y. Then the chain rule will look like

$$\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$

Challenge. In order to give an idea as to how the chain rule comes about for single-variable functions, we gave an intuitive explanation based on the interpretation of the derivative as the slope of a linear approximation. Can you extend that idea to the more general form of the chain rule given above?

The Chain Rule. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives with respect to all its variables. Suppose also that we have n differentiable functions $x_i : \mathbb{R} \to \mathbb{R}$ where $i = 1, \ldots, n$. Then

$$\frac{d}{dt}(f(x_1(t), x_2(t), \ldots, x_n(t)) = f_1(\boldsymbol{x}(t))x_1'(t) + f_2(\boldsymbol{x}(t))x_2'(t) + \cdots + f_n(\boldsymbol{x}(t))x_n'(t)$$

where $\boldsymbol{x}(t)$ stands for $(x_1(t), \ldots, x_n(t))$.

6.5 Concavity and convexity of $\mathbb{R}^n \to \mathbb{R}$ functions

The notions of concavity and convexity can be extended to $\mathbb{R}^n \to \mathbb{R}$ functions in a fairly straightforward fashion, however f'' is a much more complicated concept when f is a multi-variable function. Without getting into what f'' might really mean for multi-variable functions, let's quickly define the concepts of concavity and convexity more generally.

Convex sets in $\mathbb{R}^m$

A set K in $\mathbb{R}^n$ is called **convex** if for every $\boldsymbol{x}, \boldsymbol{y} \in K$ and for every $\lambda \in (0,1)$, the point $\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}$ is also in K. Any such point is called a convex combination of $\boldsymbol{x}$ and $\boldsymbol{y}$. It is an algebraic exercise to verify that if we take any k points $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k$ in a convex set K, and arbitrary real numbers $0 < \lambda_1, \ldots, \lambda_k < 1$ such that $\lambda_1 + \cdots + \lambda_k = 1$, then the point $\lambda_1\boldsymbol{x}^1 + \cdots + \lambda_k\boldsymbol{x}^k$ is also in K. Any such point is called a **convex combination** of the k points $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k$.

Concavity of functions

Suppose f is a real-valued function defined on a convex subset K of $\mathbb{R}^n$. We say f is **concave** over K if for every $\boldsymbol{a}, \boldsymbol{b} \in K$ and every $\lambda \in (0,1)$:

$$f(\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}) \quad \geq \quad \lambda f(\boldsymbol{a}) + (1-\lambda) f(\boldsymbol{b}).$$

If the inequality is strict for all $\boldsymbol{a}, \boldsymbol{b} \in K$ and all $\lambda \in (0,1)$, then we say f is **strictly concave** over K.

Theorem. Suppose K is a convex set. If $f : K \to \mathbb{R}$ is strictly concave over K, then f takes its maximum value at most once in K.

Proof. Suppose, for a contradiction, that f attained its maximum value at two different points a and b in K. The fact that K is convex implies $a/2 + b/2$ is in K. The fact that f is strictly concave over K implies $f(a/2 + b/2) > f(a)/2 + f(b)/2 = f(a) = f(b)$. That is, f takes an even higher value at $a/2 + b/2 \in K$, a contradiction with its taking its maximum value at a and b. □

Convexity of functions

Suppose f is a real-valued function defined on a convex subset K of $\mathbb{R}^n$. We say f is **convex** over K if for every $\boldsymbol{a}, \boldsymbol{b} \in K$ and every $\lambda \in (0,1)$:

$$f(\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}) \quad \leq \quad \lambda f(\boldsymbol{a}) + (1-\lambda) f(\boldsymbol{b}).$$

If the inequality is strict for all $\boldsymbol{a}, \boldsymbol{b} \in K$ and all $\lambda \in (0,1)$, then we say f is **strictly convex** over K.

Theorem. Suppose K is a convex set. If $f : K \to \mathbb{R}$ is strictly convex over K, then f takes its minimum value at most once in K.

7 Optimisation of $\mathbb{R}^n \to \mathbb{R}$ functions

Consider the profit-maximising firm's problem of choosing K and L to maximise $\pi : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\pi(K, L) = pf(K, L) - rK - wL$$

Note that the firm's choices of K and L both have to be non-negative. Hence, we can write the problem as

$$\text{Choose } K \geq 0 \text{ and } L \geq 0 \text{ to maximise} \quad pf(K, L) - rK - wL$$

Once we formulate the problem as a two-variable maximisation exercise, we note that the optimal solution (K^*, L^*) must satisfy the following two properties

(1) K^* must be maximising the single variable function

$$K \longmapsto pf(K, L^*) - rK - wL^* \qquad \text{where } K \geq 0 \text{ is the domain}$$

(2) L^* must be maximising the single variable function

$$L \longmapsto pf(K^*, L) - rK^* - wL \qquad \text{where } L \geq 0 \text{ is the domain}$$

Now let's think of these two properties as two single-variable optimisation problems. Thanks to our understanding of maximising single-variable functions: in problem (1), if the point of maximisation (i.e., K^*) is an interior point of the domain, then the first order condition (FOC) applies in that problem: the first derivative of the objective function in problem (1) is zero at that point.

What does it mean for K^* to be an interior solution of problem (1)? Since the domain of K is $[0, \infty)$, for K^* to be in the interior, it must simply be positive. Likewise, L^* being interior means $L^* > 0$ because L also has the domain $[0, \infty)$.

Combining all of these we are able to conclude that if (K^*, L^*) is a solution to

$$\max \pi(K, L) \quad \text{subject to} \quad K, L \geq 0$$

and if K^* and L^* are both positive, then

$$\begin{aligned} \pi_K(K^*, L^*) &= 0 \\ \pi_L(K^*, L^*) &= 0 \end{aligned}$$

So, solutions to these last two equations will provide us with candidates for the solutions to our problem. Why do we say candidates? Well, it might be that the solution involves $K^* = 0$ or $L^* = 0$ or both. In that case, the solution does not necessarily satisfy the first order conditions, and just looking at the FOC won't actually deliver the solution to the original problem. Secondly, it might be that the FOC lead to multiple answers. If so, we will also need to compare those answers to find out which one really maximises the objective function.

Before we proceed to the analysis of non-interior solutions, let's state the FOC in general:

FOC for unconstrained optimisation. Suppose $\boldsymbol{x}^* \in \mathbb{R}^n$ is a local max or a local min of a function f whose domain is $D \subseteq \mathbb{R}^n$. Assume also that $\boldsymbol{x}^*$ in an interior point of D, that is, there exists a ball centred at $\boldsymbol{x}^*$ all of which is contained in D. If f is partially differentiable at $\boldsymbol{x}^*$ for each variable, then

$$\frac{\partial f}{\partial x_i}(x_1^*, \ldots, x_n^*) = 0 \quad \text{for all } i.$$

A more concrete problem

That's all cool, but let's now turn to an even more concrete problem: maximise $f(x, y) = xy$ on the domain $\mathbb{R}^2_{>0}$. First of all, we don't need to worry about the solution not being an interior one

because all points in $\mathbb{R}^2_{>0}$ are interior. So, we can trust the FOC to catch all possible candidate solutions. If (x^*, y^*) is a solution to the maximisation problem, we must have

$$f_x(x^*, y^*) = 0 \quad \text{and} \quad f_y(x^*, y^*) = 0$$

If we compute the first partial derivatives of f and evaluate them at (x^*, y^*), we obtain

$$f_x(x^*, y^*) = y^* \quad \text{and} \quad f_y(x^*, y^*) = x^*$$

Setting these to 0 would imply $(x^*, y^*) = (0, 0)$ which is not part of the domain of f. Therefore, no point in the domain of f satisfies the FOC. In particular, we conclude that f cannot be maximised on this domain. It cannot be minimised here, because, again, if there was a minimising point, the FOC must be holding there. (Remember that all points in the domain are interior, so there are no boundaries to speak of.)

Without further constraints on the set of alternatives, this function does not have optima. But we recognise this as a commonly studied example of a utility function over bundles, and a natural constraint on the domain of a utility function is that the domain is bounded due to the budget being a finite number, and the prices of the goods being positive.

A constrained optimisation problem

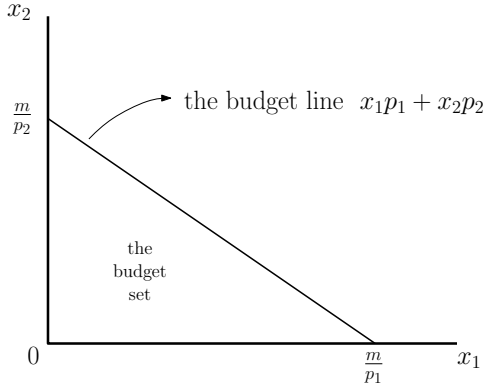
So, now impose a budget constraint on the problem by requiring the solution to satisfy the familiar affordability condition. Given a fixed income $m > 0$ and unit prices p_1 and p_1 for goods 1 and 2, respectively, we have the problem

$$\max f(x_1, x_2) \qquad \text{subject to} \qquad \underbrace{x_1 \geq 0 \ , \ x_2 \geq 0}_{\text{non-negativity constraints}} \ , \ \underbrace{x_1 p_1 + x_2 p_2 \leq m}_{\text{budget constraint}}$$

If f is an increasing function, then higher values of x_1 and x_2 lead to higher values of f. That means, the optimal solution (if exists) is indeed constrained by the constraint $x_1 p_1 + x_2 p_2 \leq m$. If that happens, i.e., the optimal solution turns out to satisfy $x_1 p_1 + x_2 p_2 = m$, then we say the **budget constraint binds**. So, for an increasing utility function f, the budget constraint will bind. That means, we know the optimal solution (if exists) will be a solution to the following problem:

$$\max f(x_1, x_2) \qquad \text{subject to} \quad x_1 \geq 0 \ , \ x_2 \geq 0 \ , \ x_1 p_1 + x_2 p_2 = m$$

So the actual set of alternative bundles over which we are trying to maximise f is a line segment on the coordinate plane connecting the point $(0, \frac{m}{p_2})$ to $(\frac{m}{p_1}, 0)$. This is a compact set. If f is continuous on this set, then we know by the EVT that there is a solution to our problem!



For concreteness, let's continue our discussion with $f(x_1, x_2) = x_1 x_2$ which is clearly continuous and increasing for $x_1, x_2 \geq 0$. We have already observed that when we impose the budget constraint, there exists a maximising point for this function. But does the FOC approach help? Looking for the solutions to $f_1(x_1, x_2) = f_2(x_1, x_2)$ will again yield $(0, 0)$ which is not even on the budget line,

where we know the solution must lie. In the previous section, we said the FOC didn't deliver because the function didn't even have a maximum. Here we have a compact domain, and we are guaranteed a solution, but can't seem to be able to make use of the FOC. What's going on?

Well, let's remember what the FOC was about. It was making use of the fact that if (a_1, a_2) is indeed a point at which the function is maximised, and the domain of f contains a neighbourhood around (a_1, a_2), i.e., this point is in the interior of the domain of the optimisation problem, then moving away from this point in any direction will lower the value attained by f. In particular

- moving away from this point west or east lowers the value (hence $f_1(a_1, a_2) = 0$)
- moving away from this point north or south lowers the value (hence $f_2(a_1, a_2) = 0$)

Well, it's not surprising that these conditions are not satisfied by any point on our domain (the budget line). From every point on this budget line, if we move north or east, the function would achieve higher values! But these conditions can't be expected to hold, because if we move off the budget line, we are leaving the domain of the optimisation problem.

7.1 Lagrange

The FOC was simply a consequence of the fact that if you are at a maximising point in the interior of the domain, then the value of f will suffer when you move away in any direction. Can we come up with another condition which is concise (in the spirit of the FOC) and makes the value of f suffer when we move away from the maximising point?

Remember that for an increasing function f which we'd like to maximise, the rewards were from going northeast. In other words, by violating the constraint

$$x_1 p_1 + x_2 p_2 \leq m$$

we can achieve higher values of f.

Perhaps, says Lagrange, we can work with an adjusted version of f which incorporates a "punishment" for exceeding the budget. That is we add a term to the function f such that whenever $x_1 p_1 + x_2 p_2$ exceeds m, this added term takes negative values to counteract the increase in the value of f. But how can that help? Well, if you set up this additional punishing term in such a way that

- the new function (f+punishment term) is allowed to be evaluated at every (x_1, x_2) (and not just on the budget),
- the optimal solution (a_1, a_2) to maximising f on the budget line is also an optimal (or at least locally optimal) solution to maximising the new function
- the punishment term is differentiable

then we know that the FOC must hold for the new function at the point (a_1, a_2).

Here's an idea

$$f(x_1, x_2) + \lambda(m - x_1 p_1 - x_2 p_2) \qquad (\mathcal{L})$$

The punishment term ensures that whenever (x_1, x_2) exceed the budget by an amount $t > 0$, the function $\mathcal{L}$ suffers a punishment of λt.

But what is λ? It seems like it not only punishes for exceeding the budget, but also rewards for not spending all of the budget. For example whenever (x_1, x_2) leaves an amount t unspent from the budget (i.e., whenever $m - x_1 p_1 - x_2 p_2 = t > 0$), the new function gets a reward of λt. How do we determine the correct λ such that the local optima of $(\mathcal{L})$ are not moving off the budget line towards the interior of the budget set?

It seems like the choice of the correct λ will be a delicate deal. If we set it too low, the punishment for violating the constraint won't be high enough, and the maximising points of $(\mathcal{L})$ will occur outside the budget. If, on the other hand, we set λ too high, then the rewards for saving your income will exceed the value achieved by f, and the maximising points of $(\mathcal{L})$ will occur in the interior of the budget. In fact, this logic suggests that the correct λ will have to be a knife-edge

value, like a point where a non-constant function is maximised. So, the correct λ must satisfy the property that if we move away from the correct λ in either direction (up or down), we move away from the knife-edge on which we want to be. *What is that knife-edge?* The budget line!

So it must be that:

IF (a_1, a_2) maximises $f(x_1, x_2)$ subject to $x_1p_1 + x_2p_2 - m = 0$,

THEN there must exist a λ^* such that the function $\mathcal{L} : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda(m - x_1p_1 - x_2p_2)$$

satisfies the FOC at (a_1, a_2, λ^*) for its all three variables:

$$\begin{aligned} \mathcal{L}_1(a_1, a_2, \lambda^*) &= 0 \\ \mathcal{L}_2(a_1, a_2, \lambda^*) &= 0 \\ \mathcal{L}_\lambda(a_1, a_2, \lambda^*) &= 0 \end{aligned}$$

Note that the FOC for λ is

$$\mathcal{L}_\lambda(a_1, a_2, \lambda^*) = 0 = m - a_1p_1 - a_2p_2$$

which is nothing but the requirement that (a_1, a_2) is on the budget line.

On the Lagrangian multiplier λ^*

λ^* is sometimes called the **Lagrangian multiplier** and it admits the following interpretation:

(1) Violating the budget constraint by a small amount, say by h, increases the value achieved by f. This increase is cancelled exactly by the Lagrangian punishment term. And that punishment is λ^*h.

(2) Relaxing the budget constraint by h, that is, allowing the budget to increase by h means the optimiser can violate the old budget by h without suffering any punishment. What is the added value (increase in f) resulting from this relaxation? It must be the amount of punishment which would exactly cancel out such added value. As argued in point (1), that amount is λ^*h.

Therefore λ^* is the increase in f due to a marginal increase in the budget. If we interpret f as a utility function, and m as income, then λ^* is the marginal utility of income!

7.2 The Lagrangian Recipe

The idea we explained above can be used even when the constraint is more complicated than a budget line. As long as a constraint can be expressed via a differentiable function, we can write a punishment term associated with violating that constraint.

If we are trying to maximise or minimise $f : \mathbb{R}^n \to \mathbb{R}$ subject to the constraint that $g(x_1, \ldots, x_n) = 0$, then we will set the associated Lagrangian function $\mathcal{L} : \mathbb{R}^{n+1} \to \mathcal{L}$ as

$$\mathcal{L}(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n)$$

Lagrange's theorem. Suppose $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$ is a local maximum or a local minimum of $f(\boldsymbol{x})$ subject to the constraint $g(x) = 0$.

Suppose also f and g both have partial derivatives at $\boldsymbol{x}^*$. Then there exists λ^* such that

$$\begin{aligned} \mathcal{L}_i(\boldsymbol{x}^*, \lambda^*) &= 0 \qquad \text{for all } i = 1, \ldots, n \\ \mathcal{L}_\lambda(\boldsymbol{x}^*, \lambda^*) &= 0 \end{aligned}$$

The recipe

Well, the theorem suggests that we should search for the maxima and minima of f by solving the $n + 1$ equations given by the FOC for $\mathcal{L}$. Typically there will be a small number of solutions and we will need to compare them with each other, and perhaps with values attained in their close neighbourhood to determine which one corresponds to the minimum or the maximum of the function.

An example with a non-linear constraint

Suppose a firm's production is given by $f(K,L) = (K^2L^3)^{1/6}$ and unit prices of its inputs are $p_K = 1$ and $p_L = 3$, respectively. What is the cost of producing 1 unit for the cost-minimising firm? So, we would like to solve:

$$\text{minimise} \quad K + 3L \qquad \text{subject to} \quad (K^2L^3)^{1/6} = 1$$

The constraint is $(K^2L^3)^{1/6} = 1$, but we should probably rewrite it in the easiest to differentiate form to make the subsequent algebra less tedious. Let's express the constraint as $K^2L^3 - 1 = 0$, so the Lagrangian is

$$\mathcal{L}(K,L,\lambda) = K + 3L - \lambda(K^2L^3 - 1)$$

A moment of reflection on λ. We are trying to make $K + 3L$ which obviously will get smaller as we push K and L down. But we want to ensure the production target of q to be met. So the job of λ is to punish attempts to lower K and L off the constraint. If $(K^2L^3)^{1/6}$ drops below 1 by an amount h, the Lagrangian punishment will change by λh, counteracting the drop in the cost function. So we are expecting the correct λ to be positive.

Back to the recipe. The FOC says the optimal (K^*, L^*, λ^*) must be a solution to

$$\mathcal{L}_K(K,L,\lambda) = 1 - 2KL^3\lambda = 0 \implies KL^3 = (2\lambda)^{-1}$$

$$\mathcal{L}_L(K,L,\lambda) = 3 - 3K^2L^2\lambda = 0 \implies K^2L^2 = \lambda^{-1}$$

$$\mathcal{L}_\lambda(K,L,\lambda) = K^2L^3 - 1 = 0 \implies K^2L^3 = 1$$

The last two equations imply $L = \lambda$ and $K = \lambda^{-3/2}$. Plugging these into the first equation yields $\lambda^* = 2^{-2/5}$, and hence $K^* = 2^{3/5}$ and $L^* = 2^{-2/5}$. Plugging these optimal values of K and L into the cost function, we conclude that the cost of producing one unit is $2^{3/5} + 3 \times 2^{-2/5}$.

But, wait a minute! What if this (K^*, L^*, λ^*) is not the cost minimiser, but the cost maximiser? Or neither? How did we know that it was indeed the minimiser?

7.3 The second order conditions for two-variable optimisation

The Lagrangian recipe helps us identify a number of points which are the candidates for global maxima/minima, but is not conclusive as to whether we have one or the other. In the case of single-variable optimisation, we appealed to the second derivative to help determine whether a critical point is a local max or a local min. For example, if c is a critical point of a differentiable $f : \mathbb{R} \to \mathbb{R}$, then we know $f'(c) = 0$. If, in addition, we know that f is twice differentiable at c and if $f''(c) < 0$, then we conclude that c is a local max. If it happens that $f'' < 0$ everywhere, then we know f is concave, and there is at most one point where $f'(c) = 0$, and that point is a global maximum.

What is the correct analogue of a second derivative for multi-variable functions? If there is such an analogue, is it useful in identifying convexity/concavity of the function? Is it at least useful to determine whether a critical point (a point where the FOC are satisfied) is a local max or a local min?

We will answer these questions for the special case of two-variable functions.

First, remember that we defined the second partial derivatives, as the partial derivative of the partial derivative:

$$f_{ij} \quad = \quad (f_i)_j \quad = \quad \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) \quad = \quad \frac{\partial^2 f}{\partial x_j \partial x_i}$$

So, in the case of a $\mathbb{R}^2 \to \mathbb{R}$ functions, we have four second partial derivatives:

$$f_{11}, \quad , f_{12}, \quad f_{21}, \quad f_{22}$$

But if these second partial derivatives are continuous, then Young's theorem says $f_{12} = f_{21}$; a fact sometimes referred to as "mixed partials commuting".

The **Hessian** of $f : \mathbb{R}^2 \to \mathbb{R}$ (or more generally $f : B \to \mathbb{R}$ where B is an open ball in $\mathbb{R}^2$) at the point $\boldsymbol{c}$ is defined as the following 2×2 table of four values[5]

$$\begin{bmatrix} f_{11}(\boldsymbol{c}) & f_{12}(\boldsymbol{c}) \\ f_{21}(\boldsymbol{c}) & f_{22}(\boldsymbol{c}) \end{bmatrix}$$

If f has continuous second partial derivatives, we know that $f_{12}(\boldsymbol{c}) = f_{21}(\boldsymbol{c})$.

Let $f : D \to \mathbb{R}$ have continuous second partial derivatives, where D is a convex subset of $\mathbb{R}^2$.

- If $f_{11}(\boldsymbol{x})f_{22}(\boldsymbol{x}) - (f_{12}(\boldsymbol{x}))^2 > 0$ and $f_{11}(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in D$, then f is convex on D.
- If $f_{11}(\boldsymbol{x})f_{22}(\boldsymbol{x}) - (f_{12}(\boldsymbol{x}))^2 > 0$ and $f_{11}(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \in D$, then f is concave on D.
- If $\boldsymbol{c}$ is a critical point of f, and if $f_{11}(\boldsymbol{c})f_{22}(\boldsymbol{c}) - (f_{12}(\boldsymbol{c}))^2 > 0$ and $f_{11}(\boldsymbol{c}) > 0$, then $\boldsymbol{c}$ is a local min of f.
- If $\boldsymbol{c}$ is a critical point of f, and if $f_{11}(\boldsymbol{c})f_{22}(\boldsymbol{c}) - (f_{12}(\boldsymbol{c}))^2 > 0$ and $f_{11}(\boldsymbol{c}) < 0$, then $\boldsymbol{c}$ is a local max of f.

An example.

Here is Question 2 from 2013 Exam: An agent has the utility function

$$U(c_1, c_2) \;=\; 2 + 3(c_1^{1/3} - 1) + 3\beta(c_2^{1/3} - 1)$$

where $c_1, c_2 \geq 0$ are consumption in periods 1 and 2, respectively. Finally, β is the factor with which the agent discounts the future.

(a) Solve the agent's utility maximisation problem subject to the constraint that $c_1 + c_2 = 1$, i.e., solve for the optimal values of c_1 and c_2.

(b) Assume that the agent lives for two periods, but that in addition to his utility above, he also derives utility from leaving a bequest z. In other words, that

$$U(c_1, c_2, z) \;=\; 2 + 3(c_1^{1/3} - 1) + 3\beta(c_2^{1/3} - 1) + z$$

Also, assume that $\beta = 0.4$ and that utility is maximised subject to the constraint $c_1 + c_2 + z = 1$ and that $c_1, c_2, z \geq 0$. Solve the agent's utility maximisation problem, i.e. for the optimal values of c_1, c_2, z.

Solution. (a) Let's begin with the observation that maximising the expression U is equivalent to the problem of maximising

$$V(c_1, c_2) \;=\; 3c_1^{1/3} + 3\beta c_2^{1/3}$$

since the constant terms have no effect on how one should choose c_1 and c_2 to solve the original problem.

Given the constraint $c_1 + c_2 = 1$, the associated Lagrangian is

$$\mathcal{L} \;=\; V(c_1, c_2) + \lambda(1 - c_1 - c_2)$$

with the following FOC

$$\begin{aligned} \mathcal{L}_1 &= c_1^{-2/3} - \lambda = 0 \\ \mathcal{L}_2 &= \beta c_2^{-2/3} - \lambda = 0 \\ \mathcal{L}_\lambda &= 1 - c_1 - c_2 = 0 \end{aligned}$$

[5] At this stage we refer to this as a 2×2 table, but this is really a special case of a mathematical object called matrix which we will study in more detail later in the course.

The first two equations, together, imply

$$c_1^{-2/3} = \beta c_2^{-2/3}$$

Plugging this into the third equation, we obtain

$$c_1(1+\beta^{3/2}) = 1$$

which yields

$$c_1^* = \frac{1}{1+\beta^{3/2}} \qquad c_2^* = \frac{\beta^{3/2}}{1+\beta^{3/2}}$$

But, we are not done yet, because we haven't really checked that the problem should have an interior solution. One way to verify this is indeed the solution is to look at the Hessian

$$\begin{pmatrix} -\frac{2}{3}c_1^{-5/3} & 0 \\ 0 & -\frac{2\beta^2}{3}c_2^{-5/3} \end{pmatrix}$$

which is clearly negative definite at all interior points (c_1, c_2). Therefore V is strictly concave, and the unique point identified by the FOC is necessarily the point V is maximised. [6]

(b) Now we are maximising $3c_1^{1/3} + 3\beta c_2^{1/3} + z$ subject to $c_1 + c_2 + z = 1$ and $c_1, c_2, z \geq 0$.

The associated Lagrangian is

$$\mathcal{L} = 3c_1^{1/3} + 3\beta c_2^{1/3} + z + \lambda(1 - c_1 - c_2 - z)$$

If there is a solution with $c_1, c_2, z > 0$, then we must have

$$\begin{aligned} \mathcal{L}_1 &= c_1^{-2/3} - \lambda = 0 \\ \mathcal{L}_2 &= \beta c_2^{-2/3} - \lambda = 0 \\ \mathcal{L}_z &= 1 - \lambda = 0 \\ \mathcal{L}_\lambda &= 1 - c_1 - c_2 - z = 0 \end{aligned}$$

The third equation implies $\lambda = 1$. Plugging this into the first two equations, we obtain $c_1 = 1$ and $c_2 = \beta^{3/2}$. And finally plugging these into the last equation, we find $z = -\beta^{3/2}$ which contradicts with the assumption that the optimal z is positive.

Thus, the beginning assumption of a solution with $c_1, c_2, z > 0$ is wrong. At least one of c_1, c_1, z is zero at the optimal choice. But which one? Before we try every possibility, we can take a moment of reflection to have a more economic feel for the question.

The problem is set up as if a consumer has income 1, splitting this income over three goods c_1, c_2, z, where each good has the same unit price of 1. But note that the MU of the first good is $c_1^{-2/3}$ which is greater than 1 if $c_1 < 1$ whereas the MU of z is always 1. That means, this consumer wouldn't spend a penny on z until she is consuming more than 1 unit of c_1. The budget constraint says, she can't spend more than 1 on c_1 anyway, so the optimal choice of z must be 0. But then the problem reduces to the case in part (a).

[6] The idea of imposing the condition of MU_i/p_i being the same for all goods is also at heart of the Lagrangian FOC. For these approaches to yield the solution to the optimisation problem, the solution should actually be an interior one. By interior, we mean that the solution should be away from the boundary not reflected in the Lagrangian function. For example, in the above analysis, $\mathcal{L}$ features the boundary of the domain captured by the budget constraint, and we will be looking for solutions which actually lie on this boundary. On the other hand, the other two boundaries, namely c_1 and c_2 axes do not feature in the $\mathcal{L}$ expression, and the above Lagrangian approach cannot guarantee picking up solutions which might lie on those boundaries (i.e., potential solutions with $c_1 = 0$ or $c_2 = 0$). There is a more generalised approach (the so-called Kuhn-Tucker Lagrangian) which can incorporate inequality constraints along with equality constraints, but is beyond the scope of this course.

7.4 Multiple Constraints

The Lagrangian approach to optimisation can handle a problem which involves multiple constraints as long as each of those constraints can be expressed by a differentiable function, and these functions satisfy some appropriate technical condition called the constraint qualification (CQ) the details of which we won't get into.

Consider the problem

$$\max f(x_1, \ldots, x_n) \qquad \text{subject to} \qquad \begin{array}{c} g^1(x_1, \ldots, x_n) = 0 \\ g^2(x_1, \ldots, x_n) = 0 \\ \vdots \\ g^m(x_1, \ldots, x_n) = 0 \end{array}$$

So we have an objective function with n variables, and we require solutions to satisfy m different constraints.[7] In our first example, we had a single constraint which required the solution to the maximisation problem to also satisfy a budget equation. That meant, we weren't allowed to use any point in $\mathbb{R}^n$ to maximise the objective function; rather the solution had to lie in the subset described by the budget equation. In the more general case we have just introduced, each constraint $g^i(x_1, \ldots, x_n) = 0$ requires solutions to be in the subset of $\mathbb{R}^n$ described by the equation $g^i(x_1, \ldots, x_n) = 0$. m constraints describe m different subsets of $\mathbb{R}^n$, and for $\boldsymbol{x}$ to satisfy these constraints, it must be in the intersection of those m sets. In order to "discipline" our choice of x, we must have a multiplier associated with each constraint which will counteract the costs/benefits of violating that constraint. So the Lagrangian will look like

$$\mathcal{L}(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_m) \quad = \quad f(x_1, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j g^j(x_1, \ldots, x_n)$$

To make this look tidier, we can use the more concise notation of $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ for $(x_1, \ldots, x_n)$ and $(\lambda_1, \ldots, \lambda_m)$, respectively, so $\mathcal{L} : \mathbb{R}^{n+m} \to \mathbb{R}$ will look like

$$\mathcal{L}(\boldsymbol{x}; \boldsymbol{\lambda}) \quad = \quad f(\boldsymbol{x}) + \sum_{j=1}^{m} \lambda_j g^j(\boldsymbol{x})$$

where λ_j corresponds to the cost of violating (or benefit of relaxing) constraint g^j.

Lagrange's theorem. Suppose $\boldsymbol{x}^*$ is a local max or a local min of f subject to the m equality constraints given by $g(\boldsymbol{x}) = 0$. Suppose also f and each g^i are differentiable at $\boldsymbol{x}^*$, and CQ is satisfied at $\boldsymbol{x}^*$. Then there are numbers $\lambda_1^*, \ldots, \lambda_m^*$ such that

$$\begin{aligned} \frac{\partial \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)}{\partial x_i} &= 0 \qquad \text{for } i = 1, \ldots, n \\ \frac{\partial \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)}{\partial \lambda_j} &= 0 \qquad \text{for } j = 1, \ldots, m. \end{aligned}$$

[7] Is this problem guaranteed to have a solution? If the constraints are described by continuous functions, i.e., if g^i is continuous for all $i = 1, \ldots, m$, and if the set of points satisfying these constraints is bounded, then the constraint set is compact. If, moreover, f is continuous, then yes, we are guaranteed to have a solution.

8 $\mathbb{R}^n$ as a vector space

Reading: Sydsaeter & Hammond (4th edition), Sections 15.7 (Vectors), 15.8 (Geometric Interpretation of Vectors), 15.1 (Systems of Linear Equations), 15.2 (Matrices and Matrix Operations)

We have been using the notion of n-tuples for a while now. When we talked about bundles of n goods, we first indexed all n goods as good 1, good 2, good 3, and so on. For example, apple is good 1, banana is good 2, etc. Then we denoted the amount of goods in a bundle with a string of n numbers:

$$(x_1, x_2, \ldots, x_n)$$

which indicated that the bundle $\boldsymbol{x}$ contained x_1 apples, x_2 bananas, and so on. We have been referring to this n-tuple as an n-dimensional point which is an element of the n-dimensional real space $\mathbb{R}^n$. Now we would like to develop a bit more machinery to use these n-dimensional points effectively in various applications. If we can afford to write our symbols in bold face, it would be a good practice to write multi-dimensional points in bold face, so we would write $\boldsymbol{x}$ instead of x when we talk about $(x_1, \ldots, x_n)$. We won't always do that partly because it's not easy to stick with that practice in handwriting. In these notes, we will try to use the bold face notation for the beginning when we are keen to emphasise the distinction between numbers in $\mathbb{R}$ and points in $\mathbb{R}^n$. Later, we will trust that the reader can infer from the context when x stands for an n-dimensional point $(x_1, x_2, \ldots, x_n)$.

As usual, in order to take advantage of visual intuition, we will first introduce concepts for two-dimensional vectors, and then extend them to n dimensions.

Special case: the two-dimensional vector space $\mathbb{R}^2$

When we are dealing with two types of goods only, the bundles represented by pairs (x_1, x_2) enjoy a convenient geometric representation: points on the coordinate plane that depicts $\mathbb{R}^2$.

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$$

Another word for a point in $\mathbb{R}^2$ is **vector**, and sometimes it is useful to visualise $\boldsymbol{x} = (x_1, x_2)$ as an arrow which begins at the origin $(0, 0)$ and ends at (x_1, x_2). While we can use the words point and vector interchangeably, the word vector can be particularly useful in highlighting the fact that we are talking about a tuple, not a single number.

For vectors $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^2$, and scalars c in $\mathbb{R}$ we define the following operations

- **vector addition:** $(x_1, x_2) + (y_1, y_2) = (x_1 + x_2, x_1 + x_2)$
- **scalar multiplication:** $c(x_1, x_2) = (cx_1, cx_2)$

The symbol c stands for an arbitrary number in $\mathbb{R}$ and in order to emphasise that it is a number, not a vector, we sometimes use the word **scalar** when we refer to numbers which we multiply with vectors.

We have already been using these operations extensively. For example, if we put together bundles $(3, 5)$ and $(7, 1)$, we get another bundle $(10, 6)$. Vector addition generalises this operation to allow for negative entries for x_1 and x_2.

Multiplying a bundle by 2 is like doubling the contents of the bundle, i.e., scaling it up by 2. That's what scalar multiplication refers to, but of course we now have a more general operation, because we can multiply a vector by a negative scalar. This notion might not have an immediate interpretation in terms of scaling up a bundle (like negative numbers didn't have an obvious interpretation for cavemen), but algebraically there is nothing to stop us to multiply vector $(5, -2)$ by -4. (*Actually, one can come up with an interpretation for multiplying* $(5, -2)$ *by* -4*, and we will leave that exercise to the reader.*)

With these operations of **vector addition** and **scalar multiplication**, the set $\mathbb{R}^2$ gets a structure, called **vector space**, which allows us not only to add but also subtract one vector from another. After all, subtracting $\boldsymbol{y}$ from $\boldsymbol{x}$ can be achieved by first multiplying $\boldsymbol{y}$ with the scalar -1 to obtain $-\boldsymbol{y}$, and then add $-\boldsymbol{y}$ with $\boldsymbol{x}$. The vector $(0, 0)$ is the identity element of vector addition, denoted $\mathbf{0}$, or simply 0.

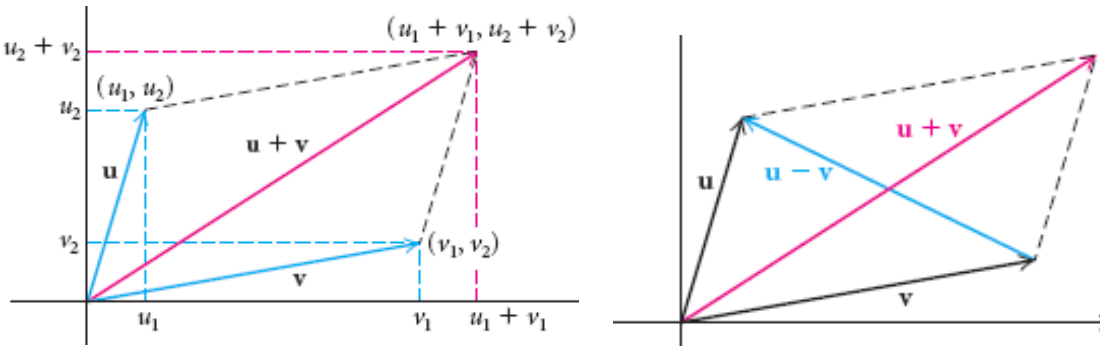


Figure 2: Adding vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$.

More generally: vectors in $\mathbb{R}^n$

In the same fashion as above, we can define the following algebraic operations for vectors.

Vector addition. Given two vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$,

$$\boldsymbol{x} + \boldsymbol{y} \quad = \quad (x_1 + y_1, \ldots, x_n + y_n)$$

Scalar multiplication. Given a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ and a scalar $c \in \mathbb{R}$,

$$c\boldsymbol{x} \quad = \quad \boldsymbol{x}c \quad = \quad (cx_1, \ldots, cx_n)$$

It is straightforward to verify that these operations satisfy the following properties

- $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ (Commutative Law)
- $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ (Associative Law)
- $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$, where $\mathbf{0} = (0, \ldots, 0)$ is the zero vector
- $c(\boldsymbol{x} + \boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$, where c is a scalar (Distributive Law)

The inner product (dot product) of vectors

The vector space $\mathbb{R}^n$ admits a third operation, whose geometric interpretation is less obvious, but is very convenient in capturing widely used operations in applications. Given two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we define their **inner product** (also called **dot product**) as

$$\boldsymbol{x} \cdot \boldsymbol{y} \quad = \quad x_1y_2 + x_2y_2 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i$$

Note the "dot" sign in between $\boldsymbol{x}$ and $\boldsymbol{y}$. This is another reminder that $\boldsymbol{x}$ and $\boldsymbol{y}$ refer to vectors (not scalars). Secondly, the outcome of a dot product is not another vector, but instead a scalar, i.e., a number in $\mathbb{R}$.

What is this good for? Well, if nothing, to express concisely the cost of a bundle. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ stands for a bundle, and if $\boldsymbol{p} = (p_1, \ldots, p_n)$ is the vector of prices for the n goods we have in mind, then the cost of the bundle is nothing but the dot product of $\boldsymbol{p}$ with $\boldsymbol{x}$.

The following properties of the dot product are not hard to establish:

For all vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$

- $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$
- $\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}$
- $(c\boldsymbol{x}) \cdot \boldsymbol{y} = c(\boldsymbol{x} \cdot \boldsymbol{y}) = \boldsymbol{x} \cdot (c\boldsymbol{y})$

The standard unit vectors for $\mathbb{R}^2$

Consider the unit vectors $\boldsymbol{e}^1 = (1, 0)$ and $\boldsymbol{e}^2 = (0, 1)$ on the coordinate plain. Together, they have the nice property of being able to combine to express all vectors in $\mathbb{R}^2$. Namely, given any vector $\boldsymbol{x} = (x_1, x_2)$ we can express it as a **linear combination** of $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$:

$$\boldsymbol{x} = (x_1, x_2) = x_1\boldsymbol{e}^1 + x_2\boldsymbol{e}^2$$

We refer to $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$ as the **unit vectors** in $\mathbb{R}^2$.

The standard unit vectors for $\mathbb{R}^n$

Extending the idea of unit vectors from two dimensions to n dimensions, we can use the **unit vectors** in $\mathbb{R}^n$ listed below

$$\boldsymbol{e}^1 = (1, 0, 0, 0, \ldots, 0, 0) \quad \boldsymbol{e}^2 = (0, 1, 0, 0, \ldots, 0, 0) \quad \boldsymbol{e}^3 = (0, 0, 1, 0, \ldots, 0, 0) \quad \cdots \quad \boldsymbol{e}^n = (0, 0, 0, \ldots, 0, 1)$$

to express any given vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ as a linear combination of $\boldsymbol{e}^1, \boldsymbol{e}^2, \ldots, \boldsymbol{e}^n$:

$$\boldsymbol{x} \quad = \quad x_1\boldsymbol{e}^1 + x_2\boldsymbol{e}^2 + \cdots + x_n\boldsymbol{e}^n$$

9 Linear transformations

A **linear transformation** (also called a **linear map**) is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ which satisfies

- $T(\mathbf{0}) = \mathbf{0}$
- $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
- $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$

Examples of linear transformations

1. The identity function from $\mathbb{R}^n$ to $\mathbb{R}^n$.
2. The zero function from $\mathbb{R}^n$ to $\mathbb{R}^m$ which maps every vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$.
3. The vertical-projection function from $\mathbb{R}^2$ to $\mathbb{R}$ which maps (x_1, x_2) to x_1. Note that, geometrically, this projects each point vertically onto the horizontal axis (assuming that we denote the horizontal axis to capture x_1).
4. "Scaling up by 5" from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps (x_1, x_2) to $(5x_1, 5x_2)$.
5. Reflection (mirror image) with respect to the vertical axis in $\mathbb{R}^2$ is a transformation which maps (x_1, x_2) to $(-x_1, x_2)$.
6. Rotations of a fixed degree around the origin in $\mathbb{R}^2$. This one is harder to see immediately, but visualising it helps explain why it is indeed a linear transformation. For example, let $R : \mathbb{R}^2 \to \mathbb{R}^2$ rotate vectors in $\mathbb{R}^2$ around the origin $\mathbf{0}$ by ninety degrees counterclockwise.

   That is, we fix the origin, and rotate the whole plane ninety degrees counterclockwise. We need to check the three required properties:

   - $R(\mathbf{0}) = \mathbf{0}$

     Satisfied, since our rotation is fixing the origin, and rotating everything else around this point.
   - $R(\boldsymbol{x} + \boldsymbol{y}) = R(\boldsymbol{x}) + R(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

     Satisfied since rotating the whole plane would preserve shapes, and addition of two vectors can be viewed as a triangle. (See the last two figures to get a feel.)

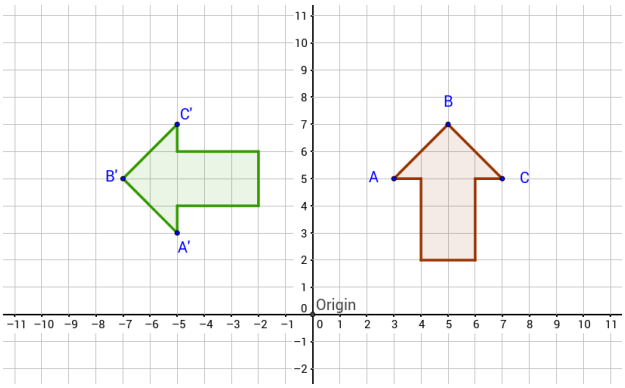


Figure 3: R maps $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ to $\boldsymbol{A}', \boldsymbol{B}', \boldsymbol{C}'$, respectively.

- $R(c\boldsymbol{x}) = cR(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$.
  Whether we scale it before rotating or after rotating does not matter.

Just to be clear, the brief explanations we provided above do not constitute a proper proof, but rather an intuitive support for the validity of the suggestion that R is indeed a linear map.

Understanding linear transformations a bit better

Here is the key observation: in order to describe a linear transformation T, it is sufficient to describe the image of each unit vector $\boldsymbol{e}^j$. This is because given any $\boldsymbol{x}$, we know we can express it as

$$\boldsymbol{x} = x_1\boldsymbol{e}^1 + \cdots + x_n\boldsymbol{e}^n$$

and using linearity of T, we can compute $T(\boldsymbol{x})$ as

$$\begin{aligned} T(\boldsymbol{x}) &= T(x_1\boldsymbol{e}^1 + \cdots + x_n\boldsymbol{e}^n) \\ &= x_1T(\boldsymbol{e}^1) + \cdots + x_nT(\boldsymbol{e}^n) \end{aligned}$$

Example: $90°$ counterclockwise rotation about the origin

Let's apply this simple observation to the $90°$ counterclockwise rotation mapping $R : \mathbb{R}^2 \to \mathbb{R}^2$. We need to figure out what the rotations of $(1, 0)$ and $(0, 1)$ are. Well, it is not hard to see on the coordinate plane that

- $R(\boldsymbol{e}^1) = R(1, 0) = (0, 1)$, and
- $R(\boldsymbol{e}^2) = R(0, 1) = (-1, 0)$

So, given any vector $\boldsymbol{x} = (x_1, x_2)$, rotating it $90°$ counterclockwise would result in

$$\begin{aligned} R(\boldsymbol{x}) &= x_1R(\boldsymbol{e}^1) + x_2R(\boldsymbol{e}^2) \\ &= x_1(0, 1) + x_2(-1, 0) \\ &= (-x_2, x_1) \end{aligned}$$

So, now we have an algebraic description of our visual notion of rotating points counterclockwise around the origin by $90°$.

10 Matrices: a tidy notation for linear transformations

We'd like a concise and practical notation for linear transformations which will help us gather speed in our calculations when, for example,

- we want to evaluate a linear transformation T at a particular point $\boldsymbol{x}$
- for a given $\boldsymbol{v} \in \mathbb{R}^m$, we want to find out if there is any $\boldsymbol{x} \in \mathbb{R}^n$ such that $T(\boldsymbol{x}) = \boldsymbol{v}$
- given two linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^m \to \mathbb{R}^k$, we want to understand the composition

$$\begin{array}{rccccc} U \circ T & : & \mathbb{R}^n & \xrightarrow{T} & \mathbb{R}^m & \xrightarrow{U} & \mathbb{R}^k \\ & & \boldsymbol{x} & \longmapsto & T(\boldsymbol{x}) & \longmapsto & U(T(\boldsymbol{x})) \end{array}$$

Let's begin with an example and then generalise in a natural fashion.

Example: $90°$ counterclockwise rotation about the origin

For this linear map $R : \mathbb{R}^2 \to \mathbb{R}^2$ we already have a neat and concise description:

$$R(x_1, x_2) = (-x_2, x_1)$$

But we will introduce another one which soon will be our standard notation. First we will adopt a convention to express vectors as columns, so instead of writing (x_1, x_2), we will write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

With this new notation, the unit vectors of $\mathbb{R}^2$ will be written as

$$\boldsymbol{e}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{e}^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The expression $R(x_1, x_2) = (-x_2, x_1)$ can now be rewritten as

$$R\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

But remember how we came to conclude $R(x_1, x_2) = (-x_2, x_1)$. We first applied R to the unit vectors $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$:

$$R(\boldsymbol{e}^1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad R(\boldsymbol{e}^2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Next, since we were able to express $\boldsymbol{x}$ as $x_1\boldsymbol{e}^1 + x_2\boldsymbol{e}^2$, we could compute $R(\boldsymbol{x})$ as

$$R(\boldsymbol{x}) \quad = \quad R(x_1\boldsymbol{e}^1) + R(x_2\boldsymbol{e}^2) \quad = \quad x_1 R(\boldsymbol{e}^1) + x_2 R(\boldsymbol{e}^2) \quad = \quad x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

We will remember this description of R by referring to it by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

where the first column is the image of $\boldsymbol{e}^1$ under the mapping R, and the second column is the image of $\boldsymbol{e}^2$ under the mapping R.

More generally

In general, when we are describing a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, it is sufficient to describe the images of the unit vectors $\boldsymbol{e}^1, \ldots, \boldsymbol{e}^n$. Our matrix notation will summarise that information in the form

$$\left[\begin{array}{c|c|c|c|c} T(\boldsymbol{e}^1) & T(\boldsymbol{e}^2) & \cdots & T(\boldsymbol{e}^j) & \cdots & T(\boldsymbol{e}^n) \end{array}\right]$$

$\boldsymbol{e}^j$ is the jth unit vector in $\mathbb{R}^n$
and
$T(\boldsymbol{e}^j) \in \mathbb{R}^m$ is written as a column.

Note that each column is a vector in $\mathbb{R}^m$, so the columns are of height m. Therefore, there are m rows in the above matrix.

Second, there one column for each unit vecrtor of $\mathbb{R}^n$. Hence, there are n columns.

Thus, we can express a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ simply by a matrix with m **rows** and n **columns**. This is an "m **by** n **matrix**."

Another example

Using the matrix

$$\begin{bmatrix} 3 & -2 & 1 \\ -7 & 0 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$

we can represent a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ which maps

$$\boldsymbol{e}^1 \text{ to } \begin{bmatrix} 3 \\ -7 \\ 0 \\ 2 \end{bmatrix} \quad , \quad \boldsymbol{e}^2 \text{ to } \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \boldsymbol{e}^3 \text{ to } \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

We could have written

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -7 \\ 0 \\ 2 \end{bmatrix} \quad , \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

And given an arbitrary vector $\boldsymbol{x} = (x_1, x_2, x_3)$ we can express it as a linear combination of unit vectors of $\mathbb{R}^3$, and use linearity property of T to obtain

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) &= x_1 T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + x_3 T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= x_1 \begin{bmatrix} 3 \\ -7 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 3x_1 - 2x_2 + x_3 \\ -7x_1 + 3x_3 \\ x_2 + x_3 \\ 2x_1 \end{bmatrix} \end{aligned}$$

We really want to economise on notation, and we will express all of the above more concisely as follows:

$$\begin{bmatrix} 3 & -2 & 1 \\ -7 & 0 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 + x_3 \\ -7x_1 + 3x_3 \\ x_2 + x_3 \\ 2x_1 \end{bmatrix}$$

From linear transformations to matrices

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we will represent it with an $m \times n$ matrix A such that the jth column of the matrix A is $T(\boldsymbol{e}^j)$, that is, the image of the jth unit vector $\boldsymbol{e}^j$ under T:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ A_{31} & A_{32} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \qquad \text{where} \qquad T(\boldsymbol{e}^j) = \underbrace{\begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \\ \vdots \\ A_{mj} \end{bmatrix}}_{\text{the } j\text{th column of } A}$$

When we say this is an m **by** n **matrix**, we mean it has m **rows** and n **columns**. Once again, remember that the jth column of the matrix will be the image of $\boldsymbol{e}^j \in \mathbb{R}^n$ under the mapping T. The number of columns is the dimension of the domain of T, because there is a column for each unit vector $\boldsymbol{e}^j$ of the domain $\mathbb{R}^n$. The number of rows is the height of the columns, which is nothing but the dimension of the codomain $\mathbb{R}^m$, because after all, column j is the image of $\boldsymbol{e}^j$, and the images under T are in the codomain.

We will refer to A_{ij} as the ij-**entry** of matrix A.

Given matrix A representing T, let's now write explicitly what $T(\boldsymbol{x})$ is in the matrix notation:

$$A\left[\boldsymbol{x}\right] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ A_{31} & A_{32} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \text{the} \\ \text{first} \\ \text{column} \\ \text{of } A \end{bmatrix} x_1 + \begin{bmatrix} \text{the} \\ \text{second} \\ \text{column} \\ \text{of } A \end{bmatrix} x_2 + \cdots + \begin{bmatrix} \text{the} \\ n\text{th} \\ \text{column} \\ \text{of } A \end{bmatrix} x_n$$

$$= \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n \\ A_{31}x_1 + A_{32}x_2 + \ldots + A_{3n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n \end{bmatrix}$$

We will think of a column vector in $\mathbb{R}^n$ as an $n \times 1$ matrix. Likewise, a column vector in $\mathbb{R}^m$ will be an $m \times 1$ matrix. And we will refer to the following expression as multiplying the $m \times n$ matrix A with the $n \times 1$ matrix $[\boldsymbol{x}]$ to obtain the $m \times 1$ matrix $A\left[\boldsymbol{x}\right]$:

$$A\left[\boldsymbol{x}\right] = \underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ A_{31} & A_{32} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}}_{\text{An } m \times n \text{ matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\substack{\text{An } n \times 1 \\ \text{matrix}}} = \underbrace{\begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n \\ A_{31}x_1 + A_{32}x_2 + \ldots + A_{3n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n \end{bmatrix}}_{\text{An } m \times 1 \text{ matrix}}$$

Using matrix notation effectively

In order to justify introducing this whole new vocabulary and notation, we should probably show a few more uses of matrices in capturing meaningful tasks. Let's begin with an easy one.

Adding two linear transformations

Suppose T and U are two linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Since these functions have the same domain and codomain, we can talk about the function $T+U : \mathbb{R}^n \to \mathbb{R}^m$. Is this function a linear transformation as well? *The answer is yes, and verifying this takes checking the three simple requirements we laid out in defining linear transformations. (This is an exercise left to the reader.)*

In order to understand better how $T+U$ operates, it is sufficient to understand the image of each unit vector under the transformation $T+U$. That is, we need to figure out what $(T+U)(\boldsymbol{e}^j)$ is, and simply write this as the jth column of a matrix which will then represent the linear transformation $T+U$. Denoting the matrices representing T and U by A and B, respectively:

$$(T+U)(\boldsymbol{e}^j) = \underbrace{T(\boldsymbol{e}^j)}_{j\text{th column of matrix } A} + \underbrace{U(\boldsymbol{e}^j)}_{j\text{th column of matrix } B}$$

Thus, the jth column of the matrix which represents $T+U$ is simply the sum of the jth columns of the matrices representing T and U. Therefore the ij-entry of the matrix which represents $T+U$ is $A_{ij}+B_{ij}$, so $T+U$ is represented by

$$\begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1n}+B_{1n} \\ A_{21}+B_{21} & A_{22}+B_{22} & \cdots & A_{2n}+B_{2n} \\ A_{31}+B_{31} & A_{32}+B_{32} & \cdots & A_{3n}+B_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}+B_{m1} & A_{m2}+B_{m2} & \cdots & A_{mn}+B_{mn} \end{bmatrix}$$

We will refer to this matrix above as the sum of matrices A and B and will denote this summation by $A+B$. In other words, by adding the corresponding entries of the matrices A and B which represent the linear maps T and U, respectively, we get the matrix $A+B$ that represents the linear map $T+U$.

So, from now on, when we talk about the sum $A+B$ of two matrices A and B, we are necessarily assuming that they have the same number of rows, and they have the same number of columns. The ij-entry of the matrix $A+B$ is $A_{ij}+B_{ij}$. Not surprisingly $A+B=B+A$.[8]

Multiplying two linear transformations?

Again, we have T and U which are two linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Above, we talked about adding these functions. Can we also talk about multiplying them? Not really! The values taken by these functions are vectors in $\mathbb{R}^m$, and what does it mean to multiply vectors in $\mathbb{R}^m$ and obtain another vector? We might be tempted to think of the dot product of vectors, but that would result in scalars. In other words, if we wanted to come up with a notion of multiplying T and U via the use of dot product, we would not get another linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Bottom line: we won't try to make sense of "multiplying linear transformations".

Composing two linear transformations

When can we talk about composing two linear transformations? Since they are functions, we can do so when the codomain of one is the domain of the other. So, when we have two linear

[8] Since a matrix is nothing but a way of representing a linear transformation, the properties satisfied by linear transformations will have counterparts for matrices. And vice versa. For example, we now know that if T and U refer to linear transformations, then we have $T+U=U+T$.

transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^k \to \mathbb{R}^n$, we can talk about the composition $T \circ U$ defined in the usual way we talk about composing functions:

$$\begin{array}{rcccccc} T \circ U & : & \mathbb{R}^k & \xrightarrow{U} & \mathbb{R}^n & \xrightarrow{T} & \mathbb{R}^m \\ & & \boldsymbol{x} & \longmapsto & U(\boldsymbol{x}) & \longmapsto & T(U(\boldsymbol{x})) \end{array}$$

Is $T \circ U : \mathbb{R}^k \to \mathbb{R}^m$ a linear transformation? *The answer is yes, and again, verifying this takes checking the three simple requirements we laid out in defining linear transformations. (Another exercise left to the reader.)* Therefore, $T \circ U$ can be represented by an $m \times k$ matrix.

As before, let's represent T and U by matrices A and B, respectively. How can we derive the entries of the matrix which represents $T \circ U$ using the entries of matrices A and B? Let's simply go back to what matrices stand for: the jth column represents the image of $\boldsymbol{e}^j$. So the jth column of the matrix that corresponds to $T \circ U$ will be

$$(T \circ U)(\boldsymbol{e}^j) = T(U(\boldsymbol{e}^j)) = T\left(\begin{array}{c} \text{the } j\text{th column} \\ \text{of the matrix } U \end{array} \right) = T\left(\begin{bmatrix} B_{1j} \\ B_{2j} \\ B_{3j} \\ \vdots \\ B_{nj} \end{bmatrix} \right)$$

Sure, but we are not done yet. We have at the right hand side T evaluated at the vector $(B_{1j}, B_{2j}, B_{3j}, \ldots, B_{nj})$. We know how to use the matrix form of T to evaluate it at any vector.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ A_{31} & A_{32} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{bmatrix} = \begin{bmatrix} A_{11}B_{1j} + A_{12}B_{2j} + \cdots + A_{1n}B_{nj} \\ A_{21}B_{1j} + A_{22}B_{2j} + \cdots + A_{2n}B_{nj} \\ A_{31}B_{1j} + A_{32}B_{2j} + \cdots + A_{3n}B_{nj} \\ \vdots \\ A_{m1}B_{1j} + A_{m2}B_{2j} + \cdots + A_{mn}B_{nj} \end{bmatrix}$$

the jth column of the matrix representing the linear map $T \circ U$

Note that the big expression at the right hand side is just a column vector in $\mathbb{R}^m$, and writing down the matrix that corresponds to $T \circ U$ will involve writing its k columns corresponding, respectively, to the k vectors $(T \circ U)(\boldsymbol{e}^1), (T \circ T)(\boldsymbol{e}^2), \ldots, (T \circ U)(\boldsymbol{e}^k)$, respectively. We can economise on notation, and simply write the ij-entry of that matrix to describe $T \circ U$:

$$ij\text{-entry of the matrix for } T \circ U \quad = \quad A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$$

10.1 Matrix multiplication

Given an $m \times n$ matrix A and a $n \times k$ matrix B, we define the product matrix AB as the $m \times k$ matrix whose ij-entry is given by

$$(AB)_{ij} \quad = \quad A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} \qquad \text{(MM)}$$

Note that this is the same ij-entry of the matrix representing $T \circ U$. Thus the notion of matrix multiplication captures the composition of their corresponding linear transformations:

If matrices A and B represent linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^k \to \mathbb{R}^n$, respectively, then the product matrix AB represents $T \circ U : \mathbb{R}^k \to \mathbb{R}^m$.

A few rules regarding matrix multiplication

1. Matrix multiplication is **associative**: $A(BC) = (AB)C$
2. Matrix multiplication is **distributive** over matrix addition: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
3. If α is a scalar (i.e., a real number), then $\alpha(AB) = (\alpha A)B = A(\alpha B)$

Note 1: We didn't write the above rules precisely, in that, we didn't express the conditions for A, B, C to hold for the above multiplication and addition operations to be sensible. Use the space below to write those conditions (regarding the dimensions of the matrices) carefully.

Note 2: If we take matrix multiplication as a mechanical rule defined by (MM) on the previous page, verifying the associative law will be a tedious exercise. (*Give it a go if you like!*) On the other hand, an understanding of matrices as representations of linear transformations makes the associative law an almost straightforward observation. (*How so?*)

Note 3: Unlike multiplication of scalars, matrix multiplication is not commutative, in other words, AB is not necessarily equal to BA. In fact, BA might not even make sense when AB is a sensible expression. Just think about the dimensions of matrices and what they must be like for matrix multiplication to make sense. But even when we work with $n \times n$ matrices (so we can talk about both AB and BA), it is not hard to come up with examples of A and B such that $AB \neq BA$. *How so?* Again, think along linear transformations, and you might be able to find two linear transformations T and U from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T \circ U \neq U \circ T$.

10.2 Square matrices

If a matrix has the same number of rows and columns, then we call it a **square matrix**. An $n \times n$ matrix corresponds to a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$, and we can raise the question of whether that transformation has an inverse. Suppose it does, and let's denote that inverse T^{-1}. It is only natural to denote the matrix corresponding to T^{-1} by A^{-1}.

If we compose T with its inverse, we will get the identity transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ which maps $\boldsymbol{x}$ to $\boldsymbol{x}$. In particular, the identity transformation maps $\boldsymbol{e}^j$ to $\boldsymbol{e}^j$ for each $j = 1, \ldots, n$, and

therefore is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

which we will call the $n \times n$ **identity matrix**, denoted I_n.

Since composition of linear transformations correspond to multiplication of matrices, we immediately have

$$T^{-1} \circ T = T \circ T^{-1} = \text{the identity function from } \mathbb{R}^n \text{ to } \mathbb{R}^n \quad \Longrightarrow \quad A^{-1}A = AA^{-1} = I_n$$

Another word for a matrix to be invertible is **non-singular**.

A non-singular (i.e., invertible) matrix represents an one-to-one and onto linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Being a linear transformation, it maps 0 to 0. It maps lines to lines. So does its inverse.

A square matrix which is not non-singular, i.e., not invertible is called **singular**. Such a matrix cannot correspond to a one-to-one linear transformation. In particular, there must exist $\boldsymbol{v}$ and $\boldsymbol{w}$ such that

$$T(\boldsymbol{v}) = T(\boldsymbol{w}) \qquad \text{where} \quad \boldsymbol{v} \neq \boldsymbol{w}$$

and therefore

$$T(\boldsymbol{v} - \boldsymbol{w}) = 0 \qquad \text{where} \quad \boldsymbol{v} - \boldsymbol{w} \neq 0$$

Renaming the non-zero vector $\boldsymbol{v} - \boldsymbol{w}$ as $\boldsymbol{z}$, we then have

$$T(\lambda \boldsymbol{z}) = 0 \qquad \text{for all} \quad \lambda \in \mathbb{R}$$

That is, the line described by scaling the vector $\boldsymbol{z}$ is mapped to the point 0 by T.

More generally, the linear transformation which corresponds to a singular $n \times n$ matrix maps the n-dimensional space to a lower dimensional space. Projection from $\mathbb{R}^2$ to the vertical $\mathbb{R}$ is a standard example which can be represented by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For a different example, imagine $T : \mathbb{R}^n \to \mathbb{R}^n$ which maps every $\boldsymbol{e}^j$ to $\boldsymbol{e}^1$. This mapping collapses the whole n-dimensional space to a line, and can be represented by the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

10.3 Systems of 2 linear equations with 2 unknowns

Consider the following system of equations

$$\begin{aligned} ax_1 + bx_2 &= y_1 \\ cx_1 + dx_2 &= y_2 \end{aligned}$$

where a, b, c, d, y_1, y_2 are given; and x_1 and x_2 are unknowns. Does this system of equations have a solution? A key observation is that we can rephrase this question in the terminology of linear transformations:

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Is there a vector $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ such that $T(\boldsymbol{x}) = \boldsymbol{y} = (y_1, y_2)$?

In other words, given $a, b, c, d, y_1, y_2 \in \mathbb{R}$, we are wondering if there are $x_1, x_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

If the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse A^{-1}, then we can multiply the above expression by this A^{-1} to obtain:

$$\begin{aligned} A^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= A^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= A^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{aligned}$$

So solving the above system of equations is simply a matter of computing the inverse of matrix A.

Does A have an inverse?

What does it mean for A to have an inverse? It is very useful to think along linear transformations. If T is the linear transformation that corresponds to A, then we know that T maps $(1, 0)$ to (a, c), and it maps $(0, 1)$ to (b, d).

What we'd like to know is whether T maps some (x, z) to $(1, 0)$, and some (y, t) to $(0, 1)$, because if it does, then the linear transformation which maps $(1, 0)$ to (x, z), and $(0, 1)$ to (y, t) must be the inverse of T. This is the case where T maps $\mathbb{R}^2$ onto $\mathbb{R}^2$, i.e., the two-dimensional plane to two-dimensional plane in a one-to-one fashion.

Failing that, T must be mapping the two-dimensional plane to either a one-dimensional line, or to the point 0 alone. The latter case corresponds to A being the zero matrix: the matrix with all zero entries. The case where T maps the plane to a line is where $T(1, 0)$ and $T(0, 1)$ are scalar multiples of each other. In other words

$$(a, c) = \lambda(b, d) \quad \text{for some} \quad \lambda$$

which means

$$ad - bc = 0$$

Thus we have a simple way of checking whether A is non-singular or not:

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.

If A is not invertible, the system of equations will not have a solution (x_1, x_2) for every (y_1, y_2).

10.4 Gaussian elimination

Now we explain an algorithm (a step-by-step procedure) to simplify systems of linear equations to find their solution(s). At each step, we will convert the system of equations we have at hand to another system of equations which appears easier to solve than the previous one. Our procedure will eventually take us to a final system of equations where the first equation is $x_1 = a_1$, the second equation is $x_2 = a_2$, and so on. In other words, the final system of equations will simply tell us the solution. Since each system of linear equations admits a matrix representation, we will translate the steps of our procedure to steps for matrices. The tidy nature of matrices will make the procedure more efficient to carry out in matrix notation.

Let's illustrate the procedure in an example. Suppose we are given three linear equations with three unknowns:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - 2x_3 &= 2 \end{aligned}$$

Below, we have the equations on the left hand side, their matrix formulation in the middle. At the right hand side, we describe what we do with these equations to get to the next set of equations.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 - 2x_3 &= 2 \end{aligned} \quad \left\| \quad \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} \right\| \quad \begin{array}{l} \textbf{Step 1:} \\ \text{Subtract } (3 \times \text{Eq1}) \text{ from Eq2} \\ \text{Subtract } (2 \times \text{Eq1}) \text{ from Eq3} \end{array}$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 6 \\ 7x_2 - 10x_3 &= -16 \\ 7x_2 - 8x_3 &= -10 \end{aligned} \quad \left\| \quad \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -10 \\ 0 & 7 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -16 \\ -10 \end{bmatrix} \right\| \quad \begin{array}{l} \textbf{Step 2:} \\ \text{Subtract Eq2 from Eq3} \end{array}$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 6 \\ 7x_2 - 10x_3 &= -16 \\ 2x_3 &= 6 \end{aligned} \quad \left\| \quad \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -10 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -16 \\ 6 \end{bmatrix} \right\| \quad \begin{array}{l} \textbf{Step 3:} \\ \text{Divide Eq3 by 2} \end{array}$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 6 \\ 7x_2 - 10x_3 &= -16 \\ x_3 &= 3 \end{aligned} \quad \left\| \quad \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -16 \\ 3 \end{bmatrix} \right\| \quad \begin{array}{l} \textbf{Step 4:} \\ \text{Subtract } (3 \times \text{Eq3}) \text{ from Eq 1} \\ \text{Add } (10 \times \text{Eq3}) \text{ to Eq 2} \end{array}$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 7x_2 &= 14 \\ x_3 &= 3 \end{aligned} \quad \left\| \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 14 \\ 3 \end{bmatrix} \right\| \quad \begin{array}{l} \textbf{Step 5:} \\ \text{Divide Eq2 by 7} \end{array}$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned} \quad \left\| \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} \right\| \quad \begin{array}{l} \textbf{Step 6:} \\ \text{Add } (2 \times \text{Eq2}) \text{ to Eq1} \end{array}$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned} \quad \left\| \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|$$

To summarise the above procedure, we employed simple algebraic operations of *adding* (or *subtracting*) and *re-scaling* on the system of equations which eventually simplified them to reveal the solution (x_1, x_2, x_3). The corresponding operations on the matrices are **row adding** and **row re-scaling**. We could have swapped the order in which present the equations, and that would correspond to **row swapping** in the matrix. These three operations on matrices are called **elementary row operations**.

Of course there are many ways of employing these operations to reach the solution. Here's one possible guide: If necessary, swap rows to make sure the top left entry is non-zero. Then add

(positive or negative) multiples of this row to clear (i.e., make zero) all other entries on the first column. Then, if necessary, swap among rows 2 to n to make sure the second diagonal entry is non-zero. Next, add (positive or negative) multiples of this row to clear (i.e., make zero) all other entries on the second column. Continue until you reach a diagonal matrix. Finally, rescale each row to make sure each diagonal entry is 1. If this process does not yield a non-zero entry for each diagonal entry, then the system of equations cannot be guaranteed to have a solution. This also means the original matrix is not invertible. If it is invertible, then we will end up with the identity matrix following the above procedure named after Gauss.

10.5 Computing the inverse of a non-singular matrix

Gaussian elimination does more than answering the question of whether the matrix we begin with is invertible or not. It can be used to compute the inverse matrix if it exists. To see how, let's rewrite the matrix equation $A\boldsymbol{x} = \boldsymbol{y}$ by inserting the $n \times n$ identity matrix I_n in front of the column matrix $\boldsymbol{y}$:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & & \vdots & \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

Now, each elementary row operation we apply to A can be thought as a linear transformation. So, if we list these operations as $r_1, r_2, \ldots, r_k$ in the order we carry them out, and if we denote the corresponding linear transformations by $R_1, R_2, \ldots, R_k$, respectively, then the effect of carrying out this sequence is nothing but computing the composition

$$R_k \circ R_{k-1} \circ \cdots R_2 \circ R_1 \circ T$$

where T is the linear transformation corresponding to the matrix A (i.e., the original set of linear equations).

These row operations applied in this order transform T into the identity mapping. Well, if we denote the matrix corresponding to R_i by B_i, we will have

$$\underbrace{B_k B_{k-1} \cdots B_1 A}_{\text{we know this is } I_n} \boldsymbol{x} \quad = \quad \underbrace{B_k B_{k-1} \cdots B_1 I_n}_{\text{hence this must be } A^{-1}} \boldsymbol{y}$$

Therefore, carrying out the same elementary row operations $r_1, r_2, \ldots, r_k$ on the $n \times n$ identity matrix I, we will obtain the inverse matrix A^{-1}.

A tidy way of computing the inverse of matrix is to begin with

$$\left[\begin{array}{ccc:ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

and to carry out the elementary row operations as we did on the previous page to transform the left half of the above matrix into the 3×3 identity matrix. As we do so, the right half of the above matrix will be transformed into A^{-1}.

Cramer's method

Another systematic procedure (algorithm) to compute the inverse of a non-singular matrix is the so-called *Cramer's method*. In terms of its computational requirements, it is probably not simpler nor faster than the above method. Arguably, it is less intuitive since the procedure does not make it clear why it should indeed lead to inverting a matrix. Consult Sydsaeter & Hammond or any other textbook for the details of Cramer's method, feel free to adopt it if you find it an attractive method to solve matrix inversion problems.

Example: Inverting a 2×2 non-singular matrix

$$\left[\begin{array}{cc:cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

Assuming $a \neq 0$, add $-c/a$ times row 1 to row 2, and then rescale row 1 by $1/a$:

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

Rescale row 2 by $\frac{a}{ad-bc}$:

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Add $-b/a$ times row 2 to row 1:

$$\left[\begin{array}{cc:cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Thus, as long as $ad - bc \neq 0$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Note that if $a = 0$, then the condition $ad - bc \neq 0$ implies $c \neq 0$ and $b \neq 0$, and we would begin the elementary row operations by swapping rows first, so the top left entry is non-zero, and proceed in the same fashion.

10.6 Determinant

The determinant of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is denoted by} \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and is equal to} \quad ad - bc.$$

As we can see from the example above, for this 2×2 matrix to be invertible it is necessary and sufficient for its determinant to be non-zero. In that case, we can write it inverse as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The determinant of a matrices with higher dimensions is much more complicated to define. For most purposes, we will not need the concept of a determinant, and a proper treatment of determinants in general is beyond the scope of this course. For those who are curious, we list below a number key facts about determinants.

10.7 Rules for determinants

1. The determinant of a singular matrix is 0. The determinant of a non-singular matrix is non-zero.
2. Given any two square matrices A and B of the same dimension, $\det(AB) = \det(A)\det(B)$.
3. Swapping two rows of a matrix changes the sign of the determinant. That is, if B is obtained from A by swapping two rows of A, then $\det(B) = -\det(A)$.
4. Given a matrix A and a scalar λ, if B is obtained from A by multiplying one row of A by λ, then $\det(B) = \lambda\det(A)$.
5. Given a matrix A, if B is obtained from A by adding one row of A to another row, then $\det(B) = \det(A)$.
6. Given a matrix A and a scalar λ, if B is obtained from A by adding λ times one row of A to another row, then $\det(B) = \det(A)$. (*Can you derive this rule from the previous two rules?*)
7. The determinant of a diagonal matrix (i.e., a square matrix whose entries off the diagonal are all zero) is the product of the diagonal entries:

$$\det\begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 \\ 0 & 0 & A_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{nn} \end{bmatrix} = A_{11} \times A_{22} \times \cdots \times A_{nn}$$

   In particular, $\det(I_n) = 1$, where I_n is the identity matrix.
8. The determinant of a triangular matrix is the product of the diagonal entries:

$$\begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{bmatrix} = A_{11} \times A_{22} \times \cdots \times A_{nn}$$

   and

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ 0 & A_{22} & A_{23} & \cdots & A_{2n} \\ 0 & 0 & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{nn} \end{bmatrix} = A_{11} \times A_{22} \times \cdots \times A_{nn}$$

   (*Can you derive this rule from the previous rules?)*
9. Finally, taking advantage of the above rules, we can use Gaussian elimination to compute the determinant of any square matrix.

   Going back to the example through which we introduced Gaussian elimination, recall

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \\ 2 & 3 & -2 \end{bmatrix}$$

   Think of Steps 1 to 6 as applying transformations T_1 to T_6, with the associated matrices B_1 to B_6 so

   $B_6B_5B_4B_3B_2B_1A = I_3$ and $\det B_6 \det B_5 \det B_4 \det B_3 \det B_2 \det B_1 \det A = \det I_3 = 1$

   Only two of the operations $B_1, \ldots, B_6$ would change the determinant: dividing by 2 and 7 in Steps 3 and 5, respectively, so $\frac{1}{14}\det A = 1$, and therefore $\det A = 14$.

10.8 Positive/Negative Definiteness of a 2×2 Matrix

Studying what it means for a square matrix to be definite (or semi-definite) in arbitrary dimensions is beyond the scope of this course. Instead, we will restrict our attention to 2×2 matrices.

Positive Definite. We say a 2×2 matrix A is **positive definite** if

$$\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} > 0 \qquad \text{for all} \quad (h_1, h_2) \neq 0 \in \mathbb{R}^2$$

Positive Semi-Definite. We say a 2×2 matrix A is **positive semi-definite** if

$$\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \geq 0 \qquad \text{for all} \quad (h_1, h_2) \in \mathbb{R}^2$$

Negative Definite. We say a 2×2 matrix A is **negative definite** if

$$\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} < 0 \qquad \text{for all} \quad (h_1, h_2) \neq 0 \in \mathbb{R}^2$$

Negative Semi-Definite. We say a 2×2 matrix A is **negative semi-definite** if

$$\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \leq 0 \qquad \text{for all} \quad (h_1, h_2) \in \mathbb{R}^2$$

Now remember that we defined the Hessian of a twice-partially differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $\boldsymbol{a}$ as

$$\begin{bmatrix} f_{11}(\boldsymbol{a}) & f_{12}(\boldsymbol{a}) \\ f_{21}(\boldsymbol{a}) & f_{22}(\boldsymbol{a}) \end{bmatrix}$$

Convexity/concavity of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is closely related to its Hessian:

Theorem. Let f have continuous second partial derivatives on an open, convex set $K \subseteq \mathbb{R}^2$.

1. f is convex on K iff its Hessian matrix is positive semi-definite for all $\boldsymbol{x} \in K$.
2. If its Hessian matrix is positive definite for all $\boldsymbol{x} \in K$, then f is strictly convex on K.
3. f is concave on K iff its Hessian matrix is negative semi-definite for all $\boldsymbol{x} \in K$.
4. If its Hessian matrix is negative definite for all $\boldsymbol{x} \in K$, then f is strictly concave on K.

The second order condition (SOC) at a critical (stationary) point of f:

Theorem. Let $\boldsymbol{a}$ be a critical point of $f : \mathbb{R}^2 \to \mathbb{R}$, that is, suppose $f_1(\boldsymbol{a}) = f_2(\boldsymbol{a}) = 0$. If the second partial derivatives of f are continuous (so Young's theorem holds), then we have the following SOC:

1. if $\boldsymbol{a}$ is a local max, then the Hessian at $\boldsymbol{a}$ is negative semi-definite;
2. if the Hessian at $\boldsymbol{a}$ is negative definite, then $\boldsymbol{a}$ is a local max;
3. if $\boldsymbol{a}$ is a local min, then the Hessian at $\boldsymbol{a}$ is positive semi-definite;
4. if the Hessian at $\boldsymbol{a}$ is positive definite, then $\boldsymbol{a}$ is a local min.

10.9 Checking whether the Hessian is definite or not

The following result makes it fairly easy to determine if a 2×2 matrix is positive/negative definite or not.

Given a twice-continuously partial differentiable $f : \mathbb{R}^2 \to \mathbb{R}$, its Hessian matrix

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}$$

is **definite** if and only if $f_{11}f_{22} - (f_{12})^2 > 0$. If, in addition, $f_{11} > 0$, then it is positive definite; whereas if $f_{11} < 0$, then it is negative definite. By the way, note that if the matrix is definite, f_{11} and f_{22} has the same sign. If the inequalities are not strict, then the matrix is semi-definite.

Example

Consider the problem of identifying minima and maxima of the function $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$ defined as

$$f(x, y) = x^3 y + \frac{2}{3} \ln x + y^2 - xy$$

Note that all points of the domain of the function are interior points[9], therefore the solutions to our problem, if they exist,[10] will need to satisfy the first order conditions. In other words, any local minimum or maximum must also be a solution to the following system of equations:

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0$$

that is,

$$3x^2 y + \frac{2}{3x} - y = 0 \qquad \text{and} \qquad x^3 + 2y - x = 0$$

These are not linear equations, and therefore we cannot appeal to the linear algebraic techniques we have developed earlier. Instead, we will first use the second equation to write $y = (x - x^3)/2$, and plug this in the first equation to obtain

$$3x^2 \left(\frac{x - x^3}{2} \right) + \frac{2}{3x} - \frac{x - x^3}{2} = 0$$

Multiplying this by $6x$, we get

$$9x^3(x - x^3) + 4 - 3x(x - x^3) = 12x^4 - 9x^6 + 4 - 3x^2 = 0$$

which simplifies to

$$3x^4(4 - 3x^2) + 4 - 3x^2 \quad = \quad (3x^4 + 1)(4 - 3x^2) \quad = \quad 0$$

There are only two solutions to the above equation: $x = 2/\sqrt{3}$ or $x = -2/\sqrt{3}$. Since the domain of f is $\mathbb{R}_{>0} \times \mathbb{R}$, we need to have $x > 0$. Hence the only feasible solution is $x^* = 2/\sqrt{3}$ which implies

$$y^* = \frac{x - x^3}{2} = \frac{x}{2}(1 - x^2) = \frac{1}{\sqrt{3}} \left(1 - \frac{4}{3} \right) = -\frac{1}{3\sqrt{3}}$$

In order to check the SOC, we take the second partial derivatives

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x} \left(3x^2 y + \frac{2}{3x} - y \right) = 6xy - \frac{2}{3x^2} \\ f_{xy} &= \frac{\partial}{\partial y} \left(3x^2 y + \frac{2}{3x} - y \right) = 3x^2 - 1 \\ f_{yy} &= \frac{\partial}{\partial y} \left(x^3 + 2y - x \right) = 2 \end{aligned}$$

[9] Remember that a point P is an interior point of a domain D if there is a neighbourhood of P which is contained in the domain.

[10] The domain of f is not compact, so we are not guaranteed that f will attain its minimum or maximum values.

which yields the Hessian as

$$H(x,y) = \begin{bmatrix} 6xy - \frac{2}{3x^2} & 3x^2 - 1 \\ 3x^2 - 1 & 2 \end{bmatrix}$$

At the point $(x^*, y^*) = (2/\sqrt{3}, -1/3\sqrt{3})$

$$H(2/\sqrt{3}, -1/3\sqrt{3}) \quad = \quad \begin{bmatrix} -11/6 & 3 \\ 3 & 2 \end{bmatrix}$$

which is not semi-definite, so $(x^*, y^*) = (2/\sqrt{3}, -1/3\sqrt{3})$ is neither a local max nor a local min. Thus it is a saddle point.

Example. Determine the convexity/concavity of $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = x^4 - 6x^3 + 12x^2 + y^2 + 6y + 9$$

Solution. One approach is to write the Hessian matrix and examine where it is positive/negative definite.

$$\begin{aligned} f_{xx} &= 12x^2 - 36x + 24 \\ f_{xy} = f_{yx} &= 0 \\ f_{yy} &= 2 \end{aligned}$$

One diagonal entry, namely f_{yy}, is positive. Therefore, when the determinant is positive over a part of the domain, then we can conclude that f is convex over that region. If, however, the determinant is negative over some other part of the domain, then we conclude that f is neither convex or concave over that part of the domain. The determinant of the Hessian is $2f_{xx}$:

$$24(x^2 - 3x + 2) \quad = \quad 24(x-1)(x-2)$$

which is negative if and only if $1 < x < 2$. So, we conclude that

if $x < 1$	then $f(x,y)$ is strictly convex
if $1 < x < 2$	then the determinant of the Hessian is negative
if $2 < x$	then $f(x,y)$ is strictly convex

Example. Let $f(x,y) = (y - x^2)(y - 2x^2)$. Fixing an arbitrary (h,k), we can define a single-variable function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(t) \quad = \quad f(th, tk)$$

First, note that $g(0) = f(0,0) = 0$. Now verify the following:

(a) The function g admits a local minimum at the point $t = 0$.

(b) The function f does not admit a local minimum at the point $(0,0)$.

(c) Offer an explanation as to why (a) does not imply $(0,0)$ being a local minimum for f.

Last note on notation: transpose

The **transpose** of an $m \times n$ matrix A is defined as the $n \times m$ matrix denoted by $A^\intercal$ whose ij-entry is the ji-entry of A, that is

$$A^\intercal = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ A_{13} & A_{23} & \cdots & A_{m3} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{mn} \end{bmatrix}$$

row i of A becomes column i of $A^\intercal$
column j of A becomes row j of $A^\intercal$

11 The envelope theorems

Recommended reading and exercises: Chapter 13.7 (*Comparative Statics and the Envelope Theorem*) and Chapter 14.7 (*Comparative Statics*) of Sydsaeter & Hammond (4th Edition).

11.1 The envelope theorem for unconstrained optimisation

Consider the unconstrained optimisation problem

$$\max_x f(x;\alpha)$$

where α is an exogenous parameter, i.e., a variable of f, but not a choice variable of the optimisation problem. Therefore, solving the above problem is solving for optimal x for a given α. Hence, we will denote the solution by $x^*(\alpha)$ to keep in mind that it indeed depends on the value of the parameter α. Let us denote by $v(\alpha)$ the achieved value of the function when the maximisation problem is solved for a given α.

If the maximiser $x^*(\alpha)$ is differentiable, then:

$$\frac{dv}{d\alpha} = \left.\frac{\partial f}{\partial \alpha}\right|_{x=x^*(\alpha)}$$

> The rate of change of the optimal value with respect to the exogenous parameter
> =
> The rate of change of the objective function with respect to the parameter, evaluated at the optimal solution

Proof. Given parameter α, the optimal value of the objective function is

$$v(\alpha) = f(x^*(\alpha);\alpha)$$

where $x^*(\alpha)$ is the solution to

$$\max_x f(x;\alpha)$$

Differentiating $v(\alpha)$ using the Chain Rule

$$\frac{dv}{d\alpha} = \left.\frac{\partial f}{\partial x}\right|_{x=x^*} \frac{dx^*}{d\alpha} + \left.\frac{\partial f}{\partial \alpha}\right|_{x=x^*} \frac{d\alpha}{d\alpha}$$

We know that the FOC is satisfied at the solution, that is,

$$\left.\frac{\partial f}{\partial x}\right|_{x=x^*} = 0,$$

and $d\alpha/d\alpha = 1$, hence the expression for $dv/d\alpha$ simplifies to

$$\frac{dv}{d\alpha} = \left.\frac{\partial f}{\partial \alpha}\right|_{x=x^*}$$

□

An example

A firm's production function is

$$Q(L) = AL^2 - BL^3$$

where A and B are positive constants, and L denotes the number of workers with $L \leq A/B$.

(a) Find the size of workforce L^* that maximises output $Q(L)$. Find the maximum output Q^*.

(b) Suppose $A = 12$ and $B = 1/20$. If A increases by 0.1, what is the change in the firm's output?

(c) In order to answer the last question, calculate dQ^*/dA, and obtain an approximate value for the change in the firm's maximum output. Compare this answer with your previous answer.

(d) Compute, without using a calculator, an approximate value of the change in maximum output when B increases by 0.001.

Solution.

(a) L^* will either be a corner solution (0 or A/B) or it will have to satisfy the FOC. It is easy to verify that the corners yield zero output. Let's look at the FOC then, that is, let's solve $Q'(L) = 0$, which implies

$$Q'(L) \quad = \quad 2AL - 3BL^2 \quad = \quad 0$$

which has a unique non-zero solution $L^* = \frac{2A}{3B}$. We should now check that this choice of L results in positive production (to make sure it beats the zero output obtained at the corners).

We can thus compute the maximum output of the firm (as a function of the parameters A and B):

$$Q^*(A,B) \quad = \quad A\left(\frac{2A}{3B}\right)^2 - B\left(\frac{2A}{3B}\right)^3 \quad = \quad \left(\frac{2A}{3B}\right)^2\left(A - \frac{2A}{3}\right) \quad = \quad \frac{4A^3}{27B^2}$$

(b) Plugging in $A = 12$ and $B = 1/20$ would yield

$$L^*(12, 1/20) = 160 \quad \text{and} \quad Q^*(12, 1/20) = 102400$$

If A were to increase to 12.1, the new maximum output $Q^*(12.1, 1/20)$ can be computed with a calculator to be ≈ 104981, and thus the change in maximum output is

$$Q^*(12.1, 1/20) - Q^*(12, 1/20) \quad \approx \quad 2581$$

(The actual change is somewhere between 2581 and 2582.)

(c) On the margin, increasing A changes Q^* at a rate of dQ^*/dA which can be computed as

$$\frac{dQ^*}{dA} \quad = \quad \frac{d}{dA}\left(\frac{4A^3}{27B^2}\right) \quad = \quad \frac{4A^2}{9B^2}$$

To the extent that 0.1 is small enough for the purposes of approximation, the effect on maximum output of increasing A by 0.1 will be approximately

$$0.1 \quad \times \quad \left.\frac{dQ^*}{dA}\right|_{A=12,B=1/20} \quad = \quad 0.1 \times \frac{4 \times 12^2}{9 \times (1/20)^2} \quad = \quad 2560$$

Comparing this with our answer above, we can confirm that the linear approximation (based on the first derivative) is not too far off the real value of the change.

(d) The whole exercise of studying dQ^*/dB can be carried out without having to calculate the function $Q^*(A, B)$ if we remember the envelope theorem which states: *The rate of change of the optimal value with respect to the exogenous parameter is equal to the rate of change of the objective function with respect to the parameter, evaluated at the optimal solution.*

$$\left.\frac{dQ^*}{dB}\right|_{A=12,B=1/20} \quad = \quad \left.\frac{\partial Q(L,A,B)}{\partial B}\right|_{\substack{L=L^*(12,1/20)\\A=12,B=1/20}} \quad = \quad -\left.L^3\right|_{\substack{L=160\\A=12,B=1/20}} \quad = \quad -2^{12} \times 10^3$$

and hence the change in maximum output is approximately $0.001 \times (-2^{12} \times 10^3) \; = \; -4096$.

11.2 The envelope theorem for constrained optimisation

Consider the constrained optimisation problem

$$\max_x f(x;\alpha) \qquad \text{s.t.} \quad g(x;\alpha)=0$$

Once again, α is not a choice variable. From the perspective of solving the above problem, α is exogenously fixed. Different values of α correspond to different optimisation problems, and we denote by $x^*(\alpha)$ the solution that corresponds to α. Note that as α changes both the objective function and the constraint might change. Again, we denote by $v(\alpha)$ the achieved value of the function when the maximisation problem is solved for a given α.

If the maximiser $x^*(\alpha)$ is differentiable, then:

$$\frac{dv}{d\alpha}=\left.\frac{\partial \mathcal{L}}{\partial \alpha}\right|_{\substack{x=x^*\\ \lambda=\lambda^*}}$$

where $\mathcal{L}(x,\lambda,\alpha)=f(x;\alpha)+\lambda g(x;\alpha)$.

> The rate of change of the optimal value with respect to the parameter
> =
> The rate of change of the Lagrangian function with respect to the parameter, evaluated at the optimal solution

Proof. Given parameter α, the optimal value of the objective function is

$$v(\alpha)=f(x^*(\alpha);\alpha)$$

where $x^*(\alpha)$ is the solution to

$$\max_x f(x;\alpha) \qquad \text{subject to} \quad g(x;\alpha)=0$$

Lagrange's theorem says that there must be a $\lambda^*(\alpha)$ such that $(x^*(\alpha),\lambda^*(\alpha))$ satisfy the FOC associated with

$$\mathcal{L}(x,\lambda,\alpha)=f(x;\alpha)+\lambda g(x;\alpha)$$

that is,

$$\mathcal{L}_x \;=\; \mathcal{L}_\lambda \;=\; 0 \qquad \text{at} \;\; (x^*(\alpha),\lambda^*(\alpha),\alpha)$$

Evaluating $\mathcal{L}$ at $(x^*(\alpha),\lambda^*(\alpha),\alpha)$

$$\begin{aligned}\mathcal{L}(x^*(\alpha),\lambda^*(\alpha),\alpha) &= f(x^*(\alpha),\alpha)+\lambda^*(\alpha)g(x^*(\alpha),\alpha)\\ &= v(\alpha)\end{aligned}$$

since $v(\alpha)=f(x^*(\alpha),\alpha)$ and $x^*(\alpha)$ is known to satisfy the constraint $g(x;\alpha)=0$.

Differentiating $v(\alpha)$ using the Chain Rule

$$\frac{dv}{d\alpha}=\frac{d}{d\alpha}\mathcal{L}(x^*(\alpha),\lambda^*(\alpha),\alpha)=\left.\frac{\partial \mathcal{L}}{\partial x}\right|_{\substack{x=x^*\\ \lambda=\lambda^*}}\frac{dx^*}{d\alpha}+\left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{\substack{x=x^*\\ \lambda=\lambda^*}}\frac{d\lambda^*}{d\alpha}+\left.\frac{\partial \mathcal{L}}{\partial \alpha}\right|_{\substack{x=x^*\\ \lambda=\lambda^*}}\frac{d\alpha}{d\alpha}$$

We know that the FOC is satisfied for $\mathcal{L}$ at $(x^*(\alpha),\lambda^*(\alpha))$, that is,

$$\left.\frac{\partial \mathcal{L}}{\partial x}\right|_{\substack{x=x^*\\ \lambda=\lambda^*}}=0 \quad \text{and} \quad \left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{\substack{x=x^*\\ \lambda=\lambda^*}}=0$$

Moreover $d\alpha/d\alpha=1$, hence the expression for $dv/d\alpha$ simplifies to

$$\frac{dv}{d\alpha}=\left.\frac{\partial \mathcal{L}}{\partial \alpha}\right|_{\substack{x=x^*\\ \lambda=\lambda^*}}$$

□

An example

A profit-maximising firm uses labour, L and capital, K, to produce output, Q, according to the production function:

$$Q = L^{0.5} + K^{0.5}$$

The firm pays a wage rate w for labour and a rental rate r for capital. The unit price of the output is p.

(a) What is the firm's level of output if $(w, r, p) = (1, 3, 60)$?

(b) Suppose the firm can sell no more than 20 units of output at the price of 60. Write down the Lagrangian function for the firm's profit maximisation problem subject to the constraint on output sold. Calculate the value of the firm's profit-maximising output, and the value of the Lagrange multiplier. Offer an interpretation for the value of the Lagrangian multiplier.

(c) What is the marginal cost of producing an extra unit at $Q = 20$?

(d) Suppose the firm can sell a small amount of additional units of output in the "black market" (but can sell no more than 20 units in the "official" economy). What price would the firm be willing to accept for these additional "black market" units of output?

Solution sketch.

(a) Unconstrained optimal output is 40.

(b) Since the unique unconstrained solution violates the constraint $Q \leq 20$, the constraint must be binding, and therefore the constrained optimal output is 20. The Lagrangian for profit maximisation subject to output being 20 is

$$\mathcal{L}(L, K, \lambda, w, r, p) = (L^{0.5} + K^{0.5})p - Lw - Kr + \lambda(20 - L^{0.5} - K^{0.5})$$

which have the FOCs given as $\mathcal{L}_L = \mathcal{L}_K = \mathcal{L}_\lambda = 0$.

When $(w, r, p) = (1, 3, 60)$, the FOC lead to $\lambda^* = 30$. This is the price the firm is willing to pay to relax the output constraint by a unit. So the marginal profit at the point the firm is optimally producing (subject to the constraint) is 30.

(c) The price of the output is 60, so the marginal cost is 60 minus the marginal profit, thus 30.

(d) Any price higher than 30 for the first marginal unit. But note that the production function is DRS, so the AC is increasing. The more units to be sold, the higher the price the firm would charge.

Multiple exogenous parameters

We might have multiple endogenous variables and exogenous parameters that feature in an optimisation problem as in

$$\max_{x_1,\ldots,x_n} \; f(x_1,\ldots,x_n;\alpha_1,\ldots,\alpha_m) \qquad \text{s.t.} \quad g(x_1,\ldots,x_n;\alpha_1,\ldots,\alpha_m)=0$$

Here, the agent faces the exogenous parameters $\alpha_1,\ldots,\alpha_m$, and chooses $x_1,\ldots,x_n$ to solve the above maximisation problem. This choice depends on $\alpha_1,\ldots,\alpha_m$. If there is a unique solution to the optimisation problem, we can talk about the optimal choice as a function of the parameters, and denote it as $\boldsymbol{x}^*(\boldsymbol{\alpha}) = (x_1^*(\alpha_1,\ldots,\alpha_m),\ldots,x_n^*(\alpha_1,\ldots,\alpha_m))$.

As before, let's denote by $v(\alpha_1,\ldots,\alpha_m)$ the optimal value achieved by the agent when he faces the exogenous parameters $\alpha_1,\ldots,\alpha_m$. That is

$$v(\alpha_1,\ldots,\alpha_m) \;=\; f(x_1^*(\alpha_1,\ldots,\alpha_m),\ldots,x_n^*(\alpha_1,\ldots,\alpha_m);\alpha_1,\ldots,\alpha_m)$$

The envelope theorem is about the rate of change of v with respect to the exogenous parameters $\alpha_1,\ldots,\alpha_m$, looking at the effect of a change in one parameter at a time.

$$\frac{\partial v}{d\alpha_i} \;=\; \left.\frac{\partial \mathcal{L}}{\partial \alpha_i}\right|_{\boldsymbol{x}=\boldsymbol{x}^*(\boldsymbol{\alpha}),\lambda=\lambda^*(\boldsymbol{\alpha})}$$

where

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{\alpha},\lambda) \;=\; f(x_1,\ldots,x_n;\alpha_1,\ldots,\alpha_m)+\lambda g(x_1,\ldots,x_n;\alpha_1,\ldots,\alpha_m)$$

and $x_1^*(\alpha_1,\ldots,\alpha_m),\ldots,x_n^*(\alpha_1,\ldots,\alpha_m),\lambda^*(\alpha_1,\ldots,\alpha_m)$ satisfy the FOC for $\mathcal{L}$.

12 Series

Recall that a real-valued sequence is nothing but a function $a : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ stands for the set of natural numbers. Since the standard notation for sequences uses subindices instead of parentheses, we will write a_n instead of $a(n)$. Moreover, we will refer to a_n as the nth term of the sequence (a_n) instead of calling it the value of the function a at the point n.

Associated with the notion of a real-valued sequence (a_n), we sometimes would like to talk about the sum of its terms. But there are infinitely many terms, and it is not obvious how we add up infinitely many numbers in a meaningful way. In fact an attempt to add up all terms of a sequence will certainly fail at times. Take, for example, a constant sequence whose terms are all equal to 1. How can we talk about adding up infinitely many ones and expect to get a number as an answer?

Before we jump to any major conclusion, let's think how we may try to conceptualise a notion of adding up infinitely many terms of a sequence. Remember that *infinity*, to begin with, was a property of a set. Namely, the property of not being finite, not ever ending when we tried to count the elements of a set. But then we talked about the limit of a sequence (s_n) as n approaches to infinity. There is no such thing as s_∞, but if the sequence (s_n) is convergent, then there is such a thing as $\lim_{n\to\infty} s_n$.

Ah, maybe we should formalise what we mean by "adding up infinitely many numbers" as a "limit of adding up finitely many numbers". The terms of the sequence are neatly indexed and it feels only natural to list them beginning from the lowest index to higher indices

$$a_0, a_1, a_2, a_3, \ldots$$

so perhaps we have in mind a summation which looks like

$$a_0 + a_1 + a_2 + a_3 + \cdots$$

Of course, this last expression doesn't make sense yet since it suggests adding up infinitely many terms. On the other hand, it also suggests which "finite groups of numbers" we might begin thinking about when we are trying to make sense of an "infinite sum". Let's do the summation step-by-step, adding one term at a time, going from left to right. And let's denote the answer to step k by S_k so we have

$$\begin{aligned}
S_0 &= a_0 \\
S_1 &= a_0 + a_1 \\
S_2 &= a_0 + a_1 + a_2 \\
S_3 &= a_0 + a_1 + a_2 + a_3 \\
&\vdots \quad \vdots \quad \vdots \\
S_n &= a_0 + a_1 + a_2 + a_3 + \cdots + a_n
\end{aligned}$$

As n gets bigger and bigger, we have more and more terms added up, and we will think of the desired summation of all terms of (a_n) as the limit of these finite sums. We will refer to

$$\sum_{i=0}^{\infty} a_i = a_0 + a_1 + a_2 + a_3 + \cdots$$

as the **series** whose nth term is a_n, and whose nth partial sum is S_n.

Clearly (S_n) is a sequence on its own account, and we can talk about the limit of S_n as n approaches to infinity. This is what the series is meant to capture. If S_n converges to L, then we say the series $\sum_{i=0}^{\infty} a_i$ converges to L. If S_n diverges to ∞, we say the series $\sum_{i=0}^{\infty} a_i$ diverges to ∞. And so on.

For example, if $a_i = 1$ for all $i = 1, 2, \ldots$, then $\sum a_i$ diverges to ∞. If $b_i = (-1)^i$, then $\sum b_i$ diverges because its partial sums alternate between 1 and 0. Does a series ever converge? Well, if $z_i = 0$, then $\sum z_i$ converges to 0. Sure, but is there any series which involves adding up non-zero terms and still converges?

12.1 Geometric sequences and series

A geometric sequence is one where the consecutive terms have a fixed ratio, that is, when there exists a constant c such that

$$\text{for every } n \in \mathbb{N}, \text{ we have } \quad \frac{a_{n+1}}{a_n} = c$$

The sort of expressions which relate the $n+1$st term of a sequence to its nth (and perhaps a few earlier terms) are sometimes referred to as a *recursive expression*. For example, we say the expression $a_{n+1} = ca_n$ describes the sequence (a_n) recursively. If we also know the value of a_0, then we can figure out every term of the sequence.[11] In order to see why, simply observe that

$$\begin{aligned} a_1 &= c \times a_0 \\ a_2 &= c \times a_1 = c \times c \times a_0 = c^2 \times a_0 \\ a_3 &= c \times a_2 = c \times c^2 \times a_0 = c^3 \times a_0 \\ &\vdots \quad\quad \vdots \\ a_n &= c^n \times a_0 \end{aligned}$$

One conclusion of the above argument also provides us with an alternative definition of what it means for a sequence to be a geometric sequence. A geometric sequence is one whose terms can be listed as:

$$a, ac, ac^2, ac^3, \ldots$$

Convergence properties of a geometric sequence are fairly straightforward to explore, and they depend on the value of c. Take for example the geometric sequence given by $a_n = ac^n$, where $a \neq 0$.

- $a_n \to 0$ if $|c| < 1$
- $a_n \to a$ if $c = 1$
- a_n diverges if $c = -1$. More specifically, it fluctuates between a and $-a$.
- a_n diverges to ∞ if $c > 1$
- a_n diverges if $c < -1$, with its magnitude growing without bound, but its sign fluctuating between positive and negative.

A **geometric series** is a series associated with a geometric sequence, and hence looks like

$$\sum_{i=0}^{\infty} ac^i$$

Here are two quick observations regarding the partial sums of this geometric series:

$$S_{n+1} = S_n + ac^{n+1} \qquad \text{and} \qquad S_{n+1} = cS_n + a$$

which imply

$$S_n + ac^{n+1} \;=\; cS_n + a$$

When $c \neq 1$, this last equation allows us to solve for S_n:

$$S_n = a\frac{1 - c^{n+1}}{1 - c}$$

That is

[11] In general, one might have more complicated recursive expressions. For example, a very famous example is the Fibonacci sequence which begins as $F_0 = 0$ and $F_1 = 1$, and then described by the recursive expression $F_{n+2} = F_{n+1} + F_n$.

If $c \neq 1$, then $\sum_{i=0}^{n} ac^i = a + ac + ac^2 + \cdots + ac^n = a\frac{1-c^{n+1}}{1-c}$.

But we know that if $|c| < 1$, then $c^n \to 0$ as $n \to \infty$. Thus we have the following result:

If $|c| < 1$, then the geometric series $\sum_{i=0}^{\infty} ac^i$ converges to $\frac{a}{1-c}$.

12.2 Some applications

Geometric sequences naturally emerge in the context of compound interest. For instance, suppose an amount a is deposited into a savings account which promises an annual interest rate of $r \in [0, 1]$. (By the way, the banks cite the number $100r$ and use the word "percent". So, what the bank calls 5% means a rate of 0.05.) If this rate stays constant over n years, and if the interest earned every year is simply added to the capital, how much would the total yield be? Denoting c_i with the total yield after i years, we know that

$$c_0 = a \qquad \text{and} \qquad c_{i+1} = \underbrace{c_i}_{\text{yield after } i \text{ years}} + \underbrace{rc_i}_{\text{interest earned in year } i+1} = (1+r)c_i$$

which implies

$$c_n = (1+r)^n a$$

When the bank tells you that they'll pay your interest monthly into your account, but quote an annual rate of 12% (that is 0.12), how much do you really get at the end of, say, the first month? Do you get 1% of your deposit? You should probably read the small print. The bank's real monthly rate m is set such that if you were to deposit 1£, and left that capital to appreciate for 12 months, where the monthly interest (earned at the real monthly rate of m) were added to the pot every month (so you can get interest on those added bits as well), your original 1£ would become $(1+r)$£ after 12 months. In other words, the real monthly rate m satisfies

$$(1+m)^{12} = 1 + r = 1.12$$

While it might not be obvious from this equation that m is less than 1% (that is 0.01), we know it must be. Because, if the bank were to reward us with a monthly interest rate of 1%, and if we were to withdraw the interest earned in month i at the end of the month, and left only the original deposit to earn interest next month, we would collect exactly 12% in one year. That's without compounding, because we withdrew the monthly interest every month. So, with compounding (that is leaving the interest to earn further interest and so on), we would be earning more. And if we would be earning more than 12% with compound interest, your average bank would probably boast about that number.

Geometric series or partial sums of geometric series appear frequently in models of *exponential discounting of future* or models of regular saving (as in building a retirement pot) in the presence of one or more interest rate. For example, the value of 1£ saved in a piggy bank today will typically be less than 1£ because in one year, its purchasing power will be lower due to inflation and the today's decision maker's valuation of money available in a distant future might be lower than money available now. The first effect will incorporate the expected inflation rate, whereas the latter effect will feature the decision maker's so-called discount factor. Now, throw in also the possibility of a proper bank paying an annual interest rate, and compute the value (seen from today) of a 30-year saving plan where our decision maker pays in a fixed amount into the bank every year.

12.3 Taylor polynomials

Suppose f is differentiable at least n times around the point a. Define the n-th degree **Taylor polynomial** of f around the point a as

$$P_n(x) \;=\; f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Observation 0. The function P_n agrees with f at a. That is, $P_n(a) = f(a)$.

Observation 1. The derivative of the function P_n agrees with the derivative of f at a. That is, $P_n'(a) = f'(a)$, because

$$P_n'(x) \;=\; f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$$

Observation 2. Differentiating P_n', we get

$$P_n''(x) \;=\; f''(a) + f'''(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2}$$

and evaluating this at the point a, we end up with $f''(a)$.

$\vdots$

Observation k. Continuing in the same fashion, we will find that

$$P_n^{(k)}(a) = f^{(k)}(a) \qquad \text{for each} \quad k = 1, \ldots, n.$$

If we know the values $f(a), f'(a), f''(a), \ldots, f^{(n)}(a)$, we can construct an nth degree polynomial P_n which has this same "local information" as f in the sense that P_n and f have the same 1st to nth order derivatives at a.

QUESTION. Is using Taylor polynomials a good way to approximate a function?

Sometimes, but not always!

12.4 Taylor series

If f is infinitely differentiable at a point a, then we can construct Taylor polynomials of all degrees $n > 0$, and talk about their limit:

When f has derivatives of all orders at $x = a$, its **Taylor series** about the point $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \;=\; f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots$$

A few possibilities for the above series:

1. The series diverges for some x. If so, the Taylor polynomials P_n do not provide a good approximation to f at x.
2. The series converges for x, but to a value other than $f(x)$. If so, once again, the Taylor polynomials P_n do not provide a good approximation to f at x.

3. The series at x converges to $f(x)$. If so, the Taylor polynomials provide a method to approximate the function at x.

We will mostly deal with Case 3: when dealing with a Taylor series expansion around a point a, this series will indeed converge to $f(x)$ for those x which are in some neighbourhood of a. Such functions are called **analytic**.

How good is such an approximation? That is, can we say anything about how far off the Taylor polynomial is from the original function?

That is,

$$\text{How large is the error term} \quad R_n(x) = f(x) - P_n(x) \quad ?$$

Taylor's Remainder Theorem.

If f is n times continuously differentiable on $[a, x]$, and $n+1$ times differentiable on (a, x), then

$$R_n(x) \;=\; \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt \;=\; \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \in [a, x]$$

Example. Let's look at $f(x) = \ln(1+x)$ around the point 0.

In order to explore its Taylor series around $x = 0$, we will evaluate $f(0), f'(0), f''(0), \ldots$.

First, note that

$$\begin{aligned} f'(x) &= \frac{1}{1+x} \\ f''(x) &= \frac{-1}{(1+x)^2} \\ f'''(0) &= \frac{2}{(1+x)^3} \\ \vdots \quad &\vdots \quad \vdots \\ f^{(k)}(x) &= \frac{(-1)^{k+1}(k-1)!}{(1+x)^k} \end{aligned}$$

Evaluating the above expressions at $x = 0$, and plugging them into the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!}{k!}x^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Using Taylor polynomials for $\ln(1+x)$, we can estimate $\ln(1.01)$.

Note that $\ln(1.01) = \ln(1 + 0.01) = f(0.01)$. So, using the second degree Taylor polynomial of the function $\ln(1+x)$, the remainder theorem tells us that

$$\begin{aligned} |f(0.01) - P_2(0.01)| &= \left|\frac{f^{(3)}(c)}{3!}(0.01)^3\right| \quad \text{for some } c \in [0, 0.01] \\ &= \frac{1}{3(1+c)^3}(0.01)^3 \\ &\leq 0.34 \times 0.000001 \;=\; 0.00000034 \end{aligned}$$

$P_2(x) = x - \frac{x^2}{2}$, so we can easily compute $P_2(0.01)$:

$$P_2(0.01) \;=\; 0.01 - \frac{0.0001}{2} \;=\; 0.01 - 0.00005 \;=\; 0.00995$$

Thus we can conclude

$$\ln(1.01) \;\approx\; 0.00995 \qquad \text{with an accuracy level of } 0.00000034$$

Reading. *Sydsaeter & Hammond:* Section 7.5 (Polynomial Approximations), Section 7.6 (Taylor's Formula), Section 10.8 (A Glimpse at Difference Equations)

Example. Find an estimate for $\sqrt{100.1}$ which you can be sure to be correct up to a maximum error of 0.0001.

The Taylor series expansion of $f(x) = \sqrt{100+x}$ around the point $x = 0$ can be useful if the error term $R_n(0.1)$ indeed goes to zero quickly as n grows.

After all,

$$|\underbrace{\sqrt{100+0.1}}_{\text{actual value}} - \underbrace{P_n(0.1)}_{\text{estimate}}| \;=\; |R_n(0.1)| \;=\; \underbrace{\left|\frac{f^{(n+1)}(c)}{(n+1)!}(0.1)^{n+1}\right|}_{\text{want this to be small}} \quad \text{for some } c \in [0, 0.1]$$

How small? We are asked to provide an estimate which we know to be correct up to a maximum error of 0.0001. So, if we can make the error term smaller than 0.0001, then we should be OK.

What remains is to evaluate the derivatives of f at 0, and to provide a not-too-relaxed upper bound for $f^{(n+1)}(c)$ where $c \in [0, 0.1]$.

$$\begin{aligned} f'(x) &= \frac{1}{2}(x+100)^{-1/2} \qquad f'(0) = \frac{1}{20} \\ f''(x) &= -\frac{1}{4}(x+100)^{-3/2} \qquad f''(0) = -\frac{1}{4000} \end{aligned}$$

Looks like $R_1(0.1)$ might already be small enough for our purposes.

First, let's quickly observe that $|f''(x)| = \dfrac{1}{4(x+100)^{3/2}}$ is decreasing on $[0, 1]$.

That implies $|f''(c)| \le |f''(0)|$ for all $c \in [0, 1]$, which can be used to conclude

$$R_1(0.1) = \left|\frac{f''(c)}{2}(0.1)^2\right| \;\le\; \left|\frac{f''(0)}{2}(0.1)^2\right| = \frac{1}{800000} < 0.0001$$

Thus, our estimate for $\sqrt{100.1}$ is

$$P_1(0.1) = f(0) + f'(0) \times 0.1 = 10 + \frac{1}{200} = 10.005$$

13 Difference equations

Suppose a quantity (e.g., one's mortgage debt to a bank) changes over time where

- we measure time in discrete terms (e.g., in years or in months)
- the relationship between the quantity in time t has a cleanly expressed relationship to the quantity in time $t-1$.

For example, denoting by D_t the total debt after t years

$$D_t = (1+r)D_{t-1} - p$$

where r is the annual interest rate and p is the annual payment back.

The previous year's debt grows by the interest rate (so D_{t-1} grows by a rate r), and declines by the payment p made at the end of the year.

To make things appear more concrete, suppose the original loan (D_0) is £200K and the annual interest rate is 3%.

If the mortgage needs to be cleared in exactly 30 years, what should be the payment p be?

We can rephrase the question as follows: if we know that

$$\begin{aligned} D_0 &= 200000 \\ D_t &= 1.03D_{t-1} - p \quad \text{for all } t = 1, 2, \ldots \\ D_{30} &= 0 \end{aligned}$$

then what is p?

We will approach such problems by trying to express D_t as a function of t.

Equations of the following sort are called linear, first-order, autonomous difference equations:

$$x_t = bx_{t-1} + a \qquad \text{where } a \text{ and } b \text{ real constants}$$

- **linear**, because x_t is a linear function of x_{t-1}
- **first-order**, because the largest difference between the indices (namely t and $t-1$) is 1
- **autonomous**, because a and b are independent of t

13.1 Solving a linear, first-order, autonomous difference equation

Given the value of x_0 and constants a and b, in order to solve

$$x_t = bx_{t-1} + a$$

begin with x_1 and x_0, and try to express all x_t for $t > 1$ in terms of x_0:

$$\begin{aligned} x_1 &= bx_0 + a \\ x_2 &= bx_1 + a = b(bx_0 + a) + a = b^2x_0 + (b+1)a \\ x_3 &= bx_2 + a = b(b^2x_0 + (b+1)a) + a = b^3x_0 + (b^2 + b + 1)a \\ x_4 &= bx_3 + a = b(b^3x_0 + (b^2 + b + 1)a) + a = b^4x_0 + (b^3 + b^2 + b + 1)a \\ &\vdots \qquad \vdots \\ x_t &= b^tx_0 + (b^{t-1} + b^{t-2} + \cdots + b^2 + b + 1)a \end{aligned}$$

Note that the geometric sum $b^{t-1} + b^{t-2} + \cdots + b^2 + b + 1$ is equal to $\dfrac{1-b^t}{1-b}$ as long as $b \neq 1$.

Thus, the solution to the equation $x_t = bx_{t-1} + a$ can be expressed simply as

$$x_t = \begin{cases} b^t x_0 + \dfrac{1-b^t}{1-b} a & \text{if } b \neq 1 \\ x_0 + ta & \text{if } b = 1 \end{cases}$$

Back to the mortgage example. We had the difference equation $D_t = 1.03D_{t-1} - p$, and the specific values $D_0 = 200000$ and $D_{30} = 0$. Using these with the formula we derived above

$$0 = D_{30} = (1.03)^{30} 200000 + \frac{1-(1.03)^{30}}{1-1.03} \times (-p)$$

which implies

$$p = \frac{(1.03)^{30} 200000 \times (1-1.03)}{1-(1.03)^{30}} \approx 10204$$

as the annual payment required to clear the debt in 30 years.

An exercise. Consider a twist of the mortgage problem above: suppose the annual payments go up by the same interest rate of 3%. What is the relevant difference equation? How can we solve it?

13.2 Steady states

If $x_{n+1} = x_n$ for some n, then note that

$$x_{n+2} = bx_{n+1} + a = bx_n + a = x_{n+1}$$

and therefore $x_n = x_{n+1} = x_{n+2} = x_{n+3} = \cdots$

In other words, once t reaches n, the quantity x_t does not change any more, and this value is referred to as a **steady state** of the difference equation.

- Does every difference equation have a steady state?

  Not necessarily! Take for example $x_t = x_{t-1} + 1$.

- If a difference equation has a steady state, will x_t eventually reach that state?

  Not necessarily! Take for example $x_t = \frac{1}{2}x_{t-1}$, where $x_0 = 1$. Clearly 0 is a steady state, and the values of x_t will evolve as $1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \cdots \quad \frac{1}{2^n} \quad \cdots$ and will never take the value 0. But note that x_t converges to 0 as $t \to \infty$.

- If a difference equation has a steady state, will x_t converge to that state as $t \to \infty$?

  Not necessarily! Take for example $x_t = 2x_{t-1}$, where $x_0 = 1$. Clearly 0 is a steady state, and the values of x_t will evolve as $1 \quad 2 \quad 4 \quad 8 \quad \cdots \quad 2^n \quad \cdots$ and therefore will diverge as $t \to \infty$.

Convergence

If $x_t \to x$ as $t \to \infty$, the steady state x is easy to calculate because we can simply take the limit as $t \to \infty$ of the following

$$x_t = bx_{t-1} + a$$

to conclude that $x = bx + a$, and therefore $x = \dfrac{a}{1-b}$.

Remember the general solution to $x_t = bx_{t-1} + a$ we derived on the previous page

$$x_t = \begin{cases} b^t x_0 + \dfrac{1-b^t}{1-b}a & \text{if } b \neq 1 \\ \\ x_0 + ta & \text{if } b = 1 \end{cases}$$

Note that x_t converges to $x = \dfrac{a}{1-b}$ if and only if $|b| < 1$.

What if $|b| \geq 1$ and $a \neq 0$?

13.3 More examples on difference equations

Reading. *Pemberton & Rau, Section 23.4: Difference Equations*

Example 1. A fixed population of users subscribe to one of the two music streaming apps (*Spotify* or *Deezer*) such that at any given time each user is subscribed to exactly one app. At the end of every month, each user has the option to cancel her contract and switch to the other app, and indeed it just happens that every month 20% of Spotify users switch to Deezer, while 10% of Deezer users switch to Spotify. Suppose at time 0, everybody is a Spotify user.

Denoting by (S_0, D_0) the respective percentages of Spotify and Deezer users at time 0, let (S_k, D_k) be those percentages after k months. So

$$\begin{aligned} S_{k+1} &= 0.8S_k + 0.1D_k \\ D_{k+1} &= 0.2S_k + 0.9D_k \end{aligned}$$

The dynamics of the changes in user subscription percentages can then be summarised as

$$\begin{bmatrix} S_{k+1} \\ D_{k+1} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} S_k \\ D_k \end{bmatrix}$$

which allows us to obtain a general formula as

$$\begin{bmatrix} S_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}^n \begin{bmatrix} S_0 \\ D_0 \end{bmatrix}$$

Is there a steady state for the market shares of these apps? Secondly, do they converge to this steady state as $n \to \infty$?

Let's first check for a steady state. If (S^*, D^*) is one, then

$$\begin{aligned} S^* &= 0.8S^* + 0.1D^* \\ D^* &= 0.2S^* + 0.9D^* \end{aligned}$$

Note that both equations imply $2S^* = D^*$. Since we know these percentages add up to 1, we conclude that the unique steady state is $(1/3, 2/3)$ where one third of the population use Spotify, and the rest use Deezer.

Will we reach this steady state beginning from an initial state in which everyone used Spotify?

Put differently, does (S^n, D^n) converge to $(1/3, 2/3)$?

Note that $S_k + D_k = 1$ at every k, so

$$S_{k+1} = 0.8S_k + 0.1D_k = 0.8S_k + 0.1(1 - S_k) = 0.7S_k + 0.1$$

By using the formula we have developed earlier, we can write the solution immediately:

$$S_n = (0.7)^n S_0 + 0.1 \times \frac{1-(0.7)^n}{1-0.7}$$

which converges to $1/3$. Therefore D_n converges to $2/3$.

The cobweb model

In the so-called *cobweb model*, we have a market where supply decisions of producers follow the observed prices with a lag due to the time required in adjusting supplies. In particular, supply decisions in period t are in response to prices in period $t-1$:

$$S_t = g(p_{t-1}) \qquad \text{where} \qquad \begin{array}{l} g \text{ is the aggregate supply function} \\ S_t \text{ is the aggregate quantity supplied in period } t \\ p_{t-1} \text{ is the market price observed in period } t-1 \end{array}$$

Unlike production, consumers' demand decisions respond to prices immediately, so the demand in period t is in response to price observed in that period, and this relationship is summarised by the aggregate market demand function

$$D_t = f(p_t) \qquad \text{where} \qquad \begin{array}{l} f \text{ is the aggregate demand function} \\ D_t \text{ is the aggregate quantity demanded in period } t \\ p_t \text{ is the market price observed in period } t \end{array}$$

And finally, in each period the market clears, meaning demand equals supply:

$$\begin{aligned} D_t &= S_t \\ f(p_t) &= g(p_{t-1}) \end{aligned}$$

which is nothing but a difference equation for price p_t.

A steady state for p_t is obtained when the supply and the demand have stabilised. A natural question is whether this system would converge to such a steady state if it begins from a non-steady state.

Special case: linear demand and supply.

Let's try to answer this question for a special case where the aggregate supply and demand functions are linear:

$$f(p) = A + Bp \qquad \text{and} \qquad g(p) = M + Np$$

so the market clearing condition $D_t = S_t$ becomes

$$A + Bp_t = M + Np_{t-1}$$

which can then be written in the form of a difference equation we are familiar with:

$$p_t = \frac{N}{B} p_{t-1} + \frac{M-A}{B}$$

whose solution is

$$p_t = \left(\frac{N}{B}\right)^t p_0 + \frac{M-A}{B} \times \frac{1-(N/B)^t}{1-N/B}$$

Rewrite the solution as

$$p_t = \left(\frac{N}{B}\right)^t p_0 + \frac{M-A}{B-N} \times (1-(N/B)^t)$$

and remember that

- demand $D(p) = A + Bp$ decreases in price, so $B < 0$
- supply $S(p) = M + Np$ increases in price, so $N > 0$

Therefore $N/B < 0$.

As far as convergence of price is concerned, we have three separate cases for N/B:

- if $N/B < -1$, then p_t diverges because p_t is unbounded.
- if $-1 < N/B < 0$, then p_t converges to $p^* = \dfrac{M-A}{B-N}$.
- if $N/B = -1$, then price oscillates between p_0 and $2p^* - p_0$.

Note that convergence is obtained when the demand function is steeper than the supply function. (Put differently, when the inverse supply curve is steeper than the inverse demand curve.)

Another special case: constant elasticity demand and supply.

Assume a constant price elasticity of demand $\varepsilon < 0$, and a constant price elasticity of supply $\varphi > 0$.

That means (to be verified as an exercise) demand and supply are given by

$$D(p) = Bp^{\varepsilon} \qquad \text{and} \qquad S(p) = Np^{\varphi}$$

Price dynamics and market clearing imply

$$Bp_t^{\varepsilon} \;=\; Np_{t-1}^{\varphi}$$

Taking the natural log:

$$\ln B + \varepsilon \ln p_t \;=\; \ln N + \varphi \ln p_{t-1}$$

Denoting $y_t = \ln p_t$ and rearranging yield the following familiar form of a difference equation:

$$y_t \;=\; \frac{\varphi}{\varepsilon} y_{t-1} + \frac{\ln(N/B)}{\varepsilon}$$

which converges to a steady state if and only if $|\varphi| < |\varepsilon|$, that is, demand is more price elastic than supply.

14 Differential equations

Reading. *Sydsaeter & Hammond:* Section 9.8 (A Glimpse at Difference Equations). *Pemberton & Rau:* Section 23.1 (Differential equations), Section 23.2 (Linear differential equations with constant coefficients)

A differential equation is a "functional equation" which relates a function to its derivatives. For example

$$f(x)\ln x + 2f'(x) + xf''(x) = x^2$$

is one such equation. We refer to this as a *functional* equation since this is assumed to hold not only at some particular point x, but for all x in the domain of f. It is a *second degree* differential equation because it involves f and its second derivative f'', but no higher order derivative.

Solving the above differential equation means identifying all functions f for which the equation holds. We will be dealing with simpler differential equations which happen to appear frequently in economic applications. Having said that, an economic problem may well involve a differential equation for which we don't necessarily have a recipe. So it is worth keeping in mind that there are differential equations which might require techniques beyond the scope of this course (or even beyond what's known to mathematicians). Part IIA Paper 6 will explore further than what we cover in this course.

Example 1. Suppose the price elasticity of the aggregate demand for a good is constant and equal to ε. What can we say about the aggregate demand function?

Remembering the definition of price elasticity of demand for a demand function D, we can write

$$\lim \frac{\frac{\Delta D}{D}}{\frac{\Delta p}{p}} \quad = \quad \frac{D'(p)}{D(p)} \times p \quad = \quad \varepsilon$$

which gives us the following first degree differential equation for the function D:

$$\frac{D'(p)}{D(p)} \quad = \quad \frac{\varepsilon}{p}$$

Integrating both sides with respect to p yields

$$\ln(D(p)) \quad = \quad \varepsilon \ln p + C \qquad \text{for some constant } C.$$

Taking the exp of both sides in the above equation leads to:

$$D(p) \quad = \quad p^{\varepsilon} e^{C} \qquad \text{for some constant } C.$$

Thus a demand function whose price elasticity is the constant ε must be of the form

$$D(p) \quad = \quad Ap^{\varepsilon} \qquad \text{where } A \text{ is a positive constant.}$$

Example 2. The ratio $x'(t)/x(t)$ is sometimes referred to as the *relative rate of change* of x and it captures the proportional change in $x(t)$ since it is nothing but

$$\frac{x'(t)}{x(t)} \quad = \quad \lim_{h \to 0} \frac{1}{h} \frac{x(t+h) - x(t)}{x(t)}$$

Now, suppose the relative rate of change of x is constant and equal to r. Can we obtain a functional expression for $x(t)$ which does not feature its derivative?

That is, we need to solve the differential equation $\frac{x'(t)}{x(t)} = r$, which is even easier than the example above. Integrating both sides yields

$$\ln(x(t)) = rt + C$$

and therefore $x(t) = e^{rt+C}$.

Thus, a quantity $x(t)$ which exhibits constant relative rate of change r has the form

$$x(t) \quad = \quad Ae^{rt} \qquad \text{where } A \text{ is a positive constant.}$$

14.1 Linear, autonomous, first-order differential equations

This is a class of simple differential equations of the form

$$(\star) \qquad x'(t) = bx(t) + a \qquad \text{where } a \text{ and } b \text{ are constants.}$$

Observation 1. The constant function $-a/b$ is a solution to $(\star)$.

Observation 2. If $x(t)$ and $y(t)$ are two different solutions to the equation $(\star)$, then $z(t) = x(t) - y(t)$ satisfies

$$(\star\star) \qquad z'(t) = bz(t)$$

But we have already solved equations of type $(\star\star)$:

$$z'(t) = bz(t) \implies \frac{z'(t)}{z(t)} = b \implies z(t) = Ae^{bt} \quad \text{where } A > 0 \text{ is a constant.}$$

So, putting together the above observations. If $x(t)$ is a solution to the equation $(\star)$, then both $x(t) + a/b$ and $-a/b - x(t)$ satisfy $(\star\star)$. That suggests

$$x(t) + \frac{a}{b} = Ae^{bt} \qquad \text{OR} \qquad -\frac{a}{b} - x(t) = Ae^{bt} \quad \text{where } A > 0 \text{ is a constant.}$$

Or put differently

$$x(t) = -\frac{a}{b} \pm Ae^{bt} \quad \text{where } A > 0 \text{ is a constant.}$$

Hence, solutions to $(\star)$ look like:

$$x(t) = -\frac{a}{b} + Ae^{bt} \qquad \text{where } A \text{ is an arbitrary constant}$$

Note that Example 1 above is not in this class of differential equations, but we still managed to solve it.

14.2 Stationary solutions of differential equations

A constant solution of a differential equation is called a **stationary solution** (or **stationary state**). Obviously, such a solution will have its derivatives equal to 0 everywhere, and therefore can be easily computed from the differential equation. For example, given a differential equation $f'(t) = bf(t) + a$, a stationary solution necessarily satisfies $f'(t) = 0$ for all t. Plugging this in to the differential equation, we obtain the only stationary state as $-a/b$.

Recall that a quantity, say $f(t)$, which changes with time t is said to converge when $\lim_{t\to\infty} f(t)$ exists. If a differential equation has a *unique* stationary state, and if every solution approaches to this stationary state as $t \to \infty$, then we say the stationary state is **stable**.

We have already seen that the solution to the differential equation $f'(t) = bf(t) + a$ has the form $f(t) = Ae^{bt} - \frac{a}{b}$, where A is a constant (which will be uniquely pinned down if we know the value of f at some point). Clearly, $\lim_{t\to\infty} f(t)$ is equal to the stationary state $-a/b$ if $b < 0$, and this state is stable.

Example. As the bid price p^D for a stock exceeds the asking price p^S, then we'd expect to see the volume of trade of this stock to go up. Suppose these prices change with time t, resulting in a volatile volume of trade $q(t)$ given by

$$q'(t) = \alpha(p^D(t) - p^S(t))$$

For this to make sense, let's first observe that $\alpha > 0$. In other words, as the wedge between the bid price and the asking price goes up, so does the trade volume.

Let's suppose the prices p^D and p^S are related to the volume q of trade as follows:

$$p^D = c + dq \qquad p^S = g + hq$$

Note that these describe the demand function (with slope $1/d$) and the supply function (with slope $1/h$ for the stock. So realistically we'd expect $d < 0$ and $h > 0$.

If the initial volume of trade is denoted by q_0, what can we say about the volume at time t? Is there convergence? What does convergence mean?

Putting all of the above together

$$q'(t) = \alpha(c + dq(t) - g - hq(t)) = \alpha(d-h)q(t) + \alpha(c-g)$$

whose solution is

$$q(t) = Ae^{\alpha(d-h)t} - \frac{c-g}{d-h}$$

where the constant A can be evaluated by using the assumption that $q(0) = q_0$:

$$q_0 = A - \frac{c-g}{d-h} \implies A = q_0 + \frac{c-g}{d-h}$$

Hence

$$q(t) = \underbrace{\left(q_0 + \frac{c-g}{d-h}\right)e^{\alpha(d-h)t}}_{\text{converges if } d-h<0} - \frac{c-g}{d-h}$$

If indeed $d < 0$ and $h > 0$, convergence will be obtained as $t \to \infty$.

Note that stability in this application corresponds to $q'(t)$ being equal to zero, which means the bid price p^D converging to ask price p^S.

An example of a non-autonomous differential equation

Those who have worked on the twist of the mortgage question from Lecture 16 might remember that the relevant differential equation was

$$D_t = 1.03D_{t-1} - (1.03)^{t-1}p$$

where p is the first payment made at the end of period 1. The formula we developed for first-degree, linear, autonomous difference equations would not apply here because the expression $(1.03)^{t-1}p$ depends on t. But a solution could still be obtained.[12]

Likewise, for differential equations, suppose we have to tackle one of the form

$$f'(t) = bf(t) + a(t)$$

This equation differs from the one we solved earlier in that $a(t)$ is not a constant. It turns out this equation is solvable with a little trick: multiply the equation with e^{-bt} and rearrange to obtain

$$f'(t)e^{-bt} - bf(t)e^{-bt} = a(t)e^{-bt}$$

Exercise. Observe that the LHS is the derivative of $f(t)e^{-bt}$, and proceed with integrating the equation to reach the solution

$$f(t) = Ce^{bt} + e^{bt}\int a(t)e^{-bt}dt$$

[12]Simply write D_1, and then D_2, and D_3, and so on until you notice a pattern emerging which allows you pin down a formula for D_n.

A past exam question

Consider a market for an ordinary good where the price in each period is determined by the standard market clearing condition: supply equals demand. Let the demand curve for this good be $f(p)$ and the supply curve be $g(p)$.

(a) You are told that at every price p, the price elasticity of market demand for this good is ap^2 and the price elasticity of market supply is bp^2 where a and b are given constants. If both the market demand and the market supply at $p = 1$ are 100, derive the demand curve f and the supply curve g.

(b) Suppose demand D_t and supply S_t in period t are determined according to

$$D_t \;=\; f(p_t) \qquad \text{and} \qquad S_t \;=\; g(p_{t-1})$$

where p_t is the market price in period t. Obtain a difference equation for p_t, and evaluate all steady states if there are any.

(c) Obtain a formula which expresses the price in period t as a function of t and p_0.

(d) Determine the conditions under which p_t converges, and whether there is a stable long run equilibrium. Comment on the nature of the convergence path.

Answer.

(a) The price elasticity of demand is $f'(p)p/f(p)$. The price elasticity of supply is $g'(p)p/g(p)$. These are given to be equal to ap^2 and bp^2, respectively. Hence we have two differential equations

$$\frac{f'(p)p}{f(p)} = ap^2 \qquad \text{and} \qquad \frac{g'(p)p}{g(p)} = bp^2$$

which can be rewritten as

$$\frac{f'(p)}{f(p)} = ap \qquad \text{and} \qquad \frac{g'(p)}{g(p)} = bp$$

Integrating with respect to p, we obtain

$$\ln f(p) \;=\; \frac{ap^2}{2} + \alpha \qquad \text{and} \qquad \ln g(p) \;=\; \frac{bp^2}{2} + \beta$$

where α and β are constants. Taking the exponential yields

$$f(p) \;=\; A\exp(ap^2/2) \qquad \text{and} \qquad g(p) \;=\; B\exp(bp^2/2)$$

where A and B are positive constants.

We are told that $f(1) = g(1) = 100$, which imply

$$A \;=\; 100\exp(-a/2) \qquad \text{and} \qquad B \;=\; 100\exp(-b/2)$$

and therefore

$$f(p) \;=\; 100\exp(-a/2 + ap^2/2) \qquad \text{and} \qquad g(p) \;=\; 100\exp(-b/2 + bp^2/2)$$

(b) Simply imposing $D_t = S_t$, i.e., $f(p_t) = g(p_{t-1})$ yields

$$\exp\left(-\frac{a}{2} + \frac{ap_t^2}{2}\right) \;=\; \exp\left(-\frac{b}{2} + \frac{bp_{t-1}^2}{2}\right)$$

which implies

$$a(1 - p_t^2) \;=\; b(1 - p_{t-1}^2)$$

A steady state p^* implies $p_t = p_{t-1} = p^*$, and hence

$$(a-b)(1-(p^*)^2) \quad = \quad 0$$

Demand is downward sloping, so $a < 0$, whereas supply is upward sloping, so $b > 0$. In particular $a - b \neq 0$, and therefore $(p^*)^2 = 1$, and $p^* = 1$ since price cannot be negative.

(c) In order to make the above difference equation look like a first order linear autonomous difference equation, we can set $q_t = 1 - p_t^2$, so

$$aq_t \quad = \quad bq_{t-1}$$

and therefore $q_t = (b/a)^t q_0$, which means

$$1 - p_t^2 \quad = \quad (b/a)^t (1 - p_0^2)$$

and therefore

$$p_t \quad = \quad \left(1 - \left(\frac{b}{a}\right)^t (1 - p_0^2)\right)^{1/2} \quad = \quad \left(\left(\frac{b}{a}\right)^t p_0^2 + 1 - \left(\frac{b}{a}\right)^t\right)^{1/2}$$

(d) p_t converges if and only if $|b/a| < 1$ and the limit is the unique stationary point $p^* = 1$.

The stationary price $p^* = 1$ is stable if there is convergence $p_t \to p^*$, and this happens if and only if $|b| < |a|$. The demand and the supply functions have the "same shape", and when $|b| < |a|$, demand is more elastic than supply. If supply were to be more elastic than demand ($|a| < |b|$) the prices would go further and further away from p^* as $t \to \infty$.

Note also that $a < 0 < b$ since we are talking about an ordinary good: demand is downward sloping, supply is upward sloping. The price will fluctuate about p^*. For $b < |a|$, the movement of price-quantity pairs on supply and demand curves lead to the so-called cobweb picture spiralling around the intersection point of the demand and supply curves (which happens at $(1, 100)$.)