Matrices are typically used to represent big systems of linear equations. Of course one can use calculus in such systems. Let \( f(X) \) be a scalar function of the \((n \times m)\) matrix \(X\). Then 
\[
\frac{\partial f(X)}{\partial X} \text{ is an } (n \times m) \text{ matrix whose } (i,j)-\text{th element is } \frac{\partial f(X)}{\partial x_{ij}}. 
\]

**Exercise 1** Let \( x, a \) denote two \((n \times 1)\) vectors. Denote \( f(x) = a'x \). Show that 
\[
\frac{\partial f(x)}{\partial x} = a
\]

**Exercise 2** If \( A \) denotes \((n \times m)\) matrix and \( x \) is \((m \times 1)\) vector, verify that 
\[
\frac{\partial (Ax)}{\partial x} = A
\]

**Exercise 3** Take \( A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \). Verify 
\[
\frac{\partial x'Ax}{\partial x} = (A + A')x
\]
where \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) is an \((2 \times 1)\) vector.

**Exercise 4** Take matrix 
\[
X = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -1 & 0 \end{bmatrix}
\]
Compute \( X'X \).

**Exercise 5** Show that \( X'X \) is symmetric. Compute 
\[
\frac{\partial b'(X'X)b}{\partial b}
\]
using formula from exercise 3.

**Exercise 6** Suppose that for given \((n \times 1)\) vector \( y \) and \((n \times k)\) full column rank matrix \( X \) we want to find \((k \times 1)\) vector of coefficients \( b \), such that the linear combination of column vectors of \( X \) (i.e. \( Xb \)) is closest to vector \( y \), i.e. we want to minimise \( \|y - Xb\|^2 \equiv \langle y - Xb, y - Xb \rangle = (y - Xb)'(y - Xb) \). We can proceed in two ways:

(a) Use calculus: Compute \( \frac{\partial \|y - Xb\|^2}{\partial b} \) and find \( b \) for which this derivation is zero.
(b) Use so called orthogonal projection theorem, which says that:

\[
\hat{b} = \arg\min_{\mathbf{b} \in \mathbb{R}^k} \|\mathbf{y} - \mathbf{Xb}\|^2 \iff (\mathbf{y} - \mathbf{X}\hat{b}) \text{ is orthogonal to any linear combination } \mathbf{Xb}, \mathbf{b} \in \mathbb{R}^k
\]

I.e. we have to find vector \(\hat{b}\), which satisfies following orthogonality conditions

\[
\langle \mathbf{y} - \mathbf{X}\hat{b}, \mathbf{X} \rangle \equiv (\mathbf{y} - \mathbf{X}\hat{b})' \mathbf{X} = 0_k
\]