Static Optimisation
Problem Set 3

1. A competitive profit-maximising firm uses two inputs, z_1 and z_2, to produce output y according to the production function $y = F(z_1, z_2)$. F is twice continuously differentiable, satisfies the sufficient condition for strict concavity, and the marginal products of both inputs are strictly positive. The firm faces exogenously given prices w_1 and w_2 for the two inputs and p for output.

   (a) What are the first-order conditions that characterise a solution to the firm's profit-maximisation problem?

   (b) Suppose that z_1^0 and z_2^0 solve the first-order conditions at (p^0, w_1^0, w_2^0). Use the implicit function theorem to argue that there exist differentiable functions $z_i(p, w_1, w_2)$, $i = 1, 2$, in the neighbourhood of (p^0, w_1^0, w_2^0) which give the firm's profit-maximising input choices.

   (c) Find an expression for $\partial z_1(p^0, w_1^0, w_2^0)/\partial p$. Under what conditions is $\partial z_1(p^0, w_1^0, w_2^0)/\partial p < 0$?

2. Consider the problem

$$\underset{\mathbf{x}}{Max}\ f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) = c$$

   where $\mathbf{x} = (x_1, ..., x_n)$. Let $V(c)$ be the optimum value function for this problem. Show that $\partial V(c)/\partial c$ is given by the value of the Lagrange multiplier that is found as part of the solution to the problem. Give a brief interpretation of this result.

3. Show that the value of the Lagrange multiplier that is found as part of the solution to the cost-minimisation problem in Q8 of Problem Set 1 is the marginal cost of output.

4. Consider again the competitive profit-maximising firm of question 1 above. Let the vector $\mathbf{p} = (p, w_1, w_2)$ denote the exogenously given prices faced by this firm, and let $\mathbf{y} = (y, -z_1, -z_2)$, so that profit is given by $\mathbf{p} \cdot \mathbf{y}$. The firm's output supply function is $y(p, w_1, w_2) = F(z_1(p, w_1, w_2), z_2(p, w_1, w_2))$, where $z_i(p, w_1, w_2)$ is the firm's unconditional demand function for input i, $i = 1, 2$.

   (a) The optimum value function for the firm's profit-maximisation problem is the profit function $\pi(\mathbf{p})$. Show that the profit function is a convex function of prices.

   (b) Show that

$$\frac{\partial \pi(p, w_1, w_2)}{\partial p} = y(p, w_1, w_2) \text{ and } \frac{\partial y(p, w_1, w_2)}{\partial p} \geq 0.$$

   (c) Show that

$$\frac{\partial y(p, w_1, w_2)}{\partial w_i} = -\frac{\partial z_i(p, w_1, w_2)}{\partial p} \quad i = 1, 2.$$

   (d) Show that the profit function is homogeneous of degree one in (p, w_1, w_2). Hence show that

$$\frac{\partial y(p, w_1, w_2)}{\partial p} p + \frac{\partial y(p, w_1, w_2)}{\partial w_1} w_1 + \frac{\partial y(p, w_1, w_2)}{\partial w_2} w_2 = 0.$$