1. A competitive profit-maximising firm uses two inputs, $z_1$ and $z_2$, to produce output $y$ according to the production function $y = F(z_1, z_2)$. $F$ is twice continuously differentiable, satisfies the sufficient condition for strict concavity, and the marginal products of both inputs are strictly positive. The firm faces exogenously given prices $w_1$ and $w_2$ for the two inputs and $p$ for output.

(a) What are the first-order conditions that characterise a solution to the firm’s profit-maximisation problem?

(b) Suppose that $z_1^0$ and $z_2^0$ solve the first-order conditions at $(p^0, w_1^0, w_2^0)$. Use the implicit function theorem to argue that there exist differentiable functions $z_i(p, w_1, w_2)$, $i = 1, 2$, in the neighbourhood of $(p^0, w_1^0, w_2^0)$ which give the firm’s profit-maximising input choices.

(c) Find an expression for $\frac{\partial z_1(p^0, w_1^0, w_2^0)}{\partial p}$. Under what conditions is $\frac{\partial z_1(p^0, w_1^0, w_2^0)}{\partial p} < 0$?

2. Consider the problem

$$\max_x f(x) \text{ subject to } g(x) = c$$

where $x = (x_1, ..., x_n)$. Let $V(c)$ be the optimum value function for this problem. Show that $\frac{\partial V(c)}{\partial c}$ is given by the value of the Lagrange multiplier that is found as part of the solution to the problem. Give a brief interpretation of this result.

3. Show that the value of the Lagrange multiplier that is found as part of the solution to the cost-minimisation problem in Q8 of Problem Set 1 is the marginal cost of output.

4. Consider again the competitive profit-maximising firm of question 1 above. Let the vector $p = (p, w_1, w_2)$ denote the exogenously given prices faced by this firm, and let $y = (y, -z_1, -z_2)$, so that profit is given by $p \cdot y$. The firm’s output supply function is $y(p, w_1, w_2) = F(z_1(p, w_1, w_2), z_2(p, w_1, w_2))$, where $z_i(p, w_1, w_2)$ is the firm’s unconditional demand function for input $i$, $i = 1, 2$.

(a) The optimum value function for the firm’s profit-maximisation problem is the profit function $\pi(p)$. Show that the profit function is a convex function of prices.

(b) Show that

$$\frac{\partial \pi(p, w_1, w_2)}{\partial p} = y(p, w_1, w_2) \text{ and } \frac{\partial y(p, w_1, w_2)}{\partial p} \geq 0.$$ 

(c) Show that

$$\frac{\partial y(p, w_1, w_2)}{\partial w_i} = -\frac{\partial z_i(p, w_1, w_2)}{\partial p} \quad i = 1, 2.$$ 

(d) Show that the profit function is homogeneous of degree one in $(p, w_1, w_2)$. Hence show that

$$\frac{\partial y(p, w_1, w_2)}{\partial p} p + \frac{\partial y(p, w_1, w_2)}{\partial w_1} w_1 + \frac{\partial y(p, w_1, w_2)}{\partial w_2} w_2 = 0.$$