Static Optimisation
Problem Set 1

1. Suppose that the point (x_0, y_0, u_0, v_0) satisfies the two equations

$$\begin{aligned} F(x,y,u,v) &= x^2 - y^2 + uv - v^2 + 3 = 0 \\ G(x,y,u,v) &= x + y^2 + u^2 + uv - 2 = 0. \end{aligned}$$

   State a condition that is sufficient for this equation system to define u and v as implicit functions of x and y, $u = f(x,y)$, $v = g(x,y)$, in a neighbourhood of this point. Show that this condition is satisfied when $(x_0, y_0, u_0, v_0) = (2, 1, -1, 2)$. Compute $\partial f(2,1)/\partial x$ and $\partial g(2,1)/\partial y$.

2. A monopolist that produces a single output has two types of customers. Customers of type 1 are willing to pay a price of $50 - 5q_1$ per unit if q_1 units are sold to them. Customers of type 2 are willing to pay a price of $100 - 10q_2$ per unit if q_2 units are sold to them. The monopolist's cost of producing $q(= q_1 + q_2)$ units of output is $90 + 20q$. How much should the monopolist sell to each type of customer in order to maximise its profit? (Be careful to show that the solution gives a maximum of profit.)

3. Graph each of the following sets and indicate whether it is convex.

   (a) $\{(x,y) : y = e^x\}$
   (b) $\{(x,y) : y \geq e^x\}$
   (c) $\{(x,y) : y \leq 13 - x^2\}$

4. Check whether the following functions are concave, convex, or neither.

   (a) $f = x^2$
   (b) $f = x_1^2 + 2x_2^2$

5. A competitive firm produces a single output y according to the production function $y = 40z - z^2$, where z is the single input. The price of y is denoted by p and the price of z is denoted by w. It is necessary that $z \geq 0$.

   (a) What is the first-order condition for profit-maximisation?
   (b) For what values of w and p will the profit-maximising z be zero?
   (c) For what values of w and p will the profit-maximising z be 20?
   (d) What are the output supply and input demand functions?

6. Solve the problem

$$\underset{x_1, x_2}{Min} \ (x_1)^2 + (x_2)^2 \ \text{ subject to } \ (x_2)^2 = (x_1 - 1)^3$$

7. (a) Solve the problem

$$\underset{x_1,x_2}{Max}\ (x_1)^2 x_2 \text{ subject to } 2x_1^2 + x_2^2 = 3$$

(b) Solve the problem

$$\underset{x_1,x_2}{Min}\ (x_1)^2 x_2 \text{ subject to } 2x_1^2 + x_2^2 = 3$$

8. A producer uses two inputs, z_1 and z_2, to produce a single output y according to the production function

$$y = z_1^{1/2} + z_2^{1/2}$$

The producer faces exogenously given prices w_1 and w_2 for the two inputs, and wishes to minimise the cost of producing a specified level of output, $\overline{y} > 0$.

(a) Write down the Lagrangean for the producer's cost-minimisation problem, and use the first-order conditions to obtain expressions for the producer's cost-minimising input choices in terms of the parameters of the problem.

(b) Use the expressions obtained in (a) to find the cost-minimising choices of z_1 and z_2 when $w_1 = 1$, $w_2 = 2$, and $\overline{y} = 30$. Use the second-order condition to confirm that this solution is a true minimum.