Problem Sets

Problem Set 1: Elements of Probability Theory

1. Two fair coins, a 10p and a 50p, are tossed. Find the probabilities of:
   (a) Both showing heads
   (b) Different faces showing up
   (c) At least one head
   (d) You are told that the 10p shows heads. What is the probability that both show heads?
   (e) You are told that at least one of the two coins shows heads. What is the probability that both show heads?
   (f) What is the probability of 2 sixes when 2 fair dice are rolled?
   (g) What is the probability of at least one six when 2 fair dice are rolled?

2. What is the probability of at least one boy in a family of 4 children? What is the probability that all the children are of the same sex? Hint: Assume boys and girls and equally likely.

3. Find $\Pr\left[A^c \cup (A \cap B)\right]$, if $\Pr[A] = 0.2$, $\Pr[B] = 0.6$, $\Pr[A \cup B] = 0.7$.

4. Consider a piece-wise linear function as shown in the figure. If this function is to be used to define a probability density on the sample space $\Omega = [-2, 2]$, what value does b have to have? Calculate $\Pr([1, 2])$.

1

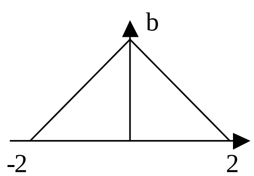

Figure1

$_1$.*pdf*

5. Two fair dices are rolled. Find
   (a) $\Pr[2nd = 6 \,|1st = 1]$,
   (b) $\Pr[sum > 6 \,|1st = 2]$.

Problem Set 2: Distributions and Expectations

Univariate random variables

1. Consider a continuous random variable X which is uniformly distributed between a and b, with $b > a$, so $X \sim U[a,b]$ with $f(x) = (b-a)^{-1}$ for $x \in [a,b]$ and 0 elsewhere. Show that $f(x)$ is a density function. Derive the distribution function

$$F(x) = \int_{-\infty}^{x} f(t)\ dt.$$

   Calculate the mean $\mathrm{E}[X]$ and the second moment $\mathrm{E}\left[X^2\right]$ and hence derive the variance of X.

2. A plane has an estimated arrival time of noon, plus or minus 15 minutes, and within this interval *no time should be given preference over any other time*. If we label noon as time 0 and measure time in minutes, then the state space is $[-15, 15]$.
   (a) What is the probability density function?
   (b) What is the distribution function?
   (c) What is the probability that the plane arrives after 12:05?
   (d) What is the probability that the plane arrives before 11:50?

3. Show that
   (a) $\mathrm{E}[a + bX] = a + b\mathrm{E}[X]$
   (b) $\mathrm{Var}[a + bX] = b^2\mathrm{Var}[X]$
   (c) $\mathrm{E}\left[(X-\mu)^2\right] = \mathrm{E}\left[X^2\right] - \mu^2$, where $\mu : \mathrm{E}[X]$.

4. A random variable X has the distribution function

$$F(x) = \begin{cases} x^2 & \text{for} \quad 0 \le x \le 1 \\ 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 1. \end{cases}$$

   Define the probability density of X, and find its mean and standard deviation.

5. A random variable X has a density function given by

$$f(x) = \begin{cases} cx^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

   for some appropriately defined constant c.

(a) Find the value of c which ensures that f is indeed a density function and determine

$$\Pr\left[X \leq \frac{1}{2}\right] \text{ and } \Pr\left[\frac{1}{4} \leq X \leq \frac{3}{4}\right].$$

(b) Find the mean and standard deviation of X.

Random vectors

1. The joint density function for two random variables, X and Y, is given by

$$f(x,y) = a(x+y) \text{ for } x, y \in [0,1].$$

where a is a constant.

(a) Find the value of a.
(b) Find the marginal distributions $f_x(x)$and $f_y(y)$.
(c) Find $\mathrm{E}[X]$, $\mathrm{E}[Y]$ and $\mathrm{Cov}[X,Y]$.
(d) Find the conditional distributions $f(y|x)$and $f(x|y)$.
(e) Are X and Y independent?

2. Consider the joint distribution function:

$$F(x,y) = 1 - e^{-\lambda x} - e^{-\lambda y} + e^{-\lambda(x+y)}, \quad \lambda, x, y > 0.$$

Derive the marginal distributions of X and Y and hence show that X and Y are independent. What does your result tell you about $\mathrm{E}[XY]$? Derive the conditional distribution of X given $Y = y$.

Problem Set 3: Distribution Theory

1. Suppose that the load X required to break a 1×10 board is normally distributed with mean 2.50, and a standard deviation of .24. What's the probability that the board breaks at a load of 2.61 or less[1]?

Problem Set 4: Statistical Inference

Point estimation

1. Show that the MSE , i.e. $\mathrm{E}\left[\left(\hat{\theta}-\theta\right)^2\right] = \mathrm{Var}(\widehat{\theta}) + bias^2$, where $bias = \mathrm{E}\left[\hat{\theta}-\theta\right]$.

2. Let X be a random variable with unknown mean μ and variance σ^2. Consider the problem of estimating μ from a random sample of observations on $X_1 ... X_n$. Three estimators are proposed:

[1] Note that statistical tables are required to solve this question. Should you not have them with you just go as far as you can.

(a) $\overline{X} = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$.

(b) $\widehat{\mu} = \frac{1}{n-1}\sum\limits_{i=1}^{n}(X_i + 2)$.

(c) $\widetilde{\mu} = \frac{1}{2}X_1 + \frac{1}{2n}\sum\limits_{i=3}^{n} X_i$.

Which of these are unbiased?

3. Suppose $X_1, ..., X_n$ is a random sample from a Poisson distribution, where the density for each observation is given by,

$$f(x_i;\theta) = \frac{e^{-\theta}\theta^{x_i}}{x_i!}$$

   (a) Derive the maximum likelihood estimator (MLE) of θ.
   (b) Find the bias (if any) and the variance of the MLE.
   (c) Derive the CRLB. Is there any other unbiased estimator of θ with smaller variance?

4. Suppose $X_1, ..., X_n$ is a random sample from a Gaussian distribution with mean μ and variance σ^2.

   (a) Derive the maximum likelihood estimators (MLE) for these parameters.
   (b) Are these estimators biased?
   (c) Derive the variance of the MLE of μ. Does it achieve the Cramer-Rao lower bound?

5. **Advanced.** Prove that, for n normally distributed random observations from the distribution $N(\mu, \sigma^2)$, s^2 is an unbiased estimator of σ^2, but that s is a biased estimator of σ, where:

   (a) s^2 is the sample variance given by

$$s^2 = \frac{\sum_i (x_i - m)^2}{(n-1)} = \frac{\sum_i x_i^2 - nm^2}{(n-1)}.$$

   (b) m is the sample mean

$$m = \frac{\sum_i x_i}{n}.$$

6. **Advanced.** Prove

$$CRLB = \left(-E\left[\frac{\partial^2 \ln L(\theta)}{\partial\theta^2}\right]\right)^{-1} = \left(E\left[\left(\frac{\partial \ln L(\theta)}{\partial\theta}\right)^2\right]\right)^{-1},$$

assuming that the observations $(x_1, \ldots, x_n)$ are the outcome of a random sample.

Hypothesis testing[2]

1. The reaction times for 8 police officers were found to be

$$0.28, 0.23, 0.21, 0.26, 0.29, 0.21, 0.25, 0.22.$$

   Determine a 95% confidence interval for the mean reaction time of all police officers. **Hint**: Assume the population is normally distributed.

2. What do you understand by the following terms?

   (a) The null hypothesis.

   (b) Type I and Type II errors.

   (c) The significance level of a test.

   (d) The power of a test.

3. Suppose that in a test, the null hypothesis is $H_0 : \quad \theta = 0$ and the alternative is $H_1 : \quad \theta = 2$. Suppose that you use a statistic $\hat{\theta}$ that is normal with mean θ and variance 1. Find an expression for the Type I error and an expression for the power of the test. Use this to show that the power of the test is largest if you use a one sided test instead of a two sided test.

4. A sample of full professors of economics at several Scottish universities had the following salaries (in £):

$$41,200,\ 43,100,\ 42,600,\ 44,200,\ 50,700,\ 53,000,$$
$$45,700,\ 51,400,\ 48,600,\ 60,900,\ 63,700.$$

   Suppose that an advocacy group knows that the national average (outside London) for full professors is £55,200, and the standard deviation is 7,000. Does this group have a case that full professors in Scotland are underpaid? **Hint**: Assume the population is normally distributed.

Problem Set 5: Ordinary Least Squares

1. Consider the Classical Linear Regression Model (CLRM) in matrix form

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

   and assume

$$\boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \boldsymbol{\Sigma}\right) \quad \boldsymbol{\Sigma} \neq \sigma^2 \mathbf{I}_N \ .$$

   Is the OLS estimator

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{y}$$

   still an unbiased estimator of $\boldsymbol{\beta}$? Show that covariance matrix of $\hat{\boldsymbol{\beta}}$ is no longer given by

$$\mathrm{Cov}\left[\hat{\boldsymbol{\beta}}\right] = \sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1} .$$

[2] In order to solve the questions in this section statistical tables are need. Should you not have them with you just go as far as you can.