Exercises in Differential Equations and Difference Equations

Exercises

1. By separating variables, find solutions to the following differential equations:

   (a)
   $$\frac{dy}{dx} = -\alpha \frac{x}{y}$$

   (b)
   $$\frac{dy}{dx} = -\alpha \frac{y}{x}$$

   (c)
   $$\frac{dy}{dx} = -\alpha \frac{y^2}{x^2}$$

2. Find a solution to the differential equation $\dot{x} = x\left(\alpha - x\right)$

   (a) by separation of variables and expressing the resulting integral in terms of partial fractions.

   (b) by re-writing the equation in the form $\dot{x} = \alpha x - x^2$ (a Bernoulli equation) and making the substitution, $z = kx^{-1}$.

3. For the differential equation:

   $$\dot{x} + \frac{1}{1+t}x = 1$$

   with $x(0) = 1$,

   (a) show that the above equation is inexact.

   (b) find an integrating factor, $I(t)$, that makes the equation exact.

   (c) solve the resulting exact equation and obtain a definite solution, given the starting value.

4. Find the general solution of:

   (a) $\dot{x} - ax = h$ where a and h are constants.
   (b) $\dot{x} - ax = he^{\sigma t}$ where a, h, σ are constants.
   (c) For $\dot{x} - ax = f(t)$ show that a particular solution is $z = e^{at}b(t)$ where $b(t) = \int e^{-at} f(t)dt$.

5. For a 2×2 matrix $\mathbf{A}$ in the differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, show that the following relationships between the eigenvalues hold:

$$\lambda_1 + \lambda_2 = tr\mathbf{A}, \text{ and } \lambda_1\lambda_2 = |\mathbf{A}|$$

6. $IS - LM$ dynamics. For the exercise given in Section 7 of the notes, establish the conditions for local stability. Express the conditions in terms of the relative slopes of the IS and LM curves. Sketch the phase diagram in r, y space.

7. Find the general solution to the pair of differential equations:

$$\begin{aligned} \dot{x} &= -2x + y + 2 \\ \dot{y} &= x - 2y + 2 \end{aligned}$$

   Sketch on a phase diagram the properties of the solution. How are the solution paths related to the eigenvectors of the dynamic system?

8. Use the method of undetermined coefficients to find a particular solution to the differential equation $\ddot{z} + \dot{z} - 12z + 12t^2 = 0$.

9. Establish the stability of: $\ddot{z} + \dot{z} - 12z = 0$. Express this equation as a coupled 1st order linear differential equation of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Find the general solution for this system and sketch the solution on a phase diagram. If $y(0) = 10$, what value must $x(0)$ take for the subsequent path of x and y to converge to a stable equilibrium? (x and y are the elements of the vector $\mathbf{x}$).

10. Find the general solution to the following pair of difference equations:

$$\begin{aligned} x_{t+1} &= -x_t - 2y_t + 3t - 2 \\ y_{t+1} &= -2x_t + 2y_t + t + 1 \end{aligned}$$

   with initial conditions: $x_0 = 95/36$ and $y_0 = 53/18$.

11. In the standard cobweb model of supply and demand for a commodity, the functions are:

$$\begin{aligned} q_t &= a - bp_t \qquad \text{Demand} \\ q_t &= c + dp_{t-1} \qquad \text{Supply} \end{aligned}$$

where q_t is quantity of commodity and p_t is the price, with a, b, c, d parameters.

(a) Using a phase diagram relating q_t and q_{t+1} (or p_t and p_{t+1}), what is the condition on the supply and demand functions to generate a cobweb cycle that converges to an equilibrium?

(b) Assuming that the supply curve has positive slope, what condition is required on the demand curve for the price and quantity to converge to an equilibrium without a cobweb cycle?

12. Find the stability property of: $\ddot{z} + z = 0$ and show that it may alternatively be expressed as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Show that the solution of $\ddot{z} + z = 0$, subject to $z(0) = z_{0,}$ $\dot{z}(0) = \dot{z}_0$ is:

$z(t) = z_0 \cos t + \dot{z}_0 \sin t$

13. For the differential equation, describing how the stock of external debt evolves over time (see section 10.1 of the notes):

$$\dot{d} = -b(t) + r(t)d$$

If the initial debt at time t_0 is d_0, show from the general solution that the stock of debt at time t is:

$$d(t) = d_0 e^{\int_{t_0}^{t} r(u)du} - \int_{t_0}^{t} b(s) e^{\int_{s}^{t} r(u)du} ds$$