Abstract

A framework for estimation and hypothesis testing of functional restrictions against general alternatives is proposed. The parameter space is a reproducing kernel Hilbert space (RKHS). The null hypothesis does not necessarily define a parametric model. The test allows us to deal with infinite dimensional nuisance parameters. The methodology is based on a moment equation similar in spirit to the construction of the efficient score in semiparametric statistics. The feasible version of such moment equation requires to consistently estimate projections uniformly in the space of RKHS and it is shown that this is possible using the proposed approach. This allows us to derive some tractable asymptotic theory and critical values by fast simulation. The methodology is applied to the study of the trade impact of crude oil futures. For the studied high frequency dataset, various hypotheses are tested which also relate to the propensity of market abuse. Simulation results show that the finite sample performance of the test is consistent with the asymptotics and that ignoring the effect of nuisance parameters highly distorts the size of the tests.

Key Words: Constrained estimation, convergence rates, functional restriction, high frequency trading, hypothesis testing, market impact, nonlinear model, reproducing kernel Hilbert space, spoofing.

JEL: C12; C14; C58.
1 Introduction

Suppose you are a market maker in the crude oil futures market. You are interested in estimating the number of trade arrivals $Y$ in the next one minute, conditioning on a vector of market variables $X$ known at the start of the interval. You decide to minimize the negative log-likelihood for Poisson arrivals with conditional intensity $\exp \{ \mu (X) \}$ for some function $\mu$. For observation $i$, the negative loglikelihood is proportional to

$$\exp \{ \mu (X_i) \} - Y_i \mu (X_i).$$

(1)

You suppose that $\mu$ lies in some infinite dimensional space. For example, to avoid the curse of dimensionality, you could choose

$$\mu (X) := \sum_{k=1}^{K} f^{(k)} (X^{(k)})$$

(2)

where $X^{(k)}$ denotes the $k^{th}$ covariate (the $k^{th}$ element of the $K$-dimensional covariate $X$), and the $f^{(k)}$'s are univariate functions in some possibly infinite dimensional space. However, you suppose that $f^{(1)}$ is a linear function. Here, $X^{(1)}$ could be a volume imbalance, a quantity of interest to high frequency traders and to be defined in Section 6. You want to test whether linearity with respect to the first variable holds under the alternative of a general additive model. You could also test against the alternative of a general continuous multivariate function, not necessarily additive. This paper addresses practical problems such as the above. The paper is not restricted to this Poisson problem or additive models on real valued variables, which are only used as motivation.

From the example above, we need to (i) estimate $\mu$, which in this example we chose to be additive with $f^{(1)}$ linear under the null; we need to (ii) test this additive restriction, against a more general non-parametric alternative. Under the null, the remaining $K-1$ functions in (2) are not specified. Problem (i) is standard, though the actual numerical estimation can pose problems. Having solved problem (i), solution of problem (ii) requires to test a non-parametric hypothesis (linearity for $f^{(1)}$ plus additivity) with infinite dimensional nuisance parameters (the remaining unknown $K-1$ functions) against a more general non-parametric alternative. In this paper, we shall call the restriction under the null semi-parametric. This does not necessarily mean that the parameter of interest is finite dimensional, as often the case in the semiparametric
Semiparametric inference requires that the infinite dimensional parameter and the finite dimensional one are orthogonal in population (e.g., Andrews, 1994, eq. (2.12)). In our Poisson motivating example this is not the case. Even if the restriction is parametric, we do not need to suppose that the parameter value is known under the null. This requires us to modify the test statistic in order to achieve the required orthogonality. Essentially, we project the test statistic on some space that is orthogonal to the infinite dimensional nuisance parameter. This is the procedure involved in the construction of the efficient score in semiparametric statistics. The reader is referred to van der Vaart (1998) for a review of the basic idea. Recent advances in the literature under great degree of generality can be found in Belloni et al. (2017) and references therein. However, here the focus is different as we are concerned with functional restrictions and are able to obtain critical values by fast simulation instead of relying on resampling methods. In the context of our example, the problem also needs to be tackled allowing for possibly dependent observations. The extension to dependence is not particularly complicated, but will require us to join together various results in a suitable way.

Throughout, we shall use the framework of reproducing kernel Hilbert spaces. Estimation in these spaces has been studied in depth and is flexible and intuitive from a theoretical point of view. RKHS also allow us to consider multivariate problems in a very natural way. In consequence of these remarks, this paper’s main contribution to the literature is related to testing rather than estimation. Nevertheless, as far as estimation is concerned, we do provide results that are partially new. For example, we establish insights regarding the connection between constrained and penalized estimation together with convergence rates.

Estimation in RKHS can run into computational issues when the sample size is large, as it might be the case in the presence of large datasets. We will address practical computational issues. Estimation of the model can be carried out via a greedy algorithm, possibly imposing LASSO kind of constraints under additivity.

It is worth nothing that for the results derived here to hold, the RKHS setup is necessary. We need to derive consistency results for the empirical projection under the uniform operator topology. Under the null hypothesis, this allows us to find a representation for the limiting asymptotic distribution which is amenable of fast simulation. In consequence critical values do not need to be generated using resampling procedures. While the discussion of the asymptotic validity of the procedure is involved, the implementation of the test is relatively simple. The Matlab code for greedy
estimation and to perform the test and its critical values is available from the URL: <https://github.com/asancetta/ARKHS/>. We shall refer to the specific functions in the repository in the text of the paper.

Part of this research was motivated by the need to test functional restrictions as done in the empirical section of the present paper. There, we study the market impact of trades on crude oil futures prices. As a byproduct of our analysis we infer that the CME crude oil futures market is not prone to spoofing using orders deep in the book, though top of book orders could still be used to manipulate the market. Spoofing is an illegal practice used to provide a false view of market demand and supply. A set of simulations confirms that the procedure works well, and illustrates the well known fact that nuisance parameters can considerably distort the size of a test if not accounted for using our proposed procedure. The reader can have a preliminary glance at Table 1 in Section 2.1, and Table 5 in Section 7.1 to see this more vividly.

1.1 Relation to the Literature

Estimation in RKHS has been addressed in many places in the literature (Wahba, 1990, and Steinwart and Christmann, 2008, for a textbook treatment). Inference is usually confined to consistency (e.g., Mendelson, 2002, Christmann and Steinwart, 2007), though there are exceptions (Hable, 2012, in the frequentist framework). A common restriction used in the present paper is additivity and estimation in certain subspaces of additive functions. Estimation of additive models has been extensively studied by various authors using different techniques (e.g., Buja et al., 1989, Linton and Nielsen, 1995, Mammen et al., 1999, Meier et al., 2009, Christmann and Hable, 2012). The last reference considers estimation in RKHS which allows for a more general concept of additivity. Here, the assumptions and estimation results are not overall necessarily comparable to existing results, and tend to be more tailored to econometric applications with no independence requirement and concept of true model. Moreover, we establish rates of convergence and the link between constrained versus penalized estimation in RKHS. The two are not always equivalent.

The problem of parametric inference in the presence of non-orthogonal nuisance parameters has been addressed by various authors by modification of the score function or equivalent quantities. Belloni et al. (2017) provide very general results in the context of high dimensional models. There, the reader can also find the main references in that literature. The asymptotic distribution usually requires the use of the bootstrap in
order to compute critical values.

The problem of testing parametric restrictions with no infinite dimensional nuisance parameter under the null against general nonparametric alternatives is well known (Härdle and Mammen, 1993) and requires the use of the bootstrap in order to derive confidence intervals. Fan et al. (2001) have developed a Generalized Likelihood Ratio test of the null of parametric or nonparametric additive restrictions versus general nonparametric ones. This is based on a Gaussian error model (or parametric error distribution) for additive regression, and estimation using smoothing kernels. Fan and Jiang (2005) has extended this approach to the nonparametric error distribution, but the asymptotic distribution is Chi-square with degrees of freedom equal to some (computable) function of kernel, the smoothing parameter and the data. Chen et al.(2014) considers the framework of sieve estimation and derives a likelihood ratio statistic with asymptotic Chi-square distribution (see also Shen and Shi, 2005).

The approach considered here is complementary to the above references. It allows the parameter space to be a RKHS of smooth functions. Estimation in RKHS is well understood and can cater for many circumstances of interest in applied work. The testing procedure is based on a corrected moment condition. Hence, does not rely on likelihood estimation. The conditions used are elementary, as they just require existence of real valued derivatives of the loss function and mild regularity conditions on the covariance kernel. We also allow for dependent errors. The correction is estimated by ridge regression or just ordinary least square using pseudo-inverse.

For moderate sample sizes (e.g. less than 10,000) estimation in RKHS does not pose particular challenges and it is trivial for the regression problem under the square error loss. For large sample sizes, computational aspects in RKHS have received a lot of attention in the literature (e.g., Rasmussen and Williams, 2006, Ch.8, Banerjee et al., 2008, Lazaro-Gredilla et al., 2010).

Here we discuss a greedy algorithm, which is simple to implement (e.g., Jaggi, 2013, Sancetta, 2016) and, apparently, has not been applied to the RKHS estimation framework of this paper.

1.2 Outline

The plan for the paper is as follows. Section 2 reviews some basics of RKHS and defines the problem and model used in the paper. Section 3 contains the statistical analysis of the estimation problem and the proposed testing procedure in the presence
of nuisance parameters. Section 4 provides the details to implement the constrained estimation. Section 5 applies the theoretical results of Sections 3 and 4 to the analysis market impact in the crude oil futures markets. Section 6 contains further remarks with numerical examples via simulations, and some partial extensions to non-smooth loss functions. The proofs, and additional results are in the Appendices as supplementary material.

2 The Inference Problem

The explanatory variables $X^{(k)}$ take values in $\mathcal{X}$, a compact subset of a separable Banach space $(k = 1, 2, ..., K)$. The most basic example of $\mathcal{X}$ is $[0, 1]$. The vector covariate $X = (X^{(1)}, ..., X^{(K)})$ take values in the Cartesian product $\mathcal{X}^K$, e.g., $[0, 1]^K$. The dependent variable takes values in $\mathcal{Y}$ usually $\mathbb{R}$. Let $Z = (Y, X)$ and this takes values in $\mathcal{Z} = \mathcal{Y} \times \mathcal{X}^K$. If no dependent variable $Y$ can be defined (e.g., unsupervised learning, or certain likelihood estimators), $\mathcal{Z} = \mathcal{X}^K$. Let $P$ be the law of $Z$, and use linear functional notation, i.e., for any $f : \mathcal{Z} \to \mathbb{R}$, $Pf = \int_{\mathcal{Z}} f(z) dP(z)$. Let $P_n = \frac{1}{n} \sum_{i=1}^n \delta_{Z_i}$, where $\delta_{Z_i}$ is the point mass at $Z_i$, implying that $P_n f = \frac{1}{n} \sum_{i=1}^n f(Z_i)$ is the sample mean of $f(Z)$. For $p \in [1, \infty]$, let $|\cdot|_p$ be the $L_p$ norm (w.r.t. the measure $P$), e.g., for $f : \mathcal{Z} \to \mathbb{R}$, $|f|_p = (P|f|^p)^{1/p}$, with the obvious modification to sup norm when $p = \infty$.

2.1 Motivation

The problem can be described as follows, though in practice we will need to add extra regularity conditions. Let $\mathcal{H}^K$ be a vector space of functions on $\mathcal{X}^K$, equipped with a norm $|\cdot|_{\mathcal{H}^K}$. Consider a loss function $L : \mathcal{Z} \times \mathbb{R} \to \mathbb{R}$. We shall be interested in the case when the second argument is $\mu(x) : L(z, \mu(x))$. Therefore, to keep notation compact, let $\ell_\mu(Z) = L(Z, \mu(X))$. For the special case of the square error loss we would have $\ell_\mu(z) = L(z, \mu(x)) = |y - \mu(x)|^2 (z = (y, x))$. The use of $\ell_\mu$ makes it more natural to use linear functional notation. The unknown function of interest is the minimizer $\mu_0$ of $P\ell_\mu$, and it is assumed to be in $\mathcal{H}^K$. We find an estimator $\mu_n = \arg \inf \mu P_n \ell_\mu$ where the infimum is over certain functions $\mu$ in $\mathcal{H}^K$. The main goal it to test the restriction that $\mu_0 \in \mathcal{R}_0$ for some subspace $\mathcal{R}_0$ of $\mathcal{H}^K$ (for example a linear restriction). The restricted
estimator in $\mathcal{R}_0$ is denoted by $\mu_{0n}$. To test the restriction we can look at how close

$$\sqrt{n} P_n \partial \ell_{\mu_{0n}} h = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \partial \ell_{\mu_{0n}} (Z_i) h (X_i)$$

is to zero for suitable choice of $h \in \mathcal{H}^K \setminus \mathcal{R}_0$. Throughout, $\partial^k \ell_{\mu} (z) = \partial^k L (z, t) / \partial t^k |_{t=\mu(x)}$ is the $k^{th}$ partial derivative of $L (z, t)$ with respect to $t$ and then evaluated at $\mu (x)$. The validity of this derivative and other related quantities will be ensured by the regularity conditions we shall impose. The compact notation on the left hand side (l.h.s.) of (3) shall be used throughout the paper. If necessary, the reader can refer to Section 9.5 in the Appendix (supplementary material) for more explicit expressions when the compact notation is used in the main text.

If $\mathcal{R}_0$ is finite dimensional, or $\mu_{0n}$ is orthogonal to the functions $h \in \mathcal{H}^K \setminus \mathcal{R}_0$ (e.g., Andrews, 1994, eq. 2.12), the above display is $\sqrt{n} P_n \ell_{\mu_{0n}} h$ to first order, under regularity conditions. However, unless the sample size is relatively large, this approximation may not be good. In fact, supposing stochastic equicontinuity and the null that $\sqrt{n} P \partial \ell_{\mu} h = 0$, it can be shown that (e.g., Theorem 3.3.1 in van der Vaart and Wellner, 2000),

$$\sqrt{n} P_n \partial \ell_{\mu_{0n}} h = \sqrt{n} P_n \partial \ell_{\mu_{0}} h + \sqrt{n} P \partial^2 \ell_{\mu_0} (\mu_{0n} - \mu_0) h + o_p (1).$$

The orthogonality condition in Andrews (1994, eq., 2.12) guarantees that the second term on the right hand side (r.h.s.) is zero (see Andrews, 1994, eq. 2.8 assuming Fréchet differentiability). Hence, we aim to find/construct functions $h \in \mathcal{H}^K \setminus \mathcal{R}_0$ such that the second term on the r.h.s. is zero. In fact this term can severely distort the asymptotic behaviour of $\sqrt{n} P_n \partial \ell_{\mu_{0n}} h$.

An example is given in Table 1 which is an excerpt from the simulation results in Section 7.1. Here, the true model is a linear model model with 3 variables plus Gaussian noise such that the signal to noise ratio is one. We call this model Lin3. We use a sample of $n = 1000$ observations with $K = 10$ variables, where only the first three variables enter the true model. We correctly suppose the null hypothesis that the first three variables are linear against an alternative that all $K = 10$ variables enter the true model in an additive nonlinear form. The subspace of these three linear functions is $\mathcal{R}_0$ while the full model is $\mathcal{H}^K$. The test functions $h$ are restricted to a polynomials with no linear term. Details can be found in Section 7.1. The nuisance parameters are the three estimated linear functions, which are low dimensional. It might be plausible
Table 1: Frequency of rejections. Results from 1000 simulations when the number of covariates $K = 10$ and the true model is Lin3 (only the first three variables enter the model and they do so in a linear way). The column No $\Pi$ denotes test results using instruments in $H^K \setminus R_0$. The column $\Pi$ denotes test results using instruments in $H^K \setminus R_0$ that have been made orthogonal to the functions in $R_0$ using the empirical procedure discussed in this paper. The signal to noise ratio is denoted by $\sigma^2_{\mu/\varepsilon}$, while all the variables have equal pairwise correlation equal to $\rho$. The column Size denotes the theoretical size of the test. A value in columns No $\Pi$ and $\Pi$ smaller than 0.05 indicates that the test procedure rejects less often than it should be.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma^2_{\mu/\varepsilon}$</th>
<th>Size</th>
<th>No $\Pi$</th>
<th>$\Pi$</th>
<th>No $\Pi$</th>
<th>$\Pi$</th>
</tr>
</thead>
<tbody>
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<td>0.75</td>
<td>1</td>
<td>0.05</td>
<td>0.03</td>
<td>0.06</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.05</td>
<td>0.02</td>
<td>0.05</td>
<td>0.03</td>
<td>0.05</td>
</tr>
</tbody>
</table>

to think that estimation of the three linear functions should not affect the asymptotic distribution. When the variables are uncorrelated, this is clearly the case as confirmed by the 5% size of the test in Table 1. It does not matter whether we use instruments $h \in H^K \setminus R_0$ that are orthogonal to the linear functions or not. However, as soon as the variables become correlated, Table 1 shows that the asymptotic distribution can be distorted. This happens even in such a simple finite dimensional problem. Nevertheless, the test that uses instruments that are made orthogonal to functions in $R_0$ is not affected. The paper will discuss this empirical procedure in details and study its properties via asymptotic analysis and simulations.

The situation gets really worse with other simulation designs that can be encountered in applications and details are given in Section 7.1. More generally, $R_0$ can be a high dimensional subspace of $H^K$ or even an infinite dimensional one, e.g. the space of additive functions when $H^K$ does not impose this additive restriction. In this case, it is unlikely that functions in $H^K \setminus R_0$ are orthogonal to functions in $R_0$ and the distortion due to the nuisance parameters will be larger than what is shown in Table 1.

Here, orthogonal functions $h \in H^K \setminus R_0$ are constructed to asymptotically satisfy

$$P \partial^2 \ell_{\mu_0} \nu h = 0$$

for any $\nu \in R_0$. The above display will allow us to carry out inferential procedures as in cases previously considered in the literature. The challenge is that the set of such orthogonal functions $h \in H^K \setminus R_0$ needs to be estimated. It is not clear before hand
that estimation leads to the same asymptotic distribution as if this set were known. We show that this is the case. Suppose that \( \{ \hat{h}^{(r)} : r = 1, 2, \ldots, R \} \) is a set of such estimated orthogonal functions using the method to be spelled out in this paper. The test statistic is

\[
\hat{S}_n = \frac{1}{R} \sum_{r=1}^{R} \left( \sqrt{n} P_n \partial \ell_{\mu_0} \hat{h}^{(r)} \right)^2.
\]  

We show that its asymptotic distribution can be easily simulated.

Next, some basics of RKHS are reviewed and some notation is fixed. Restrictions for functions in (2) are discussed and finally the estimation problems is defined.

### 2.2 Additional Notation and Basic Facts about Reproducing Kernel Hilbert Spaces

Recall that a RKHS \( \mathcal{H} \) on some set \( \mathcal{X} \) is a Hilbert space where the evaluation functionals are bounded. A RKHS of bounded functions is uniquely generated by a centered Gaussian measure with covariance \( C \) (e.g., Li and Linde, 1999) and \( C \) is usually called the (reproducing) kernel of \( \mathcal{H} \). We consider covariance functions with representation

\[
C(s, t) = \sum_{v=1}^{\infty} \lambda_v^2 \varphi_v(s) \varphi_v(t),
\]

for linearly independent functions \( \varphi_v : \mathcal{X} \to \mathbb{R} \) and coefficients \( \lambda_v \) such that \( \sum_{v=1}^{\infty} \lambda_v^2 \varphi_v^2(s) < \infty \). Here, linear independent means that if there is a sequence of real numbers \( (f_v)_{v \geq 1} \) such that \( \sum_{v=1}^{\infty} f_v^2 / \lambda_v^2 < \infty \) and \( \sum_{v=1}^{\infty} f_v \varphi_v(s) = 0 \) for all \( s \in \mathcal{X} \) (in the support of \( P \)), then \( f_v = 0 \) for all \( v \)'s. The \( \lambda_v^2 \)'s would be the eigenvalues of (6) if the \( \varphi_v \)'s were orthonormal, but this is not implied by the above definition of linear independence. The representation (6) is not unique. However, it always exists when the covariance is continuous (Karhunen–Loève theorem, e.g., Bosq, 2000). The representation makes explicit the basis \( \{ \lambda_1 \varphi_1, \lambda_2 \varphi_2, \ldots \} \). In fact, \( \mathcal{H} \) is the completion of the set of functions representable as \( f(x) = \sum_{v=1}^{\infty} f_v \varphi_v(x) \) for real valued \( f_v \)'s such that \( \sum_{v=1}^{\infty} f_v^2 / \lambda_v^2 < \infty \). Equivalently, \( f(x) = \sum_{j=1}^{\infty} \alpha_j C(s_j, x) \), for \( s_j \)'s in \( \mathcal{X} \) and coefficients \( \alpha_j \)'s satisfying \( \sum_{j=1}^{\infty} \alpha_j \lambda_j^2 C(s_j, s_j) < \infty \). Moreover, for \( C \) in (6),

\[
\sum_{j=1}^{\infty} \alpha_j C(s_j, x) = \sum_{v=1}^{\infty} \left( \sum_{j=1}^{\infty} \alpha_j \lambda_j^2 \varphi_v(s_j) \right) \varphi_v(x) = \sum_{v=1}^{\infty} f_v \varphi_v(x)
\]

(7)
by obvious definition of the \( f_v \)'s. The change of summation is possible by the aforementioned restrictions on the \( \lambda_v \)'s and \( \varphi_v \)'s. The inner product in \( \mathcal{H} \) is denoted by \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and satisfies \( f(x) = \langle f, C(x, \cdot) \rangle_{\mathcal{H}} \). This implies the reproducing kernel property \( C(s,t) = \langle C(s, \cdot), C(t, \cdot) \rangle_{\mathcal{H}} \). Therefore, the square of the RKHS norm is defined in the two following equivalent ways

\[
|f|^2_{\mathcal{H}} = \sum_{v=1}^{\infty} \frac{f_v^2}{\lambda_v^2} = \sum_{i,j=1}^{\infty} \alpha_i \alpha_j C(s_i, s_j)
\] (8)

Throughout, the unit ball of \( \mathcal{H} \) will be denoted by \( \mathcal{H}(1) := \{ f \in \mathcal{H} : |f|_{\mathcal{H}} \leq 1 \} \).

The additive RKHS is generated by the Gaussian measure with covariance function \( C_{K^\mathcal{H}}(s,t) = \sum_{k=1}^{K} C^{(k)}(s^{(k)}, t^{(k)}) \), where \( C^{(k)}(s^{(k)}, t^{(k)}) \) is a covariance function on \( \mathcal{X} \times \mathcal{X} \) (as \( C \) in (6)) and \( s^{(k)} \) is the \( k \)th element in \( s \in \mathcal{X}^K \). The RKHS of additive functions is denoted by \( \mathcal{H}^K \), which is the set of functions as in (2) such that \( f^{(k)} \in \mathcal{H} \) and \( \sum_{k=1}^{K} |f^{(k)}|^2_{\mathcal{H}} < \infty \). For such functions, the inner product is \( \langle f, g \rangle_{\mathcal{H}^K} = \sum_{k=1}^{K} \langle f^{(k)}, g^{(k)} \rangle_{\mathcal{H}} \), where to avoid overloading the notation, the individual RKHS are supposed to be the same. However, in some circumstances, it can be necessary to make the distinction between the spaces (see Example 6 in Section 3.3). The norm \(|\cdot|_{\mathcal{H}^K} \) on \( \mathcal{H}^K \) is the one induced by the inner product.

Within this scenario, the space \( \mathcal{H}^K \) restricts functions to be additive, where these additive functions in \( \mathcal{H} \) can be multivariate functions.

**Example 1** Suppose that \( K = 1 \) and \( \mathcal{X} = [0,1]^d \) (\( d > 1 \)) (only one additive function, which is multivariate). Let \( C(s,t) = \exp \left\{ -a \sum_j |s_j - t_j|^2 \right\} \) where \( s_j \) is the \( j \)th element in \( s \in [0,1]^d \), and \( a > 0 \). Then, the RKHS \( \mathcal{H} \) is dense in the space of continuous bounded functions on \( [0,1]^d \) (e.g., Christmann and Steinwart, 2007). A (kernel) \( C \) with such property is called universal.

Additive versions of the conditional CAPM (Example 7) provide an application of multivariate additivity. The framework also covers the case of functional data because \( \mathcal{X} \) is a compact subset of a Banach space (e.g., Bosq, 2000). Most problems of interest where the unknown parameter \( \mu \) is a smooth function are covered by the current scenario.
2.3 The Estimation Problem

Estimation will be considered for models in \( \mathcal{H}^K (B) := \{ f \in \mathcal{H}^K : |f|_{\mathcal{H}^K} \leq B \} \), where \( B < \infty \) is a fixed constant. The goal is to find

\[
\mu_n = \arg \inf_{\mu \in \mathcal{H}^K (B)} P_n \ell_\mu, \tag{9}
\]

i.e. the minimizer with respect to \( \mu \in \mathcal{H}^K (B) \) of the loss function \( P_n \ell_\mu \).

Example 2 Let \( \ell_\mu (z) = |y - \mu (x)|^2 \) so that

\[
P_n \ell_\mu = \frac{1}{n} \sum_{i=1}^{n} \ell_\mu (Z_i) = \frac{1}{n} \sum_{i=1}^{n} |Y_i - \mu (X_i)|^2.
\]

By duality, we can also use \( P_n \ell_\mu + \rho_{B,n} |\mu|_{\mathcal{H}^K}^2 \) with sample dependent Lagrange multiplier \( \rho_{B,n} \) such that \( |\mu|_{\mathcal{H}^K} \leq B \).

For the square error loss the solution is just a ridge regression estimator with (random) ridge parameter \( \rho_{B,n} \). Interest is not restricted to least square problems.

Example 3 Consider the negative log-likelihood where \( Y \) is a duration, and \( \mathbb{E} [Y|X] = \exp \{ \mu (X) \} \) is the hazard function. Then, \( \ell_\mu (z) = y \exp \{ \mu (x) \} - \mu (x) \) so that \( P_n \ell_\mu = \frac{1}{n} \sum_{i=1}^{n} Y_i \exp \{ \mu (X_i) \} - \mu (X_i) \).

Even though the user might consider likelihood estimation, there is no concept of “true model” in this paper. The target is the population estimate

\[
\mu_0 = \arg \inf_{\mu \in \mathcal{H}^K (B)} P \ell_\mu. \tag{10}
\]

We shall show that this minimizer always exists and is unique under regularity conditions on the loss because \( \mathcal{H}^K (B) \) is closed.

For arbitrary loss functions, the (Nonparametric) Representer Theorem (Schölkopf et al., 2001, Theorem 1) says that the solution to the penalised problem takes the form

\[
\mu_n (x) = \sum_{i=1}^{n} \alpha_i C (X_i, x)
\]

for some \( \alpha_i \)'s. Hence, even if the parameter space where the estimator lies is infinite dimensional, \( \mu_n \) is not. This fact will be used without further mention in the matrix implementation of the testing problem.
2.4 The Testing Problem

Inference needs to be conducted on the estimator in (9). To this end, consider inference on functional restrictions possibly allowing \( \mu \) not to be fully specified under the null. Within this framework, tests based on the moment equation \( P_n \partial \ell_\mu h \) for suitable test functions \( h \) are natural (recall (5)). Let \( \mathcal{R}_0 \subset \mathcal{H}^K \) be the RKHS with kernel \( C_{\mathcal{R}_0} \). Suppose that we can write \( C_{\mathcal{H}^K} = C_{\mathcal{R}_0} + C_{\mathcal{R}_1} \), where \( C_{\mathcal{R}_1} \) is some suitable covariance function. Under the null hypothesis we suppose that \( \mu_0 \in \mathcal{R}_0 \) (\( \mu_0 \) in (10)). Under the alternative, \( \mu_0 \notin \mathcal{R}_0 \). Define

\[
\mu_n := \arg \inf_{\mu \in \mathcal{R}_0(B)} P_n \ell_\mu \tag{11}
\]

where \( \mathcal{R}_0(B) = \mathcal{R}_0 \cap \mathcal{H}^K(B) \). This is the estimator under the null hypothesis. For this estimation, we use the kernel \( C_{\mathcal{R}_0} \). The goal is to consider the quantity in (3) with \( h \in \mathcal{R}_1 \).

2.4.1 Matrix Implementation

We show how to construct the statistic in (5) using matrix notation. Consider the regression problem under the square error loss: nonlinear least squares. Let \( C \) be the \( n \times n \) matrix with \((i, j)\) entry equal to \( C_{\mathcal{H}^K} (X_i, X_j) \), \( y \) the \( n \times 1 \) vector with \( i^{th} \) entry equal to \( Y_i \). The penalized estimator is the \( n \times 1 \) vector \( a := (C + \rho I)^{-1} y \). Here, \( \rho \) can be chosen such that \( a^T C a \leq B^2 \) so that the constraint is satisfied: \( \mu_n(\cdot) = \sum_{i=1}^n a_i C_{\mathcal{H}^K}(X_i, \cdot) \) is in \( \mathcal{H}^K(B) \); here \( a_i \) is the \( i^{th} \) entry in \( a \). For other problems the solution is still linear, but the coefficients usually do not have a closed form. For the regression problem under the square error loss, the \( \rho \) that satisfies the constraint is given by the solution of

\[
\sum_{i=1}^n (y^T Q_i)^2 \frac{r_i}{r_i + \rho} = B^2
\]

where \( Q_i \) is the \( i^{th} \) eigenvector of \( C \) and \( r_i \) is the corresponding eigenvalue.

The restricted estimator has the same solution with \( C \) replaced by \( C_0 \) which is the matrix with \((i, j)\) entry \( C_{\mathcal{R}_0}(X_i, X_j) \). For the square error loss, let \( e_0 = Y - C_0 a_0 \) be the vector or residuals under the null. (For other problems, \( e_0 \) is the vector of generalised residuals, i.e. the \( i^{th} \) entry in \( e_0 \) is \( \partial \ell_{\mu_0,n}(Z_i) \).) Under the alternative we have the covariance kernel \( C_{\mathcal{R}_1} \). Denote by \( C_1 \) the matrix with \((i, j)\) entry \( C_{\mathcal{R}_1}(X_i, X_j) \). Let \( S \) be the diagonal matrix with \((i, i)\) diagonal entry equal to \( \partial^2 \ell_{\mu_0,n}(Z_i) \). In our case,
this entry can be taken to be one, as the second derivative of the square error loss is a constant. However, the next step is the same regardless of the loss function and the problem, as we need to project the functions in $R_1$ onto $R_0$ and only consider the orthogonal part so that the sample version of the orthogonality condition (4) is satisfied. We regress each column in $C_1$ on the columns of $C_0$. We denote by $C_1^{(r)}$ the $r^{th}$ column in $C_1$. We approximately project $C_1^{(r)}$ onto the column space spanned by $C_0$ minimising the loss function

$$
\left( C_1^{(r)} - C_0 b^{(r)} \right)^T S \left( C_1^{(r)} - C_0 b^{(r)} \right) + \rho \left( b^{(r)} \right)^T C_0 b^{(r)}.
$$

Here $\rho$ is chosen to go to zero with the sample size (Theorem 3 and Corollary 2). In applications, we may just use a subset of $R$ columns from $C_1$ and to avoid notational trivialities, say the first $R$. The solution for all $r = 1, 2, ..., R$ is

$$
b^{(r)} = \left( C_0 + \rho S^{-1} \right)^{-1} C_1^{(r)}
$$

and can be verified substituting in the first order conditions. Let the residual vector from this regression be $e_1^{(r)}$. This is (in sample) orthogonal to the column space of $C_0$ when $\rho = 0$. We define the $r^{th}$ instruments by $\hat{h}^{(r)} = e_1^{(r)}$. The test statistic is $\hat{S}_n = \sum_{r=1}^{R} \left( e_0^T \hat{h}^{(r)} \right)^2 / R$. Under regularity conditions, if the true parameter $\mu_0$ lies inside $R_1 \cap H^K (B)$, the $R \times 1$ vector $s = \left( e_0^T \hat{h}^{(1)}, e_0^T \hat{h}^{(2)}, ..., e_0^T \hat{h}^{(R)} \right)^T$ is asymptotically Gaussian for any $R$ and its covariance matrix is consistently estimated by

$$(n^{-1} e_0^T e_0) \sum_{k,l=1}^{R} \left[ n^{-1} \left( \hat{h}^{(k)} \right)^T \hat{h}^{(l)} \right] .$$

The distribution of $\hat{S}_n$ can be simulated from the process $\sum_{l=1}^{R} \omega_{n,l} N_l^2$, where the $N_j$’s are i.i.d. standard normal random variables and the $\omega_{n,j}$’s are eigenvalues of the estimated covariance matrix.

**Operational remarks.**

1. If $C_{R_1}$ is not explicitly given, we can set $C_{R_1} = C_{H^K}$ in the projection step.

2. Instead of $C_1 \times n$ we can use a subset of the columns of $C_1$, e.g. $R < n$ columns.

   Each column is an instrument.

3. The column $r$ of $C_1$ can be replace by an $n \times 1$ vector with $i^{th}$ entry $C_{R_1} (X_i, z_r)$ where $z_r$ is an arbitrary element in $X^K$. 

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4. When the series expansion (6) for the covariance is known, we can use the elements in the expansion because the functions $\lambda_v^{1/2} \varphi_v(\cdot)$ are in $\mathcal{H}^K(1)$ for each $v = 1, 2, \ldots$. Hence, we can project the elements in the series expansion of $C_{R_1}$ onto the elements in the series expansion of $C_{R_0}$. Hence, the procedure covers sieve estimators (under restrictions on their coefficients).

**Additional remarks.** The procedure can be seen as a J-Test where the instruments are given by the $\hat{h}^{(r)}$'s. Given that the covariance matrix of the vector $s$ can be high dimensional (many instruments for large $R$) we work directly with the unstandardized statistic. This is common in some high dimensional problems, as it is the case of functional data (Horváth and Kokoszka, 2012, Theorem 5.1).

We could replace $\hat{S}_n$ with $\max_{r \leq R} \mathbb{E}_0^T \hat{h}^{(r)}$. The maximum of correlated Gaussian random variables can be simulated or approximated but it might be operationally challenging (Hartigan, 2014, Theorem 3.4).

The rest of the paper provides details and justification for the estimation and testing procedure. The theoretical justification beyond simple heuristics is technically involved. Section 7.1 (Table 5) will show that failing to use the projection procedure discussed in this paper leads to poor results. Additional details can be found in Appendix 2 (supplementary material).

## 3 Asymptotic Results

### 3.1 Conditions for Basic Analysis

Throughout the paper, $\lesssim$ means that the l.h.s. is smaller than the r.h.s., up to a finite multiplicative constant.

**Condition 1** The set $\mathcal{H}$ is a RKHS on a compact subset of a separable Banach space $\mathcal{X}$, with continuous uniformly bounded kernel $C$ admitting an expansion (6), where $\lambda_v^2 \lesssim v^{-2\lambda}$ with $\lambda > 1$ and with (linearly independent) continuous uniformly bounded $\varphi_v(s)$'s for $s \in \mathcal{X}$, such that $\inf_v \mathbb{P}_{\varphi_v^2} = 1$. If each additive component has difference covariance kernel, the condition is meant to apply to each of them individually.

Attention is restricted to loss functions satisfying the following, though generalizations will be considered in Section 7.2. Recall the loss $L(z, t)$ from Section 2.1. Let $\bar{B} :=$
where \( c_K := \max_{s \in \mathcal{X}_K} \sqrt{C_K(s,s)} \). Define \( \Delta_k(z) := \max_{|t| \leq \bar{B}} |\partial^k L(z,t)/\partial t^k| \) for \( k = 0, 1, 2, \ldots \).

**Condition 2** The loss \( L(z,t) \) is non-negative, twice continuously differentiable for real \( t \) in an open set containing \([-\bar{B}, \bar{B}]\), and \( \inf_{z,t} d^2 L(z,t)/dt^2 > 0 \) for \( z \in \mathcal{Z} \) and \( t \in [-\bar{B}, \bar{B}] \). Moreover, \( P(\Delta_0 + \Delta_1^p + \Delta_2^p) < \infty \) for some \( p > 2 \).

The data are allowed to be weakly dependent, but restricted to uniform regularity.

**Condition 3** The sequence \((Z_i)_{i \in \mathbb{Z}} \) (\( Z_i = (Y_i, X_i) \)) is stationary, with beta mixing coefficient \( \beta(i) \lesssim (1 + i)^{-\beta} \) for \( \beta > p/ (p - 2) \), where \( p \) is as in Condition 2.

Remarks on conditions can be found in Section 4.1.

### 3.2 Basic Results

The results in this section are concerned with consistency and some basic convergence in distribution. In part they can be viewed as a review except for the fact that we allow for dependent random variables. We also provide details regarding the relation between constrained and penalized estimators and convergence rates. The usual penalized estimator is defined as

\[
\mu_{n,\rho} = \arg \inf_{\mu \in \mathcal{H}_K} P_n \ell_\mu + \rho |\mu|_{\mathcal{H}_K}^2. \tag{12}
\]

As mentioned in Example 2, suitable choice of \( \rho \) leads to the constrained estimator. Throughout, \( \text{int} (\mathcal{H}_K(B)) \) will denote the interior of \( \mathcal{H}_K(B) \).

**Theorem 1** Suppose that Conditions 1, 2, and 3 hold. The population minimizer in (10) is unique up to an equivalence class in \( L_2 \).

1. There is a random \( \rho = \rho_{B,n} \) such that \( \rho = O_p(n^{-1/2}) \), \( \mu_{n,\rho} = \mu_n \) and if \( \mu_0 \in \mathcal{H}_K(B) \), \( |\mu_n - \mu_0|_\infty \to 0 \) in probability where \( \mu_n \) is as in (9).

2. For (9) we also have that \( |\mu_n - \mu_0|_2 = O_p(n^{-(2\lambda-1)/(4\lambda)}) \) and if \( \mathcal{H}_K \) is finite dimensional the r.h.s. is \( O_p(n^{-1/2}) \).

3. Consider possibly random \( \rho = \rho_n \) such that \( \rho \to 0 \) and \( \rho n^{1/2} \to \infty \) in probability. Suppose that there is a finite \( B \) such that \( \mu_0 \in \text{int} (\mathcal{H}_K(B)) \) (i.e. \( \mu_0 \in \mathcal{H}_K \)). Then, \( |\mu_{n,\rho}|_{\mathcal{H}_K} < B \) and \( |\mu_{n,\rho} - \mu_0|_{\mathcal{H}_K} \to 0 \) in probability.
4. If \( \mathcal{H}^K \) is infinite dimensional, there is a \( \rho = \rho_n \) such that \( \rho \to 0, \ \rho n^{1/2} \to \infty \), and \( |\mu_{n,\rho} - \mu_0|_\infty \to 0 \) in probability, but \( |\mu_{n,\rho} - \mu_0|_{\mathcal{H}^K} \) does not converge to zero in probability.

All the above consistency statements hold if \( \mu_n \) and \( \mu_{n,\rho} \) in (9) and (12) are approximate minimizers in the sense that the following hold

\[
P_n \ell_{\mu_n} \leq \inf_{\mu \in \mathcal{H}^K (B)} P_n \ell_{\mu} + o_p (1)
\]

and

\[
P_n \ell_{\mu_{n,\rho}} + \rho |\mu_{n,\rho}|_{\mathcal{H}^K} \leq \inf_{\mu \in \mathcal{H}^K} \{ P_n \ell_{\mu} + \rho |\mu|_{\mathcal{H}^K} \} + o_p (\rho).
\]

The above result establishes the clear connection between the constrained estimator \( \mu_n \) in (9) and the penalized estimator \( \mu_{n,\rho} \) in (12). It is worth noting that whether \( \mathcal{H}^K \) is finite or infinite dimensional, the estimator \( \mu_n \) is equivalent to a penalized estimator with penalty parameter \( \rho \) going to zero relatively fast (i.e. \( \rho n^{1/2} \to \infty \) does not hold). However, this only ensures uniform consistency and not consistency under the RKHS norm \( |\cdot|_{\mathcal{H}^K} \) (Point 3 in Theorem 1). For the testing procedure discussed in this paper, we need the estimator to be equivalent to a penalized one with penalty that converges fast. This is achieved working with the constrained estimator \( \mu_n \).

Having established consistency, interest also lies in the distribution of the estimator. We shall only consider the constrained case \( \mu_n \). To ease notation, for any arbitrary, but fixed real valued functions \( g \) and \( g' \) on \( \mathcal{Z} \) define \( P_{1,j} (g, g') = \mathbb{E} g (Z_1) g' (Z_{1+j}) \). For suitable \( g \) and \( g' \), the quantity \( \sum_{j \in \mathbb{Z}} P_{1,j} (g, g') \) will be used as short notation for sums of population covariances. We shall also use the additional condition \( |\Delta_3|_{\infty} < \infty \), where \( \Delta_k (z) \) is as in Section 3.1.

**Theorem 2** Suppose Conditions 1, 2, and 3 hold. If \( \mu_0 \in \text{int} (\mathcal{H}^K (B)) \), then

\[
\sqrt{n} P_n \partial \ell_{\mu_0} h \to G (h), \ h \in \mathcal{H}^K (1)
\]

weakly, where \( \{ G (h) : h \in \mathcal{H}^K (1) \} \) is a mean zero Gaussian process with covariance function

\[
\mathbb{E} G (h) G (h') = \sum_{j \in \mathbb{Z}} P_{1,j} (\partial \ell_{\mu_0} h, \partial \ell_{\mu_0} h')
\]

for any \( h, h' \in \mathcal{H}^K (1) \).
Now, in addition to the above, also suppose that $|\Delta_3|_\infty < \infty$. If $\mu_n \in \mathcal{H}^K (B)$ is an asymptotic minimizer such that $P_n \ell_{\mu_n} \leq \inf_{\mu \in \mathcal{H}^K (B)} P_n \ell_\mu + o_p (n^{-1})$, and $\sup_{h \in \mathcal{H}^K (1)} P_n \partial \ell_{\mu_n} h = o_p \left( n^{1/2} \right)$, then,

$$\sqrt{n} P \partial^2 \ell_{\mu_0} (\mu_n - \mu_0) h = \sqrt{n} P \partial \ell_{\mu_0} h + o_p (1), \quad h \in \mathcal{H}^K (1).$$

The second statement in Theorem 2 cannot be established for the penalized estimator with penalty satisfying $\rho n^{1/2} \to \infty$. The restriction $\sup_{h \in \mathcal{H}^K (1)} P_n \partial \ell_{\mu_n} h = o_p \left( n^{-1/2} \right)$ holds for finite dimensional models as long as $\mu_0 \in \text{int} (\mathcal{H}^K (B))$. When testing restrictions, this is often of interest. However, for infinite dimensional models this is no longer true as the constraint is binding even if $\mu_0 \in \text{int} (\mathcal{H}^K (B))$. Then, it can be shown that the $o_p \left( n^{-1/2} \right)$ term has to be replaced with $O_p \left( n^{-1/2} \right)$ (Lemma 8, in the Appendix). This has implications for testing. Additional remarks can be found in Section 4.2.

### 3.3 Testing Functional Restrictions

This section considers inference on functional restrictions possibly allowing $\mu$ not to be fully specified under the null. As previously discussed, we write $C_{\mathcal{H}^K} = C_{\mathcal{R}_0} + C_{\mathcal{R}_1}$ as in Section 2.3. It is not necessary that $\mathcal{R}_0 \cap \mathcal{R}_1 = \emptyset$, but $\mathcal{R}_0$ must be a proper subset of $\mathcal{H}^K$ as otherwise there is no restriction to test. Hence, $\mathcal{R}_1$ is not necessarily the complement of $\mathcal{R}_0$ in $\mathcal{H}^K$. A few examples clarify the framework. We shall make use of the results reviewed in Section 2.2 when constructing the covariance functions and in consequence the restrictions.

#### 3.3.1 Examples

**Example 4** Let $C_{\mathcal{H}^K} (s, t) = \sum_{k=1}^K C (s^{(k)}, t^{(k)})$ so that $\mu (x) = \sum_{k=1}^K f^{(k)} (x^{(k)})$ as in (2), though $x^{(k)}$ could be $d$-dimensional as in Example 1. Consider the subspace $\mathcal{R}_0$ such that $f^{(1)} = 0$. This is equivalent to $C_{\mathcal{R}_0} (s, t) = \sum_{k=2}^K C (s^{(k)}, t^{(k)})$. In consequence, we can set $C_{\mathcal{R}_1} (s, t) = C (s^{(1)}, t^{(1)})$.

Some functional restrictions can also be naturally imposed.

**Example 5** Suppose that $\mathcal{H}^K$ is an additive space of functions, where each univariate function is an element in the Sobolev Hilbert space of index $V$ on $[0, 1]$, i.e. functions with $V$ square integrable weak derivatives. Then, $C_{\mathcal{H}^K} (s, t) = \sum_{k=1}^K C (s^{(k)}, t^{(k)})$ where
The semiparametric model discussed in Connor et al. (2012) that they take values in a compact set. This framework can easily be modified to include misspecification rather than pricing. Moreover, we would winsorize the $X$ variables so that they take values in a compact set. This framework can easily be modified to include the semiparametric model discussed in Connor et al. (2012).

\[ C(s^{(k)}, t^{(k)}) = \sum_{v=1}^{V-1} \lambda_v^2 (s^{(k)} t^{(k)})^v + H_V(s^{(k)}, t^{(k)}) \] and where $H_V$ is the covariance function of the $V$-fold integrated Brownian motion (see Section 9.1, in the supplementary material, or Wahba, 1990, p.7-8, for the details). Consider the subspace $\mathcal{R}_0$ that restricts the univariate RKHS for the first covariate to be the set of linear functions, i.e. $f^{(1)}(x^{(1)}) = cx^{(1)}$ for real $c$. Then, $C_{\mathcal{R}_0} = \lambda_1^2 s^{(1)} t^{(1)} + \sum_{k=2}^{K} C(s^{(k)}, t^{(k)})$. Hence we can choose $C_{\mathcal{R}_1} = \sum_{v=2}^{V-1} \lambda_v^2 (s^{(1)} t^{(1)})^v + H_V(s^{(k)}, t^{(k)})$.

In both examples above, $\mathcal{R}_1$ is the complement of $\mathcal{R}_0$ in $\mathcal{H}^K$. However, in some cases, the decomposition can be involved, and we just consider spaces $\mathcal{R}_0$ and $\mathcal{R}_1$, to define the model under the null and the space of instruments under the alternative, respectively.

**Example 6** Suppose $C_K(s, t)$ is a universal kernel on $[0, 1]^K \times [0, 1]^K$ (see Example 1). We suppose that $C_{\mathcal{R}_0} = \sum_{k=1}^{K} C(s^{(k)}, t^{(k)})$, while $C_{\mathcal{R}_1} = C_K(s, t)$. If $C$ is continuous and bounded on $[0, 1] \times [0, 1]$, then, $\mathcal{R}_0 \subset \mathcal{R}_1$ and we test an additive model against a general nonlinear one.

It is worth noting that Condition 1 restricts the individual covariances in $C_{\mathcal{H}^K}$. The same condition is inherited by the individual covariances that comprise $C_{\mathcal{R}_0}$ (i.e., it is assumed to apply to the individual components of $C_{\mathcal{R}_0}$). In a similar vein, in Example 6, the covariance $C_{\mathcal{R}_1}$ can be seen as the individual covariance of a multivariate variable $X^{(K+1)} := (X^{(1)}, ..., X^{(K)})$ and $C_{\mathcal{R}_1}$ will have to satisfy (6) where the features $\varphi_v$'s are functions of the variable $X^{(K+1)}$. Hence, also Example 6 fits into our framework, though additional notation is required (see Section 9.1, in the supplementary material for more details).

The examples above do address important questions in a variety of econometrics problems.

**Example 7** In the conditional CAPM, \( \mu(X_i) = b X_i^{(1)} + \beta \left( X_i^{(2)}, ..., X_i^{(K)} \right) X_i^{(1)} \), where the last $K - 1$ covariates are known at time $i - 1$ (e.g. treasury bill, dividend yield etc., at time $i - 1$) and $X_i^{(1)}$ is the market excess return. The function $\beta \left( X_i^{(2)}, ..., X_i^{(K)} \right)$ can be restricted to linear or additive under the null $\mu \in \mathcal{R}_0$. In the additive case, $C_{\mathcal{R}_0}(s, t) = \lambda_0^2 + s^{(1)} t^{(1)} + \sum_{k=1}^{K} C(s^{(k)}, t^{(k)})$. The addition of a constant $\lambda_0^2$ is equivalent to estimating the CAPM with the intercept, as interest is in function misspecification rather than pricing. Moreover, we would winsorize the $X$ variables so that they take values in a compact set. This framework can easily be modified to include the semiparametric model discussed in Connor et al. (2012).
### 3.3.2 Correction for Nuisance Parameters

Recall that $\mathcal{R}_0 (B) := \mathcal{R}_0 \cap \mathcal{H}^K (B)$ for any $B > 0$ and similarly for $\mathcal{R}_1 (B)$. Suppose that $\mu_0$ in (10) lies in the interior of $\mathcal{R}_0 (B)$. Then, the moment equation $P \partial \ell_{\mu_0} h = 0$ holds for any $h \in \mathcal{R}_1$. This is because, by definition of (10), $\partial \ell_{\mu_0}$ is orthogonal to all elements in $\mathcal{H}^K$. By linearity, one can restrict attention to $h \in \mathcal{R}_1 (1)$ (i.e., $\mathcal{R}_1 (B)$ with $B = 1$). For such $h$’s, the statistic $P_n \partial \ell_{\mu_0} h$ is normally distributed by Theorem 2. In practice, $\mu_0$ is rarely known and it is replaced by $\mu_n$ in (11). The estimator $\mu_n$ does not need to satisfy $P_n \partial \ell_{\mu_n} h = 0$ for any $h$ in $\mathcal{H}^K (1)$ under the null. Moreover, the nuisance parameter affects the asymptotic distribution because it affects the asymptotic covariance. These problems can be addressed testing only the part of the moment condition $P_n \partial \ell_{\mu_n} h, h \in \mathcal{R}_1 (1)$, which is asymptotically independent of $P_n \partial \ell_{\mu_n} h$ for all $h \in \mathcal{R}_0$, using the correction to be described next. From now on, we suppose that the restriction is true, i.e. $\mu_0$ in (10) lies in $\mathcal{R}_0 (B)$, throughout.

Let $\Pi_\rho$ be the penalized population projection operator such that

$$\Pi_\rho h = \arg \inf_{\nu \in \mathcal{R}_0} P \partial \ell_{\mu_0}^2 (h - \nu)^2 + \rho |\nu|^2_{\mathcal{H}^K}$$

for any $h \in \mathcal{H}^K$. Let the population projection operator be $\Pi_0$, i.e. (13) with $\rho = 0$. We need the following conditions to ensure that we can construct a test statistic that is not affected by the estimator $\mu_n$.

**Condition 4** On top of Conditions 1, 2 and 3, the following are also satisfied:

1. $P \Delta_{1p}^2 < \infty$, $|\Delta_2|_{\infty} + |\Delta_3|_{\infty} < \infty$;

2. Under the null, the sequence of scores at the true value is uncorrelated in the sense that

   $$\sup_{j > 1, h \in \mathcal{H}^K (1)} |P_{1,j} (\partial \ell_{\mu_0} h, \partial \ell_{\mu_0} h)| = 0;$$

3. Using the notation in (2), for any $\mu \in \mathcal{H}^K (B)$ such that $|\mu|^2 > 0$, there is a constant $c > 0$ independent of $\mu = \sum_{k=1}^K f^{(k)}$ such that $|\mu|^2 \geq c \sum_{k=1}^K |f^{(k)}|^2_2$.

Remarks on these conditions can be found in Section 4.1. The following holds.

**Theorem 3** Suppose that Condition 4 holds and that $\mu_n \in \mathcal{R}_0 (B)$ is such that $P_n \ell_{\mu_n} \leq \inf_{\mu \in \mathcal{R}_0 (B)} P_n \ell_{\mu} + o_p (n^{-1})$. Under the null $\mu_0 \in \text{int} (\mathcal{R}_0 (B))$,

$$P_n \partial \ell_{\mu_n} (h - \Pi_0 h) \to G (h - \Pi_0 h), h \in \mathcal{H}^K (1),$$
weakly, where the r.h.s. is a mean zero Gaussian process with covariance function

\[ \Sigma (h, h') := \mathbb{E} G (h - \Pi_0 h) G (h' - \Pi_0 h') = P \partial \ell_{\mu_0}^2 (h - \Pi_0 h) (h' - \Pi_0 h') \]

for any \( h, h' \in \mathcal{H}^K (1) \).

Theorem 3 says that if we knew the projection (13), we could derive the asymptotic distribution of the moment equation. Additional comments on Theorem 3 are postponed to Section 4.2.

Considerable difficulties arise when the projection is not known. In this case, we need to find a suitable estimator for the projection and construct a test statistic using the moment conditions, whose number does not need to be bounded. Next we show that it is possible to do so as if we knew the true projection operator.

### 3.3.3 The Test Statistic

For the moment, to avoid distracting technicalities, suppose that the projection \( \Pi_0 h \) and the covariance \( \Sigma \) are known. Then, Theorem 3 suggests the construction of the test statistic for any finite set \( \hat{\mathcal{R}}_1 \subseteq \mathcal{R}_1 \cap \mathcal{H}^K (1) \). Let the cardinality of \( \hat{\mathcal{R}}_1 \) be \( R \). Define the test statistic

\[ S_n := \frac{1}{R_1} \sum_{h \in \hat{\mathcal{R}}_1} \left[ P_n \partial \ell_{\mu_0} (h - \Pi_0 h) \right]^2. \]

The cardinality of \( \hat{\mathcal{R}}_1 \) does not need to be bounded or restricted by the sample size. Section 2.4 provided the explicit example of \( \hat{\mathcal{R}}_1 \) in terms of linear combinations of basis functions \( \varphi_{R_1,v} \)'s. Let \( \omega_k \) be the \( k \)th scaled eigenvalue of the covariance with entries \( \Sigma (h, h') \) for \( h, h' \in \hat{\mathcal{R}}_1 \), i.e., \( \omega_k \psi_k (h) = \frac{1}{\hat{R}_1} \sum_{h' \in \hat{\mathcal{R}}_1} \Sigma (h, h') \psi_k (h') \), where the \( k \)th eigenfunction \( \{ \psi_k (h) : h \in \hat{\mathcal{R}}_1 \} \) is orthonormal in the sense that \( \frac{1}{\hat{R}_1} \sum_{h \in \hat{\mathcal{R}}_1} \psi_k (h) \psi_l (h) = 1 \) if \( k = l \) and zero otherwise.

**Remark 1** Given that \( R_1 \) is finite, we can just compute the eigenvalues (in the usual sense) of \( \Sigma / R_1 \), where \( \Sigma \) is a matrix with \( (r_1, r_2) \) entry \( \Sigma (h^{(r_1)}, h^{(r_2)}) \) and \( \hat{\mathcal{R}}_1 = \{ h^{(1)}, ..., h^{(R_1)} \} \).

**Corollary 1** Let \( \{ \omega_k : k > 1 \} \) be the set of scaled eigenvalues of the covariance with entries \( \Sigma (h, h') \) for \( h, h' \in \hat{\mathcal{R}}_1 \), from Theorem 3. Suppose that they are ordered in descending value. Under Condition 4, \( S_n \to S \), in distribution, where \( S = \sum_{k>0} \omega_k N_k^2 \), and the \( N_k \)'s are independent standard normal random variables.
To complete this section, it remains to consider an estimator of the projection $\Pi_0 h$ and of the covariance function $\Sigma$. The population projection operator can be replaced by a sample version

$$\Pi_{n,\rho} h = \arg \inf_{\nu \in \mathbb{R}_0} P_n \partial^2 \ell_{\mu_0} (h - \nu)^2 + \rho |\nu|_{HK}^2,$$

(14)

which depends on $\rho = \rho_n \to 0$. To ease notation, write $\Pi_n = \Pi_{n,\rho}$ for $\rho = \rho_n$. The estimator of $\Sigma$ is given by $\Sigma_n$ such that

$$\Sigma_n (h, h') = P_n \partial^2 \ell_{\mu_0} (h - \Pi_n h) (h' - \Pi_n h').$$

(15)

It is not at all obvious that we can use the estimated projection (over as sequence of increasing subsets) uniformly in $h$ in place of the population one. The following shows that this is the case.

**Theorem 4** In Condition 1, $\lambda > 3/2$, and in (14), choose $\rho$ such that $\rho n^{1/(2\lambda)} \to 0$ and $\rho n^{(2\lambda - 1)/(4\lambda)} \to \infty$, and define

$$\hat{S}_n := \frac{1}{R_1} \sum_{h \in \mathcal{R}_1} [P_n \partial \ell_{\mu_0} (h - \Pi_n h)]^2.$$

(16)

Let $\hat{S} := \sum_{k>0} \omega_{nk} N_k^2$ where $\omega_{nk}$ is the $k^{th}$ scaled eigenvalue of $\{\Sigma_n (h, h') : h, h' \in \mathcal{R}_1\}$ (see Remark 1). Under Conditions 4, $\hat{S}_n$ and $\hat{S}$ converge in distribution to $S$, where the latter is as given in Corollary 1.

Note that the condition on $\rho$ can only be satisfied if in Condition 1, $\lambda > 3/2$, as otherwise the condition on $\rho$ is vacuous.

Let $P (y|x)$ be the distribution of $Y_i$ given $X_i$. Under the null, $w (x) := \int \partial^2 \ell_{\mu_0} (y, x) dP (y|x)$ might be known ($\partial^2 \ell_{\mu_0} (y, x) = \partial^2 \ell_{\mu_0} (z)$ for $z = (y, x)$). In this case, $\partial^2 \ell_{\mu_0}$ in (14) can be replaced by $w (x)$, i.e., define the empirical projection as the arg inf of

$$P_n w (h - \nu)^2 + \rho |\nu|_{HK}^2 = \frac{1}{n} \sum_{i=1}^n w (X_i) (h (X_i) - \nu (X_i))^2 + \rho |\nu|_{HK}^2$$

(17)

w.r.t. $\nu \in \mathcal{R}_0$. For example, for the regression problem, using the square error loss, $w (x) = 1$. 

21
Corollary 2 Suppose \( x \mapsto w(x) \) is known. Replace \( \Pi_n h \) with the minimizer of (17) in the construction of the test statistic \( \hat{S}_n \) and \( \Sigma_n \). Suppose Condition 4 and \( \rho \) such that \( \rho n^{1/(2\lambda)} \to 0 \) and \( \rho n^{1/2} \to \infty \). Then, the conclusion of Corollary 4 continues to hold.

Despite the technicalities required to justify the procedure, Section 2.4 showed that its implementation is straightforward. In fact \( \partial \ell_{\mu_0} \) evaluated at \((Y_i, X_i)\) is the score for the \( i \)th observation and it is the \( i \)th entry in \( e_0 \). On the other hand the vector \( \hat{h}^{(r)} \) has \( i \)th entry \( (h^{(r)}(X_i) - \Pi_n h^{(r)}(X_i)) \) and \( \mathcal{R}_1 = \{ h^{(1)}, ..., h^{(R)} \} \), for example \( \{ C_{\mathcal{R}_1} (\cdot, z_r) : z_r \in \mathcal{X}^K, r = 1, 2, ..., R \} \).

4 Remarks on the Results and the Conditions

4.1 Remarks on Conditions

A minimal condition for the \( \lambda_v \)'s would be \( \lambda_v \lesssim v^{-\lambda} \) with \( \lambda > 1/2 \) as this is essentially required for \( \sum_{v=1}^{\infty} \lambda_v^2 \varphi_v^2(s) < \infty \) for any \( s \in [0, 1] \). Mendelson (2002) derives consistency under this minimal condition in the i.i.d. case, but no convergence rates. Here, the condition is strengthened to \( \lambda > 1 \), but it is not necessarily so restrictive. The covariance in Example 1 satisfies Condition 1 with exponentially decaying \( \lambda_v \)'s (e.g. Rasmussen and Williams, 2006, Ch. 4.3.1); the covariance in Example 5 satisfies \( \lambda_v \lesssim v^{-\lambda} \) with \( \lambda \geq V \) (see Ritter et al., 1995, Corollary 2, for this and more general results).

It is not difficult to see that many loss functions (or negative log-likelihoods) of interest satisfy Condition 2 using the fact that \( |\mu|_\infty \leq B \) (square loss, logistic, negative log-likelihood of Poisson, etc.). Nevertheless, interesting loss functions such as absolute deviation for conditional median estimation do not satisfy Condition 2. The extension to such loss functions requires arguments that are specific to the problem together with additional restrictions to compensate for the lack of smoothness. Some partial extension to the absolute loss will be discuss in Section 7.2.

Condition 3 is standard in the literature. More details and examples can be found in Section 9.4 in the Appendix.

In Condition 4, the third derivative of the loss function and the strengthening of the moment conditions (Point 1) are used to control the error in the expansion of the moment equation. The moment conditions are slightly stronger than needed. The proofs show that we use the following in various places \( P (|\Delta_1|^{2p} + |\Delta_2|^{2p} + |\Delta_3|^{2p}) < \infty, |\partial^2 \ell_{\mu_0}|_\infty + |\Delta_3|_\infty < \infty \), and these can be weakened somehow, but at the cost of introducing
dependence on \( \lambda \) (\( \lambda \) as in Condition 1). The condition is satisfied by various loss functions. For example, the following loss functions have bounded second and third derivative w.r.t. \( t \in [-cB, cB] \): \( (y - t)^2 \) \( y \in \mathbb{R} \) (regression), \( \ln(1 + \exp\{-yt\}) \) \( y \in \{-1, 1\} \) (classification), \( -yt + \exp\{t\} \) \( y \in \{0, 1, 2, \ldots\} \) (counting).

Time uncorrelated moment equations in Condition 4 are needed to keep the computations feasible. This condition does not imply that the data are independent. The condition is satisfied in a variety of situations. In the Poisson example given in the introduction this is the case as long as \( \mathbb{E} \{I_{i-1}Y_i = \exp\{\mu_0(X_i)\} \} \) (which implicitly requires \( X_i \) being measurable at time \( i - 1 \)). In general, we can still have misspecification as long as the conditional expectation is not misspecified.

If the scores at the true parameter are correlated, the estimator of \( \Sigma \) needs to be modified to include additional covariance terms (e.g., Newey-West estimator). Also the projection operator \( \Pi_0 \) has to be modified such that

\[
\Pi_0h = \arg \inf_{v \in \mathbb{R}_0} \sum_{j \in \mathbb{Z}} P_{1,j} \left( \partial \ell_{\mu_0}(h - v), \partial \ell_{\mu_0}(h - v) \right).
\]

This can make the procedure rather involved and it is not discussed further.

It is simple to show that Point 4 in Condition 4 means that for all pairs \( k, l \leq K \) such that \( k \neq l \), and for all \( f, g \in \mathcal{H} \) such that \( \mathbb{E} \left| f \left( X^{(k)} \right) \right|^2 = \mathbb{E} \left| g \left( X^{(l)} \right) \right|^2 = 1 \), then \( \mathbb{E} f \left( X^{(k)} \right) g \left( X^{(l)} \right) < 1 \) (i.e., no perfect correlation when the functions are standardized).

### 4.2 Remarks on Theorem 2

The asymptotic distribution of the estimator is immediately derived if \( \mathcal{H}^K(B) \) is finite dimensional.

**Example 8** Consider the rescaled square error loss so that \( \partial^2 \ell_{\mu_0} = 1 \). Defining \( \nu = \lim_n \sqrt{n} (\mu_n - \mu_0) \), Theorem 2 gives

\[
G(h) = \nu h,
\]

in distribution, where \( G \) is as in Theorem 2 as long as \( \mu_0 \in \text{int} \left( \mathcal{H}^K(B) \right) \) (i.e., in the estimation, the constraint is not asymptotically binding). The distribution of \( \nu \) is then given by the solution to the above display when \( \mathcal{H}^K(B) \) is finite dimensional.

In the infinite dimensional case, Hable (2012) has shown that \( \sqrt{n} (\mu_{n,\rho}(x) - \mu_{0,\rho}(x)) \) converges to a Gaussian process whose covariance function would require the solution
of some Fredholm equation of the second type. Recall that \( \mu_{n, \rho} \) is as in (12), while we use \( \mu_{0, \rho} \) to denote its population version. The penalty \( \rho = \rho_0 \) needs to satisfy \( \sqrt{n} (\rho_n - \rho_0) = o_p(1) \) and \( \rho_0 > 0 \) is a fixed constant. When \( \mu_0 \in \text{int} (\mathcal{H}^K (B)) \), we have \( \mu_0 = \arg \min_{\mu \in \mathcal{H}} P(\mu) \). Hence, there is no \( \rho_0 > 0 \) such that \( \mu_0 = \mu_{0, \rho_0} \) because the constraint is not binding. The two estimators are both of interest with different properties. When \( \rho > 0 \) the approximation error is non-negligible, e.g., for the square loss the estimator is biased.

Theorem 2 requires \( \mu_0 \in \text{int} (\mathcal{H}^K (B)) \). In the finite dimensional case, the distribution of the estimator when \( \mu_0 \) lies on the boundary of \( \mathcal{H}^K (B) \) is not standard (e.g., Geyer, 1994). In consequence the p-values are not easy to find.

### 4.3 Alternative Constraints

As an alternative to the norm \( |.|_H^K \), define the norm \( |f|_{L^K} := \sum_{k=1}^{K} |f^{(k)}|_H^K \). Estimation in \( L^K (B) := \{ f \in \mathcal{H}^K : |f|_{L^K} \leq B \} \) is also of interest for variable screening. The following provides some details on the two different constraints.

**Lemma 1** Suppose an additive kernel \( C_{H^K} \) as in Section 2.2. The following hold.

1. \( |.|_{H^K} \) and \( |.|_{L^K} \) are norms on the (additive) \( \mathcal{H}^K \).
2. We have the inclusion

\[
K^{-1/2} \mathcal{H}^K (1) \subset L^K (1) \subset \mathcal{H}^K (1) .
\]

3. For any \( B > 0 \), \( \mathcal{H}^K (B) \) and \( L^K (B) \) are convex sets.
4. Let \( c := \max_{s \in X} \sqrt{C(s, s)} \). If \( \mu \in \mathcal{H}^K (B) \), then, \( \sup_{\mu \in \mathcal{H}^K (B)} |\mu|_p \leq c \sqrt{K} B \) for any \( p \in [1, \infty] \), while \( \sup_{\mu \in L^K (B)} |\mu|_p \leq c B \).

By the inclusion in Lemma 1, all the results derived for \( \mathcal{H}^K (B) \) also apply to \( L^K (K^{1/2} B) \). In this case, we still need to suppose that \( \mu_0 \in \text{int}(\mathcal{H}^K (B)) \). Both norms are of interest. When interest lies in variable screening and consistency only, estimation in \( L^K (B) \) inherits the properties of the \( l_1 \) norm (as for LASSO). The estimation algorithms discussed in Section 5 cover estimation in both subsets of \( \mathcal{H}^K \).

### 5 Computation Algorithm

By duality, when \( \mu \in \mathcal{H}^K \) and the constraint is \( |\mu|_{H^K} \leq B \) the estimator is the usual one obtained from the Representer Theorem (e.g., Steinwart and Christmann, 2008).
Estimation in an RKHS poses computational difficulties when the sample size \( n \) is large. Simplifications are possible when the covariance \( C_{HK} \) admits a series expansion as in (6) (e.g., Lazaro-Gredilla et al., 2010).

Estimation for functions in \( \mathcal{L}^K(B) \) rather than in \( \mathcal{H}^K(B) \) is even more challenging. Essentially estimation in \( \mathcal{L}^K(B) \) resembles LASSO, while estimation in \( \mathcal{H}^K(B) \) resembles ridge regression (in the case of the square error loss).

A greedy algorithm can be used to solve both problems. In virtue of Lemma 1 and the fact that estimation in \( \mathcal{H}^K(B) \) has been considered extensively, only estimation in \( \mathcal{L}^K(B) \) will be addressed in details. The minor changes required for estimation in \( \mathcal{H}^K(B) \) will be discussed in Section 5.2.

### 5.1 Estimation in \( \mathcal{L}^K(B) \)

Estimation of \( \mu_n \) in \( \mathcal{L}^K(B) \) is carried out according the following Frank-Wolfe algorithm. Let \( f^{(s(m))}_m = f^{(s(m))}_m \) be the solution to

\[
\min_{k \leq K} \min_{f^{(k)} \in \mathcal{H}(1)} P_n \partial \ell_{F_{m-1}} f^{(k)}
\]  

(18)

where \( F_0 = 0, F_m = (1 - \tau_m) F_{m-1} + c_m f^{(s(m))}_m \), and \( c_m = B \tau_m \), where \( \tau_m \) is the solution of the line search

\[
\min_{\tau \in [0,1]} P_n \ell \left((1 - \tau) F_{m-1} + \tau B f^{(s(m))}_m \right),
\]  

(19)

writing \( \ell(\mu) \) instead of \( \ell_\mu \) for typographical reasons. Details on how to solve (18) will be given in Section 5.1.1; the line search in (18) is elementary. The algorithm produces functions \( \{f^{(s(j))}_j : j = 1, 2, ..., m\} \) and coefficients \( \{c_j : j = 1, 2, ..., m\} \). Note that \( s(j) \in \{1, 2, ..., K\} \) identifies which of the \( K \) additive functions will be updated at the \( j^{th} \) iteration.

To map the results of the algorithm into functions with representation in \( \mathcal{H}^K \), one uses trivial algebraic manipulations. A simpler variant of the algorithm sets \( \tau_m = 1/m \). In this case, the solution at the \( m^{th} \) iteration, takes the particularly simple form

\[
F_m = \sum_{j=1}^{m} \frac{B}{m} f^{(s(j))}_j
\]  

e.g., Sancetta, 2016

and the \( k^{th} \) additive function can be written as \( \tilde{f}^{(k)} = \frac{B}{m} \sum_{j \leq m; s(j)=k} f^{(s(j))}_j \).

To avoid cumbersome notation, the dependence on the sample size \( n \) has been suppressed in the quantities defined in the algorithm. The algorithm can find a solution with arbitrary precision as the number of iterations \( m \) increases.
Theorem 5 For $F_m$ derived from the above algorithm,

$$P_n \ell_{F_m} \leq \inf_{\mu \in \mathcal{L}^K(B)} P_n \ell_{\mu} + \epsilon_m$$

where,

$$\epsilon_m \lesssim \begin{cases} 
\frac{B^2 \sup_{|t| \leq B} P_n d^2 L(\cdot, t)/dt^2}{B^2 \sup_{|t| \leq B} [P_n d^2 L(\cdot, t)/dt^2] \ln(1+m)} & \text{if } \tau_m = \frac{2}{m+2} \text{ or line search in (19)} \\
\frac{m \ln(1+m)}{m} & \text{if } \tau_m = \frac{1}{m}
\end{cases}.$$  

For the sake of clarity, note that $P_n d^2 L(\cdot, t)/dt^2 = \frac{1}{n} \sum_{i=1}^n d^2 L(Z_i, t)/dt^2$.

5.1.1 Solving for the Additive Functions

The solution to (18) is found by minimizing the Lagrangian

$$P_n \partial \ell_{F_{m-1}} f^{(k)} + \rho \left| f^{(k)} \right|^2_H.$$  

(20)

Let $\Phi^{(k)}(x^{(k)}) = C(\cdot, x^{(k)})$ be the canonical feature map (Lemma 4.19 in Steinwart and Christmann, 2008); $\Phi^{(k)}$ has image in $H$ and the superscript $k$ is only used to stress that it corresponds to the $k^{th}$ additive component. The first derivative w.r.t. $f^{(k)}$ is $P_n \partial \ell_{F_{m-1}} \Phi^{(k)} + 2\rho f^{(k)}$, using the fact that $f^{(k)}(x^{(k)}) = \langle f^{(k)}, \Phi^{(k)}(x^{(k)}) \rangle_H$, by the reproducing kernel property. Then, the solution is

$$f^{(k)} = -\frac{1}{2\rho} P_n \partial \ell_{F_{m-1}} \Phi^{(k)},$$

where $\rho$ is such that $f^{(k)} = 1$. If $P_n \partial \ell_{F_{m-1}} \Phi^{(k)} = 0$, set $\rho = 1$. Explicitly, using the properties of RKHS (see (8))

$$\left| f^{(k)} \right|^2_H = \frac{1}{(2\rho)^2} \sum_{i,j=1}^n \frac{\partial \ell_{F_{m-1}}(Z_i) \partial \ell_{F_{m-1}}(Z_j)}{n} C(X^{(k)}_i, X^{(k)}_j)$$

which is trivially solved for $\rho$. With this choice of $\rho$, the constraint $f^{(k)} \leq 1$ is satisfied for all $k$'s, and the algorithm, simply selects $k$ such that $P_n \partial \ell_{F_{m-1}} f^{(k)}$ is minimized. Additional practical computational aspects are discussed in Section 9.3 in the Appendix (supplementary material).

The above calculations together with Theorem 5 imply the following, which for
simplicity, it is stated using the update $\tau_m = m^{-1}$.

**Theorem 6** Let $\rho_j$ be the Lagrange multiplier estimated at the $j^{th}$ iteration of the algorithm in (18) with $\tau_m = m^{-1}$ instead of the line search (19). Then,

$$\mu_n = \lim_{m \to \infty} \sum_{j=1}^{m} \frac{-B}{2m\rho_j} P_n \partial \ell_{F_{j-1}} \Phi^{s(j)},$$

where $\Phi^{s(j)} = \Phi^{(s(j))}$ and $\Phi^{(k)}(x^{(k)}) = C(\cdot, x^{(k)})$, is the solution in $L^K(B)$.

The Matlab function `estimateAdditiveRKHS.m` performs this estimation under the square error loss.

5.2 The Algorithm for Estimation in $\mathcal{H}^K(B)$

When estimation is constrained in $\mathcal{H}^K(B)$, the algorithm has to be modified. Let $\Phi(x) = C_{\mathcal{H}^K}(\cdot, x)$ be the canonical feature map of $\mathcal{H}^K$ (do not confuse $\Phi$ with $\Phi^{(k)}$ in the previous section). Then, (18) is replaced by

$$\min_{f \in \mathcal{H}^K(B)} P_n \partial \ell_{F_{m-1}} f$$

and denote by $f_m \in \mathcal{H}^K(B)$ the solution at the $m^{th}$ iteration. This solution can be found replacing minimization of (20) with minimization of $P_n \partial \ell_{F_{m-1}} f + \rho |f|^2_{\mathcal{H}^K}$. The solution is then $f_m = -\frac{1}{2\rho} P_n \partial \ell_{F_{m-1}} \Phi$ where $\rho$ is chosen to satisfy the constraint $|f|^2_{\mathcal{H}^K} \leq 1$. No other change in the algorithm is necessary and the details are left to the reader.

6 Estimation of the Trade Impact Cost Function for the Crude Oil Futures Market

This section considers the problem of market impact for the front month futures contract (i.e., the most liquid contract) on crude oil, traded on the Chicago Mercantile Exchange (CME). The sample dataset comprises of the full order book and trade updates for 1-Apr-2013 (9:30-16:15 EST). Market impact refers to the price movement that results from buy or sell orders of a financial instrument. The movement is due to the fact that an (aggressive) order crosses the bid and ask spread and depletes liquidity. The impact is not only due to liquidity depletion, but also emphasized by other factors.
such as information transmission (Grossman and Stiglitz, 1980, Kyle, 1985), inventory management on the side of liquidity providers (e.g., Lyons, 1997), etc..

Estimation of the impact cost of trades on futures is an important topic in algorithmic trading and execution. Various conditioning variables are found to predict market impact. In particular, the traded and quoted sizes, the bid and ask spread, etc., are examples of variables that are often considered by practitioners. The relation between traded size and market impact is often reported to be non-linear (e.g., Hasbrouck, 1991, Lillo et al., 2003) and this may possibly be for other variables.

The goal of the study is twofold. On one hand it will identify the first few variables that are most relevant to explain market impact. This is done using some version of stability selection (Meinshausen and Bühlmann, 2010). The second goal is to formally test a set of hypotheses via functional restrictions.

Throughout the empirical study, the analysis regarding the most important variables uses estimators in $\mathcal{L}^K (B)$ as we want to take advantage of the variable screening properties of the $l_1$ norm, while for testing various hypotheses, estimation is carried out in $\mathcal{H}^K (B)$.

6.1 The Model and the Hypotheses to be Tested

The variables. The dependent variable is $Y_i = Ret_{t_i+5sec}$, where $Ret_{t_i+5sec}$ is the 5 seconds price change (in levels) standardized by the minimum tick size. The variable $Ret_{t_i+5sec}$ is computed as the difference between the mid price quote 5 seconds after the $i^{th}$ trade occurred at time $t_i$, and the mid price quote just before the trade. The dependent variables are $RetLag_i$, $Spread_i$, $VolImb_{1i}$, $VolImb_{2to5i}$, $VolImb_{6to10i}$, $Duration_i$, $TrdSize_i$. All the variables are computed just before the $i^{th}$ trade at time $t_i$. In particular, $RetLag_i$ is the lagged dependent variable. $Spread_i$ is the top of book bid-ask spread standardized by the minimum tick size. $VolImb_{1i}$ is the volume imbalance computed as the quoted volume on the top of book bid minus the quoted volume on the top of book ask divided by the average of the two volumes. To compute $VolImb_{2to5i}$, aggregate the volumes quoted on the 2nd level to the 5th on the bid and on the ask and call these the aggregated sizes. Then, subtract the aggregated ask size from the aggregated bid size and divide by their average. The $VolImb_{6to10i}$ is computed similarly aggregating the sizes on the 6th to the 10th level. The duration is $Duration_i = \ln (1 + t_i - t_{i-1})$ where $t_i - t_{i-1}$ is the time elapsed from the last trade, measured in seconds and rounded to the nearest second. The $TrdSize_i$ is the total
Table 2: Explanatory Variables: X

<table>
<thead>
<tr>
<th>RetLog</th>
</tr>
</thead>
<tbody>
<tr>
<td>TrdSign</td>
</tr>
<tr>
<td>VolImb1</td>
</tr>
<tr>
<td>VolImb2to5</td>
</tr>
<tr>
<td>VolImb6to10</td>
</tr>
<tr>
<td>TrdSign</td>
</tr>
<tr>
<td>TrdSign</td>
</tr>
</tbody>
</table>

EWMA’s of all the above with $\alpha = 0.75$, and $0.95$

Total number of variables: 21

Trade size reported in a single CME message. The trade sign $TrdSign_i$ will also be used. This is computed as buy or sell depending on whether the trade was above or below the mid price. The data contain all the messages sent by the CME, and are time stamped with a nanosecond resolution and adjusted when the order book does not update because there are too many trades sent over the network. Time stamping and adjustments and in consequence the sign of the trades were all implemented by a proprietary high frequency trading group in Chicago.

Market impact tends to exhibit time series dependence. Hence, the model with no dynamics is augmented adding exponential moving averages (EWMA) of all the variables. For a variable $X^{(k)}$, the EWMA is defined as $EM_1(X^{(k)}) = X_1^{(k)}$, $EM_i(X^{(k)}) = \alpha EM_{i-1}(X^{(k)}) + (1 - \alpha) X_i^{(k)}$, where the smoothing parameter $\alpha \in [0, 1)$ implies more persistence on the past when $\alpha$ increase, while $\alpha = 0$ recovers $X_i^{(k)}$. EWMA’s with parameters $\alpha = 0.75$, and 0.95 are used. The total number of variables is 21 and Table 2 reports the actual variables used. Each explanatory variable is divided by its standard deviation and mapped into $[-1, 1]$ capping it by 9, and dividing by the cap size. Calculations were carried out using smaller cap values, but the conclusion of the analysis remained essentially the same.

The kernel. The additive kernel is $C_{H^{(K)}}(s, t) = \lambda_0^2 + \sum_{k=1}^{K} C (s^{(k)}, t^{(k)})$, where $\lambda_0 = 1/10$, $C (s^{(k)}, t^{(k)}) = \sum_{v=1}^{9} v^{-2.2} (s^{(k)} t^{(k)})^v$. Hence, the covariance has series expansion as in (6). The constant $\lambda_0$ is used with a small value to account for the possibility of the mean of the variables to be possibly different from zero in sample.

Functional restrictions are imposed in order to test the following hypotheses.
The hypotheses.

1. VolImb2to5 and VolImb6to10 do not have any price impact, i.e., the functions $f^{(k)}$’s associated to VolImb2to5 and VolImb6to10 are zero;

2. The impact of volume imbalances is linear and imbalances beyond the 5th level have no impact, i.e. linearity of $f^{(k)}$ for VolImb1 and VolImb2to5, and the $f^{(k)}$ associated to VolImb6to10 is zero;

3. The impact of trade size on price is linear: $f^{(6)}(TrdSign \times TrdSize) = c \times TrdSign \times TrdSize$ for some real $c$.

4. The true model is linear.

Hypothesis 1 tests the resilience of the market to any gaming activity. It is well known that market participants can manipulate the market by placing large orders deeper in the book to create a false impression of the true level of demand and supply. A market where the imbalances deeper in the book contribute marginally to market impact is a market less prone to spoofing. Spoofing is a practice that is forbidden on the CME, and is a criminal offense in many countries including the US.

Hypothesis 2 is related to hypothesis 1. Spoofing is a practice that involves placing quotes that have no chances to be executed. When comparing the largest trade sizes to the volumes quoted in the order book, the levels beyond the 5th appear to have no chance to be executed. Hence, one can restrict a test of sensitivity to spoofing to levels beyond the 5th only. At the same time, the hypothesis also tries to characterize the functional form of the market impact relative to the volume imbalances.

Many authors find a concave relationship between signed trade size and market impact. Given that data from various studies confirm that the impact is concave, Hypothesis 3 checks whether a linear relationship could be rejected. From the results in the literature, rejection then implies that the relation must be strictly concave. The shape of the market impact is critical for execution and to devise algorithms that minimizes trading costs.

Finally, hypothesis 4 tests whether estimation of market impact by a linear model could be a good choice. Inference for a liner model is simpler and it is preferable first order approximation, unless rejected.
Table 3: The three variables most often selected when estimating 50 times the model using the algorithm in Section 5 and stopping once 5 variables have been selected.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spread</td>
<td>0.58</td>
</tr>
<tr>
<td>VolImb</td>
<td>1</td>
</tr>
<tr>
<td>TrdSize</td>
<td>0.48</td>
</tr>
</tbody>
</table>

6.2 Results and Discussion

We are interested in estimating and testing functional restrictions on $E[Y|X] = \sum_{k=1}^{K} f^{(k)}(X^{(k)})$. In consequence, the loss function used is the square error loss $|y - \mu|^2$ and the model is estimated using the variables and feature space discussed above. Results are derived as follows.

Variable screening. The most relevant variables are identified via subsampling and stability selection (Meinshausen and Bühlmann, 2010). A block of consecutive data points of size $0.2 \times n$ is sampled randomly and a “large” value of $B$ is used when running the algorithm in Section 5 (details are in the Appendix in Section 9.6). The algorithm is stopped when 5 variables have been selected. The procedure is repeated 50 times. Using the sets of 5 variables selected in each of the 50 iterations, the proportion of times each variable was selected, if any, is computed. Table 3 shows the variables that have been selected at least 50% of the times. The results are surprising because the trade size is not the most often selected variable. The included variables are not a surprise. For example, despite being often used by practitioners, volume imbalances have only been recently considered in the academic literature (Cont et al., 2014, Sancetta, 2017).

Functional restrictions. Constraints on space of functions are imposed and tested using the framework from Section 3.3 and the test statistic from Theorem 4. The details on how the test statistic and the p-values are computed are reported in Section 9.6, in the Appendix. The results are reported in Table 4.

Hypothesis 1 falls within the framework of Example 4. It is rejected at any conventional level of significance. Hence, it seems that the front month of crude oil futures is driven by demand and supply beyond the first level.

Hypothesis 2 falls within the context of Examples 4 and 5, as it tests both the importance of some variables as well as the functional restriction of some others. This
Table 4: P-values for tests of the restriction hypotheses

<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>&lt;0.01</td>
</tr>
<tr>
<td>2</td>
<td>0.23</td>
</tr>
<tr>
<td>3</td>
<td>0.68</td>
</tr>
<tr>
<td>4</td>
<td>&lt;0.01</td>
</tr>
</tbody>
</table>

hypothesis is not rejected. Hence, we can infer that volume imbalances have a linear impact on returns and that the market could be sensitive to moderate market manipulation, though information deep in the book (i.e., beyond the 5th level) is not relevant.

As far as hypothesis 3, concerning the linearity of trade impact, the results are surprising. The impact of trades is usually found to be concave, though no formal tests are usually carried out, but large datasets provide overwhelming evidence. The null of linear impact of trade sizes cannot be rejected for CME Crude Oil futures for the specific date used here. Cont et al. (2014) provide some explanation for the square root phenomena of trade impact, which might be in line with the present finding once we use other controls such as volume imbalances.

Finally, hypothesis 4, concerning linearity of the model, is rejected at any conventional level of significance. This means that market impact appears to be driven by different market variables in a possibly complex way.

7 Further Remarks

In this last section additional remarks of various nature are included. A simulation example is used to shed further light on the importance of the projection procedure. The simulation design tries to bear some relation to the framework of the empirical study in Section 6. The choice of the parameter $B$ is not discussed in here, but details can be found in Section 9.6 in the Appendix (supplementary material). The paper will conclude with an example on how Condition 2 can be weakened in order to accommodate other loss functions, such as the absolute loss.

7.1 Some Finite Sample Evidence via Simulation Examples

Simulation Design: True Models. Consider the regression problem where $Y_i = \mu_0(X_i) + \varepsilon_i$, the number of covariates $X^{(k)}$ is $K = 10$, and the sample size is $n =$
100, and 1000. The covariates are i.i.d. standard Gaussian random variables truncated to the interval $\mathcal{X} = [-2, 2]$. The cross-sectional correlation between $X^{(k)}$ and $X^{(l)}$ is $\rho^{k-l}$ with $\rho = 0, 0.25$. The error terms are i.i.d. mean zero, Gaussian with variance such that the signal to noise ratio $\sigma^2_{\mu/\varepsilon}$ is equal to 1 and 0.2. This is equivalent to an $R^2$ of 0.5 and 0.167, i.e., a moderate and low $R^2$. The following specifications for $\mu_0$ are used: Lin3: $\mu_0 (X) = \sum_{k=1}^{3} b_k X^{(k)}$ with $b_k = 1/10$; LinAll: $\mu_0 (X) = \sum_{k=1}^{10} b_k X^{(k)}$ with $b_k = 1/10$; NonLinear: $\mu_0 (X) = X^{(1)} + \sum_{v=1}^{9} b_{4,v} (X^{(4)}/2)^v$ where the $b_{4,v}$’s are uniformly distributed in $[-20/v, 20/v]$. In NonLinear the first variable enters the model linearly, the forth variable enters it in a nonlinear fashion, while the remaining variables do not enter the model. The choice of random coefficient for NonLinear is along the lines of Friedman (2001) to mitigate the dependence on a specific nonlinear functional form. The number of simulations is 1000.

**Estimation Details.** For estimation, a polynomial additive kernel is used: $C_{\mathcal{H}^{10}} = \sum_{k=1}^{10} C \left( s^{(k)}, t^{(k)} \right)$, where $C \left( s^{(k)}, t^{(k)} \right) = \sum_{v=1}^{9} v^{-2.2} (s^{(k)} t^{(k)})^v$. For such kernel, the true models in the simulation design all lie in a strict subset of $\mathcal{H}^{10}$. Estimation is carried out in $\mathcal{L}^{10} (B)$ using the algorithm in Section 5 with number of iterations $m$ equal to 500. This should also allow us to assess whether there is a distortion in the test results when the estimator minimizes the objective function on $\mathcal{L}^{10} (B)$ only approximately. The parameter $B$ is chosen equal to $10 \hat{\sigma}_Y$ where $\hat{\sigma}_Y$ is the sample standard deviation of $Y$, which is a crude approach to keep simulations manageable.

**Hypotheses.** Hypotheses are tested within the framework of Section 3.3. We estimate Lin1, Lin2, Lin3 and LinAll, using the restricted kernel $C_{\mathcal{R}_0} (s, t) = \sum_{k=1}^{J} s^{(k)} t^{(k)}$ with $J = 1, 2, 3, 10$, i.e., a linear model with 1, 2, 3 and 10 variables respectively. We also estimate LinPoly using the restricted kernel $C_{\mathcal{R}_0} (s, t) = s^{(1)} t^{(1)} + \sum_{k=2}^{10} C \left( s^{(k)}, t^{(k)} \right)$ with $C \left( s^{(k)}, t^{(k)} \right)$ as defined in the previous paragraph, i.e., the first variable enters the model linearly, all other functions are unrestricted. In all cases we test against the full unrestricted model with kernel $C_{\mathcal{H}^{10}} (s, t)$.

**Results.** Table 5 reports results for $n = 1000$, a signal to noise level $\sigma^2_{\mu/\varepsilon} = 1$, and $\rho = 0$ under the three different true designs: Lin3, LinAll, and NonLin. The column heading “No II” means that no correction was used in estimating the test statistic (i.e., test statistic ignoring the presence of nuisance parameters). The results for the other configurations of sample size, signal to noise ratio and correlation in the variables were
Table 5: Simulation results for \( n = 1000, \sigma^2_{\mu/\varepsilon} = 1, \rho = 0 \). The column heading “Size” stands for the nominal size.

<table>
<thead>
<tr>
<th>Size</th>
<th>Lin3</th>
<th>LinAll</th>
<th>LinPoly</th>
</tr>
</thead>
<tbody>
<tr>
<td>No II</td>
<td>No II</td>
<td>No II</td>
<td>II</td>
</tr>
<tr>
<td>True model: Lin3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>0.09</td>
<td>0.11</td>
<td>0.07</td>
</tr>
<tr>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.04</td>
</tr>
<tr>
<td>True model: LinAll</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>1.00</td>
<td>1.00</td>
<td>0.49</td>
</tr>
<tr>
<td>0.05</td>
<td>1.00</td>
<td>1.00</td>
<td>0.23</td>
</tr>
<tr>
<td>True model: NonLin</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>1.00</td>
<td>1.00</td>
<td>0.92</td>
</tr>
<tr>
<td>0.05</td>
<td>1.00</td>
<td>1.00</td>
<td>0.9</td>
</tr>
</tbody>
</table>

similar. The LinPoly model is only estimated when the true model is NonLin. The detailed and complete set of results is in Section 9.7 in the Appendix (supplementary material). Failing to use the projection adjustment, the size of the test can be highly distorted, as expected. The results reported in Table 5 show that the test (properly constructed using the projection adjustment) has coverage probability relatively close to the nominal one when the null holds, and that the test has a good level of power.

7.2 Weakening Condition 2: Partial Extension to the Absolute Loss

Some loss functions are continuous and convex, but they are not differentiable everywhere. An important case is the absolute loss and its variations used for quantile estimation. The following considers an alternative to Condition 2 that can be used in this case. Condition 3 can be weakened, but Condition 1 has to be slightly tightened. The details are stated next, but for simplicity for the absolute loss only. More general losses such as the one used for quantile estimation can be studied in a similar way.

**Condition 5** Suppose that \( \ell_{\mu} (z) = |y - \mu (x)| \), \( P\ell_{\mu} < \infty \), and that \( P (y, x) = P (y|x) P (x) \) where \( P (y|x) \) (the conditional distribution of \( Y \) given \( X \)) has a bounded density pdf \( (y|x) \) w.r.t. the Lebesgue measure on \( \mathcal{Y} \), and \( P (x) \) is the distribution of \( X \in \mathcal{X}^K \). Moreover, pdf \( (y|x) \) has derivative w.r.t. \( y \) which is uniformly bounded for any \( x \in \mathcal{X}^K \), and \( \min_{|t| \leq B, x \in \mathcal{X}^K} \text{pdf} (t|x) > 0 \) (\( B \) as in Section 3.1). The sequence \((Z_i)_{i \in \mathbb{Z}}\) is stationary with summable beta mixing coefficients. Finally, Condition 1 holds with \( \lambda > 3/2 \).
Theorem 7 Under Condition 5, Theorems 1 and 2 hold, where \(\partial\ell_{\mu_0}(z) = 2\times 1_{\{y-\mu_0(x) \geq 0\}} - 1\) (\(1_{\cdot}\) is the indicator function) and

\[
P\partial^2\ell_{\mu_0}(\mu_n - \mu_0) h = 2 \int pdf(\mu_0(x) | x) \sqrt{n} (\mu_n(x) - \mu_0(x)) h(x) dP(x).
\]

The result depends on knowledge of the probability density function of \(Y\) conditioning on \(X\). Hence, inference in the presence of high dimensional nuisance parameters is less feasible within the proposed methodology.

References


Supplementary Material

8 Appendix 1: Proofs

Recall that \( \ell_\mu(Z) = L(Z, \mu(X)) \) and \( \partial^k \ell_\mu(Z) = \partial^k L(Z, t)|_{t=\mu(X)}, \ k = 1, 2, \ldots \). Condition 2 implies that \( P\ell_\mu \) is two times Fréchet differentiable (w.r.t. \( \mu \in \mathcal{H}^K \)) at \( \mu \) in the direction of \( h \in \mathcal{H}^K \). It can be shown that these two derivatives are \( P\partial \ell_\mu h \) and \( P\partial^2 \ell_\mu hh \), respectively. For this purpose, we view \( P\ell_\mu \) as a map from the space of uniformly bounded functions on \( \mathcal{X}^K (L_\infty(\mathcal{X}^K)) \) to \( \mathbb{R} \). The details can be derived following the steps in the proof of Lemma 2.21 in Steinwart and Christmann (2008) or the proof of Lemma A.4 in Hable (2012). The application of those proofs to the current scenario, essentially requires that the loss function \( L(Z, t) \) is differentiable w.r.t. real \( t \), and that \( \mu \) is uniformly bounded, together with integrability of the quantities \( \Delta_0 \) and \( \Delta_1 \), as implied by Condition 2. At some point it will also be necessary to take the Fréchet derivative of \( P_n \ell_\mu \) and \( P_n \partial \ell_\mu h \) conditioning on the sample data. By Condition 2 this will also hold because \( \Delta_0 \) and \( \Delta_1 \) are finite. This will also allow us to apply Taylor’s Theorem in Banach spaces. Following the aforementioned remarks, when the loss function is three times differentiable, we also have that for any \( h \in \mathcal{H}^K \), the Fréchet derivative of \( P\partial^2 \ell_\mu h \) in the direction of \( h' \in \mathcal{H}^K \) is \( P\partial^3 \ell_\mu hh' \). These facts will be used throughout the proofs with no further mention. Moreover, throughout, for notational simplicity, we tacitly suppose that \( \sup_{x \in \mathcal{X}^K} \sqrt{C_{\mathcal{H}^K}(x, x)} = 1 \) so that \( h \in \mathcal{H}^K(B) \) implies that \( |h|_\infty \leq B \) for any \( B > 0 \).

8.1 Complexity and Gaussian Approximation

The reader can skip this section and refer to it when needed. Recall that the \( \epsilon \)-covering number of a set \( \mathcal{F} \) under the \( L_p \) norm (denoted by \( N(\epsilon, \mathcal{F}, |\cdot|_p) \)) is the minimum number of balls of \( L_p \) radius \( \epsilon \) needed to cover \( \mathcal{F} \). The entropy is the logarithm of the covering number. The \( \epsilon \)-bracketing number of the set \( \mathcal{F} \) under the \( L_p \) norm is the minimum number of \( \epsilon \)-brackets under the \( L_p \) norm needed to cover \( \mathcal{F} \). An \( L_p \) \( \epsilon \)-bracket \( [f_L, f_U] \) is the set of all functions \( f \in \mathcal{F} \) such that \( f_L \leq f \leq f_U \) and \( |f_L - f_U|_p \leq \epsilon \). Denote the \( L_p \) \( \epsilon \)-bracketing number of \( \mathcal{F} \) by \( N(\epsilon, \mathcal{F}, |\cdot|_p) \). Under the uniform norm, \( N(\epsilon, \mathcal{F}, |\cdot|_\infty) = N(\epsilon, \mathcal{F}, |\cdot|_\infty) \).

In this section, let \( (G(x))_{x \in \mathcal{X}} \) be a centered Gaussian process on \( \mathcal{X} \) with covariance
C as in (6). For any $\epsilon > 0$, let

$$\phi (\epsilon) = - \ln \Pr (\|G\|_\infty < \epsilon).$$

The space $\mathcal{H}$ is generated by the measure of the Gaussian process $(G(x))_{x \in \mathcal{X}}$ with covariance function $C$. In particular, $G(x) = \sum_{v=1}^{\infty} \lambda_v \xi_v \varphi_v (x)$, where the $\xi_v$’s are i.i.d. standard normal random variables, and the equality holds in distribution. For any positive integer $V$, the $l$-approximation number $l_V (G)$ w.r.t. $\|\cdot\|_\infty$ (e.g., Li and Linde, 1999, see also Li and Shao, 2001) is bounded above by $\left( \mathbb{E} \left| \sum_{v>V} \lambda_v \xi_v \varphi_v \right|_\infty^2 \right)^{1/2}$. Under Condition 1, deduce that

$$l_V (G) \lesssim \sum_{v>V} \lambda_v \lesssim V^{-(\lambda-1)}. \quad (21)$$

There is a link between the $l_V (G)$ approximating number of the centered Gaussian process $G$ with covariance $C$ and the $L_\infty \epsilon$-entropy number of the class of functions $\mathcal{H} (1)$, which is denoted by $\ln N (\epsilon, \mathcal{H} (1), \|\cdot\|_\infty)$. These quantities are also related to the small ball probability of $G$ under the sup norm (results hold for other norms, but will not be used here). We have the following bound on the $\epsilon$-entropy number of $\mathcal{H} (1)$.

**Lemma 2** Under Condition 1, $\ln N (\epsilon, \mathcal{H} (1), \|\cdot\|_\infty) \lesssim \epsilon^{-2/(2\lambda-1)}$.

**Proof.** As previously remarked, the space $\mathcal{H} (1)$ is generated by the law of the Gaussian process $G$ with covariance function $C$. For any integer $V < \infty$, the $l$-approximation number $l_V (G)$ is bounded as in (21). In consequence, $\phi (\epsilon) \lesssim \epsilon^{-1/(\lambda-1)}$, by Proposition 4.1 in Li and Linde (1999). Then, Theorem 1.2 in Li and Linde (1999) implies that $\ln N (\epsilon, \mathcal{H} (1), \|\cdot\|_\infty) \lesssim \epsilon^{-2/(2\lambda-1)}$. □

**Lemma 3** Under Condition 1, the $\epsilon$-entropy number of $\mathcal{H}^K (B)$ under $\|\cdot\|_\infty$ is given by

$$\ln N (\epsilon, \mathcal{H}^K (B), \|\cdot\|_\infty) \lesssim (B/\epsilon)^{2/(2\lambda-1)} + K \ln \left( \frac{B}{\epsilon} \right).$$

**Proof.** Functions in $\mathcal{H}^K (B)$ can be written as $\mu (x) = \sum_{k=1}^{K} b_k f^{(k)} (x^{(k)})$ where $f^{(k)} \in \mathcal{H} (1)$. Hence, the covering number of $\{ \mu \in \mathcal{H}^K (B) \}$ is bounded by the product of the covering number of the sets $\mathcal{F}_1 := \{(b_1, b_2, \ldots, b_K) \in \mathbb{R}^K : \sum_{k=1}^{K} b_k^2 \leq B^2 \}$ and $\mathcal{F}_2 := \{ f^{(k)} \in \mathcal{H} (B) \}$. The $\epsilon$-covering number of $\mathcal{F}_1$ is bounded by a constant multiple of $(B/\epsilon)^K$ under the supremum norm. The $\epsilon$-covering number of $\mathcal{F}_2$ is given by Lemma 2, i.e. $\exp \left\{ (B/\epsilon)^{2/(2\lambda-1)} \right\}$. The lemma follows by taking logs of these quantities. □
Next, link the entropy of $\mathcal{H}(1)$ to the entropy with bracketing of $\ell_\mu h$.

**Lemma 4** Suppose Condition 1 holds. For the set $\mathcal{F} := \{\partial \ell_\mu h : \mu \in \mathcal{H}^K(B), h \in \mathcal{H}^K(1)\}$, for any $p \in [1, \infty]$ satisfying Condition 2, the $L_p$ $\epsilon$-entropy with bracketing is

$$\ln N_{\|\,} (\epsilon, \mathcal{F}, |\cdot|_p) \lesssim (B/\epsilon)^{2/(2\lambda - 1)} + K \ln \left( \frac{B}{\epsilon} \right).$$

The same exact result holds for $\mathcal{F} := \{\ell_\mu : \mu \in \mathcal{H}^K(B)\}$ under Condition 2.

**Proof.** The proof uses the smoothness of the loss function to show that the complexity of $\mathcal{F}$ can be bounded by the complexity of $\mathcal{H}^K(B)$. For simplicity, only prove the result for $\mathcal{F} := \{\partial \ell_\mu h : \mu \in \mathcal{H}^K(B), h \in \mathcal{H}^K(1)\}$, as the full result follows from the same argument. To this end, note that by Condition 2 and the triangle inequality,

$$|\partial \ell_\mu h - \partial \ell_{\mu'} h' | \leq |\partial \ell_\mu h - \partial \ell_{\mu'}| \sup_{h \in \mathcal{H}^K(1)} |h| + \sup_{\mu \in \mathcal{H}^K(B)} |\partial \ell_{\mu'}| |h - h'|.$$

By Condition 2, $|\partial \ell_\mu (z)| \leq \Delta_1(z)$, and $|\partial \ell_\mu (z) - \partial \ell_{\mu'} (z) | \leq \Delta_2(z) |\mu (x) - \mu' (x)|$, and $P(\Delta_1^p + \Delta_2^p) < \infty$. By Lemma 1, $|h(x)| \lesssim 1$. Hence, by these remarks, explicitly writing the arguments of the functions, the previous display is bounded by

$$\Delta_2(z) |\mu - \mu'|_\infty + \Delta_1(z) |h - h'|_\infty.$$

Theorem 2.7.11 in van der Vaart and Wellner (2000) says that the $L_p$ $\epsilon$-bracketing number of class of functions satisfying the above Lipschitz kind of condition is bounded by the $L_\infty$ $\epsilon'$-covering number of $\mathcal{H}^K(B) \times \mathcal{H}(1)$ with $\epsilon' = \epsilon/ \left[ 2 (P |\Delta_1 + \Delta_2|^p)^{1/p} \right]$. Using Lemma 3, the statement of the lemma is deduced because the product of the covering numbers is the sum of the entropy numbers.

We shall also need the following.

**Lemma 5** Suppose Condition 1 holds. For the set $\mathcal{F} := \{\partial \ell^2 hh' : \mu \in \mathcal{H}^K(B), h, h' \in \mathcal{H}^K(1)\}$, and any $p \in [1, \infty]$ satisfying Condition 2 with the addition that $P(\Delta_1^{2p} + \Delta_2^p) < \infty$, the $L_p$ $\epsilon$-entropy with bracketing is

$$\ln N_{\|\,} (\epsilon, \mathcal{F}, |\cdot|_p) \lesssim (B/\epsilon)^{2/(2\lambda - 1)} + K \ln \left( \frac{B}{\epsilon} \right),$$

where $N_{\|\,} (\epsilon, \mathcal{F}, |\cdot|_p)$ is the bracketing number.
If also $P (\Delta_2^b + \Delta_3^b) < \infty$, $\{\partial^2 \ell_{\mu} hh' : \mu \in \mathcal{H}^K (B), h, h' \in \mathcal{H}^K (1)\}$ has $L_p \epsilon$-entropy with bracketing as in the above display.

**Proof.** The proof is the same as the one of Lemma 4. By Condition 2 and the triangle inequality, for $g, g' \in \mathcal{H}^K (1)$

$$|\partial \ell^2_{\mu} hh' - \partial \ell^2_{\mu} gg'| \leq |\partial \ell^2_{\mu} - \partial \ell^2_{\mu}'| \sup_{h \in \mathcal{H}^K (1)} |h|^2 + \sup_{\mu \in \mathcal{H}^K (B)} |\partial \ell^2_{\mu}| |hh' - gg'|.$$ 

By Condition 2, $|\partial \ell^2_{\mu} (z)| \leq \Delta^2_1 (z), |\partial \ell^2_{\mu} (z) - \partial \ell^2_{\mu'} (z)| \leq 2\Delta_1 (z) \Delta_2 (z) |\mu (x) - \mu' (x)|$, and $P (\Delta_1^{2p} + \Delta_2^2) < \infty$. By Lemma 1, $|h (x)| \lesssim 1$. Hence, by these remarks, explicitly writing the the arguments of the functions, the previous display is bounded by

$$2\Delta_1 (z) \Delta_2 (z) |\mu - \mu'|_\infty + \Delta^2_1 (z) |h - h'|_\infty.$$ 

Theorem 2.7.11 in van der Vaart and Wellner (2000) says that the $L_p \epsilon$-bracketing number of class of functions satisfying the above Lipschitz kind of condition is bounded by the $L_\infty \epsilon'$-covering number of $\mathcal{H}^K (B) \times \mathcal{H}^K (1)$ with $\epsilon' = \epsilon / \left[2 \left(P \left|\Delta_1^{2p} + \Delta_2^2\right|^{1/p}\right]\right].$

The last statement in the lemma is proved following step by step the proof of Lemma 4 with $\partial \ell_{\mu}$ replaced by $\partial^2 \ell_{\mu}$ and $h$ by $hh'$.

**Lemma 6** Under Conditions 1, 2, 3,

$$\sqrt{n} (P_n - P) \partial \ell_{\mu} h \to G (\partial \ell_{\mu}, h)$$

weakly, where $G (\partial \ell_{\mu}, h)$ is a mean zero Gaussian process indexed by $(\partial \ell_{\mu}, h) \in \{\partial \ell_{\mu} : \mu \in \mathcal{H}^K (B)\} \times \mathcal{H}^K (1)$, with a.s. continuous sample paths and covariance function

$$\mathbb{E} G (\partial \ell_{\mu}, h) G (\partial \ell_{\mu}', h') = \sum_{j \in \mathbb{Z}} P_{1,j} (\partial \ell_{\mu} h, \partial \ell_{\mu} h')$$

**Proof.** The proof shall use the main result in Doukhan et al. (1995). Let $\mathcal{F} := \{\partial \ell_{\mu} h : \mu \in \mathcal{H}^K (B), h \in \mathcal{H}^K (1)\}$. The elements in $\mathcal{F}$ have finite $L_p$ norm because $P |\partial \ell_{\mu}|^p \leq P \Delta_1^p$ by Condition 2, and $|h|_\infty \lesssim 1$ by Lemma 1. To avoid extra notation, it is worth noting that the entropy integrability condition in Doukhan et al. (1995, Theorem 1, eq. 2.10) is implied by

$$\int_0^1 \sqrt{\ln N \left(\epsilon, \mathcal{F}, |.|_p\right)} d\epsilon < \infty. \quad (22)$$
and $\beta (i) \lesssim (1 + i)^{-\beta}$ with $\beta > p / (p - 2)$ and $p > 2$. Then, Theorem 1 in Doukhan et al. (1995) shows that the empirical process indexed in $\mathcal{F}$ converges weakly to the Gaussian one given in the statement of the present lemma. By Condition 3, it is sufficient to show (22). By Lemma 4, the integral is finite because $\lambda > 1$ by Condition 1.

8.2 Proof of Theorem 1

The proof is split into the part concerned with the constrained estimator and the one that studies the penalized estimator.

8.2.1 Consistency of the Constrained Estimator

At first we show Point 1 verifying the conditions of Theorem 3.2.5 van der Vaart and Wellner (2000) which we will refer to as VWTh herein. To this end, by Taylor’s Theorem in Banach spaces,

$$P\ell_\mu - P\ell_{\mu_0} = P \partial_\mu \ell_{\mu_0} (\mu - \mu_0) + \frac{1}{2} P \partial^2 \ell_{\mu_t} (\mu - \mu_0)^2$$

for $\mu_t = \mu + t (\mu_0 - \mu)$ with some $t \in [0, 1]$ and arbitrary $\mu \in \mathcal{H}^K (B)$. The variational inequality $P \partial_\mu \ell_{\mu_0} (\mu - \mu_0) \geq 0$ holds by definition of $\mu_0$ and the fact that $\mu \in \mathcal{H}^K (B)$. Therefore, the previous display implies that $P\ell_\mu - P\ell_{\mu_0} \gtrsim P (\mu - \mu_0)^2$ because $P \partial^2 \ell_{\mu_t} (\mu - \nu)^2 \geq P (\mu - \nu)^2 \geq 0$ by Condition 2. The latter holds with equality if and only if $\mu = \mu_0$ in $L_2$. This verifies the first condition in VWTh. Given that the loss function is convex and coercive and that $\mathcal{H}^K (B)$ is a closed convex set, this also shows that the population minimizer $\mu_0$ exists and is unique up to an $L_2$ equivalence class, as stated in the theorem. Moreover, given that $\mu, \mu_0 \in \mathcal{H}^K (B)$, then both $\mu$ and $\mu_0$ are uniformly bounded by a constant multiple $B$, hence for simplicity suppose they are bounded by $B$. This implies the following relation

$$B^{2-p} |\mu - \mu_0|_p \leq |\mu - \mu_0|_2 \leq |\mu - \mu_0|_p$$

for any $p \in (2, \infty)$. Hence, for any finite real $\delta$,

$$\sup_{|\mu - \mu_0|_2 < \delta} \mathbb{E} |(P_n - P) (\ell_\mu - \ell_{\mu_0})| \leq \sup_{|\mu - \mu_0|_p < B^{p-2}\delta} \mathbb{E} |(P_n - P) (\ell_\mu - \ell_{\mu_0})|$$

for any $p \in (2, \infty)$. Hence, for any finite real $\delta$,
To verify the second condition in VWTh, we need to find a function \( \phi(\delta) \) that grows slower than \( \delta^2 \) such that the r.h.s. of the above display is bounded above by \( n^{-1/2}\phi(\delta) \).

To this end, note that we are interested in the following class of functions \( \mathcal{F} := \{ \ell_\mu - \ell_\mu_0 : |\mu - \mu_0|_p \leq \delta' \} \) with \( \delta' = B^{p-2}\delta \). This class of functions satisfies \( |\ell_\mu - \ell_\mu_0|_p \leq (\Delta^p)^{1/p} \delta' \) using the differentiability and the bounds implied by Condition 2. Theorem 3 in Doukhan et al. (1995) says that that for large enough \( n \), eventually (see their page 410),

\[
\phi(\delta) \lesssim \int_0^{B^{p-2}\delta} \sqrt{\ln N[\epsilon, \mathcal{F}, |·|_p]} \, d\epsilon.
\]

Note that we have \( L_p \) balls of size \( B^{p-2}\delta \) rather than \( \delta \) and for this reason we have modified the limit in the integral. Moreover, as remarked in the proof of Lemma 6, the entropy integral in Doukhan et al. (1995) uses the bracketing number based on another norm. However, their norm is bounded by the \( L_p \) norm used here under the restrictions we impose on the mixing coefficients via Condition 3. To compute the integral we use Lemma 4, so that the l.h.s. of the display is a constant multiple of \( B^{(1-\alpha)+\alpha(p-2)}\delta^\alpha \) with \( \alpha = (2\lambda - 2) / (2\lambda - 1) \). The third condition in VWTh requires to find a sequence \( r_n \) such that \( r_n^2 N_\alpha(r_n^{-1}) \leq n^{1/2} \). Given that \( \phi(\delta) \lesssim A\delta^\alpha \) with \( A := B^{(1-\alpha)+\alpha(p-2)} \), deduce that we can set \( r_n \approx n^{(2\lambda-1)/(4\lambda)} \). Then VWTh states that \( |\mu_n - \mu_0|_2 = O_p(r_n^{-1}) \). Of course, if \( \mathcal{H}^K \) is finite dimensional, it is not difficult to show that \( r_n \approx n^{1/2} \). The space \( \mathcal{H}^K \) is finite dimensional if (6) has a finite number of terms.

We also show that \( \sup_{\mu \in \mathcal{H}^K(B)} |(P_n - P) \ell_\mu| \to 0 \) a.s. which shall imply \( |\mu_n - \mu_0|_2 \to 0 \) a.s. (Corollary 3.2.3 in van der Vaart and Wellner, 2000, replacing the in probability result with a.s.). This only require the loss function to be integrable (if the loss is positive), but does not allow us to derive convergence rate. For any fixed \( \mu \), \( |(P_n - P) \ell_\mu| \to 0 \) a.s., by the ergodic theorem, because \( P |\ell_\mu| < \infty \) by Condition 2. Hence, it is just sufficient to show that \( \{ \ell_\mu : \mu \in \mathcal{H}^K(B) \} \) has finite \( \epsilon \)-bracketing number under the \( L_1 \) norm (e.g., see the proof of Theorem 2.4.1 in van der Vaart and Wellner, 2000). This is the case by Lemma 4, because by Condition 1, \( \lambda > 1 \). Hence, \( |\mu_n - \mu_0|_2 \to 0 \) a.s..

To turn the \( L_2 \) convergence into uniform, note that \( \mathcal{H}^K(B) \) is compact under the uniform norm and functions in \( \mathcal{H}^K(B) \) are defined on a compact domain \( \mathcal{X}^K \). Hence, \( \mathcal{H}^K(B) \) is a subset of the space of continuous bounded function equipped with the uniform norm. In consequence, any convergent sequence in \( \mathcal{H}^K(B) \) converges uniformly.

We now turn to the relation between the constrained and penalized estimator, which
will also conclude the proof of Theorem 1.

8.2.2 The Constraint and the Lagrange Multiplier

The following lemma puts together crucial results for estimation in RKHS (Steinwart and Christmann, 2008, Theorems 5.9 and 5.17 for a proof). The cited results make use of the definition of integrable Nemitski loss of finite order $p$ (Steinwart and Christmann, 2008, Def. 2.16). However, under Condition 2, the proofs of those results still hold.

**Lemma 7** Under Condition 2,

\[
|\mu_{0,\rho} - \mu_{n,\rho}|_{\mathcal{H}^K} \leq \frac{1}{\rho} \left| P \partial_{\mu_{0,\rho}} \Phi - P_n \partial_{\mu_{0,\rho}} \Phi \right|_{\mathcal{H}^K},
\]

(23)

where $\Phi(x) = C_{\mathcal{H}^K}(\cdot, x)$ is the canonical feature map. Moreover, if $\mu_{0,\rho}$ is bounded for any $\rho \to 0$, then $|\mu_{0,\rho} - \mu_0|_{\mathcal{H}^K} \to 0$.

We apply Lemma 7 and the results in Section 8.1 to derive the following.

**Lemma 8** Suppose Conditions 1, 2 and 3. The following statements hold.

1. There is a finite $B$ such that $\mu_0 \in \text{int}(\mathcal{H}^K(B))$.

2. For any $\rho > 0$ possibly random, $|\mu_{n,\rho} - \mu_{0,\rho}|_{\mathcal{H}^K}^2 = O_p(\rho^{-2}n^{-1})$, and $|\mu_{n,\rho}|_{\mathcal{H}^K} \leq B$ eventually in probability for any $\rho \to 0$ such that $\rho n^{1/2} \to \infty$.

3. There is a $\rho = O_p\left(n^{-1/2}\right)$ such that $|\mu_{n,\rho}|_{\mathcal{H}^K} \leq B$ and

\[
\sup_{h \in \mathcal{H}^K(1)} P_n \partial_{\mu_{n,\rho}} h = O_p\left(n^{-1/2}B\right).
\]

**Proof.** Given that $K$ is finite and the kernel is additive, there is no loss in restricting attention to $K = 1$ in order to reduce the notational burden. We shall need a bound for the r.h.s. of (23). By (6), it follows that the canonical feature map satisfies $\Phi(x) = \sum_{v=1}^{\infty} \lambda_v^2 \varphi_v(\cdot) \varphi_v(x)$. This implies that,

\[
(P_n - P) \partial_{\mu_{0,\rho}} \Phi(x) = \sum_{v=1}^{\infty} \left[ \lambda_v^2 (P_n - P) \partial_{\mu_{0,\rho}} \varphi_v \right] \varphi_v(x).
\]

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By Lemma 7, \((8)\), and the above,

\[
\left| (P_n - P) \partial \ell_{\mu_0, \rho} \Phi \right|^2_{\mathcal{H}^K} = \sum_{v=1}^{\infty} \frac{\lambda_v^2 (P_n - P) \partial \ell_{\mu_0, \rho} \varphi_v^2}{\lambda_v^2} = \sum_{v=1}^{\infty} \lambda_v^2 [(P_n - P) \partial \ell_{\mu_0, \rho} \varphi_v]^2.
\]

In consequence of the above display, by the triangle inequality,

\[
\left| \mu_{0, \rho} - \mu_{n, \rho} \right|_{\mathcal{H}^K} \leq \frac{1}{\rho} \left[ \sum_{v=1}^{\infty} \lambda_v^2 \left| (P_n - P) \partial \ell_{\mu_0, \rho} \varphi_v \right|^2 \right]^{1/2} \leq \frac{1}{\rho} \sum_{v=1}^{\infty} \lambda_v \left| (P_n - P) \partial \ell_{\mu_0, \rho} \varphi_v \right|.
\]

Given that \(\mu_0 \in \mathcal{H}^K\), there is a finite \(B\) such that \(\mu_0 \in \text{int} (\mathcal{H}^K (B))\) (this proves Point 1 in the lemma). By this remark, it follows that, uniformly in \(\rho \geq 0\), there is an \(\epsilon > 0\) such that \(\left| \mu_{0, \rho} \right|_{\mathcal{H}^K} \leq B - \epsilon\). Hence, for any \(v \geq 1\), the maximal inequality of Theorem 3 in Doukhan et al. (1995) implies that

\[
\mathbb{E} \sup_{\mu \in \mathcal{H}^K (B)} \left| \sqrt{n} (P_n - P) \partial \ell_{\mu} \varphi_v \right| \leq c_1 \quad (24)
\]

for some finite constant \(c_1\) because, for each \(v\), the entropy integral \((22)\) is finite in virtue of Lemma 4. Define

\[
L_n := \sum_{v=1}^{\infty} \lambda_v \sup_{\mu \in \mathcal{H}^K (B)} \left| \sqrt{n} (P_n - P) \partial \ell_{\mu_0, \rho} \varphi_v \right|.
\]

Given that the \(\lambda_v\)'s are summable by Condition 1, deduce from \((24)\) that \((L_n)\) is a tight random sequence. Moreover, we have shown that \((23)\) is bounded by \(L_n/ \left( \rho n^{1/2} \right)\). This proves Point 2 in the lemma. Moreover, for any fixed \(\epsilon > 0\), we can choose \(\rho = \rho_n := L_n/ \left( \epsilon n^{1/2} \right)\) so that \(\left| \mu_{0, \rho} - \mu_{n, \rho} \right|_{\mathcal{H}^K} \leq \epsilon\) in probability. By the triangle inequality and the above calculations, deduce that, in probability,

\[
\left| \mu_{n, \rho} \right|_{\mathcal{H}^K} \leq \left| \mu_{0, \rho} \right|_{\mathcal{H}^K} + \left| \mu_{0, \rho} - \mu_{n, \rho} \right|_{\mathcal{H}^K} \leq B
\]

for \(\rho = \rho_n\). By tightness of \(L_n\), deduce that \(\rho_n = O_{\rho} \left( n^{-1/2} \right)\). Also, the first order condition for the sample estimator \(\mu_{n, \rho}\) reads

\[
P_n \partial \ell_{\mu_{n, \rho}} h = -2 \rho \left( \mu_{n, \rho}, h \right)_{\mathcal{H}^K} \leq 2 \rho \left| \mu_{n, \rho} \right|_{\mathcal{H}^K} \left| h \right|_{\mathcal{H}^K} \quad (25)
\]
for any $h \in \mathcal{H}^K (1)$. In consequence, $\sup_{h \in \mathcal{H}^K (1)} P_n \partial \ell_{\mu_n, \rho} h \leq 2 \rho |\mu_{n, \rho}|_{\mathcal{H}^K}$. These calculations prove Point 3 in the lemma when $\rho = O_p \left(n^{-1/2}\right)$.

The penalized objective function is increasing with $\rho$. In the Lagrangian formulation of the constrained minimization, interest lies in finding the smallest value of $\rho$ such that the constraint is still satisfied. When $\rho$ equals such smallest value $\rho_{B,n}$, we have $\mu_n = \mu_{n, \rho}$. From Lemma 8 deduce that $\rho_{B,n} = O_p \left(n^{-1/2}\right)$. Also, if $\mathcal{H}^K$ is infinite dimensional, the constraint needs to be binding so that $|\mu_n|_{\mathcal{H}^K} = B$. Hence, if $\mu_0 \in int \left(\mathcal{H}^K (B)\right)$ there is an $\epsilon > 0$ such that $|\mu_0|_{\mathcal{H}^K} = B - \epsilon$. Then, we must have

$$
|\mu_n - \mu_0|^2_{\mathcal{H}^K} = |\mu_n|^2_{\mathcal{H}^K} + |\mu_0|^2_{\mathcal{H}^K} - 2 \langle \mu_n, \mu_0 \rangle_{\mathcal{H}^K} = \left(B^2 + (B - \epsilon)^2 - 2 \langle \mu_n, \mu_0 \rangle_{\mathcal{H}^K}\right).
$$

But $\langle \mu_n, \mu_0 \rangle_{\mathcal{H}^K} \leq |\mu_n|_{\mathcal{H}^K} |\mu_0|_{\mathcal{H}^K} \leq B (B - \epsilon)$. Hence, the above display is greater or equal than

$$
B^2 + (B - \epsilon)^2 - 2B (B - \epsilon) \geq \epsilon^2.
$$

This means that $\mu_n$ cannot converge under the norm $|\cdot|_{\mathcal{H}^K}$.

The statement concerning approximate minimizers will be proved in Section 8.4.

### 8.3 Proof of Theorem 2

It is convenient to introduce additional notation and concepts that will be used in the remaining of the paper. By construction the minimizer of the population objective function is $\mu_0 \in \mathcal{H}^K (B)$. Let $l^\infty (\mathcal{H}^K)$ be the space of uniformly bounded functions on $\mathcal{H}^K$. Let $\Psi (\mu)$ be the operator in $l^\infty (\mathcal{H}^K)$ such that $\Psi (\mu) h = P \partial \ell_{\mu} h$, $h \in \mathcal{H}^K$. If the objective function is Fréchet differentiable, the minimizer of the objective function $P \ell_{\mu}$ in $\mathcal{H}^K (B)$ satisfies the variational inequality: $\Psi (\mu) h \geq 0$ for any $h$ in the tangent cone of $\mathcal{H}^K (B)$ at $\mu_0$. This tangent cone is defined as $\limsup_{t \downarrow 0} \left(\mathcal{H}^K (B) - \mu_0\right) / t$. If $\mu_0$ is in the interior of $\mathcal{H}^K (B)$, this tangent cone is the whole of $\mathcal{H}^K$. Hence by linearity of the operator $\Psi (\mu)$, attention can be restricted to $h \in \mathcal{H}^K (1)$. When $\mu_0 \in int \left(\mathcal{H}^K (B)\right)$, it also holds that $\Psi (\mu_0) h = 0$, for any $h \in \mathcal{H}^K (1)$. Then, in the following calculations, $\Psi (\mu)$ can be restricted to be in $l^\infty (\mathcal{H}^K (1))$. The empirical counterpart of $\Psi (\mu)$ is the operator $\Psi_n (\mu)$ such that $\Psi_n (\mu) h = P_n \partial \ell_{\mu} h$. Finally, write $\hat{\Psi}_{\mu_0} (\mu - \mu_0)$ for the Fréchet derivative of $\Psi (\mu)$ at $\mu_0$ tangentially to $(\mu - \mu_0)$, where $\mu, \mu_0 \in \mathcal{H}^K (B)$. Then, $\hat{\Psi}_{\mu_0}$ is an operator from $\mathcal{H}^K$ to $l^\infty (\mathcal{H}^K)$. As for $\Psi (\mu)$, the operator $\hat{\Psi}_{\mu_0} (\mu - \mu_0)$ can
be restricted to be in \( l^\infty (H^K(1)) \). These facts will be used without further notice in what follows. Most of these concepts are reviewed in van der Vaart and Wellner (2000, ch.3.3) where this same notation is used.

Deduce that \( P\ell_\mu \) is Fréchet differentiable and its derivative is the map \( \Psi (\mu) \). By the conditions of Theorem 2, \( \mu_0 \in \text{int} (H^K(B)) \), hence by the first order conditions, \( \Psi (\mu_0) h = 0 \) for any \( h \in H^K(1) \). By this remark, and basic algebra,

\[
\sqrt{n} \Psi_n (\mu_n) = \sqrt{n} \Psi_n (\mu_0) + \sqrt{n} [\Psi (\mu_n) - \Psi (\mu_0)]
\]

\[
+ \sqrt{n} [\Psi_n (\mu_n) - \Psi (\mu_n)] - \sqrt{n} [\Psi_n (\mu_0) - \Psi (\mu_0)].
\]  

(26)

To bound the last two terms, verify that

\[
\sup_{h \in H^K(1)} \sqrt{n} [\Psi_n (\mu_n) - \Psi (\mu_n)] - (\Psi_n (\mu_0) - \Psi (\mu_0)) h = o_p (1).
\]

This follows if (i) \( \sqrt{n} (\Psi_n (\mu) - \Psi (\mu)) h \), \( \mu \in H^K(B) \), \( h \in H^K(1) \), converges weakly to a Gaussian process with continuous sample paths, (ii) \( H^K(B) \) is compact under the uniform norm, and (iii) \( \mu_n \) is consistent for \( \mu_0 \) in \( |\cdot|_\infty \). Point (i) is satisfied by Lemma 6, which also controls the first term on the r.h.s. of (26). Point (ii) is satisfied by Lemma 3. Point (iii) is satisfied by Theorem 1. Hence, by continuity of the sample paths of the Gaussian process, as \( \mu_n \to \mu_0 \) the above display holds true.

To control the second term on the r.h.s. of (26), note that the Fréchet derivative of \( \Psi (\mu) \) at \( \mu_0 \) is the linear operator \( \dot{\Psi}_{\mu_0} \) such that \( \dot{\Psi}_{\mu_0} (\mu - \mu_0) h = P\partial^2 \ell_{\mu_0} (\mu - \mu_0) h \), which can be shown to exist based on the remarks at the beginning of Section 8. For any \( h \in H^K(1) \),

\[
\left| [\Psi (\mu_n) - \Psi (\mu_0)] h - \dot{\Psi}_{\mu_0} (\mu_n - \mu_0) h \right| \leq \sup_{t \in (0,1)} \left| P\partial^3 \ell_{\mu_0 + t (\mu_n - \mu_0)} (\mu_n - \mu_0)^2 h \right|
\]

(27)

using differentiability of the loss function and Taylor’s theorem in Banach spaces. By Condition 4, and the fact that \( h \) is uniformly bounded, the r.h.s. is a constant multiple of \( P (\mu - \mu_0)^2 \). By Theorem 1 this quantity is \( O_p \left( n^{-(2\lambda - 1)/(2\lambda)} \right) \). Given that \( \lambda > 1 \), these calculations show that

\[
\sqrt{n} [\Psi (\mu_n) - \Psi (\mu_0)] = \sqrt{n} \dot{\Psi}_{\mu_0} (\mu_n - \mu_0) + o_p (1).
\]
In consequence, from (26) deduce that
\[
\sqrt{n} \Psi_n (\mu_n) - \sqrt{n} \Psi_n (\mu_0) = \sqrt{n} (\Psi (\mu_n) - \Psi (\mu_0)) + o_p (1)
\]
\[
= \sqrt{n} \Psi_{\mu_0} (\mu_n - \mu_0) + o_p (1). \tag{28}
\]

By Lemma 6, \( \sqrt{n} \Psi_n (\mu_0) = O_p (1) \). For the moment, suppose that \( \mu_n \) is the exact solution to the minimization problem, i.e. as in (9). Hence, by Lemma 8, \( \sup_{h \in \mathcal{H}} \sqrt{n} \Psi_n (\mu_n) h = O_p (1) \), implying that \( \sup_{h \in \mathcal{H}} \sqrt{n} \Psi_{\mu_0} (\mu_n - \mu_0) h = O_p (1) \). Finally, if \( \sup_{h \in \mathcal{H}} \sqrt{n} \Psi_n (\mu_n) h = o_p (1) \), (28) together with the previous displays imply that \( -\lim_n \sqrt{n} (\Psi_n (\mu_0) - \Psi (\mu_0)) = \lim_n \Psi_{\mu_0} \sqrt{n} (\mu_n - \mu_0) \) in probability, where the l.h.s. is the Gaussian process \( G \) as given in the statement of the theorem. It remains to show that if we use an approximate minimizer say \( \nu_n \) to distinguish it here from \( \mu_n \) in (9), the result still holds. The lemma in the next section shows that this is true, hence completing the proof of Theorem 2.

8.4 Approximate Minimizers

Lemma 9 Let \( (\epsilon_n) \) be an \( o_p (1) \) sequence. Suppose that \( \nu_n \) satisfies \( P_n \ell_{\nu_n} \leq P_n \ell_{\mu_n} - O_p (\epsilon_n) \), where \( \mu_n \) is as in (9). Also suppose that \( \nu_{n, \rho} \) satisfies \( P_n \ell_{\nu_{n, \rho}} + \rho |\nu_{n, \rho}|_{\mathcal{H}_K}^2 \leq P_n \ell_{\mu_{n, \rho}} + \rho |\mu_{n, \rho}|_{\mathcal{H}_K}^2 - O_p (\rho \epsilon_n) \), where \( \mu_{n, \rho} \) is as in (12) and \( \rho n^{1/2} \to \infty \).

1. Under the conditions of Theorem 1, \( |\mu_n - \nu_n|_\infty = o_p (1), |\mu_n - \nu_n|_2 = o_p (\epsilon_n) \) and \( |\mu_{n, \rho} - \nu_{n, \rho}|_{\mathcal{H}_K} = O_p (\epsilon_n) \) in probability, and there is a finite \( B \) such that \( |\nu_{n, \rho}|_{\mathcal{H}_K} \leq B \) eventually in probability.

2. If \( \epsilon_n = o_p (n^{-1/2}) \), under the Conditions of Theorem 2, \( \sup_{h \in \mathcal{H}} |\Psi_n (\mu_n) h - \Psi_n (\nu_n) h| = o_p (n^{-1/2}) \).

Proof. At first consider the penalized estimator. To this end, follow the same steps in the proof of 5.14 in Theorem 5.9 of Steinwart and Christmann (2008). Mutatis mutandis, the argument in their second paragraph on page 174 gives
\[
\langle \nu_{n, \rho} - \mu_{n, \rho}, P_n \partial \ell_{\mu_{n, \rho}} \Phi + 2 \rho \mu_{n, \rho} \rangle_{\mathcal{H}_K} + \rho |\mu_{n, \rho} - \nu_{n, \rho}|_{\mathcal{H}_K}^2
\leq P_n \ell_{\nu_{n, \rho}} + \rho |\nu_{n, \rho}|_{\mathcal{H}_K}^2 - (P_n \ell_{\mu_{n, \rho}} + \rho |\mu_{n, \rho}|_{\mathcal{H}_K}^2).
\]

Derivation of this display requires convexity of \( L (z, t) \) w.r.t. \( t \), which is the case by Condition 2. By assumption, the r.h.s. is \( O_p (\rho \epsilon_n) \). Note that \( \mu_{n, \rho} \) is the exact minimizer of the penalized empirical risk. Hence, eq. (5.12) in Theorem 5.9 of Steinwart and
Christmann (2008) says that \( \mu_{n,\rho} = -(2\rho)^{-1} P_n \partial \ell_{\mu_{n,\rho}} \Phi \) for any \( \rho > 0 \), implying that the inner product in the display is zero. By these remarks, deduce that the above display simplifies to \( \rho |\mu_{n,\rho} - \nu_{n,\rho}|_{H^K} = O_p(\rho \epsilon_n) \). Deduce that \( |\mu_n - \nu_n|_{H^K} = o_p(1) \) so that by the triangle inequality, and Lemma 8, \( |\nu_{n,\rho}|_{H^K} \leq B \) eventually, in probability for for some \( B < \infty \).

Now, consider the constrained estimator. Conditioning on the data, by definition of \( \mu_n \), the variational inequality \( P_n \partial \ell_{\mu_n}(\nu_n - \mu_n) \geq 0 \) holds because \( \nu_n - \mu_n \) is an element of the tangent cone of \( H^K(B) \) at \( \mu_n \). Conditioning on the data, by Taylor’s theorem in Banach spaces, and the fact that \( \inf_{z \in Z, |t| \leq B} \partial^2 L(z, t) > 0 \) by Condition 2, deduce that \( |P_n \ell_{\nu_n} - P_n \ell_{\mu_n}| \geq P_n (\mu_n - \nu_n)^2 \). By the conditions of the lemma, and the previous inequality deduce that \( P_n (\mu_n - \nu_n)^2 = O_p(\epsilon_n) \). The \( L_2 \) convergence is then turned into uniform using the same argument used in the proof of Theorem 1. Now, conditioning on the data, by Fréchet differentiability,

\[
|\Psi_n(\mu_n) h - \Psi_n(\nu_n) h| = |P_n \partial \ell_{\nu_n} - P_n \partial \ell_{\mu_n}| \leq P_n \sup_{\mu \in H^K(B)} \partial^2 \ell_{\mu}(\nu_n - \mu_n) h.
\]

By Holder’s inequality, and the fact that \( h \in H^K(1) \) is bounded, the r.h.s. is bounded by

\[
\left[ P_n \left( \sup_{\mu \in H^K(B)} \partial^2 \ell_{\mu} \right)^2 \right]^{1/2} \leq \left[ P_n (\nu_n - \mu_n)^2 \right]^{1/2} \leq \left[ P_n \Delta_2^2 \right]^{1/2} \left[ P_n (\nu_n - \mu_n)^2 \right]^{1/2}.
\]

By Condition 2, deduce that \( P_n \Delta_2^2 = O_p(1) \) so that, by the previous calculations, the result follows.

**8.5 Proof of Theorem 3**

We use the operators \( \Pi_{\rho}, \Pi_{n,\rho}, \tilde{\Pi}_{n,\rho} \) such that for any \( h \in H^K \):

\[
\Pi_{\rho} h := \arg \inf_{\nu \in \mathcal{R}_0} P \partial^2 \ell_{\mu_0} (h - \nu)^2 + \rho |\nu|_{H^K}^2 \quad \text{as in (13)}
\]

\[
\Pi_{n,\rho} h := \arg \inf_{\nu \in \mathcal{R}_0} P_n^2 \partial \ell_{\mu_{n0}} (h - \nu)^2 + \rho |\nu|_{H^K}^2 \quad \text{as in (14)}
\]

\[
\tilde{\Pi}_{n,\rho} h := \arg \inf_{\nu \in \mathcal{R}_0} P \partial \ell_{\mu_{n0}}^2 (h - \nu)^2 + \rho |\nu|_{H^K}^2.
\]

(29)
To ease notation, we may write $\Pi_n = \Pi_{n, \rho}$ when $\rho = \rho_n$.

The proof uses some preliminary results. In what follows, we shall assume that $K = 1$ in the proofs. This is to avoid notational complexities that could obscure the main steps in the derivations. Because of additivity, this should not be restrictive as long as $K$ is bounded. We need to account for the fact the in the projection $\mu_{n0}$ might be used in place of $\mu_0$ even when we use the true law $P$.

**Lemma 10** Suppose that $h \in \mathcal{H}^K (1)$. Then, $|\Pi_0 h|_{\mathcal{H}^K} \leq 1$.

**Proof.** By construction, the linear projection $\Pi_0 h$ satisfies $\Pi_0 h \in \mathcal{R}_0$ and $\Pi_0 (h - \Pi_0 h) = 0$. Hence, the space $\mathcal{H}^K$ is the direct sum of $\mathcal{R}_0$ and its complement in $\mathcal{H}^K$, say $\mathcal{R}_0^c$ and their intersection is zero. Note that we do not necessarily have $\mathcal{R}_0^c = \mathcal{R}_1$ unless the basis that spans $\mathcal{R}_1$ is already linearly independent of $\mathcal{R}_0$. Then, by Lemma 9.1 in van der Vaart and van Zanten (2008) $|h|_{\mathcal{H}^K} = |\Pi_0 h|_{\mathcal{R}_0} + |h - \Pi_0 h|_{\mathcal{R}_0^c}$. The norms are the ones induced by the inner products in the respective spaces. Hence, we trivially have that $|\Pi_0 h|_{\mathcal{R}_0} \leq |h|_{\mathcal{H}^K} = 1$ when $h \in \mathcal{H}^K (1)$. ■

**Lemma 11** Under Condition 4, if $\rho n^{(2\lambda-1)/(4\lambda)} \to \infty$, then, $\sup_{h \in \mathcal{H}^K (1)} \left| \left( \Pi_\rho - \tilde{\Pi}_{n, \rho} \right) h \right|_{\mathcal{H}^K} \to 0$ in probability.

**Proof.** Let $\tilde{P}$ and $\tilde{P}_n$ be finite positive measures such that $d\tilde{P}/dP = \partial^2 \ell_{\mu_0}$ and $d\tilde{P}_n/dP = \partial^2 \ell_{\mu_{0,n}}$. By Lemma 7, using the same arguments as in the proof of Lemma 8,

$$\left| \left( \Pi_\rho - \tilde{\Pi}_{n, \rho} \right) h \right|_{\mathcal{H}^K} \leq \frac{1}{\rho} \sum_{v=1}^{\infty} \lambda_{\mathcal{R}_0,v} \left| \left( \tilde{P}_n - \tilde{P} \right) (h - \Pi_\rho h) \varphi_{\mathcal{R}_0,v} \right|. \quad (30)$$

Taking derivatives, we bound each term in the absolute value by

$$|P \left( \partial^2 \ell_{\mu_{0,n}} - \partial^2 \ell_{\mu_0} \right) (h - \Pi_\rho h) \varphi_{\mathcal{R}_0,v}| \leq \left| P \sup_{\mu \in \mathcal{H}^K (B)} \left| \partial^3 \ell_\mu \right| (\mu_{0,n} - \mu_0) (h - \Pi_\rho h) \varphi_{\mathcal{R}_0,v} \right|. $$

By Lemma 10 and the definition of penalised estimation, $|\Pi_\rho h|_{\mathcal{H}^K} \leq |\Pi_0 h|_{\mathcal{H}^K} \leq 1$ independently of $\rho$. Hence, $|h - \Pi_\rho h|_{\infty} \leq 2$. Moreover, the $\varphi_{\mathcal{R}_0,v}$’s are uniformly bounded. Therefore, the r.h.s. of the above display is bounded by a constant multiple of

$$\sqrt{P |\mu_{0,n} - \mu_0|^2} \sqrt{P \sup_{\mu \in \mathcal{H}^K (B)} \left| \partial^3 \ell_\mu \right|^2} = |\mu_{0,n} - \mu_0|_2 \sqrt{P \Delta^2_3}. $$

51
The term $P\Delta_3^2$ is finite by Condition 2. By Theorem 1, we have that $|\mu_{0,n} - \mu_0|_2 = O_p\left(n^{-(2\lambda - 1)/(4\lambda)}\right)$. Using the above display to bound (30), deduce that the lemma holds true if $\rho^{-1}\left(n^{-(2\lambda - 1)/(4\lambda)}\right) = o_p(1)$ as stated in the lemma. Taking supremum w.r.t. $h \in H^K(1)$ in the above steps, deduce that the result holds uniformly in $h \in H^K(1)$. ■

**Lemma 12** Under Condition 4, we have that $\sup_{h \in H^K(1)} \left| \left( \Pi_{n,\rho} - \tilde{\Pi}_{n,\rho} \right) h \right|_{H^K} \to 0$ in probability for any $\rho$ such that $\rho n^{1/2} \to \infty$ in probability.

**Proof.** Following the same steps as in the proof of Lemma 11, deduce that

$$\left| \left( \Pi_{n,\rho} - \tilde{\Pi}_{n,\rho} \right) h \right|_{H^K} \leq \frac{1}{\rho} \sum_{v=1}^{\infty} \lambda_v \left| (P_n - P) \partial^2 \ell_{\mu_0,n} \left( h - \tilde{\Pi}_{n,\rho} h \right) \varphi_v \right|.$$  

Each absolute value term is bounded in $L_1$ by

$$\mathbb{E} \sup_{h \in H^K(1), \mu \in H^K(B), \nu \in H^K(1)} \left| (P_n - P) \partial^2 \ell_{\mu} (h - \nu) \varphi_v \right| \leq 2\mathbb{E} \sup_{h \in H^K(1), \mu \in H^K(B)} \left| (P_n - P) \partial^2 \ell_{\mu} h \varphi_v \right|.$$  

Define the class of functions $\mathcal{F} := \{ \partial^2 \ell_{\mu} h \varphi_k : \mu \in H^K(B), h \in H^K(1) \}$. Given that $\varphi_v$ is uniformly bounded, it can be deduced from Lemma 5 that $\ln N(\epsilon, \mathcal{F}, |\cdot|_p) \lesssim \left( B/\epsilon \right)^{2(2\lambda - 1)} + K \ln \left( \frac{B}{\epsilon} \right)$. Hence, to complete the proof of the lemma, we can follow the same exact steps as in the proof of Lemma 8. ■

**Lemma 13** Suppose Conditions 4, and $\mu_0 \in \text{int}(H^K(B))$. Then, for $\rho$ such that $\rho n^{(2\lambda - 1)/(4\lambda)} \to \infty$ in probability, and for $n \to 0$, the following hold

$$\sup_{h \in H^K(1)} \left| \left( \Pi_{\rho} - \Pi_{n,\rho} \right) h \right|_{H^K} = o_p(1),$$

and

$$\sup_{h \in H^K(1)} \left| \sqrt{n} \Psi_n \left( \mu_{n0} \right) \left( \Pi_{\rho} - \Pi_{n,\rho} \right) h \right| = o_p(1).$$  

Finally, if $w(x) = \int_y \partial^2 \ell_{\mu_0}(y, x) dP(y|x)$ is a known function, the above displays hold for $\rho$ such that $\rho n^{1/2} \to \infty$ in probability.
Proof. By the triangle inequality
\[
\sup_{h \in \mathcal{H}^K(1)} \| (\Pi_\rho - \Pi_{n,\rho}) h \|_{\mathcal{H}^K} \leq \sup_{h \in \mathcal{H}^K(1)} \left| \left( \tilde{\Pi}_{n,\rho} - \Pi_\rho \right) h \right|^2 + \sup_{h \in \mathcal{H}^K(1)} \left| \left( \Pi_{n,\rho} - \tilde{\Pi}_{n,\rho} \right) h \right|^2. \tag{32}
\]

The first statement in the lemma follows by showing that the r.h.s. of the above is \(o_p(1)\). This is the case by application of Lemmas 11 and 12.

By the established convergence in \(\| \cdot \|_{\mathcal{H}^K}\), for any \(h \in \mathcal{H}^K(1)\), \((\Pi_\rho - \Pi_{n,\rho}) h \in \mathcal{H}^K(\delta)\) with probability going to one for any \(\delta > 0\). Therefore, to prove (31), we can restrict attention to a bound for
\[
\lim_{\delta \to 0} \sup_{h \in \mathcal{H}^K} \sqrt{n} |h|^2 \lesssim \rho.
\]

From Lemma 8 and (25) in its proof, deduce that the above is bounded by
\[
\lim_{\delta \to 0} \sup_{h \in \mathcal{H}^K} \sqrt{n} \Psi_n(\mu_{n,0}) h = \lim_{\delta \to 0} \sup_{h \in \mathcal{H}^K} \sqrt{n} \partial h. \tag{33}
\]

The first term in the product is zero so that (31) holds.

Finally, to show the last statement in the lemma, note that it is Lemma 11 that puts an additional constraint on \(\rho\). However, saying that \(w(x)\) is known, effectively amounts to saying that we can replace \(\mu_{0,n}\) with \(\mu_0\) in the definition of \(\tilde{\Pi}_{n,\rho}\) in (29). This means that \(\tilde{\Pi}_{n,\rho} = \Pi_\rho\) so that the second term in (32) is exactly zero and we do not need Lemma 11. Therefore, \(\rho\) is only constrained for the application of Lemma 12.

We also need to bound the distance between \(\Pi_\rho\) and \(\Pi_0\), but this cannot be achieved in probability under the operator norm.

Lemma 14 Under Condition 4, we have that \(\sup_{h \in \mathcal{H}^K(1)} \tilde{P}(\Pi_\rho h - \Pi_0 h)^2 \leq \rho\).

Proof. At first show that
\[
\tilde{P}(\Pi_\rho h - \Pi_0 h)^2 \leq \tilde{P}(h - \Pi_\rho h)^2 - \tilde{P}(h - \Pi_0 h)^2. \tag{33}
\]

To see this, expand the r.h.s. of (33), add and subtract \(2\tilde{P}(\Pi_0 h)^2\), and verify that the r.h.s. of (33) is equal to
\[
-2\tilde{P}\Pi_\rho h + 2\tilde{P}\Pi_0 h (h - \Pi_0 h) + \tilde{P}\left[ (\Pi_\rho h)^2 + (\Pi_0 h)^2 \right].
\]
However, $\Pi_0 h$ is the projection of $h \in \mathcal{H}^K(1)$ onto the subspace $\mathcal{R}_0$. Hence, the middle term in the above display is zero. Then, add and subtract $2\tilde{P}\Pi_0 h \Pi_0 h$ and rearrange to deduce that the above display is equal to

$$2\tilde{P}\Pi_0 h (\Pi_0 h - h) + \tilde{P} (\Pi_0 h - \Pi_0 h)^2.$$  

Given that $\Pi_0 h \in \mathcal{R}_0$ and $(\Pi_0 h - h)$ is orthogonal to elements in $\mathcal{R}_0$ by definition of the projection $\Pi_0$, we have shown that (33) holds true. Following the proof of Corollary 5.18 in Steinwart and Christmann (2008),

$$\tilde{P} (h - \Pi_0 h)^2 - \tilde{P} (h - \Pi_0 h)^2 \leq [\tilde{P} (h - \Pi_0 h)^2 + \rho |\Pi_0 h|_{\mathcal{H}^K}^2] - \tilde{P} (h - \Pi_0 h)^2 = \rho |\Pi_0 h|_{\mathcal{H}^K}^2$$

because $|\Pi_0 h|_{\mathcal{H}^K}$ is positive and $\Pi_0 h$ is the minimizer of the penalized population loss function (see (29)). Now note that the r.h.s. of the above display is bounded by $\rho$ using Lemma 10 and (29). Hence the r.h.s. of (33) is bounded by $\rho$ uniformly in $h \in \mathcal{H}^K(1)$, and the lemma is proved. \[ \Box \]

**Lemma 15** Under Condition 4, we have that $\dot{\Psi}_{\mu_0} \sqrt{n} (\mu_{n0} - \mu_0) (\Pi_0 h - \Pi_0 h) = o_p (1)$ for any $\rho$ such that $n^{1/(2\lambda)} \rho \to 0$ in probability.

**Proof.** By definition,

$$\dot{\Psi}_{\mu_0} \sqrt{n} (\mu_{n0} - \mu_0) (\Pi_0 h - \Pi_0 h) = P \partial^2 \ell_{\mu_0} \sqrt{n} (\mu_{n0} - \mu_0) (\Pi_0 h - \Pi_0 h).$$

By Holder inequality, the absolute value of the display is bounded by

$$\sqrt{n} \left[ P \partial^2 \ell_{\mu_0} (\mu_{n0} - \mu_0)^2 \right]^{1/2} \left[ P \partial^2 \ell_{\mu_0} (\Pi_0 h - \Pi_0 h)^2 \right]^{1/2}. \tag{34}$$

By Condition 4, $|\partial^2 \ell_{\mu_0}|_\infty < \infty$, so that

$$\sqrt{n} \left[ P \partial^2 \ell_{\mu_0} (\mu_{n0} - \mu_0)^2 \right]^{1/2} \lesssim \sqrt{n} \left[ P (\mu_{n0} - \mu_0)^2 \right]^{1/2} = O_p \left( n^{1/(4\lambda)} \right)$$

using Point 2 in Theorem 1. Hence, by Lemma 14, deduce that (34) is bounded above by $O_p \left( n^{1/(4\lambda)} \rho^{1/2} \right) = o_p (1)$ for the given choice of $\rho$. \[ \Box \]

**Lemma 16** Suppose that $\mu_0 \in \text{int} \left( \mathcal{H}^K(B) \right)$. Under Condition 4, if $\rho \to 0$ in probability,
\[ \sqrt{n} \Psi_n (\mu_0) (h - \Pi_\rho h) \to G (h - \Pi_0 h), \; h \in \mathcal{H}^K (1), \]

weakly, where the r.h.s. is a mean zero Gaussian process with covariance function
\[ \Sigma (h, h') := \mathbb{E} G (h - \Pi_0 h) G (h' - \Pi_0 h') = P \vartheta^2 \mu_0 (h - \Pi_0 h) (h' - \Pi_0 h') \]

for any \( h, h' \in \mathcal{H}^K (1) \).

**Proof.** Any Gaussian process \( G (h) \) - not necessarily the one in the lemma - is continuous w.r.t. the pseudo norm \( d (h, h') = \sqrt{\mathbb{E} |G (h) - G (h')|^2} \) (Lemma 1.3.1 in Adler and Taylor, 2007). Hence, \( d (h, h') \to 0 \) implies that \( G (h) - G (h') \to 0 \) in probability. By Lemma 10, deduce that \( (h - \Pi_\rho h) \in \mathcal{H}^K (2) \). Hence, consider the Gaussian process \( G (h) \) in the lemma with \( h \in \mathcal{H}^K (2) \). By direct calculation,
\[ d^2 (h, h') = P \vartheta^2 \mu_0 h (h - h') + P \vartheta^2 \mu_0 h' (h' - h). \] (35)

Recall the notation \( \tilde{P} = P \vartheta^2 \mu_0 \). Multiply and divide the r.h.s. of the display by \( \sqrt{\vartheta^2 \mu_0} \) and use Holder inequality, to deduce that (35) is bounded above by
\[ \sqrt{\tilde{P} \left( \vartheta^4 \mu_0 \right)} \sqrt{P \vartheta^2 \mu_0 (h - h')^2} \lesssim \sqrt{\tilde{P} (h - h')^2} \]

using the fact that \( \vartheta^2 \mu_0 \) is bounded away from zero and \( \vartheta^4 \mu_0 \) is integrable. Hence, to check continuity of the Gaussian process at arbitrary \( h \to h' \), we only need to consider \( \tilde{P} (h - h')^2 \to 0 \). By Theorem 2 which also holds for any \( h \in \mathcal{H}^K (2) \), \( \sqrt{n} \Psi_n (\mu_0) h \) converges weakly to a Gaussian process \( G (h), h \in \mathcal{H}^K (2) \). Hence, \( \sqrt{n} \Psi_n (\mu_0) (h - \Pi_\rho h) \) converges weakly to \( G (h - \Pi_0 h) \) if for any \( h \in \mathcal{H}^K (1) \)
\[ \sup_{h \in \mathcal{H}^K (1)} \lim_{\rho \to 0} |G (h - \Pi_\rho h) - G (h - \Pi_0 h)| = 0 \]

in probability. The above display holds true if \( \sup_{h \in \mathcal{H}^K (1)} \tilde{P} (\Pi_0 h - \Pi_\rho h)^2 \to 0 \) in probability as \( \rho \to 0 \). This is the case by Lemma 14. ■

Furthermore one needs to be able to estimate the eigenvalues \( \omega_k \)’s in order to compute critical values.
**Lemma 17** Under the conditions of Theorem 3, if in Condition 2, \( P \Delta_1^P (\Delta_1^P + \Delta_2^P) < \infty \), the following holds in probability:

1. \( \sup_{h,h' \in \mathcal{H}^K(1)} | \Sigma_n (h, h') - \Sigma (h, h') | \to 0 \);
2. \( \sup_{k>0} | \omega_{nk} - \omega_k | \to 0 \), where \( \omega_{nk} \) and \( \omega_k \) are the \( k^{th} \) eigenvalues of the covariance functions with entries \( \Sigma_n (h, h') \) and \( \Sigma (h, h') \), \( h, h' \in \tilde{\mathcal{R}}_1 \); moreover, both the sample and population eigenvalues are summable and there is an \( \epsilon > 0 \) such that \( \sup_{k>0} \omega_{nk} k (\ln k)^{1+\epsilon} < \infty \) in probability.

**Proof.** To show Point 1, use the triangle inequality to deduce that

\[
| \Sigma_n (h, h') - \Sigma (h, h') | \leq | (P_n - P) (\partial_{\mu_0}^2 \partial_{\mu_0}^2) (h - \Pi_n h) (h' - \Pi_n h') | \\
+ | P (\partial_{\mu_0}^2 - \partial_{\mu_0}^2) (h - \Pi_n h) (h' - \Pi_n h') | \\
+ | P \partial_{\mu_0}^2 (\Pi_0 h - \Pi_n h) (h' - \Pi_n h') | \\
+ | P \partial_{\mu_0}^2 (h - \Pi_0 h) (\Pi_0 h' - \Pi_n h') |.
\]

(36)

It is sufficient to bound each term individually uniformly in \( h, h' \in \mathcal{H}^K (1) \).

To bound the first term in (36), note that, with probability going to one, \( (h - \Pi_n h) \in \mathcal{H}^K (1 + r) \) for some finite constant \( r \), by Lemma 13, as \( n \to \infty \). By this remark, to bound the first term in probability, it is enough to bound \( | (P_n - P) \partial_{\mu}^2 h h' | \) uniformly in \( \mu \in \mathcal{H}^K (B) \) and \( h, h' \in \mathcal{H}^K (1 + r) \). By Lemma 5 and the same maximal inequality used to bound (24), deduce that this term is \( O_p (n^{-1/2}) \).

To bound the second term in (36), note that \( P \partial_{\mu}^2 \) is Fréchet differentiable w.r.t. \( \mu \). To see this, one can use the same arguments as in the proof of Lemma 2.21 in Steinwart and Christmann (2008) as long as \( P \sup_{\mu \in \mathcal{H}^K (B)} | \partial_{\mu} \partial_{\mu}^2 \partial_{\mu} | < \infty \), which is the case by the assumptions in the lemma. Hence,

\[
| P (\partial_{\mu_0}^2 - \partial_{\mu_0}^2) (h - \Pi_n h) (h' - \Pi_n h') | \\
\leq 2 | P \partial_{\mu_0}^2 \partial_{\mu_0} \partial_{\mu_0}^2 (\mu_{n_0} - \mu_0) (h - \Pi_n h) (h' - \Pi_n h') | + o_p (1)
\]

using the fact that \( | \mu_{n_0} - \mu_0 |_\infty = o_p (1) \) by Theorem 1. By an application of Lemma 13, again, a bound in probability for the above is given by a bound for

\[
2 \sup_{h, h' \in \mathcal{H}^K (1+r)} | P \partial_{\mu_0}^2 \partial_{\mu_0} \partial_{\mu_0}^2 (\mu_{n_0} - \mu_0) h h' |.
\]

By Theorem 1 and \( P | \partial_{\mu_0}^2 \partial_{\mu_0} | \leq P \Delta_1 \Delta_2 < \infty \), implying that the above is \( o_p (1) \).
The third term in (36) is bounded by

\[
P \left| \partial^2_{\mu_0} (\Pi_0 h - \Pi_n h)(h' - \Pi_n h') \right| \\
\leq |\Pi_0 h - \Pi_n h|_{\infty} \times P \left| \partial^2_{\mu_0} (h' - \Pi_n h') \right|.
\]

By Lemma 13, the first term in the product on the r.h.s. is \( o_p(1) \). Using the triangle inequality, the second term in the product, is bounded by

\[
P \left| \partial^2_{\mu_0} (h' - \Pi_0 h') \right| + P \left| \partial^2_{\mu_0} (\Pi_0 - \Pi_n) h' \right| \\
\leq P \left| \partial^2_{\mu_0} (h' - \Pi_0 h') \right| + |(\Pi_0 - \Pi_n) h'|_{\infty} \times P \left| \partial^2_{\mu_0} \right| = O_p(1).
\]

The last term in (36) is bounded similarly. The uniform convergence of the covariance is proved because all the bounds converge to zero uniformly in \( h, h' \in \mathcal{H}(1) \).

It remains to show Point 2. This follows from the inequality

\[
\sup_{k > 0} |\omega_{nk} - \omega_k| \leq \frac{1}{R_1} \sum_{h \in \tilde{R}_1} |\Sigma_n (h, h) - \Sigma (h, h)|,
\]

which uses Lemma 4.2 in Bosq (2000) together with the fact that the operator norm of the covariance functions is bounded by the nuclear norm (e.g., Bosq, 2000). Clearly, the r.h.s. is bounded by \( \sup_{h \in \mathcal{H}(1)} |\Sigma_n (h, h) - \Sigma (h, h)| \) which converges to zero in probability. Finally, by definition of the eigenvalues and eigenfunctions, \( \Sigma (h, h) = \sum_{k=1}^{\infty} \omega_k \psi_k (h) \psi_k (h) \) so that

\[
\frac{1}{R_1} \sum_{h \in \tilde{R}_1} \Sigma (h, h) = \sum_{k=1}^{\infty} \omega_k \leq \sup_{h \in \mathcal{H}(1)} \Sigma (h, h) < \infty
\]

implying that the eigenvalues are summable. The sum of the sample eigenvalues is equal to

\[
\frac{1}{R_1} \sum_{h \in \tilde{R}_1} \Sigma_n (h, h) \leq \frac{1}{R_1} \sum_{h \in \tilde{R}_1} \Sigma (h, h) + \frac{1}{R_1} \sum_{h \in \tilde{R}_1} |\Sigma_n (h, h) - \Sigma (h, h)| \\
\leq \sup_{h \in \mathcal{H}(1)} \Sigma (h, h) + \sup_{h \in \mathcal{H}(1)} |\Sigma_n (h, h) - \Sigma (h, h)|.
\]

As shown above, the first term on the r.h.s. is finite and the second term converges.
to zero in probability. Hence, the sample eigenvalues are summable in probability. In particular, from these remarks deduce that for any \( c < \infty \) such that \( c \geq 1 + \sum_{k=1}^{\infty} \omega_k \),
\[
\Pr \left( \sum_{k=1}^{\infty} \omega_{nk} > c \right) = o(1).
\]
The event \( \{ \sup_{k \geq 1} \omega_{nk} (\ln k)^{1+\epsilon} = \infty \ \forall \epsilon > 0 \} \) is contained in the event \( \{ \sum_{k=1}^{\infty} \omega_{nk} > c \} \) whose probability goes to zero. This implies that there is an \( \epsilon > 0 \) such that \( \sup_{k \geq 1} \omega_{nk} (\ln k)^{1+\epsilon} < \infty \) in probability. 

To avoid repetition, the results in Section 3.3 are proved together. Mutatis mutandis, from (28), we have that
\[
\sqrt{n} \Psi_n (\mu_0) = \sqrt{n} \Psi_n (\mu_0) + \dot{\Psi}_{\mu_0} \sqrt{n} (\mu_0 - \mu_0) + o_p (1) . \tag{37}
\]

Trivially, any \( h \in \mathcal{H}^K (1) \) can be written as \( h = \Pi_p h + (h - \Pi_p h) \). By Lemma 10, replace \( h \in \mathcal{H}^K (1) \) with \( (h - \Pi_p h) \in \mathcal{H}^K (2) \) in Lemma 6. Then, \( \sqrt{n} (\Psi_n (\mu) - \Psi (\mu)) (h - \Pi_p h) \) for \( \mu \in \mathcal{H}^K (B) \), \( h \in \mathcal{H}^K (1) \) converges weakly to a Gaussian process with a.s. continuous sample paths. Therefore, (37) also applies to \( \Psi_n (\mu) \) as an element in the space of uniformly bounded functions on \( \mathcal{H}^K (2) \). Because of this, using Lemma 13,
\[
\sqrt{n} \Psi_n (\mu_0) (h - \Pi_p h) \\
= \sqrt{n} \Psi_n (\mu_0) (h - \Pi_p h) + \sqrt{n} \Psi_n (\mu_0) (\Pi_p - \Pi_n) h \\
= \sqrt{n} \Psi_n (\mu_0) (h - \Pi_p h + o_p (1) \\
= \sqrt{n} \Psi_n (\mu_0) (h - \Pi_p h) + \sqrt{n} \dot{\Psi}_{\mu_0} (\mu_0 - \mu_0) (h - \Pi_p h) + o_p (1) .
\]

By linearity, rewrite
\[
\dot{\Psi}_{\mu_0} \sqrt{n} (\mu_0 - \mu_0) (h - \Pi_p h) = \dot{\Psi}_{\mu_0} \sqrt{n} (\mu_0 - \mu_0) (h - \Pi_0 h) \\
+ \dot{\Psi}_{\mu_0} \sqrt{n} (\mu_0 - \mu_0) (\Pi_0 h - \Pi_p h) .
\]

The first term on the r.h.s. is \( P \partial^2 \ell_{\mu_0} \sqrt{n} (\mu_0 - \mu_0) (h - \Pi_0 h) \). This is zero because \( (\mu_0 - \mu_0) \) is in the linear span of elements in \( \mathcal{R}_0 \), and \( (h - \Pi_0) \) is orthogonal to any element in \( \mathcal{R}_0 \) (w.r.t. the linear functional \( P \partial^2 \ell_{\mu_0} (\cdot) \) by definition of the projection (see (13) with \( \rho = 0 \)). Lemma 34 shows that the absolute value of the second term on the r.h.s. of the display is \( o_p (1) \).

Deduce that the asymptotic distribution of \( \sqrt{n} \Psi_n (\mu_0) (h - \Pi_p h) \) is given by the one of \( \sqrt{n} \Psi_n (\mu_0) (h - \Pi_p h) \) for \( \rho \to 0 \). By Lemma 16, the latter converges weakly to a centered Gaussian process as in the statement of Theorem 3.
The test statistic \( \hat{S}_n \) is the square of \( \sqrt{n} \Psi_n (\mu_n h) (h - \Pi_n h) \) averaged over a finite number of \( h \)’s, i.e., the average of squared asymptotically centered Gaussian random variables. By the singular value decomposition, its distribution is given by \( S \). The distribution of the approximation to \( S \) when the sample eigenvalues are used is \( \hat{S} \). By the triangle inequality,

\[
|\hat{S} - S| \leq \sum_{k=1}^{\infty} |\omega_{nk} - \omega_k| N_k^2. \tag{38}
\]

The sum can be split into two parts, one for \( k \leq L \) plus one for \( k > L \) where here \( L \) is a positive integer. Hence, deduce that the above is bounded by

\[
L \sup_{k \leq L} |\omega_{nk} - \omega_k| N_k^2 + \sum_{k > L} (\omega_{nk} + \omega_k) N_k^2
\]

Using Lemma 17, the first term is \( o_p(1) \) for any fixed integer \( L \). By Lemma 17 again, there is an \( \epsilon > 0 \) such that the second term is bounded by

\[
\left( \sup_{k > 0} \omega_{nk} k (\ln k)^{1+\epsilon} \right) \sum_{k > L} \left( \frac{1}{k (\ln k)^{1+\epsilon}} \right) N_k^2 + \sum_{k > L} \omega_k N_k^2,
\]

where \( \sup_{k > 0} \omega_{nk} k (\ln k)^{1+\epsilon} = O_p(1) \). Given that

\[
\mathbb{E} \left[ \sum_{k > L} \left( \frac{1}{k (\ln k)^{1+\epsilon}} \right) N_k^2 + \sum_{k > L} \omega_k N_k^2 \right] \leq \sum_{k > L} \left( \frac{1}{k (\ln k)^{1+\epsilon} + \omega_k} \right) \rightarrow 0
\]

as \( L \rightarrow \infty \), deduce that letting \( L \rightarrow \infty \) slowly enough, (38) is \( o_p(1) \).

### 8.6 Proof of Theorem 7

The proof of Theorem 7 follows the proof of Theorems 1 and 2. The following lemmas can be used for this purpose.

Note that the estimator \( \mu_n \) satisfies \( P_n \partial \ell_{\mu_n} h = -2 \rho \langle \mu_n, h \rangle_{\mathcal{H}^K} \) for \( h \in \mathcal{H}^K \), where \( \partial \ell_{\mu}(z) = \text{sign} (y - \mu(x)) \). This is a consequence of the following results together with the fact that \( \mu_0 \in \text{int} (\mathcal{H}^K(B)) \) minimizes the expected loss \( P \ell_{\mu} \). Hence, first of all, find an expression for \( P \ell_{\mu} \).

**Lemma 18** Suppose that \( \mathbb{E} |Y| < \infty \) and \( \mu \) has bounded range. Then, for \( \ell_{\mu}(z) = \)
\[ |y - \mu(x)|, \]

\[ P\ell_\mu = \int \left[ \int_0^\infty \Pr(Y \geq s|x) \, ds + \int_{-\infty}^{\mu(x)} \Pr(Y < s|x) \, ds \right] dP(x), \]

where \( \Pr(Y \leq s|x) \) is the distribution of \( Y \) conditional on \( X = x \), and similarly for \( \Pr(Y \geq s|x) \).

**Proof.** Note that for any positive variable \( a \),

\[ a = \int_0^a ds = \int_0^\infty 1_{\{s \leq a\}} ds \quad \text{and} \quad a = \int_{-\infty}^0 1_{\{s > -a\}} ds. \]

Since \( |y - \mu(x)| = (y - \mu(x)) 1_{\{y - \mu(x) > 0\}} + (\mu(x) - y) 1_{\{\mu(x) - y \geq 0\}} \), by the aforementioned remark

\[ |y - \mu(x)| = \int_0^\infty 1_{\{s \leq y - \mu(x)\}} ds + \int_{-\infty}^0 1_{\{s > y - \mu(x)\}} ds. \]

Write \( P(y, x) = P(y|x) P(x) \) and take expectation of the above to find that

\[ \int \int |y - \mu(x)| dP(y|x) dP(x) = \int \int \left[ \int_0^\infty 1_{\{s \leq y - \mu(x)\}} + \int_{-\infty}^0 1_{\{s > y - \mu(x)\}} \right] ds dP(y|x) dP(x). \]

By the conditions of the lemma, the expectation is finite. Hence, apply Fubini’s Theorem to swap integration w.r.t. \( s \) and \( y \). Integrating w.r.t. \( y \), the the above display is equal to

\[ \int \left[ \int_0^\infty \Pr(Y \geq \mu(x) + s|x) \, ds + \int_{-\infty}^0 \Pr(Y < \mu(x) + s|x) \, ds \right] dP(x). \]

By change of variables, this is equal to the statement in the lemma. \( \blacksquare \)

The population loss function is Fréchet differentiable and strictly convex. This will also ensure uniqueness of \( \mu_0 \).

**Lemma 19** Under Condition 5, the first, second and third Fréchet derivatives of \( P\ell_\mu \) are

\[ \partial \ell_\mu P h = \int \left[ 2 \Pr(Y \leq \mu(x) | x) - 1 \right] h(x) \, dP(x) \]

\[ \partial^2 P\ell_\mu h^2 = 2 \int \text{pdf} (\mu(x) | x) h^2(x) \, dP(x), \]

\[ \partial^3 P\ell_\mu h^3 = 2 \int \text{pdf}' (\mu(x) | x) h^3(x) \, dP(x), \]
where pdf' (y|x) = dpdf (y|x) /dy. Moreover, P\partial^2 \ell_\mu h^2 \succeq Ph^2 and
\[ \sup_{\mu \in \mathcal{H}^K (B)} |\partial^3 P\ell_\mu h (\mu - \mu_0)| \lesssim P (\mu - \mu_0)^2. \]

**Proof.** Define
\[ I (t) := \int_{t}^{\infty} \Pr (Y \geq s|x) \, ds + \int_{-\infty}^{t} \Pr (Y < s|x) \, ds. \]
By Lemma 18, \(P\ell_\mu = PI (\mu) = \int I (\mu (x)) \, dP (x).\) For any sequence \(h_n \in \mathcal{H}^K (B)\) converging to 0 under the uniform norm,
\[ I' (\mu (x)) := \lim_{n \to \infty} \frac{I (\mu (x) + h_n (x)) - I (\mu (x))}{h_n (x)} = -\Pr (Y \geq \mu (x) |x) + \Pr (Y < \mu (x) |x) \]
by standard differentiation, as \(\mu (x)\) and \(h_n (x)\) are just real numbers for fixed \(x.\)
By Condition 5 the probability is continuous so that the above can be written as
\[ 2 \Pr (Y \leq \mu (x)|x) - 1. \]
It will be shown that the Fréchet derivative of \(P\ell_\mu\) is \(\int I' (\mu (x)) \, dP (x).\)
To see this, define
\[ U_n (x) := \left| \frac{I (\mu (x) + h_n (x)) - I (\mu (x))}{h_n (x)} - I' (\mu (x)) \right|, \]
if \(h_n \neq 0,\) otherwise, \(U_n (x) := 0.\) By construction, \(U_n (x)\) converges to zero pointwise.
By a similar method as in the proof of Lemma 2.21 in Steinwart and Christmann (2008),
it is sufficient to show that the following converges to zero,
\[ \lim_{|h_n|_\infty \to 0} \left| \frac{P\ell_{\mu + h_n} - P\ell_\mu - \int I' (\mu (x)) \, dP (x)}{|h_n|_\infty} \right| \leq \int U_n (x) \, dP (x). \]
The upper bound follows replacing \(|h_n|_\infty\) with \(|h_n (x)|\) because \(1/|h_n|_\infty \leq 1/h_n (x).\)
The above goes to zero by dominated convergence if we find a dominating function. To this end, the mean value theorem implies that for some \(t_n \in [0, 1],\)
\[ \left| \frac{I (\mu (x) + h_n (x)) - I (\mu (x))}{h_n (x)} \right| \leq 2 \Pr (Y \leq \mu (x) + t_n h_n (x) |x) - 1. \]
The sequence \(h_n (x) \in \mathcal{H}^K (B)\) is uniformly bounded by \(B.\) By monotonicity of probabilities, this implies that the above display is bounded by \(2 \Pr (Y \leq \mu (x) + B |x) - 1\)
uniformly for any \( h_n(x) \). This is also an upper bound for \( I'(\mu(x)) \). Hence, using the definition of \( U_n(x) \), it follows that

\[
|U_n(x)| \leq 4 \Pr(Y \leq \mu(x) + B|x),
\]

where the r.h.s. is integrable. This implies the existence of a dominating function and in consequence the first statement of the lemma. To show that also \( \partial P^\ell \mu \) is Fréchet differentiable one can use a similar argument as above. Then, the Fréchet derivative of \( I'(\mu(x)) \) can be shown to be \( I''(\mu(x)) = 2pdf(\mu(x)|x) \). The third derivative is found similarly as long as \( pdf(y|x) \) has bounded derivative. The final statements in the lemma follow by the condition on the conditional density as stated in Condition 5.

The last statement in Lemma 19 establishes the bound in (27). For the central limit theorem, an estimate of complexity tailored to the present case is needed.

**Lemma 20** Suppose that Condition 5 holds. Consider \( \ell_\mu(z) = |y - \mu(x)| \) (recall the notation \( z = (y,x) \)). For the set \( \mathcal{F} := \{\partial \ell_\mu h : \mu \in \mathcal{H}_K(B), h \in \mathcal{H}_K(1)\} \), the \( L_1 \) \( \epsilon \)-entropy with bracketing is

\[
\ln N_\mathcal{L}(\epsilon, \mathcal{F}, |\cdot|_1) \lesssim \epsilon^{-2/(2\lambda-1)} + K \ln \left( \frac{B}{\epsilon} \right),
\]

**Proof.** The first derivative of the absolute value of \( x \in \mathbb{R} \) is \( d|x|/dx = sign(x) = 2 \times 1_{\{x \geq 0\}} - 1 \). In consequence, it is sufficient to find brackets for sets of the type \( \{y - \mu(x) \geq 0\} \). For any measurable sets \( A \) and \( A' \), \( \mathbb{E}|1_A - 1_{A'}| = \Pr(A \Delta A') \); here \( \Delta \) is the symmetric difference. Hence, under Condition 5, for \( A = \{Y - \mu(X) \geq 0\} \), \( A' = \{Y - \mu'(X) \geq 0\} \),

\[
\Pr(A \Delta A') = \Pr(Y - \mu(X) \geq 0, Y - \mu'(X) < 0) + \Pr(Y - \mu(X) < 0, Y - \mu'(X) \geq 0) = \Pr(\mu(X) \leq Y < \mu'(X)) + \Pr(\mu'(X) \leq Y < \mu(X)).
\]

Using Condition 5, conditioning on \( X = x \) and differentiating the first term on the r.h.s., w.r.t. \( \mu' \),

\[
\Pr(\mu(x) \leq Y < \mu'(x)|x) \lesssim |\mu(x) - \mu'(x)| \leq |\mu - \mu'|_\infty.
\]

From the above two displays, deduce that \( \mathbb{E}|1_A - 1_{A'}| \lesssim |\mu - \mu'|_\infty \). Hence, the \( L_1 \) bracketing number of \( \{\partial \ell_\mu : \mu \in \mathcal{H}_K(B)\} \) is bounded above by the \( L_\infty \) bracketing...
number of $\mathcal{H}^K(B)$, which is given in Lemma 3. The proof is completed using the same remarks as at the end of the proof of Lemma 4.

The following provides the weak convergence of $P_n \partial \ell_\mu h, \mu \in \mathcal{H}^K(B), h \in \mathcal{H}^K(1)$.

**Lemma 21** Let $\mu \in \mathcal{H}^K(B)$. Under Condition 5

$$\sqrt{n}(P_n - P) \partial \ell_\mu h \to G(\partial \ell_\mu h)$$

weakly, where $G(\partial \ell_\mu h)$ is a mean zero Gaussian process indexed by

$$\partial \ell_\mu h \in \{\partial \ell_\mu h : \mu \in \mathcal{H}^K(B), h \in \mathcal{H}^K(1)\},$$

with a.s. continuous sample paths and covariance function

$$\mathbb{E}G(\partial \ell_\mu, h)G(\partial \ell_\mu', h') = \sum_{j \in \mathbb{Z}} P_{1,j}(\partial \ell_\mu h, \partial \ell_\mu h')$$

**Proof.** This just follows from an application of Theorem 8.4 in Rio (2000). That theorem applies to bounded classes of functions $\mathcal{F}$ and stationary sequences that have summable mixing coefficients. It requires that $\mathcal{F}$ satisfies

$$\int_0^1 \sqrt{\ln N[|\epsilon,\mathcal{F},|.|_1|]} / |\epsilon| d\epsilon < \infty.$$

When $\mathcal{F}$ is as in Lemma 20, this is the case when $\lambda > 3/2$, as stated in Condition 5.

Using the above results, Theorem 7 can be proved following step by step the proofs of Theorems 1 and 2.

### 8.7 Proof of Theorem 5

Only here, for typographical reasons, write $\ell(\mu)$ instead of $\ell_\mu$ and similarly for $\partial \ell(\mu)$. Let

$$h_m := \arg\min_{h \in \mathcal{L}^K(B)} P_n \partial \ell(F_{m-1}) h.$$

Note that by linearity, and the $l_1$ constraint imposed by $\mathcal{L}^K(B)$, the minimum is obtained by an additive function with $K - 1$ additive components equal to zero and a
non-zero one in $\mathcal{H}$ with norm $|.|_{\mathcal{H}}$ equal to $B$, i.e. $B f^{s(m)}$, where $f^{s(m)} \in \mathcal{H}$. Define,

$$D(F_{m-1}) := \min_{h \in \mathcal{L}^{K}(B)} P_n \partial \ell(F_{m-1}) (h - F_{m-1}),$$

so that for any $\mu \in \mathcal{L}^{K}(B),$

$$P_n \ell(\mu) - P_n \ell(F_{m-1}) \geq D(F_{m-1})$$

by convexity. For $m \geq 0$, define $\tilde{\tau}_m = 2/(m + 2)$ if $\tau_m$ is chosen by line search, or $\tilde{\tau}_m = \tau_m$ if $\tau_m = m^{-1}$. By convexity, again,

$$P_n \ell(F_m) = \inf_{\tau \in [0,1]} P_n \ell(F_{m-1} + \tau (h_m - F_{m-1})) \leq P_n \ell(F_{m-1}) + P_n \partial \ell(F_{m-1}) (h_m - F_{m-1}) \tilde{\tau}_m + \frac{Q}{2} \tilde{\tau}_m^2$$

where

$$Q := \sup_{h,F \in \mathcal{L}^{K}(B), \tau \in [0,1]} \frac{2}{\tau^2} [P_n \ell(F + \tau(h - F)) - P_n \ell(F) - \tau P_n \partial \ell(F)(h - F)].$$

The above two displays together with the definition of $D(F_{m-1}) = P_n \partial \ell(F_{m-1}) (h_m - F_{m-1})$ imply that for any $\mu \in \mathcal{L}^{K}(B),$

$$P_n \ell(F_m) \leq P_n \ell(F_{m-1}) + \tilde{\tau}_m D(F_{m-1}) + \frac{Q}{2} \tilde{\tau}_m^2$$

$$\leq P_n \ell(F_{m-1}) + \rho_m (P_n \ell(\mu) - P_n \ell(F_{m-1})) + \frac{Q}{2} \rho_m^2,$$

where the second inequality follows from (39). Subtracting $P_n \ell(\mu)$ on both sides and rearranging, we have the following recursion

$$P_n \ell(F_m) - P_n \ell(\mu) \leq (1 - \rho_m) (P_n \ell(F_{m-1}) - P_n \ell(\mu)) + \frac{Q}{2} \rho_m^2.$$

The result is proved by bounding the above recursion for the different choices of $\tilde{\tau}_m$. When, $\tilde{\tau}_m = 2/(m + 1)$, the proof of Theorem 1 in Jaggi (2013) bounds the recursion by $2Q/(m + 2)$. If $\rho_m = m^{-1}$, then, Lemma 2 in Sancetta (2013) bounds the recursion by $4Q \ln(1 + m)/m$ for any $m > 0$. It remains to bound $Q$. By Taylor expansion of $\ell(F + \tau(h - F))$ at $\tau = 0$,

$$\ell(F + \tau(h - F)) = \ell(F) + \partial \ell(F)(h - F) \tau + \frac{\partial^2 \ell(F + t(h - F))(h - F)^2 \tau^2}{2}$$

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for some $t \in [0, 1]$. It follows that

$$Q \leq \max_{t \in [0,1]} \sup_{h,F \in \mathcal{L}^K(B), \tau \in [0,1]} P_n \partial^2 \ell (F + t (h - F)) (h - F)^2$$

$$\leq 4B^2 \sup_{|t| < B} P_n \partial^2 \ell (t).$$

### 8.8 Proof of Lemma 1

Let $C_{HK} (s,t)$ be the (additive) covariance function of a mean zero Gaussian process $G$ on $\mathcal{X}^K$. Consider the Gaussian random variable $\sum_{j=1}^{\infty} \alpha_j G (s_j)$. By Theorem 2.1 in van der Vaart and van Zanten (2008), functions in $\mathcal{H}^K$ have the representation

$$E G (x) \sum_{j=1}^{\infty} \alpha_j G (s_j) = \sum_{k=1}^{K} \sum_{j=1}^{\infty} \alpha_j C (s_j^{(k)}, x^{(k)}) = \sum_{k=1}^{K} f^{(k)} (x^{(k)})$$

defining $f^{(k)} (x^{(k)}) = \sum_{j=1}^{\infty} \alpha_j C (s_j^{(k)}, x^{(k)})$. Clearly, $|.|_{HK}$ and $|.|_{LK}$ are norms on $\mathcal{H}^K$.

By the relation between the $l_1$ and $l_2$ norms (derived using Minkowski and the Cauchy-Schwarz inequality), $|\mu|_{HK} \leq |\mu|_{LK} \leq \sqrt{K} |\mu|_{HK}$ and this shows the inclusion.

Every subspace of a Hilbert space is uniformly convex, hence, the second part is proved. For the third part, by the RKHS property $f^{(k)} (x^{(k)}) = \langle f^{(k)}, C (\cdot, x^{(k)}) \rangle_{\mathcal{H}}$ for $\mu (x) = \sum_{k=1}^{K} f^{(k)} (x^{(k)})$,

$$|\mu (x)| = \left| \sum_{k=1}^{K} \langle f^{(k)}, C (\cdot, x^{(k)}) \rangle_{\mathcal{H}} \right|.$$  

When $\mu \in \mathcal{L}^K (B)$, by the Cauchy-Schwarz inequality and the RKHS property again, the display is bounded by

$$\sum_{k=1}^{K} |f^{(k)}|_{\mathcal{H}} |C (\cdot, x^{(k)})|_{\mathcal{H}} \leq \sum_{k=1}^{K} |f^{(k)}|_{\mathcal{H}} \sqrt{C (x^{(k)}, x^{(k)})} \leq cB,$$

using the definition of $\mathcal{L}^K (B)$ and the assumed bound on the kernel. The above two displays imply that $|\mu|_{\infty} \leq cB$. This shows the result for $p = \infty$. For any $p \in [1, \infty)$, use the trivial inequality $P |\mu|^p \leq |\mu|_{\infty}^p P (\mathcal{X}^K) = |\mu|_{\infty}^p$. When $\mu \in \mathcal{H}^K (B)$, by Cauchy-Schwarz inequality it is simple to deduce from the above two displays that $|\mu|_{\infty} \leq c\sqrt{K} B$. 

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9 Appendix 2: Additional Details and Numerical Results

9.1 Additional Details for Examples in Section 3.3.1

The function $H_V$ in Example 5 is

$$H_V(\cdot, \cdot) = \int_0^1 G_V(\cdot, u) G_V(\cdot, u) \, du \text{ with } G_V(r, u) := \max \left\{ \frac{(r - u)^{V-1}}{(V - 1)!}, 0 \right\},$$

where $r, u \in [0, 1]$ (Wahba, 1990, p.7-8).

To see that Example 6 fits in the framework of the paper, let $\mathcal{X}^{K+1} = \prod_{k=1}^{K+1} \mathcal{X}^{(k)}$ and $\mathcal{H}^{K+1} = \bigoplus_{k=1}^{K+1} \mathcal{H}^{(k)}$. Here, $\mathcal{H}^{(k)}$ is a RKHS on $\mathcal{X}^{(k)} = [0, 1]$ for $k \leq K$, and $\mathcal{H}^{(K+1)}$ is a RKHS on $\mathcal{X}^{(K+1)} = [0, 1]^K$. (Formally, this would also require us to define $X = (X^{(1)}, ..., X^{(K)}, X^{(K+1)})$ with $X^{(K+1)} = (X^{(1)}, ..., X^{(K)})$.) As the example shows, in practice, we can directly consider $\mathcal{R}_0$ and $\mathcal{R}_1$ rather than $\mathcal{H}^{K+1}$.

9.2 Selection of $B$ and Variable Screening

The parameter $B$ uniquely identifies the Lagrange multiplier $\rho_{B,n}$ in the penalized version of the optimization problem (9) (see Example 2). If the loss is non-negative, $|\mu_n|_{\mathcal{H}^K}^2 \leq \rho_{B,n}^{-1} P_n \ell_0$ (e.g., Steinwart and Christmann, 2008, Section 5.1). The exact same argument holds for $\mathcal{L}^K(B)$ in place of $\mathcal{H}^K(B)$. When the constraint $\mu \in \mathcal{L}^K(B)$ is considered, the solution via the greedy algorithm in Section 5 allows us to keep track of the iterations at which selected variables are included. Variables included at the early stage of the algorithm will be clearly included even when $B$ is increased. Hence, exploration for the purpose of feature selection (using the constraint $\mu \in \mathcal{L}^K(B)$) can be carried out using a large $B$ to reduce the computational burden.

Selection of $B$ is usually based on cross-validation or penalized estimation, where the penalty estimates the “degrees of freedom”.

9.3 Additional Representations for Practical Computations

At each iteration $m$, the Lagrange multiplier in (20) is derived as follows. Define
\[ \rho_m^{(k)} := \left[ \frac{1}{4} \sum_{i,j=1}^{n} \frac{\partial \ell_{F_{m-1}}(Z_i)}{n} \frac{\partial \ell_{F_{m-1}}(Z_j)}{n} C \left( X_i^{(k)}, X_j^{(k)} \right) \right]^{1/2}. \]

Let \( s(m) = \arg \max_{k \leq K} \rho_m^{(k)} \), and \( \rho_m = \max_{k \leq K} \rho_m^{(k)} \). In consequence, \( f^{s(m)} = -\frac{1}{2\rho_m} P_n \partial \ell_{F_{m-1}} \Phi^{s(m)}. \)

When the series representation (6) holds, with a finite number \( V \) of terms (either exactly, or approximately, by Condition 1), define

\[ a_v^{(k)} = \sum_{i=1}^{n} \frac{\partial \ell_{F_{m-1}}(Z_i) \lambda_{v} \varphi_v(X_i^{(k)})}{n} \]

and note that \( \rho_m^{(k)} = \frac{1}{2} \sqrt{\sum_{v=1}^{V} |a_v^{(k)}|^2} \), so that

\[ f^{s(m)}(x^{s(m)}) = -\sum_{v=1}^{V} \frac{a_v^{s(m)}}{\sqrt{\sum_{v=1}^{V} |a_v^{s(m)}|^2}} \lambda_v \varphi_v(x^{s(m)}) \]

and as before \( s(m) = \arg \max_{k \leq K} \rho_m^{(k)} \). This representation is suited for large sample \( n \) when ready access memory (RAM) is limited. For example, if \( n = O(10^6) \), which is not uncommon for high frequency applications, naive matrix methods to estimate a regression function under RKHS constraints requires to store a \( n \times n \) matrix of doubles, which is equivalent to about half a terabyte of RAM.

### 9.4 The Beta Mixing Condition

To avoid ambiguities, recall the definition of beta mixing. Suppose that \( (Z_i)_{i \in \mathbb{Z}} \) is a stationary sequence of random variables and let \( \sigma(Z_i : i \leq 0) \), \( \sigma(Z_i : i \geq k) \) be the sigma algebra generated by \( \{Z_i : i \leq 0\} \) and \( \{Z_i : i \geq k\} \), respectively, for integer \( k \). For any \( k \geq 1 \), the beta mixing coefficient \( \beta(k) \) for \( (Z_i)_{i \in \mathbb{Z}} \) is

\[ \beta(k) := \mathbb{E} \sup_{A \in \sigma(Z_i : i \geq k)} |\Pr(A|\sigma(W_i : i \leq 0)) - \Pr(A)| \]

(see Rio, 2000, section 1.6, for other equivalent definitions). In the context of Condition 3, set \( Z_i = (Y_i, X_i) \). Condition 3 is a convenient technical restriction and is satisfied by any model that can be written as a Markov chain with smooth conditional distribution.
(e.g., Doukhan, 1995, for a review; Basrak et al., 2002, for GARCH). Models with non-smooth innovations, unless taking values in a finite set are usually not covered (e.g., Rosenblatt, 1980, Andrews, 1984, Bradley, 1986, for a well known example).

**Example 9** Suppose that \( Y_i = \sum_{k=1}^{K} f^{(k)}(X_i^{(k)}) + \varepsilon_i \), where the sequence of \( \varepsilon_i \)'s and \( X_i \)'s are independent. By independence, deduce that the mixing coefficients of \( \{Y_i, X_i : i \in \mathbb{Z}\} \) are bounded by the sum of the mixing coefficients of the \( \varepsilon_i \)'s and \( X_i \)'s (e.g., Bradley, 2005, Theorem 5.1). Suppose that the \( \varepsilon_i \)'s and \( X_i \)'s are positive recurrent Markov chains with innovations with continuous conditional density function. Under additional mild regularity conditions, Condition 3 is satisfied with geometric mixing rates (e.g., Mokkadem, 1987, Doukhan, 1995, section 2.4.0.1). Examples include GARCH and others, as in the aforementioned references.

**Example 10** Suppose that \( Y_i \in \{-1, 1\} \). A classification model based on the regressors \( X_i \) can be generated via the random utility model

\[
Y_i^* = \mu(X_i) + \varepsilon_i
\]

where \( Y_i = \text{sign}(Y_i^*) \). The sigma algebra generated by \( \{Y_i : i \in \mathcal{A}\} \) for any subset \( \mathcal{A} \) of the integers is contained in the sigma algebra generated by \( \{Y_i^* : i \in \mathcal{A}\} \). Hence, for the errors \( \varepsilon_i \)'s and the \( X_i \)'s as in Example 9, the variables are beta mixing with geometric mixing rate.

### 9.5 Explicit Expressions Implied by the Compact Notation

The following examples should be nearly exhaustive in making quantities more readable:

\[
P_n \partial \ell_{\mu_0} (h - \Pi_n h) = \frac{1}{n} \sum_{i=1}^{n} \partial \ell_{\mu_0} (Y_i, X_i) (h(X_i) - \Pi_n h(X_i)) ,
\]

\[
P_n \partial^2 \ell_{\mu_0} (h - \nu)^2 = \frac{1}{n} \sum_{i=1}^{n} \partial^2 \ell_{\mu_0} (Y_i, X_i) (h(X_i) - \nu(X_i)) .
\]

\[
P_n \partial^2 \ell_{\mu_0} (h - \Pi_n h) (h' - \Pi_n h') = \frac{1}{n} \sum_{i=1}^{n} \partial^2 \ell_{\mu_0} (Y_i, X_i) (h(X_i) - \Pi_n h(X_i)) (h'(X_i) - \Pi_n h'(X_i)) .
\]
As mentioned in the main test of the paper, with abuse of notation in the last equality, we write \( \partial \ell_\mu (z) = \partial \ell_\mu ((y, x)) = \partial \ell_\mu (y, x) \) where \( z = (y, x) \). Moreover, with the same abuse of notation, we can write

\[
P \partial^2 \ell_{\mu_0} (\mu_n - \mu_0) h = \int_{\mathcal{X}^K} \int_{\mathcal{Y}} \partial^2 \ell_{\mu_0} (y, x) (\mu_n (x) - \mu_0 (x)) h (x) dP (y, x)
\]

where \( P \) is the law of \( Z = (Y, X) \).

9.6 Additional Details in the Calculation in Sections 6 and 7.1

When \( \mu \in \mathcal{L}^K (B) \) estimation was carried out using the algorithm in Section 5 with \( m = 500 \) number of iterations.

In the empirical application in Section 6, the value of \( B \) is chosen by cross-validation with \( \mu \) constrained in \( \mathcal{L}^K (B) \). In particular, we fix a \( B = 2^J \times \hat{\sigma}_Y \), \( j = 4, 5, ..., 10 \), where \( \hat{\sigma}_Y \) is the sample standard deviation of the dependent variable \( Y \). We resample \( n \) observations (with replacement), and compute the estimator on this bootstrap sample.

We then evaluate the estimator on the data that have not been resampled and compute its mean square error on the data not used for the estimation. We repeat the procedure ten times and average. We choose the \( B \) that produces the smallest estimate of mean square error. For stability selection, \( \mu \) is constrained in \( \mathcal{L}^K (B) \). For testing the hypotheses, the derived value of \( B \) via cross-validation in used with \( \mu \) constrained in \( \mathcal{H}^K (B) \) and estimated using ridge regression with (sample dependent) penalty \( \rho \) such that the constraint is satisfied (i.e. the dual of the constrained optimization). For the simulations, the value of \( B \) is chosen as \( 10 \times \hat{\sigma}_Y \). In the simulations, \( \mu \) is constrained in \( \mathcal{L}^K (B) \) rather than \( \mathcal{H}^K (B) \).

The set of test functions \( \tilde{R}_1 \) were restricted as in Section 2.4 with \( A \) equal to the identity. We also considered the case of \( A \) such that \( \Gamma_1 A \) has \((i, j)\) entry \( C_{\mathcal{R}_1} (X_i, X_j) / \sqrt{C_{\mathcal{R}_1} (X_j, X_j)} \).

The results were similar, though this last method had higher power against non-linear models; for conciseness, these results are not reported.

The sample projection was obtained by approximation using ordinary least square and pseudo inverse and setting \( \rho_n = 0 \). Given that the covariance operator is finite dimensional, though with a relatively large number of components, this is reasonable and speeds up calculations. The same set of test functions was also used in the calculation of the sample covariance of the limiting Gaussian process. The eigenvalues from the sample covariance were then used to simulate the limiting process (see Lemma 17),
Table 6: Simulation results when \( n = 100, K = 10 \) and the true model is Lin3.

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<th>( \rho )</th>
<th>( \sigma^2_{\mu/\varepsilon} )</th>
<th>Size</th>
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Table 7: Simulation results when \( n = 1000, K = 10 \) and the true model is Lin3.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \sigma^2_{\mu/\varepsilon} )</th>
<th>Size</th>
<th>Lin1</th>
<th>Lin2</th>
<th>Lin3</th>
<th>LinAll</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.10</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.09</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.05</td>
</tr>
<tr>
<td>0</td>
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<td>0.10</td>
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<td>1.00</td>
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</tr>
<tr>
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<td>1.00</td>
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<td>0.05</td>
</tr>
<tr>
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<td>0.05</td>
<td>1.00</td>
<td>1.00</td>
<td>0.32</td>
<td>0.45</td>
</tr>
</tbody>
</table>

from which the p-values were derived using \( 10^4 \) simulations.

### 9.7 Additional Numerical Results

The following tables report more simulation results. The column heading “No \( \Pi \)” means that no correction was used in estimating the test statistic and the covariance function, i.e. in Theorem 3, the terms \((h - \Pi_0h)\) are replaced with \(h\), which is the naive estimator in the presence of a nuisance parameter.
Table 8: Simulation results when $n = 100$, $K = 10$ and the true model is LinAll.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma_{\mu/\varepsilon}^2$</th>
<th>Size</th>
<th>Lin1</th>
<th>Lin2</th>
<th>Lin3</th>
<th>LinAll</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.10</td>
<td>1.00</td>
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<td>1.00</td>
</tr>
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<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0</td>
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<td>0.10</td>
<td>0.84</td>
<td>0.88</td>
<td>0.80</td>
<td>0.85</td>
</tr>
<tr>
<td>0.75</td>
<td>1</td>
<td>0.10</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.75</td>
<td>0.2</td>
<td>0.10</td>
<td>0.95</td>
<td>0.97</td>
<td>0.89</td>
<td>0.93</td>
</tr>
<tr>
<td>0.75</td>
<td>0.2</td>
<td>0.05</td>
<td>0.89</td>
<td>0.94</td>
<td>0.80</td>
<td>0.89</td>
</tr>
</tbody>
</table>

Table 9: Simulation results when $n = 1000$, $K = 10$ and the true model is LinAll.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma_{\mu/\varepsilon}^2$</th>
<th>Size</th>
<th>Lin1</th>
<th>Lin2</th>
<th>Lin3</th>
<th>LinAll</th>
</tr>
</thead>
<tbody>
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</tr>
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</tr>
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</tr>
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</tr>
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<td>1.00</td>
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</tr>
</tbody>
</table>

Table 10: Simulation results when $n = 100$, $K = 10$ and the true model is NonLin.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma_{\mu/\varepsilon}^2$</th>
<th>Size</th>
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<th>Lin2</th>
<th>Lin3</th>
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<th>LinPoly</th>
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<td>0.85</td>
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</table>
Table 11: Simulation results when $n = 1000$, $K = 10$ and the true model is NonLin.

<table>
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<tr>
<th>$\rho$</th>
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<th>Size</th>
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<th>Lin2</th>
<th>Lin3</th>
<th>LinAll</th>
<th>LinPoly</th>
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