Social Discounting and Long-Run Discounting

Tangren Feng[*] Shaowei Ke[†‡]

April 19, 2017

Abstract

The most critical issue in evaluating policies and projects that affect generations of individuals is the choice of social discount rate. This paper shows that there exist social discount rates such that the planner can simultaneously be (i) time-consistent; (ii) intergenerationally Pareto—i.e., if all individuals from all generations prefer one policy/project to another, the planner agrees; and (iii) strongly non-dictatorial—i.e., no individual from any generation is ignored. Moreover, to satisfy (i)–(iii), if the time horizon is long enough, it is generically sufficient and necessary for social discounting to be more patient than the most patient individual long-run discounting, in which an individual's long-run discount rate is his asymptotic average and relative discount rate.

1 Introduction

Most economic decisions are inherently dynamic, such as investment-saving decisions, intertemporal taxation, durable public good provision, etc. These decisions crucially depend on one parameter, the *social discount rate*, which encapsulates the trade-off between current

[*]Department of Economics, University of Michigan. Email: tangren@umich.edu.

[†]Department of Economics, University of Michigan. Email: shaoweik@umich.edu.

[‡]The authors thank Tilman Börgers, Christopher Chambers, Caicai Chen, Federico Echenique, Srihari Govindan, Marina Halac, John Leahy, Yusufcan Masatlioglu, Marciano Siniscalchi, and Joel Sobel for helpful comments.

benefit and future benefit from the society's point of view. Unfortunately, there is no consensus on which social discount rate should be used. This disagreement has led to a debate, for example, about the cost-benefit analysis of environmental projects that affect many, if not all, future generations, and the evaluation of those projects is sensitive to the choice of social discount rate. The famous Stern review uses a near-zero social discount rate, and suggests that we should take strong and immediate actions on the climate change (see Stern (2007)). Nordhaus (2007) argues that Stern's conclusion no longer holds if a market rate is used instead. Stern, on the other hand, thinks that using a high discount rate (such as the market rate) is ethically indefensible.

Clearly, the social discount rate that a policy maker chooses should depend on the discount rates that individuals use. In the social discounting literature, many economists have argued that social discounting should be more patient than individual discounting (see Pigou (1920), Caplin and Leahy (2004), and Farhi and Werning (2007)). The idea is that if social discounting takes into account how future generations will feel about their consumption, then because future generations will value future consumption more than the current generation values future consumption, social discounting will also value future consumption more than the current generation does. However, even with this insight, exactly how much more patient social discounting should be—and which individual's discounting social discounting should be more patient than—remains unanswered.

This paper attempts to provide a new perspective on these issues using a classic approach. To begin with, note that what is common among these dynamic economic decisions is that there is a benevolent planner who needs to make choices for generations of individuals, and, as in environmental projects and many other examples, payoff uncertainty is usually involved. In such a setting, first, economists often assume that the planner has a standard time-consistent preference; that is, the planner should have an exponential discounting (expected) utility function. This assumption is widely used in economics, and is normatively appealing because if the planner's choices change over time in an inconsistent way, she may decide to undertake some expensive project but quit halfway. Second, economists often assume that a benevolent planner respects individuals' preferences. In other words, some notion of the Pareto property should hold: If "12 all" individuals agree that one project is better than another, the planner should agree that the former is better.

Despite the fact that these two assumptions are fundamental to economics, economists have established that these two assumptions cannot be satisfied simultaneously (see Zuber (2011) and Jackson and Yariv (2015)). Even if every individual has an exponential discounting utility function, a time-consistent planner has to be a dictator to ensure that her preference satisfies some Pareto property. The negative result also raises a challenge to the previous studies that conclude that social discounting should be more patient than individual discounting. Those studies usually focus on the case with only one individual (the representative agent). In light of the negative result, however, with multiple heterogeneous individuals, perhaps the planner/dictator is more patient than the only individual she cares about?

To address these issues, this paper introduces a new Pareto property, and characterizes the range of social discount rates that are compatible with the new Pareto property. Our setting is close to Jackson and Yariv's (2014, 2015). The Pareto property that they use, which we call *current-generation Pareto*, is the key to the negative result. In Jackson and Yariv's model, there is only one generation of individuals. The current-generation Pareto property requires that whenever a consumption sequence $\mathbf{p}$ is preferred to another sequence $\mathbf{q}$ by every current-generation individual, then the planner prefers $\mathbf{p}$ to $\mathbf{q}$. In many problems that we are interested in, especially the environmental projects, multiple generations of individuals are involved. To determine the social discount rate, the planner should not only respect how the current generation discounts the future, but also care about the actual well-being of future generations; that is, how future generations will feel about their consumption and how they will discount the future. The Pareto property that we introduce, *intergenerational Pareto*, captures this. It requires that whenever a consumption sequence $\mathbf{p}$ is preferred to $\mathbf{q}$

by every individual from every generation, then the planner prefers $\mathbf{p}$ to $\mathbf{q}$.

Specifically, each generation-t individual i lives for one period, and has an arbitrary discount function[1] $\delta_i(\tau)$ to discount the $(t + \tau)^{\text{th}}$ period consumption.[2] The planner is time-consistent and intergenerationally Pareto. To contrast with Jackson and Yariv's (2015) result, we require the planner to be *strongly non-dictatorial* in the sense that she never ignores the preference of any individual from any generation. Under these assumptions, we show that the range of social discount rates depends on the following factors: (a) individual relative discounting, average discounting, and long-run discounting, (b) the time horizon, and (c) the linear dependency of individual instantaneous utility functions.

Our main results (Theorems 2–4) highlight the roles of (b) and (c). We first examine a benchmark case in which individuals share the same instantaneous utility function (so that we can exclusively focus on discounting), and the time horizon is finite. We show that there exist two cutoffs for the social discount factor.[3] One is the lowest (across individuals) maximal (across time) relative discount factor, $\min_i \max_\tau \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, and the other is the lowest (across individuals) asymptotic average discount factor, $\min_i \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$. If the social discount factor is above the first cutoff, we show that the planner must be intergenerationally Pareto and strongly non-dictatorial. Checking whether a planner's utility function is compatible with the Pareto property is generally difficult, but this result provides an easy way to do it. Conversely, if the social discount factor is below the second cutoff, we show that the planner must violate the intergenerationally Pareto property as long as the time horizon is long enough; that is, by definition, there exist two consumption sequences, such that every individual from every generation thinks that one is better than the other, but the planner disagrees. We provide examples to show that these two cutoffs are tight.

The two cutoffs merge into one cutoff when individuals exhibit *present bias*. The unique cutoff is equal to the *least patient* individual *long-run discount factor*, in which each indi-

[1] In contrast, individuals have exponential discounting functions in Jackson and Yariv (2014, 2015).

[2] Each individual altruistically cares about the future generations' consumption, as is the case when we think about environmental projects.

[3] A discount rate is equal to one minus the discount factor.

vidual i's long-run discount factor δ_i^* is defined to be the asymptotic relative discount factor and the asymptotic average discount factor.

Since the least patient individual's long-run discount factors could be quite low, the benchmark case does not say much about which social discount factor is reasonable. However, if we replace the finite-horizon assumption with an infinite-horizon assumption, the cutoff for the social discount factor jumps to the *most patient* individual long-run discount factor. Roughly speaking, in the first case, the planner can give arbitrarily small weights to individuals with high discount factors when the time horizon is finite. When the time horizon is infinite, fixing any weights, the highest discount factor $(\max_i \delta_i^*)^\tau$ eventually dominates as τ goes to infinity.

Most importantly, the range of social discount factors also crucially depends on individual instantaneous utility functions. Unlike in the benchmark case, generically, individual instantaneous utility functions are linearly independent in the functional space. We show that even if the time horizon is finite, if individual instantaneous utility functions are linearly independent, the cutoff for the social discount factor again jumps to the most patient individual long-run discount factor, *independent of the planner's instantaneous utility function.* Moreover, the cutoff for the social discount factor can change gradually, as the number of extreme points of the convex hull of individual instantaneous utility functions increases. If that number is 1, we are back to the case in which the cutoff is equal to the least patient individual long-run discount factor. As that number increases, the cutoff moves nonmonotonically to the most patient individual long-run discount factor.

Relating back to the debate on the cost-benefit analysis of environmental projects, our main result says that when the time horizon is infinite, or individual instantaneous utility functions are linearly independent, social discounting has to be more patient than the *most patient* individual long-run discounting. This means that using a social discount rate that is lower than the market rate, or even a near-zero social discount rate, is justifiable in our framework.

1.1 Related Literature

This paper is not the first to aggregate preferences of multiple generations of individuals. In those studies, however, there is usually only one individual (the representative agent) in each generation. Caplin and Leahy (2004) and Farhi and Werning (2007) both show that with one individual, social discounting should be more patient than the only individual's discounting. Our results show that having multiple heterogeneous individuals in each generation makes an important difference. Depending on the assumptions on time horizon and individual instantaneous utility functions, social discounting may be more patient than the least patient individual discounting, or the most patient individual discounting.

Many papers have analyzed the aggregation of heterogeneous individual dynamic preferences. Weitzman (2001) conducts a survey on economists' discount rates to motivate a gamma discounting model, in which social discounting is some weighted average of individual discounting. Gollier and Zeckhauser (2005) study a dynamic efficient allocation problem with heterogeneous individuals and show that even when individuals have constant discount rates, the representative agent has a decreasing discount rate. In a different setting, Zuber (2011) first establishes that a planner cannot be time-consistent and (current-generation) Pareto. Jackson and Yariv (2014, 2015) continue to show that a Pareto planner who aggregates exponential discounting individuals will either exhibit present bias or be a dictator. A key difference between these papers and ours is that they aggregate only one generation of individuals, whereas ours aggregates multiple generations. This distinction is important in economic decisions that have long-term impact, such as environmental policies.

All of the studies discussed above, except for Gollier and Zeckhauser (2005), assume that individuals have exponential discounting functions,. It is well known that individuals are often time-inconsistent (see Strotz (1955), Laibson (1997), and Frederick et al. (2002), among others). Hence, it is important to understand whether the results continue to hold when we allow individuals to have more general discount functions.

There are other approaches to study social discounting. Our paper emphasizes the relation between social discounting and individual discounting implied by the intergenerational Pareto property. Echenique and Chambers (2016) study three models on discount rates. In the first, they characterize when a sequence of utility is always preferred to another sequence, for any discount rate between zero and one. The second model is similar to Weitzman (2001) and Jackson and Yariv (2014, 2015): The aggregate discount function is a weighted average of a set of exponential discount functions. In the last model, in order to discount a sequence of utility, the aggregate preference selects from a set of discount rates the most pessimistic one. Zuber and Asheim (2012), Asheim and Zuber (2014), and Fleurbaey and Zuber (2015) study models in which social discounting is due to intergenerational inequality aversion. Those models do not use the information on how individuals discount the future. Jonsson and Voorneveld (2016) study a welfare criterion for multiple generations. Each generation has one individual. As in the previous papers related to inequality, only the individual's utility matters. How an individual discounts the future does not. In the limit of Josson and Voorneveld's criterion, different generations are treated equally, which is consistent with a classic argument by Ramsey (1928).

Our paper is also related to Mongin (1998). Mongin establishes that under a standard form of the Pareto property, as long as the individuals' subjective probabilities are linearly independent or their instantaneous utility functions are affinely independent, the planner has to be a dictator. Similar results can be found in Mongin (1995) and Chambers and Hayashi (2006). In our model, if we view each period as a state, and discount factors as subjective probabilities, then Mongin's result seems to apply to our case. However, our time-consistent and intergenerationally Pareto planner does not have to be a dictator. The technical reason why we can bypass Mongin's negative result is that, for example, our Theorem 1 assumes that all individuals share the same instantaneous utility function. In the proof of Theorem 3, we have two aggregation steps. In the first, we aggregate some individuals sharing the same instantaneous utility functions. In the second, we aggregate individuals with identical discount functions (subjective probabilities).

Lastly, related to the preference aggregation literature, our Lemma 1 identifies technical conditions that allow us to extend Harsanyi's (1955) theorem to the case with infinitely many individuals.

The paper proceeds as follows. In Sections 2 and 3, we describe the individuals' and the planner's preferences. We then introduce a variant of Jackson and Yariv (2015) and the intergenerationally Pareto property. Section 4 studies the benchmark case, in which we characterize the range of social discount factors that are compatible with the intergenerationally Pareto property. Our main results in Section 5 and 6 show how time horizon matters, and how individual instantaneous utility functions interact with social discounting. Section 7 concludes.

2 Preferences

There are $2 < T \leq \infty$ generations/periods. In each generation, $N < \infty$ individuals live for one period. With an abuse of notation, we use $N := \{1, \ldots, N\}$ and $T := \{1, \ldots, T\}$ to denote the set of individuals and the set of time periods, respectively. The generation-t individual i is the parent of the generation-$(t+1)$ individual i. In each period, there is a (public) consumption good, which possibly involves uncertainty.[4] The set of consumption goods is $\Delta(X)$, where $\Delta(X)$ is set of probability measures on a compact set $X \subset \mathbb{R}^m$. A typical consumption sequence is denoted by $\mathbf{p} = (p_1, \ldots, p_T) \in \Delta(X)^T$.

Each generation-t individual $i \in N$ has a preference $\succsim_{i,t}$ over the consumption sequences. As is the case when we think about environmental projects, each individual altruistically cares about his own consumption and the future generations' consumption. Following the seminal work by Strotz (1955), we assume throughout the paper that the generation-t indi-

[4] All results we derive in this paper apply to the case in which each individual has his own consumption. We only need to view the public consumption as an N-tuple of individual consumption, and let each individual care only about his own component.

vidual i has the following utility function,

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i(\tau - t)u_i(p_\tau), \tag{1}$$

where $\delta_i : \{0, \ldots, T-1\} \to \mathbb{R}_{++}$ with $\delta_i(0) = 1$ is called the *discount function*,[5] and the *instantaneous utility function* $u_i : \Delta(X) \to \mathbb{R}$ is a continuous expected utility function. We call (1) a *discounting utility* function. The well-known exponential, hyperbolic, and quasi-hyperbolic discounting utility functions are special cases. It is standard to assume that $U_{i,t}(\mathbf{p})$ does not depend on past consumption.[6] When a generation-t individual comes into existence, the past is sunk; that is, comparing $\mathbf{p}$ and $\mathbf{q}$ from his point of view is the same as comparing $(p_t, \ldots, p_T)$ and $(q_t, \ldots, q_T)$.

The generation-$(t+1)$ individual i inherits the generation-t individual i's preference in the sense that they share the same discount function and instantaneous utility function. This assumption is not essential to our results, but it simplifies the analysis. This assumption does not imply that a parent and his offspring have the same preference, because the generation-$(t+1)$ individuals' discount functions are shifted one period forward.

Another way to interpret this assumption is that even if the offspring do not inherit the parent's utility function, overall the set of preferences of the entire population stays the same across generations. Most importantly, with either interpretation, *this assumption allows the planner to forecast the preferences of future generations.*

In each period $t \in T$, the planner has a preference $\succsim_t$ over the consumption sequences. As in most dynamic models in economics, we want the planner to be time-consistent. It is well known that if the planner's utility function is a special case of $U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta(\tau - t)u(p_\tau)$, the planner is time-consistent if and only if the planner's discount function is exponential. Therefore, we assume throughout the paper (except in Lemma 1) that in each period t, the

[5]When $T = +\infty$, for (1) to make sense, we require δ_i to be an absolutely summable sequence.

[6]See Caplin and Leahy (2004) and Ray and Wang (2015) for models that allow for backward discounting for past consumption. In the Appendix, we show that our results continue to hold in a simple generalization of Jackson and Yariv (2015) that allows for backward discounting.

planner has a utility function of the following form:

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau), \tag{2}$$

where $\delta > 0$ is the *social discount factor* and u is the planner's instantaneous utility function.[7] Note that the above equation holds for every $t \in T$; that is, the time-consistency requirement implies that the social discount rate and the planner's instantaneous utility function are time-invariant.

Throughout the paper, we normalize the instantaneous utility functions by assuming that for some fixed consequences $x_*, x^* \in X$, $u_i(x_*) = u(x_*) = 0$ and $u_i(x^*) = u(x^*) = 1$ for all $i \in N$. When this holds, we say that the instantaneous utility functions are *normalized*.

3 Intergenerational Pareto

We want to assume that the planner's preference $(\succsim_t)_{t \in T}$ satisfies some Pareto property. In a dynamic setting, however, there are multiple ways to define the Pareto property. For instance, recently Jackson and Yariv (2015) establish that if a time-consistent planner follows their Pareto property, the planner must be a dictator.[8] To motivate our new Pareto property, it is useful to first understand their negative result. Below, we introduce a version of their main result.

3.1 Incompatibility Between Pareto and Time Consistency: A Variant of Jackson and Yariv (2015)

Below is a variant of the Pareto property used by Jackson and Yariv (2015) that fits into our setting.

[7] When $T = +\infty$, for (2) to make sense, we require $\delta < 1$.
[8] Zuber (2011) introduces a similar result in a different setting.

Definition 1 *The planner's preference* $(\succsim_t)_{t\in T}$ *is current-generation Pareto if for any consumption sequences* $\mathbf{p}, \mathbf{q} \in \Delta(X)^T$*, in each period* $t \in T$*,* $\mathbf{p} \succsim_{i,t} \mathbf{q}$ *for all* $i \in N$ *implies* $\mathbf{p} \succsim_t \mathbf{q}$*, and* $\mathbf{p} \succ_{i,t} \mathbf{q}$ *for all* $i \in N$ *implies* $\mathbf{p} \succ_t \mathbf{q}$.

This notion of the Pareto property says that in any period t, if all current-generation individuals agree that a consumption sequence $\mathbf{p}$ is preferred to another sequence $\mathbf{q}$, then the planner should agree that $\mathbf{p} \succsim_t \mathbf{q}$. The same applies when the preferences are all strict, which will rule out the uninteresting case in which the planner is always indifferent.

Consider a simple situation in which every generation-t individual i has an exponential discounting utility function. Jackson and Yariv show that if the planner is current-generation Pareto, the planner has to be dictatorial. Let us present below a variant of this finding that fits into our setting. We say that the generation-t individual i has an *exponential discounting utility* (EDU) function if $\delta_i(\tau) = \delta_i^\tau$ for some discount factor $\delta_i > 0$; that is,

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i^{\tau-t} u_i(p_\tau).$$

When $T = +\infty$, we require $\delta_i < 1$.

The result below demonstrates that time consistency and the current-generation Pareto property are incompatible.

Proposition 1 *Suppose each generation-t individual i has an EDU function with discount factor* δ_i *and instantaneous utility function* u_i*. For a generic set of discount factors* $\{\delta_i\}_{i\in N}$ *and a generic set of instantaneous utility functions* $\{u_i\}_{i\in N}$*, the planner is current-generation Pareto if and only if for each* $t \in T$*, there exists a unique* $i \in N$ *such that* $U_t = U_{i,t}$.

The result says that if we require the time-consistent planner to be current-generation Pareto, the planner's preference has to be identical to exactly one individual's preference. Jackson and Yariv's result is different from this proposition; they require instantaneous utility functions to be defined on a one-dimensional space and be twice continuously differentiable.

Our instantaneous utility functions are expected utility functions, which allows us to avoid their assumptions, and to prove the negative result in a much simpler way.

The intuition is quite simple. First of all, the planner is current-generation Pareto if and only if her discounting utility function is equal to a weighted sum of the individuals' EDU functions. This is an implication of Harsanyi (1955). For simplicity, suppose there are only two individuals with identical instantaneous utility functions $u_1 = u_2$. The planner attaches a weight ω to the first individual and $1-\omega$ to the second individual. Now, for the planner to not be a dictator, there must be some $\omega \in (0,1)$ and $\delta > 0$ such that

$$\omega\delta_1 + (1-\omega)\delta_2 = \delta,$$

and

$$\omega\delta_1^2 + (1-\omega)\delta_2^2 = \delta^2.$$

However, one cannot find such a δ, unless $\omega = 0$ or 1.

3.2 Intergenerational Pareto

The key feature of environmental projects and many other economic policies is that the decisions affect multiple generations. The current-generation Pareto property only takes into account the preferences of the current generation. The current generation does altruistically care about future consumption, and we do want the planner to respect how individuals discount the future; otherwise, social discounting could be, in principle, anything. However, how the current generation thinks about the future may well differ from how future generations will think. Since future generations will be affected by the planner's decision, the planner should take into account their actual well-being. Moreover, the planner should continue to respect how future generations will discount their future. The following Pareto property captures these ideas.

Definition 2 *The planner's preference $(\succsim_t)_{t\in T}$ is intergenerationally Pareto if for any con-*

sumption sequences $\mathbf{p}, \mathbf{q} \in \Delta(X)^T$*, in each period* $t \in T$*,* $\mathbf{p} \succsim_{i,s} \mathbf{q}$ *for all* $i \in N$ *and all* $s \geq t$ *implies* $\mathbf{p} \succsim_t \mathbf{q}$*, and* $\mathbf{p} \succ_{i,s} \mathbf{q}$ *for all* $i \in N$ *and all* $s \geq t$ *implies* $\mathbf{p} \succ_t \mathbf{q}$*.*

The intergenerationally Pareto property says that in any period t, if all current-generation and future-generation individuals agree that a consumption sequence $\mathbf{p}$ is preferred to another sequence $\mathbf{q}$, then the planner should agree that $\mathbf{p} \succsim_t \mathbf{q}$. For example, suppose all of the current-generation individuals are extremely selfish: They are willing to sacrifice the environment in order to increase their own consumption. If the planner is current-generation Pareto, the planner must agree with them, and let them destroy the environment. However, if the planner is intergenerationally Pareto, the planner is allowed to disagree with them, because what they prefer hurts future generations.

The intergenerationally Pareto property does not require the planner to consider past generations in any given period. If the planner is current-generation Pareto, she is also intergenerationally Pareto.

We are not the first to assume that the planner should take future generations into account (see Caplin and Leahy (2004) and Farhi and Werning (2007)). However, in the literature, this idea is often studied under the assumption that there is only one individual (the representative agent), and the literature does not allow the individuals to have general discount functions. As will be seen in the next section, having multiple individuals with heterogeneous preferences yields new insights into social discounting. Moreover, the findings do not rely on particular assumptions on the individual discount functions.

The intergenerationally Pareto property seems to allow the planner's decisions to be rather discretionary, because, for instance, the planner is allowed to disagree with all the current-generation individuals if she finds that some future-generation individual may benefit from this decision. The following lemma characterizes what the planner is allowed to do, and shows that in fact the planner's hands are somewhat tied. To illustrate this point more thoroughly, the lemma covers a more general case than necessary for our analysis. By analyzing the more general case, we emphasize that the following result has nothing to do with our assumptions that the planner is time-consisent, and that the instantaneous utility functions are time-invariant.

Lemma 1 *Suppose each generation-t individual i's utility function takes that following form:*

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i(\tau - t) u_i(p_\tau, \tau),$$

and the planner's utility function in period t takes the following form:

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta(\tau - t) u(p_\tau, \tau),$$

where δ_i and δ are discount functions, and $u_i(\cdot, \tau)$ and $u(\cdot, \tau)$ are normalized instantaneous utility functions.

1. *Suppose $T < +\infty$. The planner's preference $(\succsim_t)_{t \in T}$ is intergenerationally Pareto if and only if for each $t \in T$, there exists a set of nonnegative numbers $\{\omega_{i,t}(s)\}_{i \in N, s \geq t}$ such that $\sum_{i=1}^{N} \sum_{s=t}^{T} \omega_{i,t}(s) > 0$, and*

$$U_t = \sum_{i=1}^{N} \sum_{s=t}^{T} \omega_{i,t}(s) U_{i,s}; \tag{3}$$

2. *Suppose $T = +\infty$ and for each i, $\{u_i(\cdot, \tau)\}_{\tau=1}^{\infty}$ is Lipschitz equicontinuous.*[9] *The planner's preference $(\succsim_t)_{t \in T}$ is intergenerationally Pareto if and only if for each $t \geq 1$, there exists a set of nonnegative numbers $\{\omega_{i,t}(s)\}_{i \in N, s \geq t}$ such that $0 < \sum_{i=1}^{N} \sum_{s=t}^{\infty} \omega_{i,t}(s) < \infty$, and*

$$U_t = \sum_{i=1}^{N} \sum_{s=t}^{\infty} \omega_{i,t}(s) U_{i,s}. \tag{4}$$

The first part of the lemma essentially follows from Harsanyi (1955). The second part is a countably infinite version of the Harsanyi theorem, which, to the best of our knowledge,

[9]The formal definition can be found in the Appendix.

has not been established in the literature. The lemma says that if the planner is intergenerationally Pareto, in each period t, her utility function must be equal to a weighted sum of all the current-generation and future-generation individuals' utility functions. This lemma depends on the fact that $U_{i,t}$'s and U_t's are expected utility functions.

4 Social Discounting and Individual Long-Run Discounting: The Benchmark Case

There are two important questions to be addressed. First, which social discount factors are reasonable? In particular, we want to understand which social discount factors, under our assumptions, are compatible with the intergenerationally Pareto property. Notice that in light of Jackson and Yariv (2015) and our Proposition 1, it is possible that there is again no social discount factor that is compatible with the intergenerational Pareo property.

Second, recall that in the social discounting literature, it is often argued that the social discount factor should be higher (more patient) than the individual discount factor. We want to understand which individual's discount factor the social discount factor should be higher than.

To contrast with the negative results due to the current-generation Pareto property, we introduce a strong notion of non-dictatorship.

Definition 3 *We say that the planner is strongly non-dictatorial if for each $t \in T$,*

$$U_t(\mathbf{p}) = f_t\left(U_{1,t}(\mathbf{p}), \ldots, U_{1,T}(\mathbf{p}), U_{2,t}(\mathbf{p}), \ldots, U_{2,T}(\mathbf{p}), \ldots, U_{N,T}(\mathbf{p})\right)$$

for some strictly increasing function f_t.

Thus, we not only want to ensure that the planner is not a dictator, but also that every individual from every generation has a say. In light of Lemma 1, this means that the planner's utility function can be written as a weighted sum of individual utility functions in which the weights are strictly positive.

The intergenerationally Pareto property is weaker than the current-generation Pareto property. However, when combined with the strongly non-dictatorial assumption, this is not true. Under the intergenerationally Pareto property, the planner does have more weights to assign, but also a more complicated task to accomplish. An analogy of this is the following. In Proposition 1, if we increase the number of individuals N, the planner has more weights to assign. However, this does not make it easier for the planner to be time-consistent. If the planner is required to give strictly positive weights to the newly added individuals, that also brings in their new discount factors, which makes it more difficult for the planner to be time-consistent. In fact, the easiest way for the planner to be time-consistent is when there is only one individual and one weight to be trivially assigned.

Another obvious assumption that makes our problem more difficult than Jackson and Yariv's is that, in our model, individuals have general discount functions. Under this assumption, it does not even make sense to talk about individual discount factors; our results show how the social discount factor depends on general individual discount functions.

4.1 The Benchmark Case

We first examine the simplest case to illustrate how social discounting is related to individual discounting. First, to focus on discounting, we assume that all individual instantaneous utility functions are identical; that is, there is some continuous expected utility function $u : \Delta(X) \to \mathbb{R}$ such that each generation-t individual i's utility function is

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i(\tau - t)u(p_\tau). \tag{5}$$

We will study the case without this assumption in Section 6. When all individuals share the same instantaneous utility function, that is, $u_i = u_j$ for all $i, j \in N$, it is standard to show that the planner's instantaneous utility function has to satisfy $u = u_i$ in order to satisfy the Pareto properties.

Second, we assume that T is finite. We will characterize the case without this assumption in Section 5.

Although T is finite, we assume that the discount function δ_i is well defined over $\{0, 1, \dots\}$. This is true for most discount functions that have been used by economists, such as exponential discounting, hyperbolic discounting, and quasi-hyperbolic discounting. For instance, with quasi-hyperbolic discounting, $\delta_i(T+\tau) = \beta_i \delta_i^{T+\tau}$ for any $\tau > 0$.

For each individual discount function $\delta_i(\tau)$, we call $\sqrt[\tau]{\delta_i(\tau)}$ his *average discount function*,[10] and $\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ his *relative discount function*. The average discount function measures the equivalent exponential discount factor for τ-period-ahead consumption. The relative discount function captures the additional instantaneous discounting for consumption that is $\tau + 1$ periods ahead, relative to consumption that is τ periods ahead.

We make two weak assumptions on the individual discount functions. The first assumption says that the average discount function has a limit; that is,

$$\lim_{\tau \to \infty} \sqrt[\tau]{\delta_i(\tau)} \text{ exists.} \tag{6}$$

This assumption is even weaker than assuming that the relative discount function has a limit. The second assumption says that the relative discount function is bounded; that is,

$$\text{there is some } \alpha > 0 \text{ such that } \frac{\delta_i(\tau+1)}{\delta_i(\tau)} < \alpha \text{ for all } \tau \geq 0. \tag{7}$$

The following theorem characterizes the set of compatible social discount factors under these assumptions.

Theorem 1 *Suppose $T < +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function $\delta_i(\tau)$ such that (6) and*

[10] When $\tau = 0$, we set the average discount function's value to be 1.

(7) hold. Then,

1. *if* $\delta > \min_i \max_{\tau \in \{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$*, the planner is intergenerationally Pareto and strongly non-dictatorial;*

2. *For each* $\delta < \min_i \lim_{\tau \to \infty} \sqrt[\tau]{\delta_i(\tau)}$*, there exists some* $T^* > 0$ *such that if* $T \geq T^*$*, the planner is not intergenerationally Pareto.*

The theorem shows how social discounting depends on individual discounting when there are multiple individuals, and individuals have general discount functions. We can find two cutoffs for the social discount factor. If the social discount factor is above the *least patient* (lowest across individuals) individual maximal relative discount factor, the planner's preference must be intergenerationally Pareto and strongly non-dictatorial. If the social discount factor is below the *least patient* (lowest across individuals) individual asymptotic average discount factor, the planner's preference must have violated the intergenerationally Pareto property *as long as* T *is large enough.* Also note that the planner has a utility function in each period t. Cutoffs for the social discount factor apply uniformly in all periods.

In general, when we choose a social discount factor, it is not obvious whether the planner is Pareto or not. The first part of the theorem allows us to check whether a social discount factor is consistent with the intergenerationally Pareto property. Moreover, it shows that even if we allow individuals to have arbitrary discount functions, and require the planner to be time-consistent, the planner can still be intergenerationally Pareto without being a dictator. In fact, the planner can even be strongly non-dictatorial.

Conversely, the second part of the theorem says that if the social discount factor is too low, then there must be two consumption sequences such that all individuals from all generations prefer one over the other, but the planner disagrees. We do not want to use a social discount factor that allows this to happen.

The first part of this theorem can be proved in two steps. First, focus on one arbitrary individual and his offspring. We show that there are strictly positive weights such that the weighted sum of their utility functions is an EDU function whose discount factor is strictly higher than $\max_{\tau\in\{0,\dots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. Thus, without loss of generality, assume that every individual i has an EDU function with a discount factor that is strictly higher than $\max_{\tau\in\{0,\dots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. Next, fix any social discount factor δ that is higher than the first cutoff in the theorem. Let all individuals' weights be equal to some small number $\varepsilon > 0$, except for the least patient individual and his offspring. We show that we can find strictly positive weights for the least patient individual and his offspring such that the weighted sum of utility functions of all individuals from all generations is an EDU function with the social discount factor δ. Intuitively, we want to push down the weights for all individuals except for the least patient ones. The first part of the theorem shows that it is possible to push down the weights without getting into the impossibility theorems in the previous literature. The intuition for the second part of the theorem is that the weighted average of individual discount factors must be higher than the lowest individual discount factor. However, in the proof we need to ensure that this intuition holds after the instantaneous utility functions are well normalized.

Although this theorem does tell us which individual (the least patient individual) the planner should be more patient than, this is not extremely helpful in pinning down social discount factors, because the least patient individual's relative and average discount factors can be quite low. Thus, a lot of social discount rates can satisfy our requirements. However, as will be shown below, this is no longer the case once we relax some unrealistic assumptions in the benchmark case.

The two cutoffs in the theorem are "tight" bounds for the social discount factor in the following sense. For any social discount factor $\delta < \min_i \max_{\tau\in\{0,\dots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, there must exist some T and some individual discount functions $\delta_i(\tau)$'s such that the planner is not intergenerationally Pareto. Similarly, for any social discount factor $\delta > \min_i \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$, we can find some individual discount functions $\delta_i(\tau)$'s such that for all finite T, the planner is intergenerationally Pareto and strongly non-dictatorial.

To understand more concretely what the two cutoffs are, we examine two popular special cases. We also use them to illustrate why the cutoffs are tight.

4.2 Individual Quasi-Hyperbolic Discounting and Exponential Discounting

We say that the generation-t individual i has a *quasi-hyperbolic discounting utility* (QHDU) function if his discount function satisfies

$$\delta_i(\tau) = \begin{cases} 1, & \text{if } \tau = 0 \\ \beta_i \delta_i^{\tau}, & \text{if } \tau \in \{1, \ldots, T-1\} \end{cases}$$

for some $\beta_i \in (0, 1]$ and $\delta_i > 0$. The QHDU function often appears in the time inconsistency literature. Here, each generation only lives for one period, and there is no time-inconsistency issue for individuals. However, there will be an interesting connection between our result and the time-inconsistency literature.

It is immediate to see that if a generation-t individual i has a QHDU function, then

$$\max_{\tau \in \{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)} = \lim_{\tau \to \infty} \sqrt[\tau]{\delta_i(\tau)} = \delta_i.$$

Let $\bar{\delta} := \max_i \delta_i$, and $\underline{\delta} := \min_i \delta_i$. The following result is an application of Theorem 1.

Corollary 1 *Suppose $T < +\infty$, and each generation-t individual i has a QHDU function with an instantaneous utility function u, $\beta_i \in (0, 1)$, and $\delta_i > 0$ such that $\bar{\delta} > \underline{\delta}$. Then,*

1. *If $\delta > \underline{\delta}$, the planner is intergenerationally Pareto and strongly non-dictatorial;*
2. *For each $\delta < \underline{\delta}$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

In the quasi-hyperbolic discounting literature, economists sometimes ignore the β_i parameter, and use an EDU function with a discount factor δ_i as the individual i's welfare criterion. The parameter β_i is ignored because it is the cause of time inconsistency, and hence β_i should not enter the welfare criterion.

However, there is no foundation for this practice. Now, if we interpret the generation-$(t+1)$ individual i in our model as the future self of the generation-t individual i, Corollary 1 provides some foundation for the use of this welfare criterion. Assume that the individual i is the only individual and apply Corollary 1. We immediately know that the EDU function with a discount factor δ_i is a criterion that is consistent with the intergenerationally Pareto property; that is, if the individual i in every period t agrees that one consumption sequence is better than another, then the welfare criterion says that the utility of the former is greater than the latter. Moreover, δ_i is the smallest discount factor such that the corresponding EDU function is consistent with the intergenerationally Pareto property.[11]

This corollary also shows that the second cutoff of Theorem 1 is tight, because $\min_i \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)} = \underline{\delta}$, and Corollary 1 shows that for any social discount factor above $\underline{\delta}$, the planner must be intergenerationally Pareto and strongly non-dictatorial.

Next, we present a similar but stronger result in which all individuals have EDU functions.

Proposition 2 *Suppose $T < +\infty$, and each generation-t individual i has an EDU function with an instantaneous utility function u and $\delta_i > 0$ such that $\bar{\delta} > \underline{\delta}$. Then, the planner is intergenerationally Pareto and strongly non-dictatorial if and only if $\delta > \underline{\delta}$.*

This result is different from Theorem 1, because in Theorem 1, the second cutoff works under the assumption that T is sufficiently large. Proposition 2 does not require this.

Proposition 2 can be directly compared to Jackson and Yariv (2015). Assuming that individuals have EDU functions, Proposition 2 shows that under the intergenerationally Pareto property, the planner can simultaneouly be time-consistent and strongly non-dictatorial.

Why, in Jackson and Yariv (2015), does adding more current-generation individuals not help, but in our case adding future-generation individuals helps? The difference is that

[11] Note that in fact Corollary 1 does not directly say anything about the boundary case δ_i. However, in the Appendix, we show that when individuals have QHDU functions, if the social discount factor $\delta = \delta_i$, then the planner is intergenerationally Pareto and strongly non-dictatorial.

future generations will not care about past consumption as much as the past generations did. In our model, this is captured by the assumption that past consumption does not enter the future generations' utility functions. In the Appendix, we show—in a model that is directly comparable to Jackson and Yariv—that when individuals backward discount past consumption, our main results continue to hold.

This proposition also provides an example that shows that the first cutoff of Theorem 1 is tight. To see this, note that $\min_i \max_{\tau \in \{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)} = \underline{\delta}$. From Proposition 2, we know that any social discout factor below $\underline{\delta}$ implies that the planner is not intergenerationally Pareto.

4.3 Individual Long-Run Discount Factors

In the two special cases above, the two cutoffs from Theorem 1 merge into one. This is not a coincidence. Let us introduce the following assumption:

$$\text{the relative discount function } \frac{\delta_i(\tau+1)}{\delta_i(\tau)} \text{ is increasing in } \tau. \tag{8}$$

In the time inconsistency literature, when an individual has an increasing relative discount function, the individual has *present bias.*

Now, since $\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ is increasing and bounded, we know that $\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ exists, and is always above $\max_{\tau \in \{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ for any finite T. Moreover, it can be shown that if $\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ exists, the average discount factor has a limit, and the asymptotic relative discount factor and the asymptotic average discount factor coincide,

$$\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)} = \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}.$$

Therefore, assumptions (7) and (8) imply (6).

Definition 4 *When* $\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ *exists, we call* $\delta_i^* := \lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)} = \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$ *individual i's long-run discount factor.*

We immediately have the following corollary.

Corollary 2 *Suppose $T < +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function $\delta_i(\tau)$ such that (7) and (8) hold. Then,*

1. *For each $\delta > \min_i \delta_i^*$, the planner is intergenerationally Pareto and strongly non-dictatorial;*

2. *For each $\delta < \min_i \delta_i^*$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

From here on, to simplify the statement of our results, we focus on the case in which each individual i's long-run discount factor δ_i^* is well defined.

5 Social Discounting and the Time Horizon

In the benchmark case, Theorem 1 shows that if (i) the time horizon is finite, and (ii) all individuals share the same instantaneous utility functions, then the social discount factor only has to be higher than the lowest individual long-run discount factor. With a large number of individuals, the lowest individual long-run discount factor could possibly be quite low, in which case this result does not tell us a lot about which social discount factors are reasonable.

In many economic models (and perhaps in reality), the time horizon is infinite. The result below shows that if we believe that this is a better assumption, the set of social discount factors that are compatible with the intergenerationally Pareto property will become smaller.

Theorem 2 *Suppose $T = +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function $\delta_i(\tau)$ such that (7) and (8) hold. Then,*

1. *For each* $\delta > \max_i \delta_i^*$*, the planner is intergenerationally Pareto and strongly non-dictatorial;*

2. *For each* $\delta < \max_i \delta_i^*$*, the planner is not simultaneously intergenerationally Pareto and strongly non-dictatorial.*

This result says that if $T = +\infty$, the cutoff for the social discount factor jumps from $\min_i \delta_i^*$ to $\max_i \delta_i^*$; that is, social discounting has to be more patient than the *most patient* individual long-run discounting. If the social discount factor is not higher than the highest individual long-run discount factor, then by definition there are two consumption sequences such that all individuals from all generations prefer one to the other, but the planner disagrees.

If there is a large number of individuals with a wide range of long-run discount factors, this result may imply that the planner has to be very patient in order to be intergenerationally Pareto. If so, perhaps the extremely low social discount rate used by Stern (2007) can be justified. If one thinks that a market rate is higher than the lowest individual discount rate, this result does rule out the use of a market rate as the social discount rate.

The intuition for this discontinuity is the following. Fixing an arbitrarily large but finite T, the planner can always attach small enough weights to individuals with high δ_i^*. This way, the planner can keep her social discount factor small. However, fixing any set of strictly positive weights, as T increases to infinity, $(\max_i \delta_i^*)^T$ dominates all the other individuals' long-run discount factors. Therefore, the social discount factor has to exceed $\max_i \delta_i^*$ at some point. Because we want to have a set of strictly positive weights, in this theorem, the negative part (the second part) of the theorem says that the planner cannot be simultaneously intergenerationally Pareto and strongly non-dictatorial. In the previous theorem, the negative part of the theorem is stronger: It says that under certain condition, the planner cannot be intergenerationally Pareto.

6 Social Discounting and Individual Instantaneous Utility Functions

Although Theorem 2 provides a sharper result on which social discount factors are reasonable, it still relies on the assumption that all individuals share the same instantaneous utility function. This clearly can be an unreasonable assumption. In fact, generically, the instantaneous utility functions should not only be different, but also linearly independent.

Definition 5 *A set of continuous expected utility functions* $\{u_i\}_{i\in N}$ *is said to be linearly independent if there are no constants* $\{\alpha_i\}_{i\in N}$ *that are not all zero, and* $\sum_i \alpha_i u_i(p) = 0$ *for all* $p \in \Delta(X)$*. If a set of continuous expected utility functions is not linearly independent, it is linearly dependent.*

It turns out that whether T is finite or infinite, as long as individual instantaneous utility functions are linearly independent, the cutoff for the social discount factor also has to be equal to the highest individual long-run discount factor. Moreover, this result holds independently of the planner's instantaneous utility function.

For an arbitrary set of individual instantaneous utility functions, the case in which $T = +\infty$ follows from the same argument as Theorem 2. We omit this case. The theorem below focuses on the case in which $T < +\infty$ to contrast with Theorem 1.

Theorem 3 *Suppose* $T < +\infty$*, and each generation-t individual i's discounting utility function has an instantaneous utility function* u_i *and a discount function* $\delta_i(\tau)$ *such that (7) and (8) hold and* $\{u_i\}_{i\in N}$ *is linearly independent. Let the planner's instantaneous utility function* u *be an arbitrary strict convex combination of* $\{u_i\}_{i\in N}$*.*[12] *Then,*

1. *For each* $\delta > \max_i \delta_i^*$*, the planner is intergenerationally Pareto and strongly non-dictatorial;*

[12] By a strict convex combination of $\{u_i\}_{i\in N}$, we mean that u is in the interior of $\text{co}(\{u_i\}_{i\in N})$.

2. *For each $\delta < \max_i \delta_i^*$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

Notice that the planner's instantaneous utility function—in other words, her risk attitude—is independent of the cutoff for the social discount factor. This is somewhat surprising. Suppose there are two individuals, 1 and 2. Individual 2 is more patient. The above theorem says that if the social discount factor is close to individual 2's discount factor, it is not necessarily the case that the planner's risk attitude is also close to individual 2's risk attitude. We can have a planner whose risk attitude is close to individual 1's, but the social discount factor is close to individual 2's.

The theorem also shows that the cutoff for the social discount factor in Theorem 1 is not robust even if we insist that $T < +\infty$. To see this, when $u_i = u_j$ for all $i, j \in N$, the cutoff is $\min_i \delta_i^*$. If we introduce a small perturbation to each u_i, and generically the perturbations will lead to a set of linearly independent individual instantaneous utility functions, then the cutoff jumps discontinuously to $\max_i \delta_i^*$.

Again, this is a sharp result that says the social discount factor should be high enough to satisfy the basic intergenerationally Pareto property. This finding again supports the use of a low discount rate.

To prove the first part of this theorem, again, we can show that there are strictly positive weights for each individual i and his offspring such that the weighted sum of their utility functions is an EDU function with a new discount factor that is higher than that individual's maximal relative discount factor. In particular, let the new discount factor be $\delta >\max_i \delta_i^*$. Therefore, we have an EDU function with the same discount factor δ for each i. They only differ in the instantaneous utility function. We can aggregate these EDU functions easily. The second part of this theorem relies on the fact that each instantaneous utility function that belongs to the linear space spanned by $\{u_i\}_{i\in N}$ can be written as a *unique* weighted sum of $\{u_i\}_{i\in N}$. Therefore, the planner's instantaneous utility function u now imposes additional restrictions on individuals' weights. It turns out that under the new restrictions, the lowest social discount factor that is compatible with intergenerational Pareto is again $\max_i \delta_i^*$.

In Section 5, the cutoff for the social discount factor changes discontinuously when the time horizon becomes infinite. Here, the cutoff can change "gradually" from the case in which $u_i = u$ for all $i \in N$ to the case in which $\{u_i\}_{i\in N}$ is linearly independent. To see this, note that the number of extreme points of $\{u_i\}_{i\in N}$ is 1 when $u_i = u$ for all $i \in N$, and the number of extreme points of $\{u_i\}_{i\in N}$ is N when $\{u_i\}_{i\in N}$ is linearly independent. When the number of extreme points of $\{u_i\}_{i\in N}$ is $1 \leq m \leq N$, Theorems 1 and 3 can be applied to find the cutoff for the social discount factor.

To state this result, we first introduce some notations. A subset of individual instantaneous utility functions $\left\{u_{i_j}\right\}_{j=1}^m \subset \{u_i\}_{i\in N}$ is called the *extreme points* of $\{u_i\}_{i\in N}$ if

$$\mathrm{co}\left(\left\{u_{i_j}\right\}_{j=1}^m\right) = \mathrm{co}\left(\{u_i\}_{i\in N}\right).$$

Let $\mathbb{U}(\{u_i\}_{i\in N}) := \left\{\left\{u_{i_j}\right\}_{j=1}^m \subset \{u_i\}_{i\in N} : \left\{u_{i_j}\right\}_{j=1}^m \text{ are the extreme points of } \{u_i\}_{i\in N}\right\}$.

Theorem 4 *Suppose $T < +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u_i and a discount function $\delta_i(\tau)$ such that (7) and (8) hold. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $\{u_i\}_{i\in N}$. Then,*

1. *For each*

$$\delta > \min_{\left\{u_{i_j}\right\}_{j=1}^m \in \mathbb{U}(\{u_i\}_{i\in N})} \max_j \delta_{i_j}^*,$$

the planner is intergenerationally Pareto and strongly non-dictatorial;

2. *For each*

$$\delta < \min_{\left\{u_{i_j}\right\}_{j=1}^m \in \mathbb{U}(\{u_i\}_{i\in N})} \max_j \delta_{i_j}^*,$$

there exists some $T^* > 0$ and some such that if $T \geq T^*$, the planner is not intergenerationally Pareto.

For each set of extreme points $\{u_{i_j}\}_{j=1}^{m}$ of $\{u_i\}_{i\in N}$, we apply Theorem 3 to show that the cutoff it implies is $\max_j \delta^*_{i_j}$. Then, there can be multiple sets of extreme points with distinct discount functions. We use the extreme points that deliver the lowest $\max_j \delta^*_{i_j}$. This becomes the cutoff for the social discount factor.

7 Conclusion

The value of a policy or a public project that affects generations of individuals often crucially depends on which social discount rate is used for the evaluation. However, there is no consensus on which social discount rate is the right one to use. This paper considers a few important and widely used assumptions in economics, and characterizes the set of social discount rates that are compatible with those assumptions. The key assumptions are (i) individuals discount future consumption in a general and heterogeneous way, (ii) the planner is time-consistent, (iii) the planner takes into account every individual's preference from every generation strictly, and (iv) the planner is intergenerationally Pareto, which means that if all individuals from all generations agree that one consumption sequence is better than another, then the planner must agree.

We show that when the time horizon is infinite, or individual instantaneous utility functions are linearly independent (which is generically true), the social discount rate has to be lower than the *lowest* individual's long-run discount rate. Therefore, using a near-zero social discount rate is justifiable.

If the time horizon is finite and individual instantaneous utility functions are not linearly independent, then depending on the number of extreme points of the set of individual instantaneous utility functions, the social discount rate's upper bound ranges from the lowest individual long-run discount rate to the highest individual long-run discount rate.

References

Asheim, G. and S. Zuber (2014). Escaping the Repugnant Conclusion: Rank-Discounted Utilitarianism with Variable Population. *Theoretical Economics 9*(3), 629–650.

Caplin, A. and J. Leahy (2004). The Social Discount Rate. *Journal of Political Economy 112*(6), 1257–1268.

Chambers, C. and T. Hayashi (2006). Preference Aggregation under Uncertainty: Savage vs. Pareto. *Games and Economic Behavior 54*(2), 430–440.

Craven, B. and J. Koliha (1977). Generalizations of Farkas' Theorem. *SIAM Journal on Mathematical Analysis 8*(6), 983–997.

Echenique, F. and C. Chambers (2016). On Multiple Discount Rates. *mimeo*.

Farhi, E. and I. Werning (2007). Inequality and Social Discounting. *Journal of Political Economy 115*(3), 365–402.

Fleurbaey, M. and S. Zuber (2015). Discounting, Risk and Inequality: A General Approach. *Journal of Public Economics 128*, 34–49.

Frederick, S., G. Loewenstein, and T. O'donoghue (2002). Time Discounting and Time Preference: A Critical Review. *Journal of Economic Literature 40*(2), 351–401.

Gollier, C. and R. Zeckhauser (2005). Aggregation of Heterogeneous Time Preferences. *Journal of Political Economy 113*(4), 878–896.

Harsanyi, J. (1955). Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparison of Utility. *Journal of Political Economy 63*(4), 309–321.

Jackson, M. and L. Yariv (2015). Collective Dynamic Choice: The Necessity of Time Inconsistency. *American Economic Journal: Microeconomics 7*(4), 150–178.

Jonsson, A. and M. Voorneveld (2016). The Limit of Discounted Utilitarianism. *mimeo*.

Laibson, D. (1997). Golden Eggs and Hyperbolic Discounting. *Quarterly Journal of Economics 112*(2), 443–477.

Mongin, P. (1995). Consistent Bayesian Aggregation. *Journal of Economic Theory 66*(2), 313–351.

Mongin, P. (1998). The Paradox of the Bayesian Experts and State-Dependent Utility Theory. *Journal of Mathematical Economics 29*(3), 331–361.

Nordhaus, W. (2007). A Review of the Stern Review on the Economics of Climate Change. *Journal of Economic Literature 45*(3), 686–702.

Pigou, A. (1920). *The Economics of Welfare*. Macmillan.

Ramsey, F. (1928). A Mathematical Theory of Saving. *The Economic Journal 38*(152), 543–559.

Ray, D. and R. Wang (2015). Backward discounting. *mimeo*.

Stern, N. (2007). *The Economics of Climate Change: The Stern Review.* Cambridge University Press.

Strotz, R. (1955). Myopia and Inconsistency in Dynamic Utility Maximization. *Review of Economic Studies 23*(3), 165–180.

Weitzman, M. (2001). Gamma Discounting. *American Economic Review 91*(1), 260–271.

Zuber, S. (2011). Can Social Preferences Be Both Stationary and Paretian? *Annals of Economics and Statistics* (101/102), 347–360.

Zuber, S. and G. Asheim (2012). Justifying Social Discounting: The Rank-Discounted Utilitarian Approach. *Journal of Economic Theory 147*(4), 1572–1601.

A Appendix

A.1 Proof of Proposition 1

Proof. If part If for all $t \in T$, there exists a unique $i \in N$ such that $U_t = U_{i,t}$, then the planner takes no one but individual i into account in period t. The corresponding weights in period t are $\omega_i = 1$, and $\omega_j = 0$ for all $j \neq i$. Therefore, the planner's preference $(\succsim_t)_{t \in T}$ is current-generation Pareto.

Only if part Suppose the planner's preference $(\succsim_t)_{t \in T}$ is current-generation Pareto. Then, there exists a set of nonnegative weights $\{\omega_i\}_{i \in N}$, such that

$$\sum_{i=1}^{N} \omega_i \sum_{\tau=1}^{T} \delta_i^{\tau-1} u_i(p_\tau) = \sum_{\tau=1}^{T} \delta^{\tau-1} u(p_\tau),$$

i.e., for $\tau = 1, ..., T-1$,

$$\sum_{i=1}^{N} \omega_i \delta_i^{\tau-1} u_i(p_\tau) = \delta^{\tau-1} u(p_\tau).$$

Let $\tau = 1, 2$. We have

$$\begin{cases} \sum_{i=1}^{N} \omega_i u_i = u, \\ \sum_{i=1}^{N} \omega_i \delta_i u_i = \delta u \end{cases}.$$

Combining the above two equations,

$$\sum_{i=1}^{N} \omega_i \delta_i u_i = \sum_{i=1}^{N} \omega_i \delta u_i. \tag{9}$$

For a set of generic instantaneous utility functions $\{u_i\}_{i \in N}$, $\{u_i\}_{i \in N}$ is linearly independent. Then equation (9) is equivalent to

$$\omega_i \delta_i = \omega_i \delta$$

for all $i \in N$.

Moreover, for a set of generic discount factors $\{\delta_i\}_{i \in N}$, $\delta_i \neq \delta_j$ for all $i \neq j$. Hence, there cannot exist more than one individual whose weight is greater than zero. Otherwise, their discount factors have to be the same, which contradicts $\delta_i \neq \delta_j$ for all $i \neq j$. Therefore, there exists a unique $i \in N$ such that $\omega_i = 1$, and $\omega_j = 0$ for all $j \neq i$, which means that $U_t = U_{i,t}$ under our normalization assumption. ■

A.2 Proof of Lemma 1

Proof. Part 1 of the lemma is an application of the Harsanyi (1955) theorem, because the individuals' and the planner's utility functions are expected utility functions. We only need to prove part 2 of the lemma. The "if" part is straightforward to verify. The verification does not require Lipschitz equicontinuity. To show the "only if" part, we first define Lipschitz equicontinuity. Let us use $\|\cdot\|$ to denote the Prohorov metric on $\Delta(X)$. For each $i \in N$, we say that $\{u_i(\cdot,\tau)\}_{\tau=1}^{\infty}$ is Lipschitz equicontinuous if there exists some constant L_i such that for any $p, q \in \Delta(X)$,

$$|u_i(p,\tau) - u_i(q,\tau)| \leq L_i \cdot \|p - q\|.$$

Only if part Suppose the generation-t individual i's utility function is $U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{\infty} \delta_i(\tau - t)u_i(p_\tau, \tau)$, the planner's utility function is $U_t(\mathbf{p}) = \sum_{\tau=t}^{\infty} \delta(\tau - t)u(p_\tau, \tau)$, and intergenerational Pareto holds. For each $t > 0$ and each vector $\vec{\omega}_t = (\omega_{1,t}(t), \ldots, \omega_{N,t}(t), \omega_{1,t}(t+1), \ldots, \omega_{N,t}(t+1), \ldots)'$ such that $\sum_{i=1}^{N} \sum_{s=t}^{\infty} \omega_{i,t}(s)$ exists, consider the following mapping from $\vec{U}_{i,t} = \{U_{i,t}\}_{i \in N, s \geq t}$ to

$$\phi_{\vec{\omega}_t}(\{U_{i,t}\}_{i \in N, s \geq t}) = \vec{U}_{i,t} \cdot \vec{\omega}_t = \sum_{i=1}^{N} \sum_{s=t}^{\infty} \omega_{i,t}(s) U_{i,s}.$$

We want to show that the following equation

$$U_t = \phi_{\vec{\omega}_t}(\{U_{i,t}\}_{i \in N, s \geq t}) = \vec{U}_{i,t} \cdot \vec{\omega}_t \tag{10}$$

has a nonnegative solution; that is, there exists some $\vec{\omega}_t$ such that $\omega_{i,t}(s) \geq 0$ for all $i \in N$ and $s \geq t$ that solves (10). If we can find such a vector $\vec{\omega}_t$, it is obvious that $\sum_{i=1}^{N} \sum_{s=t}^{\infty} \omega_{i,t}(s) > 0$, because intergenerational Pareto requires U_t to be nontrivial.

We want to apply an infinite dimensional Farkas' lemma to show that we can find a nonnegative solution $\vec{\omega}_t$ to (10). First, since X is compact in R^m and $\{u_i(\cdot,\tau)\}_{\tau=1}^{\infty}$ is Lipschitz equicontinuous, $\{U_{i,s}\}_{s\geq t}$ must be uniformly bounded. To see this, let us use $||\Delta(X)||$ to denote the diameter of $\Delta(X)$. Since X is compact, $||\Delta(X)|| < \infty$. Then,

$$\begin{aligned} |U_{i,s}| &\leq \sum_{\tau=s}^{\infty} \delta_i(\tau - s) \max_{p\in\Delta(X)} |u_i(p,\tau)| \\ &\leq \sum_{\tau=0}^{\infty} \delta_i(\tau) \cdot L_i \cdot ||\Delta(X)||. \end{aligned}$$

Thus $\phi_{\vec{\omega}_t}$ is a well-defined linear mapping, because $\sum_{i=1}^{N}\sum_{s=t}^{\infty}\omega_{i,t}(s)$ exists and $\vec{U}_{i,t}$ is bounded. Similarly, it is straightforward to verify that, since $\{U_{i,s}\}_{s\geq t}$ is uniformly bounded, $\phi_{\vec{\omega}_t}$ must be a continuous mapping.

Thus, we can apply Theorem 2 of Craven and Koliha (1977) to show that (10) has a nonnegative solution if and only if for any finite signed Borel measure μ on X^{∞},

$$\int_{X^{\infty}} U_{i,s}\ d\mu \geq 0 \tag{11}$$

for all $i \in N$ and $s \geq t$ implies $\int_{X^{\infty}} U_t\ d\mu \geq 0$. To see this, first note that by the Hahn-Jordan decomposition theorem, μ can be uniquely decomposed into $\alpha\mu_+ - \beta\mu_-$ where $\alpha, \beta \geq 0$ and $\mu_+, \mu_- \in \Delta(X^{\infty})$. Thus, (11) becomes

$$\alpha \int_{X^{\infty}} U_{i,s}\ d\mu_+ \geq \beta \int_{X^{\infty}} U_{i,s}\ d\mu_-$$

for all $i \in N$ and $s \geq t$. Notice that $U_{i,s}$'s are time-additively separable. Therefore, probability measures μ_+ and μ_- over X^{∞} can be identified with sequences of probability measures $\mathbf{p} \in \Delta(X)^{\infty}$ and $\mathbf{q} \in \Delta(X)^{\infty}$, respectively, by taking the marginal distributions of μ_+ and

μ_-. Hence, (11) becomes

$$\alpha U_{i,s}(\mathbf{p}) \geq \beta U_{i,s}(\mathbf{q})$$

for all $i \in N$ and $s \geq t$.

Suppose $\alpha \geq \beta$. The other case can be proved in a similar way. Let us use $\mathbf{x}_*$ to denote the sequence $(x_*, x_*, \dots)$. Since instantaneous utility functions are all normalized, $U_{i,s}(\mathbf{x}_*) = 0$ for all $i \in N$ and $s \geq t$. Then, (11) becomes

$$U_{i,s}(\mathbf{p}) \geq \frac{\beta}{\alpha} U_{i,s}(\mathbf{q}) + \left(1 - \frac{\beta}{\alpha}\right) U_{i,s}(\mathbf{x}_*)$$

for all $i \in N$ and $s \geq t$. Thus, for every $i \in N$ and $s \geq t$, the generation-s individual i prefers $\mathbf{p}$ to $\frac{\beta}{\alpha}\mathbf{q} + \left(1 - \frac{\beta}{\alpha}\right)\mathbf{x}_*$, where $\frac{\beta}{\alpha}\mathbf{q} + \left(1 - \frac{\beta}{\alpha}\right)\mathbf{x}_* \in \Delta(X)^\infty$ is the period-by-period mixture between $\mathbf{q}$ and $\mathbf{x}_*$. By intergenerational Pareto, this means that

$$\begin{aligned} U_t(\mathbf{p}) &\geq \frac{\beta}{\alpha} U_t(\mathbf{q}) + \left(1 - \frac{\beta}{\alpha}\right) U_t(\mathbf{x}_*) \\ \alpha U_t(\mathbf{p}) &\geq \beta U_t(\mathbf{q}) \\ \int_{X^\infty} U_t \, d\mu &\geq 0. \end{aligned}$$

Then, from the infinite dimensional Farkas' lemma, we know that (10) has a nonnegative solution. ■

A.3 Proof of Proposition 2

Proof. The following lemma will be useful in proving Proposition 2.

Lemma 2 *Given a positive n-tuple $(\delta_1, \delta_2, ..., \delta_n)$ in which $\bar{\delta} > \underline{\delta}$, if $\delta > \underline{\delta}$, then there exists a set of numbers $\{\omega_{i,t}(s) > 0\}_{i \in N, s \geq t}^{t \in T}$ such that the following equation holds*

$$\sum_{i=i}^{N} \sum_{s=t}^{\tau} \omega_{i,t}(s) \delta_i^{\tau - t} = \delta^{\tau - t} \tag{12}$$

for $t = 1, 2, ..., T,\ \tau = t, t+1, ..., T.$

Proof. Without loss of generality, we assume that $\delta_1 = \underline{\delta}$.

First, we fix all the weights other than individual 1's: For $i \geq 2$, we set $\omega_{i,t}(s) = \epsilon_t(s) > 0$, for $t \geq 1$,$s \geq t$. The remaining part is finding $\{\omega_{1,t}(s)\}_{s \geq t}^{t \in T}$ such that

1. Equation (12) holds;

2. $\omega_{1,t}(s) > 0$, for $t \geq 1$, $s \geq t$.

Construct $\{\omega_{1,t}(s)\}_{s \geq t}^{t \in T}$ by the following recursive formula:

$$\begin{cases} \omega_{1,t}(t) = 1 - \sum_{i=2}^{n} \omega_{i,t}(t), \\ \omega_{i,t}(s+1) = \delta^{s+1-t} - \sum_{i=1}^{n} \omega_{i,t}(t)\delta_i^{s+1-t} - \cdots - \sum_{i=1}^{n} \omega_{i,t}(s)\delta_i - \sum_{i=2}^{n} \omega_{i,t}(s+1), s \geq t+1. \end{cases} \tag{13}$$

It can be verified that the construction of (13) ensures that equation (12) holds for $t = 1, 2, ..., T,\ \tau = t, t+1, ..., T$. The remaining part is to show that $\{\omega_{1,t}(s)\}_{s \geq t}^{t \in T}$ derived from (13) are strictly greater than zero, if $\{\epsilon_t(s)\}_{s \geq t}^{t \in T}$ are small enough. We prove it in two steps.

Step 1 Setting $\epsilon_t(s) = 0$, the recursive formula (13) becomes

$$\begin{cases} \omega_{1,t}(t) = 1, \\ \omega_{i,t}(s) = \delta^{s-t-1}(\delta - \delta_1), s \geq t+1. \end{cases}$$

for each $t \in T$. This can be proved by induction. Since $\delta > \delta_1$, we have $\omega_{1,t}(s) > 0$.

Step 2 Plugging $\epsilon_t(s)$ into formula (13), we have,

$$\begin{cases} \omega_{1,t}(t) = 1 - (n-1)\epsilon_t(t), \\ \omega_{1,t}(t+1) = \delta - \delta_1 - (n-1)\epsilon_t(t+1) - \big[\sum_{i=2}^{n}(\delta_i - \delta_1)\big]\epsilon_t(t), \\ \omega_{i,t}(t+2) = (\delta - \delta_1)\delta - (n-1)\epsilon_t(t+2) - \big[\sum_{i=2}^{n} \delta_i - \delta_1 + \delta_1^2\big]\epsilon_t(t+1) + \big[\sum_{i=2}^{n} \delta_i\delta_1 - \delta_1^2 - \delta_i^2\big]\epsilon_t(t) \\ \vdots \end{cases}$$

i.e., $\omega_{1,t}(s) = F_t^{(s)}\big(\epsilon_t(t), ..., \epsilon_t(s)|\delta, \delta_1, ..., \delta_n\big)$, where $F_t^{(s)}$ is an affine (and hence continuous) function of $(\epsilon_t(t), ..., \epsilon_t(s))$. By the continuity of $F_t^{(s)}$, $\{\omega_{1,t}(s)\}_{s\geq t}^{t\in T}$ are strictly greater than zero, if $\{\epsilon_t(s)\}_{s\geq t}^{t\in T}$ are small enough. ■

Now we are able to prove Proposition 2.

If Part Since the planner's instantaneous utility function u is identical to individual instantaneous utility function u, the if part follows from lemma 2 immediately.

Only If Part Suppose the planner's preference is intergenerationally Pareto and strongly non-dictatorial. For each $t \in T$, there exists a set of strictly positive numbers $\{\omega_{i,t}(s)\}_{i\in N, s\geq t}$ such that

$$\begin{aligned} U_t &= \sum_{s=t}^{T}\sum_{i=1}^{N} \omega_{i,t}(s) U_{i,s} \\ &= \sum_{s=t}^{T}\sum_{i=1}^{N} \omega_{i,t}(s) \sum_{\tau=s}^{T} \delta_i^{\tau-s} u(p_\tau) \\ &= \sum_{\tau=t}^{T}\sum_{s=t}^{\tau}\sum_{i=1}^{N} \omega_{i,t}(s) \delta_i^{\tau-s} u(p_\tau). \end{aligned}$$

Then for $\forall t$, $\forall \tau \geq t$, the following equality holds,

$$\sum_{s=t}^{\tau}\sum_{i=1}^{N} \omega_{i,t}(s) \delta_i^{\tau-s} u(p_\tau) = \delta^{\tau-t} u(p_\tau). \tag{14}$$

Let $\tau = t, t+1$ in (14). We have

$$\begin{cases} \sum_{i=1}^{N} \omega_{i,t}(t) u(p_t) = u(p_t), \\ \sum_{i=1}^{N} \omega_{i,t}(t) \delta_i u(p_{t+1}) + \sum_{i=1}^{N} \omega_{i,t}(t+1) u(p_{t+1}) = \delta u(p_{t+1}). \end{cases}$$

Combining the above two equations,

$$\sum_{i=1}^{N} \omega_{i,t}(t)\delta = \sum_{i=1}^{N} \omega_{i,t}(t)\delta_i + \sum_{i=1}^{N} \omega_{i,t}(t+1).$$

Rearranging the above equation, we have

$$\delta = \frac{\sum_{i=1}^{N} \omega_{i,t}(t)\delta_i + \sum_{i=1}^{N} \omega_{i,t}(t+1)}{\sum_{i=1}^{N} \omega_{i,t}(t)}$$

$$> \frac{\sum_{i=1}^{N} \omega_{i,t}(t)\delta_i}{\sum_{i=1}^{N} \omega_{i,t}(t)} > \frac{\sum_{i=1}^{N} \omega_{i,t}(t)\underline{\delta}}{\sum_{i=1}^{N} \omega_{i,t}(t)} = \underline{\delta}$$

■

A.4 An Alternative Version of Corollary 1 and Its Proof

Proposition 3 *Suppose $T < +\infty$, and each generation-t individual i has a QHDU function with an instantaneous utility function u, $\beta_i \in (0,1)$, and $\delta_i \in (0,1)$ such that $\bar{\delta} > \underline{\delta}$. Then,*

1. *If $\delta \geq \underline{\delta}$, the planner is intergenerationally Pareto and strongly non-dictatorial;*
2. *For each $\delta < \underline{\delta}$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

To prove this proposition, we consider the one-individual case first.

Lemma 3 *Fix some $i \in N$. Suppose $T < +\infty$, and individual i has a QHDU function with parameters $\beta_i \in (0,1)$, $\delta_i \in (0,1)$, and u. Then, there exists a cutoff $\delta(T)$ for each T such that the planner is intergenerationally Pareto and strongly non-dictatorial if and only if the social discount rate $\delta > \delta(T)$. In addition, $\{\delta(T)\}_{T=3}^{+\infty}$ is a strictly increasing sequence with a limit δ_i.*

Proof. The planner is intergenerationally Pareto and strongly non-dictatorial if and only if there exists a set of strictly positive weights $\{\omega_{i,t}(s)\}_{s \geq t}^{t \in T}$ such that the following equation holds:

$$\sum_{s=t}^{\tau} \omega_{i,t}(s)\beta_i^{\mathbb{1}(\tau>s)}\delta^{\tau-s}u(p_\tau) = \delta^{\tau-t}u(p_\tau). \tag{15}$$

for $t = 1, 2, ..., T$, $\tau = t, t+1, ..., T$.

Since there is only one individual, we can solve $\{\omega_{i,t}(s)\}_{s\geq t}^{t\in T}$ from (15),

$$\begin{cases} \omega_{i,t}(t) = 1, \\ \omega_{i,t}(t+m) = \delta^m - \dfrac{\beta_i}{1-\beta_i}\sum_{h=1}^{m}(1-\beta_i)^h\delta_i^h\delta^{m-h},\ 1 \leq m \leq T-t. \end{cases} \tag{16}$$

Note that $\omega_{i,t}(t) = 1 > 0$, the planner is intergenerationally Pareto and strongly non-dictatorial if and only if all $\{\omega_{i,t}(t+m)\}_{1\leq m\leq T-t}^{t\in T}$ are strictly positive.

We can rewrite the second equation of (16) as $\omega_{i,t}(t+m) = F_m(\delta|\beta_i, \delta_i)$, where F is a degree-m polynomial of a single indeterminate δ with parameters β_i, δ_i. Define the set of compatible social discount rates by

$$S(\beta_i, \delta_i, T) := \left\{\delta \in \mathbb{R}_+ \middle| F_m(\delta|\beta_i, \delta_i) > 0, \forall m \leq T-1\right\}$$

Therefore, the planner's preference (with the discount rate δ) is intergenerationally Pareto and strongly non-dictatorial if and only if $\delta \in S(\beta_i, \delta_i, T)$.

We want to show that $S(\beta_i, \delta_i, T)$ is an interval that (strictly) shrinks to $[\delta_i, +\infty)$ as T goes to infinity.

First, we prove that there exists a unique root/cutoff $x_m \in (0, \delta_i]$ for $F_m(\delta|\beta_i, \delta_i)$, such that $F_m(x_m|\beta_i, \delta_i) = 0$, $F_m(\delta|\beta_i, \delta_i) < 0$ for $\delta < x_m$, and $F_m(\delta|\beta_i, \delta_i) > 0$ for $\delta > x_m$.

We know that $F_m(0|\beta_i, \delta_i) = -(1-\beta_i)^{m-1}\delta_i^m < 0$, $F_m(\delta_i|\beta_i, \delta_i) = (1-\beta_i)^m\delta_i^m > 0$, and F_m is continuous. Therefore, the existence of x_m is guaranteed by the Bolzano's theorem.

Also note that the function $G_m(\delta|\beta_i, \delta_i) := \delta^{-m}F_m(\delta|\beta_i, \delta_i)$ has the same root as $F_m(\delta|\beta_i, \delta_i)$, and $G_m(\delta|\beta_i, \delta_i)$ is strictly increasing in δ.[13] By Rolle's theorem, there cannot be more than

[13] $\frac{dG_m(\delta)}{d\delta} = \frac{\beta_i}{1-\beta_i}\sum_{k=1}^{m} k\frac{(1-\beta_i\delta_i)^k}{\delta^{k+1}} > 0$.

one root; hence, the uniqueness is proved.

Second, we prove that the cutoff sequence $\{x_m\}$ is strictly increasing to δ_i as m goes to infinity.

The increasing part: Noting that $G_{m+1}(\delta|\beta_i, \delta_i) - G_m(\delta|\beta_i, \delta_i) = -\frac{\beta_i}{1-\beta_i}[\frac{(1-\beta_i)\delta_i}{\delta}]^{m+1} < 0$, we have $G_{m+1}(x_m|\beta_i, \delta_i) - G_m(x_m|\beta_i, \delta_i) < 0$. By the definition of $\{x_m\}$, $G_m(x_m|\beta_i, \delta_i) = G_{m+1}(x_{m+1}|\beta_i, \delta_i) = 0$. Therefore, $G_{m+1}(x_m|\beta_i, \delta_i) < G_m(x_m|\beta_i, \delta_i) = G_{m+1}(x_{m+1}|\beta_i, \delta_i)$. We also know that $G_m(\delta|\beta_i, \delta_i)$ is increasing. Hence, $x_{m+1} > x_m$. Now that we have proved $\{x_m\}$ is bounded and increasing, the convergence is guaranteed by the monotone convergence theorem.

The only remaining part is to prove that the limit of cutoff sequence is δ_i. Suppose $\lim_{m\to\infty} x_m = x$. Then $x_m < x$ for all $m > 1$. Since $G_m(\delta|\beta_i, \delta_i)$ is strictly increasing, we have

$$\begin{aligned}
&G_m(x_m|\beta_i, \delta_i) < G_m(x|\beta_i, \delta_i) \\
\Leftrightarrow\, &0 < 1 - \frac{\beta_i}{1-\beta_i}\sum_{h=1}^{m}(1-\beta_i)^h\delta_i^h x^{-h} \\
\Leftrightarrow\, &\frac{\beta_i}{1-\beta_i}\sum_{h=1}^{m}(1-\beta_i)^h\delta_i^h x^{-h} < 1 \\
\Leftrightarrow\, &\sum_{h=1}^{m}\left[\frac{(1-\beta_i)\delta_i}{x}\right]^h < \frac{1-\beta_i}{\beta_i}
\end{aligned} \tag{17}$$

for all $m > 1$.

Since $\frac{(1-\beta_i)\delta_i}{x} > 0$, $\frac{(1-\beta_i)\delta_i}{x} < 1$; otherwise, $\sum_{h=1}^{m}\left[\frac{(1-\beta_i)\delta_i}{x}\right]^h$ diverges as m increases. Now let $m \to +\infty$ at (17), we have

$$\begin{aligned}
&\sum_{h=1}^{+\infty}\left[\frac{(1-\beta_i)\delta_i}{x}\right]^h \leq \frac{1-\beta_i}{\beta_i} \\
\Leftrightarrow\, &\frac{(1-\beta_i)\delta_i}{x}\frac{1}{1-\frac{(1-\beta_i)\delta_i}{x}} \leq \frac{1-\beta_i}{\beta_i} \\
\Leftrightarrow\, &\delta_i \leq x.
\end{aligned} \tag{18}$$

In addition, since $x_m < \delta_i$ for all m, we have $x \leq \delta_i$. Therefore, $x = \delta_i$. ■

Lemma 3 states that for any finite T, the planner can aggregate each individual across generations so that the aggregated utility is an EDU function with a discount rate slightly below δ_i. Then, we can apply the "if" part of Theorem 2 for N exponential discounting individuals, and obtain a social discount rate $\delta \geq \underline{\delta}$.

A.5 Proof of Theorem 1

A.5.1 Part I

Proof. We prove part I in two steps. First, we prove a lemma for the one-individual case. Then we apply Proposition 2 to complete the proof.

Lemma 4 *Fix any $i \in N$. Suppose $T < +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function $\delta_i(\tau)$ such that (6) and (7) hold. For any $\delta > \hat{\delta}_i := \max_{\tau \in \{0,\dots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, the planner is intergenerationally Pareto and strongly non-dictatorial.*

Proof. We want to show that for any $\delta > \hat{\delta}_i$, there exists a set of strictly positive numbers $\{\omega_{i,t}(s)\}_{s \geq t}^{t \in T}$ such that

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau) = \sum_{s=t}^{T} \omega_{i,t}(s) U_{i,s}(\mathbf{p}).$$

Given any $\delta > \hat{\delta}_i$, we can construct $\{\omega_{i,t}(s)\}_{s \geq t}$ according to the following formula recursively

$$\begin{cases} \omega_{i,t}(t) = 1, \\ \omega_{i,t}(s) = \delta^{s-t-1}(\delta - \hat{\delta}_i) + \sum_{\tau=t}^{s-1} [\hat{\delta}_i \delta_i(s-1) - \delta_i(s)] \omega_{i,t}(\tau), s \geq t+1. \end{cases} \tag{19}$$

Note that by assuming $\delta > \hat{\delta}_i$, the first term of $\omega_{i,t}(s)$ is strictly greater than 0. According to the definition of $\hat{\delta}_i$, the second term of $\omega_{i,t}(s)$ is greater than 0. Hence, we can be sure that $\omega_{i,t}(s) > 0$ for all $s \geq t$.

Then

$$
\begin{aligned}
U_t(\mathbf{p}) &= \sum_{s=t}^{T} \omega_{i,t}(s) U_{i,s}(\mathbf{p}) \\
&= \sum_{s=t}^{T} \omega_{i,t}(s) \sum_{\tau=s}^{T} \delta_i(\tau - s) u(p_\tau) \\
&= \sum_{\tau=t}^{T} \left(\sum_{s=t}^{\tau} \delta_i(\tau - s) \omega_{i,t}(s) \right) u(p_\tau).
\end{aligned}
$$

We want to prove that $U_t(p) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau)$. Clearly for $\tau = t$, $\sum_{s=t}^{\tau} \delta_{i,s}(\tau)\omega_{i,t}(s) = \omega_{i,t}(t) = 1 = \delta^0$. Suppose for some $\tau \geq t$, we have proven that $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_{i,t}(s) = \delta^{\tau-t}$. We want to prove that for $\tau + 1$,

$$
\sum_{s=t}^{\tau+1} \delta_i(\tau + 1 - s)\omega_{i,t}(s) = \delta^{\tau-t+1}. \tag{20}
$$

To prove (20), we only need to notice that according to (19),

$$
\begin{aligned}
\sum_{s=t}^{\tau+1} \delta_i(\tau + 1 - s)\omega_{i,t}(s) &= \omega_{i,t}(\tau + 1) + \sum_{s=t}^{\tau} \delta_i(\tau + 1 - s)\omega_{i,t}(s) \\
&= \omega_{i,t}(\tau + 1) + \hat{\delta}_i \left[\delta^{s-t} + \sum_{s=t}^{\tau} \frac{\delta_i(\tau + 1 - s)}{\hat{\delta}_i} \omega_{i,t}(s) - \delta^{s-t} \right] \\
&= \omega_{i,t}(\tau + 1) + \hat{\delta}_i \left[\delta^{s-t} + \sum_{s=t}^{\tau} \frac{\delta_i(\tau + 1 - s)}{\hat{\delta}_i} \omega_{i,t}(s) - \sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_{i,s}^{(t)} \right] \\
&= \omega_{i,t}(\tau + 1) + \hat{\delta}_i \delta^{s-t} + \sum_{s=t}^{\tau} \left[\frac{\delta_i(\tau + 1 - s)}{\hat{\delta}_i} - \hat{\delta}_i \delta_i(\tau - s) \right] \omega_{i,t}(s) \\
&= \delta^{\tau-t+1}.
\end{aligned}
$$

By induction, we know that $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_{i,t}(s) = \delta^{\tau-t}$ for all $\tau \geq t$. Now we know that $U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u_i(p_\tau)$. ■

Lemma 4 states that the planner can aggregate one individual's utility functions across generations to derive an EDU function, with any discount factor greater than $\hat{\delta}_i$. Then by Proposition 2, she can aggregate N exponential discounting individuals one more time, and obtain an EDU function with any social discount factor greater than $\min_i \hat{\delta}_i$. ■

A.5.2 Part II

Proof. Define $\tilde{\delta}_i := \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$. Without loss of generality, we assume that $\tilde{\delta}_1$ is the unique minimum of $\{\tilde{\delta}_i\}_{i\in N}$. The proof can easily be extended to the case with multiple minimums.

Proof by contradiction: Suppose the planner is intergenerationally Pareto. For each $t \in T$, there exists a set of nonnegative numbers $\{\omega_{i,t}(s)\}_{i\in N, s\geq t}$ such that the following equality holds:

$$\sum_{s=t}^{\tau}\sum_{i=1}^{N} \omega_{i,t}(s)\delta_i(\tau-s)u(p_\tau) = \delta^{\tau-t}u(p_\tau) \tag{21}$$

for $\forall t$, $\forall \tau \geq t$.

The above equality (21) gives

$$\begin{aligned} \delta^{\tau-t} &= \frac{\sum\limits_{s=t}^{\tau}\sum\limits_{i=1}^{N} \omega_{i,t}(s)\delta_i(\tau-s)}{\sum\limits_{i=1}^{n}\omega_{i,t}(t)} \\ &\geq \frac{\sum\limits_{i=1}^{N}\omega_{i,t}(t)\delta_i(\tau-t)}{\sum\limits_{i=1}^{N}\omega_{i,t}(t)}. \end{aligned} \tag{22}$$

Since $\tilde{\delta}_1 = \min\limits_i \tilde{\delta}_i$, there exists $T_1 > 0$ such that for $\forall \tau > T_1$, $\delta_1(\tau - t) = \min_i \delta_i(\tau - t)$.

Hence, (22) becomes

$$\begin{aligned} \delta^{\tau-t} &\geq \frac{\sum\limits_{i=1}^{N}\omega_{i,t}(t)\delta_1(\tau-t)}{\sum\limits_{i=1}^{N}\omega_{i,t}(t)} \\ &= \delta_1(\tau-t). \end{aligned} \tag{23}$$

According to our assumptions, $\delta < \tilde{\delta}_1$. Then, there exists $T_2 > 0$ such that for $\forall \tau > T_2$,

$$\delta^{\tau-t} < \delta_1(\tau - t). \tag{24}$$

Let $T^* = \max\{T_1, T_2\}$. Then (23) and (24) contradict each other. ■

A.6 Proof of Theorem 2

Proof. Part I First, since we require $\delta < 1$ here, Lemma 4 extends to the infinite-horizon case naturally. Hence, for each individual i, for any $\delta \geq \max_i \delta_i$, there exists a set of strictly positive weights $\{\omega_{i,t}(s)\}_{s\geq t}^{t\in T}$ such that

$$\sum_{s=t}^{+\infty} \omega_{i,t}(t) U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{+\infty} \delta^{\tau-t} u(p_\tau)$$

for each $i \in N$.

Second, under these weights, the planner's utility function becomes

$$\begin{aligned} U_t(\mathbf{p}) &= \frac{1}{N} \sum_{i=1}^{N} \sum_{s=t}^{+\infty} \omega_{i,t}(s) U_{i,s}(\mathbf{p}) \\ &= \frac{1}{N} \sum_{i=1}^{N} \sum_{\tau=t}^{+\infty} \delta^{\tau-t} u(p_\tau) \\ &= \sum_{\tau=t}^{+\infty} \delta^{\tau-t} \frac{1}{N} \sum_{i=1}^{N} u(p_\tau) \\ &= \sum_{\tau=t}^{+\infty} \delta^{\tau-t} u(p_\tau). \end{aligned}$$

Part II Without loss of generality, we assume that δ_n^* is the unique maximal of $\{\delta_i^*\}_{i\in N}$. The proof can easily be extended to the case with multiple minimums.

Proof by contradiction: Suppose the planner is intergenerationally Pareto, then for each $t \in T$, there exists a set of nonnegative numbers $\{\omega_{i,t}(s)\}_{i\in N, s\geq t}$ such that equality (21) holds;

that is, for $\forall t$, $\forall \tau \geq t$,

$$\sum_{s=t}^{\tau}\sum_{i=1}^{N}\omega_{i,t}(s)\delta_i(\tau-s)u(p_\tau)=\delta^{\tau-t}u(p_\tau).$$

Then we have

$$\begin{aligned}\delta^{\tau-t} &= \frac{\sum\limits_{s=t}^{\tau}\sum\limits_{i=1}^{N}\omega_{i,t}(s)\delta_i(\tau-s)}{\sum\limits_{i=1}^{N}\omega_{i,t}(t)}\\ &\geq \frac{\sum\limits_{i=1}^{N}\omega_{i,t}(t)\delta_i(\tau-t)}{\sum\limits_{i=1}^{N}\omega_{i,t}(t)} \\ &\geq \frac{\omega_{N,t}(t)\delta_N(\tau-t)}{\sum\limits_{i=1}^{N}\omega_{i,t}(t)} := c\cdot\delta_N(\tau-t).\end{aligned} \tag{25}$$

where $c=\frac{\omega_{N,t}^{(t)}}{\sum_{i=1}^{N}\omega_{i,t}^{(t)}}$ is a constant. Since $\delta<\delta_N^*$, $\lim\limits_{s\to\infty}\dfrac{\delta}{\delta_N(s)/\delta_N(s-1)}<1$. Furthermore, $\lim\limits_{\tau\to\infty}\prod\limits_{s=t}^{\tau-t}\dfrac{\delta}{\delta_N(s)/\delta_N(s-1)}=0$. Therefore, for $\forall c>0$, there exists $T^*>0$ that is large enough such that for $\forall\tau>T^*$, $\prod\limits_{s=t}^{\tau-t}\dfrac{\delta}{\delta_N(s)/\delta_N(s-1)}<c$. This contradicts (25). ■

A.7 Proof of Theorem 3

Proof. Part I We prove this theorem in two steps. First, we consider the special case in which there is only one individual i to be aggregated across generations. Since the relative discounting is increasing, $\delta_i^*\geq\hat{\delta}_i:=\max\limits_{\tau\in\{0,\ldots,T-1\}}\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. Hence, by Lemma 4, for any $\delta>\delta_i^*$, we can find some positive $\{\omega_{i,t}(s)\}_{s\geq t}$ such that

$$\sum_{s=t}^{T}\omega_{i,t}(s)U_{i,s}(\mathbf{p})=\sum_{\tau=t}^{T}\delta^{\tau-t}u_i(p_\tau).$$

In other words, we can aggregate each individual's utility functions across generations into an EDU function with any discount factor δ greater than δ_i^*.

Next, for any social discount factor $\delta > \max_i \delta_i^*$, we can find $\{\omega_{i,t}(s)\}_{i\in N, s\geq t}$ such that

$$\sum_{s=t}^{T} \omega_{i,t}(s) U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u_i(p_\tau)$$

for each $i \in N$.

Consider any set of strictly positive numbers $\{\lambda_i\}_{i\in N}$ such that $\sum_{i\in N} \lambda_i = 1$. Together with the weights $\{\omega_{i,t}(s)\}_{i\in N, s\geq t}$ we have found above, the planner's utility function becomes

$$\begin{aligned}
U_t(\mathbf{p}) &= \sum_{i=1}^{N} \sum_{s=t}^{T} \lambda_i \omega_{i,t}(s) U_{i,s}(\mathbf{p}) \\
&= \sum_{i=1}^{N} \sum_{\tau=t}^{T} \delta^{\tau-t} \lambda_i u_i(p_\tau) \\
&= \sum_{\tau=t}^{T} \delta^{\tau-t} \sum_{i=1}^{N} \lambda_i u_i(p_\tau) \\
&= \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau),
\end{aligned}$$

in which $u(p_\tau) = \sum_{i\in N} \lambda_i u_i(p_\tau)$ can be any strictly convex combination of $\{u_i\}_{i\in N}$.

Part II Proof by contradiction: Suppose there exists an intergenerationally Pareto planner with the social discount factor $\delta < \max_i \delta_i^*$. By the intergenerationally Pareto property, for each $t \in T$, there exists a set of positive numbers $\{\omega_{i,t}(s)\}_{i\in N, s\geq t}$ such that the following equality holds for each $t \in T$

$$\sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau) = \sum_{\tau=t}^{T} \sum_{i=1}^{N} \sum_{s=t}^{\tau} \omega_{i,t}(s) \delta_i(\tau - s) u_i(p_\tau).$$

Let $t = 0$, $\tau = 0$, and $\tau = \tau$, respectively. We have

$$\begin{cases} \sum_{i=1}^{N} \omega_{i,0}(0) u_i(p_0) = u(p_0), \\ \sum_{s=0}^{\tau} \sum_{i=1}^{N} \omega_{i,0}(s) \delta_i(\tau - s) u_i(p_\tau) = \delta^\tau u(p_\tau). \end{cases} \tag{26}$$

Recall that u is a strictly convex combination of $\{u_i\}_{i \in N}$. The first equation of (26) shows that $\omega_{i,0}(0) > 0$ for each i. Combining the above two equations of (26), we have

$$\sum_{i=1}^{N} \omega_{i,0}(0) \delta^\tau \delta u_i = \sum_{i=1}^{N} \sum_{s=0}^{\tau} \omega_{i,0}(s) \delta_i(\tau - s) u_i.$$

Since $\{u_i\}_{i=1}^N$ is linearly independent, the above equation is equivalent to

$$\omega_{i,0}(0) \delta^\tau \delta u_i = \sum_{s=0}^{\tau} \omega_{i,0}(s) \delta_i(\tau - s) u_i.$$

for $\forall i \in N$.

Rearrange the above equation. We have

$$\begin{aligned} \delta^\tau &= \frac{\sum_{s=0}^{\tau} \omega_{i,0}(s) \delta_i(\tau - s)}{\omega_{i,0}(0)} \\ &= \frac{\omega_{i,0}(0) \delta_i(\tau) + \sum_{s=1}^{\tau} \omega_{i,0}(s) \delta_i(\tau - s)}{\omega_{i,0}(0)} \\ &\geq \frac{\omega_{i,0}(0) \delta_i(\tau)}{\omega_{i,0}(0)} = \delta_i(\tau) \end{aligned}$$

for $\forall i \in N$.

Hence, for any $i \in N$, and any $\tau \leq T$,

$$\delta \geq \sqrt[\tau]{\delta_i(\tau)}. \tag{27}$$

Without loss of generality, we assume δ_n^* is the maximal of $\{\delta_i^*\}_{i=1}^n$. Since $\delta < \delta_n^* = \lim_{\tau \to \infty} \sqrt[\tau]{\delta_n(\tau)}$, there exists T^* such that for any $\tau \geq T^*$, $\delta < \sqrt[\tau]{\delta_n(\tau)}$, which contradicts (27). ■

A.8 Proof of Theorem 4

Proof. Let $\check{\delta} := \min_{\{u_{i_j}\}_{j=1}^{m} \in \mathbb{U}(\{u_i\}_{i \in N})} \max_j \delta_{i_j}^*$. It can be shown that there exists a set $\mathcal{M}_1 = \{1, 2, ..., K\}$ such that

1. $\text{co}\left(\{u_i\}_{i \in \mathcal{M}_1}\right) = \text{co}\left(\{u_i\}_{i \in N}\right)$, $\{u_i\}_{i \in \mathcal{M}_1}$is a set of extreme points;

2. For $\forall i \neq j \in \mathcal{M}_1$, $u_i \neq u_j$, $\{u_i\}_{i \in \mathcal{M}_1}$is a *minimal* set of extreme points;

3. The largest long-run discount factor in $\{\delta_i^*\}_{i \in \mathcal{M}_1}$, $\max_{i \in \mathcal{M}_1} \delta_i^* = \check{\delta}$, is $\check{\delta}$.

The existence of a set M_1 that satisfies properties 1 and 3 is guaranteed by definition. Now suppose there exist $i \neq j \in \mathcal{M}_1$ such that $u_i = u_j$; we can delete one of them, while properties 1 and 3 still hold. Therefore, the existence of a set M_1 that satisfies all three properties is guaranteed. We define $\mathcal{M}_2 := \{K+1, K+2, \cdots, N\}$ as the complementary set of $\mathcal{M}_1$. Without loss of generality, we assume $\{K\} = \arg\max_{i \in \mathcal{M}_1} \delta_i^*$.

Part I We prove this part in three steps.

Step 1 For any $j \in \mathcal{M}_2$, there exists a set of non-negative weights $\left\{\alpha_i^{(j)}\right\}_{i \in \mathcal{M}_1}$ such that $\sum_{i \in \mathcal{M}_1} \alpha_i^{(j)} = 1$ and $u_j = \sum_{i \in \mathcal{M}_1} \alpha_i^{(j)} u_i$. Let us consider an aggregation problem for the following two individuals with the same instantaneous utility function:

$$\left\{(u_j, \delta_j(\tau)), \left(\sum_{i \in \mathcal{M}_1} \alpha_i^{(j)} u_i, \sum_{i \in \mathcal{M}_1} \alpha_i^{(j)} \delta_i(\tau)\right)\right\}.$$

We know that

$$\lim_{\tau \to +\infty} \left[\sum_{i \in \mathcal{M}_1} \alpha_i^{(j)} \delta_i(\tau)\right]^{1/\tau} = \max_{\{i : \alpha_i^{(j)} \neq 0\}} \delta_i^* \leq \check{\delta}.$$

Then, we can apply Theorem 1 to show that, for any $\delta > \check{\delta} \geq \min\left\{\delta_j^*, \lim_{\tau \to +\infty} \left[\sum_{i \in \mathcal{M}_1} \alpha_i^{(j)} \delta_i(\tau)\right]^{1/\tau}\right\}$, there exists a set of strictly positive numbers $\{\omega_{j,t}(s)\}_{s \geq t}$ and non-negative numbers $\left\{\omega_{i,t}^{(j)}(s)\right\}_{i \in \mathcal{M}_1, s \geq t}$ such that

$$\sum_{s=t}^{T} \left[\omega_{j,t}(s) U_{j,s}(\mathbf{p}) + \sum_{i \in \mathcal{M}_1} \omega_{i,t}^{(j)}(s) \alpha_i^{(j)} U_{i,s}(\mathbf{p})\right] = \sum_{\tau=t}^{T} \delta^{\tau-t} u_j(p_\tau).$$

Step 2 For any $i \in \mathcal{M}_1$ and any $\delta > \check{\delta}$, by Lemma 4, there exists a set of strictly positive numbers $\{\omega_{i,t}(s)\}_{s \geq t}$ such that

$$\sum_{s=t}^{T} \omega_{i,t}(s) U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u_i(p_\tau).$$

Step 3 Consider any set of strictly positive numbers $\{\lambda_i\}_{i \in N}$ such that $\sum_{i \in N} \lambda_i = 1$. We can construct a new set of weights $\{\bar{\omega}_{i,t}(s)\}_{i \in N, s \geq t}$ such that

$$\begin{cases} \bar{\omega}_{i,t}(s) = \lambda_i \omega_{i,t}(s) + \sum\limits_{j \in \mathcal{M}_2} \lambda_j \alpha_i^{(j)} \omega_{i,t}^{(j)}(s), & i \in \mathcal{M}_1, \\ \bar{\omega}_{j,t}(s) = \lambda_j \omega_{j,t}(s), & j \in \mathcal{M}_2. \end{cases}$$

By construction, every $\bar{\omega}_{i,t}(s)$ is strictly positive. Moreover,

$$\begin{aligned}
&\sum_{i \in N} \sum_{s=t}^{T} \bar{\omega}_{i,t}(s) U_{i,s}(\mathbf{p}) \\
=& \sum_{i \in \mathcal{M}_1} \sum_{s=t}^{T} \bar{\omega}_{i,t}(s) U_{i,s}(\mathbf{p}) + \sum_{j \in \mathcal{M}_2} \sum_{s=t}^{T} \bar{\omega}_{j,t}(s) U_{j,s}(\mathbf{p}) \\
=& \sum_{i \in \mathcal{M}_1} \sum_{s=t}^{T} \big[\lambda_i \omega_{i,t}(s) + \sum_{j \in \mathcal{M}_2} \lambda_j \alpha_i^{(j)} \omega_{i,t}^{(j)}(s)\big] U_{i,s}(\mathbf{p}) + \sum_{j \in \mathcal{M}_2} \sum_{s=t}^{T} \lambda_j \omega_{j,t}(s) U_{j,s}(\mathbf{p}) \\
=& \sum_{i \in \mathcal{M}_1} \sum_{s=t}^{T} \lambda_i \omega_{i,t}(s) U_{i,s}(\mathbf{p}) + \sum_{j \in \mathcal{M}_2} \sum_{s=t}^{T} \lambda_j \big[\omega_{j,t}(s) U_{j,s}(\mathbf{p}) + \sum_{i \in \mathcal{M}_1} \alpha_i^{(j)} \omega_{i,t}^{(j)}(s) U_{i,s}(\mathbf{p})\big] \\
=& \sum_{i \in \mathcal{M}_1} \sum_{\tau=t}^{T} \delta^{\tau-t} \lambda_i u_i(p_\tau) + \sum_{j \in \mathcal{M}_2} \sum_{\tau=t}^{T} \delta^{\tau-t} \lambda_j u_j(p_\tau) \\
=& \sum_{\tau=t}^{T} \delta^{\tau-t} \sum_{i \in N} \lambda_i u_i(p_\tau) \\
=& \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau).
\end{aligned}$$

Part II Proof by contradiction: Suppose there exists an intergenerationally Pareto planner with the discount factor $\delta < \check{\delta}$. Due to the intergenerationally Pareto property, for each

$t \in T$, there exists a set of positive numbers $\{\omega_{i,t}(s)\}_{i\in N, s\geq t}$ such that the following equality holds for each $t \in T$:

$$\sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau) = \sum_{\tau=t}^{T} \sum_{i=1}^{N} \sum_{s=t}^{\tau} \omega_{i,t}(s)\delta_i(\tau - s)u_i(p_\tau).$$

Let $t = 1$. Consider the first period and the τ^{th} period consumption, respectively,

$$\begin{cases} \sum_{i=1}^{N} \omega_{i,1}(1)u_i(p_1) = u(p_1), \\ \sum_{s=1}^{\tau} \sum_{i=1}^{N} \omega_{i,1}(s)\delta_i(\tau - s)u_i(p_\tau) = \delta^{\tau-1}u(p_\tau). \end{cases} \tag{28}$$

We consider two cases.

Case 1 Suppose $\nexists j \neq K$ such that $u_j = u_K$. In other words, u_K is a unique extreme point in $\{u_i\}_{i\in N}$. Recall that u is a strict convex combination of $\{u_i\}_{i\in N}$. The first equation of (28) shows that $\omega_{K,1}(1) > 0$.

Decompose each $u_j, j \in \mathcal{M}_2$ as convex combinations of $\{u_i\}_{i\in\mathcal{M}_1}$. The second equation of (28) becomes

$$\begin{aligned} \delta^{\tau-1}u(p_\tau) &= \sum_{i\in N} \sum_{s=1}^{\tau} \omega_{i,1}(s)\delta_i(\tau - s)u_i(p_\tau) \\ &= \sum_{i\in\mathcal{M}_1} \sum_{s=1}^{\tau} \omega_{i,1}(s)\delta_i(\tau - s)u_i(p_\tau) + \sum_{j\in\mathcal{M}_2} \sum_{s=1}^{\tau} \omega_{j,1}(s)\delta_j(\tau - s)u_j(p_\tau) \\ &= \sum_{i\in\mathcal{M}_1} \sum_{s=1}^{\tau} \omega_{i,1}(s)\delta_i(\tau - s)u_i(p_\tau) + \sum_{j\in\mathcal{M}_2} \sum_{s=1}^{\tau} \omega_{j,1}(s)\delta_j(\tau - s)\big[\sum_{i\in\mathcal{M}_1} \alpha_i^{(j)} u_i(p_\tau)\big] \\ &= \sum_{i\in\mathcal{M}_1} \sum_{s=1}^{\tau} \big[\omega_{i,1}(s)\delta_i(\tau - s) + \sum_{j\in\mathcal{M}_2} \alpha_i^{(j)} \omega_{j,1}(s)\delta_j(\tau - s)\big] u_i(p_\tau). \end{aligned}$$

Multiply $\delta^{\tau-1}$ to both sides of the first equation of (28), and decompose each $u_j, j \in \mathcal{M}_2$ as convex combinations of $\{u_i\}_{i\in\mathcal{M}_1}$ again. We have

$$\sum_{i\in\mathcal{M}_1} \delta^{\tau-1}\big[\omega_{i,1}(1) + \sum_{j\in\mathcal{M}_2} \alpha_i^{(j)} \omega_{j,1}(1)\big] u_i(p_1) = \delta^{\tau-1}u(p_1).$$

Combining the above two equations, we have

$$\sum_{i\in\mathcal{M}_1}\delta^{\tau-1}\big[\omega_{i,1}(1)+\sum_{j\in\mathcal{M}_2}\alpha_i^{(j)}\omega_{j,1}(1)\big]u_i=\sum_{i\in\mathcal{M}_1}\sum_{s=1}^{\tau}\big[\omega_{i,1}(s)\delta_i(\tau-s)+\sum_{j\in\mathcal{M}_2}\alpha_i^{(j)}\omega_{j,1}(s)\delta_j(\tau-s)\big]u_i. \tag{29}$$

Since u_K is a unique extreme point in $\{u_i\}_{i\in N}$, equation (29) implies that

$$\delta^{\tau-1}\big[\omega_{K,1}(1)+\sum_{j\in\mathcal{M}_2}\alpha_K^{(j)}\omega_{j,1}(1)\big]=\sum_{s=1}^{\tau}\big[\omega_{K,1}(s)\delta_i(\tau-s)+\sum_{j\in\mathcal{M}_2}\alpha_K^{(j)}\omega_{j,1}(s)\delta_j(\tau-s)\big].$$

Rearrange the above equation. We have

$$\begin{aligned}\delta^{\tau-1}&=\frac{\sum_{s=1}^{\tau}\big[\omega_{K,1}(s)\delta_K(\tau-s)+\sum_{j\in\mathcal{M}_2}\alpha_K^{(j)}\omega_{j,1}(s)\delta_j(\tau-s)\big]}{\omega_{K,1}(1)+\sum_{j\in\mathcal{M}_2}\alpha_K^{(j)}\omega_{j,1}(1)}\\&\geq\frac{\omega_{K,1}(1)\delta_K(\tau-1)+\sum_{j\in\mathcal{M}_2}\alpha_K^{(j)}\omega_{j,1}(1)\delta_j(\tau-1)}{\omega_{K,1}(1)+\sum_{j\in\mathcal{M}_2}\alpha_K^{(j)}\omega_{j,1}(1)}\\&\geq\frac{\omega_{K,1}(1)}{\omega_{K,1}(1)+\sum_{j\in\mathcal{M}_2}\alpha_K^{(j)}\omega_{j,1}(1)}\delta_K(\tau-1)\\&=:c\cdot\delta_K(\tau-1).\end{aligned}$$

Hence, for any $\tau\leq T$,

$$\delta\geq\sqrt[\tau-1]{\delta_K(\tau-1)}. \tag{30}$$

However, we have assumed that $\delta<\delta_K^*=\lim_{\tau\to\infty}\sqrt[\tau]{\delta_K(\tau)}$, which means there exists T_1^* such that for any $\tau\geq T_1^*$, $\delta<\sqrt[\tau]{\delta_K(\tau)}$. This contradicts (30).

Case 2 Suppose there exists (at least) one $K'\neq K$ such that $u_{K'}=u_K$. Since $\delta_K^*=\check{\delta}$, $\delta_{K'}^*\geq\delta_K^*$. Otherwise, we can consider the set $\mathcal{M}_1'=\{1,2,\dots\cdots,K-1,K'\}$. Then, $\{u_i\}_{i\in\mathcal{M}_1'}$ is a minimal set of extreme points of $\{u_i\}_{i\in N}$ as well, and $\max_{i\in\mathcal{M}_1'}\delta_i^*<\check{\delta}$. This contradicts the definition of $\check{\delta}$. Following the same steps as in Case 1, we have the following inequality:

$$\begin{aligned}\delta^{\tau-1}&\geq c_1\delta_K(\tau-1)+c_2\delta_{K'}(\tau-1)\\&\geq(c_1+c_2)\min\{\delta_K(\tau-1),\delta_{K'}(\tau-1)\}.\end{aligned} \tag{31}$$

However, we have assumed $\delta < \delta^*_K \leq \delta^*_{K'}$. Thus, there exists T^*_2 such that for any $\tau \geq T^*_2$, $\delta < \min\left\{\sqrt[\tau]{\delta_K(\tau)}, \sqrt[\tau]{\delta_{K'}(\tau)}\right\}$, which contradicts (30). ■

A.9 The Case with Backward Discounting

Suppose there are $T < +\infty$ generations. The result we introduce below shows that if individuals exponentially forward and backward discount consumption, our main results continue to hold.

Before proceeding, let us note that although backward discounting has appeared in Strotz (1955), Caplin and Leahy (2004), and Ray and Wang (2015), one important drawback of backward discounting is that it has no revealed preference foundation. Whenever we observe an individual choosing, the past is sunk. There are no choices (yet) that allow the individual to alter the past. Therefore, we never know how individuals think about the past from actual choice data. Perhaps, because the past cannot be changed, individuals make choices only based on what may still be changed; that is, current and future consumption. Moreover, in many widely used models in economics, such as hyperbolic discounting and quasi-hyperbolic discounting, backward discounting is not allowed; it is not clear how those models could be extended to include backward discounting.

However, in this part, we want to analyze our aggregation problem by allowing individuals to backward discount. This exercise highlights the distinction between Jackson and Yariv's (2014, 2015) results and our results.

Instead of assuming that $U_{i,t}(\mathbf{p})$ does not depend on past consumption, we assume that the generation-t individual i discounts both past and future by the same discounting factor δ_i.

Definition 6 *The generation-t individual i has an exponential forward and backward dis-*

counting utility function if his utility function has the following form:

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=1}^{T} \delta_i^{|\tau-t|} u_i(p_\tau), \tag{32}$$

where the discount factor $\delta_i \in (0,1)$, and u_i is the individual i's instantaneous utility function.

Note that Jackson and Yariv's negative results obviously continue to hold if each generation-t individual i's utility function is

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=1}^{T} \delta_i^{\tau-t} u_i(p_\tau).$$

In this case, the individual i's offspring has exactly the same preference as the indivdual i. This is problematic, However, because then the generation-2 individual i will value period-1 consumption even more than period-2 consumption.

The result below demonstrates that time consistency and the intergenerationally Pareto property are compatible when individuals exponentially forward and backward discount consumption. Jackson and Yariv only consider the planner's aggregation problem in period 1. The following result also only considers the period-1 aggregation problem to highlight the difference.

Proposition 4 *Suppose $T \leq +\infty$, and each generation-t individual i has an exponential forward and backward discounting utility function with discount factor δ_i and instantaneous utility function u_i such that $0 < \underline{\delta} < \bar{\delta} < 1$. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $\{u_i\}_{i \in N}$. Then, for each $\delta \in \left(\bar{\delta}, \bar{\delta}^{-1}\right)$, the planner in period 1 is intergenerationally Pareto and strongly non-dictatorial.*

In particular, if we require the social discount factor δ to be less than 1, then this result has the same implication as our main results.

To prove the proposition, we consider the one-individual case first.

Lemma 5 *Fix any $i \in N$. Suppose $T < +\infty$, and each generation-t individual i has an exponential forward and backward discounting utility function with discount factor $\delta_i \in (0,1)$ and instantaneous utility function u. Then, for each $\delta \in (\delta_i, \delta_i^{-1})$, the planner in period 1 is intergenerationally Pareto and strongly non-dictatorial.*

Proof. We want to show that for any $\delta \in (\delta_i, \delta_i^{-1})$, there exists a vector of strictly positive weights $\vec{\omega} = (\omega_{i,1}(1), \omega_{i,1}(2), \cdots, \omega_{i,1}(T))$ such that the following equation holds

$$U_1(\mathbf{p}) = \sum_{\tau=1}^{T} \delta^{\tau-1} u(p_\tau) = \sum_{s=1}^{T} \omega_{i,1}(s) U_{i,s}(\mathbf{p}). \tag{33}$$

Plugging in $U_1(\mathbf{p})$ and $U_{i,s}(\mathbf{p})$, equation (33) becomes

$$\sum_{\tau=1}^{T} \delta^{\tau-1} u(p_\tau) = \sum_{s=1}^{T} \omega_{i,1}(s) \sum_{\tau=1}^{T} \delta_i^{|\tau-s|} u(p_\tau) = \sum_{\tau=1}^{T} \sum_{s=1}^{T} \omega_{i,1}(s) \delta_i^{|s-\tau|} u(p_\tau), \tag{34}$$

i.e., for each $\tau \geq 1$,

$$\delta^{\tau-1} = \sum_{s=1}^{T} \omega_{i,1}(s) \delta_i^{|s-\tau|}. \tag{35}$$

We can rewrite equation (35) into the following matrix form:

$$A \cdot \vec{\omega} = \vec{\delta}, \tag{36}$$

where $\vec{\delta} = (1, \delta, \delta^2, \cdots, \delta^{T-1})$ and

$$A = \begin{pmatrix} 1 & \delta_i & \delta_i^2 & \dots & \delta_i^{T-1} \\ \delta_i & 1 & \delta_i & \dots & \delta_i^{T-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_i^{T-1} & \delta_i^{T-2} & \delta_i^{T-3} & \dots & 1 \end{pmatrix}.$$

Note that since A is invertible, we have $\vec{\omega} = A^{-1} * \vec{\delta}$. In particular,

$$A^{-1} = \frac{1}{1-{\delta_i}^2} \begin{pmatrix} 1 & -\delta_i & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\delta_i & 1+\delta_i^2 & -\delta_i & 0 & & & & \vdots \\ 0 & -\delta_i & 1+\delta_i^2 & -\delta_i & \ddots & & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & -\delta_i & 1+\delta_i^2 & -\delta_i & 0 \\ \vdots & & & & 0 & -\delta_i & 1+\delta_i^2 & -\delta_i \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\delta_i & 1 \end{pmatrix}$$

If we can show that $\vec{\omega} \gg 0$, the lemma is proved. Showing $\vec{\omega} \gg 0$ is equivalent to showing that

$$\begin{cases} \omega_{i,1}(1) = 1 - \delta_i\delta > 0, \\ \omega_{i,1}(s) = \delta^{s-2}[-\delta_i + (1+\delta_i^2)\delta - \delta_i\delta^2] > 0, 2 \le s \le T-1 \\ \omega_{i,1}(T) = -\delta_i\delta^{T-2} + \delta^{T-1} > 0. \end{cases} \tag{37}$$

Since $\delta \in (\delta_i, {\delta_i}^{-1})$, one can verify that (37) holds. ■

Lemma 5 shows that we can aggregate each individual's utility functions across generations into an EDU function with any discount factor δ within $(\delta_i, {\delta_i}^{-1})$.

Now we can prove Proposition 4. For any social discount rate $\delta \in (\bar{\delta}, \bar{\delta}^{-1})$, we can find $\{\omega_{i,1}(s)\}_{i \in N, s \ge 1}$ such that

$$\sum_{s=1}^{T} \omega_{i,1}(s) U_{i,s}(\mathbf{p}) = \sum_{\tau=1}^{T} \delta^{\tau-1} u_i(p_\tau)$$

for each $i \in N$.

Consider any set of strictly positive numbers $\{\lambda_i\}_{i \in N}$ such that $\sum_{i \in N} \lambda_i = 1$. Together with the weights $\{\omega_{i,t}(s)\}_{i\in N, s\geq t}$ we have found above, the planner's utility function becomes

$$\begin{aligned} U_1(\mathbf{p}) &= \sum_{i=1}^{N}\sum_{s=1}^{T}\lambda_i\omega_{i,1}(s)U_{i,s}(\mathbf{p}) \\ &= \sum_{i=1}^{N}\sum_{\tau=1}^{T}\delta^{\tau-1}\lambda_i u_i(p_\tau) \\ &= \sum_{\tau=t}^{T}\delta^{\tau-1}\sum_{i=1}^{N}\lambda_i u_i(p_\tau) \\ &= \sum_{\tau=t}^{T}\delta^{\tau-1}u(p_\tau), \end{aligned}$$

in which $u(p_\tau) = \sum_{i\in N}\lambda_i u_i(p_\tau)$ is an arbitrary strict convex combination of $\{u_i\}_{i\in N}$.