Multi-Purpose Consumption and Functional Differentiation :
Why has the Vibrant Galleria replaced the Good Old Fashioned Department Store?*

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Abstract

This paper introduces a new form of product differentiation called *functional differentiation* to analyse the increased degree of specialisation of durable goods, which has been a very striking change in product selection over the last century. The importance of this new analytical concept is illustrated in an application to an entry game between an incumbent monopolist and an entrant. It is shown that several standard results are reversed. For example the monopoly would maximise differentiation, whereas the duopoly would choose efficient characteristics. The analysis furthermore provides a novel argument in favour of bundling of functionally differentiated goods.

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Consumers use durable goods for various purposes under different conditions. For example a pair of shoes can be used to walk in the street, run in the forest or hike in the mountain under different weather conditions. A century ago most people would have used the same pair of shoes for all activities, whereas an increased supply of specialised varieties indicate that consumers to a larger extent use different shoes when walking, running, cycling or hiking nowadays. Since it is annoying to use a good that is not fit for purpose but costly to get several specialised varieties to match different conditions, the hypothesis would be that the higher the real income of the consumer, the more specialised will his consumption be, e.g. a special outfit for each activity he does. Thus an increase in the real income should result in an increase in demand for functional differentiation even if consumers are identical. The question is how the supply side will respond to this. Is it the department store or the galleria that will take the lead? Will there be any distortions in product selection? Why are mobile phones\(^1\) bundles of an increasing number of functionally differentiated varieties, whereas shoes are not? What should the policy towards bundling of functionally differentiated goods be?

Hitherto the literature on product differentiation has confined attention to single purpose consumption, i.e. situations when the utility from a good for one particular consumer is independent of circumstances. Functional differentiation, in contrast, is a response to the multi-purpose nature of consumption of many durable goods. Existing models of product differentiation are therefore not directly applicable. However, I show that one can construct a general framework for the analysis of functional differentiation by introducing an annoyance function which is sub additive in the number of varieties. This function is constructed using elements from models of horizontal (Hotelling (1929)) and vertical differentiation (Mussa and Rosen (1978)) which makes the analysis directly comparable with previous work.

\(^1\)Phone, text message, photograph and email are all examples of different varieties of a good that allows information to be communicated. Which one is preferred depends on circumstances.
Functional differentiation differs from all other models of product differentiation in one important respect: it cannot be generated with a characteristics model and heterogeneous consumers. On the contrary, this paper shows that it does matter whether the demand for differentiation is due to individuals having different needs to be met, or whether it is due to consumers being different. The reason for this difference is that all other models are based on the assumption of single-purpose consumption, including the love of variety approach and the spokes model Chen and Riordan (2007). With single-purpose consumption there will be no functional overlap and therefore it will not matter whether different consumers or the same consumer consumes the different varieties.

In particular, I show that two standard results are reversed. First, the incentive to differentiate too much horizontally, arises in a monopoly rather than in a duopoly. Second, socially efficient locations are chosen in a duopoly rather than in a monopoly.

Dynamically this has interesting implications. Whereas a monopoly will offer too few varieties unless it can bundle, a market with free entry will be characterised by too much variety. Since an entrant does not take into account the negative externality

\[^{2}\text{In an important paper Anderson et al. (1989) showed that all existing models used in empirical and analytical work could be generated with a characteristics model and heterogeneous consumers, implying that the source to the demand for heterogeneity would not matter.}\]

\[^{3}\text{Spence (1976) and Dixit-Stiglitz (1977) modeled preferences for love of variety. For these preferences all varieties are equally substitutable, whereas functionally differentiated goods are more or less substitutable depending on their functional overlap.}\]

\[^{4}\text{If the assumption of heterogenous consumers in the spokes model is replaced by an assumption of a consumer having different needs, the spokes model could be used to model functional differentiation in more dimensions, since it is a generalisation of the Hotelling model.}\]

\[^{5}\text{The result with maximum differentiation occurs when the cost of transport is quadratic as was shown by d’Aspremont, Gabszewicz and Thisse (1979). Whereas the result in this paper is more general and thus applies both to the linear and the quadratic cases.}\]

\[^{6}\text{For a heterogeneous population a monopoly would offer optimal locations. See Tirole (1989).}\]

\[^{7}\text{This result is interesting since it differs from the conclusions in endogenous growth models based on the love of variety approach. Innovation of horizontally differentiated goods (Judd (1985), Romer}\]

3
on the price that can be charged for other varieties it will be more profitable for an entrant to offer a specialised variety than for an incumbent to increase the number of varieties even if the incumbent can bundle. An incumbent firm can therefore not credibly deter entry by bundling. However, in the presence of barriers to entry the ability to bundle has positive effects on welfare. In a monopoly it takes away all the distortions in product space, leaving consumers’ surplus unchanged. In the case of a duopoly there is a positive effect on welfare as well as consumers’ surplus. This is because bundling has implications for the firm’s choice of product characteristics, and which combinations can be supported as a Nash equilibrium as well as the number of varieties on offer.

This is a novel argument in favour of bundling. When there are high barriers to entry due to for example research and development, there will be less distortions in product selection over time if firms can bundle. Hence, by bundling their software Microsoft will have less distorted incentives to optimise the characteristics of each individual software, and to add new software at an efficient rate. This is an argument which complements the literature on bundling following Adams and Yellen (1976). This literature has looked at various ways in which bundling enables a firm to extract more surplus for given product characteristics through leverage or price discrimination, or through strategic effects on price or quantity. However the result (1990), Grossman and Helpman (1989,1991), Aghion and Howitt (1990)) results in too few varieties over time. However, a shared feature of my model, Reinganum (1982) one-shot patent race model, and Grossman and Helpman (1991)) is that the incentive to innovate is stronger for an entrant than the incumbent.

This is because there is a negative effect on prices from more varieties, if this is not the case, Klemperer and Padilla (1997), showed that when consumers prefer to concentrate their purchases at a single supplier their new products may be introduced to foreclose competing firms from the market.

Carbajo, de Meza and Seidmann (1990) look at the strategic effects on price and quantity, and find that the welfare effects will depend on the nature of product market competition. Whinston (1990) explored a justification for the leverage theory through its strategic implications.
that bundling implies that the characteristics and number of varieties will be closer to optimal and thus have positive effects on welfare has not been shown before.\textsuperscript{10}

The results in my paper are more general than standard models of horizontal differentiation based on Hotelling’s approach since they only require the utility function to be sub-additive in the number of varieties. Hence, it does not matter whether the annoyance is linear or convex in the distance. The most important implication from sub-additivity is that it is not possible to extract all consumers’ surplus from a set of varieties, since the consumer can choose to buy any subset. Hence, an equilibrium condition for the prices is that they have to be compatible with the consumer buying all varieties. This gives rise to a binding incentive compatibility constraint, not because of heterogeneous consumers, but because of the option of buying a subset.\textsuperscript{11} The price that can be charged for a given variety will therefore be more constrained the larger the functional overlap with other available varieties.

The monopoly internalises the externality and therefore chooses maximum differentiation if it differentiates. The firms competing in a duopoly, on the other hand, do not internalise this externality on the competitor’s price and as a result end up with optimal characteristics. However, the binding incentive compatibility constraint gives rise to another distortion in this case, which is that there will be too many varieties with free entry. This is because the price that an entrant can charge if entry is accommodated is higher than the increase in welfare from more variety.

The outline of the paper is as follows. Preferences for multi-purpose consumption and the demand for functionally differentiated goods are derived in Section ??.

\textsuperscript{10}On the contrary, when bundling is used to deter entry, as in Choi (1996) and Choi and Stefandis (2001) who showed that the leverage theory could be understood through its impact on innovation in the case of complementary goods, there will be a negative effect on consumers’ surplus as well as welfare.

\textsuperscript{11}In the literature on quantity discrimination with high and low types, there will be different sized bundles and therefore an opportunity to buy a subset of the high quantity bundle. In particular, when the possibility to buy several small sized packages is allowed as in Alger (1999), who showed that this was necessary to get an unambiguous result regarding quantity discounts.
This is followed by a derivation of the social optimum in Section ???. Results regarding price, characteristics and number of varieties are derived in a strategic game between an incumbent monopoly and an entrant in Section ???. This game is solved backwards in three stages: the price in subsection ???, characteristics in the case of monopoly and duopoly in subsection ???, and finally number of varieties in subsection ???. The subgames of the entry game reveal what would happen in a market that were a monopoly or a duopoly and are therefore interesting, as well as the final equilibrium of the game. Section ?? analyses bundling of functionally differentiated goods. Section ?? addresses the effects on consumers’ surplus. Section ?? discusses the results. The appendix contains proofs, a summary of payoffs and how to calculate total annoyance.

1 Preferences for Multi-purpose Consumption

Consider needs such as protecting and supporting one’s feet in various activities like dancing, cycling, hiking and walking under different conditions such as temperature, humidity and surface. Let these needs be summarized as exogenously given states $s \in [0, 1]$ with distribution function $F(s)$ and continuous density $f(s)$. Durable goods have characteristics which make them more suitable in some states and less suitable in other states. For example, suede soles are perfect for ballroom dancing, but are a disaster outdoors on a wet surface, whereas rubber soles with good grip are ideal for hiking but would spoil the dancing experience. Let these characteristics be represented by a parameter $\theta \in [0, 1]$. To use a good in a state that it is not perfectly suited for is annoying. The annoyance experienced by a consumer is a function of the state and the characteristics $a(z)$, where $z = |s - \theta|$. The consumer is happy.

\[12\text{Thus the analysis does not include situations where the distribution of states is endogenously determined.}\]

\[13\text{Hence, the ranking of two varieties depends on the state, and no variety is preferred to all other varieties regardless of state.}\]
when the characteristic exactly matches the state, \( a(0) = 0 \) and becomes increasingly annoyed the poorer the match, \( a'(z) > 0 \) at a constant or increasing rate \( a''(z) \geq 0 \).

Let \( V \) be the ultimate comfort, \( m \) denote the inverse of the marginal utility of income\(^{15} \), and assume that \( a(1) \leq V \).\(^{16} \) The consumer is willing to pay more for comfort both on the margin and in absolute terms the higher his income. Thus it is assumed that the willingness to pay for comfort is monotone in income. On the assumption that the expenditures on e.g. shoes is relatively small compared with the total income, the utility from getting one pair of shoes, that will be used in all states, with characteristics \( \theta \) at a price \( p \), can be written

\[
m[V - \int_0^1 a(z)f(s)ds] - p.
\]

Now consider the possibility of buying several pairs of shoes with different characteristics. If there are \( n \) available varieties, their characteristics will be a subset \( \Theta_n \subset \Theta \) of the set of all possible combinations of product characteristics for any number of varieties. Let \( \theta_{nj} \in \Theta_n \), be variety \( j \) when there are \( n \) varieties in total. For analytical convenience let \( \theta_{nj} < \theta_{nj+1} \). Thus variety \( \theta_1 \), is the variety with the smallest \( \theta \) of available varieties. When there is no scope for misunderstanding \( n \) is dropped, and the characteristics are referred to as \( \theta_j \). The average characteristics of two varieties is denoted \( \bar{\theta}_{ij} = \frac{\theta_i + \theta_j}{2} \). To minimise annoyance, each pair of shoes will be used in the states for which they are best suited. Hence, variety \( \theta_1 \) will be used in states \( s \in [0, \bar{\theta}_{1,2}] \) and so on. If the individual has \( n \) varieties with characteristics \( \Theta_n \)

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\(^{14}\)If the marginal annoyance increases with distance, e.g. the sunnier it is the more annoying it is on the margin to wear a raincoat, the function is convex \( a''(z) > 0 \). Note that the annoyance function is a more general representation of the cost of transport in models of spatial differentiation, and therefore encapsulates both the standard linear and quadratic cases.

\(^{15}\)This parameter is equivalent to the taste parameter in models of vertical differentiation. One interpretation (see e.g. Tirole (1989)) is that it is the inverse of the marginal utility of income, such that it will be higher the wealthier the consumer.

\(^{16}\)The individual gets non-negative utility even for the worst possible match, e.g. the individual is always better off using shoes whatever their characteristics to being barefoot in all states.
the total annoyance will be given by

\[ A(\Theta_n) = \left[ \int_{\theta_1}^{\theta_2} a(z_1)f(s)ds + \int_{\theta_2}^{\theta_3} a(z_2)f(s)ds + \cdots + \int_{\theta_{n-1}}^{\theta_n} a(z_n)f(s)ds \right] . \]

Let \( p_i \) denote the price for variety \( i \). Then the utility from buying all of these varieties can be written

\[ m[V - A(\Theta_n)] - \sum_{i=1}^{n} p_i . \]  \hspace{1cm} (2)

If an individual were to get an infinite number of different varieties to match all possible states, the individual would never be annoyed and gain maximum utility \( mV \). However, for a finite number of varieties durable goods will be multi-purpose in use.

Let \( U(\Theta_n) = m[V - A(\Theta_n)] \). This is a set function which is sub-additive.

**Lemma 1** Let \( \Theta_k \subset \Theta_n \). \( U(\Theta_n) \) is sub-additive, i.e. \( U(\Theta_k) + U(\Theta_n \setminus \Theta_k) > U(\Theta_n) \).

**Proof in the appendix.**

This is an important property which follows from the multi-purpose nature of consumption. Since each variety could potentially be used in all states there will be a functional overlap between each variety. When a consumer owns several varieties, each variety will therefore be used less frequently but in states where it causes less annoyance. Hence, the consumer’s utility from a set of goods depends on how well various states can be matched given the set of available varieties and their characteristics, rather than the goods *per se.*

17 Functionally differentiated goods are therefore both complements and substitutes. The smaller the functional overlap the more complementary the varieties become, e.g. if a consumer owns a pair of cycling shoes that cannot be used to walk in, the more the consumer will value having a second pair of shoes for walking. Whereas when the functional over-lap is large the varieties are close substitutes.

\[ ^{17} \text{This is conceptually in line with Lancaster’s (1966) characteristics approach to consumer demand.} \]
The substitutability furthermore implies that the marginal value of a variety with characteristics $\theta$ is dependent on what else the individual is getting. Suppose the individual is getting a subset $\Theta_k \subset \Theta_n$. Let $\theta_j \in \Theta_k$. The marginal contribution of $j$ is
\[ M(\theta_j \mid \Theta_k) = U(\Theta_k) - U(\Theta_k \setminus \theta_j). \] (3)

If the individual buys a larger subset of the number of available varieties the marginal utility of adding $\theta_j$ will be unchanged or lower.

**Lemma 2** Let $\{\theta_i, \theta_j\} \subset \Theta_n$. Then $M(\theta_j \mid \Theta_n) \leq M(\theta_j \mid \Theta_n \setminus \theta_i)$.

Proof in the Appendix.

The marginal value of an additional pair of shoes depends on how close the nearest varieties are. If a smaller subset contains the closest varieties the marginal reduction is the same, whereas if it does not, the marginal reduction in annoyance will be higher.

Preferences for multi-purpose consumption brings together three strands of the literature on product differentiation. The utility being a set function which is sub-additive, captures the notion introduced by Lancaster (1966) that consumers value a set of goods for the characteristics they jointly possess, rather than the goods *per se*. Furthermore the notion that consumers use durable goods under different conditions implies that demand for different locations, i.e. horizontal differentiation, can arise even in a homogeneous population.\(^{18}\) Finally, a bundle of specialised goods versus a general purpose good, is equivalent to choosing between a high quality good and a low quality good, in terms of overall comfort from the two bundles. Thus it is the same principle as in Mussa and Rosen’s (1979) model with vertically differentiated goods.

\(^{18}\)Note that the annoyance function is a more general formulation which includes both the linear and the quadratic cost of transportation case in Hotelling’s (1929) Linear city model.
2 Social Optimum

Deriving the social optimum entails two steps. First a derivation of the optimal characteristics for a given set of \( n \) varieties; second, a derivation of the number of varieties which generates the highest surplus net of costs of producing \( n \) varieties.

Suppose that there is a constant marginal cost \( c \) for producing any variety. Hence, all locations cost the same and there are no economies of scale or scope. Furthermore, the location of any given variety can be changed at no cost.

Optimal characteristics solve

\[
\Theta_n^* = \arg \min A(\Theta_n)
\]

since the utility is maximised for

\[
\max_{\Theta_n \in \Theta} U(\Theta_n) = m[V - A(\Theta_n)]
\]

with first order conditions

\[
-mA_{\theta_i}(\Theta_n) = 0.
\]

To illustrate let us consider the case where the marginal annoyance is constant and calculate the optimal characteristics for one, two and \( n \) characteristics. Let \( \Theta_n = \{\theta_1, \theta_2, \ldots, \theta_n\} \) denote a set with \( n \) varieties. Total annoyance can then be written

\[
A(\Theta_n) = a \left[ \int_0^{\theta_1} (\theta_1 - s) f(s) ds + \int_{\theta_1}^{\theta_1,2} (s - \theta_1) f(s) ds + \int_{\theta_1,2}^{\theta_2} (\theta_2 - s) f(s) ds + \int_{\theta_2}^{\theta_2,3} (s - \theta_2) f(s) ds + \cdots + \int_{\theta_{n-1},n}^{\theta_n} (\theta_n - s) f(s) ds + \int_{\theta_n}^1 (s - \theta_n) f(s) ds \right].
\]

Integration by parts yields

\[
A(\Theta_n) = a \left[ \int_0^{\theta_1} F(s) ds - \int_{\theta_1}^{\theta_1,2} F(s) ds + \int_{\theta_1,2}^{\theta_2} F(s) ds - \int_{\theta_2}^{\theta_2,3} F(s) ds + \cdots + (1 - \theta_n) + \int_{\theta_{n-1},n}^{\theta_n} F(s) ds - \int_{\theta_n}^1 F(s) ds \right].
\]

The first order conditions for a minimum are

\[
a \left[ 2F(\theta_1) - F(\bar{\theta}_{1,2}) \right] = 0, \quad (7)
\]

\[
a \left[ 2F(\theta_2) - F(\bar{\theta}_{1,2}) - F(\bar{\theta}_{2,3}) \right] = 0, \quad (8)
\]
\[ a \left[ 2F(\theta_n) - F(\bar{\theta}_{n-1,n}) - 1 \right] = 0. \]  
\( (9) \)

Combining first order conditions one gets
\[ a \left[ 2 \sum_{i=1}^{n} (-1)^{i+1} F(\theta_i) + (-1)^n \right] = 0 \]
\( (10) \)

For \( n = 1 \) this gives
\[ F(\theta^*) = \frac{1}{2}. \]  
\( (11) \)

And for \( n = 2 \)
\[ F(\theta_2) - F(\theta_1) = \frac{1}{2}. \]  
\( (12) \)

The solution to this system of equations is denoted \( \Theta^*_n \), and is unique if the second order conditions for a minimum are satisfied. These are that the Hessian of second derivatives is positive definite, i.e. that all sub matrices along the diagonal are positive,
\[ a \left[ 2f(\theta_1) - \frac{1}{2} f(\bar{\theta}_{1,2}) \right] > 0, \]  
\( (13) \)

e tc. and
\[ a \left[ 2f(\theta_n) - \frac{1}{2} f(\bar{\theta}_{n-1,n}) \right] > 0. \]  
\( (14) \)

These conditions are satisfied as long as the distribution function does not have spikes.

When some states are more frequent than other states, the optimal set of varieties will be such that all varieties are used with the same frequency. For example a consumer who lives in a hot climate and thus mainly experience hot weather will optimally be equipped with several varieties to cope with small variations in hot weather and perhaps only one pair of shoes to cater for all needs on the rare occasions of cold weather.

For a uniform distribution optimal characteristics are such that \( z_i \leq 1/2n \). Hence
\[ A(\Theta^*_n) = a \frac{1}{2} \left( \frac{1}{2n} \right)^{2n} = a \frac{1}{4n}, \]
since total annoyance is then made up from \( 2n \) triangles of width and height \( 1/2n \). The marginal value of an additional variety is thus
\[ A(\Theta^*_n) - A(\Theta^*_n+1) = a \left[ \frac{1}{4n} - \frac{1}{4(n+1)} \right] = a \frac{1}{4n(n+1)}. \]  
\( (15) \)
which is clearly decreasing in \( n \). Hence the marginal benefit from adding an additional variety is

\[
A(\Theta_1^*) - A(\Theta_2^*) = a\frac{1}{4} - a\frac{1}{8} = a\frac{1}{8} \tag{16}
\]

\[
A(\Theta_2^*) - A(\Theta_3^*) = a\frac{1}{8} - a\frac{1}{12} = a\frac{1}{24} \tag{17}
\]

\[
A(\Theta_3^*) - A(\Theta_4^*) = a\frac{1}{12} - A\frac{1}{16} = a\frac{1}{48} \tag{18}
\]

etc. This result is stated more generally in the following lemma.

**Lemma 3** The marginal reduction in annoyance from increasing the number of varieties is strictly decreasing in number of varieties when characteristics are chosen optimally, \( A(\Theta_{n-1}^*) - A(\Theta_n^*) > A(\Theta_n^*) - A(\Theta_{n+1}^*) \).

Proof in the appendix.

This result is quite intuitive since a larger number of varieties implies that the individual’s needs are already quite well catered for, hence the marginal value of an additional specialised variety will be smaller.

### 2.1 Optimal number of varieties

The optimal number of varieties solves

\[
n^* = \arg \max \{ U(\Theta_n^*) - nc \}. \tag{19}
\]

The optimal \( n^* \) thus depends on \( r = m/c \), that is the ratio between how much the individual values a reduction in annoyance, to what it would cost to reduce it by producing an additional variety. To see this note that one variety is better than no variety if

\[
U(\Theta_1^*) - c > 0. \tag{20}
\]

Two varieties is better than one if

\[
U(\Theta_2^*) - 2c \geq U(\Theta_1^*) - c. \tag{21}
\]
Or more generally,
\[ U(\Theta^*_n) - nc \geq U(\Theta^*_{n-1}) - (n - 1)c. \] (22)
This implies that there exist critical values of \( r \), which is when \( r \) equals the inverse of the marginal change in annoyance from an additional variety.

\[
\begin{align*}
    r^*_1 &= \frac{1}{V - A(\Theta^*_1)} \\
    r^*_2 &= \frac{1}{A(\Theta^*_1) - A(\Theta^*_2)} \\
    r^*_n &= \frac{1}{A(\Theta^*_{n-1}) - A(\Theta^*_n)}.
\end{align*}
\] (23) (24) (25)
From Lemma ?? it follows that \( r^*_{n+1} > r^*_n \). Hence for \( r \in \{r^*_n, r^*_{n+1}\} \), it is optimal to produce \( n \) varieties.

For the uniform distribution one gets
\[ r^*_n = \frac{4n(n - 1)}{a}. \] (26)
As real income \( r \) goes up, consumers should optimally consume a larger number of specialised varieties. However will there be any distortions in product selection over time?

### 3 Strategic product selection

What are the incentives to innovate in imperfectly competitive markets when durable goods are multi-purpose? Does an incumbent monopolist have an incentive to innovate and offer specialised varieties with optimal characteristics when real income goes up? Furthermore does the possibility of selling a specialised variety to match states in which the currently available varieties do poorly, open up opportunities for profitable entry? If, yes, what is the optimal response of an incumbent monopolist to such entry?

All these issues can be covered in a game between an incumbent monopolist and a potential entrant, where some of the answers appear in the solution to a subgame.
Consider an incumbent $I$ who is producing coats. Coats can be functionally differentiated to match different weather conditions. There is another firm $E$ that can decide to enter the market for coats. Once $E$ has decided whether or not to enter, $E$ and $I$ independently decide on how many varieties of coats they will supply, $X_j \in \{0, 1, 2\}, j = I, E$. A firm can either specialise on light breathable coats, $\theta_1$ small, or heavy duty windproof, $\theta_3$ high, or a general purpose coat, $\theta_2$ average. Once they have observed the product strategy of their competitor, they can modify the product characteristics $\theta_j$.\(^{19}\) The firms observe characteristics and independently decide on price $p_j$ for each variety. The objective of each firm is to maximise its profit given the strategy of the other player.

Let $\Pi_j(X_j, X_i)$ denote the reduced form profit function. It is derived by solving for the subgame perfect equilibrium for every combination of varieties. The size of the population is normalised to one since consumers are identical and the marginal cost is assumed to be constant and independent of number of varieties produced by the same firm.\(^{20}\) The demand for each variety is thus $d_k = \{0, 1\}$. The reduced form profit function can thus be written,

$$\Pi_j(X_j, X_i) = \sum_{k=1}^{X_i} (p_k - c)d_k. \quad (27)$$

The game is solved using backward induction. Thus we start by determining the profit maximising price in equilibrium.

### 3.1 Price

In this section it is shown that the equilibrium prices depend on the number of varieties and their individual characteristics, and that these prices are independent

\(^{19}\)If there is entry with light breathable coats, the incumbent can accommodate this entry by replacing a general purpose coat with a heavy coat. Or the incumbent could try to deter entry by offering an even lighter coat in combination with a heavy duty one, to which the entrant would optimally respond by making its coat more general-purpose. Hence, the final specification is assumed to be more flexible than the decision how many different varieties the firm is set up to make.

\(^{20}\)I.e. there are no economies of scale or scope.
of market structure if firms cannot bundle the goods.

**Proposition 1 (Price)** Let the set of available varieties be $\Theta_n \in \Theta$, and suppose that bundling is not feasible. Then there is a unique equilibrium price for variety $\theta_{ni}$, which is the marginal contribution of each variety,

$$\hat{p}_{ni} = M(\theta_i \mid \Theta_n).$$

(28)

This applies to a monopoly as well as a duopoly.

**Proof:** If $p_{ni} < \hat{p}_{ni}$ the firm could increase the price with no effect on demand, i.e. $d_i = 1$. Whereas if $p_{ni} > \hat{p}_{ni}$, $d_i = 0$, since a bundle not containing variety $i$ would then give a higher surplus. Buying all varieties except $\theta_{ni}$ is preferable to buying all if

$$U(\Theta_n \setminus \theta_{ni}) - \sum_{j=1}^{n} p_{nj} + p_{ni} > U(\Theta_n) - \sum_{j=1}^{n} p_{nj}$$

(29)

that is if $p_{ni} > M(\theta_{ni} \mid \Theta_n)$. Hence, the price that will make the consumer willing to buy variety $\theta_{ni}$ is a function of the set $\Theta_n$ and not the prices of other varieties. This in turn implies that the profit maximising equilibrium prices for a given set $\Theta_n$ is independent of whether they are supplied by the same or different firms. Q.E.D.

From this proposition follows that, if there is only one variety available the consumer will buy it as long as

$$U(\theta) - p \geq 0.$$ 

(30)

Hence, a one good monopoly can extract all surplus. Whereas if there are two varieties

$$p_j = U(\theta_1, \theta_2) - U(\theta_i), j \neq i,$$

(31)

total revenue is

$$p_1 + p_2 = 2U(\theta_1, \theta_2) - U(\theta_1) - U(\theta_2) < U(\theta_1, \theta_2).$$

(32)

This is less than total gross surplus, since $U(\theta_1, \theta_2) < U(\theta_1) + U(\theta_2)$. Hence, once the consumer can choose to buy a subset it is no longer possible to extract all surplus.
Furthermore the larger the number of varieties, the less surplus can be extracted, since
the utility that can be derived from alternative bundles will be higher the greater the
variety of specifications.

The marginal contribution and thus the equilibrium price is equivalent to the
reduction in annoyance in the states where the additional variety will be used in place
of the existing ones that would otherwise have been used. This can be illustrated in
the case of linear annoyance. For variety $\theta_1$ the price is

$$p_1 = ma2 \int_{\theta_1}^{\bar{\theta}_1} F(s) ds$$

(33)

for variety $\theta_j$

$$p_j = ma2 \left[ \int_{\theta_j}^{\bar{\theta}_{j,j+1}} F(s) ds - \int_{\bar{\theta}_{j-1,j}}^{\bar{\theta}_{j-1,j+1}} F(s) ds \right]$$

(34)

whereas for variety $\theta_n$

$$p_n = ma \left[ \theta_n - \theta_{n-1} - 2 \int_{\theta_{n-1,n}}^{\theta_n} F(s) ds \right].$$

(35)

Whilst the equilibrium price for a given $\Theta_n \in \Theta$ is independent of market structure,
the prices will indirectly depend on market structure through firms’ strategic choices
of characteristics and thus the relevant set $\Theta_n$. These choices are determined in the
next subsection.

3.2 Characteristics

After having observed how many varieties the competitor has chosen to produce, firms
simultaneously decide on characteristics of these varieties in anticipation of their effect
on equilibrium prices. There are two classes of sub-games which will be considered in
this section. First, the case with no entry, in which case the incumbent is a monopoly.
Second the case with entry in which case there is a duopoly.

21Complete derivations can be found in the appendix.
3.2.1 Monopoly

First, consider the sub games with no entry, that is $X_E = 0$. If the monopoly has set up production lines for either one or two varieties, $X_I \in \{1, 2\}$, it will then choose characteristics of these varieties to maximise the price that it can charge for each variety.

**Proposition 2** If the monopoly offers only one variety it will have optimal characteristics, $\theta^M = \theta^*$, whereas if it offers two, they will be too specialised, $\theta^M_1 < \theta^*_1$, and $\theta^M_2 > \theta^*_2$.

This result can be illustrated for linear annoyance,

$$\Pi(2,0) = ma\left[\sum_{i=1}^{2} \left( \int_{0}^{\theta_i} (\theta_i - s) f(s) ds + \int_{\theta_i}^{1} (s - \theta_i) f(s) ds \right) - 2a \left( \int_{0}^{\bar{\theta}_{12}} (\theta_1 - s) f(s) ds + \int_{\bar{\theta}_{12}}^{\theta_1} (s - \theta_1) f(s) ds + \int_{\theta_2}^{\bar{\theta}_{12}} (s - \theta_2) f(s) ds + \int_{\bar{\theta}_{12}}^{1} (s - \theta) f(s) ds \right) \right] - 2c$$

which can be simplified to

$$\Pi(2,0) = ma \left[ \int_{0}^{\theta_1} (\theta_2 - \theta_1) f(s) ds + \int_{\theta_1}^{\bar{\theta}_{12}} (\theta_1 + \theta_2 - 2s) f(s) ds + \int_{\theta_2}^{\bar{\theta}_{12}} (2s - \theta_1 - \theta_2) f(s) ds + \int_{\bar{\theta}_{12}}^{1} (\theta_2 - \theta_1) f(s) ds \right] - 2c.$$ 

Integration by parts gives

$$\Pi(2,0) = ma \left[ \theta_2 - \theta_1 + 2 \left( \int_{\theta_1}^{\theta_2} F(s) ds - \int_{\bar{\theta}_{12}}^{\theta_2} F(s) ds \right) \right] - 2c$$

Product characteristics maximise the profit if first order conditions are satisfied. These can be written

$$F(\bar{\theta}_{12}) - F(\theta_1) = \frac{1}{2}, \quad F(\theta_2) - F(\bar{\theta}_{12}) = \frac{1}{2}.$$ 

Combining these gives

$$F(\theta_2) - F(\theta_1) = 1.$$ 

17
Hence, the profit maximising combination of characteristics is $\theta_1 = 0$ and $\theta_2 = 1$. Maximum differentiation allows the monopoly to charge the highest price for each variety since it minimises functional overlap. However, the monopoly may be better off not differentiating at all if it is not possible to bundle.

Compare the price the monopoly can charge if it offers one variety with the prices it can charge if it offers two. Can it charge more in total when it offers two varieties? Thus is

$$m[A(\theta_1^M) + A(\theta_2^M) - 2A(\theta_1^M, \theta_2^M) - V + A(\theta^*)] > 0? \quad (42)$$

This is positive if

$$A(\theta^*) - A(\theta_1^M, \theta_2^M) > V + A(\theta_1^M, \theta_2^M) - A(\theta_1^M) - A(\theta_2^M). \quad (43)$$

Hence, a necessary condition for differentiation in a monopoly is that the increase in social surplus i.e. reduction in annoyance, the left hand side, is greater than the rent to the consumer.

If this condition is satisfied there exists an $r^M$, such that the monopoly differentiates for $r \geq r^M$.

Now consider a situation where the real income goes up in the economy, thus $r$ increases. When $r$ has reached $r_2^*$, it would be socially optimal to differentiate, however, since the monopoly requires an even higher $r$, that is $r^M > r_2^*$ the monopoly will innovate with a delay. Note that $r^M > r_2^*$ if

$$A(\theta^*) - A(0, 1) - [V + A(0, 1) - A(0) - A(1)] < A(\theta^*) - A(\theta_1^*, \theta_2^*) \quad (45)$$

which is equivalent to stating that the maximum increase in social surplus, the right hand side, has to be greater than the increase in surplus for the monopoly minus the rent to the consumer, which by definition is true.

However, there are instances when the necessary condition will not be satisfied.
Proposition 3 (A conservative monopoly) If all states are equally likely, the monopoly has no incentive to offer more specialised durable goods if the real income goes up.

Proof: With a uniform distribution of states there is no increase in social surplus when the monopoly differentiates since the annoyance is unchanged $A(0,1) = A(1/2)$. Hence, the monopoly will always be strictly worse off differentiating in this case. QED.

For a uniform distribution of states, maximum differentiation implies that there will be no social gains from differentiation. In the absence of social gains, the monopoly could never charge more for these two varieties in total than for one general purpose variety, since he will be able to extract less surplus in the latter case. Thus there is no incentive to differentiate.

However, when the social gains are big enough the monopoly will differentiate but with a delay if real income $r$ is increasing over time. To see this consider a slight modification to the uniform distribution. Suppose that the various states are distributed according to the following distribution function $s \in [0,1]$

\[
F(s) = \begin{cases} 
    \frac{1 - h(1 - 2z)}{2z} & \text{if } s \in [0, z) \\
    \frac{1 - h(1 - 2s)}{2} & \text{if } s \in [z, 1 - z] \\
    1 - \frac{1 - h(1 - 2z)}{2z} & \text{if } s \in (1 - z, 1] 
\end{cases} 
\]

(46)

with density

\[
f(x) = \begin{cases} 
    \frac{1 - h(1 - 2z)}{2z} & \text{if } x \in [0, z) \\
    h & \text{if } x \in [z, 1 - z] \\
    \frac{1 - h(1 - 2z)}{2z} & \text{if } x \in (1 - z, 1], 
\end{cases} 
\]

(47)

where $h > 0$ and $z \in (\frac{h-1}{2h}, \frac{1}{2})$. This is a symmetric distribution which becomes the uniform distribution for $h = 1$. For $h < 1$, it has two flat tails in the regions $s < z$ and $s > 1 - z$ with higher density than the flat middle section. Hence, it represents a situation where the consumer tends to be more frequently using the good in states with special needs rather than states with average needs. For example if the individual
lives in a country where it is either seriously cold, or very hot, rather than constant drizzling rain. Whereas if $h > 1$ the opposite applies.

For this class of distribution functions total annoyance becomes

$$A(\theta_1) = \int_0^z (\theta_1 - s) \frac{1 - h(1 - 2z)}{2z} ds + \int_{\theta_1}^{\theta_1} (\theta_1 - s) h ds + \int_{\theta_1}^{\theta_1} (s - \theta_1) h ds + \int_{1-z}^{1} (s - \theta_1) \frac{1 - h(1 - 2z)}{2z} ds$$

in the case of one variety and

$$A(\theta_1, \theta_2) = \int_0^{\theta_1} (\theta_1 - s) \frac{1 - h(1 - 2z)}{2z} ds + \int_{\theta_1}^{\theta_2} (s - \theta_1) \frac{1 - h(1 - 2z)}{2z} ds + \int_{\theta_1}^{\theta_2} (s - \theta_1) h ds + \int_{1-z}^{1} (s - \theta_2) \frac{1 - h(1 - 2z)}{2z} ds$$

in the case of two.

If the tails of the distribution are thick and short enough, the monopolist will be able to extract enough surplus provided that the real income $r$ is high enough. In a situation where $r$ is increasing over time, this would imply that the monopoly would innovate but that it would happen with a delay relative to social optimum.

**Proposition 4** Let $z < \frac{2}{7}$ and $h < \frac{4}{5} - \frac{14z}{4 - 14z}$ then there exists an

$$r^M = \frac{1}{a \left[ (1 - h)(1 - \frac{3}{2}z) - \frac{h}{4} \right]}$$

such that a monopoly offers two specialised varieties for $r > r^M$.

It is only when the social benefits from specialisation are high due to some states being much more frequent than other states, that a monopoly will differentiate. Thus an uneven distribution of states is a necessary condition for functional differentiation in a monopoly when bundling is not feasible.

### 3.2.2 Duopoly

Now suppose that entry occurred, i.e. $X_E \geq 1$, so that there are two competing firms in the market. There are two relevant subgames. First, the subgame in which both
firms have set up production lines for one variety. Second, the subgame in which one firm has set up two production lines, and the other firm one.\textsuperscript{22}

If each firm produces one variety, characteristics are chosen simultaneously by each firm to maximise

$$\max_{\theta_i} \pi_i = m[A(\theta_j) - A(\theta_i, \theta_j)] - c$$

(52)

which gives first order conditions

$$-mA_{\theta_i}(\theta_i, \theta_j) = 0,$$

(53)

i.e. the socially efficient ones.\textsuperscript{23}

Thus there is no distortion in characteristics. Each firm tries to minimise annoyance since this will maximise the consumers willingness to pay. The maximised payoff to each firm $j$ is therefore $\Pi_j(1, 1) = m[A(\theta^*_i) - A(\theta^*_1, \theta^*_j)] - c$.

A firm makes non-negative profit if

$$r \geq r^E_2 = \frac{1}{[A(\theta^*_j) - A(\Theta^*_2)]}.$$  

(54)

If $r < r^*_2$ it is optimal with one variety. Thus since $r^E_2 < r^*_2$, there will be too much variety in equilibrium. This is indeed the case since

$$\frac{1}{[A(\theta^*_j) - A(\Theta^*_2)]} < \frac{1}{[A(\theta^*_{11}) - A(\Theta^*_2)]}$$

(55)

which simplifies to $A(\theta^*_j) > A(\Theta^*_2)$. This is true by definition since $\theta^*_{11}$ minimises annoyance for one variety.

However, it is not going to be optimal to enter with a third variety before it is socially optimal to offer two, that is $r^*_2 < r^E_3$ which is equivalent to

$$A(\Theta^*_2) < \frac{A(\theta^*_2) + A(\Theta^*_3)}{2}.$$ 

(56)

\textsuperscript{22}There is also a possibility that both firms have set up lines to produce two varieties. This case does not add additional insights and is not needed to derive the subgame perfect Nash equilibrium of the game, and has therefore been dropped from the presentation.

\textsuperscript{23}This result generalises to $n$ firms each offering one variety.
This again is satisfied since annoyance is reduced at a decreasing rate when more varieties are added. If \( r \) is increasing over time this implies that at the time of entry of a new specialised variety the number of varieties will be optimal, and thus the new variety will be excessive.

Next consider a sub game in which one of the firms offers two varieties and the other one.

**Proposition 5 (Characteristics)** If one firm offers two varieties and the other firm only one, there is a unique subgame perfect Nash equilibrium in characteristics in which the firm with two production lines caters for extreme conditions \( \theta_1^* \) and \( \theta_3^* \), and the firm with one production line offers one general purpose variety \( \theta_2^* \), making respective profits

\[
\Pi^*(2,1) = m [A(\theta_1^*, \theta_2^*) + A(\theta_2^*, \theta_3^*) - 2A(\theta_1^*, \theta_2^*, \theta_3^*)] - 2c, \quad (57)
\]

\[
\Pi^*(1,2) = m [A(\theta_1^*, \theta_3^*) - A(\theta_1^*, \theta_2^*, \theta_3^*)] - c. \quad (58)
\]

Proof in the appendix.

The reason why there will not be an equilibrium in which the firm with two production lines offers two adajacent varieties, is because it would then have an incentive to differentiate these as much as possible. If it does so, the best response, for the firm who has only one production line would be to offer a general purpose variety rather than a specialised variety, whereas an equilibrium would require that a best response to this would be to offer a specialised variety on the other side of the market.

The results in this section are summarised in the matrix below.

<table>
<thead>
<tr>
<th>( X_I, X_E )</th>
<th>Incumbent</th>
<th>Entrant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,0</td>
<td>( \theta^* )</td>
<td>-</td>
</tr>
<tr>
<td>2,0</td>
<td>( \theta_1^M, \theta_2^M )</td>
<td>-</td>
</tr>
<tr>
<td>1,1</td>
<td>( \theta_1^* )</td>
<td>( \theta_2^* )</td>
</tr>
<tr>
<td>1,2</td>
<td>( \theta_2^* )</td>
<td>( \theta_1^<em>, \theta_3^</em> )</td>
</tr>
<tr>
<td>2,1</td>
<td>( \theta_1^<em>, \theta_3^</em> )</td>
<td>( \theta_2^* )</td>
</tr>
</tbody>
</table>
The intuition for these results is that when firms are not choosing adjacent characteristics they do not internalise the negative externality from their choices on the price that can be charged for adjacent varieties. The price that they can charge is therefore maximised for the socially optimal characteristics, which minimises the annoyance $A(\Theta_n)$. The only instance where a firm chooses adjacent varieties is when it is a monopoly producing two varieties. In this case product characteristics will be distorted and result in too much differentiation.

### 3.3 Equilibrium

Having derived subgame perfect equilibria for all relevant combinations of number of varieties, we are ready to consider each firm’s decision regarding how many varieties to supply. Given that the number of varieties will influence whether entry would be profitable or not the question is whether entry will be accommodated or not.

If an incumbent could credibly commit to offering two varieties if entry occurs this would deter entry for $r$ low enough. Suppose that the incumbent offers light and heavy coats. The payoff to an entrant who offers a general purpose coat is

$$\Pi(1, 2) = m[A(\theta^{*}_{11}, \theta^{*}_{31}) - A(\Theta^{*}_3)] - c. \quad (59)$$

If this payoff is negative, entry is deterred. Thus if

$$r < r^D = \frac{1}{A(\theta^{*}_{11}, \theta^{*}_{31}) - A(\Theta^{*}_3)} \quad (60)$$

entry is deterred. However, $r$ needs to be high enough to make such a strategy credible, that is $\Pi(2, 1) > \Pi(1, 1)$. Hence,

$$m[A(\theta^{*}_{31}, \theta^{*}_{32}) + A(\theta^{*}_{32}, \theta^{*}_{33}) - 2A(\Theta^{*}_3)] - 2c > m[A(\theta^{*}_{22}) - A(\Theta^{*}_2)] - c. \quad (61)$$

This is positive if

$$r > \bar{r} = \frac{1}{A(\theta^{*}_{31}, \theta^{*}_{32}) + A(\theta^{*}_{32}, \theta^{*}_{33}) - 2A(\Theta^{*}_3) - A(\theta^{*}_{22}) + A(\Theta^{*}_2)}. \quad (62)$$
Thus entry could potentially be deterred for \( \bar{r} < r < r^D \). However, this condition may not be satisfied. For example if the states are uniformly distributed and the inconvenience is linear one variety is strictly better than two for all parameter values,

\[
\Pi(1,1) - \Pi(2,1) = ma \left[ \frac{3}{16} - \frac{1}{6} \right] + c = \frac{ma}{48} + c > 0.
\]  

(63)

For \( r \in [r^E, \bar{r}] \) entry is profitable if accommodated, and it is not credible to deter so it will be accommodated. That this interval is non-empty can be seen by confirming that \( \bar{r} > r^E \) if

\[
A(\theta^*_{21}) + A(\theta^*_{22}) - 2A(\Theta^*_2) > A(\theta^*_{31}, \theta^*_{32}) + A(\theta^*_{32}, \theta^*_{33}) - 2A(\Theta^*_3)
\]  

(64)

which follows from Lemma ??.

**Proposition 6** Suppose that \( E \) faces no cost of entry and let \( r^E < r \leq \bar{r} \). Then there is a unique sub game perfect equilibrium in which \( E \) enters, and \( I \) and \( E \) chooses one variety each with socially efficient characteristics \( \theta^*_1 \) and \( \theta^*_2 \), and equilibrium prices \( p^*_1 = M(\theta^*_1 \mid \Theta^*_2) \), \( p^*_2 = M(\theta^*_2 \mid \Theta^*_2) \), and profits \( \Pi^*(1,1) = p^*_j - c > 0 \).

**Proof:** The reduced form profit function was derived by solving for the subgame perfect equilibrium for different combinations of number of varieties. Thus to solve for the Nash equilibrium, what remains is to show that

\[
\Pi^*(1,1) \geq \max\{\Pi(0,1), \Pi(2,1)\},
\]  

(65)

are satisfied. If \( r < \bar{r} \) entry will be accommodated since \( \Pi(1,1) > \Pi(2,1) \). If \( r > r^E \) it will be profitable to enter, since \( \Pi(1,1) > \Pi(0,1) \). Q.E.D.

This highly stylised game reveals an important mechanism in markets with functionally differentiated goods by illustrating that when real income goes up products will become more specialised, not because of attempts to deter entry, but because functional differentiation makes entry profitable even when consumers are homogeneous. Note that entry with the same variety would result in zero profit since we then get Bertrand competition. In a scenario where \( r \) is increasing over time, entry pushes prices down towards marginal cost, but they stay above on average.
The model also gives a clear prediction as to where entry would happen if a third firm were to enter.

**Proposition 7** An entrant who positions himself in a fringe will get a higher profit than an entrant who positions himself in between the existing firms in product space.

**Proof:** An entrant who positions himself with a specialised variety $\theta_3$ in product space gets a higher profit than an entrant who would offer a general purpose variety $\theta_2$ since

$$m[A(\theta_1^*, \theta_2^*) - A(\Theta_3^*)] - c > m[A(\theta_1^*, \theta_3^*) - A(\Theta_3^*)] - c$$  \hspace{1cm} (66)

$$A(\theta_1^*, \theta_2^*) - A(\theta_1^*, \theta_3^*) = a \left[ \theta_3 - \theta_2 + 2 \left( \int_{\theta_1,2}^{\theta_2} F(s) ds - \int_{\theta_2}^{\theta_3} F(s) ds \right) \right]$$  \hspace{1cm} (67)

Q.E.D.

This result is interesting if compared with the Hotelling model when consumers are heterogeneous. A third firm would in that case wish to locate in the middle to maximise demand.

4 Bundling

The results in the previous section were derived on the assumption that bundling was not feasible. In reality one can find examples of functionally differentiated goods that are bundles, such as mobile phones, as well as goods that are not such as shoes.

This section investigates what will happen when bundling is feasible, in the case of monopoly and duopoly respectively. It turns out that the most important effect from bundling is on product characteristics and the incentive to offer specialised goods, rather than its effect on price.

**Lemma 4** If a firm can bundle it will choose socially efficient characteristics regardless of whether it is a monopoly or a duopoly.

**Proof:** The monopoly can charge $p_B(\theta_1, \theta_2) = U(\theta_1, \theta_2)$ for a bundle. First order conditions to profit maximisation are therefore $-mA_{\theta_1}(\Theta_2) = 0$, i.e. socially efficient.
The firm offering a bundle in a duopoly can charge \( p_B(\theta_j, \theta_k) = m[A(\theta_i) - A(\theta_j, \theta_k, \theta_i)] \). First order conditions are again the socially optimal ones \(-mA_{\theta_j}(\Theta_3) = 0\), since this is what minimises the annoyance. Q.E.D.

Bundling removes an incentive constraint, and therefore takes away the incentive to distort product characteristics. This has two important implications. The first is that the number of varieties will be closer to optimal.

**Corollary 1** A firm who can bundle has a less distorted incentive to innovate and offer more specialised varieties.

The second is that it makes it possible for a firm to offer two specialised varieties on one side of the market, which is something that could not be supported as a Nash equilibrium if sold separately due to the incentive to distort product characteristics as was shown in the previous section.

**Proposition 8** Let firm \( j \) be set up to produce two varieties, and firm \( i \) be set up to produce one variety. If firm \( j \) can bundle, offering any bundle with efficient characteristics \((\theta_i^*, \theta_j^*)\) is then a Nash equilibrium.

This result is vital, since it is only in the ‘new’ equilibria that bundling results in a higher price, for the firm who bundles as well as the firm who does not. The latter result follows from the fact that the marginal contribution of one specified variety, in the case of three varieties with socially optimal characteristics, will be higher than the marginal contribution of a general purpose variety. If the firm offering two varieties cannot bundle, the only subgame perfect equilibrium in characteristics is such that the firm who offers one variety will offer the general purpose variety. If, on the other hand, the firm can bundle, there will exist equilibria in characteristics space that Pareto dominates this equilibrium.

Thus it is not bundling *per se* that allows firms to extract more surplus in this case, but the fact that bundling makes efficient bundles strategically feasible.

**Proposition 9** The firm can only extract more surplus if the bundle contains adjacent varieties.
Proof: The prices that can be charged by the firm who offers a bundle are

\[ p_B(\theta_1^*, \theta_2^*) = m [A(\theta_2^*) - A(\Theta_3^*)] \] (68)

\[ p_B(\theta_1^*, \theta_3^*) = m [A(\theta_3^*) - A(\Theta_3^*)] \] (69)

Comparing these with the individual prices \( p(\theta_i) = m[A(\theta_i^*, \theta_k^*) - A(\Theta_3^*)] \) it can be verified that \( p_B(\theta_1^*, \theta_3^*) = p(\theta_1^*) + p(\theta_3^*) \) since

\[ A(\theta_3^*) - A(\theta_2^*, \theta_3^*) = A(\theta_2^*, \theta_3^*) - A(\Theta_3^*) \] (70)

and that \( p_B(\theta_1^*, \theta_2^*) > p(\theta_1^*) + p(\theta_2^*) \) since

\[ A(\theta_3^*) - A(\theta_2^*, \theta_3^*) > A(\theta_1^*, \theta_3^*) - A(\Theta_3^*) \] (71)

which follows from Lemma ??.

Q.E.D.

The difference in price between a bundle and the price that can be charged for each good when sold separately is the difference between adding that variety to a bundle of one and two goods respectively. If the closest variety in the smaller and the larger bundle is the same the marginal value of adding variety one is the same, and there are no gains from bundling. This will be the case if the varieties that are bundled are not adjacent. If they are adjacent, the price for the bundle will be higher since the marginal value of adding variety two to a small bundle will be higher than the value of adding it to a larger bundle. This is because there will be closer substitutes in the larger bundle.

When firms can bundle there is thus a stronger incentive to offer more specialised varieties in a duopoly. In particular bundling changes the prediction for the uniform case. For a uniform distribution the firm who offers a bundle can charge

\[ p_B(\theta_1, \theta_2) = a \left[ \frac{13}{36} - \frac{1}{12} \right] = ma \frac{5}{18}. \] (72)

It is better to offer a bundle than one variety if \( \Pi_B(2, 1) - \Pi^*(1, 1) > 0 \) i.e.

\[ ma \frac{5}{18} - 2c - \left[ ma \frac{3}{16} - c \right] = ma \frac{13}{(12)^2} - c \geq 0. \] (73)
Thus if $r$ is high enough.

More generally the incumbent prefers to offer a bundle with $(\theta^*_1, \theta^*_2)$ if $r > r_B$ where

$$r_B = \frac{1}{A(\theta^*_{33}) - A(\Theta^*_3) - A(\theta^*_{22}) + A(\Theta^*_2)}. \quad (74)$$

Suppose that $r$ is increasing over time. The possibility to bundle implies a monopoly will specialise at an optimal rate, whereas a duopoly will specialise earlier than it otherwise would have, but still with a delay relative to social optimum.

**Proposition 10** If a monopoly can bundle it will offer a bundle $\Theta^*_n$ for $r \in [r^*_n, r^*_n+1]$.

*Proof:* Since the monopoly can extract all consumers’ surplus when bundling it will specialise when $U(\theta^*_1, \theta^*_2) \geq U(\theta^*)$, i.e. when it is socially optimal. Q.E.D.

Similarly the possibility to bundle makes it more profitable for one firm to offer two varieties in a duopoly. For $r \geq r^B$ it will be optimal to offer three varieties in a duopoly when one firm can bundle. However, $r_B > r^*_3$, since

$$\frac{1}{A(\theta^*_{33}) - A(\Theta^*_3) - A(\theta^*_{22}) + A(\Theta^*_2)} > \frac{1}{A(\Theta^*_2) - A(\Theta^*_3)} \quad (75)$$

This simplifies to

$$A(\theta^*_{33}) > A(\theta^*_{22}) \quad (76)$$

which is true by definition, since annoyance for one variety is higher the more specialised the variety, and $\theta^*_{33} > \theta^*_{22}$ e.g. more specialised. Hence, bundling reduces the distortions in product selection in the monopoly and the duopoly. The question is whether this implies that entry will be deterred.

**Proposition 11** An incumbent monopolist will not deter entry even if it can bundle for $r \in [r^*_2, r_B]$.

*Proof:* It is only credible to deter entry if $r > r_B$. Entry will occur for $r \geq r^*_2$, but will not be deterred for $r < r_B$. Hence, we need to show that $r_B > r^*_2$, which holds since $r^*_2 < r^*_3 < r_B$. Q.E.D.
If \( r \) is increasing over time this proposition implies that, when \( r \) has become sufficiently high to make it credible to deter entry it will already have happened.

Even though bundling makes entry deterrence more profitable, it is still not sufficient to make up for the overall reduction in prices needed to support three varieties in equilibrium.

If there are no barriers to entry, and there are no economies of scope, functionally differentiated goods will not be bundled, since there is an incentive to enter before it becomes profitable to differentiate through bundling. However, if there are barriers to entry, the analysis shows that there is a strong incentive to bundle functionally differentiated goods. This incentive would be further strengthened by economies of scope. The added functions on mobile phones is an excellent example of a good satisfying both of these criteria. There are clearly economies of scope, and due to high sunk costs of research and development it is also an industry with high barriers to entry.

5 Consumers’ Surplus

Whereas social welfare as well as the firm’s incentive to specialise will depend on \( r \), the consumers’ surplus will be independent of this factor. This is because there are two effects from more variety that influences consumers’s surplus which work in the same direction. First, the effects on gross surplus, which is positive. Second the effect on prices, which is negative, since more variety implies that the firms can extract less surplus, even if consumers are homogeneous.

Consumers’ surplus is given by,

\[
S = U(\Theta_n) - \sum_{i} p_i. \tag{77}
\]

First, it should be noted that even though a monopoly who can bundle will maximise welfare since there will be no distortions in product selection, the consumer is left with zero surplus.
When there are more than one firm three scenarios are of particular interest. First the one where there has been entry and each firm produces one variety,$$S(1, 1) = U(\theta_1^*) + U(\theta_2^*) - U(\theta_2^*) = m [V + A(\theta_2^*) - A(\theta_1^*) - A(\theta_2^*)] .$$ (78)

Second a situation where one firm offers a bundle, and the other firm one specialised variety,$$S_B(2, 1) = U(\theta_3^*) + U(\theta_1^*, \theta_2^*) - U(\theta_3^*) = m [V + A(\theta_3^*) - A(\theta_1^*) - A(\theta_2^*)] .$$ (79)

Third a situation where three varieties are supplied due to entry of a third firm
$$S(1, 1, 1) = U(\theta_1^*, \theta_3^*) + U(\theta_1^*, \theta_2^*) + U(\theta_2^*, \theta_3^*) - 2U(\theta_3^*) = m [V + 2A(\theta_3^*) - A(\theta_1^*, \theta_3^*) - A(\theta_2^*, \theta_3^*) - A(\theta_1^*, \theta_2^*)] .$$ (80)

These can be calculated for our running example which gives,
$$S(1, 1) = m \left[ V - a \frac{1}{2} \right]$$ (82)
$$S(2, 1) = m \left[ V - a \frac{4}{9} \right]$$ (83)
$$S(1, 1, 1) = m \left[ V - a \frac{11}{36} \right]$$ (84)

Thus $S(1, 1, 1) > S(2, 1) > S(1, 1).$ Since the surplus is monotone in $m,$ consumers prefer more variety to less, and there surplus will be higher if the goods are supplied by independent suppliers. However, there is an interesting difference between the effects from bundling in the duopoly and monopoly cases respectively.

If there are barriers to entry, due to for example research and development or advertising, the market may either be a monopoly or a duopoly. If it is a monopoly, bundling increase welfare with no effect on consumers’ surplus. Whereas if it is a duopoly, bundling increases both welfare and consumers’ surplus.

6 Discussion

This paper has shown that functional differentiation is distinct from other forms of product differentiation. Already in the simplest possible one dimensional case with
homogeneous consumers and constant returns to scale technology, several interesting results were derived. Hence, it is a model which enables further progress to be made in the understanding of functionally differentiated goods. Apart from the fact that this is an important form of product differentiation empirically, it has also been proven in this paper to be of theoretical interest.

The model could easily be generalised to several dimensions using a novel approach by Chen and Riordan (2007) which they call the spokes model. They do indeed suggest that their model could be used to study general purpose and specialised goods. This paper makes such an application straightforward by providing an analytical principle for modelling the demand side for functionally differentiated goods.24

The paper also makes a conceptual contribution to the literature on product innovation, by adding a third class of successful innovations which encourages consumers to buy more goods rather than switching suppliers.25 Functional differentiation is the introduction of specialised goods that are less suitable for general purpose, but more suitable for specific purposes, such as cycling shoes.26 For the devoted cyclist this is a quality improvement and could therefore be treated as a combined horizontal and vertical improvement. However, the element not captured in a standard model of product innovation is that the consumer is likely to increase the overall consumption of shoes, rather than switching, since buying a pair of shoes that can only be used under one specific set of conditions for which shoes are required increases the total

---

24 In its present form their model is based on the assumption of heterogeneous consumers, which as has been shown in this paper is not equivalent to demand for different locations by the same consumer in the case of functional differentiation.

25 The nature of successful product innovations is an issue that brings industrial economists and business strategists together (see Caves (1984)). Porter (1980) points out two ways for an innovation to be successful, either by improving quality (see e.g. Fudenberg et.al.(1983) ) or to better match the taste of a market segment. In both cases some consumers will switch from one supplier to the new one.

26 There are plenty of other examples e.g. a palm. Consumers own more and more computers of different sizes that match specific needs.
number of shoes needed to perform various functions. Thus there is an element of complementarity between goods specialised to match specific conditions, which arises as a result of the multi-purpose nature of consumption of several durable goods, such as computers, cycles, clothes and shoes. In this case successful product innovations entail identifying the various conditions under which the good will be used, and make varieties that match those conditions. Such innovations have two effects. First an increase in total demand for the good, and second a shift from general purpose to specific purpose goods. If a consumer can afford two pairs of shoes, it is better to buy two specific purpose that complement one another, than one general purpose and one specific purpose that partly overlap in function.

This is a crucial difference from the love of variety approach that was initiated by Spence (1976) and Dixit and Stiglitz (1977) to study optimal product diversity under monopolistic competition. These preferences can be represented by the CES sub utility function\(^{27}\), in which variety is valued per se and thus can be used to explain the increase in differentiation of goods that perform the same function, e.g. light summer clothes. There is in this case no reason to switch from one variety to two different new varieties, since all are equally substitutable. Thus there is a demand for variety as a result of taste for variety rather than a demand for variety to reduce the annoyance from using a variety under conditions for which it is less well suited, e.g. sandals in rain.

Since multi-purpose consumption provides the micro foundations for a 'taste' for variety it explains the success of companies who are able to identify the various ways in which a durable good could potentially be used, and manage to invent a variety which makes it perfect under one of those specific conditions. The introduction of such varieties lowers the profitability of existing goods, but increases the number of durable goods an individual decides to own.

Furthermore, it highlights why the good old-fashioned department store may be too conservative offering only general purpose varieties, whilst entry of independent

\(^{27}\)See e.g. Helpman and Krugman (1985) chapter 6.
suppliers with new specialised varieties in a galleria will induce existing suppliers to 
change their characteristics as well as result in an increase in the overall degree of 
specialisation. The prices in the galleria will also be more competitive in equilibrium. 
These are factors which can explain the success of gallerias at the expense of the 
department store.

The model also opens up for other interesting applications, such as the role of spe-
cialised goods when consumers are heterogeneous. For example, what is the optimal 
design, size range and pricing of baby clothes when consumers differ in terms of how 
much they are willing to pay for a perfect fit during their baby’s first year? These 
are questions which are of theoretical as well as practical importance.

A Proofs

Proof of Lemma ??: Let \( \theta_i \in \Theta_n \). For all \( \theta_i \) such that either \( \theta_{i-1} \in \Theta_k \) and/or \( \theta_{i+1} \in \Theta_k \), there is a replacement effect, which comes from the fact that once there are more varieties 
available some varieties will no longer be used in states that they are less well suited for.

This can be proven for \( \Theta_k = \{ \theta_1, \theta_2 \} \), when the total number of varieties is \( \Theta_4 \) then

\[
A(\Theta_k) + A(\Theta_4 \setminus \Theta_k) - A(\Theta_4) = \int_{\theta_{1,2}}^{\theta_{1,2}} a(z_1)f(s)ds + \int_{\theta_{1,2}}^{1} a(z_2)f(s)ds \\
+ \int_{\theta_{3,4}}^{\theta_{3,4}} a(z_3)f(s)ds + \int_{\theta_{3,4}}^{1} a(z_4)f(s)ds \\
- \left[ \int_{\theta_{1,2}}^{\theta_{1,2}} a(z_1)f(s)ds + \cdots + \int_{\theta_{3,4}}^{\theta_{3,4}} a(z_4)f(s)ds \right] \\
= \int_{\theta_{2,3}}^{1} a(z_2)f(s)ds + \int_{\theta_{2,3}}^{\theta_{2,3}} a(z_3)f(s)ds > 0.
\]

Hence, the annoyance in states where variety \( \theta_2 \) and \( \theta_3 \) are no longer in use when the 
individual has all four varieties. The same principle applies to any combination of \( k \) and \( n \).

Q.E.D.

Proof of Lemma ??:

Note that since the marginal contribution is

\[
U(\Theta_n) - U(\Theta_n \setminus \theta_j) = m[A(\Theta_n \setminus \theta_j) - A(\Theta_n)]
\]

(85)
One can without loss of generality consider the marginal reduction in annoyance from adding variety 1, when all available varieties are being used except 1, which is given by

\[ A(\Theta_n \setminus \theta_1) - A(\Theta_n) = \int_0^{\bar{\theta}_{1,2}} [a(z_2) - a(z_1)] f(s) ds \]  

(86)

If \( \theta_1 \) is added to a subset \( \Theta_{n-1} \subset \Theta_n \) instead, the reduction will be the same or larger depending on whether the closest substitute \( \theta_2 \in \Theta_{n-1} \) or not.

\[ A(\Theta_{n-1} \setminus \theta_1) - A(\Theta_{n-1}) = \begin{cases} 
\int_0^{\bar{\theta}_{1,2}} [a(z_2) - a(z_1)] f(s) ds & \text{if } \theta_2 \in \Theta_{n-1} \\
\int_0^{\bar{\theta}_{1,3}} [a(z_3) - a(z_1)] f(s) ds & \text{otherwise.} 
\end{cases} \]  

(87)

The same principle applies to any other characteristic in the subset. Q.E.D.

**Proof of Lemma 8**: Adding variety \( i \) will reduce annoyance in states where it is used instead of variety \( i - 1 \) and variety \( i + 1 \). Hence, the benefit is

\[ \int_{\theta_{i-1,i}}^{\theta_i^*} [a(z_{i-1}) - a(z_i)] + \int_{\theta_i^*}^{\theta_{i+1,i}} [a(z_{i+1}) - a(z_i)] f(s) ds. \]  

(88)

When \( n \) increases

\[ \theta_{i+1}^* - \theta_{i-1}^* \]

decreases, if \( a(\cdot) \) and \( f(\cdot) \) are continuous. This has two implications. First annoyance will be reduced for a smaller number of states. Second, the varieties the new variety replaces will be closer substitutes, e.g. \( |z_k - z_i| \) for \( k = i - 1, i + 1 \). Hence, the marginal benefit in each state will also be smaller. Q.E.D.

**Proof of Proposition 9**: For one variety the profit maximisation problem of the monopoly is

\[ \max_{\theta} \Pi(1, 0) = m[V - A(\theta)] - c. \]  

(89)

F.o.c.

\[ -mA_0(\theta_1) = 0 \]  

(90)

which is satisfied for the socially efficient choice \( \theta^* \).

For two varieties it becomes,

\[ \max_{\theta_1, \theta_2} \Pi(2, 0) = p_1 + p_2 - 2c = m[A(\theta_1) + A(\theta_2) - 2A(\theta_1, \theta_2)] - 2c. \]  

(91)
F.o.c.

\[ m[A_{\theta_1}(\theta_1) - 2A_{\theta_1}(\theta_1, \theta_2)] = 0, \]  
\[ m[A_{\theta_2}(\theta_1) - 2A_{\theta_2}(\theta_1, \theta_2)] = 0. \]  

The second term is zero at \( \theta^*_1, \theta^*_2 \), whereas the first term reaches its minimum in between these two values. Hence, the first order condition reveals that the profit is decreasing in \( \theta_1 \) whereas it is increasing in \( \theta_2 \) at \( \theta^*_1, \theta^*_2 \). Thus, \( \theta^*_1 < \theta^*_2 \), and \( \theta^*_1 > \theta^*_2 \). Q.E.D.

**Proof of Proposition ??:** The monopoly will offer two specialised varieties if \( \Pi(1,0) < \Pi(2,0) \). For the stepped distribution we have

\[ A\left(\frac{1}{2}\right) = a \left[ \frac{1}{2} - \frac{h}{4} - \frac{1-h}{2} \right] \]  
\[ A(0) = a \left[ \frac{1}{2} - \frac{1-h}{2} \right] \]  
\[ A(1) = a \left[ \frac{1}{2} - \frac{1-h}{2} \right] \]

\[ A(0,1) = a \left[ \frac{h}{4} + \frac{1-h}{2} \right] \]  
\[ A(0,1/2) = a \left[ \frac{4-h}{16} \right] \]

Monopoly profit with one good is

\[ \Pi(1,0) = \left(\frac{1}{2}\right) = m \left( V - a \left[ \frac{1}{2} - \frac{h}{4} - \frac{1-h}{2} \right] \right) - c \]

Monopoly profit with two varieties sold separately at prices \( p_1 = m [A(1) - A(0,1)] \), \( p_2 = m [A(0) - A(0,1)] \) is

\[ \Pi^M (0,1) = 2ma \left[ \frac{1}{2} - \frac{h}{4} - \frac{1-h}{2} \right] - 2c. \]

\[ \Pi(2,0) - \Pi(1,0) = m \left[ a \left( \frac{3}{2} - \frac{3}{4}h - \frac{5}{2}(1-h)z \right) - V \right] - c \geq 0 \]

Two specialised varieties generates higher profit than one general purpose if

\[ r > \frac{1}{a \left[ \frac{3}{2} - \frac{3}{4}h - \frac{5}{2}(1-h)z \right] - V} \]
the denominator is positive if

\[
\begin{align*}
  h & \leq \frac{2 \left( 3 - \frac{2V}{a} \right) - 10z}{3 - 10z} \\
  z & \leq \frac{3a - 2V}{5a}
\end{align*}
\] (104) (105)

Q.E.D.

**Proof of Proposition ??**: If one firm has chosen to produce two varieties and the other one only one, there are two possibilities: the one who only sells one offers a 'general purpose' i.e. variety \( \theta_2 \), or a very specialised one i.e. \( \theta_1 \). The proof involves showing that, the first is a Nash equilibrium whereas the latter is not.

In the first case, for the firm offering one general purpose variety with characteristics \( \theta_2^* \), there is no profitable deviation for \( \theta \in [\theta_1^*, \theta_3^*] \) since the profit is by definition maximised for \( \theta_2^* \). Could the firm increase profit by offering something more specialised such as \( \theta' < \theta_1^* \)? Such a deviation would result in a profit

\[
\Pi^{dev}(1, 2) = m \left[ A(\theta_1^*, \theta_3^*) - A(\theta', \theta_1^*, \theta_3^*) \right] - c.
\] (106)

Since \( A(\Theta_3) \) is minimised for \( \Theta_3^* \), this profit is strictly less than the equilibrium profit.

Similarly for the firm offering two varieties, \( \theta_1^* \) and \( \theta_3^* \) maximise the profit for \( \theta_2^* \) by definition. The other alternative deviation would be to offer two specialised varieties in a niche, e.g. one ultra light and one light coat. Thus consider \( \theta', \theta'' \) such that \( \theta' < \theta_1^* < \theta'' < \theta_2^* \). This would result in strictly lower prices since

\[
\begin{align*}
  p(\theta') & = m[ A(\theta', \theta_2^*) - A(\theta', \theta'' , \theta_2^*) ] < m[ A(\theta_2^*, \theta_3^*) - A(\Theta_3^*) ] = p(\theta_1^*) \\
  p(\theta'') & = m[ A(\theta', \theta_2^*) - A(\theta', \theta'' , \theta_2^*) ] < m[ A(\theta_1^*, \theta_2^*) - A(\Theta_3^*) ] = p(\theta_3^*)
\end{align*}
\]

Hence, the socially optimal characteristics do form a Nash equilibrium.

Can a situation where one firm offers one specialised variety and the other two specialised varieties at the other end be a Nash equilibrium? Optimal characteristics if the firms anticipate that the single variety firm will produce a light coat, \( \theta_1 \) and the multi-product firm produce one general purpose and one heavy, \( \theta_2, \theta_3 \), solve the following problem:

\[
\begin{align*}
  \max_{\theta_2, \theta_3} p_2 + p_3 - 2c & = ma \left[ \theta_3 - \theta_2 + 2 \left( \int_{\theta_2}^{\theta_3} F(s)ds - \int_{\theta_1}^{\theta_3} F(s)ds - \int_{\theta_2}^{\theta_3} F(s)ds \right) \right] - 2c \\
  \max_{\theta_1} p_1 - c & = ma2 \int_{\theta_1}^{\theta_3} F(s)ds - c.
\end{align*}
\]
First order conditions in this case are:

\[
ma \left[ F(\theta_{1,2}) - 2F(\theta_1) \right] = 0 \quad (107)
\]

\[
ma \left[ -1 + 2F(\theta_{2,3}) - 2F(\theta_2) + F(\theta_{1,2}) \right] = 0 \quad (108)
\]

\[
ma \left[ 1 + 2F(\theta_{2,3}) - 2F(\theta_3) - F(\theta_{1,3}) \right] = 0 \quad (109)
\]

Combining these conditions gives

\[
1 - F(\theta_3) + F(\theta_2) - F(\theta_1) = \frac{F(\bar{\theta}_{1,3})}{2}. \quad (110)
\]

For \( F(\theta) = \theta \) the solution is

\[
\theta_1 = \frac{1}{12}, \quad (111)
\]

\[
\theta_2 = \frac{1}{4}, \quad (112)
\]

\[
\theta_3 = \frac{11}{12}. \quad (113)
\]

This is not a Nash equilibrium. The single good firm could do better for these choices of characteristics to produce a general purpose variety with characteristics \( \theta = \frac{7}{12} \) instead of \( \theta_1 = \frac{1}{12} \), since

\[
\Pi(\frac{7}{12}, \frac{1}{4}, \frac{11}{12}) = ma \left[ \int_{\frac{7}{12}}^{\frac{1}{4}} \theta d\theta - \int_{\frac{7}{12}}^{\frac{11}{12}} \theta d\theta \right] - c = \frac{ma}{18} - c \quad (114)
\]

which is clearly higher than

\[
\Pi(\frac{1}{12}, \frac{1}{4}, \frac{11}{12}) = ma \int_{\frac{1}{12}}^{\frac{1}{4}} \theta d\theta - c = \frac{ma}{48}. \quad (115)
\]

Q.E.D.

\textit{Proof of Proposition ??:} From Proposition ?? it follows that \( \theta_2^* \) is a best response to \( \theta_1^* \) and \( \theta_3^* \). This is also true if they are sold as a bundle, since the price that can be charged for variety 2 in equilibrium solely depends on the characteristics of the other available varieties and not their price.

Consider the firm with two production lines. If it offers a bundle with two specialised varieties at the same end of the market e.g. \( \theta_1^*, \theta_2^* \), the question is whether the firm two who offers \( \theta_3^* \) could do better by offering a variety \( \theta' \in (\theta_1^*, \theta_2^*) \) instead. This would result in a price

\[
p(\theta') = ma[A(\theta_1^*, \theta_2^*) - A(\theta_1^*, \theta', \theta_2^*)] \quad (116)
\]
which is strictly less than $p(\theta^*_5) = m[A(\theta^*_1, \theta^*_2) - A(\Theta^*_3)]$ since $A(\Theta^*_3)$ is minimised for the set of varieties with optimal characteristics, $\Theta^*_3$. Q.E.D.

B Appendix

This appendix contains a summary of payoffs for the three stage game, and derivations of annoyance and prices for the linear example.

The payoff matrix for the three stage game is

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0,0</td>
<td>$\Pi^*(1,0)$</td>
<td>0, $\Pi^*(2,0)$</td>
</tr>
<tr>
<td>1</td>
<td>$\Pi^*(1,0)$, 0</td>
<td>$\Pi^<em>(1,1)$,$\Pi^</em>(1,1)$</td>
<td>$\Pi^<em>(1,2)$,$\Pi^</em>(2,1)$</td>
</tr>
<tr>
<td>2</td>
<td>$\Pi^*(2,0)$,0</td>
<td>$\Pi^*(2,1)$</td>
<td>$\Pi^<em>(2,2)$,$\Pi^</em>$</td>
</tr>
</tbody>
</table>

where

\[
\Pi^*(0,X) = 0 \tag{117}
\]

\[
\Pi^*(1,0) = U(\theta^*) - c \tag{118}
\]

\[
\Pi^*(2,0) = 2U(\theta^*_1, \theta^*_2) - U(\theta^*_1) - U(\theta^*_2) - 2c \tag{119}
\]

\[
\Pi^*(1,1) = U(\theta^*_2, \theta^*_2) - U(\theta^*_2) - c \tag{120}
\]

\[
\Pi^*(1,2) = U(\Theta^*_3) - U(\theta^*_3, \theta^*_3) - c \tag{121}
\]

\[
\Pi^*(2,1) = 2U(\Theta^*_3) - U(\theta^*_3, \theta^*_3, \theta^*_3) - U(\theta^*_3, \theta^*_3, \theta^*_3) - 2c \tag{122}
\]

\[
\Pi^*(2,2) = 2U(\Theta^*_3) - U(\theta^*_4, \theta^*_4, \theta^*_4) - U(\theta^*_4, \theta^*_4, \theta^*_4) - 2c \tag{123}
\]

To calculate total annoyance for linear annoyance and a uniform distribution the following formulas can be used:

\[
A(\theta) = \frac{1}{2} - \theta(1 - \theta) \tag{125}
\]

\[
A(\theta_1, \theta_2) = \frac{1}{2} + \theta_1^2 - \theta_2(1 - \theta_2) - \left(\frac{\theta_1 + \theta_2}{2}\right)^2 \tag{126}
\]

\[
A(\theta_1, \theta_2, \theta_3) = \frac{1}{2} - \theta_3 + \sum_{j=1}^{3} \theta_j^2 - \frac{1}{4} \left[(\theta_1 + \theta_2)^2 + (\theta_2 + \theta_3)^2\right] \tag{127}
\]
For the running example this gives,

\[
A(\theta_{11}^*) = A\left(\frac{1}{2}\right) = a \frac{1}{4}
\]

(128)

\[
A(\theta_{21}^*) = A\left(\frac{1}{4}\right) = a \frac{5}{16}
\]

(129)

\[
A(\theta_{31}^*) = A\left(\frac{1}{6}\right) = a \frac{13}{36}
\]

(130)

\[
A(\theta_{21}^*, \theta_{22}^*) = A\left(\frac{1}{4}, \frac{3}{4}\right) = a \frac{1}{8}
\]

(131)

\[
A(\theta_{31}^*, \theta_{32}^*) = A\left(\frac{1}{6}, \frac{1}{2}\right) = a \frac{1}{6}
\]

(132)

\[
A(\theta_{31}^*, \theta_{33}^*) = A\left(\frac{1}{6}, \frac{5}{6}\right) = a \frac{5}{36}
\]

(133)

\[
A(\theta_3^*) = A\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right) = a \frac{1}{12}
\]

(134)

These can be used to calculate prices for bundles of varieties.

\[
p_B(\theta_1, \theta_3) = ma \left[\frac{1}{4} - \frac{1}{12}\right] = a \frac{1}{6}
\]

(135)

\[
p_B(\theta_1, \theta_2) = ma \left[\frac{13}{36} - \frac{1}{12}\right] = a \frac{5}{18}
\]

(136)

\[
p(\frac{1}{4}) = ma \left[\frac{5}{16} - \frac{1}{8}\right] = a \frac{3}{16}
\]

(137)

The price for linear annoyance, and arbitrary distribution function, can be calculated as follows.

\[
p_j = ma \left[\int_{\theta_{j-1}}^{\theta_{j-1} + 1} (s - \theta_{j-1}) f(s) ds + \int_{\theta_{j-1} + 1}^{\theta_{j+1}} (\theta_{j+1} - s) f(s) ds - \int_{\theta_{j-1}}^{\theta_{j-1} + 1} (s - \theta_{j-1}) f(s) ds \right]
\]

\[
- \int_{\theta_{j-1}}^{\theta_{j-1} + 1} (\theta_j - s) f(s) ds - \int_{\theta_j}^{\theta_{j+1}} (s - \theta_j) f(s) ds + \int_{\theta_j}^{\theta_{j+1}} (\theta_{j+1} - s) f(s) ds\]

\[
= ma \left[\int_{\theta_{j-1}}^{\theta_{j-1} + 1} (s - \theta_{j-1}) f(s) ds + \int_{\theta_{j-1} + 1}^{\theta_{j+1}} (\theta_j - s) f(s) ds - \int_{\theta_j}^{\theta_{j+1}} (s - \theta_j) f(s) ds\right]
\]

\[
= ma \left[\int_{\theta_{j-1}}^{\theta_{j+1}} F(s) ds - \int_{\theta_{j-1} + 1}^{\theta_{j+1}} F(s) ds\right]
\]

Integration by parts yields

\[
p_j = ma \left[\int_{\theta_j}^{\theta_{j+1}} F(s) ds - \int_{\theta_{j-1} + 1}^{\theta_{j+1}} F(s) ds + \int_{\theta_{j-1} + 1}^{\theta_{j+1}} F(s) ds\right]
\]

\[
= ma2 \left[\int_{\theta_j}^{\theta_{j+1}} F(s) ds - \int_{\theta_{j-1} + 1}^{\theta_{j+1}} F(s) ds\right]
\]

which is the expression that appears in the paper.
References


