Online Supplement
Tests of Policy Interventions in DSGE Models
by
M. Hashem Pesaran and R.P. Smith

S1 Statement and proof of lemmas

Lemma S1 *Let* $\mathbf{A}$ *be a* $k \times k$ *matrix and* $\mathbf{x}_{T+h-j}$ *a* $k \times 1$ *vector, and suppose that* $\mathbf{I}_k - \mathbf{A}$ *is invertible, then*

$$
\begin{aligned}
H^{-1}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathbf{A}^{j}\mathbf{x}_{T+h-j} &= H^{-1}\sum_{j=1}^{H}\left(\mathbf{I}_k+\mathbf{A}+\ldots+\mathbf{A}^{H-j}\right)\mathbf{x}_{T+j} \\
&= H^{-1}\left(\mathbf{I}_k-\mathbf{A}\right)^{-1}\sum_{j=1}^{H}\left(\mathbf{I}_k-\mathbf{A}^{H-j+1}\right)\mathbf{x}_{T+j} \\
&= \left(\mathbf{I}_k-\mathbf{A}\right)^{-1}\left(H^{-1}\sum_{j=1}^{H}\mathbf{x}_{T+j}\right)-\left(\mathbf{I}_k-\mathbf{A}\right)^{-1}\left(H^{-1}\sum_{j=1}^{H}\mathbf{A}^{H-j+1}\mathbf{x}_{T+j}\right).
\end{aligned}
$$

Proof. The result follows by direct manipulation of the terms. ■

Lemma S2 *Suppose that the* $k \times k$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *have bounded spectral norms* $\|\mathbf{A}\| \leq \lambda$ *and* $\|\mathbf{B}\| \leq \lambda$, *for some fixed positive constant* λ. *Then*

$$
\left\|\mathbf{A}^{h}-\mathbf{B}^{h}\right\| \leq h\lambda^{h-1}\left\|\mathbf{A}-\mathbf{B}\right\|, \text{ for } h=1,2,\ldots. \tag{A.1}
$$

Proof. We establish this result by backward induction. It is clear that it holds for $h = 1$. For $h = 2$, using the identity

$$
\mathbf{A}^2-\mathbf{B}^2=\mathbf{A}(\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{B})\mathbf{B},
$$

the result for $h = 2$ follows

$$
\left\|\mathbf{A}^2-\mathbf{B}^2\right\| \leq \left(\|\mathbf{A}\|+\|\mathbf{B}\|\right)\|\mathbf{A}-\mathbf{B}\| = 2\lambda\|\mathbf{A}-\mathbf{B}\|.
$$

More generally, we have the identity

$$
\mathbf{A}^h-\mathbf{B}^h=\mathbf{A}^h(\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{B})\mathbf{B}^h+\mathbf{A}(\mathbf{A}^{h-2}-\mathbf{B}^{h-2})\mathbf{B}.
$$

Now suppose now that (A.1) holds for $h-2$, and using the above note that

$$
\begin{aligned}
\left\|\mathbf{A}^h-\mathbf{B}^h\right\| &\leq \left\|\mathbf{A}^{h-1}\right\|\|\mathbf{A}-\mathbf{B}\|+\|\mathbf{A}-\mathbf{B}\|\left\|\mathbf{B}^{h-1}\right\|+\|\mathbf{A}\|\left\|\mathbf{A}^{h-2}-\mathbf{B}^{h-2}\right\|\|\mathbf{B}\| \\
&\leq \|\mathbf{A}\|^{h-1}\|\mathbf{A}-\mathbf{B}\|+\|\mathbf{A}-\mathbf{B}\|\|\mathbf{B}\|^{h-1}+\|\mathbf{A}\|\left\|\mathbf{A}^{h-2}-\mathbf{B}^{h-2}\right\|\|\mathbf{B}\| \\
&\leq 2\lambda^{h-1}\|\mathbf{A}-\mathbf{B}\|+\lambda^2\left\|\mathbf{A}^{h-2}-\mathbf{B}^{h-2}\right\| \\
&\leq 2\lambda^{h-1}\|\mathbf{A}-\mathbf{B}\|+\lambda^2\left[(h-2)\lambda^{h-3}\|\mathbf{A}-\mathbf{B}\|\right] \\
&\leq h\lambda^{h-1}\|\mathbf{A}-\mathbf{B}\|.
\end{aligned}
$$

Hence, if (A.1) holds for $h-2$, then it must also hold for h. But since we have established that (A.1) holds for $h=1$ and $h=2$, then it must hold for any h. ■

Lemma S3 *Consider the $k \times k$ matrix $\mathbf{A}(\boldsymbol{\theta}) = (a_{ij}(\boldsymbol{\theta}))$, where k is a finite integer and $a_{ij}(\boldsymbol{\theta})$, for all $i, j = 1, 2, .., k$, are continuously differentiable real-valued functions of the $p \times 1$ vector of parameters, $\theta \in \boldsymbol{\Theta}$. Suppose that $a_{ij}(\theta)$ has finite first order derivatives at all points in $\boldsymbol{\Theta}$, and assume that $\hat{\boldsymbol{\theta}}_T$ is a $\sqrt{T}$ consistent estimator of $\boldsymbol{\theta}^0$. Then*

$$\left\|\mathbf{A}(\hat{\boldsymbol{\theta}}_T) - \mathbf{A}(\boldsymbol{\theta}^0)\right\| \leq a_T \left\|\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^0\right\|, \tag{A.2}$$

$$\left\|\mathbf{A}(\hat{\boldsymbol{\theta}}_T)\right\| \leq \left\|\mathbf{A}(\boldsymbol{\theta}^0)\right\| + a_T \left\|\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^0\right\|, \tag{A.3}$$

where $a_T = \left\|\partial \mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)/\partial \boldsymbol{\theta}'\right\|$ is bounded in T, and elements of $\bar{\boldsymbol{\theta}}_T \in \boldsymbol{\Theta}$ lie on the line segment joining $\boldsymbol{\theta}^0$ and $\hat{\boldsymbol{\theta}}_T$

Proof. Consider the mean-value expansions

$$a_{ij}\left(\hat{\theta}_T\right) - a_{ij}\left(\theta^0\right) = \frac{\partial a_{ij}\left(\bar{\theta}_T\right)}{\partial \theta'}\left(\hat{\theta}_T - \theta^0\right), \text{ for } i, j = 1, 2, ..., k,$$

where elements of $\bar{\theta}_T$ lie on the line segment joining θ^0 and $\hat{\theta}_T$. Given that $\hat{\theta}_T$ is consistent for θ^0, it must also be that $\bar{\theta}_T \to_p \theta^0$, as $T \to \infty$. Collecting all the k^2 terms we have

$$\mathbf{A}(\hat{\boldsymbol{\theta}}_T) - \mathbf{A}(\boldsymbol{\theta}^0) = \left(\frac{\partial \mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)}{\partial \boldsymbol{\theta}'}\right)\left[\mathbf{I}_k \otimes \left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^0\right)\right],$$

where $\otimes$ denotes the Kronecker matrix product. Hence

$$\left\|\mathbf{A}(\hat{\boldsymbol{\theta}}_T) - \mathbf{A}(\boldsymbol{\theta}^0)\right\| \leq \left\|\frac{\partial \mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)}{\partial \boldsymbol{\theta}'}\right\| \left\|\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^0\right\|,$$

$$\left\|\mathbf{A}(\hat{\boldsymbol{\theta}}_T)\right\| = \left\|\mathbf{A}(\boldsymbol{\theta}^0) + \left(\frac{\partial \mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)}{\partial \boldsymbol{\theta}'}\right)\left[\mathbf{I}_k \otimes \left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^0\right)\right]\right\| \leq \left\|\mathbf{A}(\boldsymbol{\theta}^0)\right\| + \left\|\frac{\partial \mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)}{\partial \boldsymbol{\theta}'}\right\| \left\|\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^0\right\|.$$

The results (A.2) and (A.3) now follow since $\bar{\boldsymbol{\theta}}_T \to_p \boldsymbol{\theta}^0$, and by assumption the derivatives $\partial a_{ij}\left(\boldsymbol{\theta}^0\right)/\partial \boldsymbol{\theta}'$ exist and are bounded in T. ■

Lemma S4 *Suppose that $\lambda_T = \lambda + T^{-1/2} a_T$, $a_T > 0$ and bounded in T, $\lambda_T \neq 1$, $H = \kappa T^{\epsilon}$, where $\epsilon \leq 1/2$, $0 < \lambda < 1$, and κ is a positive fixed constant. Then*

$$\sum_{h=1}^{H} h \lambda_T^{h-1} = \frac{1}{(1-\lambda)^2} + O_p\left(T^{-1/2}\right) + O_p\left(H \lambda^H\right), \tag{A.4}$$

and

$$\sum_{h=1}^{H} \sum_{j=0}^{h-1} j \lambda_T^{j-1} = \frac{1}{(1-\lambda)^2}\left(H - \frac{1+\lambda}{1-\lambda}\right) + O_p\left(T^{-1/2}\right) + O_p\left(H \lambda^H\right). \tag{A.5}$$

Proof. We first note that

$$\sum_{h=1}^{H} h\lambda_T^{h-1} = \frac{\partial}{\partial\lambda_T}\left(\sum_{h=1}^{H}\lambda_T^h\right)$$
$$= \frac{1-\lambda_T^H}{(1-\lambda_T)^2} - \frac{H\lambda_T^H}{(1-\lambda_T)}, \quad (A.6)$$

Also since $\lambda_T = \lambda + O_p\left(T^{-1/2}\right)$

$$\sum_{h=1}^{H} h\lambda_T^{h-1} = \frac{1}{(1-\lambda)^2} + O_p\left(T^{-1/2}\right) + O_p\left(H\lambda_T^H\right). \quad (A.7)$$

But,

$$\lambda_T^H = \left(\lambda + T^{-1/2}a_T\right)^H = \lambda^H\left(1 + \frac{T^{-1/2}a_T}{\lambda}\right)^H = O_p\left(\lambda^H e^{d_T H/\sqrt{T}}\right), \quad (A.8)$$

where $d_T = a_T/\lambda$, which is also bounded in T. Finally, $H/\sqrt{T} = T^{1-\epsilon/2}$ and for $\epsilon \leq 1/2$, $e^{d_T H/\sqrt{T}}$ will be bounded in T. Using this result in (A.7) yields (A.4), as desired. Similarly,

$$\sum_{h=1}^{H}\sum_{j=0}^{h-1} j\lambda_T^{j-1} = \sum_{h=1}^{H}\left[\frac{\left(1-\lambda_T^h\right) - h(1-\lambda_T)\lambda_T^{h-1}}{(1-\lambda_T)^2}\right]$$
$$= \frac{1}{(1-\lambda_T)^2}\left[\sum_{h=1}^{H}\left[\left(1-\lambda_T^h\right) - h(1-\lambda_T)\lambda_T^{h-1}\right]\right]$$
$$= \frac{1}{(1-\lambda_T)^2}\left[H - \sum_{h=1}^{H}\lambda_T^h - (1-\lambda_T)\sum_{h=1}^{H}h\lambda_T^{h-1}\right].$$

Using (A.6) we have

$$\sum_{h=1}^{H}\sum_{j=0}^{h-1} j\lambda_T^{j-1} = \frac{1}{(1-\lambda_T)^2}\left\{H - \frac{\lambda_T - \lambda_T^{H+1}}{1-\lambda_T} - (1-\lambda_T)\left[\frac{1-\lambda_T^H}{(1-\lambda_T)^2} - \frac{H\lambda_T^H}{(1-\lambda_T)}\right]\right\}.$$

Now using (A.8) and recalling that $\lambda_T = \lambda + O_p\left(T^{-1/2}\right)$, we obtain (A.5). ■

S2 The numerical solution of the DGSE model used in Section 5

The unique solution of the New Keynesian model is given by (see also equation (2) in the paper):

$$\widetilde{\mathbf{q}}_t = \mathbf{\Phi}(\boldsymbol{\theta})\widetilde{\mathbf{q}}_{t-1} + \mathbf{\Gamma}(\boldsymbol{\theta})\mathbf{u}_t,$$

where $\mathbf{\Phi}(\boldsymbol{\theta})$ solves the quadratic matrix equation $\mathbf{A}_1\mathbf{\Phi}^2(\boldsymbol{\theta}-\mathbf{A}_0\mathbf{\Phi}(\boldsymbol{\theta})+\mathbf{A}_2 = \mathbf{0}$, and $\mathbf{\Gamma} = [\mathbf{A}_0 - \mathbf{A}_1\mathbf{\Phi}(\boldsymbol{\theta})]^{-1}$. $\mathbf{\Phi}(\boldsymbol{\theta})$ can be solved numerically by iterative back-substitution procedure which involves iterating on an initial arbitrary choice of $\mathbf{\Phi}(\boldsymbol{\theta})$ say $\mathbf{\Phi}(\boldsymbol{\theta}_{(0)}) = \mathbf{\Phi}_{(0)}$ using the recursive relation

$$\mathbf{\Phi}_{(r)} = [\mathbf{I}_k - (\mathbf{A}_0^{-1}\mathbf{A}_1)\mathbf{\Phi}_{(r-1)}]^{-1}(\mathbf{A}_0^{-1}\mathbf{A}_2).$$

See Binder and Pesaran (1995) for further details. The iterative procedure is continued until convergence using the criteria $\| \mathbf{\Phi}_{(r)} - \mathbf{\Phi}_{(r-1)} \|_{\max} \leq 10^{-6}$.

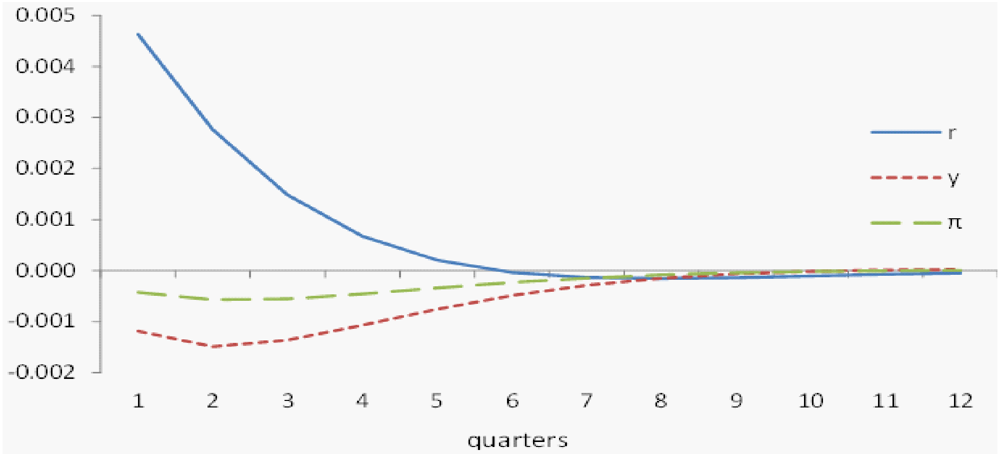
S3 Standard and Policy Impulse Response Functions for the new-Keynesian model

Here we first provide impulse response functions, IRFs for the effects of monetary policy, demand and supply shocks in the new-Keynesian model. As Figure S1 shows a contractionary monetary policy shock raises interest rates and reduces output and inflation, with output falling by more than inflation. A positive demand shock increases all three variables; output by the most, then interest rates, and then inflation. A negative supply shock, increases inflation, the interest rate rises to offset the higher inflation, but not by as much as inflation and output falls. The impact effects of the monetary policy shock are given by the first column of $\mathbf{\Gamma}(\boldsymbol{\theta}^0)$ defined by equation (50) of the paper, while the impact effects of the demand and supply shocks are given by its second and third columns. It is clear that in terms of IRFs the behaviour of the model is as expected.

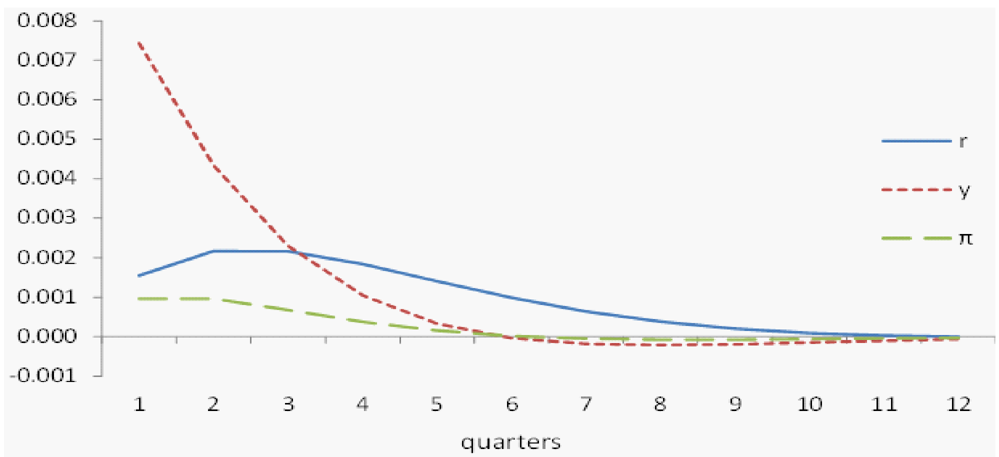
Turning to the policy impulse response function, PIRF, discussed in Section 3.1 of the paper, as noted in the text it is important that the choice of $\widetilde{\mathbf{q}}_{T_0}$ reflects a sensible combination of values of interest rate, inflation and output. One possible approach is to set $\widetilde{\mathbf{q}}_{T_0}$ equal to the impact effects of IRFs. For example, one could set $\widetilde{\mathbf{q}}_{T_0}$ to $\widetilde{\mathbf{q}}_{R,T_0} = \sigma_{uR}\mathbf{\Gamma}(\boldsymbol{\theta}^0)\mathbf{e}_R$, which is the impact effect of a monetary policy shock. Similarly, for the demand and supply shocks $\mathbf{q}_{T_0}$ can be set to $\widetilde{\mathbf{q}}_{y,T_0} = \sigma_{u\ y}\mathbf{\Gamma}(\boldsymbol{\theta}^0)\mathbf{e}_y$ and $\widetilde{\mathbf{q}}_{\pi,T_0} = \sigma_{u\pi}\mathbf{\Gamma}(\boldsymbol{\theta}^0)\mathbf{e}_\pi$, respectively, where $\mathbf{e}_y = (0,1,0)'$ and $\mathbf{e}_\pi = (0,0,1)'$. These values are given by the columns of $\mathbf{\Gamma}(\boldsymbol{\theta}^0)$ defined by equation (50) of the paper. We will also consider multiples of the effects of such shocks as representing different degrees of deviations from equilibrium. The power of the policy ineffectiveness test will then be an increasing function of the extent to which, at the time of the policy change, the economy has deviated from steady state.

Figure S2 shows PIRFs for the effects of changing the degree of persistence (or the interest rate smoothing) associated with the Taylor rule, Figure S2a shows the effect of intervention 1_A and Figure S2b of 1_B. These are the only policy changes which have much effect. This is consistent with the theoretical results that it is the dynamics that are central to policy having mean effects. Intervention 1_A increases the degree of persistence from $\delta_R = 0.7$, to $\delta_R = 0.9$. This causes the interest rate to rise and output and inflation to fall initially, with a maximum effect after about three periods before returning to zero. Intervention 1_B reduces the degree of persistence from $\delta_R = 0.7$, to $\delta_R = 0.25$. This has the opposite effect causing the interest rate to fall, by more than it rose in case 1_A, and output and inflation to rise by rather less than they fell under case 1_A. The initial effects are the same as the values of $\left[\mathbf{\Phi}\left(\boldsymbol{\theta}^1\right) - \mathbf{\Phi}\left(\boldsymbol{\theta}^0\right)\right]$ for the two cases. When the degree of persistence is low as in intervention 1_B, the variables return to zero much faster, making the mean effect of policy much smaller. This is reflected in the power of the policy ineffectiveness tests discussed in the text.

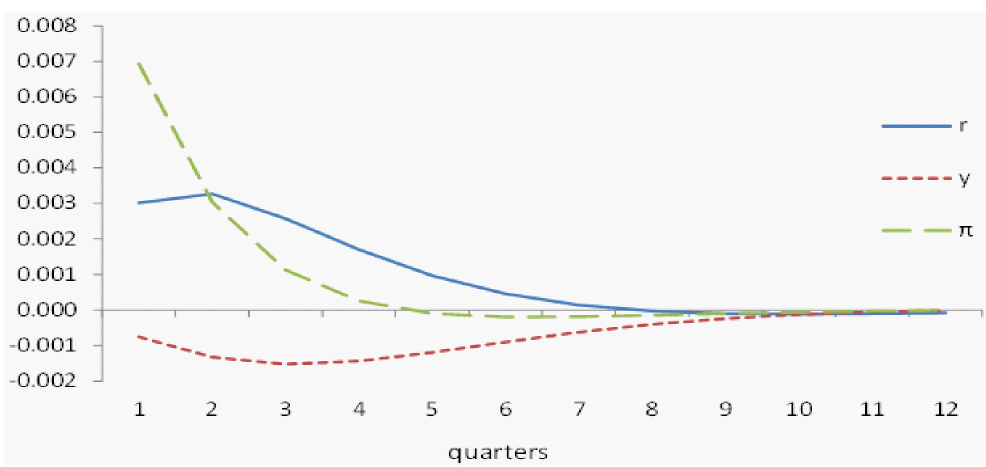
Figure S1: Impulse response functions for interest rates, $\widetilde{R}_t$, output, $\widetilde{y}_t$, and inflation $\widetilde{\pi}_t$ deviations



S1a. Monetary Policy Shock

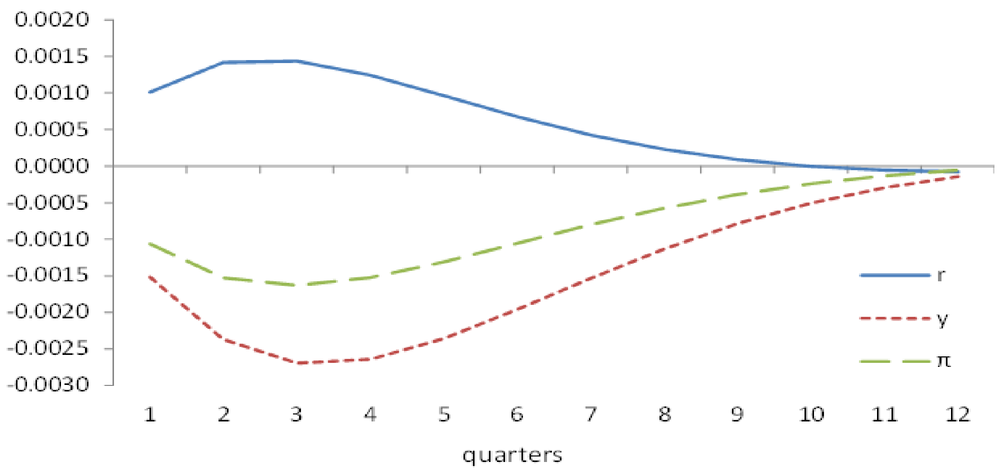


S1b. Demand Shock

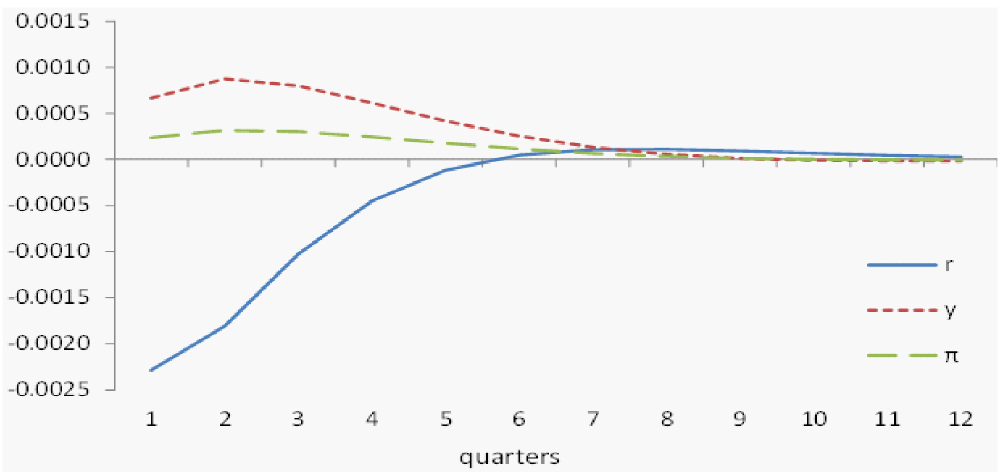


S1c. Supply Shock

Figure S2: Policy impulse response functions: $\widetilde{\mathbf{q}}_{R,T_0} = \sigma_{uR}\mathbf{\Gamma}(\boldsymbol{\theta}^0)\mathbf{e}_R$.



S2a. Intervention 1_A : $\delta_R = 0.7$, to $\delta_R = 0.9$



S2b. Intervention 1_B : $\delta_R = 0.7$, to $\delta_R = 0.25$