Solution of Finite-Horizon Multivariate Linear Rational Expectations Models and Sparse Linear Systems*

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Abstract

This paper presents efficient methods for the solution of *finite-horizon* multivariate linear rational expectations (MLRE) models, linking the solution of such models to the problem of solving sparse linear equation systems with a block-tridiagonal coefficient matrix structure. Two numerical schemes for the solution of this type of equation systems are discussed, and it is shown how these procedures can be adapted to efficiently solve finite-horizon MLRE models. As the two numerical schemes are fully recursive and only involve elementary matrix operations, they are also straightforward to implement. The numerical schemes are illustrated by applying them to a finite-horizon adjustment cost problem of expenditure shares under adding-up constraints, and to a finite-horizon linear-quadratic optimal control problem.

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GAUSS- and MATLAB-programs accompanying this paper may be downloaded from the following URL: http://www.inform.umd.edu/econ/mbinder/research.

1 Introduction

In this paper we consider the solution of finite-horizon multivariate linear rational expectations (MLRE) models. We establish a conceptual link between the solution of these models and the problem of solving sparse linear equation systems with a block-tridiagonal coefficient matrix structure, and show how methods for the solution of such sparse linear equation systems can be adapted to efficiently solve finite-horizon MLRE models.

Sparse linear equation systems with a block-tridiagonal coefficient matrix structure arise for a wide variety of scientific problems, including the numerical solution of certain classes of partial differential equations, linear-quadratic optimal control problems, and Gaussian optimal filtering problems. Not surprisingly, then, the solution of linear equation systems with such a structure has been extensively studied in the numerical analysis literature. Drawing on recent work in this literature, we describe two numerical schemes for the efficient solution of sparse linear equation systems with a block-tridiagonal coefficient matrix structure, and provide analytical conditions under which these schemes can be successfully applied. Adapting the schemes to the solution of finite-horizon MLRE models yields numerical algorithms that are efficient as well as straightforward to implement.[1] One of the schemes we discuss is applicable also to problems involving coefficient matrices with a high degree of singularity.

The solution of MLRE models has in the past few years received substantial attention in the literature. See, for example, Binder and Pesaran (1995, 1997), Broze, Gouriéroux and Szafarz (1995), Anderson, Hansen, McGrattan and Sargent (1996), Sims (1996), Anderson (1997), King and Watson (1997, 1998), Klein (1997), Uhlig (1997), and Zadrozny (1998). The primary focus of this research has been twofold: the derivation of readily applicable methods for the determination of the dimension of the solution set (Binder and Pesaran, 1995, 1997; Broze, Gouriéroux and Szafarz, 1995), and establishment of efficient algorithms for the numerical solution of MLRE models with a unique solution. In either case, the concern of this literature has been with infinite-horizon MLRE models having time-invariant solutions.[2] In this paper we consider finite-horizon MLRE models, which, as is well known, generally do not have time-invariant solutions, and therefore require a

[1] In the Appendix, we discuss application of one of these two schemes to the solution of a finite-horizon linear-quadratic optimal control problem. MLRE models that arise from finite-horizon linear-quadratic optimal control problems have been analyzed, for example, by Chow (1975), Kendrick (1981), and Aoki (1989). If the planning horizon is fixed, the numerical scheme discussed in the Appendix is likely to be significantly more efficient than the standard solution approach discussed in this literature, namely matrix Riccati equation based recursions.

[2] An exception is the work of Gilli and Pauletto (1997, 1998) that we became aware of after a first version of this paper had been completed. Gilli and Pauletto consider the solution of finite-horizon MLRE models as a step in the solution of large-scale nonlinear rational expectations models. While Gilli and Pauletto also discuss the link between the solution of finite-horizon MLRE models and the problem of solving sparse linear equation systems, the methods for the solution of such systems discussed in Gilli and Pauletto differ from those discussed in this paper. We comment on these differences in more detail in Section 5 below.

different solution approach. There is a large number of problems of substantial economic interest that give rise to finite-horizon MLRE models, such as the (log-linearized) optimality conditions of households' finite-horizon intertemporal optimization problems.[3] It is therefore important to have efficient methods available for the solution of finite-horizon MLRE models.

The remainder of this paper is organized as follows: Section 2 introduces the class of finite-horizon MLRE models with general lag- and lead-structure, and discusses in particular the various coefficient matrix singularities that can arise and that are all accommodated by our approach, at most requiring a simple generalized inverse based transformation of the original model. Section 3 describes a solution method based on backward recursions (the fully recursive method for the solution of MLRE models recently advanced in Binder and Pesaran, 1997). Section 4 shows how one may alternatively solve these models by rewriting them as sparse linear equation systems with a block-tridiagonal coefficient matrix structure. Section 5 considers two numerical schemes for the efficient solution of sparse linear equation systems with a block-tridiagonal coefficient matrix structure, and shows how they can be adapted for the efficient solution of finite-horizon MLRE models. These schemes are illustrated by applying them to a linear-quadratic adjustment cost problem involving expenditure shares in Section 6. As the expenditure shares naturally are subject to an adding-up constraint, this model is an example of an MLRE model with redundancies, that is, with a singularity in the canonical form coefficient matrix premultiplying the vector of current-period dependent variables. It is shown that one of the two numerical schemes of Section 5 is immediately applicable to this problem, and that the other is applicable upon using the generalized inverse based transformation introduced in Section 2. The application of one of the numerical schemes to a finite-horizon linear-quadratic optimal control problem is discussed in an Appendix. Some concluding remarks are offered in Section 7.

2 The Model

A general formulation of the MLRE model with a finite horizon can be written as[4]

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \mathbf{M}_{ij}\, E\left(\mathbf{z}_{t+\tau+j-i}|\Omega_{t+\tau-i}\right) - \mathbf{f}_{t+\tau} = \mathbf{0}_{r\times 1}, \qquad \tau = 0, 1, \ldots, T-t, \tag{2.1}$$

[3]For an analysis of a finite-horizon *nonlinear* rational expectations model that arises from the optimality conditions of a life-cycle consumption model possibly involving liquidity constraints, see Binder, Pesaran, and Samiei (1999). Prucha and Nadiri (1984, 1991) as well as Steigerwald and Stuart (1997) argue that firms may also operate with a finite (but fixed) planning horizon, and have static expectations beyond their planning horizon.

[4]While we formulate our model as one with a finite and shifting planning horizon (so that the terminal period is fixed), the solution method we suggest is also readily applicable to the case where the planning horizon is finite and fixed at all current and future dates. See Section 5 and the Appendix for a further discussion of the case of a fixed planning horizon.

where $E(\mathbf{z}_{t+\tau+j-i}|\Omega_{t+\tau-i})$ is given for $t+\tau+j-i>T$, $i=0,1,\ldots,n_1$, and $j=0,1,\ldots,n_2$. In (2.1), $\mathbf{z}_{t+\tau}$ denotes an $r\times 1$ dimensional vector of decision variables, $\mathbf{f}_{t+\tau}$ represents a vector of forcing variables of the same dimension, $\mathbf{M}_{ij}$ $(i=0,1,\ldots,n_1,\ j=0,1,\ldots,n_2)$ is an $r\times r$ dimensional matrix of fixed coefficients, $E(\cdot|\Omega_{t+\tau})$ is the conditional mathematical expectations operator, and $\Omega_{t+\tau}$ represents a non-decreasing information set at time $t+\tau$, containing (at least) current and lagged values of $\mathbf{z}_{t+\tau}$ and $\mathbf{f}_{t+\tau}$: $\Omega_{t+\tau}=\{\mathbf{z}_{t+\tau},\mathbf{z}_{t+\tau-1},\ldots;\mathbf{f}_{t+\tau},\mathbf{f}_{t+\tau-1},\ldots\}$. For the state variables among the decision variables in $\mathbf{z}_{t+\tau}$, we assume that the relevant initial conditions are given. We also assume that the forcing variables $\{\mathbf{f}_{t+\tau}\}$ are adapted to $\{\Omega_{t+\tau}\}$.[5]

Throughout this paper, we will base our analysis of (2.1) on the following second-order canonical form:

$$\mathbf{x}_{t+\tau}=\mathbf{A}\mathbf{x}_{t+\tau-1}+\mathbf{B}E(\mathbf{x}_{t+\tau+1}|\Omega_{t+\tau})+\mathbf{w}_{t+\tau},\qquad \tau=0,1,\ldots,T-t, \tag{2.2}$$

where

$$\mathbf{x}_{t+\tau}=\left(\mathbf{q}'_{t+\tau},\ \mathbf{q}'_{t+\tau-1},\ \ldots,\ \mathbf{q}'_{t+\tau-n_1+1}\right)',\ \mathbf{q}'_{t+\tau}=\left(\mathbf{z}'_{t+\tau},\ E\left(\mathbf{z}'_{t+\tau+1}|\Omega_{t+\tau}\right),\ \ldots,\ E\left(\mathbf{z}'_{t+\tau+n_2}|\Omega_{t+\tau}\right)\right)',$$

and $\mathbf{w}_{t+\tau}$ is an $m\times 1$ dimensional vector that consists of linear combinations of the elements of $\mathbf{f}_{t+\tau}$, with $m=n_1(n_2+1)r$. See, for example, Binder and Pesaran (1995, 1997) for a definition of the $m\times m$ dimensional matrices $\mathbf{A}$ and $\mathbf{B}$ in terms of the $\mathbf{M}_{ij}$'s $(i=0,1,\ldots,n_1,\ j=0,1,\ldots,n_2)$, as well as a precise definition of $\mathbf{w}_{t+\tau}$.

In (2.2), we have normalized the coefficient matrix associated with $\mathbf{x}_{t+\tau}$ to be the identity matrix. This is done without any loss of generality.[6] To see this, it is simpler to go back for a moment to (2.1), as it may be readily verified from the definition of $\mathbf{A}$ and $\mathbf{B}$ in terms of the $\mathbf{M}_{ij}$'s $(i=0,1,\ldots,n_1,\ j=0,1,\ldots,n_2)$ that the dimension of the null space of the coefficient matrix premultiplying $\mathbf{x}_{t+\tau}$ in (2.2) is equal to the dimension of the null space of $\mathbf{M}_{00}$ in (2.1).

If, as assumed above, $\mathbf{M}_{00}$ is a square matrix, then $\mathbf{M}_{00}$ must be nonsingular for the MLRE model (2.1) to be complete in the decision variables $\mathbf{z}_{t+\tau}$, $\tau=0,1,\ldots,T-t$: If $\mathbf{M}_{00}$ is singular, then it is not possible to uniquely solve for the elements of $\mathbf{z}_{t+\tau}$ as functions of the elements of the lagged $\mathbf{z}_{t+\tau}$'s, of the current and future expectations of the elements of $\mathbf{z}_{t+\tau}$, and of the forcing variables in $\mathbf{f}_{t+\tau}$. If one allows for the possibility that $\mathbf{M}_{00}$ is not a square matrix, but rather is a matrix of dimension, say, $q\times r$, $q>r$, then for the model equations to be consistent, it must be the case that $q-r$ rows of (2.1) are linear combinations of the remaining r rows of (2.1), and therefore are redundant.[7] If such redundancies are present, the rank of $\mathbf{M}_{00}$ must be equal to r for (2.1) to

[5] Note that this allows for many forms of nonstationarities and/or nonlinearities in the stochastic processes generating the forcing variables.

[6] Note that no rank restrictions have been imposed on the $\mathbf{M}_{ij}$'s $(i=0,1,\ldots,n_1,\ j=0,1,\ldots,n_2)$, and thus also $\mathbf{A}$ and $\mathbf{B}$.

[7] Note that the presence of an identity by itself would not cause $\mathbf{M}_{00}$ to be singular. An identity causes the

be complete. Premultiplying (2.1) by the generalized inverse of $\mathbf{M}_{00}$,

$$\mathbf{M}_{00}^{-} = \left(\mathbf{M}_{00}^{'}\mathbf{M}_{00}\right)^{-1}\mathbf{M}_{00}^{'}, \tag{2.3}$$

one then obtains

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\widetilde{\mathbf{M}}_{ij}E\left(\mathbf{z}_{t+\tau+j-i}|\Omega_{t+\tau-i}\right) - \mathbf{M}_{00}^{-}\mathbf{f}_{t+\tau} = \mathbf{0}_{r\times 1}, \qquad \tau = 0, 1, \ldots, T-t, \tag{2.4}$$

with $\widetilde{\mathbf{M}}_{ij} = \mathbf{M}_{00}^{-}\mathbf{M}_{ij}$, and thus $\widetilde{\mathbf{M}}_{00} = \mathbf{I}_r$, where $\mathbf{I}_r$ denotes the identity matrix of order r. Premultiplying (2.1) by the generalized inverse of $\mathbf{M}_{00}$ removes any redundancies in the model.[8] (Of course, in the case where $q = r$, the generalized inverse of $\mathbf{M}_{00}$ coincides with the inverse of $\mathbf{M}_{00}$.) Therefore, even in the case where a model contains redundancies, working with the canonical form (2.2) does not involve any loss of generality, and (2.2) can always be obtained from (2.1) without any model-specific transformations.[9]

3 A Backward Recursive Solution

As (2.2) is a finite-horizon MLRE model which generally does not have a time-invariant solution, the standard methods available in the literature for the solution of infinite-horizon MLRE models are not applicable to it. One approach to the solution of (2.2) would be to use backward recursions starting from time T. At time T, the solution for $\mathbf{x}_T$, given $\mathbf{x}_{T-1}$ and the terminal condition $E\left(\mathbf{x}_{T+1}|\Omega_T\right)$, is given by (2.2) for $\tau = T - t$:

$$\mathbf{x}_T = \mathbf{A}\mathbf{x}_{T-1} + \mathbf{B}E\left(\mathbf{x}_{T+1}|\Omega_T\right) + \mathbf{w}_T. \tag{3.1}$$

Proceeding recursively backward, we can obtain $\mathbf{x}_{T-1}$ as a function of $\mathbf{x}_{T-2}$, the terminal condition $E\left(\mathbf{x}_{T+1}|\Omega_T\right)$, and of $E\left(\mathbf{w}_T|\Omega_{T-1}\right)$ and $\mathbf{w}_{T-1}$. Combining (2.2) for $\tau = T - t - 1$ with (3.1), one readily obtains

$$\mathbf{x}_{T-1} = \left(\mathbf{I}_m - \mathbf{B}\mathbf{A}\right)^{-1}\left[\mathbf{A}\mathbf{x}_{T-2} + \mathbf{B}^2 E\left(\mathbf{x}_{T+1}|\Omega_T\right) + \mathbf{B}E\left(\mathbf{w}_T|\Omega_{T-1}\right) + \mathbf{w}_{T-1}\right]. \tag{3.2}$$

variance-covariance matrix of $\mathbf{f}_{t+\tau}$ to be singular, but does not present any new difficulty as far as the solution of (2.1) is concerned.

[8] For an example see Section 6.

[9] An alternative to (2.2) would be to use the first-order canonical form of Blanchard and Kahn (1980), also employed, for example, in the work of Sims (1996) and King and Watson (1997, 1998). While the Blanchard and Kahn canonical form without coefficient matrix singularity restrictions has the same level of generality as our second-order canonical form (2.2), reducing (2.1) to Blanchard and Kahn's first-order canonical form generally requires introducing new predetermined variables. If the second-order canonical form (2.2) is used, no predetermined variables need to be introduced, and the solution is directly obtained in terms of current-period decision variables.

Proceeding to period $T-2$, combining (2.2) for $\tau = T-t-2$ with (3.2), the solution for $\mathbf{x}_{T-2}$ is given by

$$\begin{aligned}\mathbf{x}_{T-2} &= \left[\mathbf{I}_m - \mathbf{B}\left(\mathbf{I}_m - \mathbf{BA}\right)^{-1}\mathbf{A}\right]^{-1} \\ &\cdot \left[\begin{array}{c} \mathbf{A}\mathbf{x}_{T-3} + \mathbf{B}\left(\mathbf{I}_m - \mathbf{BA}\right)^{-1}\mathbf{B}^2 E\left(\mathbf{x}_{T+1}|\Omega_T\right) + \mathbf{B}\left(\mathbf{I}_m - \mathbf{BA}\right)^{-1}\mathbf{B}E\left(\mathbf{w}_T|\Omega_{T-2}\right) \\ +\mathbf{B}\left(\mathbf{I}_m - \mathbf{BA}\right)^{-1}E\left(\mathbf{w}_{T-1}|\Omega_{T-2}\right) + \mathbf{w}_{T-2} \end{array}\right].\end{aligned}$$

The pattern of these backward recursions should be apparent. We thus have the following proposition, which extends Proposition 4.1 in Binder and Pesaran (1997) to finite-horizon MLRE models, and provides the solution for $\mathbf{x}_{t+\tau}$, $\tau = 0, 1, \ldots, T-t$.[10]

Proposition 3.1 [Backward Recursive Solution]
Consider the finite-horizon MLRE model (2.2). Let

$$\Phi_{T-t} = \mathbf{I}_m, \quad \Phi_{T-t-i} = \mathbf{I}_m - \mathbf{B}\Phi^{-1}_{T-t-i+1}\mathbf{A}, \qquad i = 1, 2, \ldots, T-t, \tag{3.3}$$

and

$$\Psi_T = \mathbf{B}E\left(\mathbf{x}_{T+1}|\Omega_T\right) + \mathbf{w}_T, \quad \Psi_{T-i} = \mathbf{B}\Phi^{-1}_{T-t-i+1}\Psi_{T-i+1} + \mathbf{w}_{T-i}, \qquad i = 1, 2, \ldots, T-t. \tag{3.4}$$

Suppose the matrices Φ_{T-t-i} are nonsingular for $i = 1, 2, \ldots, T-t$. Then the solution for $\mathbf{x}_{t+\tau}$ to (2.2) is given by:

$$\mathbf{x}_{t+\tau} = \Phi^{-1}_{\tau}\mathbf{A}\mathbf{x}_{t+\tau-1} + \Phi^{-1}_{\tau}E\left(\Psi_{t+\tau}|\Omega_{t+\tau}\right), \qquad \tau = 0, 1, \ldots, T-t. \tag{3.5}$$

Note that the solution in all periods is a linear combination of the initial and terminal values, and the conditional expectations of the forcing variables $\{\mathbf{f}_{t+\tau}\}$. As the forcing variables were assumed to be adapted to the information sets $\{\Omega_{t+\tau}\}$, then so will be the solution $\{\mathbf{x}_{t+\tau}\}$.

4 A Block-Tridiagonal System Representation

While Proposition 3.1 provides the solution to (2.2) as long as the matrices Φ_{T-t-i}, $i = 1, 2, \ldots, T-t$, are nonsingular, significant further insights into the solution of (2.2) can be obtained by

[10] Binder and Pesaran (1997) also discuss how this backward recursive method may be used to compute the unique stable solution (if it exists) of infinite-horizon MLRE models. Non-recursive methods may often be faster for the solution of such models than recursive solution methods (particularly for infinite-horizon MLRE models for which the solution at t could be sensitive to the value of $E\left(\mathbf{x}_{T-t+1}|\Omega_t\right)$ for large values of T). On the other hand, the fully recursive method we suggest in this paper does not involve any matrix similarity transformations, but only requires elementary matrix operations (addition, multiplication, inversion). It may therefore be easier to grasp and implement than those methods in the literature for the solution of infinite-horizon MLRE models that are based on matrix similarity transformations. Furthermore, it is often interesting to know for what values of T the solution of the infinite-horizon problem is robust to the choice of the terminal values.

relating the solution of (2.2) to the solution of sparse linear equation systems, rather than applying the backward recursive approach underlying Proposition 3.1. Accordingly, we stack the canonical form (2.2) for periods T, $T-1$, ..., t, and take conditional expectations with respect to Ω_t to obtain

$$\begin{pmatrix} \mathbf{I}_m & -\mathbf{A} & \mathbf{0}_m & \mathbf{0}_m & \cdots & \mathbf{0}_m & \mathbf{0}_m & \mathbf{0}_m \\ -\mathbf{B} & \mathbf{I}_m & -\mathbf{A} & \mathbf{0}_m & \cdots & \mathbf{0}_m & \mathbf{0}_m & \mathbf{0}_m \\ \mathbf{0}_m & -\mathbf{B} & \mathbf{I}_m & -\mathbf{A} & \cdots & \mathbf{0}_m & \mathbf{0}_m & \mathbf{0}_m \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0}_m & \mathbf{0}_m & \mathbf{0}_m & \mathbf{0}_m & \cdots & -\mathbf{B} & \mathbf{I}_m & -\mathbf{A} \\ \mathbf{0}_m & \mathbf{0}_m & \mathbf{0}_m & \mathbf{0}_m & \cdots & \mathbf{0}_m & -\mathbf{B} & \mathbf{I}_m \end{pmatrix} \begin{pmatrix} E(\mathbf{x}_T|\Omega_t) \\ E(\mathbf{x}_{T-1}|\Omega_t) \\ E(\mathbf{x}_{T-2}|\Omega_t) \\ \vdots \\ E(\mathbf{x}_{t+1}|\Omega_t) \\ \mathbf{x}_t \end{pmatrix}$$

$$= \begin{pmatrix} E(\mathbf{w}_T|\Omega_t) \\ E(\mathbf{w}_{T-1}|\Omega_t) \\ E(\mathbf{w}_{T-2}|\Omega_t) \\ \vdots \\ E(\mathbf{w}_{t+1}|\Omega_t) \\ \mathbf{A}\mathbf{x}_{t-1} + \mathbf{w}_t \end{pmatrix} + \begin{pmatrix} \mathbf{B}E(\mathbf{x}_{T+1}|\Omega_t) \\ \mathbf{0}_{m\times 1} \\ \mathbf{0}_{m\times 1} \\ \vdots \\ \mathbf{0}_{m\times 1} \\ \mathbf{0}_{m\times 1} \end{pmatrix}, \tag{4.1}$$

or, more simply,

$$\mathbf{C}\,\mathbf{g}_t = \mathbf{h}_t, \tag{4.2}$$

where the matrix $\mathbf{C}$ is of dimension $p \times p$, and $\mathbf{g}_t$ and $\mathbf{h}_t$ are $p \times 1$ dimensional vectors, with $p = m(T-t+1)$. To solve for $\mathbf{x}_t$ from (4.2), one needs to invert the matrix $\mathbf{C}$. (As we will show below, one of our two numerical schemes for the inversion of $\mathbf{C}$ in fact also yields the solutions for $\mathbf{x}_{t+\tau}$, $\tau = 1, 2, \ldots, T-t$.)

Considering the block-tridiagonal nature of the coefficient matrix $\mathbf{C}$, it is clear that (4.2) represents a sparse linear equation system. The sparsity of $\mathbf{C}$ may be yet more pronounced due to the canonical form matrices $\mathbf{A}$ and/or $\mathbf{B}$ being sparse also. The matrices $\mathbf{A}$ and/or $\mathbf{B}$ may be sparse due to the inherent dynamic structure of the model under consideration and/or because the process of constructing the canonical form (2.2) may introduce a large number of zero entries in $\mathbf{A}$ and/or $\mathbf{B}$. In the next section, we will show that by approaching the solution of (2.2) through the sparse linear equation system (4.2), one may obtain analytical conditions under which the recursions in Proposition 3.1 are well defined, and one may also achieve further gains in computational efficiency relative to carrying out the recursions of Proposition 3.1 if certain nonsingularity restrictions are satisfied.

5 Solution of Block-Tridiagonal Linear Equation Systems

Efficient inversion of block-tridiagonal matrices has been studied in the recent numerical analysis literature. We provide here - in the context of (4.2) - a brief discussion of two methods to accomplish such inversions. One of these methods leads to the same recursions as in Proposition 3.1, but readily allows to establish analytical conditions under which these recursions are well defined. The second method, when applicable and when the planning horizon is fixed, is likely to lead to further gains in computational efficiency, compared to carrying out the recursions of Proposition 3.1. Like the recursions of Proposition 3.1, both methods are straightforward to implement, as they involve only elementary matrix operations. Also the first method is applicable regardless of whether the block-subdiagonal matrix $\mathbf{B}$ is singular. The second method applies, however, only if $\mathbf{B}$ is nonsingular.[11] Before proceeding, we might also note that our discussion is not meant to be a comprehensive guide to the inversion of block-tridiagonal matrices, but rather simply intends to convey how finite-horizon MLRE models may be efficiently solved using numerical schemes from the literature on the inversion of block-tridiagonal matrices.

Case 1: $\mathbf{B}$ Possibly Singular

Consider first the case where $\mathbf{B}$ is possibly singular. The standard textbook approach to solving linear equation systems of the form $\mathbf{C}\mathbf{g}_t = \mathbf{h}_t$, defined by (4.2), is by means of Gaussian elimination, or equivalently, $\mathbf{LU}$-factorization; namely factorization of $\mathbf{C}$ into the product of a block lower triangular matrix $\mathbf{L}^*$ and a block upper triangular matrix $\mathbf{U}^*$, and then solving the resultant block-triangular equation systems

$$\mathbf{L}^*\boldsymbol{\mu}_t = \mathbf{h}_t, \tag{5.1}$$

(by forward substitution), and

$$\mathbf{U}^*\mathbf{g}_t = \boldsymbol{\mu}_t, \tag{5.2}$$

(by backward substitution). Such an algorithm does not exploit the sparse nature of $\mathbf{C}$, however, and in general can be quite inefficient. An alternative numerical procedure that utilizes the block-tridiagonal structure of $\mathbf{C}$ is the $\mathbf{LDU}$-factorization, discussed, for example, in Axelsson (1994).

[11] In the special case where $m = 1$, techniques specifically geared towards the inversion of tridiagonal matrices are also available. One of these techniques has been used by Prucha and Nadiri (1991) to solve finite-horizon univariate linear factor demand models. Some alternative schemes for the inversion of block-tridiagonal matrices to the ones outlined here are discussed in Gilli and Pauletto (1997, 1998). The methods discussed here have the advantage that analytical conditions are known under which they are operational. The nonstationary iterative methods presented in Gilli and Pauletto may have smaller storage requirements, however, and could therefore be attractive for large-scale models of the type discussed in Gilli and Pauletto.

Decompose $\mathbf{C}$ as

$$\mathbf{C} = \mathbf{D_c} + \mathbf{L_c} + \mathbf{U_c}, \tag{5.3}$$

where $\mathbf{D_c}$, $\mathbf{L_c}$, and $\mathbf{U_c}$ are the block-diagonal, block-subdiagonal, and block-superdiagonal entries in $\mathbf{C}$, and consider the factorization

$$\mathbf{C} = \mathbf{L}\mathbf{D}^{-1}\mathbf{U}, \tag{5.4}$$

where

$$\mathbf{L} = \mathbf{D} + \mathbf{L_c}, \tag{5.5}$$

and

$$\mathbf{U} = \mathbf{D} + \mathbf{U_c}. \tag{5.6}$$

Noting that

$$\mathbf{C} = (\mathbf{D} + \mathbf{L_c})\,\mathbf{D}^{-1}\,(\mathbf{D} + \mathbf{U_c}) = \mathbf{D} + \mathbf{U_c} + \mathbf{L_c} + \mathbf{L_c}\mathbf{D}^{-1}\mathbf{U_c}, \tag{5.7}$$

it is easily seen that $\mathbf{D} = \mathbf{D_c} - \mathbf{L_c}\mathbf{D}^{-1}\mathbf{U_c}$, and thus the matrix $\mathbf{D}$ satisfies the recursions

$$\mathbf{D}_1 = \mathbf{C}_{11}, \quad \mathbf{D}_i = \mathbf{C}_{ii} - \mathbf{C}_{i,i-1}\mathbf{D}_{i-1}^{-1}\mathbf{C}_{i-1,i}, \qquad i = 2,\ 3,\ \ldots,\ T-t+1, \tag{5.8}$$

where $\mathbf{C}_{ii}$ denotes the (i,i)-th block of $\mathbf{C}$, and $\mathbf{D}_i$ the i-th diagonal block of $\mathbf{D}$.

As discussed in Axelsson (1994), sufficient conditions for the recursions in (5.8) to be well defined are: (i) $\mathbf{C}$ is symmetric positive definite, or (ii) $\mathbf{C}$ is a block $\mathbf{H}$-matrix.[12]

Having factorized $\mathbf{C}$ as in (5.4), it is then a simple step to solve for $\mathbf{g}_t$ by splitting $\mathbf{C}\mathbf{g}_t = \mathbf{h}_t$ into

$$\mathbf{L}\boldsymbol{\eta}_t = \mathbf{h}_t \quad \text{or} \quad \boldsymbol{\eta}_{t,1} = \mathbf{D}_1^{-1}\mathbf{h}_{t,1},\ \boldsymbol{\eta}_{t,i} = \mathbf{D}_i^{-1}\left(\mathbf{h}_{t,i} - \mathbf{C}_{i,i-1}\boldsymbol{\eta}_{t,i-1}\right), \quad i = 2,\ 3,\ \ldots,\ T-t+1, \tag{5.9}$$

and

$$\mathbf{D}^{-1}\mathbf{U}\mathbf{g}_t = \boldsymbol{\eta}_t \quad \text{or} \quad \mathbf{g}_{t,T-t+1} = \boldsymbol{\eta}_{t,T-t+1},\ \mathbf{g}_{t,i} = \boldsymbol{\eta}_{t,i} - \mathbf{D}_i^{-1}\mathbf{C}_{i,i+1}\mathbf{g}_{t,i+1}, \quad i = T-t,\ T-t-1,\ \ldots,\ 1. \tag{5.10}$$

The solution $\mathbf{x}_t$ to the MLRE model (2.2) is given by the last m entries in $\mathbf{g}_t$. We therefore have the following proposition:

[12] See Axelsson (1994, Chapters 6 and 7) for a definition of block $\mathbf{H}$-matrices. The class of block $\mathbf{H}$-matrices encompasses, but is not restricted to, matrices that are (generalized) diagonally dominant.

Proposition 5.1 **[Solution Based on LDU-Factorization]**
Consider the finite-horizon MLRE model (2.2). Let

$$\Theta_i = \mathbf{I}_m - \mathbf{B}\Theta_{i-1}^{-1}\mathbf{A}, \qquad i = 2,\ 3,\ \ldots,\ T-t+1, \tag{5.11}$$

and

$$\Gamma_i = \Theta_i^{-1}\left(\mathbf{B}\Gamma_{i-1} + \mathbf{w}_{T+1-i}\right), \qquad i = 2,\ 3,\ \ldots,\ T-t+1, \tag{5.12}$$

with the initial conditions $\Theta_1 = \mathbf{I}_m$, *and* $\Gamma_1 = \mathbf{B}E\left(\mathbf{x}_{T+1}|\Omega_T\right) + \mathbf{w}_T$. *Suppose the matrices* Θ_i *are nonsingular for* $i = 2,\ 3,\ \ldots,\ T-t+1$. *Then the solution for* $\mathbf{x}_t$ *to (2.2) is given by:*

$$\mathbf{x}_t = \Theta_{T-t+1}^{-1}\mathbf{A}\mathbf{x}_{t-1} + E\left(\Gamma_{T-t+1}|\Omega_t\right). \tag{5.13}$$

It is easily verified that the recursions in (5.11) and (5.12) match the recursions in Proposition 3.1. The i-th recursion matrices Θ_i and Γ_i in Proposition 5.1 are related to the i-th recursion matrices $\Phi_{T-t+1-i}$ and Ψ_{T+1-i} in Proposition 3.1 as follows: $\Theta_i = \Phi_{T-t+1-i}$, $i = 1,\ 2,\ \ldots,\ T-t+1$, and $\Gamma_i = \Phi_{T-t+1-i}^{-1}\Psi_{T+1-i}$, $i = 1,\ 2,\ \ldots,\ T-t+1$. Notice that this equivalence also implies that inversion of $\mathbf{C}$ does not only yield the solution for $\mathbf{x}_t$, but also yields the solution for $\{\mathbf{x}_{t+\tau}\}_{\tau=1}^{T-t}$. We have

$$\mathbf{x}_{t+\tau} = \Theta_{T-t+1-\tau}^{-1}\mathbf{A}\mathbf{x}_{t+\tau-1} + E\left(\Gamma_{T-t+1-\tau}|\Omega_{t+\tau}\right), \qquad \tau = 1, 2, \ldots, T-t. \tag{5.14}$$

Proposition 5.1 provides a link between the recursions of Proposition 3.1 and the **LDU**-factorization, and establishes conditions under which these recursions are well defined. Notice that for these recursions to be well defined, it is by no means necessary that the coefficient matrices $\mathbf{A}$ and $\mathbf{B}$ are nonsingular.[13]

While the recursions in (5.8) to (5.10) exploit the block-tridiagonal structure of $\mathbf{C}$, the $\mathbf{D}_i$ matrices in general are full, even if the superdiagonal, diagonal and subdiagonal blocks of $\mathbf{C}$ are sparse, as the inverse of a sparse matrix is, in general, full. A fully efficient solution scheme to $\mathbf{C}\mathbf{g}_t = \mathbf{h}_t$ in the case where $\mathbf{A}$ and/or $\mathbf{B}$ are sparse will therefore also incorporate sparse approximations of the inverses $\mathbf{D}_i^{-1}$, $i = 1,\ 2,\ \ldots,\ T-t+1$, and sparse approximations of the matrix product terms in the recursions in (5.8), $\mathbf{C}_{i,i-1}\mathbf{D}_{i-1}^{-1}\mathbf{C}_{i-1,i}$. A variety of numerical schemes accomplishing this by allowing the user to control the sparse blocks are discussed in Axelsson (1994, Chapter 8).

[13] While we have not encountered in any application that we have considered near-singularity or singularity of any of the Θ_i matrices, if such (near-) singularities did arise, the recursions in Proposition 5.1 could be stabilized using techniques developed in the recent numerical analysis literature on (near-) rank-deficient problems. See, for example, Hansen (1998), for an up-to-date survey.

Case 2: **B** Nonsingular

In the case where the block-subdiagonal matrix $\mathbf{B}$ is nonsingular, one may also solve the finite-horizon MLRE model (2.2) by adapting the recursions suggested in Bowden (1989) for the inversion of block-tridiagonal matrices:

Proposition 5.2 **[Solution Based on Bowden's Procedure]**
Consider the finite-horizon MLRE model (2.2) with the coefficient matrix $\mathbf{B}$ *nonsingular. Then the solution for* $\mathbf{x}_t$ *to (2.2) is given by:*

$$\mathbf{x}_t = \Upsilon_{T-t+1}\mathbf{A}\mathbf{x}_{t-1} + \sum_{i=0}^{T-t} \Upsilon_{T-t+1-i} E\left(\mathbf{w}_{t+i}|\Omega_t\right) + \Upsilon_1 \mathbf{B} E\left(\mathbf{x}_{T+1}|\Omega_t\right), \tag{5.15}$$

where

$$\Upsilon_i = \mathbf{F}_{T-t+2}^{-1}\mathbf{F}_i, \qquad i = 1, 2, \ldots, T-t+1, \tag{5.16}$$

$$\mathbf{F}_1 = \mathbf{I}_m, \quad \mathbf{F}_2 = \mathbf{B}^{-1}, \quad \mathbf{F}_{i+1} = \left(\mathbf{F}_i - \mathbf{F}_{i-1}\mathbf{A}_{i-1}\right)\mathbf{B}^{-1}, \quad i = 2, 3, \ldots, T-t, \tag{5.17}$$

and

$$\mathbf{F}_{T-t+2} = \left(\mathbf{F}_{T-t+1} - \mathbf{F}_{T-t}\mathbf{A}\right). \tag{5.18}$$

A proof of Proposition 5.2 can be constructed following the arguments in Bowden (1989).

Note that Bowden's procedure merely requires the inversion of two $m \times m$ dimensional matrices. It is clearly an effective and straightforward method for the solution of the MLRE model, (2.2). To compute $\{\mathbf{x}_{t+\tau}\}_{\tau=1}^{T-t}$, however, the analog of (4.1) needs to be constructed for periods $t+1$, $t+2$, ..., $T-1$, before the solution technique of Proposition 5.2 can be applied. Therefore, Bowden's procedure will typically be less efficient for the computation of $\{\mathbf{x}_{t+\tau}\}_{\tau=1}^{T-t}$ than the **LDU**-factorization based procedure of Proposition 5.1. However, in the case of MLRE models with a finite and fixed planning horizon at all current and future dates, the structure of the matrix $\mathbf{C}$ in (4.2) remains unchanged in all periods, and Bowden's procedure is likely to be computationally more efficient than the **LDU**-factorization based procedure.

6 An Illustration: A Consumer's Optimal Expenditure Shares

In this section, we illustrate Proposition 5.1 and Proposition 5.2 by applying them to the solution of a model of a consumer's optimal expenditure shares if share adjustment to the target level is costly both in terms of the level of adjustment and in terms of the speed of adjustment.[14]

[14]See Pesaran (1991) for a detailed discussion of adjustment costs both for the level and the speed of adjustment. See also Price (1992) and Binder and Pesaran (1995).

Consider the following finite-horizon adjustment cost problem:

$$\min_{\{\mathbf{s}_{t+\tau}\}_{\tau=0}^{T-t}} E\left\{\sum_{\tau=0}^{T-t}\beta^{\tau}\left[\begin{array}{c}\left(\mathbf{s}_{t+\tau}-\mathbf{s}_{t+\tau}^{*}\right)^{'}\mathbf{H}\left(\mathbf{s}_{t+\tau}-\mathbf{s}_{t+\tau}^{*}\right)+\triangle\,\mathbf{s}_{t+\tau}^{'}\mathbf{G}\triangle\,\mathbf{s}_{t+\tau}\\ +\triangle^{2}\,\mathbf{s}_{t+\tau}^{'}\mathbf{K}\triangle^{2}\,\mathbf{s}_{t+\tau}\end{array}\right]|\Omega_{t}\right\} \tag{6.1}$$

for given initial and terminal conditions $\mathbf{s}_{t-1}$, $\mathbf{s}_{t-2}$, $E\left(\mathbf{s}_{T+1}|\Omega_{T-1}\right)$, $E\left(\mathbf{s}_{T+1}|\Omega_{T}\right)$, and $E\left(\mathbf{s}_{T+2}|\Omega_{T}\right)$, and subject to

$$\boldsymbol{\iota}^{'}\mathbf{s}_{t+\tau}=1,\qquad \tau=0,1,\ldots,T-t. \tag{6.2}$$

In (6.1) and (6.2), $\mathbf{s}_{t+\tau}$ is an $r\times 1$ dimensional vector of the consumer's expenditure shares, $\mathbf{H}$, $\mathbf{G}$, and $\mathbf{K}$ are $r\times r$ dimensional symmetric matrices of fixed coefficients, $\beta\in(0,1)$ is a constant discount factor, $\boldsymbol{\iota}$ is an $r\times 1$ dimensional vector of ones, and $\mathbf{s}_{t+\tau}^{*}$ is a vector of desired (target) expenditure shares, derived, for example, from the Almost Ideal Demand System of Deaton and Muellbauer (1980),

$$\mathbf{s}_{t+\tau}^{*}=\boldsymbol{\alpha}+\Gamma\ln\mathbf{p}_{t+\tau}+\boldsymbol{\delta}\ln\left(\frac{y_{t+\tau}}{p_{t+\tau}}\right),\qquad \tau=0,1,\ldots,T-t, \tag{6.3}$$

where

$$\boldsymbol{\alpha}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_r\end{pmatrix},\quad \Gamma=\begin{pmatrix}\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1r}\\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2r}\\ \vdots & \vdots & & \vdots\\ \gamma_{r1} & \gamma_{r2} & \cdots & \gamma_{rr}\end{pmatrix},$$

$$\boldsymbol{\delta}=\begin{pmatrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_r\end{pmatrix},\quad\text{and}\quad \ln\mathbf{p}_{t+\tau}=\begin{pmatrix}\ln p_{1,t+\tau}\\ \ln p_{2,t+\tau}\\ \vdots\\ \ln p_{r,t+\tau}\end{pmatrix}.$$

In (6.3), $p_{i,t+\tau}$ is the price deflator of commodity group i, $y_{t+\tau}$ is the consumer's expenditure on all the commodities, and $p_{t+\tau}$ is the general price index, approximated using the Stone formula

$$\ln p_{t+\tau}=\boldsymbol{\omega}_{0}^{'}\ln\mathbf{p}_{t+\tau}, \tag{6.4}$$

where $\boldsymbol{\omega}_0=\left(\omega_{10},\,\omega_{20},\,...,\,\omega_{r0}\right)^{'}$, and ω_{i0} is the budget share of the i-th commodity group in the base year. Consumer theory imposes the following restrictions on the parameters of the target share equations:

(a) adding-up restrictions:

$$\sum_{i=1}^{r}\alpha_i=1,\quad \sum_{i=1}^{r}\gamma_{ij}=0,\quad\text{and}\quad \sum_{i=1}^{r}\delta_i=0, \tag{6.5}$$

(b) homogeneity restrictions:

$$\sum_{j=1}^{r} \gamma_{ij} = 0, \tag{6.6}$$

and

(c) symmetry restrictions:

$$\gamma_{ij} = \gamma_{ji}, \qquad j \neq i. \tag{6.7}$$

We also assume that for given observations on $\mathbf{p}_{t+\tau}$ and $y_{t+\tau}$, the parameters in $\boldsymbol{\alpha}$, Γ, and $\boldsymbol{\delta}$ are such that the desired shares $s^*_{i,t+\tau}$ in (6.3) lie in the range $[0,1]$ for $i = 1, 2, \ldots, r$, and $\tau = 0, 1, \ldots, T-t$. To compute $E\left(\mathbf{s}^*_{t+\tau}|\Omega_t\right)$, we also assume that $\mathbf{m}_t = (\ln p_{1t}, \ln p_{2t}, \ldots, \ln p_{rt}, \ln y_t)'$ follows the vector autoregressive process of order s:

$$\mathbf{m}_t = \mathbf{a} + \sum_{i=1}^{s} \mathbf{R}_i \mathbf{m}_{t-i} + \mathbf{v}_t, \qquad \mathbf{v}_t \sim i.i.d.\ N\left(\mathbf{0}_{r\times 1}, \Sigma_{\mathbf{v}}\right), \tag{6.8}$$

for all t.

Forming the Lagrangian for (6.1) to (6.8), the Euler equations at time $t+\tau$ can after some algebraic manipulation be written as[15]

$$\mathbf{M}_{00}\mathbf{s}_{t+\tau} = \mathbf{N}\left[\begin{array}{c} \mathbf{M}_{10}\mathbf{s}_{t+\tau-1} + \mathbf{M}_{20}\mathbf{s}_{t+\tau-2} + \mathbf{M}_{01}E\left(\mathbf{s}_{t+\tau+1}|\Omega_{t+\tau}\right) \\ +\mathbf{M}_{02}E\left(\mathbf{s}_{t+\tau+2}|\Omega_{t+\tau}\right) + \mathbf{H}\mathbf{s}^*_{t+\tau} \end{array}\right] + \theta\boldsymbol{\iota}, \tag{6.9}$$

$\tau = 0, 1, \ldots, T-t$, where $\mathbf{M}_{00} = \mathbf{H} + (1+\beta)\,\mathbf{G} + \left(1 + 4\beta + \beta^2\right)\mathbf{K}$, $\mathbf{M}_{10} = \mathbf{G} + 2\,(1+\beta)\,\mathbf{K}$, $\mathbf{M}_{20} = -\mathbf{K}$, $\mathbf{M}_{01} = \beta\mathbf{M}_{10}$, $\mathbf{M}_{02} = \beta^2\mathbf{M}_{20}$, $\mathbf{N} = \mathbf{I}_r - \theta\boldsymbol{\iota}\boldsymbol{\iota}'\mathbf{M}_{00}^{-1}$, and $\theta = 1/(\boldsymbol{\iota}'\mathbf{M}_{00}^{-1}\boldsymbol{\iota}) > 0$.[16] Noting that $\boldsymbol{\iota}'\mathbf{M}_{00}^{-1}\mathbf{N} = \mathbf{0}_r$, it is easily seen that $\boldsymbol{\iota}'\mathbf{s}_{t+\tau} = 1$, for all τ, as it should. The Euler equations (6.9) constitute a special case of the finite-horizon MLRE model (2.1), and can be efficiently solved using Proposition 5.1, or, in the case where the adjustment costs are of first-order only, $\mathbf{K} = \mathbf{0}_r$, using Proposition 5.2.

Case 1: $\mathbf{K} \neq \mathbf{0}_r$

To apply Proposition 5.1 (the solution method based on the **LDU**-factorization), we just need to rewrite (6.9) so that it fits the canonical form (2.2). Let $\mathbf{x}_{t+\tau} = \left(\mathbf{s}'_{t+\tau}, \mathbf{s}'_{t+\tau-1}, E\left(\mathbf{s}'_{t+\tau+1}|\Omega_{t+\tau}\right)\right)'$. Then

$$\mathbf{x}_{t+\tau} = \mathbf{A}\mathbf{x}_{t+\tau-1} + \mathbf{B}E\left(\mathbf{x}_{t+\tau+1}|\Omega_{t+\tau}\right) + \mathbf{w}_{t+\tau}, \tag{6.10}$$

[15]We do not explicitly impose the conditions $0 \leq s_{i,t+\tau} \leq 1$, $i = 1, 2, \ldots, r$, and $\tau = 0, 1, \ldots, T-t$ in (6.9). We thus implicitly assume that the deviations of the shares from their target levels are sufficiently small. If any of the inequality constraints $0 \leq s_{i,t+\tau} \leq 1$, $i = 1, 2, \ldots, r$, and $\tau = 0, 1, \ldots, T-t$ was violated in the absence of these constraints, the model would become nonlinear, necessitating application of the type of techniques discussed in Binder, Pesaran, and Samiei (1999).

[16]Notice that $\mathbf{M}_{00}$ must be a positive definite matrix if the second-order conditions for the global optimality of the solution to the adjustment cost problem are to be met.

with

$$\mathbf{w}_{t+\tau} = \left(\left[\mathbf{M}_{00}^{-1}\mathbf{N}\mathbf{H}\mathbf{s}_{t+\tau}^{*} + \theta\mathbf{M}_{00}^{-1}\boldsymbol{\iota}\right]^{'},\ \mathbf{0}_{r\times 1}^{'},\ \mathbf{0}_{r\times 1}^{'}\right)^{'},$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{M}_{00}^{-1}\mathbf{N}\mathbf{M}_{10} & \mathbf{M}_{00}^{-1}\mathbf{N}\mathbf{M}_{20} & \mathbf{0}_r \\ \mathbf{I}_r & \mathbf{0}_r & \mathbf{0}_r \\ \mathbf{0}_r & \mathbf{0}_r & \mathbf{0}_r \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} \mathbf{M}_{00}^{-1}\mathbf{N}\mathbf{M}_{01} & \mathbf{0}_r & \mathbf{M}_{00}^{-1}\mathbf{N}\mathbf{M}_{02} \\ \mathbf{0}_r & \mathbf{0}_r & \mathbf{0}_r \\ \mathbf{I}_r & \mathbf{0}_r & \mathbf{0}_r \end{pmatrix}.$$

Proposition 5.1 can now be readily applied for the solution of (6.10).[17] Notice again that Proposition 5.1 does not require $\mathbf{A}$ and/or $\mathbf{B}$ to be nonsingular, and that $E\left(\mathbf{w}_{t+\tau+h}|\Omega_{t+\tau}\right)$ can be obtained noting that $E\left(\mathbf{s}_{t+\tau+h}^{*}|\Omega_{t+\tau}\right) = \boldsymbol{\alpha} + \left(\Gamma - \boldsymbol{\delta}\boldsymbol{\omega}_0^{'},\ \boldsymbol{\delta}\right) E\left(\mathbf{m}_{t+\tau+h}|\Omega_{t+\tau}\right)$, and using (6.8).

Case 2: $\mathbf{K} = \mathbf{0}_r$

If the adjustment costs have a first-order structure only, then the Euler equations (6.9) immediately become a special case of the canonical form (2.2):

$$\mathbf{s}_{t+\tau} = \mathbf{A}\mathbf{s}_{t+\tau-1} + \mathbf{B}E\left(\mathbf{s}_{t+\tau+1}|\Omega_{t+\tau}\right) + \mathbf{w}_{t+\tau}, \tag{6.11}$$

with $\mathbf{A} = \mathbf{M}_{\mathbf{00}}^{-\mathbf{1}}\mathbf{N}\mathbf{M}_{10}$, $\mathbf{B} = \mathbf{M}_{\mathbf{00}}^{-\mathbf{1}}\mathbf{N}\mathbf{M}_{01} = \beta\mathbf{A}$, and $\mathbf{w}_{t+\tau} = \mathbf{M}_{00}^{-1}\mathbf{N}\mathbf{H}\mathbf{s}_{t+\tau}^{*} + \theta\mathbf{M}_{00}^{-1}\boldsymbol{\iota}$. As $\mathbf{M}_{00}^{-1}\mathbf{N}$ is singular (recall that $\boldsymbol{\iota}^{'}\mathbf{M}_{00}^{-1}\mathbf{N} = \mathbf{0}_r$), so will be $\mathbf{B}$. In order to apply Proposition 5.2 (the solution method based on Bowden's procedure), we therefore first need to transform (6.11) so as to remove this singularity of $\mathbf{B}$ that is due to the constraint $\boldsymbol{\iota}'\mathbf{s}_{t+\tau}$=1. Observing this constraint, note that we can decompose the vector of expenditure shares $\mathbf{s}_{t+\tau}$ as

$$\mathbf{s}_{t+\tau} = \mathbf{e} + \mathbf{P}\tilde{\mathbf{s}}_{t+\tau}, \tag{6.12}$$

where $\tilde{\mathbf{s}}_{t+\tau} = \left(s_{1,t+\tau},\ s_{2,t+\tau},\ \ldots,\ s_{r-1,t+\tau}\right)^{'}$,

$$\mathbf{e} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & -1 \end{pmatrix},$$

[17] GAUSS- and MATLAB-programs illustrating implementation of our solution methods for the model of this section may be downloaded from the following URL: http://www.inform.umd.edu/econ/mbinder/research.

with $\mathbf{e}$ being of dimension $r \times 1$, and $\mathbf{P}$ being of dimension $r \times (r-1)$. It is clear that $\mathbf{P}$ is of rank $r-1$, and that $\mathbf{P}'\mathbf{P}$ is nonsingular. Replacing $\mathbf{s}_{t+\tau}$ from (6.12) in (6.11), we now have

$$\mathbf{P}\widetilde{\mathbf{s}}_{t+\tau} = \mathbf{A}\mathbf{P}\widetilde{\mathbf{s}}_{t+\tau-1} + \mathbf{B}\mathbf{P}E(\widetilde{\mathbf{s}}_{t+\tau+1} \mid \Omega_{t+\tau}) + \mathbf{w}^*_{t+\tau}, \tag{6.13}$$

where

$$\mathbf{w}^*_{t+\tau} = \mathbf{w}_{t+\tau} + (\mathbf{A} + \mathbf{B} - \mathbf{I}_r)\mathbf{e}.$$

Equation system (6.13) is a special case of the type of MLRE model with redundancies discussed in Section 2: The last row of (6.13) is a linear combination of the first $r-1$ rows of (6.13). To remove this redundancy, we premultiply (6.13) by the generalized inverse of $\mathbf{P}$ to obtain

$$\widetilde{\mathbf{s}}_{t+\tau} = \widetilde{\mathbf{A}}\widetilde{\mathbf{s}}_{t+\tau-1} + \widetilde{\mathbf{B}}E(\widetilde{\mathbf{s}}_{t+\tau+1} \mid \Omega_{t+\tau}) + \widetilde{\mathbf{w}}^*_{t+\tau}, \tag{6.14}$$

where $\widetilde{\mathbf{A}} = (\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'\mathbf{A}\mathbf{P}$, $\widetilde{\mathbf{B}} = (\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'\mathbf{B}\mathbf{P} = \beta\widetilde{\mathbf{A}}$, and $\widetilde{\mathbf{w}}^*_{t+\tau} = (\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'\mathbf{w}^*_{t+\tau}$. Proposition 5.2. can be readily applied to obtain the numerical solution of (6.14) if $\widetilde{\mathbf{B}}$, or, equivalently, if $\mathbf{P}'\mathbf{B}\mathbf{P}$ is nonsingular.[18]

But since (for $\mathbf{K} = \mathbf{0_r}$)

$$\mathbf{B} = \mathbf{M}_{00}^{-1}\mathbf{N}\mathbf{M}_{01} = \beta\mathbf{M}_{00}^{-1}(\mathbf{I}_r - \theta\boldsymbol{\iota}\boldsymbol{\iota}'\mathbf{M}_{00}^{-1})\mathbf{G},$$

we have

$$\mathbf{P}'\mathbf{B}\mathbf{P} = \beta\widetilde{\mathbf{P}}'(\mathbf{I}_r - \widetilde{\boldsymbol{\iota}}(\widetilde{\boldsymbol{\iota}}'\widetilde{\boldsymbol{\iota}})^{-1}\widetilde{\boldsymbol{\iota}}')\widetilde{\mathbf{G}}\widetilde{\mathbf{P}},$$

where

$$\widetilde{\mathbf{P}} = \mathbf{M}_{00}^{-1/2}\mathbf{P}, \;\; \widetilde{\boldsymbol{\iota}} = \mathbf{M}_{00}^{-1/2}\boldsymbol{\iota}, \;\text{ and }\; \widetilde{\mathbf{G}} = \mathbf{M}_{00}^{-1/2}\mathbf{G}\mathbf{M}_{00}^{1/2}.$$

It is now easily seen that $rank\,(\widetilde{\mathbf{P}}) = rank\,(\mathbf{I}_r - \widetilde{\boldsymbol{\iota}}(\widetilde{\boldsymbol{\iota}}'\widetilde{\boldsymbol{\iota}})^{-1}\widetilde{\boldsymbol{\iota}}') = r-1$. Therefore, for $\mathbf{P}'\mathbf{B}\mathbf{P}$ to be nonsingular, $\widetilde{\mathbf{G}}$ or $\mathbf{G}$ must also have at least rank $r-1$.

Once $\{\widetilde{\mathbf{s}}_{t+\tau}\}_{\tau=0}^{T-t}$ is computed, one may obtain the r-th expenditure share as

$$\{s_{r,t+\tau}\}_{\tau=0}^{T-t} = \left\{1 - \sum_{i=1}^{r-1} s_{i,t+\tau}\right\}_{\tau=0}^{T-t}.$$

7 Conclusion

In this paper, we have shown that the numerical solution of finite-horizon MLRE models can be reduced to the problem of solving sparse linear equation systems with a block-tridiagonal coefficient matrix structure. The latter problem has been discussed in the recent numerical analysis literature,

[18] If $\widetilde{\mathbf{B}}$ is singular, one can, of course, still apply Proposition 5.1 either to (6.14) or to (6.11) directly.

and there are efficient algorithms available for this purpose. We have described two such numerical schemes in this paper. Their application to finite-horizon MLRE models yields a procedure for the solution of these models that is efficient, straightforward to implement, and allows investigation of a rich array of specifications for the forcing variables of the model, as the latter are not restricted to be generated by linear and/or covariance-stationary processes. Furthermore, in the case of one of the numerical schemes we have described, the procedure is applicable also to models where the coefficient matrices in their companion canonical form involve a high degree of singularity.

Appendix: Solution of a Finite-Horizon Linear-Quadratic Optimal Control Problem

In this Appendix, we consider the solution of the following finite-horizon linear-quadratic optimal control problem:[19]

$$\min_{\{\mathbf{u}_{t+\tau}\}_{\tau=0}^{T-t-1}} \frac{1}{2} E\left[\mathbf{y}_T^{'}\mathbf{Z}\mathbf{y}_T + \sum_{\tau=0}^{T-t-1} \mathbf{y}_{t+\tau}^{'}\mathbf{Q}\mathbf{y}_{t+\tau} + \mathbf{u}_{t+\tau}^{'}\mathbf{R}\mathbf{u}_{t+\tau}|\Omega_t\right] \tag{A.1}$$

for given $\mathbf{y}_t$, and subject to

$$\mathbf{y}_{t+\tau} = \mathbf{V}\mathbf{y}_{t+\tau-1} + \mathbf{W}\mathbf{u}_{t+\tau-1} + \Xi\boldsymbol{\varepsilon}_{t+\tau}, \qquad \boldsymbol{\varepsilon}_{t+\tau} \overset{iid}{\sim} N\left(\mathbf{0}_{g\times 1}, \Sigma_{\varepsilon}\right), \quad \tau = 1, 2, \ldots, T-t, \tag{A.2}$$

where $\mathbf{y}_{t+\tau}$ is the $d \times 1$ dimensional state vector, $\mathbf{u}_{t+\tau}$ the $f \times 1$ dimensional control vector, $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{V}$ are $d \times d$ dimensional matrices of fixed coefficients, and the coefficient matrices $\mathbf{R}$, $\mathbf{W}$, and Ξ are of dimensions $f \times f$, $d \times f$, and $d \times g$, respectively. As is usually assumed, we take the coefficient matrices $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ as being symmetric and nonsingular. Terminal conditions in practice are typically imposed by choosing sufficiently large values for the appropriate elements of $\mathbf{Z}$.[20] While the objective function (A.1) does not involve cross-products of components of the state vector and the control vector, this is without loss of generality, as finite-horizon linear-quadratic optimal control problems involving such cross-product terms can always be transformed to the form of (A.1) to (A.2).[21] The information set Ω_t is specified as follows: $\Omega_t = \left\{\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots; \mathbf{u}_t, \mathbf{u}_{t-1}, \ldots; \boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_{t-1}, \ldots\right\}$. Invoking certainty equivalence, we evaluate all random components, $\boldsymbol{\varepsilon}_{t+\tau}$, $\tau = 1, 2, \ldots, T-t$, at their mean values and solve the certainty-equivalent counterpart of (A.1)-(A.2).

The standard approach to the solution of the optimal control problem (A.1)-(A.2) is the sweep method of Bryson and Ho (see, for example, Lewis and Syrmos, 1995), which yields the optimal control law

$$\mathbf{u}_{t+\tau}^{*} = -\mathbf{K}_{t+\tau}\mathbf{y}_{t+\tau}, \tag{A.3}$$

where

$$\mathbf{K}_{t+\tau} = \left(\mathbf{W}^{'}\mathbf{S}_{t+\tau+1}\mathbf{W} + \mathbf{R}\right)^{-1}\mathbf{W}^{'}\mathbf{S}_{t+\tau+1}\mathbf{V}, \tag{A.4}$$

and

$$\mathbf{S}_{t+\tau} = \left(\mathbf{V} - \mathbf{W}\mathbf{K}_{t+\tau}\right)^{'}\mathbf{S}_{t+\tau+1}\left(\mathbf{V} - \mathbf{W}\mathbf{K}_{t+\tau}\right) + \mathbf{K}_{t+\tau}^{'}\mathbf{R}\mathbf{K}_{t+\tau} + \mathbf{Q}, \tag{A.5}$$

[19] For recent surveys of discrete time optimal control problems as relating to economics see, for example, Amman (1996) and Anderson, Hansen, McGrattan, and Sargent (1996).

[20] See, for example, Kendrick (1981) and Amman (1996).

[21] See, for example, Anderson, Hansen, McGrattan, and Sargent (1996) for such a transformation.

$\tau = 0, 1, \ldots, T-t-1$, with $\mathbf{S}_T = \mathbf{Z}$. Equation (A.5) is the (Joseph stabilized version of the) matrix Riccati equation. The feedback gain matrices $\mathbf{K}_{t+\tau}$ are typically computed by backward recursions on (A.4) and (A.5). Note that these recursions involve computing $T-t$ inverse matrices, $\left(\mathbf{W}'\mathbf{S}_{t+\tau+1}\mathbf{W} + \mathbf{R}\right)^{-1}$, $\tau = 0, 1, \ldots, T-t-1$.[22]

In this Appendix, we show that the solution of the optimal control problem (A.1)-(A.2) can also be obtained by means of rewriting (A.1) and (A.2) in stacked form, and solving the resultant optimality conditions using Bowden's procedure for the inversion of a block-tridiagonal coefficient matrix. Application of Bowden's procedure can significantly reduce the computational effort needed to carry out the matrix Riccati equation recursions if the planning horizon is fixed. Alternatively, one could write the first-order conditions of (A.1)-(A.2) as a special case of the canonical form (2.2), and then use the solution techniques described in Section 5. However, it is easily verified that the resulting canonical form does not contain lagged values, and only future expectations appear on the right-hand side of the equation system. The solution method developed in the remainder of this Appendix takes this special feature of the optimal control problem (A.1)-(A.2) into account.

Following Bowden (1983), we rewrite (A.1) and (A.2) after invoking certainty equivalence as

$$\min_{\widetilde{\mathbf{u}}} \frac{1}{2}\left\{\widetilde{\mathbf{y}}'\widetilde{\mathbf{Q}}\widetilde{\mathbf{y}} + \widetilde{\mathbf{u}}'\widetilde{\mathbf{R}}\widetilde{\mathbf{u}}\right\} \tag{A.6}$$

subject to

$$\widetilde{\mathbf{V}}\widetilde{\mathbf{y}} = \widetilde{\mathbf{y}}_t + \widetilde{\mathbf{W}}\widetilde{\mathbf{u}}, \tag{A.7}$$

where

$$\widetilde{\mathbf{Q}} = \begin{pmatrix} \mathbf{Q} & \mathbf{0}_d & \mathbf{0}_d & \cdots & \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d \\ \mathbf{0}_d & \mathbf{Q} & \mathbf{0}_d & \cdots & \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d & \cdots & \mathbf{0}_d & \mathbf{Q} & \mathbf{0}_d \\ \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d & \cdots & \mathbf{0}_d & \mathbf{0}_d & \mathbf{Z} \end{pmatrix}, \quad \widetilde{\mathbf{R}} = \begin{pmatrix} \mathbf{R} & \mathbf{0}_f & \mathbf{0}_f & \cdots & \mathbf{0}_f & \mathbf{0}_f & \mathbf{0}_f \\ \mathbf{0}_f & \mathbf{R} & \mathbf{0}_f & \cdots & \mathbf{0}_f & \mathbf{0}_f & \mathbf{0}_f \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0}_f & \mathbf{0}_f & \mathbf{0}_f & \cdots & \mathbf{0}_f & \mathbf{R} & \mathbf{0}_f \\ \mathbf{0}_f & \mathbf{0}_f & \mathbf{0}_f & \cdots & \mathbf{0}_f & \mathbf{0}_f & \mathbf{R} \end{pmatrix},$$

$$\widetilde{\mathbf{V}} = \begin{pmatrix} \mathbf{I}_d & \mathbf{0}_d & \mathbf{0}_d & \cdots & \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d \\ -\mathbf{V} & \mathbf{I}_d & \mathbf{0}_d & \cdots & \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d & \cdots & -\mathbf{V} & \mathbf{I}_d & \mathbf{0}_d \\ \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d & \cdots & \mathbf{0}_d & -\mathbf{V} & \mathbf{I}_d \end{pmatrix},$$

[22] In the infinite-horizon case, $\mathbf{S}_{t+\tau}$ and thus $\mathbf{K}_{t+\tau}$ under certain conditions are time-invariant. In this case the solution of the resulting algebraic matrix Riccati equation can also be computed non-recursively. Efficient methods for this purpose are reviewed, for example, in Lewis and Syrmos (1995) and in Anderson, Hansen, McGrattan, and Sargent (1996). See also Amman and Neudecker (1997).

$$
\widetilde{\mathbf{W}} = \begin{pmatrix} \mathbf{W} & \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} & \cdots & \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} \\ \mathbf{0}_{d\times f} & \mathbf{W} & \mathbf{0}_{d\times f} & \cdots & \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} & \cdots & \mathbf{0}_{d\times f} & \mathbf{W} & \mathbf{0}_{d\times f} \\ \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} & \cdots & \mathbf{0}_{d\times f} & \mathbf{0}_{d\times f} & \mathbf{W} \end{pmatrix},
$$

$$
\widetilde{\mathbf{y}} = \begin{pmatrix} \mathbf{y}'_{t+1}, & \mathbf{y}'_{t+2}, & \ldots, & \mathbf{y}'_{T} \end{pmatrix}', \qquad \widetilde{\mathbf{u}} = \begin{pmatrix} \mathbf{u}'_{t}, & \mathbf{u}'_{t+1}, & \ldots, & \mathbf{u}'_{T-1} \end{pmatrix}',
$$

and

$$
\widetilde{\mathbf{y}}_t = \begin{pmatrix} (\mathbf{V}\mathbf{y}_t)', & \mathbf{0}'_{d\times 1}, & \ldots, & \mathbf{0}'_{d\times 1} \end{pmatrix}'.
$$

Substituting the constraints in (A.7) back into (A.6), we obtain the unconstrained optimization problem

$$
\min_{\widetilde{\mathbf{u}}} \frac{1}{2} \left\{ \left(\widetilde{\mathbf{y}}_t + \widetilde{\mathbf{W}}\widetilde{\mathbf{u}} \right)' \widetilde{\mathbf{L}}^{-1} \left(\widetilde{\mathbf{y}}_t + \widetilde{\mathbf{W}}\widetilde{\mathbf{u}} \right) + \widetilde{\mathbf{u}}' \widetilde{\mathbf{R}} \widetilde{\mathbf{u}} \right\}, \tag{A.8}
$$

where

$$
\widetilde{\mathbf{L}} = \widetilde{\mathbf{V}} \widetilde{\mathbf{Q}}^{-1} \widetilde{\mathbf{V}}'. \tag{A.9}
$$

Taking derivatives of (A.8) with respect to $\widetilde{\mathbf{u}}'$, it is readily verified that the optimal control vector $\widetilde{\mathbf{u}}^*$ is given by[23]

$$
\widetilde{\mathbf{u}}^* = -\left(\widetilde{\mathbf{R}}^{-1} - \widetilde{\mathbf{R}}^{-1} \widetilde{\mathbf{W}}' \left(\widetilde{\mathbf{W}} \widetilde{\mathbf{R}}^{-1} \widetilde{\mathbf{W}}' + \widetilde{\mathbf{L}} \right)^{-1} \widetilde{\mathbf{W}} \widetilde{\mathbf{R}}^{-1} \right) \left(\widetilde{\mathbf{W}}' \widetilde{\mathbf{L}}^{-1} \widetilde{\mathbf{y}}_t \right). \tag{A.11}
$$

Inspecting (A.11) and recalling the definitions of $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{L}}$, the main computational burden in solving for $\widetilde{\mathbf{u}}^*$ is easily seen to be given by the inversion of the block-tridiagonal matrix $\widetilde{\mathbf{J}} = \widetilde{\mathbf{W}} \widetilde{\mathbf{R}}^{-1} \widetilde{\mathbf{W}}' + \widetilde{\mathbf{L}}$,

$$
\widetilde{\mathbf{J}} = \begin{pmatrix} \mathbf{W}\mathbf{R}^{-1}\mathbf{W}' + \mathbf{Q}^{-1} & -\mathbf{Q}^{-1}\mathbf{V}' & \mathbf{0}_d & \cdots & \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d \\ -\mathbf{V}\mathbf{Q}^{-1} & \mathbf{T} & -\mathbf{Q}^{-1}\mathbf{V}' & \cdots & \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d & \cdots & -\mathbf{V}\mathbf{Q}^{-1} & \mathbf{T} & -\mathbf{Q}^{-1}\mathbf{V}' \\ \mathbf{0}_d & \mathbf{0}_d & \mathbf{0}_d & \cdots & \mathbf{0}_d & -\mathbf{V}\mathbf{Q}^{-1} & F \end{pmatrix}, \tag{A.12}
$$

where

$$
\mathbf{T} = \mathbf{W}\mathbf{R}^{-1}\mathbf{W}' + \mathbf{V}\mathbf{Q}^{-1}\mathbf{V}' + \mathbf{Q}^{-1}, \tag{A.13}
$$

[23] In deriving (A.11), we have used the result

$$
\left(\mathbf{A}_{11}^{-1} + \mathbf{A}_{12}\mathbf{A}_{22}\mathbf{A}_{21} \right)^{-1} = \mathbf{A}_{11} - \mathbf{A}_{11}\mathbf{A}_{12} \left(\mathbf{A}_{21}\mathbf{A}_{11}\mathbf{A}_{12} + \mathbf{A}_{22}^{-1} \right)^{-1} \mathbf{A}_{21}\mathbf{A}_{11}, \tag{A.10}
$$

which holds for any set of real valued matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, and $\mathbf{A}_{22}$ for which the left- and right-hand sides of (A.10) are well defined.

and

$$\digamma = \mathbf{W}\mathbf{R}^{-1}\mathbf{W}' + \mathbf{V}\mathbf{Q}^{-1}\mathbf{V}' + \mathbf{Z}^{-1}. \tag{A.14}$$

Using Bowden's Procedure to invert $\widetilde{\mathbf{J}}$ results in the following recursions: Let

$$\mathbf{N}_{T-t} = \mathbf{I}_d, \;\; \mathbf{N}_{T-t-1} = \left(\mathbf{V}\mathbf{Q}^{-1}\right)^{-1} \digamma,$$

$$\mathbf{N}_{T-t-i} = \left(\mathbf{V}\mathbf{Q}^{-1}\right)^{-1} \left(\mathbf{T}\mathbf{N}_{T-t-i+1} - \mathbf{Q}^{-1}\mathbf{V}'\mathbf{N}_{T-t-i+2}\right), \qquad i = 2, 3, \ldots, T-t-1,$$

and

$$\mathbf{P}_1 = \mathbf{I}_d, \;\; \mathbf{P}_2 = \left(\mathbf{W}\mathbf{R}^{-1}\mathbf{W}' + \mathbf{Q}^{-1}\right)\left(\mathbf{V}\mathbf{Q}^{-1}\right)^{-1},$$

$$\mathbf{P}_i = \left(\mathbf{P}_{i-1}\mathbf{T} - \mathbf{P}_{i-2}\mathbf{Q}^{-1}\mathbf{V}'\right)\left(\mathbf{V}\mathbf{Q}^{-1}\right)^{-1}, \qquad i = 3, 4, \ldots, T-t,$$

$$\mathbf{P}_{T-t+1} = \left(\mathbf{P}_{T-t}\digamma - \mathbf{P}_{T-t-1}\mathbf{Q}^{-1}\mathbf{V}'\right).$$

Then the (ij)-th block of $\widetilde{\mathbf{J}}^{-1}$ is given by

$$\widetilde{\mathbf{J}}^{-1}_{ij} = \mathbf{P}'_i \left(\mathbf{P}'_{T-t+1}\right)^{-1} \mathbf{N}'_j \qquad \text{for } j \geq i, \tag{A.15}$$

and

$$\widetilde{\mathbf{J}}^{-1}_{ij} = \mathbf{N}_i \mathbf{P}^{-1}_{T-t+1} \mathbf{P}_j \qquad \text{for } i > j, \tag{A.16}$$

$i, j = 1, 2, \ldots, T-t$.

Note that computation of $\widetilde{\mathbf{J}}^{-1}$ involves the computation of only five inverses, $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$, $\mathbf{Z}^{-1}$, $\left(\mathbf{V}\mathbf{Q}^{-1}\right)^{-1}$, and $\mathbf{P}^{-1}_{T-t+1}$. Only the solution for $\mathbf{u}_t$ in (A.11) is the solution for the stochastic optimal control problem (A.1)-(A.2), however, as the solutions for $\{\mathbf{u}_{t+\tau}\}_{\tau=1}^{T-t-1}$ do not reflect the stochastic innovations in periods $t+1$, $t+2$, $\ldots$, $T-t-1$. To compute the solution for $\{\mathbf{u}_{t+\tau}\}_{\tau=1}^{T-t-1}$, the analog of (A.8)-(A.9) needs to be constructed for the optimal control problems in periods $t+1$, $t+2$, $\ldots$, $T-t-1$. The structure of these problems is the same as that at t, (A.8)-(A.9), however, if the planning horizon is fixed at all current and future dates. Given that Bowden's procedure only involves computation of five inverses, it is likely to be significantly more efficient than the standard matrix Riccati equation recursions whenever the planning horizon is fixed at all current and future dates.

It is well known that the mathematical problems associated with the solution of the linear-quadratic optimal control problem and of the Gaussian optimal filtering problem are dual. Therefore, the Gaussian optimal filtering problem can also be reduced to the solution of a linear equation system with a block-tridiagonal matrix coefficient structure. We do not go into the details here.

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