Appendix to the paper “Determining the Number of Factors from Empirical Distribution of Eigenvalues”

Alexei Onatski
Economics Department, Columbia University

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Abstract

This technical appendix contains a proof of Lemma 3 of the paper.

To prove Lemma 3, we will need the following Lemmas A1-A4.

Lemma A1. Let $F$ be the cdf of a non-negative random variable with a finite upper boundary of support $u(F)$ and a finite positive expectation $E_F$. Let $m(z)$ be the Stieltjes transform of $F$. Then, for any $z \in \mathbb{C}^+$ such that $|z| > u(F)$ we have: $|zm(z) + 1| \leq \frac{u(F)}{|z| - u(F)}$; and for any $z \in \mathbb{C}^+$ such that $z = x + iy$, where $|x| > u(F) + 3y$ we have: $|zm(z) + 1| \geq \frac{E_F}{4(|z| + u(F))}$.

Proof: For $|z| > u(F)$, we have: $|zm(z) + 1| = |\int \frac{\lambda}{\lambda - z} dF(\lambda)| = |\int \frac{\lambda}{\lambda - z} dF(\lambda)| \leq \int |\frac{\lambda}{\lambda - z}| dF(\lambda) \leq \int |\frac{\lambda}{|x| - x}| dF(\lambda) \leq \frac{u(F)}{|z| - u(F)}$, which proves one of the lemma’s inequalities. Further, write:

$$zm(z) + 1 = \int \frac{\lambda}{\lambda - z} dF(\lambda) = \int \frac{\lambda(\lambda - x)}{|\lambda - z|^2} dF(\lambda) + i \int \frac{\lambda y}{|\lambda - z|^2} dF(\lambda).$$

Since for any real $a$ and $b$, $|a + ib|^2 = a^2 + b^2 \geq \frac{3}{2} (a + b)^2$, we have: $|zm(z) + 1| \geq \frac{E_F}{4(|z| + u(F))}$. 

(1)
\[ \frac{1}{\sqrt{2}} \int \frac{\lambda(x+y)}{|\lambda-z|^2} dF(\lambda). \]

If \( x < 0 \), then, since for any real and positive \( a \) and \( b \), \( \frac{a+b}{a+|b|} \geq (a^2 + b^2)^{-1/2} \), we have \( \frac{\lambda(x+y)}{|\lambda-z|^2} \geq \frac{1}{|\lambda-z|} \), and therefore: \( zm(z) + 1 \geq \frac{1}{\sqrt{2}} \int \frac{1}{|\lambda-z|} dF(\lambda) \geq \frac{1}{\sqrt{2}} \sum_{v} \lambda \int_{|z|} dF(\lambda) = \frac{E_{\lambda}}{\sqrt{2}(|z|+u(F))}. \]

If \( x > 0 \), then by assumption of the lemma, \( x > u(F) + 3y \), which implies that \( \frac{1}{2}(x-\lambda-y) > y \) and hence, \( x-\lambda > y + \frac{1}{2}(x-\lambda+y) \geq y + \frac{1}{2}|\lambda-z| \) so that \( x-\lambda+y \geq |\lambda-z| \). Therefore: \( zm(z) + 1 \geq \frac{1}{\sqrt{2}} \sum_{v} \lambda \int_{|z|} dF(\lambda) = \frac{E_{\lambda}}{2\sqrt{2}(|z|+u(F))}. \]

**Lemma A2.** Suppose that \( u(z), v(z) \in \mathbb{C}^+ \) are analytic functions satisfying Zhang’s system:

\[
\begin{align*}
zm(z) + 1 &= u(z)m_A(u(z)) + 1 \\
z^m(z) + 1 &= c^{-1}[v(z)m_B(v(z)) + 1] \\
z^m(z) + 1 &= -c^{-1}\frac{z}{zm(z)}
\end{align*}
\]

Let \( U = \{ z = x + iy : x > \bar{a} \text{ and } 0 < y < \bar{y} \} \). Then, for any \( x > u(F^{A,B}) \), there exists \( \bar{y} > 0 \) such that for any \( z \) from \( U : \text{Re } u(z) > u(F^A) \) and \( \text{Re } v(z) > u(F^B). \)

**Proof:** The idea of the proof is as follows. First, using Lemma A1 we prove that for any \( x > u(F) \) and \( \bar{y} > 0 \), there exists \( z_1 \in U \) such that \( \text{Re } u(z_1) > u(F^A) \) and \( \text{Re } v(z_1) > u(F^B). \) Then, we assume that Lemma A2 does not hold so that for some \( x > u(F) \) and any \( \bar{y} > 0 \) there exists \( z_2 \in U \) such that \( \text{Re } u(z_2) \leq u(F^A) \) or/and \( \text{Re } v(z_2) \leq u(F^B). \) Connecting \( z_1 \) and \( z_2 \) by a continuous path \( z(t) \in U \), we establish the existence of \( z_3 \in U \) such that \( \text{Re } u(z_3) = u(F^A) \) or/and \( \text{Re } v(z_3) = u(F^B). \) Then, we show that for small enough \( \bar{y} \), \( \text{Im } (z_3m(z_3) + 1) \) must be smaller than \( \text{Im } (u(z_3)m_A(u(z_3)) + 1) \) or than \( c^{-1}\text{Im } (v(z_3)m_B(v(z_3)) + 1) \), which contradicts the assumption that \( u(z), v(z) \) satisfy Zhang’s system.

First, we prove the existence of \( z_1 \). The last equation of Zhang’s system and the first inequality of Lemma A1 imply that \( \frac{\lambda}{\lambda} \rightarrow 0 \) as \( |z| \rightarrow \infty \). Hence, as \( |z| \rightarrow \infty \),

\[
\max \{|u|,|v|\} \rightarrow \infty \]. Suppose without loss of generality that \( |u| \rightarrow \infty \). Let us show that also \( |v| \rightarrow \infty \). Indeed, from the first equation of Zhang’s system \( zm(z) + 1 = \)
\[|um_A(u) + 1|. \] Therefore, for \( z \in U \) with large enough \( |z| \):

\[
\frac{E_{F_c,A,B}}{4(|z| + u(F_{c,A,B}))} \leq |zm(z) + 1| = |um_A(u) + 1| \leq \frac{u(F^A)}{|u| - u(F^A)}. \tag{3}
\]

where the latter inequality is obtained from Lemma A1 applied to \( um_A(u) + 1 \). This implies that \( \liminf_{|z| \to \infty} \frac{z}{|z|} > 0 \), for \( z \in U \). However, since \( |\frac{z}{|z|}| = c \left| zm(z) + 1 \right| \leq \frac{cu(F_{c,A,B})}{(|z| - u(F_{c,A,B}))} \to 0 \), we must have \( |v| \to \infty \). Hence, \( |z| \to \infty \) so that \( z \) remains in \( U \), both \( |u| \to \infty \) and \( |v| \to \infty \). Let us prove that \( \text{Re} u \to \infty \) and \( \text{Re} v \to \infty \).

First, notice that for \( z \in U \):

\[
\text{Im} (zm(z) + 1) < \tilde{y} \int \frac{\lambda}{|\lambda - z|^2} dF_{c,A,B}(\lambda) \leq \frac{\tilde{y} E_{F_{c,A,B}}}{(x - u(F_{c,A,B}))^2}, \tag{4}
\]

Further, \( \text{Im} (-\frac{z}{uv}) = \frac{z \text{Im}(uv) - y \text{Re}(uv)}{|uv|^2} \). Therefore, since Zhang’s third equation is \( zm(z) + 1 = -c^{-1} \frac{z}{uv} \), we have: \( \frac{x \text{Im}(uv) - y \text{Re}(uv)}{|uv|^2} \leq c \left( \frac{\tilde{y} E_{F_{c,A,B}}}{(x - u(F_{c,A,B}))^2} \right) \). Hence, for \( z \in U \), where \( \tilde{y} \leq \frac{\bar{y} - u(F_{c,A,B})}{3} \), we have:

\[
\text{Im}(uv) \leq \frac{|uv|}{x} \left( \frac{c \tilde{y} E_{F_{c,A,B}}}{(x - u(F_{c,A,B}))^2} \right) + \frac{y \text{Re}(uv)}{|uv|} \leq \frac{\tilde{y}}{u(F_{c,A,B})} \left( \frac{|cu|}{x^2} (F_{c,A,B} \left( x^2 + \tilde{y}^2 \right) + 1) \right). \tag{5}
\]

Now, the third equation of (2) and the second inequality of Lemma A1 imply that \( \frac{|cu|}{x^2} = \frac{1}{z(zm(z) + 1)} \leq \frac{4(|z| + u(F_{c,A,B}))}{8 E_{F_{c,A,B}}} \). Therefore, \( \frac{\text{Im}(uv)}{|uv|} \leq \frac{\tilde{y}}{u(F_{c,A,B})} \left( \frac{\bar{x}^2 + \tilde{y}^2}{8(x - u(F_{c,A,B}))^2} + 1 \right). \tag{5}
\]

Noting that \( \text{Im} u \leq \frac{\text{Im}(uv)}{|uv|} \), we have:

\[
\frac{\text{Im} u}{|u|} \leq \frac{\tilde{y}}{u(F_{c,A,B})} \left( \frac{(x^2 + \tilde{y}^2)}{8(x - u(F_{c,A,B}))^2} + 1 \right) \tag{5}
\]

for \( z \in U \), where \( \tilde{y} \leq \frac{\bar{y} - u(F_{c,A,B})}{3} \). The same inequality also holds for \( \frac{\text{Im} v}{|v|} \).

Inequality (5) and the fact that \( |u| \to \infty \) imply that \( |\text{Re} u| \to \infty \) as \( |z| \to \infty \) while \( z \) remains in \( U \). Similarly, \( |\text{Re} v| \to \infty \). But \( \text{Re} u \) and \( \text{Re} v \) must be positive for \( z \in U \) when
$|z|$ is large enough. Indeed, (1) implies that $\text{Re}(zm(z) + 1) < 0$. Hence, from the first equation of Zhang, $\text{Re}(um_A(u) + 1) < 0$. But from (1) applied to $um_A(u) + 1$ and the fact that $|\text{Re} u| \to \infty$, $\text{Re}(um_A(u) + 1)$ must be of the same sign as $-\text{Re} u$ for $|z|$ large enough. Hence, $\text{Re} u \to +\infty$ as $|z| \to \infty$ while $z$ remains in $U$. Similarly, $\text{Re} v \to +\infty$ as $|z| \to \infty$ while $z$ remains in $U$. This proves the existence of $z_1 \in U$ with properties outlined above.

Assuming that Lemma A2 does not hold, the existence of $z_3$ follows from the fact that $u(z)$ and $v(z)$ are analytic, and hence continuous, functions of $z$. Suppose without loss of generality that $\text{Re}u(z_3) = u(\mathcal{F}^A)$. Let us finish the proof of the lemma by comparing $\text{Im}(z_3m(z_3) + 1)$ with $\text{Im}(u(z_3)m_A(u(z_3)) + 1)$ when $\bar{y}$ is small. By assumption that $\liminf_{\delta \to 0} \frac{1}{\delta} \int_{|\lambda - u(\mathcal{F}^A)| \leq \delta} \lambda d\mathcal{F}^A(\lambda) = k^A > 0$, for $u(z)$ such that $\text{Re} u = u(\mathcal{F}^A)$ and $\text{Im} u$ is small enough, we have:

\[
\text{Im}(um_A(u) + 1) = \int \frac{\lambda \text{Im} u}{(\lambda - u(\mathcal{F}^A))^2 + (\text{Im} u)^2} d\mathcal{F}^A(\lambda) \geq \frac{1}{2\text{Im} u} \int_{|\lambda - u(\mathcal{F}^A)| \leq \text{Im} u} \lambda d\mathcal{F}^A(\lambda) \geq \frac{k^A}{2} > 0. \tag{6}
\]

From (6) and (5), we can choose $\bar{y}$ small enough so that $\text{Im}(u(z_3)m_A(u(z_3)) + 1) \geq \frac{k^A}{2}$. On the other hand, from the first equation of Zhang, $u(z_3)m_A(u(z_3)) + 1 = z_3m(z_3) + 1$ and hence $\text{Im}(z_3m(z_3) + 1) \geq \frac{k^A}{2}$. But from (4) we know that for small enough $\bar{y}$, $\text{Im}(z_3m(z_3) + 1)$ must be smaller than $\frac{k^A}{2}$. We have got a contradiction, which implies that the statement of Lemma A2 holds.

Lemma A3. For any real $x > u(\mathcal{F}^{c,A,B})$, there exist real limits $u(x) \equiv \lim_{z \to \infty} u(z)$ and $v(x) \equiv \lim_{z \to \infty} v(z)$. Functions $u(x)$ and $v(x)$ satisfy the limit version of Zhang's
are analytic and such that \( \lim_{x \to \infty} u(x) = \lim_{x \to \infty} v(x) = \infty. \)

**Proof:** Let \( G = \{ z \in \mathbb{C}^+ : u(\mathcal{F}_A) < x \leq \Re z \leq \bar{x} < \infty, 0 < \Im z < \bar{y} < \infty \}. \) Then \( \sup_{z \in G} \max(|u(z)|, |v(z)|) < \infty. \) Had this been not true, there would have existed a sequence \( \{z_n\} \in G \) such that \( |u(z_n)| \to \infty \) or \( |v(z_n)| \to \infty. \) Without loss of generality, let \( |u(z_n)| \to \infty. \) Lemma A.1 then would imply that \( |u(z_n) m_A(u(z_n)) + 1| \to 0, \) and hence, from Zhang's first equation, \( |z_n m(z_n) + 1| \to 0. \) But, as follows from (1),

\[
|\text{Re}(z_n m(z_n) + 1)| \geq \frac{E_{p-\mu}}{x^2+y^2} > 0,
\]

which gives a contradiction.

Since \( \sup_{z \in G} \max(|u(z)|, |v(z)|) < \infty, \) inequality (5) and a similar inequality for \( \frac{\Im v}{\max(|u|, |v|)} \) imply that for any sequence \( \{z_n\} \in G \) such that \( z_n \to x \in \mathbb{R} \) the concentration points of \( \{u(z_n)\} \) and \( \{v(z_n)\} \) must be real. Suppose that there exist subsequences of \( z_n, \{z_i\} \) and \( \{z_j\} \), such that \( u(z_i) \to u_1 \in \mathbb{R} \) and \( u(z_j) \to u_2 \in \mathbb{R} \) and \( u_1 \neq u_2. \) By Lemma A.2, \( u_1 \geq u(\mathcal{F}_A) \) and \( u_2 \geq u(\mathcal{F}_A). \) If \( u_1 = u(\mathcal{F}_A), \) then using inequalities similar to (6), we find that \( \Im(u(z_i) m_A(u(z_i)) + 1) \geq \frac{\alpha A}{2} \) for large enough \( i, \) which cannot be the case because \( \Im(u(z_i) m_A(u(z_i)) + 1) = \Im(z_i m(z_i) + 1) \to 0 \) as \( i \to \infty. \) Hence, \( u_1 > u(\mathcal{F}_A). \) Similarly, \( u_2 > u(\mathcal{F}_A). \)

Since \( m(x) \) exists and is continuous for \( x > u(\mathcal{F}_A), \) we have:

\[
\lim_{z_n \to x} (z_n m(z_n) + 1) = m(x) + 1. \]

Since \( m_A(u) \) exists and is continuous for \( u > u(\mathcal{F}_A), \) we have:

\[
\lim_{z_i \to x} (u(z_i) m_A(u(z_i)) + 1) = u_1 m_A(u_1) + 1 \text{ and } \lim_{z_j \to x} (u(z_j) m_A(u(z_j)) + 1) = u_2 m_A(u_2) + 1. \]

The first equation of Zhang's system implies that we must have:

\[
x_m(x) + 1 = u_1 m_A(u_1) + 1 = u_2 m_A(u_2) + 1.
\]
But this is not possible with \( u_1 \neq u_2 \) such that \( u_1 > u (F^A) \) and \( u_2 > u (F^A) \) because function \( um_A (u) + 1 \) is strictly increasing for \( u > u (F^A) \). Hence, there exists only one concentration point of \( \{ u(z_n) \} \), that is there exists a real limit \( u(x) \equiv \lim_{z \in C^+, z \to x} u(z) \). Similarly, there exists a real limit \( v(x) \equiv \lim_{z \in C^+, z \to x} v(z) \).

That \( u(x) \) and \( v(x) \) satisfy the limit version of Zhang’s system follows from the existence and continuity of \( m_A (u) \) for \( |u| > u (F^A) \) and from the existence and continuity of \( m_B (v) \) for \( |v| > u (F^B) \). The analyticity of \( u(x) \) follows from the analyticity of \( F(x, u) \equiv xm(x) + 1 - (um_A (u) + 1) \) for \( x > u (F^{c,A,B}) \) and \( u > u (F^A) \) and from the implicit function theorem. Similarly, the analyticity of \( v(x) \) follows from the analyticity of \( F_1 (x, v) \equiv xm(x) + 1 - c^{-1} (vm_B (v) + 1) \) for \( x > u (F^{c,A,B}) \) and \( v > u (F^B) \) and from the implicit function theorem. Finally, (7) implies that as \( x \to \infty, \; um_A (u) + 1 \to 0 \) and \( vm_B (v) + 1 \to 0 \), which can be the case only when \( \lim_{x \to \infty} u(x) = \lim_{x \to \infty} v(x) = \infty \).

Lemma A4. For \( x > u (F^{c,A,B}) \), the following system

\[
\begin{align*}
    v &= x \left( c \int \frac{\lambda u}{u-\lambda} dF_A (\lambda) \right)^{-1} \\
    u &= x \left( \int \frac{\lambda v}{v-\lambda} dF_B (\lambda) \right)^{-1}
\end{align*}
\]

has exactly two solutions \( (u_1, v_1) \) and \( (u_2, v_2) \) such that \( u_i > u (F^A) \) and \( v_i > u (F^B) \) for \( i = 1, 2 \). For \( x = u (F^{c,A,B}) \) and for \( x < u (F^{c,A,B}) \), the system has only one such solution and no such solutions, respectively.

Proof: For any \( x > u (F^{c,A,B}) \), one solution to (8) satisfying \( u(x) > u (F^A) \) and \( v(x) > u (F^B) \) is given by \( u(x) \) and \( v(x) \) defined in Lemma A3. That such \( u(x) \) and \( v(x) \) indeed provide a solution to (8) follows from the fact that (8) can be obtained from (7) by substituting the third equation into the first two. Let us now show that for \( x > u (F^{c,A,B}) \), there exists another solution to (8).

First, note that \( x \left( c \int \frac{\lambda u}{u-\lambda} dF_A (\lambda) \right)^{-1} \) as a function of \( u > u (F^A) \) is concave, tends to zero as \( u \downarrow u (F^A) \) and to \( x (cE_A)^{-1} \) as \( u \to \infty \). The concavity follows from the expression
\[
\frac{d^2}{dx^2} \left( c \int \frac{\lambda u}{u-x} d\mathcal{F}_A(\lambda) \right)^{-1} = 2x c^{-1} \left( \int \frac{\lambda u}{u-x} d\mathcal{F}_A(\lambda) \right)^{-3} \cdot \left( \int \frac{\lambda^2}{(u-x)^2} d\mathcal{F}_A(\lambda) \right)^2 - \left( \int \frac{\lambda^2}{(u-x)^2} d\mathcal{F}_A(\lambda) \right) \left( E_A + \int \frac{\lambda^2}{u-x} d\mathcal{F}_A(\lambda) \right) \quad \text{and the Cauchy inequality} \quad \int \frac{\lambda}{(u-x)^2} d\mathcal{F}_A(\lambda) \leq \left( \int \frac{\lambda^2}{(u-x)^2} d\mathcal{F}_A(\lambda) \right)^{1/2} \left( \int \frac{\lambda^2}{(u-x)^2} d\mathcal{F}_A(\lambda) \right)^{1/2}. \]

The tendency to zero follows from the fact that \( c \int \frac{\lambda u}{u-x} d\mathcal{F}_A(\lambda) \to \infty \) as \( u \downarrow u(\mathcal{F}_A) \), which is easy to show using the monotone convergence theorem and assumption \( \lim_{\delta \to 0} \frac{1}{\delta} \int_{|\lambda-u(\mathcal{F}_A)| \leq \delta} \lambda d\mathcal{F}_A(\lambda) = k^A > 0 \). Finally, the convergence to \( x(cE_A)^{-1} \) as \( u \to \infty \) is obvious. Similarly, \( x \left( \int \frac{\lambda u}{u-x} d\mathcal{F}_B(\lambda) \right)^{-1} \) as a function of \( v > u(\mathcal{F}_B) \) is concave, tends to zero as \( v \downarrow u(\mathcal{F}_B) \) and to \( x(cE_B)^{-1} \) as \( v \to \infty \).

The above properties of \( x \left( c \int \frac{\lambda u}{u-x} d\mathcal{F}_A(\lambda) \right)^{-1} \) and \( x \left( \int \frac{\lambda u}{u-x} d\mathcal{F}_B(\lambda) \right)^{-1} \) imply that the curves in the \( \{ u > u(\mathcal{F}_A), v > v(\mathcal{F}_B) \} \) subset of the \((u, v)\)-plane defined by (8) are either intersecting at two points, touching at a single point, or having no common points. Since there exists a solution to (8) for any \( x > u(\mathcal{F}_c^{A,B}) \) and since \( x \left( \int \frac{\lambda u}{u-x} d\mathcal{F}_A(\lambda) \right)^{-1} \) and \( x \left( \int \frac{\lambda u}{u-x} d\mathcal{F}_B(\lambda) \right)^{-1} \) are monotone increasing in \( x \), the curves must intersect at two points for any \( x > u(\mathcal{F}_c^{A,B}) \). Let us show that the curves are touching at a single point when \( x = u(\mathcal{F}_c^{A,B}) \).

Suppose the curves intersect at two points \((u_1, v_1)\) and \((u_2, v_2)\) when \( x = u(\mathcal{F}_c^{A,B}) \). Let \( u_2 > u_1 \) and \( v_2 > v_1 \). Define \( f_1(x, u, v) = x + cuv(u + 1) \) and \( f_2(x, u, v) = x + uv(v + 1) \). Note that system (8) is equivalent to \( f_i(x, u, v) = 0 \) for \( i = 1, 2 \). It is straightforward to check that the assumption of the proper intersection of the curves (not just a tangency at one point) is equivalent to \( \det \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} \neq 0 \) at any of the two intersection points. Then the implicit function theorem (see Krantz (1992), Theorem 1.4.11) implies that there exist holomorphic functions \( u(z), v(z) \) defined in an open neighborhood of \( z = u(\mathcal{F}_c^{A,B}) \) in \( \mathbb{C} \), which satisfy \( f_i(z, u, v) = 0 \) for \( i = 1, 2 \). To each of the two intersection points, there will correspond its own set of holomorphic functions \( u(z), v(z) \). We will consider the functions \( u(z) \) and \( v(z) \) corresponding to \((u_2, v_2)\). For such a choice, it is straightforward to check that \( \frac{d}{dz} \operatorname{Re} u(z) > 0 \) and \( \frac{d}{dz} \operatorname{Re} v(z) > 0 \).
at \( z = u(F_{c,A,B}) \).

Furthermore, using identities \( f_i(z, u(z), v(z)) = 0 \) for \( i = 1, 2 \) it is straightforward to check that in a small enough neighborhood of \( z = u(F_{c,A,B}) \in C \), \( \text{Im} \ z > 0 \) implies that \( \text{Im} \ u(z) \) and \( \text{Im} \ v(z) \) are of the same sign and are not equal to zero. Cauchy-Riemann equations for holomorphic functions imply that \( \frac{d}{d \text{Im} z} \text{Im} u(z) = \frac{d}{d \text{Re} z} \text{Re} u(z) > 0 \) and \( \frac{d}{d \text{Im} z} \text{Im} v(z) = \frac{d}{d \text{Re} z} \text{Re} v(z) > 0 \) at \( z = u(F_{c,A,B}) \). Hence, \( \text{Im} u(z) \) and \( \text{Im} v(z) \) are positive when \( \text{Im} z \) is positive and \( z \) lies in a small enough complex neighborhood of \( u(F_{c,A,B}) \). Let us define \( m(z) = -\frac{e^{-1}}{u(z)v(z)} - \frac{1}{z} \). Clearly, for \( z \) in the small complex neighborhood of \( u(F_{c,A,B}) \), \( \text{Im} m(z) > 0 \).

Zhang shows that for any \( z \in C^+ \), there is only one solution to (2) such that \( m, u \) and \( v \) belong to \( C^+ \). Hence, \( u(z), v(z) \), and \( m(z) \) defined above constitute the solution to Zhang's system (2) for \( z \) in a small neighborhood of \( u(F_{c,A,B}) \) and such that \( \text{Im} z > 0 \). Finally, for any real \( x \) which belongs to the neighborhood of \( u(F_{c,A,B}) \), we have: \( \lim_{z \to x} \text{Im} m(z) = \lim_{z \to x} \text{Im} \left( -\frac{e^{-1}}{u(z)v(z)} - \frac{1}{z} \right) = 0 \). Thus, using the Frobenius-Perron inversion formula, we get

\[
\int_{u(F_{c,A,B}) - \delta}^{u(F_{c,A,B})} dF(\lambda) = 0
\]

for small positive \( \delta \), which is impossible by definition of \( u(F_{c,A,B}) \). Hence, the curves are touching at a single point when \( x = u(F_{c,A,B}) \). This implies that they do not intersect when \( x < u(F_{c,A,B}) \).

Now we are ready to prove Lemma 3.

Proof of Lemma 3: Recall that by assumption, \( F^{AA'} \) almost surely weakly converges to \( F_A \) and \( u(F^{AA'}) \to u(F_A) \). Similarly, \( F^{BB'} \) almost surely weakly converges to \( F_B \) and \( u(F^{BB'}) \to u(F_B) \). These facts imply that if the curves in the \( \{u > u(F_A), v > v(F_B)\} \) subset of the \((u, v)\)-plane defined by (8) intersect at zero, or at two points, then the curves in the \( \{u > u(F^{AA'}), v > v(F^{BB'})\} \) subset of the \((u, v)\)-plane defined by

\[
\begin{align*}
    v &= x \left( c_n \int \frac{L_n}{u-\lambda} dF^{AA'}(\lambda) \right)^{-1} \\
    u &= x \left( \int \frac{L_n}{v-\lambda} dF^{BB'}(\lambda) \right)^{-1}
\end{align*}
\]

also intersect at zero, or at two points for large enough \( n \). Therefore, by Lemma A.4, \( u(F^{AA_n,B_n}) \) converges to \( u(F_{c,A,B}) \) and Lemma 3 follows.
References