# INSTANTANEOUS GRATIFICATION

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This Draft: June 3, 2011

Abstract. Extending Barro (1999) and Luttmer & Mariotti (2003), we introduce a new model of time preferences: the *instantaneous-gratification* model. This model applies tractably to a much wider range of settings than existing models. It applies to complete and incomplete-market settings and it works with generic utility functions. It works in settings with linear policy rules and in settings in which equilibrium cannot be supported by linear rules. The instantaneous-gratification model also generates a unique equilibrium, even in infinite-horizon applications, thereby resolving the multiplicity problem hitherto associated with dynamically inconsistent models. Finally, it simultaneously features a single welfare criterion and a behavioral tendency towards overconsumption.

JEL classification: C6, C73, D91, E21.

Keywords: quasi-hyperbolic discounting, time preference, dynamic inconsistency, continuous time, consumption, savings, buffer stock, Euler Relation, dynamic games, altruistic growth.

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*We thank Drew Fudenberg, George Mailath, Andrew Postlewaite and seminar participants at European Summer Symposium on Economic Theory (Gerzensee), Harvard University, Princeton University, Stanford University, Universita Bocconi, University of California Berkeley, University of Chicago, University College London, University of Minneapolis, University of Pennsylvania, Universitat Pompeu Fabra, Wharton, and Yale University. Laura Serban and Evdokia Nikolova provided outstanding research assistance. Harris acknowledges financial support from the British Academy. Laibson acknowledges financial support from the National Science Foundation (SBR-9510985), the Olin Foundation, the National Institute on Aging (R01-AG-1665) and the MacArthur Foundation. Christopher Harris: Cambridge University, King's College, Cambridge, United Kingdom, CB2 1ST. David Laibson: Harvard University, Department of Economics, Littauer M-14, Cambridge, MA, 02138.

## 1. Introduction

The discrete-time quasi-hyperbolic discount function $\{1, \beta\,\delta, \beta\,\delta^2, \beta\,\delta^3, ...\}$ is used to model high rates of short-run discounting.[1] With $\beta < 1$, this present-biased discount function generates a gap between a high short-run discount rate $(-\ln \beta\,\delta)$ and a low long-run rate $(-\ln \delta)$. The quasi-hyperbolic discount function has been used to study a range of behaviors, including consumption, procrastination, addiction and search.[2]

Extending the work of Barro (1999) and Luttmer and Mariotti (2003) on continuous-time models of non-exponential time preferences, the current paper shows how to operationalize quasi-hyperbolic time preferences in continuous time. Our model — which we call the instantaneous-gratification model or IG model for short — applies tractably to a much wider range of settings than existing models. For example, it applies to incomplete-market settings in which liquidity constraints arise because future labor income can't be used as collateral; and it works with an economically rich class of utility functions which includes, but is much larger than, the class with constant relative risk aversion. Consequently, we do not need to restrict our analysis to linear policy rules or to settings in which such rules support an equilibrium.

We develop the IG model in two steps. In the first step, following Barro and Luttmer-Mariotti, we assume that the present is valued discretely more than the future, mirroring the one-time drop in valuation implied by the discrete-time quasi-hyperbolic discount function. However, we assume that the transition from the present to the future occurs with a constant hazard rate $\lambda$. This assumption reduces the Bellman equation to a pair of stationary differential equations that characterize the current- and continuation-value functions. We call the model obtained after the first step the present-future model or PF model for short.

In the second step, we let the hazard rate $\lambda$ of transitions from present to future go to $\infty$. This brings us to the IG model. The Bellman equation for the IG model is even simpler than that of the PF model: it is a single ordinary differential equation.

Using convex duality we characterize the solution of the IG model. Specifically, we prove and then exploit the fact that the value function for the (dynamically *in*consistent)

[1] See Phelps and Pollak (1968) and Laibson (1997). Strotz (1956) first formalized the idea that the short-run discount rate is greater than the long-run discount rate. Loewenstein and Prelec (1992) axiomatize a true hyperboloid.

[2] For some examples, see Akerlof (1991), O'Donoghue and Rabin (1999a, 1999b), Angeletos, Laibson, Repetto, Tobacman and Weinberg (2001), DellaVigna and Malmendier (2004), and Della Vigna and Paserman (2005).

IG model is *identical* to the value function of a (dynamically consistent) optimization problem with (i) the same long-run discount rate as the IG model, and (ii) a different instantaneous utility function that depends on both the level of consumption and the level of financial assets. This optimization problem features the standard property that the instantaneous flow of utility depends on the agent's current consumption flow and the non-standard property that, holding current consumption fixed, the instantaneous flow of utility discretely jumps up when financial assets fall to zero.

Hence the IG model, which is dynamically *inconsistent*, has the same value function as a non-standard but dynamically *consistent* optimization problem. The IG model is not, however, observationally equivalent to this optimization problem: the IG model and the optimization problem share the same long-run discount rate and the same value function, but they have different instantaneous utility functions and different equilibrium policies.[3] The non-standard optimization problem is interesting, not because we think it is psychologically relevant, but rather because its partial equivalence enables us to use the machinery of optimization to study the value function of a dynamically inconsistent problem.

The IG model therefore carves out a tractable niche between dynamically inconsistent models and dynamically consistent models. On the one hand, it features dynamically inconsistent behavior and rational expectations. So, at each moment, the individual acts strategically with regard to her future preferences. On the other hand, the fact that the IG value function coincides with the value function of the related optimization problem implies that the IG model inherits many standard regularity properties.[4]

For example, the value-function-equivalence result implies that the IG model has a unique equilibrium. This uniqueness result is surprising, since the quasi-hyperbolic model is a dynamic game. Indeed, Krusell and Smith (2000) have shown that Markov-perfect equilibria are *not* unique in a deterministic discrete-time setting. In contrast, we provide two uniqueness results. First, we prove uniqueness in the case in which asset returns are stochastic. Second, we show that the unique equilibrium of the stochastic IG model converges to an equilibrium of the corresponding deterministic model as the noise in the asset returns goes to zero. In other words, we are able to select a unique equilibrium of

[3] See Laibson (1996) and Barro (1999) for the two special cases in which observational equivalence of the policy functions also holds: (1) log utility, time-varying interest interest, and no liquidity constraints; or (2) constant relative risk aversion, fixed interest rates, and no liquidity constraints.

[4] In discrete-time quasi-hyperbolic models, standard regularity properties (including differentiability and uniqueness) obtain provided that $\beta$ is close enough to 1 (Harris and Laibson 2002).

the deterministic IG model by using a natural variant of standard equilibrium-refinement procedures.[5]

Similarly, we can give a detailed characterization of the consumption function in the IG model. When the expected rate of return is below a key threshold, the equilibrium consumption function displays a discontinuity at the liquidity constraint. Consequently, consumption will fall discontinuously when a consumer spends down her assets and hits the liquidity constraint. This intuitive prediction is not possible in a dynamically-consistent consumption model. In such models, the timepath of consumption is continuous, even at the point at which the consumer hits a liquidity constraint.

Finally, the IG model features a *single* welfare criterion, even though the model involves dynamically inconsistent behavioral choices. Because the present is valued discretely more than the future, the current self has an incentive to overconsume; but the discretely higher value of the present only lasts for an instant, so this overvaluation does not affect the welfare criterion. Hence, the model simultaneously features a single welfare function and a behavioral tendency toward overconsumption.

In summary, the IG model is generalizable with regard to both market completeness and consumption preferences, supports a unique equilibrium, makes new predictions about the consumption function, and identifies a single sensible welfare criterion.

To understand the paper's overarching structure, it is helpful to divide the paper into three conceptual parts. In Sections 2, 3 and 4, we develop a formal game-theoretic foundation for the IG model. In Sections 5, 6 and 7, we analyze the IG model itself, characterizing its general properties and studying several applications. In Section 8, we show that all of the key results of earlier sections generalize to a large and flexible class of utility functions that includes all utility functions with constant relative risk aversion, and many other utility functions besides.

Turning to the detailed content, in Section 2 we present the PF model of time preferences and formulate some of its properties. In Section 3 we present the consumption problem that we use as our application. In Section 4 we describe the IG model, which arises when we let the hazard rate of transition from present to future go to infinity in the PF model. In Section 5 we show that the IG model has the same value function as

[5] Our uniqueness result even offers something new in settings in which linear policy rules support an equilibrium: it tells us that if one can find an equilibrium in linear policy rules — say by the method of undetermined coefficients — then that equilibrium is unique, not just in the set of equilibria in linear policy rules, but even in the set of *all* policy rules, linear or non-linear.

a related, but non-standard, dynamically-consistent optimization problem. We use this partial equivalence result to prove equilibrium existence and uniqueness. We also use it to derive a unique equilibrium of the limiting version of our model in which the return on the financial asset becomes deterministic. In Section 6, we characterize the equilibrium consumption function for the homogeneous limiting case of no labor income. In Section 7, we characterize the equilibrium consumption function for the inhomogeneous general case of non-zero labor income. In Section 8, we show that the value-function equivalence result of Section 5 generalizes to the class of utility functions which have variable but bounded relative risk aversion and relative prudence (or BRRA *and* BRP).[6] From this it follows that all the other results of Sections 5 and 7 generalize as well. This generalization allows us to extend the game-theoretic foundation for the IG model developed in Sections 2, 3 and 4 to include all utility functions with CRRA, and indeed all utility functions with BRRA and BRP. Section 9 concludes.

## 2. The Present-Future Model of Time Preferences

In this section, we describe a class of discount functions that model present-biased preferences in continuous time. There are two observationally equivalent representations: a stochastic discount function, which we present first, and a deterministic discount function.

**2.1. A Stochastic Discount Function.** In the *discrete-time* formulation of quasi-hyperbolic time preferences, it is natural to divide time into two intervals: the present — consisting of only the current period — and the future. All periods, present and future, are discounted *exponentially* with the discount factor $0 < \delta < 1$. Future periods are further discounted with uniform weight $0 < \beta \leq 1$. Combining these pieces, the present period (i.e. $t = 0$) receives full weight, and future periods (i.e. $t \geq 1$) are given weight $\beta\,\delta^t$.

This model can be generalized in two ways. First, the present could last for an arbitrary length of time, instead of ending after the current period. Second, the duration of the present could be stochastic, instead of being deterministic. Both of these generalizations have natural continuous-time analogues.

Consider an economic self born at time $s_0 = 0$. Call this self 'self 0'. The lifetime of self 0 is divided into two intervals: a 'present', which lasts from $s_0$ to $s_0 + \tau_0$; and a 'future', which lasts from $s_0 + \tau_0$ to $\infty$. Think of the present as the interval during

[6] Kimball (1990) introduced the concept of relative prudence.

which control is exercised by self 0, and of the future as the interval during which control is exercised by subsequent selves. The length $\tau_0$ of the present is stochastic, and is distributed exponentially with hazard rate $\lambda \in [0, \infty)$.

When the present of self 0 ends at $s_0 + \tau_0$, a new self is born and takes control of decision-making. Call this new arrival 'self 1'. The preferences of self 1, like those of self 0, can be divided into two intervals. Self 1 has a present that lasts from $s_1 = s_0 + \tau_0$ to $s_1 + \tau_1$, and a future that lasts from $s_1 + \tau_1$ to $\infty$. Continuing in this way, we obtain a sequence of selves $\{0, 1, 2, ...\}$ born at dates $\{s_0, s_1, s_2, ...\}$. For all $n \geq 1$, self $n$ has a present that lasts from $s_n = s_{n-1} + \tau_{n-1}$ to $s_n + \tau_n$, and a future that lasts from $s_n + \tau_n$ to $\infty$. Figure 1 provides a visual representation.

We assume that all selves discount exponentially with discount factor $0 < \delta < 1$. Furthermore, each self values her future discretely less than her present, discounting it by the additional factor $0 < \beta \leq 1$. More explicitly, we assume that self $n$ applies the discount factor $D_n(t)$ to the utility flow at time $s_n + t$, where

$$D_n(t) = \left\{ \begin{array}{lll} \delta^t & \text{if} & t \in [0, \tau_n) \\ \beta\,\delta^t & \text{if} & t \in [\tau_n, \infty) \end{array} \right\}. \tag{1}$$

In other words, her discount function $D_n$ decays exponentially at rate $\gamma = -\ln\delta$ up to time $\tau_n$, drops discontinuously at $\tau_n$ to a fraction $\beta$ of its level just prior to $\tau_n$, and decays exponentially at rate $\gamma$ thereafter.[7] Figure 2 plots a single realization of this discount function, with $\tau_n = 3.4$.

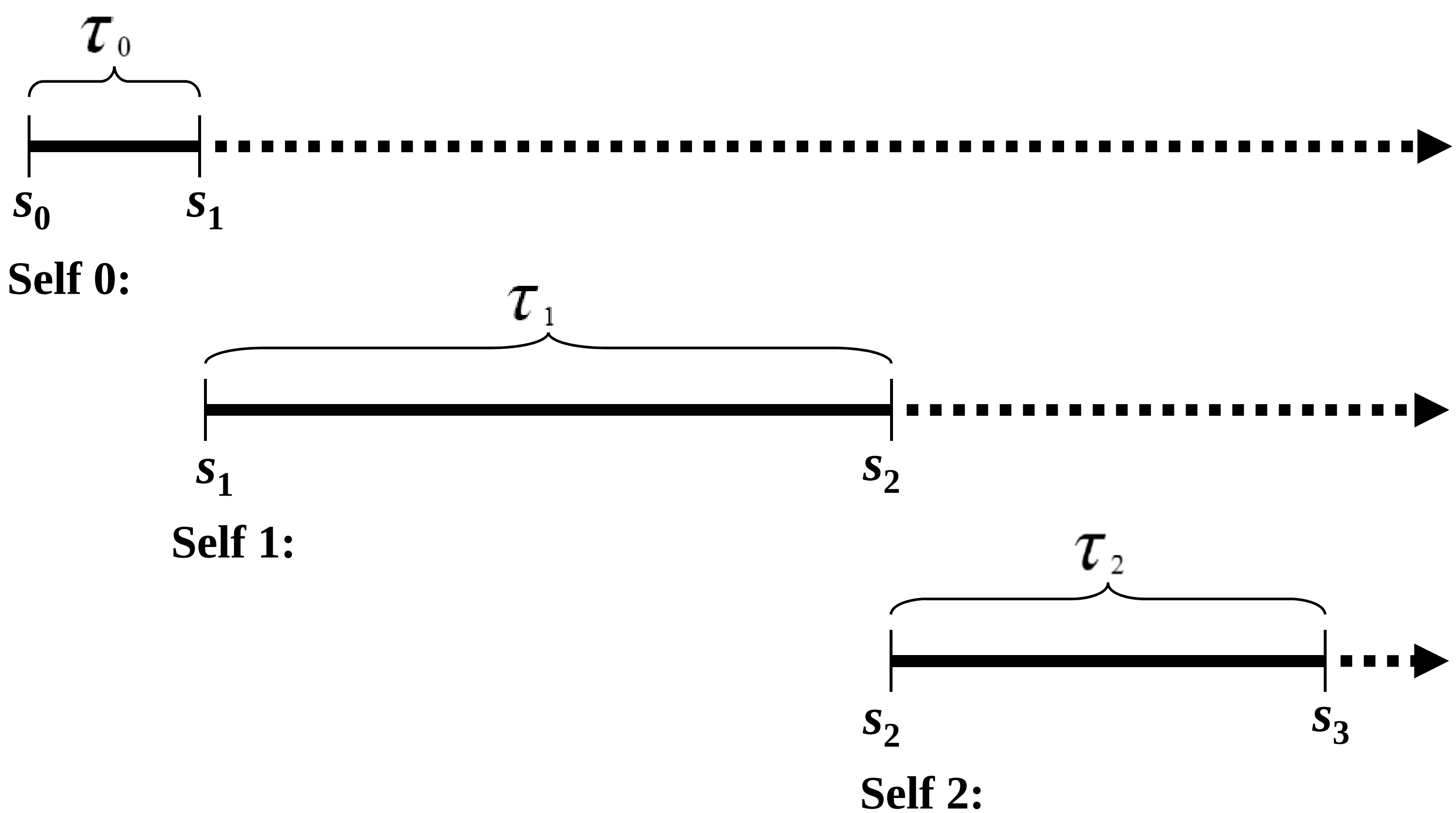
This continuous-time discount function nests classical exponential discounting: either set $\lambda = 0$, so that the future never arrives; or set $\beta = 1$, so that there is no distinction between present and future. It is similar to some of the deterministic discount functions used in Barro (1999) and Luttmer and Mariotti (2003). However, we assume that $\tau_n$ is stochastic. Among other things, this ensures that the expectation of the discount function is smooth.

When $\lambda \to \infty$, the discount function $D_n$ converges to the deterministic function $D_\infty$ given by the formula

$$D_\infty(t) = \left\{ \begin{array}{lll} 1 & \text{if} & t = 0 \\ \beta\,\delta^t & \text{if} & t \in (0, \infty) \end{array} \right\}.$$

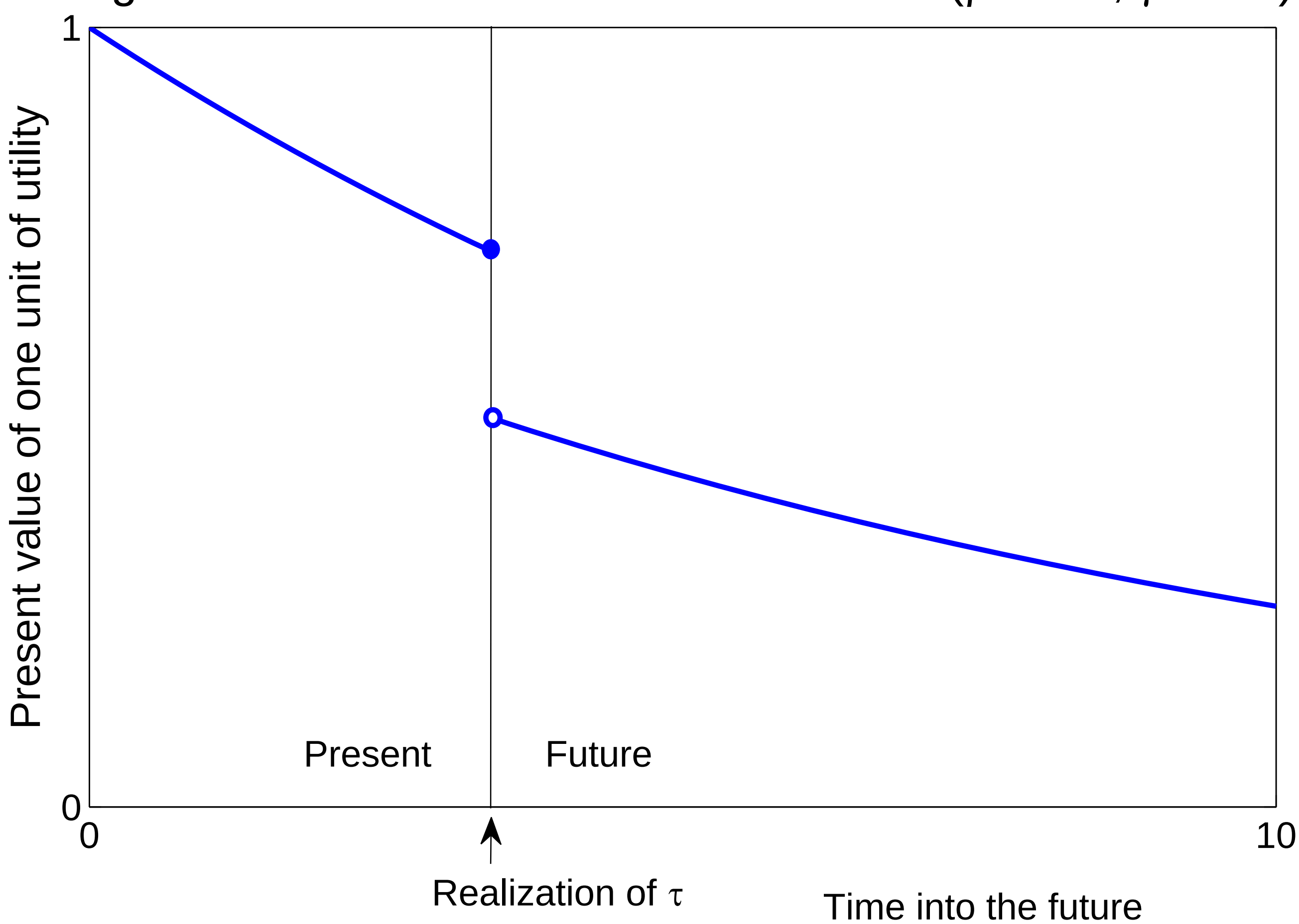
[7] The lengths $\{\tau_0, \tau_1, \tau_2, ...\}$ of the present intervals are i.i.d.

Figure 1: Sequential generations of autonomous selves.




The span of control (solid line) of self *n* lasts only from its birth at time $s_n$ to the birth of self *n*+1 at time $s_{n+1}$. The length of this control period, $s_{n+1} - s_n$, is the stochastic variable $\tau_n$, which has an exponential distribution.

Figure 2: Realization of discount function ($\beta = 0.7$, $\gamma = 0.1$)




The discount function represents the present value of one unit of future utility. The discount function discretely drops when the present ends and the future begins. This present-to-future transition occurs at a stochastic time. Figure 2 shows a particular realization of this transition.

Characterizing this limiting case is the main focus of the current paper.[8]

**2.2. A Reinterpretation Using a Deterministic Discount Function.** The arguments in this paper are consistent with a second interpretation of the time preferences described above: one can assume that a new self is born every *instant*; that the present of each self lasts only an instant; and that each self has a *deterministic* discount function $\overline{D}$ equal to the expected value of the *stochastic* discount function $D_n$ described above.[9] We describe this alternative deterministic interpretation in the current subsection and compare it to the stochastic approach in Subsection 2.3. Readers who wish to skip this material, can jump immediately to Section 3 without loss of continuity.

In the deterministic interpretation, each self uses the discount function $\overline{D}$ given by the formula

$$\overline{D}(t) = \mathrm{E}\big[D_n(t)\big] = e^{-\lambda t}\,\delta^t + (1 - e^{-\lambda t})\,\beta\,\delta^t.$$

$\overline{D}(t)$ is the sum of two terms. The first term is the probability $e^{-\lambda t}$ that the drop in $D_n$ does not occur before time $t$, times the discount factor $\delta^t$ that applies prior to the drop. The second term is the probability $1 - e^{-\lambda t}$ with which the drop in $D_n$ occurs after time $t$, times the discount factor $\beta\,\delta^t$ that applies after the drop. $\overline{D}(t)$ can also be written in the form

$$(1-\beta)\,e^{-(\gamma+\lambda)\,t} + \beta\,e^{-\gamma\,t},$$

where $\gamma = -\ln(\delta) > 0$ is the long-run discount rate. Written this way, $\overline{D}(t)$ is seen to be a convex combination of the short-run exponential discount factor $e^{-(\gamma+\lambda)\,t}$, with weight $1-\beta$, and the long-run exponential discount factor $e^{-\gamma\,t}$, with weight $\beta$.

The instantaneous discount rate associated with the deterministic discount function $\overline{D}$ is

$$-\frac{\overline{D}'(t)}{\overline{D}(t)} = \gamma + \frac{\lambda\,e^{-\lambda t}\,(1-\beta)\,\delta^t}{\overline{D}(t)}.$$

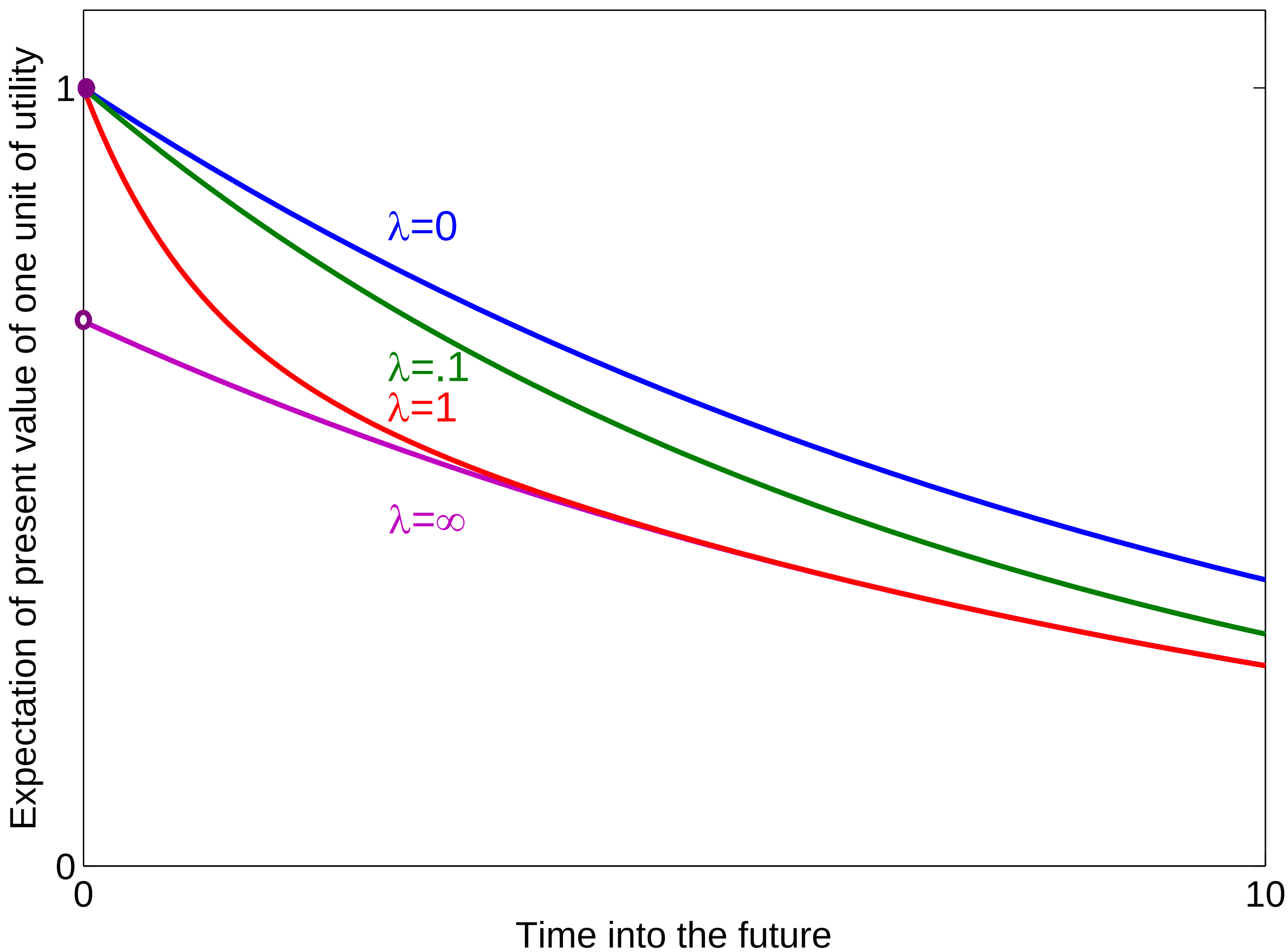
It too is the sum of two terms. The first term is the long-run (exponential) discount rate $\gamma$. The second term is the expected drop in $D$ at time $t$, namely $\lambda\,e^{-\lambda t}\,(1-\beta)\,\delta^t$, divided by the level of $\overline{D}$ at time $t$. Indeed: $\lambda\,e^{-\lambda t}$ is the flow probability with which the drop in $D$ occurs at time $t$; and $(1-\beta)\,\delta^t$ is the size of the drop in $D$ if the drop occurs at $t$.

Notice that the instantaneous discount rate decreases from $\gamma + \lambda\,(1-\beta)$ at $t = 0$ to

[8] Notice that, because $\tau_n \to 0$ as $\lambda \to \infty$, the expectation of the discount function — which is smooth when $\lambda$ is finite — has a discontinuity when $\lambda = \infty$. This will not cause any problems for us.

[9] See footnote 18 for a development of this line of argument.

Figure 3: Expectation of discount function β = 0.7, γ = 0.1, λ ∈ { 0, 0.1, 1, ∞ }




The expectation of the discount function represents the expected present value of one unit of future utility. The expectation integrates over the stochastic present-to-future transition time.

$\gamma$ at $t = \infty$. Figure 3 plots $\overline{D}$ for $\lambda \in \{0, 0.1, 1, \infty\}$.

**2.3. Comparison of the Stochastic and Deterministic Discount Functions.** The stochastic and deterministic discount functions differ in one important respect: the stochastic discount function assumes a present of non-infinitesimal duration $\tau_n > 0$, whereas the deterministic discount function assumes a present of infinitesimal duration $dt$. Hence the stochastic discount function assumes a countable number of non-infinitesimal selves, while the deterministic discount function assumes a continuum of infinitesimal selves.

The two formulations are however equivalent, in the critical sense that they generate the same equilibrium behavior. To see why, note that the current self in the stochastic formulation is dynamically consistent during her period of control between time $s_n$ and time $s_{n+1} = s_n + \tau_n$. It therefore makes no difference whether we regard her as a non-infinitesimal agent, who decides how to behave at the outset of her control interval, or as a continuum of infinitesimal agents, each of which makes a decision during its instant of control.

The stochastic formulation has two advantages over the deterministic one. First, it can be set up using only standard mathematical tools. Second, when the stochastic formulation is used, we can derive the IG model in a single step.[10] We therefore focus on the stochastic formulation.

## 3. Application to a Consumption Problem

In this section, we describe an important economic environment that we use to illustrate the implications of the discounting model. We study an infinite-horizon consumption-savings problem with liquidity constraints (cf. Deaton 1991, and Carroll 1992, 1997). We include liquidity constraints, since they make a fundamental difference to the analysis by necessitating nonlinear policy rules. On the other hand, we exclude labor-income uncertainty, since that would complicate the notation and does not affect our conclusions.

[10] In the analysis using the stochastic discount function, we let $\lambda$ go to infinity. In doing so, we simultaneously pass from non-infinitesimal to infinitesimal selves and from the finite-$\lambda$ discount function to the infinite-$\lambda$ discount function that is the ultimate focus of the paper. By contrast, in order to set up the deterministic discount function, we would first have to formalize the idea of an infinitesimal self. This would involve taking the limit as the span of control of a non-infinitesimal self goes to zero. We would then have to let $\lambda$ go to infinity, in order to pass from the finite-$\lambda$ discount function to the infinite-$\lambda$ discount function.

We also use this section to define equilibrium and to introduce the Bellman-equation representation that will organize all of our analysis.

**3.1. The Dynamics.** At any given point in time $t \in [0, \infty)$, the consumer has stock of (financial) wealth $x \in [0, \infty)$ and receives a flow of labor income $y \in (0, \infty)$. If $x > 0$ then she can choose any consumption level $c \in (0, \infty)$: wealth is a stock and consumption is a flow, so any finite consumption level is achievable provided that it is not maintained for too long. If $x = 0$ then she can only choose a consumption level $c \in (0, y]$: she has no wealth and she cannot borrow, so she cannot consume more than her labor income. In a word, she is is liquidity constrained.

Whatever the consumer does not consume is invested in an asset, the returns on which are distributed normally with mean $\mu\, dt$ and variance $\sigma^2\, dt$, where $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$. The change in her wealth at time $t$ is therefore

$$dx = (\mu\, x + y - c)\, dt + \sigma\, x\, dz,$$

where $z$ is a standard Wiener process.[11]

**3.2. Equilibrium.** Recall that the consumer is modeled as a sequence of autonomous selves (see Figure 1). Each self controls consumption during her own present and cares about — but does not control — consumption in her future. Our consumption problem is therefore an intrapersonal game. Following the literature in intergenerational games, our solution concept for this game will be stationary Markov-perfect equilibrium.[12]

Maskin and Tirole (2001) give a formal definition of Markov-perfect equilibrium (or MPE for short). MPE is a refinement of subgame-perfect equilibrium which only allows strategies to depend on information that is directly payoff relevant (i.e. information that is necessary to determine players' choice sets or payoffs). It does not allow strategies to depend on information that is only indirectly relevant (e.g. it does not allow the strategy of one player to depend on information that only becomes relevant if the strategy of another player depends on it). In our model, the only information that is directly payoff

[11] We could generalize this framework by adding a stochastic source of labor income. For example, we could assume that — in addition to her basic flow of labor income $y$ — the agent sporadically receives lump-sum bonuses. To preserve stationarity, such bonuses would need to arrive with a constant hazard rate and be drawn from a fixed distribution. We could even allow for non-stationary labor income, at the expense of an extra state variable. We do not pursue these generalizations, since they would not qualitatively change the analysis that follows.

[12] For a few important examples, see Bernheim and Ray (1987, 1989) and Leininger (1986).

relevant is the current level of wealth, so MPE restricts analysis to strategies that map current wealth to consumption.[13] We go further, restricting attention to *stationary* MPE (or SMPE for short). In other words, we study equilibria in which all selves use the same strategy.

Consider self $n$. Suppose that the future selves $n+1, n+2, \ldots$ all employ the same Markov strategy $\widetilde{c} : [0,\infty) \to (0,\infty)$. Then the dynamics of wealth from time $s_n + \tau_n$ onwards are given by

$$dx_t = (\mu\, x_t + y - \widetilde{c}(x_t))\, dt + \sigma\, x_t\, dz$$

and the continuation value of self $n$ is

$$v(x_{s_n+\tau_n}, \widetilde{c}) = \mathrm{E}_{s_n+\tau_n}\left[\int_{s_n+\tau_n}^{\infty} e^{-\gamma(t-(s_n+\tau_n))}\, u(\,\widetilde{c}(x_t))\, dt\right],$$

where: $s_n+\tau_n$ is the time at which control passes from self $n$ to self $n+1$; $x_{s_n+\tau_n}$ is wealth at time $s_n+\tau_n$; $u : (0,\infty) \to \mathbb{R}$ is the instantaneous utility function; $\gamma = -\ln(\delta) > 0$ is the long-run discount rate; and $\mathrm{E}_{s_n+\tau_n}$ denotes expectations conditional on the information available at time $s_n + \tau_n$.[14]

Suppose further that self $n$ employs the Markov strategy $c : [0,\infty) \to (0,\infty)$. Then the dynamics of wealth from time $s_n$ to time $s_n + \tau_n$ are given by

$$dx_t = (\mu\, x_t + y - c(x_t))\, dt + \sigma\, x_t\, dz$$

and the current value of self $n$ is

$$w(x_{s_n}, c, \widetilde{c}) = \mathrm{E}_{s_n}\left[\int_{s_n}^{s_n+\tau_n} e^{-\gamma(t-s_n)}\, u(c(x_t))\, dt + \beta\, e^{-\gamma\,\tau_n}\, v(x_{s_n+\tau_n}, \widetilde{c})\right],$$

where: $s_n$ is the time at which control passes from self $n-1$ to self $n$; $x_{s_n}$ is wealth at time $s_n$; and $\mathrm{E}_{s_n}$ denotes expectations conditional on the information available at time $s_n$.

The objective of self $n$ is to find a Markov strategy $c^*$ that is optimal in the sense that,

[13] In our model, the information available to self $n$ at time $t \in [s_n, s_{n+1})$ consists of the timepath $z : [0,t] \to \mathbb{R}$ of past shocks, the timepath $x : [0,t] \to [0,\infty)$ of past wealth, the sequence $\{s_0, s_1, \ldots, s_n\}$ of past transition times and the timepath $c : [0,t) \to (0,\infty)$ of past consumption. Of all this information, only the current value of wealth $x_t$ is directly payoff relevant.

[14] For a discussion of the regularity assumptions required to solve stochastic differential equations see Karatzas and Shreve (1991).

for all $x_{s_n} \geq 0$, $c^*$ maximizes $w(x_{s_n}, c, \widetilde{c})$ with respect to $c$. We denote by $\mathrm{BR}(\widetilde{c})$ the set of all such Markov strategies $c^*$. An SMPE of our model is then any Markov strategy $c$ such that $c \in \mathrm{BR}(c)$.

Two points about this objective should be noted. First, for any given $x_{s_n}$ and $\widetilde{c}$, self $n$ could in principle try various non-Markov strategies. Specifically, her consumption at time $t \in [s_n, s_n + \tau_n)$ could in principle depend on all the information available to her at time $t$. However, given that all future selves are employing Markov strategies, such non-Markov strategies never do better than an optimal Markov strategy. Second, for any pair of states $x^1_{s_n}, x^2_{s_n} \geq 0$, there is no conflict between the objective of maximizing $w(x^1_{s_n}, c, \widetilde{c})$ with respect to $c$ and the objective of maximizing $w(x^2_{s_n}, c, \widetilde{c})$ with respect to $c$. For any given $\widetilde{c}$, self $n$ can therefore find a single $c$ that simultaneously maximizes $w(x_{s_n}, c, \widetilde{c})$ with respect to $c$ for all $x_{s_n} \in [0, \infty)$.

**3.3. Bellman system for $v$, $w$, and $c$.** We can characterize SMPE using dynamic programming. First, the continuation-value function $v$ satisfies the differential equation

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - \widetilde{c}\,)\,v' - \gamma\,v + u(\,\widetilde{c}\,) \tag{2}$$

for $x \in [0, \infty)$, where we have suppressed the dependence of $v$ on $x$ and $\widetilde{c}$ and we have suppressed the dependence of $\widetilde{c}$ on $x$. The value function reflects the fact that at an optimum the following effects must sum to zero: the expected instantaneous change in the value function (namely $\frac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - \widetilde{c}\,)\,v'$); the instantaneous change in value due to discounting (namely $-\gamma\,v$); the instantaneous utility flow (namely $u(\,\widetilde{c}\,)$).

Second, the current-value function $w$ satisfies the differential equation

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,w'' + (\mu\,x + y - c)\,w' + \lambda\,(\beta\,v - w) - \gamma\,w + u(c) \tag{3}$$

for $x \in [0, \infty)$, where we have suppressed the dependence of $w$ on $x$, $c$ and $\widetilde{c}$ and the dependence of $c$ on $x$. This equation is very similar to equation (2). The only differences are: (i) the current-value function $w$ replaces the continuation-value function $v$; (ii) the Markov strategy $c$ employed by the current self replaces the Markov strategy $\widetilde{c}$ employed by future selves; and (iii) there is an additional term $\lambda\,(\beta\,v - w)$, which reflects the hazard rate $\lambda$ of making the transition from the present, valued by the current-value function $w$, to the future, valued by $\beta$ times the continuation-value function $v$.

Third, if self $n$ behaves optimally — taking the behavior of her future selves as given

— then $c$ will satisfy the instantaneous optimality condition

$$\left\{ \begin{array}{ll} u'(c) = w' & \text{if } x > 0 \\ u'(c) = \max\{u'(y), w'\} & \text{if } x = 0 \end{array} \right\}. \tag{4}$$

Intuitively, if $x > 0$, then there is no constraint on consumption. So consumption $c$ is chosen to equate the marginal utility of consumption $u'(c)$ and the marginal value of current wealth $w'$. If $x = 0$, then the liquidity constraint may or may not bind: if $w' < u'(y)$, then the constraint does bind, and $c = y$ (or, equivalently, $u'(c) = u'(y)$); if $w' \geq u'(y)$ then the constraint does not bind, and $u'(c) = w'$.

Fourth, systems of second-order ordinary differential equations like (2-4) typically require two boundary conditions. We have already supplied one boundary condition, by requiring that equations (2-4) hold at $x = 0$, and not just in the interior of the wealth space. We refer to this as the boundary condition at 0. But we need to supply a second boundary condition. This boundary condition will have two parts: global upper bounds for $v$ and $w$, and global lower bounds for $v$ and $w$. Among other things, these bounds have the effect of controlling the behavior of $v$ and $w$ near infinity.[15]

It is easy to see that $v$ is bounded above by the value function $\overline{v}$ of a consumer who: (i) has utility function $u$; and (ii) discounts the future exponentially at rate $\gamma$. Similarly, $w$ is bounded above by the value function $\overline{w}$ of a consumer who: (i) has utility function $u$; (ii) discounts the future using the stochastic discount function described in Section 2.1; but (iii) can commit her future selves to using the consumption function that she chooses for them.[16] It is also easy to see that, if $u$ is bounded below, then: $v$ is bounded below by $\frac{1}{\gamma}\, u(0)$; and $w$ is bounded below by $\frac{\gamma+\lambda\,\beta}{\gamma\,(\gamma+\lambda)}\, u(0)$.[17]

Putting these observations together, we have the following characterization of equilibrium in the PF model:

**Theorem 1.** *Suppose that $u$ is bounded below. Then the consumption function $c : [0, \infty) \to (0, \infty)$ is an SMPE of the PF model if and only if there is a continuation-*

[15] Intuitively speaking, providing appropriate global bounds is the correct way of supplying the missing boundary condition at $\infty$. However, note that such bounds are actually somewhat weaker than a standard boundary condition. For example, they do not require that $v$ and $w$ converge to specific values as $x \uparrow \infty$.

[16] The upper bound $\overline{w}$ for $w$ described here depends on $\lambda$. However, it is easy to see that, with a little more effort, this bound can be chosen to be independent of $\lambda$.

[17] The lower bound $\frac{\gamma+\lambda\,\beta}{\gamma\,(\gamma+\lambda)}\, u(0)$ for $w$ given here depends on $\lambda$. However, it is easy to see that $\frac{\gamma+\lambda\,\beta}{\gamma\,(\gamma+\lambda)}\, u(0) \geq \min\{\frac{\beta}{\gamma}\, u(0), \frac{1}{\gamma}\, u(0)\}$, which is independent of $\lambda$.

*value function* $v : [0,\infty) \to \mathbb{R}$ *and a current-value function* $w : [0,\infty) \to \mathbb{R}$ *such that* $(c,v,w)$ *together satisfy the pair of differential equations*

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - c)\,v' - \gamma\,v + u(c), \tag{5}$$
$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,w'' + (\mu\,x + y - c)\,w' + \lambda\,(\beta\,v - w) - \gamma\,w + u(c) \tag{6}$$

*for all* $x \in [0,\infty)$, *the optimality condition*

$$\left\{\begin{array}{ll} u'(c) = w' & \text{if } x > 0 \\ u'(c) = \max\{u'(y), w'\} & \text{if } x = 0 \end{array}\right\} \tag{7}$$

*and the global bounds*

$$\tfrac{1}{\gamma}\,u(0) \;\le\; v \le \overline{v}, \tag{8}$$
$$\tfrac{\gamma+\lambda\,\beta}{\gamma\,(\gamma+\lambda)}\,u(0) \;\le\; w \le \overline{w} \tag{9}$$

*for all* $x \in [0,\infty)$. *We refer to the system (5-9) as the Bellman system of the PF consumer.*[18]

The Bellman system of the IG consumer will be derived from the Bellman system of the PF consumer in three steps. In the first step, we let $\lambda \to \infty$ to obtain a preliminary version of the Bellman system of the IG consumer. In this system, the value continuation-

[18] In the model with the deterministic discount function $\overline{D} : [0,\infty) \to [0,1]$, the consumption function $c : [0,\infty) \to (0,\infty)$ is a stationary Markov-perfect equilibrium if and only if there is a value function $V : [0,\infty)^2 \to \mathbb{R}$ such that $(c,V)$ together satisfy

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,\tfrac{\partial^2 V}{\partial x^2}(t,x) + (\mu\,x + y - c(x))\,\tfrac{\partial V}{\partial x}(t,x) + \overline{D}(t)\,u(c(x)) + \tfrac{\partial V}{\partial t}(t,x)$$

for $(t,x) \in [0,\infty)^2$ and

$$\left\{\begin{array}{ll} u'(c(x)) = \tfrac{\partial V}{\partial x}(0,x) & \text{if either (i) } x > 0 \text{ or (ii) } x = 0 \text{ and } \tfrac{\partial V}{\partial x}(0,x) \ge u'(y) \\ c(x) = y & \text{if } x = 0 \text{ and } \tfrac{\partial V}{\partial x}(0,x) < u'(y) \end{array}\right\}$$

for all $x \in [0,\infty)$. Here $V(t,x)$ is the value at time 0 of consumption over the interval $[t,\infty)$, and $c(x)$ is the optimal consumption at time 0 given the marginal value of wealth at time 0, namely $\frac{\partial V}{\partial x}(0,x)$. We refer to this pair of equations as the Bellman system of the $\overline{D}$ consumer. It is valid for general $\overline{D}$. However, if $\overline{D}(t) = e^{-\lambda\,t}\,\delta^t + (1-e^{-\lambda\,t})\,\beta\,\delta^t$ as in the text, then it is easy to show that they have a solution $(c,V)$ in which $V(t,x)$ takes the form $e^{-\lambda\,t}\,\delta^t\,w(x) + (1-e^{-\lambda\,t})\,\beta\,\delta^t\,v(x)$ if and only if $(c,v,w)$ satisfies the Bellman system of the PF consumer. In other words: any solution of the Bellman system of the PF consumer generates a solution of the Bellman system of the $\overline{D}$ consumer; but the possibility remains that the Bellman system of the $\overline{D}$ consumer has other solutions that do not have this form.

value function $v$ is required to satisfy the same global lower bound $v \geq \frac{1}{\gamma}\, u(0)$ as in the Bellman system of the PF consumer. In the second step, we show that any solution of the preliminary version of the Bellman system of the IG consumer in fact satisfies the tighter global lower bound $v \geq \frac{1}{\gamma}\, u(y)$. We can therefore reformulate the Bellman system of the IG consumer to incorporate this stronger requirement. In the third and final step, we take advantage of the fact that $\frac{1}{\gamma}\, u(y)$ is finite, whether or not $u$ is bounded below, to remove the requirement that $u$ be bounded below. This is important, because it allows us to extend our theory to utility functions with constant relative risk aversion $\rho \geq 1$.[19]

**Remark 2.** *Our approach to the case in which $u$ is unbounded below is analogous to a refinement argument: where a standard refinement argument would consider the limit of games in which trembles are bounded away from $0$, we consider the limit of games in which the utility function is bounded away from $-\infty$.*[20]

**Remark 3.** *In general, the PF model can be expected to have a finite number of equilibria. Furthermore, if $\lambda$ is close to $0$ (a dynamically consistent limit case), then equilibrium is unique. Similarly, if $\beta$ is close to $1$ (another dynamically consistent limit case), then equilibrium is again unique. Much more interestingly, if $\lambda$ is close to $\infty$ (a dynamically inconsistent limit case), then equilibrium is unique.*

## 4. The Instantaneous-Gratification Model

Experimental evidence suggests that the present — in other words, the interval $[s_n, s_n{+}\tau_n)$ during which consumption is *not* down-weighted by $\beta$ — is short.[21] This is the same as saying that $\lambda$ is large, since the arrival rate of the future is $\lambda$. In the current section,

[19] If $u$ has constant relative risk aversion $\rho$ then: (i) if $\rho < 1$ then $u$ is unbounded above and bounded below: (ii) if $\rho > 1$ then $u$ is bounded above and unbounded below: and (iii) if $\rho = 1$ then $u$ is unbounded both above and below.

[20] An alternative approach to the case in which $u$ is unbounded below would be to restrict consumption to the interval $[\underline{c}, \infty)$ for some small $\underline{c} > 0$, and then consider the limit as $\underline{c} \downarrow 0$. We did not choose this approach since it makes the optimality condition somewhat more complicated.

[21] For example, McClure et al (2007) estimate a 50% discount rate over the course of an hour for food/drink rewards. In most intertemporal choice studies, sharp short-run discounting (at least 10% and usually much more) is observed at horizons of hours and days (e.g., see Ainslie 1992, Frederick, Loewenstein, and O'Donoghue 2002). [For biblio: McClure, Samuel, Keith Ericson, David Laibson, George Loewenstein, and Jonathan Cohen. 2007. Time Discounting for Primary Rewards. Journal of Neuroscience., 27: 5796–5804. ][ For biblio: Time Discounting and Time Preference: A Critical Review
Shane Frederick; George Loewenstein; Ted O'Donoghue
Journal of Economic Literature, Vol. 40, No. 2. (Jun., 2002), pp. 351-401. ] [for biblio: Ainslie, G. (1992). Picoeconomics. Cambridge: Cambridge University Press.]

we consider the limiting case $\lambda \to \infty$, which serves as an approximation of situations in which the duration of the present (namely $\tau$) is short. We refer to the limiting case as the *instantaneous-gratification* model, or IG model.

In a later section – Section 6 – we show that $\lambda = \infty$ is a good approximation for $\lambda \in [10, \infty)$. In other words, in a model with annual time units, if the duration of the present is expected to be about a month (or less), then the IG model ($\lambda \to \infty$) is a good approximation for the PF model.

In the current section, we discuss two distinct ways of deriving the IG model. The first approach is to derive the Bellman system of the IG consumer by taking the limit of the Bellman system of the PF consumer as $\lambda \to \infty$. The second approach is to derive the Bellman system of the IG consumer directly from an analysis of her objective. The first approach has the advantage that it is more compelling from a mathematical point of view.[22] The second approach has the advantage that it generates intuitive insights into the logic of the IG model. We describe both approaches. However, either approach is sufficient to follow the arguments in the paper.

**4.1. The First Approach.** Suppose that the triple $(c_\lambda, v_\lambda, w_\lambda)$ solves the Bellman system of the PF consumer. In particular, the following equations and inequalities hold for all $x \in [0, \infty)$:

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v_\lambda'' + (\mu\,x + y - c_\lambda)\,v_\lambda' - \gamma\,v_\lambda + u(c_\lambda), \tag{10}$$

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,w_\lambda'' + (\mu\,x + y - c_\lambda)\,w_\lambda' + \lambda\,(\beta\,v_\lambda - w_\lambda) - \gamma\,w_\lambda + u(c_\lambda), \tag{11}$$

[22]An equilibrium $c$ of the PF model can be represented in one of two ways: it can be represented as a fixed point of the mapping from the consumption function $\widetilde{c}$ employed by future selves to the consumption functions $c$ that are optimal for the present self; or it can be represented as the first component $c$ of a solution $(c, v, w)$ of the Bellman system of the PF consumer. The first representation is traditionally regarded as the primary definition, and the second representation is traditionally regarded as a characterization. The second of our two approaches reverses this traditional point of view. It effectively identifies the PF model with the Bellman system of the PF consumer, and it identifies an equilibrium of the PF model with the first component $c$ of a solution $(c, v, w)$ of the Bellman system of the PF consumer. It then identifies the IG model with the limit of the Bellman system of the PF consumer, and it identifies an equilibrium of the IG model with the first component $c$ of a solution $(c, v)$ of the Bellman system of the IG consumer. This approach is especially compelling in the light of the fact that the Bellman system of the IG consumer has a unique solution: not only do we have upper hemicontinuity (in the sense that the limit of any sequence of equilibria of the PF model is an equilibrium of the IG model); but we even have lower hemicontinuity (in the sense that any equilibrium of the IG model is the limit of a sequence of equilibria of the PF model).

$$\left\{ \begin{array}{ll} u'(c_\lambda) = w'_\lambda & \text{if } x > 0 \\ u'(c_\lambda) = \max\{u'(y), w'_\lambda\} & \text{if } x = 0 \end{array} \right\} \tag{12}$$

and

$$\tfrac{1}{\gamma}\, u(0) \leq v \leq \overline{v}. \tag{13}$$

Assuming that $(c_\lambda, v_\lambda, w_\lambda) \to (c, v, w)$ as $\lambda \to \infty$, the equations characterizing $(c, v, w)$ can then be derived as follows.

Note first that equation (10) does not depend directly on $\lambda$. Indeed, this equation only applies after the transition to the future has taken place, so it is not affected in any way by the arrival rate of the future. Letting $\lambda \to \infty$ therefore preserves the form of the equation, yielding

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, v'' + (\mu\, x + y - c)\, v' - \gamma\, v + u(c) \tag{14}$$

for all $x \in [0, \infty)$. In other words, just as $v_\lambda$ was the expected present discounted value obtained when consumption was chosen according to the exogenously given consumption function $c_\lambda$, so $v$ is the expected present discounted value obtained when consumption is chosen according to the exogenously given consumption function $c$.

Second, equation (11) does depend directly on $\lambda$. It can, however, be rearranged to give

$$w_\lambda - \beta\, v_\lambda = \tfrac{1}{\lambda} \left( \tfrac{1}{2}\, \sigma^2\, x^2\, w''_\lambda + (\mu\, x + y - c_\lambda)\, w'_\lambda - \gamma\, w_\lambda + u(c_\lambda) \right).$$

Letting $\lambda \to \infty$ then yields $w - \beta\, v = 0$. This reflects the fact that, as $\lambda \to \infty$, the discount function drops essentially immediately to a fraction $\beta$ of its initial value, and that the current-value function $w$ is therefore $\beta$ times the continuation-value function $v$.[23]

Third, like equation (10), equation (12) does not depend directly on $\lambda$. Letting $\lambda \to \infty$ therefore preserves the form of this equation, yielding

$$\left\{ \begin{array}{ll} u'(c) = w' & \text{if } x > 0 \\ u'(c) = \max\{u'(y), w'\} & \text{if } x = 0 \end{array} \right\}.$$

[23] In order to derive the conclusion that $w = \beta\, v$ more rigorously, one can proceed as follows. Subtracting $\beta$ times equation (10) from equation (11), we obtain

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, (w_\lambda - \beta\, v_\lambda)'' + (\mu\, x + y - c_\lambda)\, (w_\lambda - \beta\, v_\lambda)' - (\gamma + \lambda)\, (w_\lambda - \beta\, v_\lambda) + (1 - \beta)\, u(c_\lambda).$$

In other words, $w_\lambda - \beta\, v_\lambda$ is the value function of a consumer with utility function $(1-\beta)\, u$ and exogenously given consumption function $c_\lambda$. It follows that $w_\lambda - \beta\, v_\lambda \leq (1 - \beta)\, \overline{a}_\lambda$, where $\overline{a}_\lambda$ is the value function of a consumer who: (i) has utility function $\overline{u} \geq u$ given by the formula $\overline{u} = u + \max\{0, -u(0)\}$; and (ii) has discount rate $\gamma + \lambda$. It also follows that $w_\lambda - \beta\, v_\lambda \geq \frac{1-\beta}{\gamma+\lambda}\, u(0)$. Hence $w_\lambda - \beta\, v_\lambda \to 0$ pointwise as $\lambda \to \infty$.

In other words, just as $c_\lambda$ was the optimal consumption function when the current-value function was $w_\lambda$, so $c$ is the optimal consumption function when the current-value function is $w$. But we showed in the preceding paragraph that $w = \beta\, v$. We therefore have

$$\left\{ \begin{array}{ll} u'(c) = \beta\, v' & \text{if } x > 0 \\ u'(c) = \max\{u'(y), \beta\, v'\} & \text{if } x = 0 \end{array} \right\} \tag{15}$$

The logic behind this condition is the same as that behind (4). If $x > 0$, or if $x = 0$ and the liquidity constraint does not bind, then $c$ is chosen to equate the marginal utility of consumption $u'(c)$ and the marginal value of current wealth $\beta\, v'$. If, on the other hand, $x = 0$ and the liquidity constraint *does* bind, then $c = y$ (or, equivalently, $u'(c) = u'(y)$).

Fourth and last, note that inequality (13) does not depend directly on $\lambda$. We can therefore let $\lambda \to \infty$ to obtain

$$\tfrac{1}{\gamma}\, u(0) \le v \le \overline{v}. \tag{16}$$

Overall, our discussion motivates the following definition:

**Definition 4.** *The* ***Bellman system of the IG consumer (with global lower bound*** $\boldsymbol{v \ge \frac{1}{\gamma}\, u(0)}$***)*** *consists of the differential equation*

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, v'' + (\mu\, x + y - c)\, v' - \gamma\, v + u(c) \tag{17}$$

*for all* $x \in [0, \infty)$, *the optimality condition*

$$\left\{ \begin{array}{ll} u'(c) = \beta\, v' & \textit{if } x > 0 \\ u'(c) = \max\{u'(y), \beta\, v'\} & \textit{if } x = 0 \end{array} \right\} \tag{18}$$

*and the global bounds*

$$\tfrac{1}{\gamma}\, u(0) \le v \le \overline{v} \tag{19}$$

*for all* $x \in [0, \infty)$.

Notice that the Bellman system of the IG consumer differs from the Bellman system of an exponential consumer with utility function $u$ and discount rate $\gamma$ only in that the marginal value of (future) wealth $v'$ is multiplied by the factor $\beta$ in the optimality condition (18). Furthermore the former reduces to the latter if we put $\beta = 1$. The presence of the multiplicative $\beta$ term is a slight variation on the usual form of the Envelope Theorem.

This "new" Envelope Theorem is quite natural since the future arrives instantaneously, and the future has continuation value $\beta\, v$. Hence the marginal value of wealth is $w' = \beta\, v'$ and, at an equilibrium, the current self sets consumption to obtain $u'(c) = \beta\, v'$.

Notice too that the global lower bound $v \geq \frac{1}{\gamma}\, u(0)$ features explicitly in the definition. This is because, in some contexts, it is necessary to distinguish between different versions of the IG model. For example, we shall show in Theorem 5 below that, if $(c, v)$ satisfies the Bellman system of the IG consumer with lower bound $v \geq \frac{1}{\gamma}\, u(0)$, then it also satisfies Bellman system of the IG consumer with the tighter lower bound $v \geq \frac{1}{\gamma}\, u(y)$. However, for the most part, it should be clear from the context which global lower bound we are using. That is why the global lower bound is placed in parentheses.

**4.2. The Second Approach.** In the PF model, there is a sequence of selves $\{0, 1, 2, ...\}$, each of whom has a strictly positive span of control. In the IG model there is a continuum of selves $[0, \infty)$, each of whom has an infinitesimal span of control. In formulating the objective of the IG consumer, it is therefore important to bear in mind that her span of control is an instant, and that changes in her behavior have only an infinitesimal effect on her objective. In particular, careful track must be kept of such infinitesimal effects.

Consider self $s \in [0, \infty)$, and suppose that all future selves use the consumption function $\widetilde{c} : [0, \infty) \to (0, \infty)$. Then the continuation-value function of self $s$ is exactly the same as the continuation-value function of the PF consumer, namely $v$. In particular, $v$ satisfies the differential equation

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, v'' + (\mu\, x + y - \widetilde{c}\,)\, v' - \gamma\, v + u(\,\widetilde{c}\,)$$

for $x \in [0, \infty)$, where we have suppressed the dependence of $v$ and $\widetilde{c}$ on $x$.

Suppose further that self $s$ has wealth $x$, and that she chooses the consumption level $c \in (0, \infty)$. Then the current value of self $s$ is

$$w(x) = \mathrm{E}_s\big[u(c)\, dt + \beta\, \exp(-\gamma\, dt)\, v(x + dx)\big].$$

Now, Itô's Lemma implies that $\exp(-\gamma\, dt) = 1 - \gamma\, dt$ and

$$v(x + dx) = v(x) + v'(x)\, dx + \tfrac{1}{2}\, v''(x)\, (dx)^2.$$

Moreover $dx = (\mu\, x + y - c)\, dt + \sigma\, x\, dz$ and $(dx)^2 = \sigma^2\, x^2\, dt$. Hence

$$\begin{aligned} w(x) &= \mathrm{E}_s\left[u(c)\, dt + \beta\, v(x) + \beta\, \left(v'(x)\, dx + \tfrac{1}{2}\, v''(x)\, (dx)^2 - \gamma\, v(x)\, dt\right)\right] \\ &= \beta\, v(x) + \left(\beta\, \left(\tfrac{1}{2}\, \sigma^2\, x^2 v''(x) + (\mu\, x + y - c)\, v'(x) - \gamma\, v(x)\right) + u(c)\right) dt. \end{aligned}$$

In other words, there are two contributions to the current value of self $s$: the non-infinitesimal contribution $\beta\, v(x)$, and the infinitesimal contribution

$$\left(\beta\, \left(\tfrac{1}{2}\, \sigma^2\, x^2 v''(x) + (\mu\, x + y - c)\, v'(x) - \gamma\, v(x)\right) + u(c)\right) dt. \tag{20}$$

It follows at once that $w(x) = \beta\, v(x)$.

Furthermore the infinitesimal contribution (20) depends on $c$ only via the term

$$(u(c) - \beta\, v'(x)\, c)\, dt.$$

Hence, in order to maximize her current value, self $s$ need only choose $c$ to maximize this expression. Bearing in mind that self $s$ is free to choose any $c \in (0, \infty)$ when $x > 0$, and that she must choose $c \in (0, y]$ when $x = 0$, it follows that $c$ must satisfy the optimality condition

$$\left\{ \begin{array}{ll} u'(c) = \beta\, v' & \text{if } x > 0 \\ u'(c) = \max\{u'(y), \beta\, v'\} & \text{if } x = 0 \end{array} \right\},$$

where we have suppressed the dependence of $v'$ on $x$.

Next, just as in Section 3.3, $v$ is bounded above by the value function $\overline{v}$ of a consumer who: (i) has utility function $u$; and (ii) discounts the future exponentially at rate $\gamma$. Similarly, if $u$ is bounded below, then $v$ is bounded below by $\frac{1}{\gamma}\, u(0)$.

Finally, in a stationary equilibrium we must have $c = \widetilde{c}$. Overall, then, we conclude that $(c, v)$ satisfies the Bellman system of the IG consumer (with global lower bound $\frac{1}{\gamma}\, u(0)$).

## 5. Existence, Uniqueness and Value-Function Equivalence

In this section we show that the value function of the IG consumer exists and is unique. To prove this, we use a key intermediate result. We describe an *alternative* consumer with dynamically *consistent* preferences and a slightly altered utility function $\widehat{u}$. We show that a value function $v$ solves the Bellman equation of the IG consumer if and only if it solves the Bellman equation of this dynamically consistent "$\widehat{u}$-consumer". We call this result "value-function equivalence". We also emphasize that value-function equivalence is *not* the same as observational equivalence, and indeed that observational equivalence does *not* hold: the consumption function of the IG consumer is not the same as the consumption function of the $\widehat{u}$-consumer.

Value-function equivalence implies both the existence and the uniqueness of the value function of the IG consumer, for the simple reason that the $\widehat{u}$-consumer solves an optimization problem, and the value function of an optimization problem always exists and is unique.

Uniqueness is the most important property of the IG model: the IG model resolves the multiplicity problem that has plagued the literature on dynamically inconsistent preferences.

The current section also discusses a number of other issues, including the extension of the uniqueness result to the deterministic version of our model (in which asset returns are non-stochastic).

**5.1. Assumptions.** Before proceeding, we make the following simple assumptions, which temporarily restrict attention to utility functions with constant relative risk aversion:

**A1** $u(c) = \left\{ \begin{array}{ll} \frac{1}{1-\rho}\left(c^{1-\rho}-1\right) & \text{if } \rho \neq 1 \\ \ln(c) & \text{if } \rho = 1 \end{array} \right\}$;

**A2** $1-\beta < \rho$;

**A3** $\mu < \overline{\mu}$, where $\overline{\mu} = \left\{ \begin{array}{ll} \frac{1}{1-\rho}\gamma + \frac{1}{2}\rho\sigma^2 & \text{if } \rho < 1 \\ \infty & \text{if } \rho \geq 1 \end{array} \right\}$.

These assumptions can be weakened considerably, in an economically interesting way, without affecting any of our results. See Section 8 below.

Assumption A1 is standard. It means that $u$ has constant relative risk aversion $\rho$. Assumption A2 means that the dynamic inconsistency of the IG consumer (as measured by $1-\beta$) is less than the coefficient of relative risk aversion (namely $\rho$).[24] This assumption

[24] The case $1-\beta > \rho$ can also be analyzed. In this case, the consumer's desire to consume immediately (as measured by $1-\beta$) outweighs her desire to smooth consumption over time (as measured by $\rho$). The result is an equilibrium in which the current self consumes all her financial wealth during her instant of control, thereby forcing all subsequent selves to consume only their labor income $y$. Since $\frac{u(c)}{c} \to 0$

would be satisfied in a standard calibration: empirical estimates of the coefficient of relative risk aversion $\rho$ typically lie between $\frac{1}{2}$ and 5; and the short-run discount factor $\beta$ is typically thought to lie between $\frac{1}{2}$ and 1.[25] Assumption A3 is a one-sided (and therefore weaker) version of a standard integrability assumption.[26] It ensures that the consumer's expected lifetime utility is not positively infinite even when her utility function is unbounded above (i.e. when $\rho \leq 1$). It achieves this by ensuring that her wealth does not grow too fast. Notice that the upper bound on the rate of growth of wealth $\overline{\mu}$ is increasing in $\rho$ for $\rho \in (0,1)$, and goes to $\infty$ as $\rho$ goes to 1. In other words: the slower utility grows with consumption, the less stringent the restriction on $\mu$ becomes; and no restriction on $\mu$ is necessary if $\rho \geq 1$.

**5.2. The Bellman System of the IG Consumer.** Assumptions A1-A3 allow us to establish the following bootstrap result, which provides a tighter lower bound on the value function of the IG consumer.

**Theorem 5.** *Suppose that $u$ is bounded below, and that $(c, v)$ satisfies the Bellman system of the IG consumer with global lower bound $v \geq \frac{1}{\gamma}\, u(0)$, namely (17-19). Then $v$ satisfies the Bellman system of the IG consumer with the tighter global lower bound $v \geq \frac{1}{\gamma}\, u(y)$.*

**Proof.** See Appendix A. ■

In view of Theorem 5, we may as well work with the following tighter formulation of the Bellman system of the IG consumer:

**Definition 6.** *The* ***Bellman system of the IG consumer (with global lower bound $v \geq \frac{1}{\gamma}\, u(y)$)*** *consists of the differential equation*

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - c)\,v' - \gamma\,v + u(c) \tag{21}$$

as $c \to \infty$, this burst of consumption by the current self contributes nothing to the integral of lifetime utility. The value function is therefore $v = \frac{1}{\gamma}\,u(y)$. (Notice that, for this value function, we have $v' = 0$. This is consistent with the infinite consumption rate.)

[25] See Laibson et al (1998) and Ainslie (1992).

[26] In the model with $y = 0$ and $\beta = 1$, it is standard to assume that $\gamma > (1-\rho)\,(\mu - \frac{1}{2}\,\rho\,\sigma^2)$. This assumption can be broken down into two parts: if $\rho < 1$, then it can be rearranged to give $\mu < \frac{1}{1-\rho}\,\gamma + \frac{1}{2}\,\rho\,\sigma^2$; and, if $\rho > 1$, then it can be rearranged to give $\mu > \frac{1}{1-\rho}\,\gamma + \frac{1}{2}\,\rho\,\sigma^2$. Because we assume that $y > 0$, we can dispense with the second part.

*for all* $x \in [0, \infty)$*, the optimality condition*

$$\left\{ \begin{array}{ll} u'(c) = \beta\, v' & \text{if } x > 0 \\ u'(c) = \max\{u'(y), \beta\, v'\} & \text{if } x = 0 \end{array} \right\} \tag{22}$$

*and the global bounds*

$$\tfrac{1}{\gamma}\, u(y) \leq v \leq \overline{v}. \tag{23}$$

We have derived this (for our purposes definitive) formulation of the Bellman system of the IG consumer from the PF model under the joint assumptions that A1-A3 hold and that $u$ is bounded below. However, the formulation itself is economically and mathematically meaningful under Assumptions A1-A3 *alone*, whether or not $u$ is bounded below. For the analysis that follows, we shall therefore drop the assumption that $u$ is bounded below.[27] We shall show how to derive this formulation of the Bellman system of the IG consumer without the assumption that $u$ is bounded below in Section 8.

**5.3. Value-Function Equivalence.** Armed with Assumptions A1-A3, which we maintain until further notice, we now introduce the dynamically consistent $\widehat{u}$-consumer and show she has the same value function as the IG consumer.

**Theorem 7** [Value-Function Equivalence]**.** *There exist strictly increasing and concave utility functions* $\widehat{u}_+ : (0, \infty) \to \mathbb{R}$ *and* $\widehat{u}_0 : (0, y] \to \mathbb{R}$ *such that, if we*

1. *define a wealth-dependent utility function* $\widehat{u}$ *by the formula*

$$\widehat{u}(\widehat{c}, x) = \left\{ \begin{array}{ll} \widehat{u}_+(\widehat{c}) & \text{if } \widehat{c} \in (0, \infty) \text{ and } x > 0 \\ \widehat{u}_0(\widehat{c}) & \text{if } \widehat{c} \in (0, y] \text{ and } x = 0 \end{array} \right\};$$

2. *introduce a new consumer, whom we shall refer to as the* $\widehat{u}$ *consumer, who:*
   - (a) *discounts the future exponentially at rate* $\gamma$*,*
   - (b) *faces the same wealth dynamics as the IG consumer and*
   - (c) *has the utility function* $\widehat{u}$*;*

[27] This strengthens the analysis, because the results that we establish hold under weaker assumptions.

*then $v$ is a value function of the IG consumer iff $v$ is a value function of the $\widehat{u}$ consumer.*

The $\widehat{u}$-consumer has both conventional and unconventional features. On the conventional side, she is an exponential discounter with discount rate $\gamma$. In other words, she has dynamically consistent preferences. On the unconventional side, her utility function depends on her financial wealth $x$. When $x > 0$, her utility function is $\widehat{u}_+$. When $x = 0$, it is $\widehat{u}_0$. The utility function $\widehat{u}$ is constructed so as to generate value-function equivalence, and it turns out that $\widehat{u}_0(\widehat{c}) \geq \widehat{u}_+(\widehat{c})$ for all $\widehat{c}$.[28]

Using the Value-Function Equivalence Theorem, we can reduce the study of the problem of the IG consumer, which is game-theoretic, to the study of the problem of the $\widehat{u}$ consumer, which is decision-theoretic (i.e. non-strategic). There is, however, an important caveat: while the *value* function of the IG consumer coincides with *value* function of the $\widehat{u}$ consumer, it is not the case that the *consumption* function of the IG consumer coincides with the *consumption* function of the $\widehat{u}$ consumer. In particular, value-function equivalence does not translate into observational equivalence in behavior. See Appendix F.7 for a detailed exploration of the relationship between $c$ and $\widehat{c}$.

**Proof.** We begin with an overview of the proof. The first step of the proof is to eliminate $c$ from the Bellman *system* of the IG consumer to yield what we call the Bellman *equation* of the IG consumer. The second step is to eliminate $\widehat{c}$ from the Bellman *system* of the $\widehat{u}$ consumer to arrive at what we call the Bellman *equation* of the $\widehat{u}$ consumer. Third, we note that if we put

$$\widehat{u}_+(\widehat{c}) = \frac{\psi}{\beta}\, u\!\left(\frac{\widehat{c}}{\psi}\right) + \frac{\psi - 1}{\beta} \text{ for } \widehat{c} \in (0, \infty)$$

and

$$\widehat{u}_0(\widehat{c}) = \left\{ \begin{array}{ll} \widehat{u}_+(\widehat{c}) & \text{for } \widehat{c} \in (0, \psi\, y] \\ \widehat{u}_+(\psi\, y) + (\widehat{c} - \psi\, y)\, \widehat{u}'_+(\psi\, y) & \text{for } \widehat{c} \in [\psi\, y, y] \end{array} \right\},$$

where

$$\psi = \frac{\rho - (1 - \beta)}{\rho},$$

then the Bellman equation of the IG consumer is identical to the Bellman equation of the $\widehat{u}$ consumer. The two equations must therefore have the same set of solutions.

[28] Because the utility function changes from $\widehat{u}_+$ to $\widehat{u}_0$ at $x = 0$, the boundary condition of the Bellman equation of the $\widehat{u}$-consumer at $x = 0$ takes a slightly unconventional form. This does not lead to any difficulties.

Turning to the first step, for all $\alpha > 0$: let $f_+(\alpha)$ be the unique $c$ satisfying $u'(c) = \alpha$; and put $h_+(\alpha) = u(f_+(\beta\,\alpha)) - \alpha\, f_+(\beta\,\alpha)$. Similarly, for all $\alpha \in \mathbb{R}$: let $f_0(\alpha)$ be the unique $c$ satisfying $u'(c) = \max\{u'(y), \alpha\}$; and put $h_0(\alpha) = u(f_0(\beta\,\alpha)) - \alpha\, f_0(\beta\,\alpha)$. Finally, put

$$h(\alpha, x) = \left\{ \begin{array}{ll} h_+(\alpha) & \text{if } x > 0 \\ h_0(\alpha) & \text{if } x = 0 \end{array} \right\}.$$

Then we may eliminate $c$ from the Bellman *system* of the IG consumer to obtain the differential equation

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y)\,v' - \gamma\,v + h(v', x) \tag{24}$$

for $x \in [0, \infty)$ and the global bounds

$$\tfrac{1}{\gamma}\,u(y) \leq v \leq \overline{v}. \tag{25}$$

We shall refer to equation (24) with global bounds (25) as the Bellman *equation* of the IG consumer.

As for the second step, let $\widehat{u}_+$, $\widehat{u}_0$ and $\widehat{u}$ be given exactly as above. Then it can be checked that: (i) $\widehat{u}_+ < u$ on $(0, \infty)$; and (ii) $\widehat{u}_0 \leq u$ on $(0, y]$ with equality only at $y$. It follows that, like the value function of the IG consumer, the value function $\widehat{v}$ of the $\widehat{u}$ consumer satisfies $\widehat{v} \leq \overline{v}$. It can also be checked that $\lim_{\widehat{c}\to\infty} \widehat{u}_+(\widehat{c}) = \lim_{c\to\infty} u(c)$. Hence there exists $b \in (y, \infty)$ such that $\widehat{u}_+(b) = u(y)$. Hence, if the $\widehat{u}$ consumer consumes $b$ when $x > 0$ and $y$ when $x = 0$, then she will obtain a payoff of $\frac{1}{\gamma}\,u(y)$. It follows that $\widehat{v} \geq \frac{1}{\gamma}\,u(y)$. Overall, then, the Bellman *system* of the $\widehat{u}$ consumer takes the form of the differential equation

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,\widehat{v}'' + (\mu\,x + y - \widehat{c})\,\widehat{v}' - \gamma\,\widehat{v} + \widehat{u}(\widehat{c}, x) \tag{26}$$

for $x \in [0, \infty)$, the optimality condition

$$\left\{ \begin{array}{ll} \frac{\partial\widehat{u}}{\partial\widehat{c}}(\widehat{c}, x) = \widehat{v}' & \text{if } x > 0 \\ \frac{\partial\widehat{u}}{\partial\widehat{c}}(\widehat{c}, x) = \max\{\widehat{u}_0'(y), \widehat{v}'\} & \text{if } x = 0 \end{array} \right\} \tag{27}$$

and the global bounds

$$\tfrac{1}{\gamma}\,u(y) \leq \widehat{v} \leq \overline{v}. \tag{28}$$

For all $\alpha > 0$: let $\widehat{f}_+(\alpha)$ be the unique $\widehat{c}$ satisfying $\widehat{u}_+'(\widehat{c}) = \alpha$; and put $\widehat{h}_+(\alpha) = u(\widehat{f}_+(\alpha)) - \alpha\, \widehat{f}_+(\alpha)$. Similarly, for all $\alpha \in \mathbb{R}$: let $\widehat{f}_0(\alpha)$ be any $\widehat{c}$ satisfying $\widehat{u}_0'(\widehat{c}) = \max\{\widehat{u}_0'(y), \alpha\}$ (which amounts to saying that $\widehat{c} = (\widehat{u}_+')^{-1}(\alpha)$ if $\alpha > \widehat{u}_+'(\psi\, y)$, $\widehat{c} \in [\psi\, y, y]$ if $\alpha = \widehat{u}_+'(\psi\, y)$ and $\widehat{c} = y$ if $\alpha < \widehat{u}_+'(\psi\, y)$); and put $\widehat{h}_0(\alpha) = u(\widehat{f}_0(\alpha)) - \alpha\, \widehat{f}_0(\alpha)$ (which is uniquely defined even when $\widehat{f}_0(\alpha)$ is not). Finally, put

$$\widehat{h}(\alpha, x) = \left\{ \begin{array}{ll} \widehat{h}_+(\alpha) & \text{if } x > 0 \\ \widehat{h}_0(\alpha) & \text{if } x = 0 \end{array} \right\}.$$

Then we may eliminate $\widehat{c}$ from the Bellman *system* of the $\widehat{u}$ consumer, namely (26-28), to obtain the differential equation

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, \widehat{v}'' + (\mu\, x + y)\, \widehat{v}' - \gamma\, \widehat{v} + \widehat{h}(\widehat{v}', x) \tag{29}$$

for $x \in [0, \infty)$ and the global bounds

$$\tfrac{1}{\gamma}\, u(y) \leq v \leq \overline{v}. \tag{30}$$

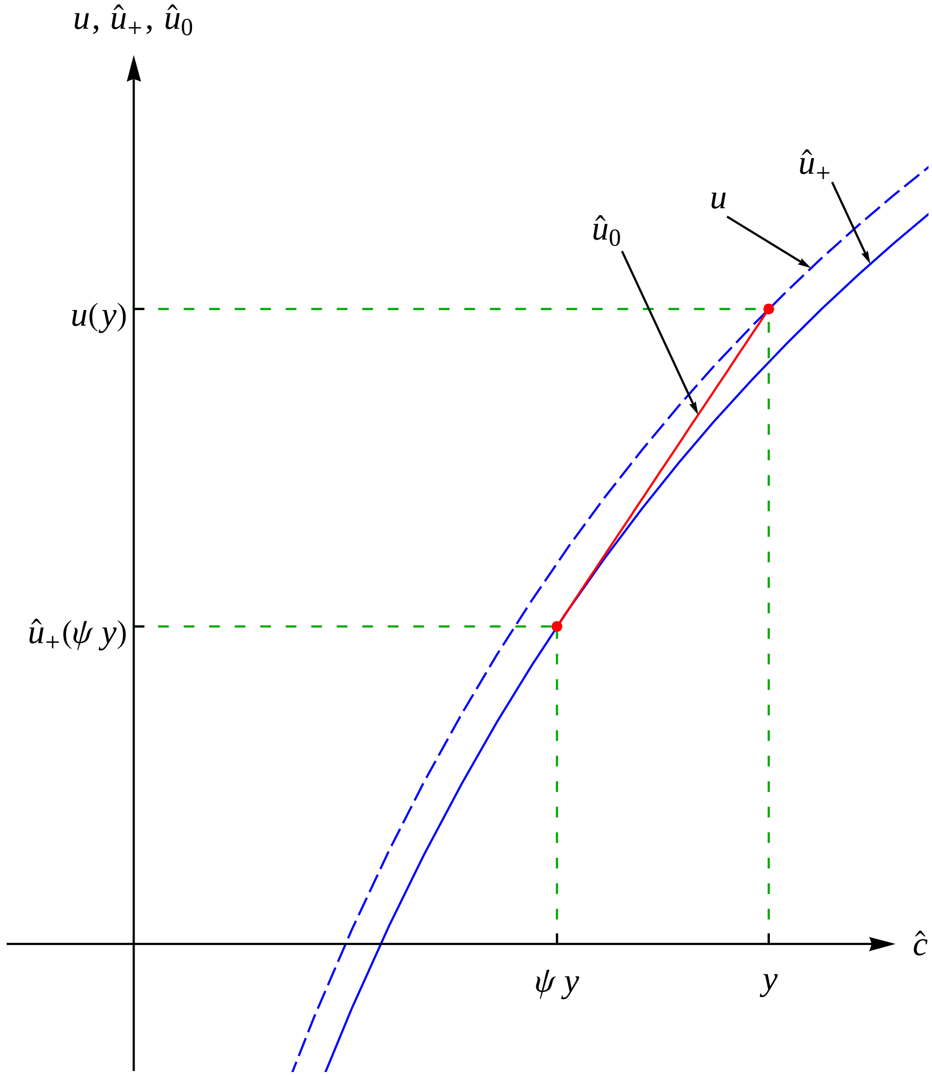
We shall refer to equation (29) with global bounds (30) as the Bellman *equation* of the $\widehat{u}$ consumer.

Finally, it is easy to see that equations (24) and (29) will be identical iff the functions $h$ and $\widehat{h}$ are the same. Moreover, as can be shown by direct calculation, this is indeed the case for the given choice of $\widehat{u}_+$, $\widehat{u}_0$ and $\widehat{u}$. Hence the Bellman equation of the IG consumer is identical to the Bellman equation of the $\widehat{u}$ consumer, as required. ■

Figure 4 depicts a portion of the graphs of $u$, $\widehat{u}_+$ and $\widehat{u}_0$ in the case in which $\beta = \frac{2}{3}$, $\rho = 2$ and $y = 1$.[29] It illustrates several important features of $u$, $\widehat{u}_+$ and $\widehat{u}_0$. First, we have $\widehat{u}_+(\widehat{c}) < u(\widehat{c})$ for all $\widehat{c} > 0$ and $\widehat{u}_0(\widehat{c}) < u(\widehat{c})$ for all $0 < \widehat{c} < y$. This makes sense: the $\widehat{u}$ consumer optimizes fully while the IG consumer does not. Hence the $\widehat{u}$ consumer must be suitably handicapped in order to prevent her from achieving a higher value than the IG consumer. Second, we have $\widehat{u}_0(y) = u(y)$. Once again this makes sense: in the liquidity constrained case, both the $\widehat{u}$ consumer and the IG consumer consume their labor income $y$ forever. So we must have $\widehat{u}_0(y) = u(y)$ if they are both to obtain the same value.

[29] In Figure 4, the graphs of the utility functions are truncated from below at a utility level of 0.4 and from the right at a consumption level of 1.1. In addition, the axes intersect at the point $(0.5, -0.5)$, and not at $(0, 0)$.

Figure 4: graph of $u$, $\hat{u}_+$ and $\hat{u}_0$



Third, the graph of $\widehat{u}_0$ coincides with that of $\widehat{u}_+$ for $\widehat{c} \in (0, \psi\, y]$, where $\psi = \frac{\rho-(1-\beta)}{\rho}$, and coincides with the tangent to the graph of $\widehat{u}_+$ at $\psi\, y$ for $\widehat{c} \in (\psi\, y, y]$. [XXX note that this figure is machine drawn]

From the Value-Function-Equivalence Theorem, it is easy to deduce the existence and uniqueness of equilibrium in the IG model:

**Theorem 8** [Existence and Uniqueness]**.** *The IG model has a unique equilibrium.*

**Proof.** Theorem 7 shows that $v$ satisfies the Bellman equation of the IG consumer iff $v$ satisfies the Bellman equation of the $\widehat{u}$ consumer. Furthermore, standard considerations show that $v$ satisfies the Bellman equation of the $\widehat{u}$ consumer iff $v$ is the value function of the optimization problem of the $\widehat{u}$ consumer. More explicitly, $v$ satisfies the Bellman equation of the $\widehat{u}$ consumer iff, for all $x \in [0, \infty)$, $v(x)$ is the supremum of all payoffs that are feasible for the $\widehat{u}$ consumer when her initial wealth is $x$. This already yields both existence and uniqueness of $v$, for the simple reason that the supremum of *any* set of numbers exists and is unique. In particular, the supremum of all the feasible payoffs of the $\widehat{u}$ consumer exists and is unique. Turning to the consumption function, we recall from the optimality condition (22) that $u'(c) = \beta\, v'$ if $x > 0$ and $u'(c) = \max\{u'(y), \beta\, v'\}$ if $x = 0$. Since $u$ is strictly concave (and therefore $u'$ is invertible), the existence and uniqueness of $c$ follows directly from the existence and uniqueness of $v$.[30] ■

We can also provide a heuristic version of this proof and the related arguments. Optimization problems have unique value functions, since there cannot be two state-contingent values that are both *best* values. Therefore, value functions of optimization problems are automatically unique. Since the Bellman Equation of the IG consumer is identical to the Bellman Equation of the $\widehat{u}$ consumer, and since the $\widehat{u}$ consumer is an optimizer, it follows that both the IG consumer and the $\widehat{u}$ consumer have a unique value function, and that this value function is the same for both consumers. Finally, one can use this common value function to derive both the equilibrium policy function of the IG consumer, which is *unique*, and the optimal policy function of the $\widehat{u}$ consumer, which is *different* from that of the IG consumer. For the IG consumer, equilibrium consumption is generated by the first-order condition $u'(c) = \beta\, v'$ if $x > 0$ and $u'(c) = \max\{u'(y), \beta\, v'\}$ if $x = 0$. For the

[30] In the case of the $\widehat{u}$ consumer, we have $\widehat{u}'_+(\widehat{c}) = \widehat{v}'$ if $x > 0$ and $\widehat{u}'_0(\widehat{c}) = \max\{\widehat{u}'_0(y), \widehat{v}'\}$ if $x = 0$. Hence $\widehat{c}$ is uniquely defined for all $x > 0$, and $\widehat{c}$ is uniquely defined for $x = 0$ provided that $\widehat{v}'(0) \neq \widehat{u}'_0(y)$. If $\widehat{v}'(0) = \widehat{u}'_0(y)$, then $\widehat{c}(0)$ can take any value in $[\psi\, y, y]$. However, all choices of $\widehat{c}(0)$ give rise to an optimal policy.

$\widehat{u}$ consumer, optimal consumption is generated by the first-order condition $\widehat{u}_+'(\widehat{c}) = v'$ if $x > 0$ and $\widehat{u}_0'(\widehat{c}) = \max\{\widehat{u}_0'(y), v'\}$ if $x = 0$. Because $\widehat{u}_+$ and $\widehat{u}_0$ differ both from $u$ and from one another, the policy functions of the two consumers are different, despite the fact that both have the same value function.

**5.4. The Deterministic Case: A Refinement.** Until now we have assumed that the standard deviation of asset returns is strictly positive ($\sigma > 0$). In other words, we have been studying the *stochastic* IG model. In the present subsection, we investigate the *deterministic* IG model ($\sigma = 0$).

We begin by defining the Bellman system of the deterministic IG consumer to be the analogue of the Bellman system of the stochastic IG consumer, namely (21-23) above, with $\sigma = 0$. We then show that, exactly as in the stochastic case, the set of value functions of the deterministic IG consumer coincides with the set of value functions of the deterministic $\widehat{u}$-consumer. Hence, the value function of the deterministic IG consumer is unique.

Then, in order to unify our deterministic and stochastic results, we show that the value function of the deterministic IG consumer is the limit of the value function of the stochastic IG consumer as noise converges to zero ($\sigma \downarrow 0$). This implies that the value function of the deterministic IG consumer is precisely the value function that would be selected by a 'trembling-hand' analysis.[31]

The deterministic model has two additional dividends. First, our results provide a way of resolving concerns that deterministic hyperbolic models may have a continuum of equilibria (cf. Krusell and Smith (2000), Ekeland and Lazrak (2006), and Karp (2007)). Our results provide a refinement that eliminates not just the possibility of a continuum of value functions, but even the possibility a finite multiplicity of value functions. Second, the deterministic case is tractable: the Bellman system of the deterministic IG consumer can be transformed into an autonomous first-order differential equation, whereas the Bellman system of the stochastic IG consumer is a second-order non-autonomous differential equation which cannot be transformed into a simpler form. An earlier draft of this paper,

[31] There is a close analogy between using the limit as $\sigma \downarrow 0$ to identify a unique equilibrium of the deterministic IG model and using trembling-hand perfection to refine the set of Nash equilibria of a finite game. In our case, the stochasticity of asset returns ensures that, starting from any interior state, every other interior state will be reached with positive probability. In the case of trembling-hand perfection, trembles by the individual players ensure that, starting from any given node, all successor nodes of the game are reached with positive probability.

Harris and Laibson (2004), expands on these points and provides a complete characterization of the value and policy functions of the deterministic case.

We begin by defining the Bellman system of the deterministic IG consumer.

**Definition 9.** *The* ***Bellman system of the deterministic IG consumer*** *consists of the differential equation*

$$0 = (\mu\, x + y - c)\, v' - \gamma\, v + u(c) \tag{31}$$

*for all* $x \in [0, \infty)$*, the optimality condition*

$$\left\{ \begin{array}{ll} u'(c) = \beta\, v' & \text{if } x > 0 \\ u'(c) = \max\{u'(y), \beta\, v'\} & \text{if } x = 0 \end{array} \right\} \tag{32}$$

*and the global bounds*

$$\tfrac{1}{\gamma}\, u(y) \leq v \leq \overline{v}. \tag{33}$$

For some parameter values, this system does not have a classical solution, and in these cases a solution should be understood to mean a viscosity solution.[32] More precisely, a consumption function $c$ is an equilibrium of the deterministic IG model iff there exists a value function $v$ such that $v$ is a viscosity solution of the Bellman *equation* of the deterministic IG consumer and $c$ satisfies equation (32). Here, the Bellman *equation* of the deterministic IG consumer is the reduced-form equation obtained from the Bellman *system* of the deterministic IG consumer by eliminating $c$. (See the proof of Theorem 7 for a detailed explanation of the distinction between the Bellman system and the Bellman equation.)[33]

[32] In the theory of classical solutions of differential equations, a function must be continuously differentiable before one can determine whether or not it satisfies a first-order differential equation. In the theory of viscosity solutions, a function need only be continuous before one can determine whether or not it satisfies a first- or even a second-order differential equation. For an introduction to viscosity solutions, see Crandall et al (1992).

[33] In practice, the Bellman equation of the stochastic IG consumer has a unique viscosity solution. This solution is (at least) twice continuously differentiable. Hence it is also a classical solution of the Bellman equation of the stochastic IG consumer. The Bellman equation of the deterministic IG consumer likewise has a unique viscosity solution. In both the low-$\mu$ case (i.e. $\mu < \gamma$) and the high-$\mu$ case (i.e. $\mu > \frac{1}{\beta}\,\gamma$), this solution is (at least) once continuously differentiable. Hence it is also a classical solution of the Bellman equation of the deterministic IG consumer. However, in the intermediate-$\mu$ case (i.e. $\gamma < \mu < \frac{1}{\beta}\,\gamma$), there

Proceeding exactly as in the stochastic case, we obtain the following two theorems. Following the convention in the math literature, we use a black box, ■, to signify that these theorems are stated without proof.

**Theorem 10** [Value-Function Equivalence]**.** *$v$ is a value function of the deterministic IG consumer iff $v$ is a value function of the deterministic $\widehat{u}$ consumer.* ■

**Theorem 11** [Existence and Uniqueness]**.** *The deterministic IG model has a unique value function.* ■

Notice that what we assert here is that the deterministic IG model has a unique value function, not that it has a unique equilibrium. This is because, in certain knife-edge cases, very mild forms of non-uniqueness can occur. These cases of non-uniqueness are certainly mild: although the deterministic IG model has more than one equilibrium consumption function, all possible equilibrium consumption functions give rise to the same dynamics and to the same value function.

In practice, the consumption function is unique in the two main cases of our model, namely the low-$\mu$ case and the high-$\mu$ case.[34] This is related to the fact that, in both these cases, the consumption function is continuous. By contrast, in the intermediate-$\mu$ case, there exists $\overline{x} \in (0, \infty)$ such that $c$ has a downward jump at $\overline{x}$. However, even in this case, $c(x)$ is still unique for all $x \neq \overline{x}$.[35] Moreover the non-uniqueness of $c(\overline{x})$ does not translate into non-uniqueness of the equilibrium dynamics: for all initial wealths $x(0) \in [0, \infty)$, there is a unique solution to the ordinary differential equation $dx = (\mu\, x + y - c(x))\, dt$ governing the evolution of wealth, and this solution does not depend on the choice of $c(\overline{x})$.[36]

exists $\overline{x} \in (0, \infty)$ such that the solution is (at least) once continuously differentiable on both $[0, \overline{x}]$ and $[\overline{x}, \infty)$, but has a convex kink at $\overline{x}$. For further explanation of the low-, intermediate- and high-$\mu$ cases, see footnote 34, Section 7 and Appendix F.

[34] In the deterministic IG model: the low-$\mu$ case is the case $\mu \in (-\infty, \gamma)$; the intermediate-$\mu$ case is the case $\mu \in (\gamma, \frac{1}{\beta}\gamma)$; and the high-$\mu$ case is the case $\mu \in (\frac{1}{\beta}\gamma, \overline{\mu})$, where $\overline{\mu} = \frac{1}{1-\rho}\gamma$ if $\rho < 1$ and $\overline{\mu} = \infty$ if $\rho \geq 1$. Cf. Harris and Laibson (2004).

[35] As in the stochastic IG model, $u$ is strictly concave. Hence the difficulty lies not in inverting the optimality condition (32), but rather in the fact that $v$ has a convex kink at $\overline{x}$, and therefore $v'(\overline{x})$ is not uniquely defined.

[36] This is because $\mu\,\overline{x} + y - c(\overline{x}) < 0$ for all choices of $c(\overline{x}) \in [c(\overline{x}+), c(\overline{x}-)]$, and the solution therefore passes through $\overline{x}$ without stopping. By contrast, in the case of the $\widehat{u}$ consumer, there are two possible choices of $\widehat{c}(\overline{x})$. They satisfy $\mu\,\overline{x} + y - \widehat{c}_1 < 0 < \mu\,\overline{x} + y - \widehat{c}_2$. Moreover: the solution to the dynamics starting at $\overline{x}$ and corresponding to the choice $\widehat{c}(\overline{x}) = \widehat{c}_2$ increases monotonically to $\infty$; and the solution

Our definition of equilibrium in the deterministic IG model implicitly rules out equilibria with value functions that are discontinuous (or continuous but not viscosity solutions of the Bellman system of the deterministic IG consumer). The following theorem makes this implicit refinement argument explicit. It shows that, by focussing on viscosity solutions of the Bellman system of the deterministic IG consumer, we are in effect restricting attention to trembling-hand perfect equilibria of the deterministic IG model.

**Theorem 12.** *For all $\sigma > 0$, let $v_\sigma$ be the value function of the stochastic IG consumer; and let $v$ be the value function of the deterministic IG consumer. Then $v_\sigma \to v$ uniformly on compact subsets of $[0, \infty)$ as $\sigma \downarrow 0$.*

**Proof.** The basic idea behind the proof is to view $v_\sigma$ as the value function of the optimization problem of the stochastic $\widehat{u}$ consumer and $v$ as the value function of the optimization problem of the deterministic $\widehat{u}$ consumer. There are several ways of implementing this idea. One way is to note that the dynamics of the problem depend continuously on $\sigma$, and that the utility function $\widehat{u}$ is upper semicontinuous. (It is continuous except at $x = 0$, where it may jump up in the limit as $x \downarrow 0$, because $\widehat{u}_0 \geq \widehat{u}_+$.) From this it follows at once that $\limsup_{\sigma\downarrow 0} v_\sigma(x) \leq v(x)$ for all $x \in [0, \infty)$. On the other hand, explicit consideration of the form of the optimal consumption function $\widehat{c}$ of the deterministic $\widehat{u}$ consumer shows that $\lim_{\sigma\downarrow 0} v_\sigma(x; \widehat{c}) = v(x)$ for all $x \in [0, \infty)$, where $v_\sigma(x; \widehat{c})$ denotes the payoff to the stochastic $\widehat{u}$ consumer when she employs the consumption function $\widehat{c}$. Specifically: if $\mu < \frac{1}{\beta}\gamma$, then there exists $\varepsilon > 0$ such that $\widehat{c}$ is continuous on $(0, \varepsilon)$ and $\widehat{c}(0+) = \psi\,\overline{c} > y$, where $\psi = \frac{\rho-(1-\beta)}{\rho}$; and, if $\mu \geq \frac{1}{\beta}\gamma$, then $\widehat{c}$ is continuous on $[0, \infty)$ and $\widehat{c}(0) \leq \psi\, y < y$. In particular, $\liminf_{\sigma\downarrow 0} v_\sigma(x) \geq v(x)$. ■

**Remark 13.** *Assumptions A1 and A2 do not involve the parameter $\sigma$, and Assumption A3 becomes more restrictive as $\sigma$ decreases. For the analysis of this section, it therefore suffices to use the special case of Assumption A3 in which $\sigma = 0$. In other words: it suffices to assume that $\mu < \overline{\mu}$, where $\overline{\mu} = \frac{1}{1-\rho}\gamma$ if $\rho < 1$ and $\overline{\mu} = \infty$ if $\rho \geq 1$.*

## 6. The Consumption Function: Homogeneous Case

In the current and the following section, we analyze the consumption function of the IG consumer. This analysis serves two purposes. First, it provides a framework for evaluating

to the dynamics starting at $\overline{x}$ and corresponding to the choice $\widehat{c}(\overline{x}) = \widehat{c}_1$ decreases monotonically to 0. There is therefore an essential non-uniqueness in the optimal strategy, corresponding to the competition between two local maxima. Somewhat remarkably, the dynamics of the deterministic IG model are therefore better behaved than those of the deterministic $\widehat{u}$ model.

the quantitative relevance of the IG limit (i.e. the limit of the PF model as the hazard rate $\lambda$ of the arrival of the future goes to $\infty$). Second, it establishes both quantitative and qualitative properties of the consumption functions that emerge in the IG model.

In the current section, we begin the analysis of the consumption function by studying the homogeneous version of our model, i.e. the version in which there is no labor income (i.e. $y = 0$).[37] This case is important for four reasons. First, the IG model (in which $\lambda = \infty$ and the present lasts only an instant) becomes highly tractable when $y = 0$: the value and policy functions corresponding to the unique equilibrium are homogeneous in wealth; and closed-form expressions can be found for both functions. Second, the PF model (in which $\lambda < \infty$ and the present has strictly positive duration) likewise becomes highly tractable when $y = 0$: it too admits an equilibrium for which the value and policy functions are homogeneous in wealth; and closed-form expressions can again be found for both functions. Third, homogeneous models are widely used in the economics literature. (See, for example, Merton 1971, Barro 1999, Luttmer and Marriotti 2003.) The case $y = 0$ is therefore an important benchmark for cross-model comparisons. Fourth, consideration of the case $y = 0$ enables us to evaluate the quantitative merits of the limit $\lambda \to \infty$. We find that the IG model is a good approximation to the PF model when $\lambda$ is at least 10. In other words, once the duration of the present is on average $\frac{1}{10}$th of a year (or less) the IG model becomes a good approximation of the PF model. Intuitively speaking, the IG model is a good approximation when the future is a month or less away.

After completing analysis of the homogeneous case $y = 0$, we turn in Section 7 to the more challenging inhomogeneous case $y > 0$.

**6.1. The Homogeneous PF Model.** The homogeneous PF model differs from the inhomogeneous PF model in two respects. First, we assume that $y = 0$ instead of assuming that $y > 0$. It is this change that makes the model homogeneous, thereby allowing us to find an explicit solution. However, this change also means that we have to strengthen Assumption A3 slightly. Instead of only assuming that $\mu < \frac{1}{1-\rho}\,\gamma + \frac{1}{2}\,\rho\,\sigma^2$ when $\rho < 1$,

[37] It should be emphasized that the current paper focusses primarily on the inhomogeneous version of our consumption model: of the six sections of the paper that deal with the consumption problem (namely Sections 3-8), it is only the current section (namely Section 6) that deals with the homogeneous model. Moreover it is a maintained assumption of the inhomogeneous model that $y > 0$. The homogeneous model of the current section is not therefore a special case of the inhomogeneous model. Our analysis of the inhomogeneous model (namely the analysis contained in Sections 3-5 and 7-8) can, however, be adapted to cover the case $y \geq 0$. The only quid-pro-quo is that we have to strengthen our integrability assumption to compensate for the weakening of the assumption $y > 0$. See Appendix B for details.

we must also assume that $\mu > \frac{1}{1-\rho}\,\gamma + \frac{1}{2}\,\rho\,\sigma^2$ when $\rho > 1$.[38] This is because, when $y = 0$, the consumer can no longer fall back on her wage income if her financial wealth runs out. We therefore need to find another way of ensuring that her expected payoff is bounded below. This is achieved by ensuring that wealth does not shrink too fast. Second, we focus exclusively on equilibria in linear consumption functions. This allows us to dispense with the global upper and lower bounds, and to drop the assumption that $u$ is bounded below.

The Bellman system of the homogeneous PF consumer consists of the pair of differential equations

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x - c)\,v' - \gamma\,v + u(c), \tag{34}$$

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,w'' + (\mu\,x - c)\,w' + \lambda\,(\beta\,v - w) - \gamma\,w + u(c) \tag{35}$$

and the optimality condition

$$u'(c) = w', \tag{36}$$

all of which are required to hold for all $x \in (0, \infty)$.

It is helpful to compare this system with the Bellman system of the inhomogeneous PF consumer, namely equations (5-9). Note first that, because the consumption function is linear, no analogue of equations (8-9) is needed. Second, again because the consumption function is linear, wealth will always remain strictly positive. Hence equations (34-36) are only required to hold for $x > 0$. Third, equations (34-35) are obtained from equations (5-6) by putting $y = 0$. Finally, equation (36) is obtained directly from equation (7) by eliminating the case $x = 0$.

It is natural to look for a solution to equations (34-36) in the form

$$v(x) = \Theta\,u(\theta\,x), \quad w(x) = \Phi\,u(\phi\,x), \quad c(x) = \alpha\,x,$$

where the constants $\Theta$, $\Phi$, $\theta$, $\phi$ and $\alpha$ are all required to be strictly positive. Making this substitution leads to the following quadratic equation for $\alpha$:

$$0 = \frac{\lambda}{1+\lambda}\,\left((\rho + \beta - 1)\;\alpha - \widetilde{\gamma}\right) + \frac{1}{1+\lambda}\left(\rho\,(1-\rho)\,\alpha^2 + (2\,\rho - 1)\,\widetilde{\gamma}\,\alpha - \widetilde{\gamma}^2\right), \tag{37}$$

[38] The strengthened version of Assumption A3 can be expressed more succinctly by requiring that $\gamma > (1-\rho)\,(\mu - \frac{1}{2}\,\rho^2\,\sigma^2)$.

where

$$\widetilde{\gamma} = \gamma - (1-\rho)\,(\mu - \tfrac{1}{2}\,\rho^2\,\sigma^2).$$

See Appendix E for details. Furthermore it can be shown that only one of the two solutions of this quadratic is relevant. This solution is always positive, varying from $\frac{\widetilde{\gamma}}{\rho}$ when $\lambda = 0$ to $\frac{\widetilde{\gamma}}{\rho+\beta-1}$ when $\lambda = \infty$.[39]

A more concrete understanding of this solution, and especially of its behavior as $\lambda \to \infty$, can be obtained by taking expansions in $\lambda^{-1}$. Indeed, we have

$$\alpha = \frac{\tilde{\gamma}}{\rho+\beta-1} - \frac{(1-\beta)\,\beta\,\tilde{\gamma}^2}{(\rho+\beta-1)^3}\,\lambda^{-1} + O\big(\lambda^{-2}\big)\,.$$

The first-order effect of increasing $\lambda$ is therefore to increase the average propensity to consume. A higher value of $\lambda$ implies that the multiplicative $\beta$-discounting associated with the passage to the future arrives more quickly. More discounting lowers the value of future consumption, thereby raising the propensity to consume today. Notice too that the $\lambda$-driven effects in the first-order term vanish when $\beta = 1$. In this case, a more rapid arrival of the future has no bearing on discounting.

**6.2. The Homogeneous IG model.** The homogeneous IG model differs from the inhomogeneous IG model in the same two respects in which the homogeneous PF model differs from the inhomogeneous PF model. First, we assume that $y = 0$ instead of assuming that $y > 0$. As before, this implies that we need to strengthen Assumption A3 by requiring in addition that $\mu > \frac{1}{1-\rho}\,\gamma + \frac{1}{2}\,\rho\,\sigma^2$ when $\rho > 1$. Second, we focus exclusively on equilibria in linear consumption functions.

The Bellman system of the homogeneous IG consumer consists of the differential equation

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x - c)\,v' - \gamma\,v + u(c) \tag{38}$$

for all $x \in (0,\infty)$ and the optimality condition

$$u'(c) = \beta\,v', \tag{39}$$

again for all $x \in (0,\infty)$.

It is helpful to compare this system with the Bellman system of the inhomogeneous IG

[39] Assumption A2 implies that $\rho + \beta - 1 > 0$, and our strengthened version of Assumption A3 implies that $\widetilde{\gamma} > 0$. Cf. footnote 38.

consumer, namely equations (17-19). Note first that, because the consumption function is linear, no analogue of equation (19) is needed. Second, again because the consumption function is linear, equations (38-39) are only required to hold for $x > 0$. Third, equation (38) is obtained from equation (17) by letting $y \to 0$. Finally, equation (39) is obtained directly from equation (18) by eliminating the case $x = 0$.

This system can be solved using the same methods as in Section 6.1. We look for a solution in the form

$$v(x) = \Theta\, u(\theta\, x), \quad c(x) = \alpha\, x,$$

where the constants $\Theta$, $\theta$ and $\alpha$ are both required to be strictly positive. Making this substitution leads to the conclusion that

$$\alpha = \frac{\widetilde{\gamma}}{\rho + \beta - 1}, \tag{40}$$

where $\widetilde{\gamma} = \gamma - (1-\rho)\,(\mu - \frac{1}{2}\,\rho^2\,\sigma^2)$ as before.[40]

It is easy to see that the right-hand side of equation (40) is the limit of the relevant solution of equation (37) as $\lambda \to \infty$. Hence the policy function of the PF model converges, as $\lambda \to \infty$, to the policy function of the IG model. It can also be shown, as one would expect in the light of the convergence of the policy functions, that the value function of the PF model converges to the value function of the IG model as $\lambda \to \infty$.

Barro (1999) and Luttmer and Marriotti (2003) also study a continuous-time homogeneous economy with a general class of dynamically inconsistent time preferences. Barro's economy has returns that vary over time due to deterministic aggregate growth dynamics, whereas we study an environment with stochastic returns that are i.i.d. Our homogeneous case and Barro's homogeneous economy both admit an equilibrium with linear policy rules. For the log utility case (i.e. $\rho = 1$), the propensity to consume in our economy is $\alpha = \frac{\gamma}{\beta}$, which matches the propensity to consume that Barro derives for log utility when converging to the continuous time analog of the quasi-hyperbolic discount function. Our propensity to consume matches Barro's despite the differences in the economic environments, since substitution effects and income effects are exactly offsetting for log utility.

Like Barro, Luttmer and Marriotti (2003) study a range of time preferences, including the continuous-time analog of the quasi-hyperbolic discount function, and restrict atten-

[40] We also obtain $\Theta = \frac{1}{\gamma}$ and $\theta^{1-\rho} = \frac{\gamma\,\alpha^{1-\rho}}{\widetilde{\gamma} + (1-\rho)\,\alpha}$.

tion to equilibria with linear policy rules. Unlike Barro, Luttmer and Marriotti study a stochastic endowment economy, which they use to characterize asset prices and risk premia.

Our paper differs from Barro (1999) and Luttmer and Marriotti (2003) in that our analysis is valid whether the economy is homogeneous or inhomogeneous, and whether equilibrium policies are linear or non-linear. In particular, our analysis can handle challenges like liquidity constraints, other forms of market incompleteness and utility functions outside of the constant relative risk aversion class. In addition, we obtain uniqueness results for both the homogeneous and inhomogeneous models without restricting the class of policy functions.

**6.3. Calibration of the Homogeneous case.** We now provide a calibration of the homogeneous model. For this calibration we fix the parameters $\gamma = 0.05$, $\beta = \frac{2}{3}$, $\sigma = 0.17$ and $\mu = 0.06$. We then vary the value of risk aversion ($\rho$) and the hazard rate at which the future arrives ($\lambda$). This calibration identifies the range of $\lambda$ for which the homogeneous PF model and the homogeneous IG model imply quantitatively similar policy functions.

Specifically, we calculate the marginal propensity to consume for the PF model for $\lambda \in \{0, 0.1, 1, 10, 100, \infty\}$. At $\lambda = 0$ the future never arrives. At the other finite extreme, $\lambda = 100$, the future arrives on average every $\frac{365}{100} = 3.65$ days. When $\lambda = \infty$ the future arrives instantaneously. We believe that the appropriate calibration is $\lambda = 1,000$, implying that the psychological future arrives on average in the span of time that passes between falling asleep and waking up the next morning. We don't report the $\lambda = 1,000$ case, since it is indistinguishable from the case $\lambda = \infty$.

| | $\rho = 0.5$ | $\rho = 1$ | $\rho = 2$ | $\rho = 5$ |
|---|---|---|---|---|
| $\lambda = 0$ | 0.100 | 0.0500 | 0.0250 | 0.0100 |
| $\lambda = 0.1$ | 0.143 | 0.0643 | 0.0290 | 0.0107 |
| $\lambda = 1$ | 0.233 | 0.0733 | 0.0299 | 0.0107 |
| $\lambda = 10$ | 0.289 | 0.0748 | 0.0300 | 0.0107 |
| $\lambda = 100$ | 0.299 | 0.0750 | 0.0300 | 0.0107 |
| $\lambda = \infty$ | 0.300 | 0.0750 | 0.0300 | 0.0107 |

Table 1: The marginal propensity to consume as a function of the coefficient of relative risk aversion ($\rho$) and the arrival rate of the future ($\lambda$).

Recall that the IG model is the case $\lambda = \infty$. Table 1 therefore shows that the IG model is a good approximation for the finite-$\lambda$ PF model as long as $\lambda \geq 10$. In other words, the IG model is a good approximation as long as the present lasts on average about $\frac{1}{10}$th of a year or less. It is helpful to express this result in terms of natural time units: if the present typically lasts a month or less, the IG model is a good approximation of the PF model.

## 7. The Consumption Function: Inhomogeneous Case

We now turn to the more challenging inhomogeneous case with general utility functions and non-zero labor income. For this case only the IG model is analytically tractable, and we therefore focus exclusively on that model. Three general properties emerge. We first provide an overview of these properties before delving into the details.

First, the consumption function is continuously differentiable in the interior of the wealth space. This is a consequence of Brownian motion in the stochastic process for wealth. The presence of Brownian motion makes the value function twice continuously differentiable in the interior of the wealth space and thereby eliminates discontinuities in the consumption function. More formally optimality implies that, when consumption is unconstrained, $u'(c) = \beta\, v'$. Differentiating this expression yields, $u''(c)\, c' = \beta\, v''$. Hence, twice continuous differentiability of the value function in the interior of the wealth space implies continuous differentiability of the consumption function there.

Second, if the expected rate of return $\mu$ is low enough, the consumption function will have an upward discontinuity when wealth $x = 0$. Intuitively, if $\mu$ is low, then the liquidity constraint binds at $x = 0$; but, even when $\mu$ is low, it cannot bind at any strictly positive $x$ (no matter how small) since $x$ is a stock and $c$ is a flow. The sudden arrival of a binding liquidity constraint as $x$ falls from any strictly positive value to 0 causes a downward jump in $c$ from $c(0+) = \overline{c} > y$ to $c(0) = y$. Moreover this downward jump can be understood in terms of the consumer's propensity to value immediate rewards discretely more than delayed rewards. It does not arise when $\mu$ is sufficiently high, since in that case the liquidity constraint does not bind at $x = 0$.

Third, it can happen that there is an interval over which the consumption function is downward sloping. This occurs if the expected rate of return $\mu$ takes on intermediate values. However, this non-monotonicity disappears when a bond is introduced, and the investor can take both long and short positions in the bond. We therefore view the first two properties as robust implications of the IG model, and the third property as an artefact of the bond-free model that we study in the present paper.

These properties contrast with the properties of the continuous-time *exponential* model, the consumption function of which is continuous everywhere, including at $x = 0$ where the liquidity constraint starts to bind, and monotonic for all choices of $\mu$.

The properties of the IG model also contrast with the properties of the *discrete-time* quasi-hyperbolic model, the consumption function of which may have *several* downward sloping regions and a *countable* number of downward jumps (cf. Laibson 1997, Morris and Postlewaite 1997, Krusell and Smith 2000, Harris and Laibson 2001, Morris 2002).[41]

**7.1. Comparative Statics on $\mu$.** In order to simplify our description of the behavior of the consumption function, we vary the expected rate of return $\mu$ and hold the other parameters fixed. It turns out that there are three qualitative cases to consider.

Recall that Assumption A3 requires that $\mu < \overline{\mu}$, where $\overline{\mu} = \frac{1}{1-\rho}\gamma + \frac{1}{2}\rho\sigma^2$ if $\rho < 1$ and $\overline{\mu} = \infty$ if $\rho \geq 1$. In the current section, we will show that there exists $\mu_1 \in (\gamma, \overline{\mu})$ such that the form of the consumption function depends on whether $\mu$ lies in the interval $(-\infty, \gamma)$, the interval $(\gamma, \mu_1)$ or the interval $(\mu_1, \overline{\mu})$.[42] We shall refer to these cases as the low-$\mu$, intermediate-$\mu$ and high-$\mu$ cases respectively.

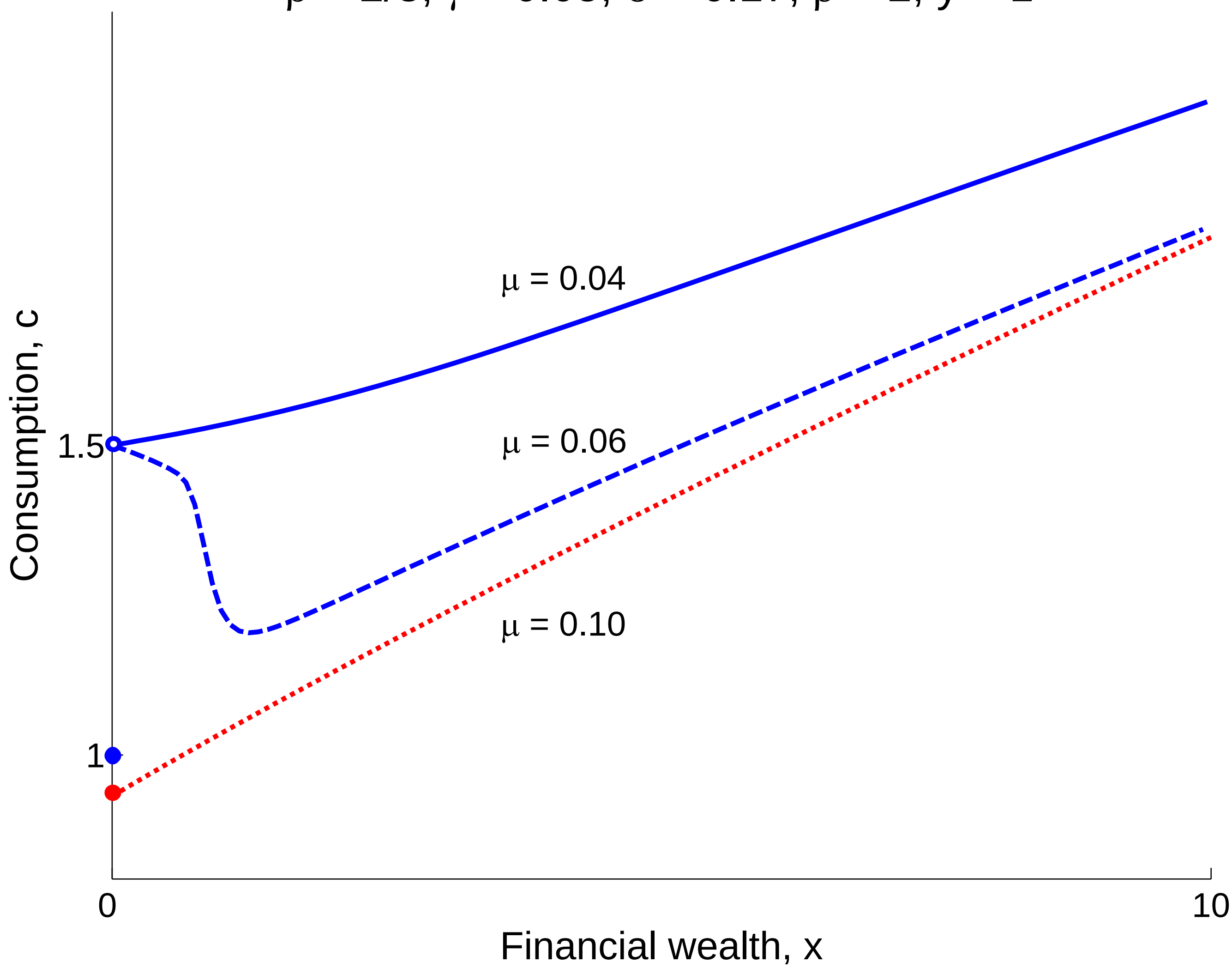
In all three cases, the consumption function is continuous everywhere except possibly when $x = 0$, at which point the liquidity constraint may bind. When $\mu$ is low, the consumption function is everywhere increasing, but the liquidity constraint is binding, which generates an upward discontinuity at $x = 0$. When $\mu$ is intermediate, there is an upward discontinuity at $x = 0$, followed first by a downward sloping region and thereafter by an upward sloping region. When $\mu$ is high, the incentive to save is strong enough to make the consumption function globally continuous and increasing (qualitatively like the exponential discounting case).

Figure 5 shows three consumption functions corresponding to the three cases for $\mu$. These functions were obtained from careful numerical simulations of our model, but we are also able to confirm their qualitative properties analytically. (See Appendix F.) All three functions use the parameter values $\beta = \frac{2}{3}$, $\gamma = 0.05$, $\sigma = 0.17$, $\rho = 2$ and $y = 1$. These values are illustrative, but they are all empirically sensible (and $y = 1$ is a normalization): they involve a present bias of about a third, a long-run discount rate of 5%, an annual standard deviation of stock returns of 17% and a coefficient of relative risk aversion of 2. The differences between the functions are the result of varying $\mu$ over the set $\{0.04, 0.06, 0.10\}$: the top consumption function corresponds to $\mu = 0.04$ (the low-$\mu$

[41] These multiple downward sloping regions and jumps are not eliminated when a bond is added to the discrete-time model.

[42] It is easy to check that $\overline{\mu} > \gamma$.

Figure 5: Consumption function
$\beta = 2/3$, $\gamma = 0.05$, $\sigma = 0.17$, $\rho = 2$, $y = 1$



case); the middle consumption function corresponds to $\mu = 0.06$ (the intermediate-$\mu$ case); and the bottom consumption function corresponds to $\mu = 0.10$ (the high-$\mu$ case). We will refer back to this figure as we work through our formal results.

**7.2. The low-$\mu$ case (discontinuity at zero wealth).** The most novel case of our model is that in which $\mu < \gamma$. In this case, the expected returns on the asset are not sufficiently attractive to induce the IG consumer to save when her wealth is zero, and the liquidity constraint binds. More precisely, let $\overline{c} \in (y, \infty)$ be the unique solution of the equation

$$u'(\overline{c}) = \beta \, \frac{u(\overline{c}) - u(y)}{\overline{c} - y}. \tag{41}$$

Then:

**Theorem 14.** *If $\mu < \gamma$ then: $c(0) = y$; $c(0+) = \overline{c} > y$; and $c' > 0$ on $(0, \infty)$.*

In other words: when the IG consumer has no wealth, she consumes all of her labor income; if she acquires even a little wealth, then her consumption jumps up from $y$ to $\overline{c}$; and her consumption increases monotonically with further increases in her wealth. In particular, her consumption function is strictly increasing.

**Proof.** See Appendix F. ■

Equation (41) can be understood as follows. Consider a consumer with strictly positive wealth. In the low $\mu$ case, the dynamics of wealth and consumption are causing wealth to trend lower. Let us refer to the (stochastic) moment at which wealth runs out as the 'crunch'. Suppose that the consumption level of the pre-crunch self is $\overline{c}$. Then the cost to the pre-crunch self of putting aside an extra $dx$ units of wealth is $u'(\overline{c})\,dx$. On the other hand, if the post-crunch self receives a windfall consisting of an extra $dx$ units of wealth, then she can raise her consumption level from $y$ to $\overline{c}$ for a length of time $dt = dx \,/\, (\overline{c} - y)$. The benefit to the post-crunch self of this increase in consumption is $(u(\overline{c}) - u(y))\,dt$, and the benefit to the pre-crunch self is $\beta\,(u(\overline{c}) - u(y))\,dt$. The pre-crunch self is therefore indifferent between putting aside the extra $dx$ units of wealth and not putting them aside if and only if

$$u'(\overline{c})\,dx = \beta\,(u(\overline{c}) - u(y))\,dt.$$

Substituting for $dt$ and dividing through by $dx$, we obtain equation (41).

As Theorem 14 leads us to expect, for the top consumption function in Figure 5: the liquidity constraint is binding, i.e. $c(0) = y = 1$; there is an upward jump in consumption at $x = 0$, from $c(0) = 1$ to $c(0+) = \overline{c} = \frac{3}{2}$; and consumption rises monotonically thereafter.

**7.3. The high-$\mu$ case.** The other polar case of our model is that in which $\mu > \mu_1$. In this case, the expected returns on the asset are sufficiently attractive to induce the IG consumer to save even when her wealth is zero, and the liquidity constraint does not bind. More precisely:

**Theorem 15.** *If $\mu > \mu_1$ then: $c(0) < y$; $c(0+) = c(0)$; and $c' > 0$ on $[0, \infty)$.*

In other words: even when the IG consumer has no wealth, she still chooses to save out of her labor income; acquiring a little wealth does not lead to a jump in her consumption; and her consumption increases steadily with further increases in her wealth. In particular, her consumption function is strictly increasing.

**Proof.** See Appendix F. ■

As Theorem 15 leads us to expect, for the bottom consumption function in Figure 5: the liquidity constraint is not binding, i.e. $c(0+) = c(0) < 1$; and consumption rises smoothly over the entire wealth space.

**7.4. The intermediate-$\mu$ case.** The remaining case of our model is that in which $\gamma < \mu < \mu_1$. Loosely speaking: when wealth is low, this case looks like the low-$\mu$ case; and when wealth is high, it looks like the high-$\mu$ case. However, the most striking feature is the behavior of the consumption function during the transition between the two regimes.

**Theorem 16.** *If $\gamma < \mu < \mu_1$ then: $c(0) = y$; $c(0+) = \overline{c} > y$; and there exists $\overline{x} \in (0, \infty)$ such that $c' < 0$ on $(0, \overline{x})$ and $c' > 0$ on $(\overline{x}, \infty)$.*

In other words, when the IG consumer has no wealth, she consumes all of her labor income. If she acquires even a little wealth, then her consumption jumps up from $y$ to $\overline{c}$. As her wealth increases from 0 to $\overline{x}$, her consumption *decreases*, but, once her wealth reaches $\overline{x}$, her consumption increases steadily with further increases in her wealth.

**Proof.** See Appendix F. ■

As Theorem 16 leads us to expect, for the middle consumption function in Figure 5, the liquidity constraint is binding, i.e. $c(0) = y = 1$. There is also an upward jump in consumption at $x = 0$, from $c(0) = 1$ to $c(0+) = \overline{c} = \frac{3}{2}$. Finally, consumption declines smoothly after the upward jump before bottoming out and rising thereafter.

Comparing Theorem 16 with Theorems 14 and 15, a simple pattern emerges. The strategic interaction between the current self and future selves induces a form of positive feedback: the higher the marginal propensity to consume of tomorrow's self, the smaller the willingness of the current self to save, and therefore the higher her own marginal propensity to consume. By the same token, the higher the marginal propensity to save of tomorrow's self, the greater the willingness of the current self to save, and therefore the higher her own marginal propensity to save.

There are therefore two possible regimes: a high-consumption regime and a low-consumption regime. When $\mu$ is low, the consumer finds herself in the high-consumption regime irrespective of her wealth. When $\mu$ is intermediate, the consumer finds herself in the high-consumption regime when her wealth is low, and in the low-consumption regime when her wealth is high. So, naturally, her consumption needs to decrease as her wealth increases in order to effect the transition between the two regimes. Finally, when $\mu$ is high, the consumer finds herself in the low-consumption regime irrespective of her wealth.

The non-monotonic consumption function in the intermediate-$\mu$ case is not a robust feature of our model. Specifically, we can show that this non-monotonicity vanishes when we introduce a risk-free bond into the economy and allow investors to take long or short positions in the bond. Intuitively, taking a large short position in the bond enables the consumer to take large gambles, enabling her to concavify her value function. This eliminates the regions of non-monotonicity of the consumption function, since a globally concave value function has a slope that is monotonically falling in wealth, and the consumer equates her marginal utility of consumption to $\beta$ times the slope of her value function.

Hence the key *robust* properties that emerge from our analysis are a continuously differentiable consumption function in the interior of the wealth space, and the potential for an upward discontinuity of the consumption function at the point where the liquidity constraint binds ($x = 0$). This latter property cannot arise with an exponential discount function.

## 8. Extension to the Case of General Preferences

The main focus of the current paper is on the IG model, as embodied in Definition 6. In Sections 5, 6, and 7, we analyzed this model on the assumption that $u$ was CRRA. The first purpose of the present section is to generalize this analysis to a much larger class of utility functions in which relative risk aversion and relative prudence vary with consumption.[43] More precisely, we shall assume that $u$ has bounded relative risk aversion (or BRRA) and bounded relative prudence (or BRP). This generalization is of intrinsic interest, not least because it shows that the techniques of the current paper do not depend on a restrictive assumption about the form of the utility function. However, it is also a central tool in allowing us to achieve the second purpose of the current section, which is to provide a game-theoretic foundation for the IG model that is valid for all possible choices of the coefficient of relative risk aversion when $u$ is CRRA.

It will be helpful to elaborate on the second purpose. Note first that the current paper contains two versions of the IG model:

1. a preliminary version, namely the IG model with global lower bound $v \geq \frac{1}{\gamma}\, u(0)$, as embodied in Definition 4; and

2. the definitive version, namely the IG model with global lower bound $v \geq \frac{1}{\gamma}\, u(y)$, as embodied in Definition 6.

In Sections 2, 3 and 4, we developed a game-theoretic foundation for the IG model with global lower bound $v \geq \frac{1}{\gamma}\, u(0)$. The crucial assumption of those sections was that $u$ was bounded below. In Sections 5 and 7., we analyzed the IG model with global lower bound $v \geq \frac{1}{\gamma}\, u(y)$. The crucial assumption of those sections was that $u$ was CRRA. This leaves an obvious gap, in that the model that we analyzed in Sections 5 and 7 is different from the model for which we built the game-theoretic foundation in Sections 2, 3 and 4.

In Section 5, we built a first bridge across this gap: Theorem 5 showed that, if $u$ is *both* bounded below *and* CRRA, then any equilibrium of the IG model with global lower bound $v \geq \frac{1}{\gamma}\, u(0)$ is also an equilibrium of the IG model with global lower bound $v \geq \frac{1}{\gamma}\, u(y)$. In other words, we did provide a game-theoretic foundation for the IG model with global lower bound $v \geq \frac{1}{\gamma}\, u(y)$ in the case in which $u$ is both bounded below and CRRA. Given the direct intuitive appeal of the IG model with global lower bound $v \geq \frac{1}{\gamma}\, u(y)$, this was more than adequate justification for proceeding with the analysis of that model. However, it should be noted that a CRRA utility function $u$ is bounded below iff its coefficient of relative risk aversion $\rho$ is less than 1. So it would be desirable to extend the bridge to include cases in which $\rho$ is greater than 1.

[43]Specifically, we generalize the arguments of Sections 5 and 7. The calculations in Section 6 do not generalize, since this limiting case ( $y = 0$) requires the exact CRRA formulation to yield closed form solutions.

Now, the most natural way to approach the case in which $u$ is unbounded below is by starting from the case in which $u$ is bounded below and taking the limit as the lower bound goes to $-\infty$. This approach already works in the case in which $u$ is CRRA: it yields the case $\rho = 1$ (i.e. log utility) as the limit of cases in which $\rho < 1$. However, it does not allow us to get at cases in which $\rho > 1$. The problem is that the set of CRRA utility functions is simply too inflexible: if we start with the set of all CRRA utility functions (i.e. $\rho \in (0, \infty)$), take the intersection of this set with the set of utility functions that are bounded below (resulting in $\rho \in (0, 1)$) and then take the closure (resulting in $\rho \in (0, 1]$), we do not recover the whole of the set with which we started (namely $\rho \in (0, \infty)$). By contrast, the set of utility functions that is BRRA and BRP is much more flexible: if we start with the set of utility functions that are BRRA and BRP, take the intersection of this set with the set of utility functions that are bounded below and then take the closure, then we recover the entire set with which we started. Of particular interest, we recover *all* CRRA utility functions.

In a nutshell, Theorem 5 extends (along with the rest of Sections 5 and 7) from the case in which $u$ is CRRA to the case in which $u$ is BRRA and BRP. This allows us to build a second, much broader, bridge across the gap. In particular, we are able to provide a game-theoretic foundation for the IG model for all possible choices of $\rho$ when $u$ is CRRA.

**8.1. Generalized Assumptions.** We now formulate three new assumptions, namely Assumptions B1-B3. These generalize and replace Assumptions A1-A3. They will be formulated in terms of the (non-constant) relative risk aversion and the (non-constant) relative prudence (Kimball 1990) of $u$, namely

$$\rho(c) \equiv \frac{-c\, u''(c)}{u'(c)} \quad \text{and} \quad \pi(c) \equiv \frac{-c\, u'''(c)}{u''(c)}.$$

Notice that both $\rho$ and $\pi$ are functions: for each consumption level $c$, they tell us the coefficient of relative risk aversion and the coefficient of relative prudence at $c$.

The basic idea behind Assumption B1 is to ensure that relative risk aversion is bounded away from 0 and $\infty$ and that relative prudence is bounded away from 1 and $\infty$. The following preliminary version of Assumption B1 assumes this directly:

**B1′** There exist constants $\underline{\rho}, \overline{\rho} \in (0, \infty)$ and $\underline{\pi}, \overline{\pi} \in (1, \infty)$ such that $\underline{\rho} \leq \rho(c) \leq \overline{\rho}$ and $\underline{\pi} \leq \pi(c) \leq \overline{\pi}$ for all $c \in (0, \infty)$.

Unfortunately, this preliminary assumption has two disadvantages. First, it involves no fewer than four constants, namely $\underline{\rho}$, $\overline{\rho}$, $\underline{\pi}$ and $\overline{\pi}$. Second, it is stronger than we need. The following result will allow us to rectify both of these deficiencies.

**Theorem 17.** *Suppose that B1′ holds. Then* $\underline{\pi} - 1 \leq \rho(c) \leq \overline{\pi} - 1$.

It should be emphasized that Theorem 17 does *not* assert that, if $\pi$ is bounded above by $\overline{\pi} < \infty$ and below by $\underline{\pi} > 1$, then $\rho$ is bounded above by $\overline{\pi} - 1$ and below by $\underline{\pi} - 1$: the fact that $\rho$ is bounded below by some $\underline{\rho} > 0$ and above by some $\overline{\rho} < \infty$ is an essential part of the hypotheses. What it does tell us is that the specific choice of $\underline{\rho}$ and $\overline{\rho}$ does not matter. More precisely, in Assumption B1′, we may take it, without loss of generality, that $\underline{\rho} \geq \underline{\pi} - 1$ and $\overline{\rho} \leq \overline{\pi} - 1$. In particular, the weakest variant of Assumption B1′ is obtained when $\underline{\rho} = \underline{\pi} - 1$ and $\overline{\rho} = \overline{\pi} - 1$. We therefore replace Assumption B1′ with

**B1** There exist constants $\underline{\rho}, \overline{\rho} \in (0, \infty)$ such that $\underline{\rho} \leq \rho(c) \leq \overline{\rho}$ and $\underline{\rho} + 1 \leq \pi(c) \leq \overline{\rho} + 1$ for all $c \in (0, \infty)$.

This assumption is also more convenient to work with than Assumption B1′ because it involves only the two constants $\underline{\rho}$ and $\overline{\rho}$ instead of the four constants $\underline{\rho}$, $\overline{\rho}$, $\underline{\pi}$ and $\overline{\pi}$.[44]

**Proof.** See Appendix C. ■

Assumption B1 is our replacement for Assumption A1. Our replacements for Assumptions A2 and A3 are:

**B2** $1 - \beta < \dfrac{\underline{\rho}}{1 + \overline{\rho} - \underline{\rho}}$.

**B3** $\mu < \dfrac{1}{1 - \underline{\rho}}\gamma + \frac{1}{2}\underline{\rho}\,\sigma^2$ if $\underline{\rho} < 1$.

Assumption B2 requires that the dynamic inconsistency of the IG consumer (as measured by $1 - \beta$) must be smaller: (i) the lower the minimum possible coefficient of relative risk aversion (as measured by $\underline{\rho}$); and (ii) the larger the fluctuations in the coefficients of relative risk aversion and relative prudence (as measured by $\overline{\rho} - \underline{\rho}$). Assumption B3 requires that the expected return on the financial asset (as measured by $\mu$) be smaller,

[44] Given that the specific bounds on $\rho$ are deduced from those on $\pi$, it would be more natural from a mathematical point of view to phrase Assumption B1 in terms of $\underline{\pi}$ and $\overline{\pi}$. However, we prefer to work with $\underline{\rho}$ and $\overline{\rho}$ since these are more familiar.

the lower the minimum possible coefficient of relative risk aversion (as measured by $\underline{\rho}$). This is because, the lower $\underline{\rho}$, the faster the potential growth of $u$.

Assumptions B1-B3 are much more general than Assumptions A1-A3, and they are only marginally more complicated to state. In fact, Assumptions A1-A3 are simply the particular case of Assumptions B1-B3 obtained when $\overline{\rho} = \underline{\rho}$.

**8.2. Extending the Analysis.** The crucial step in extending our results is to prove the required generalization of the Value-Function Equivalence Theorem (Theorem 7). In other words, we need to find a wealth-dependent utility function $\widehat{u}$ such that the Bellman equation of the IG consumer is identical to the Bellman equation of the $\widehat{u}$ consumer. We do not restate the theorem in full, since the only change is that Assumptions A1-A3 are replaced by Assumptions B1-B3. Once the required generalization of the Value-Function Equivalence Theorem is proved, the proofs of Theorems 8-16 require little alteration.

**Theorem 18** [Generalization of Value-Function Equivalence]**.** *Theorem 7 holds as stated under the weaker Assumptions B1-B3.*

**Proof.** The first step is to construct $\widehat{u}_+ : (0, \infty) \to \mathbb{R}$. As in the proof of Theorem 7, we begin by defining a function $h_+ : (0, \infty) \to \mathbb{R}$ by the formula

$$h_+(\alpha) = u(f_+(\beta\,\alpha)) - \alpha\, f_+(\beta\,\alpha),$$

where $f_+(\alpha)$ is the unique $c$ satisfying $u'(c) = \alpha$. Assumptions B1-B2 then imply:[45]

**H1** $h'_+(\alpha) < 0$ and $h''_+(\alpha) > 0$ for all $\alpha > 0$; and

**H2** there exist $0 < \underline{\theta} \le \overline{\theta} < \infty$ such that $\underline{\theta} \le -\dfrac{\alpha\, h''_+(\alpha)}{h'_+(\alpha)} \le \overline{\theta}$ for all $\alpha > 0$.

Next, we define a function $\widehat{u}_+ : (0, \infty) \to \mathbb{R}$ by the formula

$$\widehat{u}_+(\widehat{c}) = \min_{\alpha \in (0,\infty)} h_+(\alpha) + \widehat{c}\,\alpha.$$

It can be verified that

**U1** $\widehat{u}'_+(\widehat{c}) > 0$ and $\widehat{u}''_+(\widehat{c}) < 0$ for all $\widehat{c} > 0$; and

[45] For a brief explanation of why Assumptions B1-B2 imply H1-H2, see Appendix G.

**U2** $\overline{\theta}^{-1} \leq -\dfrac{\widehat{c}\,\widehat{u}_+''(\widehat{c})}{\widehat{u}_+'(\widehat{c})} \leq \underline{\theta}^{-1}$ for all $\widehat{c} > 0$.

Finally, we define a function $\widehat{h}_+ : (0, \infty) \to \mathbb{R}$ by the formula

$$\widehat{h}_+(\alpha) = \max_{\widehat{c} \in (0,\infty)} \widehat{u}_+(\widehat{c}) - \alpha\,\widehat{c}$$

for all $\alpha > 0$.

The second step is to construct $\widehat{u}_0 : (0, y] \to \mathbb{R}$. We begin by defining a function $h_0 : (-\infty, \infty) \to \mathbb{R}$ by the formula

$$h_0(\alpha) = u(f_0(\beta\,\alpha)) - \alpha\, f_0(\beta\,\alpha),$$

where $f_0(\alpha)$ is the unique $c$ satisfying $u'(c) = \max\{u'(y), \alpha\}$. It is easy to check that

$$h_0(\alpha) = \left\{ \begin{array}{ll} u(y) - \alpha\, y & \text{for } \alpha \in (-\infty, \frac{1}{\beta}\, u'(y)] \\ h_+(\alpha) & \text{for } \alpha \in [\frac{1}{\beta}\, u'(y), \infty) \end{array} \right\}.$$

Next, we define a function $\widehat{u}_0 : (0, y] \to \mathbb{R}$ by the formula

$$\widehat{u}_0(\widehat{c}) = \min_{\alpha \in (-\infty,\infty)} h_0(\alpha) + \widehat{c}\,\alpha.$$

It can be verified that

$$\widehat{u}_0(\widehat{c}) = \left\{ \begin{array}{ll} \widehat{u}_+(\widehat{c}) & \text{for } \widehat{c} \in (0, \psi(y)\, y] \\ \widehat{u}_+(\psi(y)\, y) + (\widehat{c} - \psi(y)\, y)\, \widehat{u}_+'(\psi(y)\, y) & \text{for } \widehat{c} \in [\psi(y)\, y, y] \end{array} \right\},$$

where

$$\psi(y) = \frac{\rho(y) - (1 - \beta)}{\rho(y)}$$

and $\rho(y)$ is the relative risk aversion of $u$ at $y$. Finally, we define a function $\widehat{h}_0 : (-\infty, \infty) \to \mathbb{R}$ by the formula

$$\widehat{h}_0(\alpha) = \max_{\widehat{c} \in (0,y]} \widehat{u}_0(\widehat{c}) - \alpha\,\widehat{c}$$

for all $\alpha \in (-\infty, \infty)$.

The third and final step is to note that, because $h_+$ and $h_0$ are both convex, it follows that $\widehat{h}_+ = h_+$ and $\widehat{h}_0 = h_0$. This is an instance of convex duality, the basic reference for which is Rockafellar (1970). In particular, the Bellman equation of the IG consumer and the Bellman equation of the $\widehat{u}$ consumer coincide. ■

**8.3. Approximating a CRRA Utility Function.** In this subsection, we show that it is possible to approximate any CRRA utility function $u$ with a new utility function $\widetilde{u}_\varepsilon$ that: (i) differs from $u$ only at low consumption levels; (ii) is bounded below; and (iii) satisfies Assumption B1. In the next subsection, we will explain how to use such approximations to provide a formal game-theoretic foundation for the IG model in the case in which $u$ is CRRA with any coefficient of relative risk aversion, and indeed in the much more general case in which $u$ is BRRA and BRP.

**Theorem 19.** *Suppose that the utility function $u$ is given by the formula*

$$u(c) = \left\{ \begin{array}{ll} \frac{1}{1-\rho}\left(c^{1-\rho}-1\right) & \text{if } \rho \neq 1 \\ \ln(c) & \text{if } \rho = 1 \end{array} \right\}.$$

*Then, for all $\varepsilon \in (0,\infty)$ and all $\pi_L \in (1,\infty)$, there exists a unique utility function $\widetilde{u}_\varepsilon : (0,\infty) \to \mathbb{R}$ such that:*

1. *$\widetilde{u}_\varepsilon = u$ on $[\varepsilon,\infty)$;*
2. *$\frac{-c\,\widetilde{u}_\varepsilon'''(c)}{\widetilde{u}_\varepsilon''(c)} = \pi_L$ on $(0,\varepsilon)$;*
3. *$\widetilde{u}_\varepsilon$ is twice continuously differentiable.*

*This utility function satisfies Assumption B1.*

Notice that the new utility function $\widetilde{u}_\varepsilon$ is still fairly simple: it has piecewise constant relative prudence. Specifically, it has constant relative prudence $\pi_L$ in the interval $(0,\varepsilon)$ and constant relative prudence $\pi_R = \rho + 1$ in the interval $(\varepsilon,\infty)$. In particular, it has bounded relative prudence. It does not, however, have piecewise constant relative risk aversion: its relative risk aversion on the interval $(0,\varepsilon)$ varies monotonically from $\pi_L - 1$ at 0 to $\rho$ at $\varepsilon$. But it does have bounded relative risk aversion: it has constant relative risk aversion $\rho$ on the interval $(\varepsilon,\infty)$, so its relative risk aversion is bounded between $\min\{\pi_L - 1, \rho\}$ and $\max\{\pi_L - 1, \rho\}$.

The most important case of Theorem 19 for our purposes is the case in which $\rho \geq 1$, $\pi_L \in (1,2)$ and $\varepsilon$ is small. For in this case, $u$ is *not* bounded below but $\widetilde{u}_\varepsilon$ *is* bounded below. Moreover, fixing $\pi_L$, we obtain a family of utility functions $\{\widetilde{u}_\varepsilon \mid \varepsilon > 0\}$ such that $\widetilde{u}_\varepsilon \to u$ as $\varepsilon \downarrow 0$. These utility functions differ from $u$ only on a small and, as it turns out, economically irrelevant part of consumption space. They can therefore be used to derive the Bellman system of the IG consumer from that of the PF consumer.

**Proof.** See Appendix D. ■

**8.4. Deriving the IG Model when $u$ is CRRA.** Suppose now that Assumptions A1-A3 hold.[46] Then the IG model can be derived in three steps: (i) use the fact that $\widetilde{u}_\varepsilon$ is bounded below to derive the Bellman system of the IG consumer with utility function $\widetilde{u}_\varepsilon$ and global lower bound $v \geq \frac{1}{\gamma}\widetilde{u}_\varepsilon(0)$ from the Bellman system of the PF consumer with utility function $\widetilde{u}_\varepsilon$; (ii) use the fact that Assumptions B1-B3 hold for $\widetilde{u}_\varepsilon$ to show that any solution of the Bellman system of the IG consumer with utility function $\widetilde{u}_\varepsilon$ and global lower bound $v \geq \frac{1}{\gamma}\widetilde{u}_\varepsilon(0)$ in fact satisfies the tighter global lower bound $v \geq \frac{1}{\gamma}\widetilde{u}_\varepsilon(y)$; and (iii) derive the Bellman system of the IG consumer with utility function $u$ by letting $\varepsilon \downarrow 0$ in the Bellman system of the IG consumer with utility function $\widetilde{u}_\varepsilon$.

Since the case $\rho < 1$ is trivial, we shall carry out this derivation in the case $\rho \geq 1$. Furthermore, for expositional convenience, we make the additional assumption that $1 - \beta < \frac{1}{\rho}$.[47] Put $\pi_L = 2 - \eta$ for suitable $\eta \in (0,1)$, and construct the utility functions $\widetilde{u}_\varepsilon$ as in Section 8.3. Then, $\widetilde{u}_\varepsilon$ satisfies Assumption B1 with

$$\underline{\rho} = \min\{\pi_L - 1, \rho\} = 1 - \eta \quad \text{and} \quad \overline{\rho} = \max\{\pi_L - 1, \rho\} = \rho.$$

Furthermore, by choosing $\eta$ sufficient small, we can ensure that

$$\frac{\underline{\rho}}{1 + \overline{\rho} - \underline{\rho}} = \frac{1-\eta}{\rho + \eta} > 1 - \beta$$

[46] The current derivation can be carried out under the more general Assumptions B1-B3. The main idea is to generalize Theorem 19 in the obvious way: one approximates $u$ with a function $\widetilde{u}_\varepsilon$ satisfying $\widetilde{u}_\varepsilon = u$ on $[\varepsilon, \infty)$, $\frac{-c\,\widetilde{u}_\varepsilon'''(c)}{\widetilde{u}_\varepsilon''(c)} = \pi_L$ on $(0,\varepsilon)$, $\widetilde{u}_\varepsilon(\varepsilon-) = u(\varepsilon)$, $\widetilde{u}_\varepsilon'(\varepsilon-) = u'(\varepsilon)$ and $\widetilde{u}_\varepsilon''(\varepsilon-) = u''(\varepsilon)$.

[47] The condition $1 - \beta < \frac{1}{\rho}$ is used below to ensure that B2 is satisfied. However, an analysis of the material in Appendix G shows that we do not need the full force of B2 for steps (ii) and (iii) of the derivation in the current section. It suffices to ensure that $\widetilde{\rho}_\varepsilon(c) - (1-\beta) > 0$ and $(2-\beta)\,\widetilde{\rho}_\varepsilon(c) - (1-\beta)\,\widetilde{\pi}_\varepsilon(c) > 0$ for all $c$, where $\widetilde{\rho}_\varepsilon$ and $\widetilde{\pi}_\varepsilon$ are the relative risk aversion and relative prudence of $\widetilde{u}_\varepsilon$. Now, for our choice of $\widetilde{u}_\varepsilon$, both of these conditions reduce to A2 on $(\varepsilon, \infty)$. Moreover $\widetilde{\rho}_\varepsilon$ is monotonic and $\widetilde{\pi}_\varepsilon$ is constant on $(0,\varepsilon)$. Hence it suffices to ensure that both conditions are satisfied at both 0 and $\varepsilon$. This is easy to arrange.

(where we have used the fact that $1-\beta<\frac{1}{\rho}$ in order to obtain the second relation) and

$$\frac{1}{1-\underline{\rho}}\gamma+\frac{1}{2}\underline{\rho}\,\sigma^2=\frac{1}{\eta}\gamma+\frac{1}{2}(1-\eta)\,\sigma^2>\mu$$

(where we have used the fact that $\gamma>0$ in order to obtain the second relation). In other words, we can ensure that $\widetilde{u}_\varepsilon$ satisfies Assumptions B2 and B3 as well.

We can therefore carry out steps (i) and (ii) of the derivation of the Bellman system of the IG consumer. As for step (iii), it turns out that the unique solution $(c_\varepsilon, v_\varepsilon)$ of the Bellman system of the IG consumer with with utility function $\widetilde{u}_\varepsilon$ converges to the unique solution $(c,v)$ of the Bellman system of the IG consumer with utility function $u$ in a very strong sense: there exists $\varepsilon_0<\min\{c(x)\mid x\in[0,\infty)\}$ such that, for all $\varepsilon\le\varepsilon_0$, $c_\varepsilon=c$ and $v_\varepsilon=v$. In other words, $(c_\varepsilon, v_\varepsilon)$ converges finitely to $(c,v)$. Intuitively speaking, this result is driven by the fact that equilibrium consumption is bounded away from 0, and therefore the alteration to $u$ has no effect on the equilibrium once the threshold $\varepsilon$ of the truncation falls below the lower bound on consumption. To put the same point another way, the equilibrium $(c_\varepsilon, v_\varepsilon)$ is actually independent of $\varepsilon\in(0,\varepsilon_0]$: the details of how the utility function behaves for small consumption levels are irrelevant.

## 9. Conclusions

We have described a continuous-time model of quasi-hyperbolic discounting that extends the analysis of Barro (1999) and Luttmer and Mariotti (2003). Unlike these models, our instantaneous-gratification model allows for a generic class of preferences, includes liquidity constraints and places no restrictions on equilibrium policy functions. In our model, equilibrium is unique, resolving multiplicity problems in quasi-hyperbolic models.

Our paper studies a psychologically relevant limit case: we take the phrase 'instant gratification' literally, analyzing the case in which individuals prefer gratification in the present instant discretely more than consumption in the momentarily delayed future. This limit case is analytically tractable, and can easily be adapted for a range of applications.[48] Finally, from the perspective of calibration, the instantaneous-gratification model serves as a good approximation for models in which the "present" lasts for as long as a month.

[48] A partial list of applications that use our framework include Amador (2003), Della Vigna and Paserman (2005), Grenadier and Wang (2007), Bisin and Hyndman (2009), and Hsiaw (2010, 2010a), Palacios-Huerta and Pérez-Kakabadse (2011).

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## A. Proof of Theorem 5: Tightening the Global Lower Bound from $\frac{1}{\gamma}\,u(0)$ to $\frac{1}{\gamma}\,u(y)$

This appendix assumes familiarity with the contents of Section 5.3, including the proofs of Theorems 7 and 8.

Note first that, by eliminating $c$ from the Bellman *system* of the IG consumer with global lower bound $v \geq \frac{1}{\gamma}\,u(0)$, we obtain the Bellman *equation* of the IG consumer with global lower bound $v \geq \frac{1}{\gamma}\,u(0)$, namely

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y)\,v' - \gamma\,v + h(v', x) \tag{42}$$

for $x \in [0, \infty)$ and the global bounds

$$\tfrac{1}{\gamma}\,u(0) \leq v \leq \overline{v}. \tag{43}$$

Second, if $\widehat{u}$ and $\widehat{h}$ are defined exactly as in the proof of Theorem 7, then the Bellman equation of the $\widehat{u}$ consumer takes the form

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,\widehat{v}'' + (\mu\,x + y)\,\widehat{v}' - \gamma\,\widehat{v} + \widehat{h}(\widehat{v}', x) \tag{44}$$

for $x \in [0, \infty)$ and the global bounds

$$\tfrac{1}{\gamma}\,u(y) \leq \widehat{v} \leq \overline{v}, \tag{45}$$

exactly as in the proof of that theorem.

Now, by choice of $\widehat{u}$, equations (42) and (44) are identical. Moreover $u(0) < u(y)$. Hence any solution $\widehat{v}$ of the Bellman equation of the $\widehat{u}$ consumer is also a solution of the Bellman equation of the IG consumer with global lower bound $v \geq \frac{1}{\gamma}\,u(0)$. Conversely, equations (42-43) are simply a less restrictive, but still perfectly valid, way of formulating the Bellman equation of the $\widehat{u}$ consumer. As such, they have a unique solution $v$. Moreover this solution is the value function of the $\widehat{u}$ consumer. It must therefore coincide with the unique solution $\widehat{v}$ of equations (44-45), which is likewise the value function of the $\widehat{u}$ consumer. In particular, it must satisfy the global lower bound $v \geq \frac{1}{\gamma}\,u(y)$.

## B. Extending from the Case $y > 0$ to the Case $y \geq 0$

The core ideas of this paper (namely those in Sections 3-5, 7 and 8) are developed under the assumption that $y > 0$. This simplifies the exposition. A parallel development of the same ideas is, however, possible under the weaker assumption that $y \geq 0$. In order to undertake this development, we have to make two changes. First, we need to replace the one-sided integrability assumption B3 from Section 8, which for present purposes is most conveniently expressed in the equivalent form

**B3′** $\gamma > (1 - \underline{\rho})\,(\mu - \frac{1}{2}\,\underline{\rho}\,\sigma^2)$ if $\underline{\rho} < 1$,

with the two-sided integrability assumption

**B3″** $\gamma > (1 - \underline{\rho})\,(\mu - \frac{1}{2}\,\underline{\rho}\,\sigma^2)$ if $\underline{\rho} < 1$ and $\gamma > (1 - \overline{\rho})\,(\mu - \frac{1}{2}\,\overline{\rho}\,\sigma^2)$ if $\overline{\rho} > 1$.

Second, while we can still use the IG model with global lower bound $v \geq \frac{1}{\gamma}\,u(0)$, we need to replace the IG model with global lower bound $v \geq \frac{1}{\gamma}\,u(y)$ with what might be called the IG model with global lower bound $v \geq \underline{v}$. Here $\underline{v}$ is a suitable minorant for the value function of the $\widehat{u}$ consumer (i.e. $v(x) \geq \underline{v}(x)$ for all $x \geq 0$.) One way of identifying such a minorant runs as follows. We may note that: (i) the value function of the $\widehat{u}$ consumer with wage income $y \geq 0$ is greater than or equal to that of the $\widehat{u}$ consumer with wage income 0; (ii) the value function of the $\widehat{u}$ consumer is greater than or equal to that of the $\widehat{u}_+$ consumer, since $\widehat{u}_0 \geq \widehat{u}_+$; (iii) the value function of the $\widehat{u}_+$ consumer is greater than or equal to that of the $\underline{u}$ consumer, where $\underline{u}$ is a utility function with constant relative risk aversion $\overline{\rho}$ such that $\underline{u} \leq \widehat{u}_+$; (iv) the value function of the $\underline{u}$ consumer is finite provided that the second inequality in B3″ holds. The value function of the $\underline{u}$ consumer can therefore play the role of the minorant $\underline{v}$.

This parallel development shows, first, that the homogeneous IG model (in which $u$ is not assumed to be bounded below) has a unique equilibrium in the class of all policy functions. The explicit equilibrium found in Section 6.2 is therefore *the* equilibrium of the IG model. It shows, second, how to formulate and characterize equilibrium in the PF model (in which $u$ is assumed to be bounded below). The explicit equilibrium found in Section 6.1 is therefore *an* equilibrium of the PF model.

One thing that the parallel development does not tell us is whether the explicit equilibrium found in Section 6.1 is the unique equilibrium of the PF model. However, we can offer three related observations. First, we conjecture that uniqueness holds in the PF model for all $\lambda$ sufficiently large. Second, the discussion in Section 6.1 does at least show that there is *an* equilibrium of the PF model that is close to *the* equilibrium of IG model. Third, given that the equilibrium of the IG model is linear, it is natural to try to look for linear equilibria in the approximating model.

Another thing that the parallel development does not tell us is how to formulate and characterize equilibrium in the PF model when $u$ is *not* bounded below.[49] Nevertheless, whatever formulation of equilibrium is ultimately chosen for the PF model, it is clear that the candidate equilibrium constructed in Section 6.1 will be an equilibrium according to the formulation chosen. We therefore have the same conclusion that we had when $u$ was bounded below: there is *an* equilibrium of the PF model that is close to *the* equilibrium of IG model.

## C. Proof of Theorem 17: Relationship between Bounded Relative Prudence and Bounded Relative Risk Aversion

Let $\underline{\rho} = \inf\{\rho(c) \mid c \in (0,\infty)\}$, $\overline{\rho} = \sup\{\rho(c) \mid c \in (0,\infty)\}$, $\underline{\pi} = \inf\{\pi(c) \mid c \in (0,\infty)\}$ and $\overline{\pi} = \sup\{\pi(c) \mid c \in (0,\infty)\}$; and suppose that $0 < \underline{\rho} \le \overline{\rho} < \infty$ and $1 < \underline{\pi} \le \overline{\pi} < \infty$. Then we need to show that $\underline{\rho} \ge \underline{\pi} - 1$ and $\overline{\rho} \le \overline{\pi} - 1$. We begin with the inequality $\underline{\rho} \ge \underline{\pi} - 1$.

Put $\theta = \log(\rho)$, and define the penalized objective function $\phi_\varepsilon$ by the formula

$$\phi_\varepsilon(c) = \sigma(c) + \frac{\varepsilon}{2}\,(\log(c))^2$$

for all $\varepsilon \ge 0$. Since $\sigma$ is bounded, $\phi_\varepsilon$ attains its minimum on $(0,\infty)$ for all $\varepsilon > 0$. Let the minimum be $m_\varepsilon$, and suppose that it is attained at $c_\varepsilon$. The first-order condition for the minimum gives

$$0 = \phi'_\varepsilon(c_\varepsilon) = \sigma'(c_\varepsilon) + \frac{\varepsilon\,\log(c_\varepsilon)}{c_\varepsilon} = \frac{1 - \pi(c_\varepsilon) + \rho(c_\varepsilon) + \varepsilon\,\log(c_\varepsilon)}{c_\varepsilon}.$$

[49] One way of addressing this issue would be: (i) restrict the average propensity to consume in the PF model to lie in the interval $[\underline{\alpha},\infty)$ for some small $\underline{\alpha} > 0$; (ii) consider the limit of the PF model as $\lambda \uparrow \infty$ (to obtain a preliminary version of the IG model in which the average propensity to consume was bounded below by $\underline{\alpha}$); and (iii) consider the limit of the preliminary version of the IG model as $\underline{\alpha} \downarrow 0$ (to obtain the final version of the IG model). One could also pursue a more ambitious approach: (i) restrict the average propensity to consume to lie in the interval $[\underline{\alpha},\infty)$ for some small $\underline{\alpha} > 0$ (to obtain a preliminary version of the PF model); (ii) consider the limit of the preliminary version of the PF model as $\underline{\alpha} \downarrow 0$ (to obtain the final version of the PF model); and (iii) consider the limit of the final version of the PF model as $\lambda \uparrow \infty$ (to obtain the IG model). Cf. footnote 20.

Hence

$$\rho(c_\varepsilon) = \pi(c_\varepsilon) - 1 - \varepsilon\,\log(c_\varepsilon) \geq \underline{\pi} - 1 - \varepsilon\,\log(c_\varepsilon).$$

It therefore suffices to show that $\rho(c_\varepsilon) \to \underline{\rho}$ and $\varepsilon\,\log(c_\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

Let us begin with $\varepsilon\,\log(c_\varepsilon)$. Since 1 is a possible choice for $c$, we have

$$\sigma(c_\varepsilon) + \frac{\varepsilon}{2}\,(\log(c_\varepsilon))^2 = m_\varepsilon \leq \sigma(1).$$

Hence

$$\frac{\varepsilon}{2}\,(\log(c_\varepsilon))^2 \leq \sigma(1) - \sigma(c_\varepsilon) \leq \log(\overline{\rho}) - \log(\underline{\rho}) = \log\left(\frac{\overline{\rho}}{\underline{\rho}}\right)$$

and

$$|\varepsilon\,\log(c_\varepsilon)| \leq \sqrt{2\,\varepsilon\,\log\left(\frac{\overline{\rho}}{\underline{\rho}}\right)}.$$

It follows that $\varepsilon\,\log(c_\varepsilon) = \mathrm{O}(\sqrt{\varepsilon})$ as $\varepsilon \downarrow 0$.

Turning to $\rho(c_\varepsilon)$, let $\{d_n \mid n \in \mathbb{N}\}$ be a minimizing sequence for $\sigma$. (I.e. $\sigma(d_n) \to \log(\underline{\rho})$ as $n \to \infty$.) Certainly

$$m_\varepsilon \leq \sigma(d_n) + \frac{\varepsilon}{2}\,(\log(d_n))^2.$$

Hence, holding $n$ fixed, $\underline{m} = \limsup_{\varepsilon\downarrow 0} m_\varepsilon \leq \sigma(d_n)$. Hence, letting $n \to \infty$, $\underline{m} \leq \log(\underline{\rho})$. On the other hand,

$$m_\varepsilon = \sigma(c_\varepsilon) + \frac{\varepsilon}{2}\,(\log(c_\varepsilon))^2 \geq \sigma(c_\varepsilon) \geq \log(\underline{\rho}).$$

Hence $\underline{m} \geq \log(\underline{\rho})$ and, more importantly, $\sigma(c_\varepsilon) \to \log(\underline{\rho})$ as $\varepsilon \downarrow 0$. Hence $\rho(c_\varepsilon) \to \underline{\rho}$ as $\varepsilon \downarrow 0$.

### D. Proof of Theorem 19: Approximating a CRRA Utility Function

In order to simplify the algebra, we proceed somewhat indirectly. We begin by noting that the general solution of the differential equation

$$-\frac{c\,\widetilde{u}'''(c)}{\widetilde{u}''(c)} = \pi_L$$

on $(0, \varepsilon)$ takes the form

$$\widetilde{u}_L(c) = a_0 + \varepsilon\,a_1\left(\frac{c}{\varepsilon} - 1\right) + \frac{\varepsilon\,a_2}{2 - \pi_L}\left(\left(\frac{c}{\varepsilon}\right)^{2-\pi_L} - 1\right)$$

if $\pi_2 \neq 2$ and

$$\widetilde{u}_L(c) = a_0 + \varepsilon\, a_1 \left(\frac{c}{\varepsilon} - 1\right) + \varepsilon\, a_2 \log(\frac{c}{\varepsilon})$$

if $\pi_2 = 2$. Similarly, the general solution of the differential equation

$$-\frac{c\,\widetilde{u}''(c)}{\widetilde{u}'(c)} = \rho$$

on $(\varepsilon, \infty)$ takes the form

$$\widetilde{u}_R(c) = b_0 + \frac{\varepsilon\, b_2}{1-\rho}\left(\left(\frac{c}{\varepsilon}\right)^{1-\rho} - 1\right)$$

if $\rho \neq 1$ and

$$\widetilde{u}_R(c) = b_0 + \varepsilon\, b_2 \log\left(\frac{c}{\varepsilon}\right)$$

if $\rho = 1$.

Now, $\widetilde{u}_L$ is strictly concave iff $a_2 > 0$. Moreover, if $\widetilde{u}_L$ is strictly concave then its slope is minimized at $c = \varepsilon$, at which point it takes the value $a_1 + a_2$. Hence $\widetilde{u}_L$ is strictly increasing as well, then we must have $a_1 + a_2 > 0$. Similarly, if $\widetilde{u}_R$ is strictly concave and strictly increasing iff $b_2 > 0$. Also, from the matching conditions

$$\widetilde{u}_L(\varepsilon) = \widetilde{u}_R(\varepsilon), \quad \widetilde{u}'_L(\varepsilon) = \widetilde{u}'_R(\varepsilon), \quad \widetilde{u}''_L(\varepsilon) = \widetilde{u}''_R(\varepsilon),$$

we must have

$$a_0 = b_0, \quad a_1 + a_2 = b_2, \quad a_2\,(\pi_L - 1) = b_2\,\rho.$$

The latter equations can be solved for $a_0$, $a_1$ and $a_2$ in terms of $b_0$ and $b_2$ to give

$$a_0 = b_0, \quad a_1 = \frac{(\pi_L - 1) - \rho}{\pi_L - 1}\, b_2, \quad a_2 = \frac{\rho}{\pi_L - 1}\, b_2.$$

Now:

1. We can certainly choose the constants $b_0$ and $b_2$ so that $\widetilde{u}_R = u$ on $(\varepsilon, \infty)$. For this choice of $b_0$ and $b_2$, we will have $b_2 > 0$. Hence $a_2 > 0$ and $a_1 + a_2 = b_2 > 0$.

2. By construction, we will have $-\frac{c\,\widetilde{u}'''(c)}{\widetilde{u}''(c)} = \pi_L$ on $(0, \varepsilon)$, irrespective of the choice of $b_0$ and $b_2$.

3. By construction, the derivatives of $\widetilde{u}_L$ and $\widetilde{u}_R$ match at $\varepsilon$ up to and including order 2. Hence $\widetilde{u}$ is twice continuously differentiable.

4. The relative prudence $\widetilde{\pi}$ of $\widetilde{u}$ is piecewise constant, with $\widetilde{\pi} = \pi_L$ on $(0,\varepsilon)$ and $\widetilde{\pi} = \rho + 1$ on $(\varepsilon,\infty)$. In particular, $\widetilde{\pi}$ is bounded.

It therefore remains only to verify that the relative risk aversion $\widetilde{\rho}$ of $\widetilde{u}$ is bounded. But it is easy to check that

$$\widetilde{\rho}(c) = \left\{ \begin{array}{ll} \dfrac{a_2(\pi_L - 1)}{a_1\left(\dfrac{c}{\varepsilon}\right)^{\pi_L - 1} + a_2} & \text{if } c \in (0,\varepsilon) \\ \rho & \text{if } c \in (\varepsilon,\infty) \end{array} \right\}.$$

Hence $\widetilde{\rho}$ changes monotonically from $\pi_L - 1 > 0$ at 0 to $\frac{a_2}{a_1+a_2}(\pi_L - 1) = \rho$ at $\varepsilon$, and is constant thereafter.

Instantaneous Gratification

Web Appendix

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4 June 2011

This document contains three supplementary appendices:

E. Solution of the Homogeneous PF Model

F. Proof of Theorems 14, 15 and 16: Characterization of the Consumption Function in the Inhomogeneous Case

G. Proof that Assumptions B1-B2 Imply Assumptions H1-H2

## E. Solution of the Homogeneous PF Model

Substituting for $v$ and $c$ in equation (34) and equating the constant term to 0, we get $\Theta = \frac{1}{\gamma}$. Equation (34) then simplifies to

$$0 = \mu - \alpha - \tfrac{1}{2}\,\rho\,\sigma^2 + \gamma\,u\Big(\frac{\alpha}{\theta}\Big). \tag{46}$$

Second, substituting for $v$, $w$ and $c$ in equation (35) and equating the constant term to 0, we get $\Phi = \frac{\gamma+\beta\,\lambda}{\gamma\,(\gamma+\lambda)}$. Equation (35) then simplifies to

$$0 = \mu - \alpha - \tfrac{1}{2}\,\rho\,\sigma^2 + (\gamma+\lambda)\left(\frac{\gamma}{\gamma+\beta\,\lambda}\,u\Big(\frac{\alpha}{\phi}\Big) + \frac{\beta\,\lambda}{\gamma+\beta\,\lambda}\,u\Big(\frac{\theta}{\phi}\Big)\right). \tag{47}$$

Last, substituting for $w$ and $c$ in equation (36), we get

$$u'(\alpha) = \frac{\gamma+\beta\,\lambda}{\gamma\,(\gamma+\lambda)}\,\phi\,u'(\phi). \tag{48}$$

Now, for all $\rho$, $u$ satisfies the functional equation $(1-\rho)\,u(z) = z\,u'(z) - 1$. Hence, multiplying equation (46) through by $1-\rho$, putting $m(z) = z\,u'(z)$ and rearranging, we obtain

$$\frac{m(\alpha)}{m(\theta)} = \frac{\gamma - (1-\rho)\,(\mu - \alpha - \frac{1}{2}\,\rho\,\sigma^2)}{\gamma}. \tag{49}$$

Similarly, from equation (47), we obtain

$$\frac{\gamma}{\gamma+\beta\,\lambda}\,\frac{m(\alpha)}{m(\phi)} + \frac{\beta\,\lambda}{\gamma+\beta\,\lambda}\,\frac{m(\theta)}{m(\phi)} = \frac{\gamma+\lambda-(1-\rho)\,(\mu-\alpha-\frac{1}{2}\,\rho\,\sigma^2)}{\gamma+\lambda}. \tag{50}$$

Last, multiplying (48) through by $\alpha$ and dividing through by $\phi\,u'(\phi)$, we obtain

$$\frac{m(\alpha)}{m(\phi)} = \frac{\gamma+\beta\,\lambda}{\gamma\,(\gamma+\lambda)}\,\alpha. \tag{51}$$

Using equations (49) and (51), we can eliminate $\frac{m(\alpha)}{m(\phi)}$ and $\frac{m(\theta)}{m(\phi)} = (\frac{m(\alpha)}{m(\phi)})/(\frac{m(\alpha)}{m(\theta)})$ from (50) to obtain the quadratic (37) given in the main text, namely

$$0 = \frac{\lambda}{1+\lambda}\,\left((\rho+\beta-1)\,\alpha - \widetilde{\gamma}\right) + \frac{1}{1+\lambda}\left(\rho\,(1-\rho)\,\alpha^2 + (2\,\rho-1)\,\widetilde{\gamma}\,\alpha - \widetilde{\gamma}^2\right).$$

This quadratic is a convex combination of the affine term

$$(\rho + \beta - 1)\ \alpha - \widetilde{\gamma}$$

and the quadratic term

$$\rho\,(1-\rho)\,\alpha^2 + (2\,\rho - 1)\,\widetilde{\gamma}\,\alpha - \widetilde{\gamma}^2.$$

Moreover the quadratic term is convex when $\rho \leq 1$ and concave when $\rho \geq 1$.

In the case in which $\rho < 1$, one can take advantage of the convexity of the quadratic term to show that there are two solutions of (37). The first is always positive, varying from $\frac{\widetilde{\gamma}}{\rho}$ when $\lambda = 0$ to $\frac{\widetilde{\gamma}}{\rho+\beta-1}$ when $\lambda = \infty$. The second is always negative, varying from $-\frac{\widetilde{\gamma}}{1-\rho}$ when $\lambda = 0$ to $-\infty$ when $\lambda = \infty$. Since the second solution gives rise to a negative average propensity to consume, the first solution is the only relevant one.

In the case in which $\rho = 1$, the quadratic term degenerates into an affine term and the unique solution of (37) is $\frac{\tilde{\gamma}\,(\tilde{\gamma}+\lambda)}{\tilde{\gamma}+\beta\,\lambda}$. This varies from $\widetilde{\gamma}$ when $\lambda = 0$ to $\frac{\widetilde{\gamma}}{\beta}$ when $\lambda = \infty$.

In the case in which $\rho > 1$, one can take advantage of the concavity of the quadratic term to show that there are again two solutions of (37). Both solutions are always positive. The first varies from $\frac{\widetilde{\gamma}}{\rho}$ when $\lambda = 0$ to $\frac{\widetilde{\gamma}}{\rho+\beta-1}$ when $\lambda = \infty$. The second varies from $\frac{\widetilde{\gamma}}{\rho-1}$ when $\lambda = 0$ to $+\infty$ when $\lambda = \infty$. Since the right-hand side of equation (49) can be written in the form $\frac{\widetilde{\gamma}-(\rho-1)\,\alpha}{\gamma}$, the second solution would force $m(\theta) \leq 0$ (with equality iff $\lambda = 0$). The first solution is therefore the only relevant one.

Finally, note that the relevant solution of the quadratic can be written in the form

$$\alpha = \frac{2\,\rho\,\tilde{\gamma} + \lambda\,(\beta + \rho - 1) - \tilde{\gamma} - \sqrt{(\lambda\,(\beta + \rho - 1) - \tilde{\gamma})^2 + 4\,\lambda\,\beta\,\rho\,\tilde{\gamma}}}{2\,\rho\,(\rho - 1)}.$$

Moreover equations (49) and (51) yield

$$m(\theta) = \frac{\gamma\,m(\alpha)}{\tilde{\gamma} + (1-\rho)\,\alpha}, \quad m(\phi) = \frac{\gamma\,(\gamma + \lambda)\,m(\alpha)}{(\gamma + \beta\,\lambda)\,\alpha}.$$

The behavior of the value functions $v(x) = \frac{1}{\gamma}\,u(\theta\,x)$ and $w(x) = \frac{\gamma+\beta\,\lambda}{\gamma\,(\gamma+\lambda)}\,u(\phi\,x)$ as a function of $\lambda$ can therefore be deduced from that of $\alpha$.

## F. Proof of Theorems 14, 15 and 16: Characterization of the Consumption Function in the Inhomogeneous Case

In this appendix, we outline the proof of Theorems 14, 15 and 16.

**F.1. Some background information.** In this section we state without proof three results that will help to organize the subsequent discussion on the form of the consumption function.

Note first that, by definition of equilibrium in the IG model, the value function $v$ satisfies the global bounds $\frac{1}{\gamma}\,u(y) \leq v(x) \leq \overline{v}(x)$, where $\overline{v}$ is the value function of a consumer who: (i) has utility function $u$; and (ii) discounts the future exponentially at rate $\gamma$. Our first result gives more concrete information on the form of the upper bound.

**Proposition 20.** *There exists $K > 0$ such that*

$$\tfrac{1}{\gamma}\,u(y) \leq v(x) \leq K\,(1 + u(1 + x))$$

*for all $x \geq 0$.* ■

The main significance of this result for our current purposes is that $u$ is strictly concave, and therefore $v$ cannot be convex.

Our second result concerns the smoothness of $v$.

**Proposition 21.** *Suppose that $\beta < 1$. Then $v$ is infinitely differentiable on $[0, \infty)$.* ■

In particular, the discontinuity in $\widehat{u}$ at $x = 0$ (i.e. the fact that $\widehat{u}_+ \neq \widehat{u}_0$) does not translate into a discontinuity in $v$ or any of its derivatives at $x = 0$. On the contrary, Proposition 21 actually depends on this discontinuity: when $\beta = 1$ (and therefore $\widehat{u}_+ = \widehat{u}_0$), $v$ is not smooth at $x = 0$ when $\mu < \gamma$. The discontinuity in $\widehat{u}$ at $x = 0$ does, however, give rise to a different kind of discontinuity: as we shall see below, $v'(0)$ does not always vary continuously with $\mu$. In fact, there exists $\mu_1 \in (\gamma, \overline{\mu})$ such that $v'(0)$ jumps up from $v'_L = \widehat{u}'_+(\psi\,\overline{c}) < \widehat{u}'_+(y)$ to $v'_R = \widehat{u}'_+(\psi\,y) > \widehat{u}'_+(y)$ as $\mu$ crosses $\mu_1$. (Recall that $\psi = \frac{\rho-(1-\beta)}{\rho}$ and that $\overline{c}$ is the unique solution of the equation $u'(\overline{c}) = \beta\,\frac{u(\overline{c})-u(y)}{\overline{c}-y}$.)

Our third result states that the shadow value of wealth is always strictly positive:

**Proposition 22.** $v' > 0$ *on* $[0, \infty)$. ■

This is economically obvious: the $\widehat{u}$ consumer can always consume more in its current span of control.

**F.2. A mathematical intuition.** Recall that the utility function of the $\widehat{u}$ consumer has two parts: $\widehat{u}(\widehat{c}, x) = \widehat{u}_0(\widehat{c})$ when $x = 0$; and $\widehat{u}(\widehat{c}, x) = \widehat{u}_+(\widehat{c})$ when $x > 0$. Moreover $\widehat{u}_0(\widehat{c}) \geq \widehat{u}_+(\widehat{c})$ for all $\widehat{c} \in (0, y]$, with strict inequality when $\widehat{c} \in (\psi\, y, y]$. (See Figure 4.) In other words, the $\widehat{u}$ consumer obtains a utility premium when $x = 0$.

This suggests that, at any given wealth level, the $\widehat{u}$ consumer must choose between two strategies. The first, high-consumption, strategy is to dissave until her wealth runs out, and then enjoy the utility premium that she obtains at $x = 0$. The second, low-consumption, strategy is to save forever in order to take advantage of the higher asset income associated with higher financial wealth. Which of these two strategies is better will depend on $\mu$. If $\mu$ is low, then the high-consumption strategy will be better no matter how large the wealth of the $\widehat{u}$ consumer. Similarly, if $\mu$ is high, then the low-consumption strategy will be better no matter how small the wealth of the $\widehat{u}$ consumer. However, if $\mu$ is intermediate then the high-consumption strategy will be better when wealth is low (and therefore the utility premium will be enjoyed after a relatively short wait) and the low-consumption strategy will be better when wealth is high (and therefore the prospect of the utility premium is too distant). Moreover consumption may in principle decrease with wealth over an intermediate range of wealth levels, as the $\widehat{u}$ consumer adjusts from the high-consumption strategy associated with low wealth to the low-consumption strategy associated with high wealth.

**F.3. The boundary condition at $x = 0$.** The value function $v$ must satisfy two related conditions at $x = 0$. To derive the first of these conditions, note that the Bellman equation of the $\widehat{u}$ consumer takes the form

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y)\,v' - \gamma\,v + \widehat{h}_+(v') \tag{52}$$

for $x > 0$. Letting $x \downarrow 0$ in this equation, taking advantage of Proposition 21 and rearranging yields

$$v(0) = \tfrac{1}{\gamma}\left(y\,v'(0)\; + \widehat{h}_+(v'(0))\right). \tag{53}$$

The second of these conditions is simply the Bellman equation of the $\widehat{u}$ consumer at $x = 0$ which, on rearrangement, becomes

$$v(0) = \tfrac{1}{\gamma}\left(y\,v'(0) + \widehat{h}_0(v'(0))\right). \tag{54}$$

Figure 6a: graph of $v(0) = \frac{1}{\gamma}\left(y\, v'(0) + \hat{h}_+(v'(0))\right)$

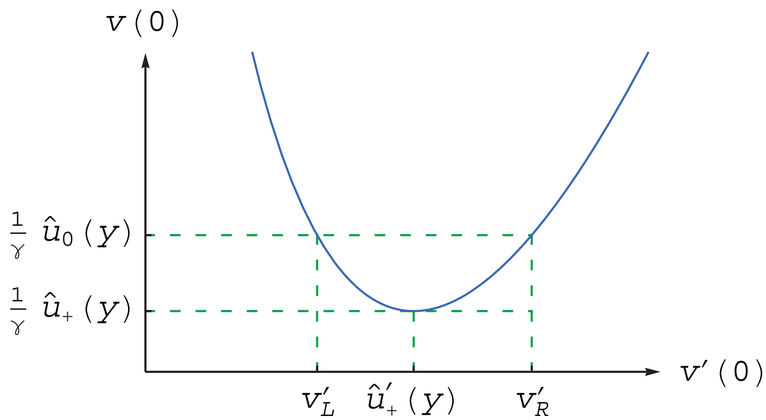



Figure 6b: graph of $v(0) = \frac{1}{\gamma}\left(y\, v'(0) + \hat{h}_0(v'(0))\right)$

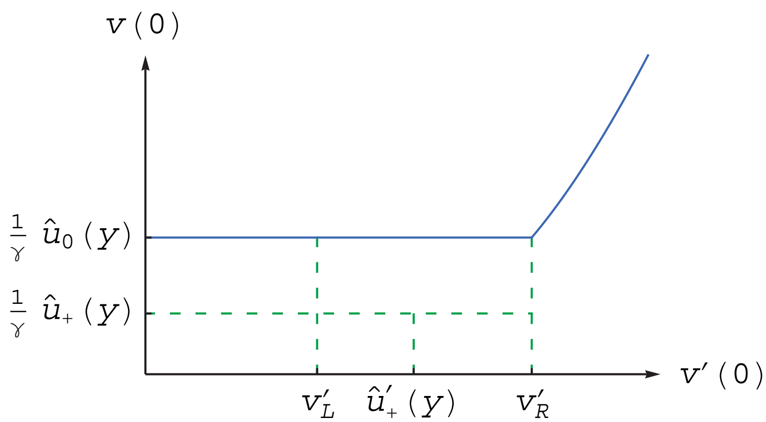
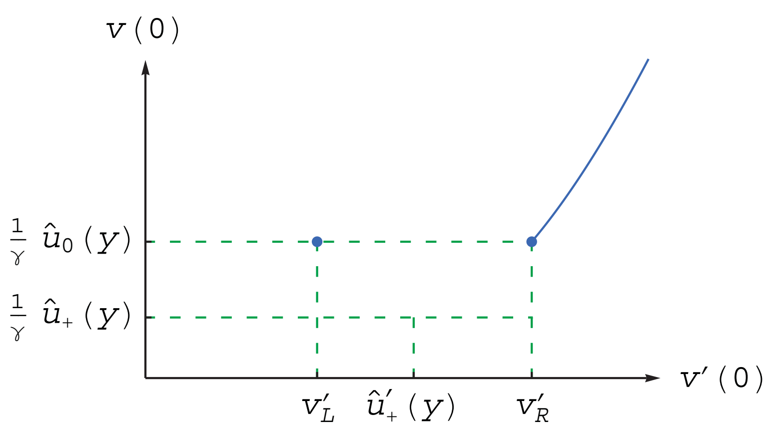



Figure 6c: feasible choices of $(v'(0), v(0))$ at $x = 0$



Figures 6a, 6b and 6c illustrate the locus of points $(v'(0), v(0))$ satisfying equation (53), the locus of points $(v'(0), v(0))$ satisfying equation (54) and the locus of points $(v'(0), v(0))$ satisfying both equations.

As Figure 6c shows, there are two possible boundary configurations. First, the $\widehat{u}$ consumer may opt for the utility premium and set $\widehat{c}(0) = y$. In this case $v(0) = \frac{1}{\gamma}\,\widehat{u}_0(y)$, and $v'(0)$ must take on the low value $v'_L = \widehat{u}'_+(\psi\,\overline{c}) < \widehat{u}'_+(y)$ in order to justify the $\widehat{u}$ consumer's high consumption level for small $x > 0$. Second, the $\widehat{u}$ consumer may forgo the utility premium and set $\widehat{c}(0) \leq \psi\, y$. In this case $v(0) \geq \frac{1}{\gamma}\,\widehat{u}_0(y)$, and we must have $v'(0) \geq v'_R = \widehat{u}'_+(\psi\, y) > \widehat{u}'_+(y)$ in order to justify the $\widehat{u}$ consumer's low consumption level at $x = 0$. We refer to these two configurations as the low-shadow-value and high-shadow-value boundary configurations respectively.

Our next major objective is to show that there exists $\mu_1 \in (\gamma, \overline{\mu})$ such that the low-shadow-value boundary configuration occurs when $\mu < \mu_1$ and the high-shadow-value boundary configuration occurs when $\mu > \mu_1$. To this end, we shall need several supporting results.

**F.4. Once convex, always strictly convex.** The following result only uses the fact that $v$ satisfies the Bellman equation of the $\widehat{u}$ consumer in the interior of the wealth space.

**Proposition 23.** *Suppose that $\mu < \gamma$ and that either*

1. *there exists $x_0 \geq 0$ such that $v''(x_0) > 0$, or*
2. *there exists $x_0 > 0$ such that $v''(x_0) \geq 0$.*

*Then $v''(x) > 0$ for all $x > x_0$.*

**Proof.** Differentiating the Bellman equation of the $\widehat{u}$ consumer (i.e. equation (52)) with respect to $x$, we obtain

$$0 = \tfrac{1}{2}\,\sigma^2\, x^2\, v''' + ((\sigma^2 + \mu)\, x + y - \widehat{c})\, v'' - (\gamma - \mu)\, v'. \tag{55}$$

Now suppose that there exists $x_0 \geq 0$ such that $v''(x_0) > 0$, and suppose for a contradiction that there exists $x_1 > x_0$ such that $v''(x_1) \leq 0$. Let $x_2$ be the leftmost point in $(x_0, x_1]$ such that $v''(x_2) \leq 0$. Since $v'' > 0$ on $[x_0, x_2)$, we must actually have $v''(x_2) = 0$. Equation (55) then yields

$$v'''(x_2) = \frac{(\gamma - \mu)\, v'(x_2)}{\frac{1}{2}\,\sigma^2\, x_2^2}, \tag{56}$$

and the latter expression is strictly positive because $\gamma > \mu$ (by assumption), $v'(x_2) > 0$ (by Proposition 22) and $x_2 > 0$ (by construction). We therefore have $v'' < 0$ to the left of $x_2$, which is the required contradiction.

It remains to consider the case in which there exist $x_0 > 0$ such that $v''(x_0) = 0$. In that case (56) imples that $v'''(x_0) > 0$. But then there exists $\widetilde{x}_0 > x_0$ such that $v''(\widetilde{x}_0) > 0$. We are then back in the previous case. ■

Combining this result with the fact that $v$ satisfies the Bellman equation of the $\widehat{u}$ consumer on the boundary of the wealth space, we obtain the following corollary.

**Corollary 24.** *Suppose that $\mu < \gamma$. Then $v'' < 0$ for all $x \geq 0$.*

In particular, $\widehat{c}' > 0$ on $(0, \infty)$. This is the case in which the $\widehat{u}$ consumer chooses the high-consumption strategy at all wealth levels.

**Proof.** Suppose for a contradiction that there exists $x_0 > 0$ such that $v''(x_0) \geq 0$. Then Proposition 23 implies that $v$ is convex on $[x_0, \infty)$. This contradicts Proposition 20, which tells us that $v$ is bounded above by a function that grows at the same rate as $u$. We therefore have $v'' < 0$ for all $x > 0$. This in turn implies that $v''(0) \leq 0$. Now, letting $x \downarrow 0$ in equation (55) and rearranging, we obtain

$$v''(0) = \frac{(\gamma - \mu)\, v'(0)}{y - \widehat{c}(0+)}.$$

So: either $v'(0) = v'_L$, in which case $y - \widehat{c}(0+) < 0$ and therefore $v''(0) < 0$; or $v'(0) \geq v'_R$, in which case $y - \widehat{c}(0+) > 0$ and therefore $v''(0) > 0$. (Recall that $\gamma - \mu > 0$ by assumption.) Since $v''(0) \leq 0$, we must be in the first of these two cases. In particular, $v''(0) < 0$. This completes the proof. ■

Actually, the proof of Corollary 24 shows more:

**Corollary 25.** *Suppose that $\mu < \gamma$. Then $v'(0) = v'_L$.* ■

In other words, if $\mu < \gamma$ then the low-shadow-value boundary configuration obtains. In particular, $\widehat{c}(0+) = \psi\, \overline{c} > \widehat{c}(0) = y$.

**F.5. Once concave, always strictly concave.** The following result likewise only uses the fact that $v$ satisfies the Bellman equation of the $\widehat{u}$ consumer in the interior of the wealth space.

**Proposition 26.** *Suppose that $\mu > \gamma$ and that either*

1. *there exists $x_0 \geq 0$ such that $v''(x_0) < 0$, or*
2. *there exists $x_0 > 0$ such that $v''(x_0) \leq 0$.*

*Then $v''(x) < 0$ for all $x > x_0$.*

**Proof.** The proof is completely analogous to that of Proposition 23. ■

Combining this result with the fact that $v$ satisfies the Bellman equation of the $\widehat{u}$ consumer on the boundary of the wealth space, we obtain the following corollary.

**Corollary 27.** *Suppose that $\mu > \gamma$. Then either*

1. *there exists $\overline{x} \in (0, \infty)$ such that $v'' > 0$ on $(0, \overline{x})$, and $v'' < 0$ on $(\overline{x}, \infty)$; or*
2. *$v'' < 0$ for all $x \geq 0$.*

In particular: either there exists $\overline{x} \in (0, \infty)$ such that $\widehat{c}' < 0$ on $(0, \overline{x})$, and $\widehat{c}' > 0$ on $(\overline{x}, \infty)$; or $\widehat{c}' > 0$ on $[0, \infty)$. The first case is the case in which the $\widehat{u}$ consumer chooses the high-consumption strategy at low wealth levels and the low-consumption strategy at high wealth levels. The second case is the case in which the $\widehat{u}$ consumer chooses the low-consumption strategy at all wealth levels.[50]

**Proof.** Let $X_0$ be the set of all $x_0 \in [0, \infty)$ such that $v''(x_0) \leq 0$. Proposition 20 implies that we cannot have $v'' > 0$ for all $x \geq 0$, so $X_0$ is non-empty. Let $\overline{x}$ be the smallest element of $X_0$. There are then two possibilities: either $\overline{x} > 0$ or $\overline{x} = 0$. If $\overline{x} > 0$, then $v'' > 0$ for all $x \in [0, \overline{x})$. Hence $v''(\overline{x}) = 0$, and Proposition 26 implies that $v'' < 0$ for all $x \in (\overline{x}, \infty)$. On the other hand, if $\overline{x} = 0$ then our construction of $\overline{x}$ yields only that $v''(0) \leq 0$. However, as in the proof of Corollary 24, we have

$$v''(0) = \frac{(\gamma - \mu)\, v'(0)}{y - \widehat{c}(0+)}.$$

[50]Notice that, in the first case, we have $\overline{x} > 0$ and $\widehat{c}'(\overline{x}) = 0$. One might therefore have expected to find a knife-edge case between the first and second cases in which $\overline{x} = 0$, $\widehat{c}'(0) = 0$ and $\widehat{c}'(x) > 0$ for all $x > 0$. This possibility is ruled out by Corollary 27. The reason why it does not arise is that $\widehat{c}'(0)$ does not vary continuously as the parameters of the model vary. Specifically, if $\overline{x} \downarrow 0$ as the parameter vector converges to an appropriate limit, then $\widehat{c}'(0)$ jumps up at the limit (and $v''(0)$ jumps down). To put the same point another way, away from the limit the low-shadow-value boundary configuration obtains, but at the limit the high-shadow-value boundary configuration obtains. Moreover it cannot happen that both boundary configurations obtain simultaneously.

Moreover: either $v'(0) = v'_L$, in which case $y - \widehat{c}(0+) < 0$ and therefore $v''(0) > 0$; or $v'(0) \geq v'_R$, in which case $y - \widehat{c}(0+) > 0$ and therefore $v''(0) < 0$. (Recall that we now have $\gamma - \mu < 0$.) Since $v''(0) \leq 0$, we must be in the second of these two cases. In particular, $v''(0) < 0$. We conclude that $v'' < 0$ for all $x \geq 0$ when $\overline{x} = 0$. ■

Actually, the proof of Corollary 27 shows slightly more:

**Corollary 28.** *Suppose that $\mu > \gamma$. Then either*

1. $v'(0) = v'_L$*, in which case* $v''(0) > 0$*; or*
2. $v'(0) \geq v'_R$*, in which case* $v''(0) < 0$. ■

In other words, if $\mu > \gamma$ then either the low-shadow-value boundary configuration occurs, in which case $\widehat{c}' < 0$ for small positive $x$, or the high-shadow-value boundary configuration occurs, in which case $\widehat{c}' > 0$ for all positive $x$. If the low-shadow-value boundary configuration obtains then $\widehat{c}(0+) = \psi\,\overline{c} > \widehat{c}(0) = y$. The main surprise in this case is the way in which: (i) the initial increase in consumption is confined to a single upward jump in $\widehat{c}$ at $x = 0$; and (ii) the decrease in consumption — as the consumer adjusts to the low-consumption strategy — begins immediately to the right of $x = 0$. On the other hand, if the high-shadow-value boundary configuration obtains, then $\widehat{c}(0+) = \widehat{c}(0) \leq \psi\, y$.

**F.6. The $\widehat{u}_+$ consumer.** Combining Corollaries 24, 25, 27 and 28, we can tentatively identify three cases:

1. $\mu < \gamma$ and the low-shadow-value boundary configuration obtains.
2. $\mu > \gamma$ and the low-shadow-value boundary configuration obtains.
3. $\mu > \gamma$ and the high-shadow-value boundary configuration obtains.

However, we have not yet identified the borderline between cases 2 and 3. In order to locate this borderline, it will be helpful to consider a consumer who

- discounts the future exponentially at rate $\gamma$,
- faces the same wealth dynamics as the IG consumer and
- has the wealth-independent utility function $\widehat{u}_+$.

We call this consumer the $\widehat{u}_+$ consumer.

Let the value function of the $\widehat{u}_+$ consumer be $\widehat{v}_+ = \widehat{v}_+(x;\mu)$, where we have made explicit the dependence of $\widehat{v}_+$ on the parameter $\mu$. Then:

**Proposition 29.**

1. *The low-shadow-value boundary configuration obtains if and only if* $\widehat{v}_+(0;\mu) < \frac{1}{\gamma}\widehat{u}_0(y)$.
2. *The high-shadow-value boundary configuration obtains if and only if* $\widehat{v}_+(0;\mu) \geq \frac{1}{\gamma}\widehat{u}_0(y)$. ■

The point here is that the $\widehat{u}$ consumer effectively has two options when $x = 0$: either exploit the utility premium available at $x = 0$ to the full, by consuming $y$ and remaining at 0; or dispense with the utility premium altogether. The first option yields $\frac{1}{\gamma}\widehat{u}_0(y)$, and the second yields $\widehat{v}_+(0;\mu)$. If the first option is strictly better than the second, then the low-shadow-value boundary configuration obtains. If the second option is at least as good as the first, then the high-shadow-value boundary configuration obtains. It should also be noted that

$$v(0;\mu) = \max\{\tfrac{1}{\gamma}\widehat{u}_0(y), \widehat{v}_+(0;\mu)\},$$

where we have made explicit the dependence of $v$ on the parameter $\mu$.

Since $\widehat{v}_+$ is the value function of a standard optimization problem, we can use standard arguments to find those of its properties that are relevant to us. These properties are summarized in the following proposition.

**Proposition 30.**

1. $\widehat{v}_+(0;\mu)$ *is non-decreasing and continuous in* $\mu$ *for* $\mu \in (-\infty, \overline{\mu})$.
2. $\widehat{v}_+(0;\mu) = \frac{1}{\gamma}\widehat{u}_+(y)$ *for all* $\mu \in (-\infty, \gamma]$.
3. $\widehat{v}_+(0;\mu)$ *is strictly increasing in* $\mu$ *for* $\mu \in [\gamma, \overline{\mu})$.
4. $\widehat{v}_+(0;\mu) \uparrow \frac{1}{\gamma}\widehat{u}_+(\infty)$ *as* $\mu \uparrow \overline{\mu}$. ■

Noting that $\widehat{u}_+(y) < \widehat{u}_0(y) < \widehat{u}_+(\infty)$, we see that there is a unique $\mu_1 \in (\gamma, \overline{\mu})$ such that: (i) $\widehat{v}_+(0;\mu) < \frac{1}{\gamma}\widehat{u}_0(y)$ for $\mu < \mu_1$; (ii) $\widehat{v}_+(0;\mu_1) = \frac{1}{\gamma}\widehat{u}_0(y)$; and (iii) $\widehat{v}_+(0;\mu) > \frac{1}{\gamma}\widehat{u}_0(y)$ for $\mu > \mu_1$. The borderline between cases 2 and 3 therefore occurs at $\mu = \mu_1$.

**F.7. From $\widehat{c}$ to $c$.** At this point we have shown that there exists $\mu_1 \in (\gamma, \overline{\mu})$ such that:

1. If $\mu < \gamma$ then the low-shadow-value boundary configuration holds. I.e. $v(0) = \frac{1}{\gamma}\,\widehat{u}_0(y)$ and $v'(0) = v'_L = \widehat{u}'_+(\psi\,\overline{c}) < \widehat{u}'_+(y)$. This implies that $\widehat{c}(0+) = \psi\,\overline{c} > \widehat{c}(0) = y$. We also have: $v''(0) < 0$; and $\widehat{c}' > 0$ on $(0, \infty)$.

2. If $\gamma < \mu < \mu_1$ then the low-shadow-value boundary configuration still holds. I.e. we still have $v(0) = \frac{1}{\gamma}\,\widehat{u}_0(y)$ and $v'(0) = v'_L = \widehat{u}'_+(\psi\,\overline{c}) < \widehat{u}'_+(y)$. This implies that $\widehat{c}(0+) = \psi\,\overline{c} > \widehat{c}(0) = y$, as before. However, we now have: $v''(0) > 0$; and there exists $\overline{x} \in (0, \infty)$ such that $\widehat{c}' < 0$ on $(0, \overline{x})$ and $\widehat{c}' > 0$ on $(\overline{x}, \infty)$.

3. If $\mu > \mu_1$ then the high-shadow-value boundary configuration holds, i.e. $v(0) > \frac{1}{\gamma}\,\widehat{u}_0(y)$ and $v'(0) > v'_R = \widehat{u}'_+(\psi\,y) > \widehat{u}'_+(y)$. This implies that $\widehat{c}(0+) = \widehat{c}(0) < \psi\,y$. We also have: $v''(0) < 0$; and $\widehat{c}' > 0$ on $(0, \infty)$.

In order to deduce the behaviour of $c$ in these three cases, note that:

- For $x > 0$, $c$ is determined by $u'(c) = \beta\,v'$ and $\widehat{c}$ is determined by $\widehat{u}'_+(\widehat{c}) = v'$. Also, the formula for $\widehat{u}_+$ given in the proof of Theorem 7 implies that

$$\widehat{u}'_+(\widehat{c}) = \tfrac{1}{\beta}\,u'(\tfrac{1}{\psi}\,\widehat{c}). \tag{57}$$

  Hence $u'(c) = \beta\,v' = \beta\,\widehat{u}'_+(\widehat{c}) = u'(\frac{1}{\psi}\,\widehat{c})$. Hence $c = \frac{1}{\psi}\,\widehat{c}$.

- For $x = 0$, $c$ is determined by $u'(c) = \max\{u'(y), \beta\,v'\}$ and $\widehat{c}$ is determined by $\widehat{c} \in \operatorname{argmax}_{\widehat{c}\in(0,y]}\{\widehat{u}_0(\widehat{c}) - v'\,\widehat{c}\}$. Now $\beta\,v' > u'(y)$ iff $v' > \widehat{u}'_+(\psi\,y)$, because $\widehat{u}'_+(\psi\,y) = \frac{1}{\beta}\,u'(y)$ by (57). And, in this case, $u'(c) = \beta\,v'$ and $\widehat{u}'_0(\widehat{c}) = \widehat{u}'_+(\widehat{c}) = v'$. Hence $c = \frac{1}{\psi}\,c$. Similarly, $\beta\,v' < u'(y)$ iff $v' < \widehat{u}'_+(\psi\,y)$. However, in this case, $c = \widehat{c} = y$.

- Provided that $\mu \neq \mu_1$, we have either

$$v'(0) = v'_L = \widehat{u}'_+(\psi\,\overline{c}) = \tfrac{1}{\beta}\,u'(\overline{c}) < \tfrac{1}{\beta}\,u'(y)$$

  or

$$v'(0) > v'_R = \widehat{u}'_+(\psi\,y) = \tfrac{1}{\beta}\,u'(y).$$

  In particular, the case $\beta\,v'(0) = u'(y)$ does not arise.

Combining these observations with points 1-3 above, we conclude that:

1. If $\mu < \gamma$ then $c(0+) = \frac{1}{\psi}\,\widehat{c}(0+) = \overline{c} > y = c(0)$ and $c' = \frac{1}{\psi}\,\widehat{c}' > 0$ on $(0, \infty)$.
2. If $\gamma < \mu < \mu_1$ then we still have $c(0+) = \frac{1}{\psi}\,\widehat{c}(0+) = \overline{c} > y = c(0)$. But now $c' = \frac{1}{\psi}\,\widehat{c}' < 0$ on $(0, \overline{x})$ and $c' = \frac{1}{\psi}\,\widehat{c}' > 0$ on $(\overline{x}, \infty)$.
3. If $\mu > \mu_1$ then $c(0+) = \frac{1}{\psi}\,\widehat{c}(0+) = \frac{1}{\psi}\,\widehat{c}(0) < y$. We also have $c' = \frac{1}{\psi}\,\widehat{c}' > 0$ on $(0, \infty)$.

The point is that, when $x > 0$, when $x \downarrow 0$ and when $x = 0$ and the high-shadow-value boundary configuration obtains, the behaviour of $c$ can be deduced from that of $\widehat{c}$ via the simple formula $c = \frac{1}{\psi}\,\widehat{c}$. (Since $\psi < 1$, this formula captures the idea that the IG consumer will overconsume compared with the $\widehat{u}$ consumer.) And, when $x = 0$ and the low-shadow-value boundary configuration obtains, we have $c = \widehat{c} = y$. This completes the proof of Theorems 14, 16 and 15.

**F.8. The borderline cases $\mu = \gamma$ and $\mu = \mu_1$.** Up to now we have said relatively little about the borderline cases. The case $\mu = \gamma$ has several interesting features. First, letting $\mu \uparrow \gamma$, we see that $v'' \leq 0$ and $\widehat{c}' \geq 0$ on $[0, \infty)$. Second, again letting $\mu \uparrow \gamma$, we obtain $\widehat{v}_+(0) = \frac{1}{\gamma}\,\widehat{u}_+(y) < \frac{1}{\gamma}\,\widehat{u}_0(y)$. It follows that the low-shadow-value boundary configuration obtains. This in turn implies that $\widehat{c}(0+) = \psi\,\overline{c} > y$. Letting $x \downarrow 0$ in equation (55) then yields $0 = (y - \widehat{c}(0+))\,v''(0)$. It follows that $v''(0) = 0$. Third, by considering higher-order analogues of equation (55), one can go on to show that $v^{(n)}(0) = 0$ for all $n \geq 3$ as well. In other words, the only non-zero coefficients in the Taylor expansion for $v$ at $x = 0$ are $v(0)$ and $v'(0)$. At first sight this would seem to suggest that $v$ is linear. However, this would contradict Proposition 20. The resolution lies in the fact that $v$ is not analytic at 0. Rather, $v'$ (and therefore $\widehat{c}$) are so called 'flat functions'. This terminology turns out to be apt: simulations show that $v'$ and $\widehat{c}$ are nearly constant for a significant interval of wealth starting at $x = 0$.

The case $\mu = \mu_1$ involves a number of subtleties. First, even though $\mu_1$ is the point at which we switch from the left- to the high-shadow-value boundary configuration, only the high-shadow-value boundary configuration can occur when $\mu = \mu_1$. This is because $v'(0)$ is essentially the limit $v'(0+)$, and as such is determined by behavior in the interior of the wealth space. Moreover, in the interior of the wealth space, the low-consumption strategy is the preferred strategy of the $\widehat{u}$ consumer. The $\widehat{u}$ consumer does, however, have two equally good options at $x = 0$: since $v'(0) = \widehat{u}'_+(\psi\,y)$ and $\widehat{u}_0$ has slope $\widehat{u}'_+(\psi\,y)$ on

$[\psi\, y, y]$, she is indifferent between $\widehat{c}(0) = \psi\, y$ and $\widehat{c}(0) = y$. (She is in fact indifferent among all $\widehat{c}(0) \in [\psi\, y, y]$, but the intermediate options should be seen as the result of strictly randomizing between $\psi\, y$ and $y$. Moreover they all lead to the same outcome as $\psi\, y$: the dynamics move immediately into the interior of the wealth space.) If she chooses $\widehat{c}(0) = \psi\, y$, then she embarks immediately on the low-consumption strategy. If she chooses $\widehat{c}(0) = y$, then she remains forever with wealth 0. Either way, she ends up with the payoff $v(0) = \frac{1}{\gamma}\, \widehat{u}_0(y)$. Second, as $\mu \uparrow \mu_1$, the length of the interval over which $\widehat{c}$ decreases — which is always an open interval with left-hand endpoint 0 — converges to 0. (So, in effect, $\widehat{c}$ jumps up from $y$ to $\psi\, \overline{c}$ at 0 and then decreases very rapidly back down to something close to $\psi\, y$.) In other words, a boundary layer develops near $x = 0$.

## G. Proof that Assumptions B1-B2 Imply Assumptions H1-H2

It can be verified by direct calculation that

$$h'_+ = -\frac{(\rho(f_+) - (1-\beta))\, f_+}{\rho(f_+)}$$

and

$$h''_+ = -\frac{\beta}{u''(f_+)\, \rho(f_+)}\left((2-\beta)\, \rho(f_+) - (1-\beta)\, \pi(f_+)\right),$$

where $h_+ = h_+(\alpha)$ and $f_+ = f_+(\beta\, \alpha)$. Hence

$$-\frac{\alpha\, h''_+}{h'_+} = \frac{(2-\beta)\, \rho(f_+) - (1-\beta)\, \pi(f_+)}{(\rho(f_+) - (1-\beta))\, \rho(f_+)},$$

where we have used the fact that

$$-\frac{\alpha\, \beta}{u''(f_+)\, f_+} = -\frac{u'(f_+)}{u''(f_+)\, f_+} = \frac{1}{\rho(f_+)}.$$

Now, considering the numerator in the expression for $h'_+$, we have

$$\rho(f_+) - (1-\beta) \geq \underline{\rho} - (1-\beta) \geq \frac{\underline{\rho}}{1 + \overline{\rho} - \underline{\rho}} - (1-\beta) > 0,$$

where the first inequality follows from Assumption B1, the second from the fact that $\overline{\rho} - \underline{\rho} \geq 0$ and the third from Assumption B2. Hence $h'_+ < 0$. Similarly, considering the term in parentheses in the expression for $h''_+$, we have

$$\begin{aligned}(2-\beta)\,\rho(f_+)-(1-\beta)\,\pi(f_+) &\geq (2-\beta)\,\underline{\rho}-(1-\beta)\,(\overline{\rho}+1)\\ &= \underline{\rho}-(1-\beta)\,(1+\overline{\rho}-\underline{\rho})\\ &> 0\end{aligned}$$

where the first relation follows from Assumption B1 and the third from Assumption B2. Hence $h''_+ < 0$. Finally, again considering the numerator in the expression for $h'_+$ and the term in parentheses in the expression for $h''_+$, we have

$$\rho(f_+)-(1-\beta) \leq \overline{\rho}-(1-\beta)$$

and

$$(2-\beta)\,\rho(f_+)-(1-\beta)\,\pi(f_+) \leq (2-\beta)\,\overline{\rho}-(1-\beta)\,(\underline{\rho}+1).$$

Hence

$$\underline{\theta} \leq -\frac{\alpha\, h''_+(\alpha)}{h'_+(\alpha)} \leq \overline{\theta},$$

where

$$\underline{\theta} = \frac{(2-\beta)\,\underline{\rho}-(1-\beta)\,(\overline{\rho}+1)}{\overline{\rho}-(1-\beta)} \quad \text{and} \quad \overline{\theta} = \frac{(2-\beta)\,\overline{\rho}-(1-\beta)\,(\underline{\rho}+1)}{\underline{\rho}-(1-\beta)}.$$