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Dominance and innovation: a returns-based beliefs approach

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Do dominant or less dominant firms innovate more? There is considerable research on this question, with theoretical studies based on game theory and empirical evidence often divided. We provide a new theory that the decisions that concern strategic choice in innovation may be influenced by expected relative returns. Our approach, which we call the returns-based beliefs approach, is based upon subjective probabilities. It combines a decision analytic solution concept and Luce's (*Individual Choice Behavior*, Wiley: New York, 1959) probabilistic choice model. We show that the returns-based beliefs approach provides support for the thesis that dominant firms invest more in research and development (R&D) within an asymmetric mixed-strategy game. Consequently, the returns-based beliefs approach accords better with recent empirical studies of innovation. Using R&D data across a range of industries in the UK from 2001 to 2006, we show that firms' spending on R&D is related more to their own profitability than to that of their competitors, which is consistent with the returns-based beliefs approach.

Keywords: innovation; game theory; decision theory

1. Introduction

Microsoft, the largest firm in the software industry, is often regarded as one of the most aggressive innovators. It can be argued that its aggressive innovative stance is partly responsible for the persistence of its large market share in the software industry. On the other hand, Kodak, the most dominant firm in traditional film photography, was relatively slow in embracing the innovations in photography due to digital technology. The relatively lethargic response of Kodak enabled Canon and Sony to capture the initial market for digital photography [1]. These observations show that the relationship between dominance and innovation is not intuitively obvious. The academic literature on this topic reflects this. One of the most contentious and widely debated issues is whether dominant firms will innovate in order to maintain their dominant position [2]. On this issue, the academic literature depicts sharply contrasting theoretical and empirical evidence.

Arrow [3] postulated that dominant firms are less innovative because of the incentive not to cannibalize their profit stream. Moorthy [4] showed that a monopolist serving multiple segments might maximize profits by offering fewer products than the available customer segments even if the fixed cost of an additional product offering is zero because of the fear of cannibalizing its existing profits. On the other hand, Schumpeter [5] had argued that the incentive to innovate is more than proportionately larger for bigger firms as they have better capability to exploit an innovation. In a similar vein, studies have shown that a large incumbent might be more innovative in order to preempt a potential entrant from competing away future profits ([6]; see also the review in [7]). In an attempt to reconcile these competing views, Henderson [8] showed that the cannibalization effect dominates if the innovation is radical and the preemptive effect dominates if the innovation is incremental.

Research also shows that there are other factors that might affect the decision of dominant incumbent firms to innovate. For example, Ghemawat [9] argued that incumbent firms might be reluctant to innovate when there is a high likelihood of spillover effects in the industry. Amaldoss and Jain [10] showed using a patent race game that in equilibrium a new entrant firm who has less to benefit from the innovation will invest more aggressively than an incumbent who has more to gain from the innovation in order to keep the latter indifferent to its different strategies. Recent research shows that whether

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dominant or less dominant firms innovate first depends on whether firms are myopic, if the investments are strategic substitutes or complements, and if there is free entry into the market [11, 12]. Moreover, empirical evidence on this issue is divided: Some researchers have shown that dominant firms are more innovative [13] whereas others have shown that less dominant firms are more innovative [2, 14]. In addition, empirical research on the relationship between dominance and innovation shows that it is a multifaceted construct where technological expectations can generate differing propensities to innovate among dominant and less dominant firms [15].

Here, we review how firms allocate resources for innovation. A recent empirical survey that examined firms' motives for innovation showed that firms allocate resources for innovation mainly on the basis of the goals set for the year and available opportunities, followed by the relative attractiveness of individual projects [16]. Competitors' spending comes a long way behind at sixth place with only 2% of firms saying that it is important to their decision making (Figure 1). Many analyses of innovation using game theory use the concept of the Nash equilibrium, which emphasizes strategic interactions in the presence of competition. In the Nash equilibrium, each firm chooses the action that maximizes their returns subject to the opponent's choice and no firm can gain by changing their strategy unilaterally. By contrast, in this paper, we use an alternative concept. We use a decision theoretic approach where firms form subjective probabilities over the actions of the firm's opponent and then choose a mixed-strategy profile over the actions on the basis of the relative returns. In doing so, we follow the previous research that proposes a decision theoretic solution concept for game theory [17–19]. We use the probabilistic choice model developed axiomatically by Luce [20]. In particular, we call this the 'returns-based beliefs' approach, which is both more sympathetic to and more consistent with the results of the innovation survey. Returns-based beliefs bring squarely into the picture the emphasis on the relative attractiveness of the firm's own returns in choosing the optimal level of investments in innovation projects within a competitive setting. In doing so, we are able to provide an alternative explanation to the extant literature in game theory that shows why dominant incumbent firms might be more innovative [13].

Game theory typically describes how managers ought to behave rather than how managers should behave given the nature of the game and their experiences [18, 21]. Recognizing this weakness, some scholars have argued that what is needed is an empirically supported psychological theory that makes probabilistic predictions about strategies firms are likely to use given the nature of the firm and the managers' psychological makeup [22]. This psychological makeup might be conditioned by the past experience of managers' beliefs about an opponent's play. The managers' past experience influences the psychological makeup, which in turn affects how the manager perceives their opponent. This is termed the 'subjective' or personal interpretation of probability. Subjective probability is the probability that a manager assigns to a possible outcome, through some process based on his own judgment [23, p. 4; 24]. An implication of the subjective probability approach is that the chosen strategy might not be consistent with the equilibrium predictions of an objectively rational outcome [25]. The firm's experience is an important determinant of the firm's expectations, which might lead to strategies being chosen that are not the Nash equilibrium prediction.

Our returns-based beliefs approach treats beliefs rather than strategies as the primary concept [26, p. 139; 27]. In doing so, we assume that firms' subjective beliefs are mutually consistent (coherent). Therefore, consistent with a decision theoretic perspective, firms adopt strategies on the basis of their respective subjective beliefs. Our approach weights the firm's own returns more than the opponent's returns in the response function. This is in contrast to the Nash equilibrium where the best response results in an equilibrium where the opponent's returns determine the optimal investment decision and not the firm's own returns. The returns-based beliefs approach provides support for the thesis that dominant firms invest more in research and development (R&D) within an asymmetric mixed-strategy game. The returns-based beliefs approach accords better with the results of the innovation survey shown in Figure 1. We also provide empirical evidence using UK

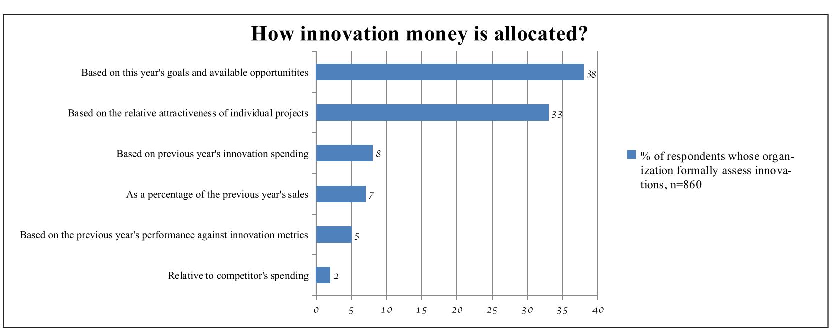

Figure 1. Allocation of innovation spending. Source: McKinsey [16].

R&D data in support of our returns-based beliefs approach and present the implications of our approach for innovation decisions within a competitive setting.

The next section revisits the patent race game between competitive firms. Section 3 discusses our concept of returns-based beliefs. In Section 4, we provide an empirical analysis of R&D spending. Section 5 compares the returns-based beliefs model with other non-Nash equilibrium models. Section 6 discusses the main implications and concludes.

2. Patent race between asymmetric firms

In this section, we develop a simple model of a patent race on the basis of the game discussed by Amaldoss and Jain [10]. Let us assume that there are two firms, a dominant and less dominant firm, $i = \{\mathrm{H}, \mathrm{L}\}$, respectively. The dominant and less dominant firms are currently earning profits of e_{H} and e_{L}, respectively, where $e_{\mathrm{H}} > e_{\mathrm{L}}$. The firms can invest in R&D to win a patent. The maximum amount they can invest is $s \in (0, c/2, c)$, where c is the constraint imposed by the capital markets [10]. Similar to the seminal paper by Gilbert and Newbery [6], we assume that the firm that invests more wins the patent. If the dominant firm wins the patent, it makes a profit of r_{H}, and if the less dominant firm wins the patent, it makes a profit of r_{L} where $r_{\mathrm{H}} > r_{\mathrm{L}}$. In addition, we assume the following: (1) $r_{\mathrm{H}} - c > e_{\mathrm{H}}$ and (2) $r_{\mathrm{L}} - c > e_{\mathrm{L}}$. We assume that the dominant firm is able to profit more from the innovation because of the existence of complementary assets such as a better brand name, distribution channels, or better marketing ability [28, 29] and therefore assume that $r_{\mathrm{H}} - e_{\mathrm{H}} > r_{\mathrm{L}} - e_{\mathrm{L}}$. We assume the Bertrand competition whereby if both firms invest the same amount in R&D, they earn zero profits. Therefore, firm i's profits if it invests s_i are $\pi_i = r_i - s_i$ if $s_i > s_k, i \neq k$, or $\pi_i = e_i - s_i$ otherwise. We assume that all profits and costs are common knowledge. The profits with respect to the various levels of investments are shown in Table 1.

There is no pure-strategy equilibrium for the game (similar to [10]). If both firms invest zero, then one firm can be better off investing $c/2$ and winning the patent. Therefore, investing zero is not a pure-strategy equilibrium. If both firms invest a positive amount, then the losing firm is better off not investing and deviating to zero investment. Therefore, both firms investing a positive amount is not a pure-strategy equilibrium either. Now consider the case when one firm does not invest whereas the other firm invests a positive amount. The firm that invests could win the patent by incurring a cost of $c/2$. However, this cannot be an equilibrium because the firm that has not invested could win the patent by investing c. Therefore, we do not have a pure-strategy equilibrium for this game. The equilibrium must involve a mixed-strategy equilibrium whereby each firm invests in the three choices with some probability. Randomization in the mixed-strategy Nash equilibrium analysis requires that the firms choose their probabilities in such as way as to make the other firm indifferent between the different strategic choices. Let us assume that the dominant firm chooses probabilities y_1 , y_2 , and y_3 for investment of 0, $c/2$, and c, respectively, where $y_1 + y_2 + y_3 = 1$. Similarly, the less dominant firm chooses probabilities x_1, x_2, and x_3 for investment of 0, $c/2$, and c, respectively, where $x_1 + x_2 + x_3 = 1$. For the less dominant firm, equilibrium implies the following:

$$e_{\mathrm{H}}x_1 + e_{\mathrm{H}}x_2 + e_{\mathrm{H}}x_3 = (r_{\mathrm{H}} - \frac{1}{2}c)x_1 + (e_{\mathrm{H}} - \frac{1}{2}c)x_2 + (e_{\mathrm{H}} - \frac{1}{2}c)x_3, \tag{1}$$

$$e_{\mathrm{H}}x_1 + e_{\mathrm{H}}x_2 + e_{\mathrm{H}}x_3 = (r_{\mathrm{H}} - c)x_1 + (r_{\mathrm{H}} - c)x_2 + (e_{\mathrm{H}} - c)x_3, \tag{2}$$

$$x_1 + x_2 + x_3 = 1. \tag{3}$$

It can be shown from Equations (1)–(3) that

$$x_1 = x_2 = \frac{c}{2(r_{\mathrm{H}} - e_{\mathrm{H}})}, \tag{4}$$

$$x_3 = \frac{r_{\mathrm{H}} - e_{\mathrm{H}} - c}{r_{\mathrm{H}} - e_{\mathrm{H}}}. \tag{5}$$

Table 1. Payoff matrix for various levels of investment in research and development.

		Less dominant firm		
		0	$c/2$	c
Dominant firm	0	$(e_{\mathrm{H}}, e_{\mathrm{L}})$	$(e_{\mathrm{H}}, r_{\mathrm{L}} - c/2)$	$(e_{\mathrm{H}}, r_{\mathrm{L}} - c)$
	$c/2$	$(r_{\mathrm{H}} - c/2, e_{\mathrm{L}})$	$(e_{\mathrm{H}} - c/2, e_{\mathrm{L}} - c/2)$	$(e_{\mathrm{H}} - c/2, r_{\mathrm{L}} - c)$
	c	$(r_{\mathrm{H}} - c, e_{\mathrm{L}})$	$(r_{\mathrm{H}} - c, e_{\mathrm{L}} - c/2)$	$(e_{\mathrm{H}} - c, e_{\mathrm{L}} - c)$

Similarly, the dominant firm will choose its mixed-strategy equilibrium in such a way as to make the less dominant firm indifferent between its strategic choices. It can be shown that

$$y_1 = y_2 = \frac{c}{2(r_L - e_L)}, \tag{6}$$

$$y_3 = \frac{r_L - e_L - c}{r_L - e_L}. \tag{7}$$

Let us assume that the following holds, $c = 2$, $r_H = 20$, $e_H = 8$, $r_L = 10$, and $e_L = 4$, which gives the payoff as in Table 2.

According to Equations (4) to (7), the corresponding equilibrium values for the less dominant firm are as follows:

$$x_1 = x_2 = 0.08, \qquad x_3 = 0.84, \tag{8}$$

and the corresponding equilibrium values for the dominant firm are as follows:

$$y_1 = y_2 = 0.17, \qquad y_3 = 0.66. \tag{9}$$

It is clear that the less dominant firm invests more in R&D compared with the dominant firm [10]. To see why this is the case, we must examine the conditions for equilibrium in asymmetric mixed-strategy equilibria such as the patent game being examined. The firms have a choice of either investing in R&D to discover the patent or not investing. In equilibrium, the firms must be indifferent between these two choices. Let us consider the returns for the dominant firm. The dominant firm has more to gain from winning the patent compared with the less dominant firm because of advantages stemming from complementary assets and superior marketing advantages. Therefore, the dominant firm needs to invest less and win the patent less often compared with the less dominant firm in order to make it indifferent between investing and not investing (see [10, p. 976]). On the other hand, the less dominant firm needs to invest more and win the patent more often to make it indifferent between investing and not investing compared with the dominant firm. Therefore, in the asymmetric mixed-strategy equilibrium, the less dominant firm invests more in R&D compared with the dominant firm.

It is clear from the above results, Equations (4)–(7), that the mixed-strategy equilibrium of the dominant firm, H, does not depend on its own returns from innovation but rather on the less dominant firm's returns, r_L and e_L (we show a more general case of this result using the mixed-strategy Nash equilibrium in Appendix A). Similarly, the mixed-strategy equilibrium of the less dominant firm, L, does not depend on its own returns from innovation but rather on the dominant firm's returns, r_H and e_H. However, this seems not in accord with the survey in Figure 1 on innovation, which shows that firms allocate resources for innovation according to the goals set for the year and available opportunities followed by relative attractiveness of individual projects rather than the actions of their competitors [16]. In that survey, competitors' spending comes in a mere sixth place, with only 2% saying that it is important (Figure 1). *Consequently, we think it important to ask whether it is possible that the mixed-strategy Nash equilibrium approach does not capture the essence of the intuition for how firms really make decisions about innovation.*

The next section elaborates an alternative approach that combines a decision analytic solution concept and Luce's [20] probabilistic choice model. This is the returns-based beliefs approach [30], which accords better with the results of the survey and does indeed provide outcomes that are different to the mixed-strategy equilibrium approach, adding to the debate about whether dominant or less dominant firms are more apt at innovation.

3. Returns-based beliefs and innovation

In the following discussion, we assume that firms are expected profit maximizers. We argue that, driven by the desire to want to avoid either both firms investing or both firms not investing in R&D in the patent race game in order to maximize expected profits, there is 'strategic uncertainty' regarding the conjecture about the choice of the other firm. We

Table 2. Numerical payoff matrix for various levels of investment in research and development.

		Less dominant firm		
		0	$c/2$	c
Dominant firm	0	(8, 4)	(8, 9)	(8, 8)
	$c/2$	(19, 4)	(7, 3)	(7, 8)
	c	(18, 4)	(18, 3)	(6, 2)

define strategic uncertainty as uncertainty that concerns the actions and beliefs (and beliefs about the beliefs) of others [31]. Researchers have argued that strategic uncertainty can arise even in cases when all possible actions and returns are completely specified and are common knowledge [32]. In such a situation, the rational firm has to form beliefs about the strategy that the other firm will adopt as a result of strategic uncertainty. As a consequence, firms form their beliefs about the probabilities that other firms play in order to determine in turn their best-response strategy. Hence, the best-response strategy of one firm is likely to be based upon the mixed strategy of the other firm. The mixture is a result of the uncertainty regarding the conjecture about the choice by the other firms [33].‡ This notion has been summarized as follows: 'In psychological games, there can be a difference between interpreting mixed strategies literally as purposeful mixing by a player versus interpreting them as uncertainty by other players' [34, p. 1286].

As discussed earlier, when randomized mixed strategies are used, a firm chooses probabilities (over their own strategies) in such a manner as to make the other firm indifferent between the different strategies. The implication of this is that each firm's equilibrium strategy depends only on the other firms' payoff and not their own. This randomized mixed-strategy approach to choosing the probabilities of the focal firm makes the other firm indifferent between the different strategies [35]. However, this approach of randomized mixed strategy would not be appropriate in the case when strategies are chosen via the use of subjective probabilities. This is because, when such outcomes are chosen, each firm maximizes their own expected values according to their conjecture of what the opponent is likely to do. Therefore, the probabilities are chosen by the focal firm over their strategies according to their conjecture of what the other firm is likely to do. Hence, because of strategic uncertainty regarding the conjecture about the choice of the other firm, the focal firm holds an opinion on the basis of the subjective probability with respect to all of the unknown contingencies affecting its payoffs. In particular, the firm is assumed to have a view about the major contingency faced, namely what the opposing firm is likely to do [18, p. 115]. Kadane and Larkey [18, p. 115] expressed the implications of this line of thinking very neatly as follows: 'If I think my opponent will choose strategy i $(i = 1, \cdots I)$ with probability p_i, I will choose any strategy j maximizing $\sum_{i=1}^{I} p_i u_{ij}$, where u_{ij} is the utility to me of the situation in which my opponent has chosen i and I have chosen j...the opponent's utilities are important only in that they affect my views $\{p_i\}$ of what my opponent may do...'.

Therefore, it follows that if firm 2 is not expected to play the mixed-strategy Nash equilibrium, then it might be optimal for firm 1 also not to play the mixed-strategy Nash equilibrium. This is because doing so could give firm 1 a better payoff as it has a profitable deviation from not playing the mixed-strategy Nash equilibrium when the other firm also does not play the mixed-strategy Nash equilibrium. The implication of this is that the Nash equilibrium is a special case when each firm is assumed to believe that the other is sure to play the Nash equilibrium strategy. The concept of objective and subjective probabilities helps to make clear the context of this discussion: the Nash equilibrium solution concept assumes rationality from the perspective of an external observer, which in effect implies objective probabilities. However, at the level of the individual firm, assumptions about the opponent's beliefs may be conditioned by the accepted management practice and value systems of the firm and hence might be different from the priors held by the rational external observer. Therefore, a focal firm who knows that the non-Nash equilibrium belief is held by the other firm could be deemed to be rational when forming a subjective assessment of the other firm's action by taking this belief into account [36]. In situations of strategic interaction such as the patent race game, the firms might hold subjective probabilities that are different from the objective probabilities demanded by the Nash equilibrium solution concept. When the subjective and objective probabilities are the same, we get the special case of the Nash equilibrium. However, there is no compelling reason *a priori* for the subjective and objective probabilities to definitely be the same. Although any distribution of probabilities could be possible on the basis of the subjective method of forming them, we shall try to propose a reasonable subjective probability belief that the firms might use when they do not know each other's respective histories. We call this the returns-based beliefs.

We posit that the firms want to cooperate in order to achieve an outcome whereby one firm invests whereas the other stays out. The disadvantage of the mutual investment outcome by both firms in the patent race game comes about from each firm trying to defend against the worst case and not considering the possibility for cooperation. However, the rational firm has to form beliefs about the opponent's strategies because of the strategic uncertainty about what the opponent is likely to invest. Velu *et al.* [30] showed that the Luce§ [20] rule encapsulates the tendency for decision makers to cooperate by taking each other's tendency to want to cooperate into account and avoid trying to defend against the worst-case

‡ *We are not assuming that the opponent is using a randomized strategy. The mixture merely reflects the representation of the dominant firm's belief about the less dominant firm and vice versa. As Wilson* [37, p. 47] *pointed out, although it makes little difference to the mathematics, conceptually this distinction between randomization and subjective beliefs to explain the mixed strategies is an important one to consider. This interpretation is also in line with Harsanyi's* [38] *purification interpretation of mixed strategy where mixing represents uncertainty in a player's mind about how other players will choose their strategies, rather than deliberate randomization* [39]. *For the purpose of our game, we exclude the consideration of cooperation between a large dominant firm and a less dominant firm whereby the former outsources its R&D to the latter with a view of buying the invention later.*

§ *In the case of an individual choice behavior, the Luce* [20] *rule implies that the individual maximizes a constant utility function subject to uncertainty in the decision process.*

outcome. Following the probabilistic choice model of Luce [20], we assume that firms form beliefs on the basis of the expected returns for a particular strategy over the total expected returns of all strategies. Therefore, it is reasonable to assume that the firm would assign probabilities on the basis of the expected returns from playing the different strategies. Similarly, we assume that the other firm also assigns probabilities on the basis of the other firm's expected returns given the probabilities of the focal firm. Following this logic, our analysis is based on a model for which the decision probabilities are proportional to the expected returns.

Our proposed approach has both theoretical and empirical support. First, for the theoretical merit, we defer to Luce [20] who showed using probability axioms that if the ratio of probabilities associated with any two decisions is independent of the payoff of any other decisions, then the choice probabilities for decision i can be expressed as a ratio of the expected payoff for that decision over the total expected payoff for all decisions: $\Pi_i^e / \sum_j \Pi_j^e$, where Π_j^e is the expected return associated with decision j. Second, this method of arriving at decision probabilities has been justified by empirical work that provides empirical support for our approach. In particular, empirical research for paired comparison data provides support for the Luce [20] method of arriving at decision probabilities such that the probability for choosing x over y, $P(x, y) = v(x)/[v(x) + v(y)]$, where $v(x)$ and $v(y)$ are the scale values of choosing x and y, respectively [40]. Our model has similarities with the logit equilibrium version of the quantal response equilibrium (QRE) model proposed by McKelvey and Palfrey [41] and the bounded rationality Nash equilibrium (BRNE) model of Chen *et al.* [42] in that all strategies with positive payoffs are played with positive probabilities in proportion to their expected payoffs. Although there are similarities, our reasoning for the decision-making process is very different to other models. We discuss these points further in the section on comparing the returns-based belief model with other non-Nash equilibrium models. We operationalize our model as follows. In this model, each firm chooses among $n = 3$ ($s_i = 0, c/2, c$) possible strategies, and the expected payoffs are given by the following summation:

$$\pi_i^e(s) = \sum_{m=1}^{n} \mathbf{u}^i(s, m)\, p_i(m) \qquad (s_i = 0, c/2, c)\,, \tag{10}$$

where $\mathbf{u}^i(s, m)$ is firm i's payoff from investing s_i when the other firm invests m and $p_i(m)$ is the belief probabilities held by firm i about the other firm playing strategy m. In turn, the decision probabilities follow the specification outlined above, which is proportional to the expected returns as follows:

$$D_i(s) = \frac{\pi_i^e(s)}{\sum_{m=1}^{n} \pi_i^e(m)}. \tag{11}$$

In our model, we assume a Nash-like equilibrium in belief formation such that the belief probabilities match the decision probabilities for both the dominant firm and the less dominant firm. This symmetry between these belief probabilities is achieved by iterating between the expected payoff in Equation (10) and the decision probabilities in Equation (11). The idea is that firm i computes the expected payoff $\Pi^i(s_i, \sigma_{\mathrm{M}})$ of each pure action s_i given a mixed action σ_{M} of the other firm. Firm i would play the mixed action,

$$\sigma_i(s_i) = \frac{\Pi^i(s_i, \sigma_{\mathrm{M}})}{\Pi^i(0, \sigma_{\mathrm{M}}) + \Pi^i(c/2, \sigma_{\mathrm{M}}) + \Pi^i(c, \sigma_{\mathrm{M}})}, \tag{12}$$

if it knew that the opponent played σ_{M}. This defines a mapping from j's mixed action to i's mixed actions, $\mathcal{M}_{ij} : \sigma_{\mathrm{M}} \mapsto \sigma_i$. The mapping $\mathcal{M}_{ji}$ of player j from i's mixed actions to j's mixed actions can be defined analogously. This leads us to define the notion of a *returns-based beliefs equilibrium* (RBBE) as follows.

Definition 1
An RBBE is a pair of mixed actions (σ_i, σ_j), such that $\mathcal{M}_{ij}(\sigma_j) = \sigma_i$ and $\mathcal{M}_{ji}(\sigma_i) = \sigma_j$. In other words, this is a solution in which both players play the Luce-type response, and each player's belief about their opponent coincides with the actual strategy the opponent adopts.

We now prove some properties of the RBBE.

Proposition 1
Any game $(i, s, \mathbf{u})$, with non-negative payoffs $\mathbf{u}$, has an RBBE.

Proof
The composite map $\mathcal{M}_{ij}(\mathcal{M}_{ji})$ is a map from mixed strategies of player i onto mixed strategies of player i and therefore has a fixed point from the fixed-point theorem. McKelvey and Palfrey [41] demonstrated in the same way that a fixed-point

equilibrium exists for the logit equilibrium (where the errors have a log Weibull distribution) version of the QRE mode. Chen *et al.* [42] provided a more general setup of the logit equilibrium model and noted that all McKelvey and Palfrey [41] results carried over to the more general case. Although the basis for our model is rather different, in that it is not based on a model of player errors, it can be seen to be equivalent to the Chen *et al.* [42] model when the error parameter is set to $\mu = 1$.

The requirement that the payoffs be non-negative follows from the fact that the mixed strategies of the players must be non-negative, that is, $\sigma^i_j \geq 0 \forall\, i$. If there are negative payoffs, then $\mathcal{M}_{ij}\,(\mathcal{M}_{ji})$ can map mixed strategies onto vectors with negative entries, which are not valid mixed strategies. However, these entries can always be made positive by replacing them with utilities. Even with negative entries, this problem can be avoided by using a modification of the Luce rule in which $\pi^e_i(s) = 0$ when $\Sigma_m \mathbf{u}^i(s,m)\,p_i(m) < 0$ but $\pi^e_i(s) = \Sigma_m \mathbf{u}^i(s,m)\,p_i(m)$ otherwise. □

Although the preceding proposition has been described for games between two players, it can be straightforwardly generalized to games involving an arbitrary number of players. However, in the case of two-player games, the RBBE has some nice properties by virtue of the following proposition.

Proposition 2

In games between two players, the RBBEs are given by eigenvectors of the matrix $\mathbf{u^1}\,(\mathbf{u^2})^{\mathrm{T}}$, where T denotes the transpose, $\mathbf{u^1} = \{u^1_{ij}\}$, $\mathbf{u^2} = \{u^2_{ij}\}$, and u^n_{ij} is the payoff to player n when player 1 chooses move i and player 2 chooses move j.

Proof

We denote the strategies of player 1 and 2 by σ^1 and σ^2, such that σ^I_i is the probability that player I chooses move i. The equilibrium conditions $\sigma^1 = \mathcal{M}_{12}(\sigma^2)$ and $\sigma^2 = \mathcal{M}_{21}(\sigma^1)$ can then be written as

$$\sigma^1_i = \lambda u^1_{ij}\sigma^2_j, \qquad \sigma^2_j = \mu u^2_{ij}\sigma^1_i, \qquad \text{where } \lambda = 1/\sum_i u^1_{ij}\sigma^2_j, \qquad \mu = 1/\sum_j u^2_{ij}\sigma^1_i. \tag{13}$$

These can be rearranged to give

$$\mathbf{u^1}\cdot\mathbf{u^2}^{\mathrm{T}}\cdot\sigma^{\mathbf{1}} = \nu\sigma^{\mathbf{1}}, \qquad \mathbf{u^2}^{\mathrm{T}}\cdot\mathbf{u^1}\cdot\sigma^{\mathbf{2}} = \nu\sigma^{\mathbf{2}}, \tag{14}$$

where $\nu = 1/(\lambda\mu)$. These are eigenvector equations for the matrices $\mathbf{u^1}\,(\mathbf{u^2})^{\mathrm{T}}$ and $(\mathbf{u^2})^{\mathrm{T}}\,\mathbf{u^1}$, but we only need to consider one or the other because the solutions are simply related.[¶] In general, an $N \times N$ matrix has N eigenvalues, but the RBBE solution must have $\sigma^I_i \geq 0\ \forall\, i$, which means we require the eigenvalue $\nu \geq 0$ and the eigenvector $\mathbf{v}$ to satisfy $v_i \geq 0\ \forall\, i$ and $\sum u^2_{ji} v_j \geq 0\ \forall\, i$. □

We can use the preceding proposition to prove a uniqueness theorem for games with positive payoffs.

Proposition 3

In a game between two players in which all the payoffs to both players are positive, that is, $u^1_{ij} > 0$, $u^2_{ij} \geq 0\ \forall\, i, j$, and there are no completely dominated strategies, the returns-based equilibrium is unique.

Proof

Suppose that there are two RBBE solutions that are eigenstates of the matrix $\mathbf{U} \equiv \mathbf{u^1}\,(\mathbf{u^2})^{\mathrm{T}}$ with eigenvalues ν_a/ν_b and eigenvectors $\mathbf{v}_a/\mathbf{v}_b$, and we label the solutions such that $\nu_a > \nu_b$. We will first prove that *the RBBE solution cannot lie on the boundary of the space of probability vectors, that is,* $v^i > 0$ *for all* i. Suppose, without loss of generality, that $v^1 = 0$. Then $\sum_{j,k} u^1_{1j} u^2_{kj} v^k = 0$, but because $u^I_{ij} > 0\ \forall\, i, j$ and $v^k \geq 0\ \forall\, k$, we need $v^k = 0\ \forall\, k$, which is a contradiction.

Now define $\kappa = \min\left\{v^i_b/v^i_a; i \in 1, \cdots, N\right\}$ and denote by K the value of i at which the minimum is realized. From the preceding result, this must be non-zero. If we then define the vector $\mathbf{x} = \mathbf{v}_b - \kappa\mathbf{v}_a$, we have $x^i \geq 0\ \forall\, i$. Because $U_{ij} > 0\ \forall\, i, j$, we also have $y^i \geq 0\ \forall\, i$ for $\mathbf{y} = \mathbf{U}\mathbf{x}$. But $\mathbf{y} = \nu_b\mathbf{v}_b - \kappa\nu_a\mathbf{v}_a$, and so $y^K < 0$, which is a contradiction.

In the above, we assumed that $\nu_a \neq \nu_b$. If we supposed instead that $\nu_a = \nu_b$, then any linear combination of $\mathbf{v}_a$ and $\mathbf{v}_b$ is also an eigenvector. Defining κ as above, the vector $\mathbf{x} = \mathbf{v}_b - \kappa\mathbf{v}_a$ now has $x^K = 0$ and so lies on the boundary of the space of probability vectors, but it is also an RBBE, which is another contradiction. So we deduce that there is exactly one RBBE solution. □

[¶] *Given two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, the eigenvalues of $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are equal, and if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{AB}$, then $\boldsymbol{Bx}$ is the corresponding eigenvector of $\boldsymbol{BA}$ with the same eigenvalue. We therefore obtain the same RBBE solution whether we consider eigenvectors of $\boldsymbol{u^1}\,(\boldsymbol{u^2})^T$ or $(\boldsymbol{u^2})^T\boldsymbol{u^1}$, as we would expect.*

For the example game given in Table 2, the matrix $\mathbf{U}$ is

$$\mathbf{U} = \begin{pmatrix} 168 & 120 & 72 \\ 195 & 153 & 111 \\ 282 & 174 & 138 \end{pmatrix}, \tag{15}$$

which has only one real eigenvalue, $\nu = 444.115$, with corresponding RBBE eigenvectors

$$\sigma^1 = (0.25215, 0.32871, 0.41913), \qquad \sigma^2 = (0.28575, 0.32240, 0.39185). \tag{16}$$

This is the unique RBBE for that particular specification of the patent race game.

It would be unreasonable to suppose that players in a game would be computing eigenstates of matrices in order to decide on their best move. However, the RBBE solution can also be derived iteratively. If a player is returns based, that is, places the Luce-type response in proportion to the expected return, and believes that his opponent is also returns based, then if he began with a guess σ_0^2 of his opponent's mixed strategy, he would play $\sigma_0^1 \propto \mathbf{u}^1\sigma_0^2$. From his belief that his opponent is returns based, the player is led to the belief that his opponent will play an alternative strategy $\sigma_1^2 \propto (\mathbf{u}^2)^{\mathrm{T}}\sigma_0^1$, so he can update his guess and repeat, iterating until he converges to a solution. The first stage of this process in the game of Table 2 is illustrated in Table 3, assuming that the player takes the initial guess $\sigma_0^2 = (0.33, 0.33, 0.33)$, that is, random choice. The far-left columns show the expected return to the dominant firm from each of his possible moves, based on this initial guess for σ_0^2, and the returns-based move probabilities the player then adopts. Even after a single iteration, these are very close to the RBBE solution.

Our initial definition of the RBBE assumed that beliefs and strategies coincided. However, this iterative convergence to the RBBE suggests an alternative, equivalent, definition for the equilibrium.

Definition 2
An RBBE is a solution in which each player plays the Luce-type response and believes that their opponent will play the Luce-type response to their strategy, that is, each player is 'returns based' and believes his opponent is also returns based.

Hence, the equilibrium can also be thought of as an equilibrium in beliefs. This is similar to Binmore's [26, p. 135] 'subjective probabilities whereby beliefs rather than strategies are treated as primary'. Camerer [43, p. 150] made a similar point that mixed-strategy equilibrium can be seen as an equilibrium in beliefs. The returns-based beliefs approach is different from the Nash equilibrium because players respond to their beliefs by placing probability on strategies in proportion to their expected payoff. The equivalence of these two definitions of the RBBE is demonstrated by the following proposition.

Proposition 4
In a game between two players in which all the payoffs to both players are positive, the iterative algorithm always converges to the unique RBBE.

Proof
To prove this, we will first prove an intermediate result: *the RBBE corresponds to the largest eigenvalue of the matrix* $\mathbf{U} \equiv \mathbf{u}^1 (\mathbf{u}^2)^{\mathrm{T}}$.

Proof: We denote the RBBE eigenvalue by ν and the eigenvector by $\mathbf{v}$. Suppose there is another eigenvector $\mathbf{v}'$ with eigenvalue ν' and $\nu' > \nu$. From the uniqueness proof $\exists\, i$ for which $(v'^i < 0$ and therefore $\kappa = \min\left\{v^i/(v'^i; i : (v'^i < 0\right\}$ is well-defined and non-zero. We denote the value of i at which the minimum occurs by K. The vector $\mathbf{x} = \mathbf{v} - \kappa\mathbf{v}'$ has $x^i \geq 0\ \forall\, i$, but $\mathbf{y} = \mathbf{U}\mathbf{x} = \nu\mathbf{v} - \kappa\nu'\mathbf{v}'$ has $y^K < 0$, which is a contradiction because $\mathbf{U}\mathbf{x}$ has to have non-negative elements. We can also prove that there cannot be $\nu' < 0$ with $|\nu'| > \nu$ by applying the above argument to the matrix $\mathbf{U}^2$, exploiting the fact that if $A\,\mathbf{x} = \lambda\mathbf{x}$ then $A^2\mathbf{x} = \lambda^2\mathbf{x}$.

Table 3. Dominant firm's probabilities during the first iteration of beliefs for the game described in Table 2.

		Less dominant firm				
		0	$c/2$	c	Total	Prob.
Dominant firm	0	2.67	2.67	2.67	8.00	0.24
	$c/2$	6.33	2.33	2.33	11.00	0.33
	c	6.00	6.00	2.00	14.00	0.43
					33.00	1.00

We can now prove the proposition. At each stage, k, of the iteration, the player updates his strategy according to $\sigma_k^1 = \mathbf{U}\sigma_{k-1}^1$ and hence $\sigma_k^1 = \mathbf{U}^k\sigma_0^1$. The eigenvectors of a matrix corresponding to distinct eigenvalues are linearly independent. If an eigenvalue, λ, is repeated with multiplicity $m > 1$, then it may be possible to find m orthogonal eigenvectors. However, this is not always possible, and if it is not, the matrix is *defective*. For a defective matrix, A, it is possible to find *generalized eigenvectors* satisfying $(A - \lambda I)^k \mathbf{v}_k = \mathbf{v}_{k-1}$, for $k = 1, \cdots, m$ with the $\mathbf{v}_k$'s orthogonal. Hence, in this way, we can construct a basis for the vector space comprising eigenvectors and generalized eigenvectors, $\{\mathbf{e}_1, \cdots, \mathbf{e}_N\}$.

The initial strategy σ_0^1 can therefore be written as a sum $\sigma_0^1 = \sum \alpha_i \mathbf{e}_i$. For an eigenvector $\mathbf{e}$ with eigenvalue ν, $\mathbf{U}^N\mathbf{e} = \nu^N\mathbf{e}$. For a generalized eigenvector $\mathbf{e}'$, corresponding to the eigenvector $\mathbf{e}$ and satisfying $(\mathbf{U} - \nu I)\mathbf{e}' = \mathbf{e}$, we see that $\mathbf{U}^N\mathbf{e}'^{N-1}(\mathbf{e} + (\lambda/N)\mathbf{e}')$, which is parallel to $\mathbf{e}$ in the limit $N \to \infty$. Therefore, $\sigma_N^1 \sim \alpha_1\nu_1^N\mathbf{e}_1 + \alpha_2\nu_2^N\mathbf{e}_2 + \cdots$, where the sum extends only over distinct eigenvectors of $\mathbf{U}$. As $N \to \infty$, this sum is dominated by the largest eigenvalue, which by previous results is the RBBE eigenvalue and is non-degenerate. We conclude that the iterative algorithm converges to the RBBE.

This argument requires the RBBE eigenvector to have a non-zero coefficient in the expansion $\sigma_0^1 = \sum \alpha_i \mathbf{e}_i$. However, if σ_0^1 has strictly positive elements, this is guaranteed by the uniqueness of the RBBE. □

For the general patent race game defined in Table 1, we can eliminate one of the variables without any loss of generality by working in units of $c/2$, that is, redefining $e_{\mathrm{H}} \to e_{\mathrm{H}}/(c/2)$ and so on, which is equivalent to setting $c = 2$ in Table 1. The characteristic equation for the eigenvalues ν of the matrix $\mathbf{U}$ is then

$$\begin{aligned} 0 &= \nu^3 - (9 + r_{\mathrm{L}}(3e_{\mathrm{H}} - 1) - r_{\mathrm{H}} - 8e_{\mathrm{L}} + 3r_{\mathrm{H}}e_{\mathrm{L}} + e_H(3e_{\mathrm{L}} - 8))\nu^2 \\ &\quad + (r_{\mathrm{H}} - e_{\mathrm{H}})(r_{\mathrm{L}} - e_{\mathrm{L}})(1 + e_{\mathrm{L}}e_{\mathrm{H}})\nu - e_{\mathrm{H}}e_{\mathrm{L}}(r_{\mathrm{H}} - e_{\mathrm{H}})^2(r_{\mathrm{L}} - e_{\mathrm{L}})^2 \\ &= \nu^3 - a_2\nu^2 + a_1\nu - a_0, \end{aligned} \tag{17}$$

where the last line defines the parameters a_i and the signs are chosen such that $a_i > 0\ \forall\, i$.

Proposition 5
For the patent race game, with parameters satisfying $\{e_{\mathrm{H}}, e_{\mathrm{L}}\} \in [2, \infty]$; $\{r_{\mathrm{H}}, r_{\mathrm{L}}\} \in [4, \infty]$; $e_{\mathrm{H}} > e_{\mathrm{L}}$ and $r_{\mathrm{H}} - e_{\mathrm{H}} > r_{\mathrm{L}} - e_{\mathrm{L}} > 2$, the matrix determining the RBBE has only one real eigenvalue and eigenvector, which is therefore the unique RBBE.

The stated restrictions on the parameters are a reasonable restriction of the game. We would expect $e_X > c = 2$ (for $X \in \{\mathrm{L}, \mathrm{H}\}$) because a firm will not be able to spend more on innovation than its current profit levels. Similarly, $r_{\mathrm{L}} > e_{\mathrm{L}} + c$ merely states that a firm will not innovate if the cost of innovation is more than the potential increase in profits due to the innovation (the constraint $r_{\mathrm{H}} > e_{\mathrm{H}} + c$ must hold for the same reason, but this is implied by the other conditions). The constraint that $r_{\mathrm{H}} - e_{\mathrm{H}} > r_{\mathrm{L}} - e_{\mathrm{L}}$ is the statement that dominant firms can profit more from innovation as described earlier. The final constraint, $e_{\mathrm{H}} > e_{\mathrm{L}}$, follows from the stipulation that H is the dominant firm. The proof is straightforward.

Proof
We denote the characteristic equation by $f(\nu) = 0$. The cubic polynomial $f(\nu)$ has turning points at $\nu_\pm = (a_2 \pm \sqrt{a_2^2 - 3a_1})/2$. If the discriminant $d^2 = a_2^2 - 3a_1 < 0$, there are no turning points, and so the characteristic equation has exactly one real root, and the proposition holds trivially. If $d > 0$, then there will only be one real root if

$$f(\nu_-) < 0 \Rightarrow a_0 > \frac{a_2a_1}{3} - \frac{2}{27}(a_2^3 - d^3) = \frac{a_2a_1}{3} - \frac{2}{27}a_2^3 + \frac{2}{27}(a_2^2 - 3a_1)d. \tag{18}$$

The term $a_2^2 - 3a_1 = d^2 > 0$ by the assumption that the discriminant is positive. Thus, if we can find $d_1 > d$ such that

$$a_0 > \frac{a_2a_1}{3} - \frac{2}{27}a_2^3 + \frac{2}{27}(a_2^2 - 3a_1)d_1 \qquad \Rightarrow \qquad a_0 > \frac{a_2a_1}{3} - \frac{2}{27}a_2^3 + \frac{2}{27}(a_2^2 - 3a_1)d. \tag{19}$$

By expanding d for $a_2 \gg a_1$, we obtain the guess $d_1 = a_2 - (3/2)a_1/a_2$, and we see that $d_1^2 = a_2^2 - 3a_1 + (9/4)a_1^2/a_2^2 = d^2 + (9/4)a_1^2/a_2^2 > d^2$. For this choice of d_1,

$$a_0 - \frac{a_2a_1}{3} - \frac{2}{27}a_2^3 + \frac{2}{27}(a_2^2 - 3a_1)d_1 = (3a_0a_2 - a_1^2)/(3a_2) = \frac{1}{3a_2}. \tag{20}$$

For the patent race game, we find

$$(3a_0a_2 - a_1^2) = \left(21e_{\mathrm{H}}e_{\mathrm{L}} - 24e_{\mathrm{H}}e_{\mathrm{L}}^2 + 3e_{\mathrm{H}}e_{\mathrm{L}}r_{\mathrm{L}}(3e_{\mathrm{H}} - 1) + 3e_{\mathrm{H}}e_{\mathrm{L}}r_{\mathrm{H}}(3e_{\mathrm{L}} - 1) - 24e_{\mathrm{H}}^2e_{\mathrm{L}} - 1\right). \tag{21}$$

We now use the fact that $r_{\rm H} > e_{\rm H}+2$ and $r_{\rm L} > e_{\rm L}+2$ to replace $r_{\rm H}$ and $r_{\rm L}$ in the right-hand side of this equation, deducing that

$$\begin{aligned}(3a_0a_2 - a_1^2) &> \big(21e_{\rm H}e_{\rm L} - 24e_{\rm H}e_{\rm L}^2 + 3e_{\rm H}e_{\rm L}(e_{\rm L}+2)(3e_{\rm H}-1)\\ &\quad +3e_{\rm H}e_{\rm L}(e_{\rm H}+2)(3e_{\rm L}-1) - 24e_{\rm H}^2e_{\rm L} - 1\big)\\ &= 9e_{\rm H}e_{\rm L}(e_{\rm H}(e_{\rm L}-1) + e_{\rm L}(e_{\rm H}-1)+1) - 1,\end{aligned} \tag{22}$$

and the latter function is clearly positive because $e_{\rm H} > 1$ and $e_{\rm L} > 1$. □

This proposition allows us to write down the solution for the eigenvalue in the RBBE, using the general expression for the roots of a cubic polynomial (see, for instance, [44])

$$\begin{aligned}\nu &= (r + \sqrt{q^3+r^2})^{1/3} + (r - \sqrt{q^3+r^2})^{1/3} + a_2/3,\\ \text{where } \; q &= a_1/3 + a_2^2/9, \qquad r = a_0/2 - a_1a_2/6 + a_2^3/27,\end{aligned} \tag{23}$$

where the a_i's are the functions of $e_{\rm H}$, $e_{\rm L}$, $r_{\rm H}$, and $r_{\rm L}$ defined by Equation (17). The corresponding choice probabilities in the RBBE are

$$\begin{aligned}\sigma^1 = \Bigg(&\frac{e_{\rm H}(3\nu(e_{\rm L}-1) + (r_{\rm H}-e_{\rm H})(r_{\rm L}-e_{\rm L})e_{\rm L})}{\nu(\nu + r_{\rm L} - 3r_{\rm L}e_{\rm H} - e_{\rm L} + 3e_{\rm H}e_{\rm L})},\\ &\frac{\nu(3 + e_{\rm L}(r_{\rm H}-3) + e_{\rm H}(2e_{\rm L}-3)) - 2e_{\rm H}e_{\rm L}(r_{\rm H}-e_{\rm H})(r_{\rm L}-e_{\rm L})}{\nu(\nu + r_{\rm L} - 3r_{\rm L}e_{\rm H} - e_{\rm L} + 3e_{\rm H}e_{\rm L})},\\ &\frac{\nu^2 + e_{\rm H}e_{\rm L}(r_{\rm H}-e_{\rm H})(r_{\rm L}-e_{\rm L}) + \nu(r_{\rm L} - 3 + 6e_{\rm H} - 3e_{\rm H}r_{\rm L} + 2e_{\rm L} - 2e_{\rm H}e_{\rm L} - r_{\rm H}e_{\rm L})}{\nu(\nu + r_{\rm L} - 3r_{\rm L}e_{\rm H} - e_{\rm L} + 3e_{\rm H}e_{\rm L})}\Bigg)\end{aligned} \tag{24}$$

$$\begin{aligned}\sigma^2 = \Bigg(&\frac{e_{\rm L}(\nu + r_{\rm L} - 3r_{\rm L}e_{\rm H} - e_{\rm L} + 3e_{\rm H}e_{\rm L})}{3\nu(e_{\rm L}-1) + (r_{\rm H}-e_{\rm H})(r_{\rm L}-e_{\rm L})e_{\rm L}},\\ &\frac{\nu^2(e_{\rm L}-1) + \nu(r_{\rm L}-e_{\rm L})(e_{\rm L}-1) + e_{\rm H}e_{\rm L}(r_{\rm H}-e_{\rm H})(r_{\rm L}-e_{\rm L})^2}{3\nu(e_{\rm L}-1) + (r_{\rm H}-e_{\rm H})(r_{\rm L}-eL)e_{\rm L}},\\ &\frac{\nu^2(e_{\rm L}-2) - e_{\rm H}e_{\rm L}(r_{\rm H}-e_{\rm H})(r_{\rm L}-e_{\rm L})^2 + \nu(r_{\rm L}-e_{\rm L})((2e_{\rm H}+r_{\rm H}-2)e_{\rm L}-1)}{3\nu(e_{\rm L}-1) + (r_{\rm H}-e_{\rm H})(r_{\rm L}-eL)e_{\rm L}}\Bigg)\end{aligned} \tag{25}$$

in which ν denotes the eigenvalue given in Equation (23).

The above expressions give us the location of the RBBE as an analytic function of the parameters. Because the parameter space is four dimensional, it is difficult to plot quantities as functions of all the parameters. Instead, we will use a Monte Carlo approach. We generate samples of the game parameters $\{e_{\rm H}, r_{\rm H}, e_{\rm L}, r_{\rm L}\}$ uniformly within the range $\{e_{\rm H}, e_{\rm L}\} \in [2, 10]$, $\{r_{\rm H}, r_{\rm L}\} \in [4, 30]$, $e_{\rm H} > e_{\rm L}$, and $r_{\rm H} - e_{\rm H} > r_{\rm L} - e_{\rm L} > 2$.[‖] We will then plot interesting quantities for all of the games in this sample. The following figures use 10,000 samples.

In Figure 2, we show, in the left-hand panel, the probability that each firm innovates in the RBBE (probability of investing in both c and $c/2$), defined as $p_{{\rm Innov},X} = \sigma^X(c/2) + \sigma^X(c)$, for $X = \{\text{H,L}\}$, computed for 10,000 Monte Carlo samples obtained as described above. The right-hand panel shows the corresponding results for the Nash equilibrium. As discussed earlier, in the Nash equilibrium, the less dominant firm always has a greater probability of investment in innovation, $p_{\rm Innov,L} > p_{\rm Innov,H}$. In the RBBE, this trend is reversed, and the more dominant firm generally has a greater probability of investment—there are more points (71%) with $p_{\rm Innov,H} > p_{\rm Innov,L}$ than vice versa. It is informative to consider how these probabilities relate to the game parameters. In particular, the quantity $(r_{\rm L} - e_{\rm L})/e_{\rm L}$ represents the fractional increase in profit to the less dominant firm from innovation and is therefore a measure of the incentive to innovate. In Figure 3, we show how the ratio of the innovation probability, $p_{\rm Innov,L}/p_{\rm Innov,H}$, varies with the ratio of the incentive to innovate, $(r_{\rm L}/e_{\rm L} - 1)/(r_{\rm H}/e_{\rm H} - 1)$. In the RBBE, the trend is that a firm will innovate more when it has a greater incentive for innovation, which is what we would expect to be the case and consistent with the survey findings in Figure 1. In particular, the less dominant firm *only* invests more in innovation when it has a greater incentive to innovate than the dominant firm (as the left-hand panel of Figure 3 does not have any points where $(r_{\rm L}/e_{\rm L} - 1)/(r_{\rm H}/e_{\rm H} - 1) < 1$ and $p_{\rm Innov,L}/p_{\rm Innov,H} > 1$). On the other hand, in the RBBE, it is not necessarily the case that the dominant firm invests more in innovation only when

‖ *This is achieved by rejection sampling—we draw points uniformly from the box* $\{e_H, e_L\} \in [2, 10]$ *and* $\{r_H, r_L\} \in [4, 30]$ *and discard points that violate the constraints* $e_H > e_L$ *or* $r_H - e_H > r_L - e_L > 2$.

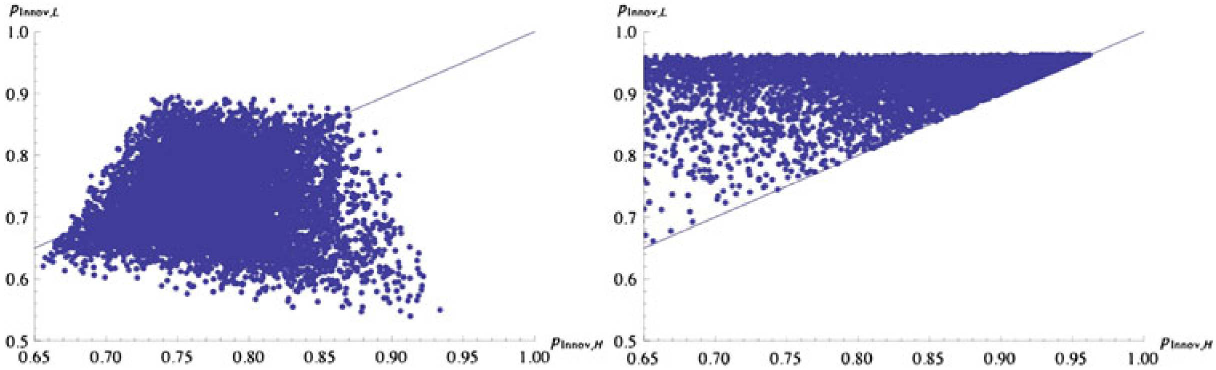

Figure 2. Probability of innovating for the dominant firm (horizontal axis) and the less dominant firm (vertical axis) in the RBBE (left panel) and the Nash equilibrium (right panel). The diagonal line in each panel corresponds to equal innovation probability $p_{\text{Innov,H}} = p_{\text{Innov,L}}$.

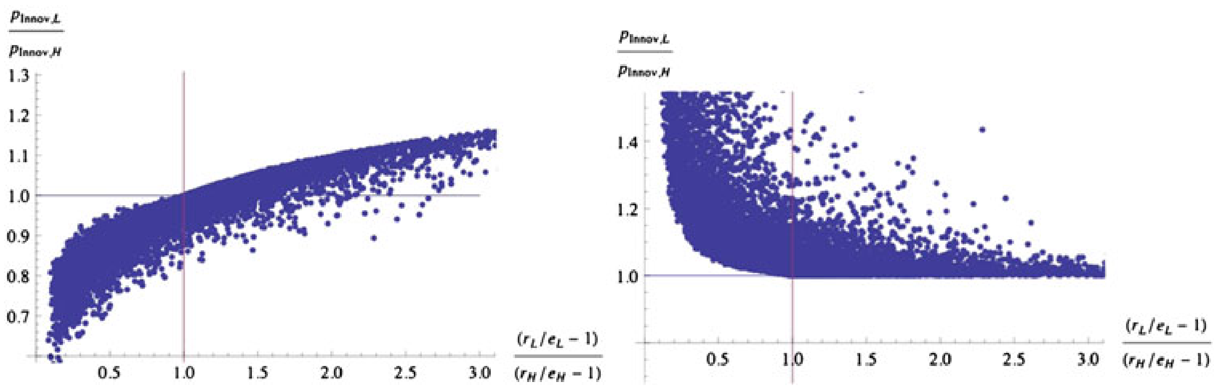

Figure 3. Ratio of innovation probabilities $p_{\text{Innov,L}}/p_{\text{Innov,H}}$ versus ratio of incentives to innovate $(r_{\text{L}}/e_{\text{L}}-1)/(r_{\text{H}}/e_{\text{H}}-1)$ in the RBBE (left panel) and in the Nash equilibrium (right panel).

it has a greater incentive to innovate than the less dominant firm (as the left-hand panel of Figure 3 does not have some points where $(r_{\text{L}}/e_{\text{L}}-1)/(r_{\text{H}}/e_{\text{H}}-1) > 1$ and $p_{\text{Innov,L}}/p_{\text{Innov,L}} < 1$). In contrast, the right-hand panel of Figure 3 shows the corresponding trend in the Nash equilibrium. In the Nash equilibrium, the results are reversed with respect to the RBBE as the firm innovates less when it has a greater incentive to innovate. As discussed earlier, the Nash equilibrium results appear to be inconsistent with the survey findings in Figure 1. Thus, we have the following result.

Result 1
In the RBBE, the dominant firm tends to invest more in innovation than the less dominant firm. The less dominant firm only invests more when it has a greater incentive to innovate, measured as the fractional increase in profits from innovation.

Moreover, the value of the RBBE for both firms is usually higher than the corresponding value of the mixed Nash equilibrium. In the game shown in Table 2, the values of the Nash equilibrium for the dominant firm and the less dominant firm are 8.0 and 4.0, respectively. In contrast, the expected values under the returns-based beliefs approach for the dominant firm and the less dominant firm are higher at 11.0 and 4.7, respectively. Therefore, the dominant firm and the less dominant firm have profitable deviations from not playing the mixed-strategy Nash equilibrium, when the other firms do not play the mixed-strategy Nash equilibrium. In Figure 4, we show how the ratio of the value of the equilibrium to a firm in the RBBE to that in the Nash equilibrium varies as we make random choices of the game parameters as before, denoting by $P_{X,Y}$ the value to firm X in the equilibrium computed using approach Y. We see that in 85% of cases, both firms are better off in the RBBE than in the Nash equilibrium. In most of the remaining 15% of cases, the dominant firm is better off, whereas the less dominant firm is marginally worse off. Only in a tiny fraction of cases, $\lesssim 0.2\%$, are both firms worse off.

We can now ask the question as to whether the probability that a firm's innovation depends more on the firm's own payoff than that of their opponent. The derivative $\partial p_{\text{Innov},X}/\partial \ln r_Y$ encapsulates this because it indicates how much the

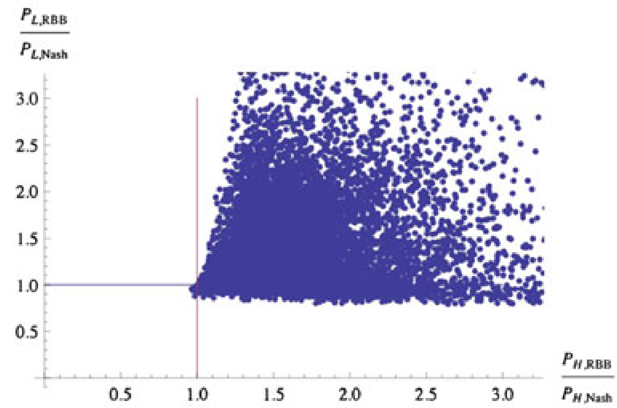

Figure 4. Ratio of the value in the RBBE to that in the Nash equilibrium for the dominant firm (horizontal axis) and the less dominant firm (vertical axis). We use $P_{X,Y}$ to denote the value to firm X in the equilibrium computed using approach Y. The vertical and horizontal lines show $V_H = P_{H,RBB}/P_{H,Nash} = 1$ and $V_L = P_{L,RBB}/P_{L,Nash} = 1$. If $V_H > 1$, the dominant firm is better off in the RBBE than in the Nash equilibrium, and if $V_L > 1$, the less dominant firm is better off; 85% of the points lie in the region with $V_H > 1$ and $V_L > 1$, in which both firms are better off in the RBBE.

probability that firm X innovates changes for a unit fractional change in the reward of innovation to firm Y. Therefore, we can define a firm's *sensitivity to reward*, S_X, as

$$S_H = \left| \frac{\partial p_{Innov,H}}{\partial \ln r_H} \bigg/ \frac{\partial p_{Innov,H}}{\partial \ln r_L} \right|, \qquad S_L = \left| \frac{\partial p_{Innov,L}}{\partial \ln r_L} \bigg/ \frac{\partial p_{Innov,L}}{\partial \ln r_H} \right|. \tag{26}$$

This can be computed analytically by differentiating the RBBE choice probabilities given in Equations (24) and (25). Because the eigenvalue ν enters these equations explicitly, we will also need $\partial\nu/\partial X$, which can be found easily from Equation (23) as

$$\frac{\partial \nu}{\partial X} = \left(\frac{\partial a_2}{\partial X}\nu^2 - \frac{\partial a_1}{\partial X}\nu + \frac{\partial a_0}{\partial X} \right) \bigg/ \left(3\nu^2 - 2a_2\nu + a_1 \right). \tag{27}$$

In Figure 5, we show how these sensitivities to reward vary as a function of the game parameters. Each point on the figure represents a different random choice of the game parameters, as described before. The horizontal and vertical lines indicate $x = 1$ and $y = 1$. We see that all the points lie in the quadrant ($x > 1$, $y > 1$), so we conclude that $S_H > 1$ and $S_L > 1$ for all these games and therefore have the following result.

Result 2

In the RBBE, the investment of firm H (firm L) depends more on its own returns, r_H (r_L), than the other firm's returns, r_L (r_H).

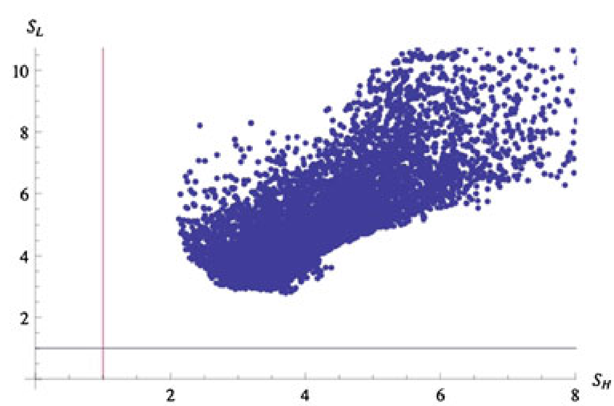

Figure 5. The sensitivity to reward of the dominant firm S_H (horizontal axis) and the less dominant firm S_L (vertical axis). Each point represents a particular choice of game parameters, chosen at random as described in the text. The vertical and horizontal lines show $x = 1$ and $y = 1$. All points lie in the region $x > 1$ and $y > 1$.

These results are in better accord with the McKinsey [16] survey on innovation spending, which highlights that the primary drivers for innovation spending are opportunities and the relative attractiveness of the innovation, consequently with a much lower emphasis on the competitor's spending (as shown in Figure 1).

Figure 5 conceals another important observation because we have defined the sensitivity as the absolute value of the derivative ratio. In fact, at all points, the derivatives take opposite signs because $\partial p_{\text{Innov},X}/\partial \ln r_X > 0$ and $\partial p_{\text{Innov},X}/\partial \ln r_Y < 0$ for $\{X, Y\} \in \{\text{H, L}\}$ and $X \neq Y$. We therefore have an additional result.

Result 3
In the RBBE, an increase in the reward to a firm leads to an increase in their probability of innovating, whereas an increase in the reward to their opponent leads to a decrease in the innovation probability.

4. Empirical analysis of the determinants of research and development investments

Our empirical analysis consists of testing whether the mixed-strategy Nash equilibrium or the returns-based beliefs approach provides a better explanation for how firms determine their R&D spending. Recall that under the mixed-strategy Nash equilibrium firms would change their R&D spending on the basis of the R&D spending of their competitors. On the other hand, the McKinsey survey [16] on innovation spending showed that R&D spending of firms are driven less by the competitors' spending but more by the available opportunities and the attractiveness of individual projects. Our proposed returns-based beliefs approach provides a closer theoretical explanation for the result of the McKinsey [16] survey than the mixed-strategy Nash equilibrium. However, in order to test the validity of these two approaches, we performed an empirical analysis on the basis of the R&D spending of 182 UK companies across 24 industry sectors.

We obtained data from the UK R&D scoreboard between 2001 and 2006, which is published jointly by the Department for Innovation, Universities, and Skills and the Department for Business, Enterprise, and Regulatory Reform. The dependent variable of interest is the annual percentage change in R&D over sales. To operationalize the competitors' R&D spending, we used the annual change in industry R&D spending excluding the individual firm's own R&D spending. To operationalize available opportunities and the attractiveness of the individual projects, we constructed a measure that captures the annual change in the difference between the industry return on sales (the industry's total operating profits for the year over the industry's total sales for the year), less the firm level return on sales (an individual firm's operating profit for the year over its sales for the year). This excess return on sales measure provides a proxy for the opportunities that the firm has to innovate its products and processes to be in line with the level of profitability of other firms in its industry. When the measure is positive, there are more opportunities to innovate to catch up with the other firms in its industry compared with when the measure is negative. For our model specifications, we use a random effects model and estimate the parameters while controlling for the size of the firm (by using the logarithm of the number of employees) and industry. Table 4 provides a summary of the measures.

Table 5 provides the correlation matrix. Table 5 shows that there are no major correlations that might contribute to multicollinearity in the explanatory variables. We also checked for scale issues with respect to the relative size of firms in the sectors. The coefficient of variation (standard deviation over mean of the log of employees) is 0.29, which provides an

Table 4. Summary of the measures.

Summary of the measures		
Conceptual variable	Measure	Variable type
Firm's innovation spending	Annual percentage change in R&D over sales	Dependent variable
Competitors' innovation spending	Annual change in industry R&D spending excluding the individual firm's own R&D spending	Explanatory variable
Available opportunities and the attractiveness of the individual projects	Excess return on sales—annual change in the difference between the industry return on sales (the industry's total operating profits for the year over the industry's total sales for the year), less the firm level return on sales (an individual firm's operating profit for the year over its sales for the year)	Explanatory variable
Firm size	Log of the number of employees	Control variable
Type of industry	Type of industry	Control variable

Table 5. Correlation matrix ($N = 1092$).

No.		1	2	3	4
1	Firm's spending	—	—	—	—
2	Competitors' spending	−0.021	—	—	—
3	Available opportunities	0.134*	−0.022	—	—
4	Firm size	−0.007	−0.199*	0.036	—

* $p < 0.05$.

indication that although there are variations in size among the firms they are not too large to make competition in R&D not worth pursuing. Formally, the model specification can be written as follows:

$$y_{it} = \beta x'_{it} + \mu_{it} \qquad \text{and} \qquad \mu_{it} = \eta_i + v_i, \tag{28}$$

where y_{it} is the dependent variable, x'_{it} is the set of regressors (independent variables), η_{it} is the unobserved individual effects, and v_{it} is the error term. In addition, we assume that $E(v_{it}) = 0$; $E(v_{it}/x_{it}) = 0$ and $E(\eta_{it}) = 0$; $E(\eta_{it}/x_{it}) = 0$. The results of the regression analysis are shown in Table 6.

We find a significant positive relationship between the change in R&D over sales for an individual firm and the opportunities and the attractiveness of the individual projects ($\beta = 0.024$, $p < 0.01$). On the other hand, there is no significant relationship between change in R&D over sales for an individual firm and the competitors' R&D spending. We generated standardized residuals from the regression. We examined the residual graphs and checked for the existence of outliers. We do not find evidence of a significant number of outliers or residuals that stand out. Therefore, we can be confident of the robustness of our regression results. The empirical analysis provides support for the returns-based beliefs approach compared with the theoretical predictions of the mixed-strategy Nash equilibrium for the patent-race-based innovation game.

In addition, we test the robustness of our results with some alternative specifications. It is possible that a firm's relatively high spending on R&D in the past could influence its spending in the future. One explanation for this is that as a consequence of having spent a relatively high amount on R&D in the past the firm's managerial experiences and knowledge are altered, which could then affect future R&D decisions. A second explanation is that firms might differ in some unmeasured variables that influence the probability of spending on R&D but that are not influenced by the R&D spending in the past. These two factors need to be accounted for in our analysis [45]. Arulampalam and Stewart [46] suggested using the Heckman estimator of the dynamic probit model to correct for these effects by incorporating lagged dependent variables. In order to perform the estimation, we coded the dependent variable as a dichotomous variable taking the value 1 if the firm's annual percentage change in R&D over sales was larger than 3.6% (3.6% is the 50 percentile level for the distribution of the dependent variable) and 0 otherwise. Following Arulampalam and Stewart [46], formally the model specification can be written as follows where $y_{it} = 1[y^*_{it} = 0]$:

$$y_{it} = \gamma y_{it} + \beta x'_{it} + \theta_t \alpha_i + \mu_{it}, \qquad t = 1, \cdots, T, \tag{29}$$

Table 6. Determinants of R&D investments.

Dependent variable: percentage change in R&D over sales	
Industry R&D	0.025 (0.016)
Excess return on sales	0.024*** (0.004)
Employee size	– 9.066*** (1.670)
Year	1.352 (1.144)
Industry	0.744 (0.054)
Intercept	−2642.651 (2293.515)
No. of observations = 1092	
p-value, chi squared test = 0.000	
$R^2 = 0.21$	

Note: Standard errors in parentheses.
R&D, research and development.
*** : $p < 0.01$, ** : $p < 0.05$, * $p < 0.1$.

where α_i denote unmeasured variables. We set $\theta_T = 1$ for identification (of σ_α^2), and the equation for the first period using error component structure is

$$y_{i0}^* = \lambda z_i' + \theta_0 \alpha_i + \mu_0, \tag{30}$$

where z_i is the vector of exogenous covariates and μ_{it} are independent of α_i. The results are robust to this alternative specification whereby we find a significant positive relationship between the change in R&D over sales for an individual firm and the opportunities and the attractiveness of the individual projects ($\beta = 0.001$, $p < 0.01$). On the other hand, the coefficient on competitor's R&D spend is not significant. Moreover, we also ran the model by including a time-invariant exogenous instrument, that is, the industry profitability variable as at the initial year for the sample. Our results remain the same with this alternative specification. These alternative specifications provide further support for the returns-based beliefs approach.

In the next section, we compare the returns-based beliefs approach with other non-Nash equilibrium models.

5. Comparing returns-based beliefs to alternative non-Nash equilibrium models

In this section, we compare our model on the basis of the returns-based beliefs to several alternative models that explain non-Nash equilibrium outcomes. The three models that are relevant include the QRE [41], the cognitive hierarchy (CH) model [47], and level-k models [48], which all have their foundations on cognitive limitations. The QRE can be interpreted as an application of stochastic choice theory to strategic games or as a generalization of the Nash equilibrium that incorporates noisy optimization [49]. The QRE assumes that the decision maker might take an action that is suboptimal and that the probability of doing so increases with the expected payoff of that action. In this model, the decision maker adopts strategies proportional to the expected payoff with some error. The error the decision maker makes could be seen as either unmodeled costs of information processing [41] or unmodeled determinants of utility from any particular strategy [42]. The use of QRE and the BRNE of Chen *et al.* [42] requires the specification of an error distribution. Many applications in the literature of such QRE models assume logit choice probabilities. It has been shown that the QRE and BRNE models' prediction converge to the Nash equilibrium as the error goes to zero for a logit specification. Our model's main similarity to the QRE and BRNE model is that all strategies with positive payoffs are played with positive probabilities in proportion to their expected payoffs. However, we do not assume that the decision maker is making errors or mistakes. Rather in the returns-based beliefs model, managers need to form subjective beliefs about each other's possible strategies. We invoke the concept of subjective probabilities and the willingness of firms to cooperate, which differs inherently from the unmodeled costs of information processing [41] or unmodeled determinants of utility from any particular strategy [42].

The level-k and CH models explain the payoff sensitivity of the deviations from equilibrium by incorporating them within the structure of the game as opposed to responses to errors. The level-k and CH models allow heterogeneous behavior in that the levels of sophistication of the decision makers can vary across the decision makers. Some decision makers are very simplistic and non-strategic. On the other hand, others are more sophisticated and best respond to the distribution of less sophisticated decision makers. The non-Nash equilibrium outcome of the game is determined by the level of sophistication of the decision makers and the proportion of decision makers at each level of sophistication. The level-k and CH models differ principally in their assumptions about how the more sophisticated decision makers respond to decision makers with sophistication levels below them. In the case of the level-k model, the more sophisticated decision maker responds only to decision makers that are one level of sophistication below them. In contrast, in the CH model, the more sophisticated decision makers respond to the distribution of decision makers at all levels of sophistication below them. The returns-based model differs from the level-k and CH models in that the former does not assume heterogeneity in the levels of sophistication in thinking by decision makers. On the other hand, the returns-based model assumes symmetry in the ability of decision makers to iterate strategically until the belief probabilities converge to the decision probabilities.

6. Discussion and conclusion

The debate about whether dominant firms or less dominant firms are more innovative is at the heart of research on innovation. Both the theoretical and empirical research on this question is still subject to great debate. We provide an additional explanation that supports the thesis that dominant firms are more innovative than less dominant firms.

Prior research on dominance and innovation using the Nash equilibrium analysis has shown that either dominant firms or less dominant firms might invest more in innovation [6, 7, 9, 10] . In this paper, we provide an explanation that is based on a non-Nash equilibrium analysis. We invoke the concept of subjective beliefs to derive our results and argue that there is an equilibrium in such beliefs when choosing actions. The basic premise of our argument is that managers' past experience might influence their subjective beliefs of what the other firm is likely to do. Therefore, managers will take these beliefs into account in making their decisions. The challenge arises as to what is a reasonable basis for forming such subjective beliefs. In this paper, we have assumed that in order to maximize expected profits, the managers formed the beliefs in order to avoid either both firms investing or both firms not investing in R&D in the patent race game. In doing so, we have assumed that this conjecture is encapsulated in the Luce [20] rule whereby beliefs are formed on the basis of the expected returns for a particular strategy over the total expected returns of all strategies. The Luce rule assumes that managers' decisions between any two actions will be determined by their relative returns and, hence, are independent of any other choice. Although we recognize that alternative beliefs are possible, our results are derived on the premise that the Luce-rule-based belief structure holds.

In assuming such a belief structure, we argue that not only is it important to understand the possible Nash solutions in formulating strategy (which frequently act as benchmarks) but equally that it is also of fundamental importance to understand the expected relative returns that could potentially influence decisions that concern strategic choice in innovation. Our approach as explained above, which we call the returns-based beliefs approach, is based on a combination of decision analytic solution concepts and Luce's [20] probabilistic choice model. The returns-based beliefs approach provides support for the argument that explains why dominant firms might innovate more—market advantage might induce the dominant firm to invest more although competition might encourage the less dominant firm to invest; this competitive effect might not be large enough to overcome the dominant firm's incentive to leverage its market advantage. Our approach accords better with the innovation survey and general observations on how firms allocate their innovation spending.

We also provide empirical evidence using UK R&D data that show support for the returns-based beliefs approach. The returns-based beliefs approach starts with the premise that firms might form subjective probabilities and hence could generate results that are out of equilibrium in actions (i.e., in the sense that they are not Nash equilibrium) but are in equilibrium in beliefs. One of the managerial implications of this view is that it is important to understand when a market could be out of equilibrium in actions. With this line of reasoning, it could be argued that industries that display continuous productivity improvements operate out of equilibrium, and hence in these contexts an approach such as the returns-based beliefs might be more appropriate than the Nash equilibrium concept. Moreover, in such industries, the past experience of managers could inform the formation of subjective probabilities. The knowledge and experience of the organization could affect the innovation strategy as the cognitive frames could impede innovation [50, 51]. For example, it has been argued in other research that Xerox did not commercialize many of its inventions from its research lab Palo Alto Research Center because the new business model that was required to commercialize these inventions did not conform to the historical business model of Xerox [50]. Therefore, the relative profitability of these innovations is what drove the investment of innovation dollars more than the desire to keep competitors indifferent to their different strategies. Finally, it is important for firms to understand how competitors might allocate resources for innovation on the basis of the relative attractiveness of the competitor's own opportunities rather than on the basis of any notion of the other firms' opportunities. Thus, the empirical study of innovation could be supported by more relevant theory, such as the returns-based beliefs approach, that establishes more accurately the theoretical underpinnings for the links between a firm's dominance and its likelihood of undertaking innovation in the market.

Finally, our returns-based beliefs approach provides a step forward in understanding innovation decisions by incumbent firms. However, a review of the innovation literature highlights that top management team decision making about resource allocation for innovation is a very complex phenomenon [52]. In particular, there are other factors, such as social and contagion effects, technological inertia, management cognition, and organizational routines, that might drive the investment decision in innovation. We hope that our approach enables more nuanced psychological aspects of management to be considered more comprehensively by using tools from game theory to analyze investment in innovation.

APPENDIX A. Mixed Nash equilibrium

We assume that there are two firms, firm 1 (dominant) and firm 2 (less dominant), and that the payoffs for firm 1 and firm 2, given move i for firm 1 and move j for firm 2, are u^1_{ij} and u^2_{ij}, respectively. Note that firm 1/2 is always the row/column firm with this convention. We assume that the firms both play mixed strategies, which we denote by $\mathbf{p^1}$ for firm 1 and $\mathbf{p^2}$ for firm 2. The notation $\mathbf{p^1}$ is a vector with components $\left\{p^1_i\right\}$, which are the probabilities that firm 1 will play move i and similarly for firm 2. The expected payoffs for firms 1 and 2, which we denote by E_1 and E_2, respectively, are then

$$E_1 = \sum_{i,j} u^1_{ij}\, p^1_i\, p^2_j, \qquad E_2 = \sum_{i,j} u^2_{ij}\, p^1_i\, p^2_j. \tag{A1}$$

In a Nash equilibrium, no firm can improve their expected return by varying their own strategy while their opponents keep their strategies fixed. Mathematically, this means that the Nash equilibrium is a turning point of E_1 with respect to variations in the p_i^1's only, subject to the constraint that the p_i^1's represent a probability distribution, that is, $\sum_i p_i^1 = 1$, and it is a simultaneous turning point of E_2 with respect to variations in the p_i^2's only, with the constraint $\sum_i p_i^2 = 1$. Maximization subject to constraints is achieved using Lagrange multipliers, that is, we extremize the function

$$\sum_{i,j} u_{ij}^1 p_i^1 p_j^2 - \lambda \sum_i p_i^1 \tag{A2}$$

with respect to the p_i^1's and extremize the function

$$\sum_{i,j} u_{ij}^2 p_i^1 p_j^2 - \mu \sum_i p_i^2 \tag{A3}$$

with respect to the p_i^2's. Differentiation of these two equations with respect to the relevant variables yields the equations

$$\sum_j u_{ij}^1 p_j^2 - \lambda = 0 \qquad \forall\ i, \qquad \sum_k u_{kj}^2 p_k^1 - \mu = 0 \qquad \forall\ j. \tag{A4}$$

We eliminate the Lagrange multipliers λ and μ using the probability condition $\sum_i p_i^1 = \sum_i p_i^2 = 1$ to deduce

$$p_i^1 = \left[\sum_j (u^2)^{-1}_{ji}\right] / \left[\sum_{ij} (u^2)^{-1}_{ij}\right], \qquad p_i^2 = \left[\sum_j (u^1)^{-1}_{ij}\right] \Big/ \left[\sum_{ij} (u^1)^{-1}_{ij}\right]. \tag{A5}$$

We can see that the mixed-strategy Nash equilibrium for firms 1 and 2 depends on the other firm's payoffs and not its own profits.

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