Continuous Cumulative Prospect Theory and Individual Asset Allocation

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Abstract

We implement the Cumulative Prospect Theory (CPT) framework (Tversky and Kahneman 1992) into a model of individual asset allocation, building on earlier work by Hwang and Satchell (2003) where they derive explicit formulae for the asset allocation decision using a loss aversion utility function. We apply Prelec’s probability weighting function (1998) to continuous distributions and derive the formulae for the optimal asset allocation between risky and safe assets. US equity returns data are used to examine the feasible parameter space. The earlier results of Hwang and Satchell are confirmed and the more complex model is compatible with observed equity proportions. The parameters are highly interconnected, but feasible combinations indicate that more inverse-S shaped deviations from linear probability weightings are associated with lower risk taking behaviour.

**JEL Classification:** G11 - Portfolio Choice; D81 - Criteria for Decision Making under Risk and Uncertainty.

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1 Introduction

The reigning paradigm of individual behaviour in finance theory is that of expected utility maximisation, combined with risk aversion. Whilst this characterisation of human choice is an enticing description of how rational agents should choose, its descriptive accuracy has come under attack in recent years as experimental psychologists have demonstrated that people systematically deviate from the choice predictions that it implies. A number of alternative behavioural models of human choice have been proposed among which one of the most fully developed and thoroughly investigated is Tversky and Kahneman’s Cumulative Prospect Theory (CPT) (1992), which combines the concepts of loss aversion (LA) and a nonlinear rank-dependent weighting of probability assessments.

CPT has been employed largely in experimental contexts with simple (often binary) gambles and there are not many attempts to use the theory in more complex settings to explain economic behaviour, or to see if the model can provide a fit to econometric data using plausible combintions of parameters. Nonetheless, there are exceptions. Barberis and Huang (2001) and Barberis, Huang and Santos (2001) use loss aversion and mental accounting (Thaler 1999) to explain aspects of stock price behaviour, but do not employ the full CPT framework, nor do they examine portfolio choice. Benartzi and Thaler (1995) utilise CPT to resolve the equity premium puzzle if investors are loss averse and evaluate their portfolios myopically with

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an horizon of approximately one year. Using the CPT parameters estimated by Kahneman and Tversky (1992) they show that this horizon is consistent with an optimal equity allocation of between 30% and 55%. Lastly, Hwang and Satchell (2003) utilise the concept of loss aversion to determine the optimal allocation between safe and risky assets for an investor with a utility function that treats gains and losses differentially and exhibits loss aversion. However, the further implications of utilising the full CPT framework, and thus whether CPT can be fit to market data using parameters with empirical and theoretical support, have not been assessed.

The purpose of this paper is to investigate asset allocation in where decisions are made using CPT, by adding nonlinear evaluation of probabilities to the LA utility function used by Hwang and Satchell. As previous work has employed only a discrete formulation, mostly over simple or binary prospects, this requires stretching the CPT framework to cover continuous distributions. Whilst there have been a number of previous empirical studies to determine appropriate values for CPT parameters, these have generally utilised highly simplified gambles in an artificial experimental setting. We provide an explicit solution for the optimal equity allocation and explore more thoroughly the CPT parameter space that is consistent with observed equity allocations given financial markets’ returns distributions over a one month horizon. This examination of viable parameter combinations in a real-world decision setting is a novel contribution of this paper, allowing us to test CPT outside of an experimental setting.

We find that existing estimates of the model parameters from experimental choice data produce viable results, although these estimates need to be treated with caution as different combinations of functional forms have been used in these estimates which will affect the resulting values. In particular, many previous estimates have assumed a single parameter to govern utility curvature for both losses and gains, which cannot be the case in our model. Many different combinations of parameter values can produce identical predictions of equity allocation so understanding the interactions
between parameters, and estimating the model as a whole becomes essential. In addition there remains the problem of the applicability of experimentally derived parameter value, often in cases where the payoffs have been hypothetical, to the real world decision making inherent in investing. There is reason to believe that observed choice behaviour is substantially different between real and hypothetical gambles, and particularly so where the outcomes are large (Holt and Laury 2004). This would lead us to expect greater degrees of risk aversion in investors behaviour than we obtain from experimental evidence. It is also plausible that a repeated choice situation, like investing, may lead to behaviour best described by parameters closer to expected utility behaviour than to those obtained in one-off experimental situations. Nonetheless, our paper shows that the restrictions on CPT parameter values imposed by both our empirical and theoretical concerns are fully consistent with observed behaviour in financial markets. That is, market behaviour can be fit with CPT using these parameter combinations.

Specifically we find that equity allocations of between 0% and 100% are consistent with a) a utility function that is concave for gains, convex for losses and, crucially, slightly less curved for losses than gains; b) Loss aversion which gives losses a psychological weighting of about twice that of gains; c) An inverse-S shaped transformation of cumulative probability for losses and of decumulative probability for gains. For these values risk-seeking behaviour (i.e., greater investment in risky assets) decreases as the probability transformation function gets increasingly non-linear in the direction predicted by previous theoretical and empirical results. That is, greater inverse-S shaped distortions of objective cumulative probabilities lead to increased risk-aversion.

The paper is organised as follows: In Section 2 we introduce the Asset allocation model of Hwang and Satchell and describe their results. Section 3 describes CPT and discusses non-linear probability evaluation. Section 4 extends CPT to continuous distributions and uses this to extend to asset allocation model. Section 5 examines the viable parameter space for this model using US and UK market
data and Section 6 discusses implications and difficulties with this approach. The final section concludes.

2 Results in a Loss Aversion World

Unlike expected utility theory, which holds that absolute wealth levels are the carriers of utility, the CPT utility framework measures utility in terms of changes in wealth from some reference level (usually the status quo) and allows our subjective experience of losses to be more sensitive than that for gains. Let $W$ represent final wealth, $W_0$ initial wealth, and $B$ the reference level (benchmark). The change in wealth is $Y = W - B$, and the CPT power utility function is defined as:

$$u(Y) = \begin{cases} Y^g, & \text{if } Y > 0, \\ -\lambda (-Y)^l, & \text{if } Y \leq 0, \end{cases}$$

(1)

where $g$, $l$, and $\lambda$ are all assumed positive\(^3\).

In this formulation, values for $g$ and $l$ equal to one result in risk neutrality in the appropriate domain of gains and losses respectively. The utility curve is concave above the reference point, and convex below if both $g$ and $l$ are less than one. This is often taken to be the standard, or at least most common form for the utility function. The combination certainly does have the appeal of having an intuitive psychological explanation: this pattern will result if individuals are decreasingly sensitive to absolute changes in the outcome value as we move further away from the reference point in either direction. However, it is by no means clear from experimental data that this is necessarily so, only that it appears to be the most common combination (see Abdellaoui 2000, Bleichrodt and Pinto 2000, Luce 2000 Ch. 3). Values greater than unity imply risk seeking for gains and risk aversion for losses.

\(^3\)This function differs slightly from that used in Hwang and Satchell’s paper in that they divided the function over gains and losses by $g$ and $l$ respectively to aid differentiation. We have not utilised this form here so as to make direct comparisons of the parameter values with those estimated from previous experimental work.
λ reflects the extent to which the subject is loss averse. Assuming $g$ and $l$ are equal, the higher the value for $\lambda$, the greater the degree of loss aversion. In the loss aversion literature evidence suggests that humans are, in general, more sensitive to losses than gains. This requires $\lambda > 1$, although in fact most estimates indicate a value of around 2 to 2.5. (see, for example, Tversky and Kaneman 1992; Kahneman, Knetsch and Thaler 1991).

We assume a world where portfolios are chosen from combinations of a risk-free asset and a risky asset, with returns of $r_f$ and $r$ respectively. The proportion of the portfolio held in the risky asset is denoted by $\theta$ (which is assumed positive, thus eliminating the possibility of short sales). Using this notation we can write final wealth at the end of the single period as

$$W = W_0(1 - \theta)(1 + r_f) + W_0\theta(1 + r)$$

$$= B + W_0\theta x$$

(2)

where $B = W_0(1 + r_f)$, $x = r - r_f$ is the excess returns to the risky asset, and thus $Y = W_0\theta x$.

The optimal portfolio is obtained by solving for the value of $\theta$ that maximises LA utility, which is given by:

$$U_{LA} = (W_0\theta)^g u^+ - \lambda (W_0\theta)^l u^-$$

(3)

where $u^+ = qE(x^g|x > 0)$ with $q = \text{prob}(x > 0)$ is the evaluation of the gains only portion of the prospect, and $\lambda u^- = \lambda (1 - q) E((-x)^l|x < 0)$ is the evaluation of the loss only portion.

The solution for $\theta$ where $g \neq l$ and initial wealth is normalised to 1 is:

$$\theta = \left( \frac{gu^+}{\lambda lu^-} \right) \frac{1}{l - g}$$

(4)

This result is true for any probability density function. Where $g = l$ the only solutions are to either purchase solely the risky asset (for $g = l < 1$), solely the risk-free asset (for $g = l > 1$). Where $g = l = 1$ the solution is indeterminate. As such, these solutions
do not provide any insight into observed portfolio choice. Lastly, under these conditions, Hwang and Satchell prove the result that $l - g > 0$ if the reasonable assumption is made that “the proportion of wealth held in equity is an increasing function of the probability that equity outperforms cash”\(^4\). This implies that risk seeking for both gains and losses is ruled out (i.e., the combination $g > 1$ and $l < 1$ is ruled out), as is risk neutrality for losses where risk seeking behaviour holds over gains (eliminating the combination $g > 1$ and $l = 1$).

Hwang and Satchell calculate the optimal $\theta$ empirically for two distributions, the normal and the Knight, Satchell and Tran (1995) (KST) distribution which captures asymmetry in upside and downside returns. The pdf of the KST distribution for excess returns is defined as follows:

$$
\text{pdf} (y_t) = \begin{cases} 
q f^+(x) & \text{for } x \geq 0 \\
(1 - q) f^-(-x) & \text{for } x < 0
\end{cases}
$$

(5)

where $f^+(x)$ and $f^-(-x)$ are conditional pdf’s and $q$ is the probability that $y_t \geq 0$. The cumulative distribution that corresponds to this pdf is

$$
F(x) = \begin{cases} 
(1 - q) + qF^+(x) & \text{if } x \geq 0 \\
(1 - q) (1 - F^-(-x)) & \text{if } x < 0
\end{cases}
$$

(6)

where $F^+(x) = \int f^+(x) \, dx$ and $F^-(-x) = \int f^-(-x) \, dx$ are the conditional cumulative distributions. In their examination, Hwang and Satchell fit $f^+$ and $f^-$ to the loss and gains data using the Gamma distribution.

These distributions are tested on both US and UK financial data to test the parameter combinations that are consistent with sensible estimates of the proportion of risky (equity) to risk free assets for these two countries. As a reference they utilise the investment proportion in domestic and foreign equity in 1993 for large pension

\(^4\)This assumption is reasonable, but will not hold in comparisons between all prospects. For instance, an increase in $q$ could be accompanied by sufficient skewing of the distribution toward losses to make the gamble look less attractive overall. Once non-linear decision weights are incorporated into the model, $q$ can no longer be factored out of $u^+$ and $u^-$ and the proof no longer follows.
funds in the UK of 83%, while the equivalent figure for the US was 46%\textsuperscript{5}.

The empirical analysis supports the above finding that $l$ is greater than $g$ for the ranges of parameter values that have been used in previous studies. It also suggests that $l$ and $g$ need to be close to each other to generate the investment proportions given above. This result is compatible with Abdellaoui’s (2000) median estimates of $g = 0.89$ and $l = 0.92$.

\section{Cumulative Prospect Theory}

The concept of loss aversion is one of the cornerstones of Cumulative Prospect Theory (CPT), a behavioural alternative to expected utility theory that has gained considerable experimental support (at least, that is, in experiments over relatively simple lotteries). CPT was proposed by Tversky and Kahneman (1992) as an improvement on, and development from, their earlier Prospect Theory (Kahneman and Tversky 1979) and is a combination of Rank Dependant Utility (RDU) (first proposed by Quiggin (1982)) with a reference point to differentiate between gains and losses. RDU models seek to address the empirical finding that no transformation of outcome values through a utility function is sufficient to account for observed choice patterns. Explaining these patterns requires an additional transformation of the probability inputs into the prospect valuation. The theory holds that individuals do not utilise objective probabilities in their decisions, but rather transform the objective probabilities using non-linear decision weighting function.

A prospect is a sequence of pairs of outcomes and probabilities $(x_i, p_i)$ which yields $x_i$ with probability $p_i$, where $x_i > x_j$ iff $i > j$. The likelihood of events can be defined by a probability distribution $f = (x_1, p_1; \cdots; x_n, p_n)$, $x_1 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots \leq x_n$.

In the valuation of the prospect, gains and losses from the reference point are treated differently (analogous to LA utility) to reflect the

psychological asymmetry of the two cases. The overall valuation \( V \) is given by

\[
V = \sum_{i=1}^{k} \left( w^- \left( \sum_{j=1}^{i} p_j \right) - w^- \left( \sum_{j=1}^{i-1} p_j \right) \right) v(x_i)
\]

\[
+ \sum_{i=k+1}^{n} \left( w^+ \left( \sum_{j=i}^{n} p_j \right) - w^+ \left( \sum_{j=i+1}^{n} p_j \right) \right) v(x_i)
\]

(7)

\( v(x_i) \) is a strictly increasing value function satisfying \( v(x_0) = v(0) = 0 \), for which the LA power utility function outlined above is most commonly utilised. \( w^+(\cdot) \) and \( w^-(\cdot) \) are strictly increasing transformations of the tail probability associated with \( x_i \). That is, transforming the cumulative distribution for losses and the decumulative for gains. \( w^+(0) = w^-(0) = 0 \), and \( w^+(1) = w^-(1) = 1 \).

Defining

\[
\pi^+_i = w^+ \left( \sum_{j=i}^{n} p_j \right) - w^+ \left( \sum_{j=i+1}^{n} p_j \right), \text{and} \pi^-_i = w^- \left( \sum_{j=1}^{i} p_j \right) - w^- \left( \sum_{j=1}^{i-1} p_j \right),
\]

enables us to simplify the notation considerably:

\[
V = \sum_{i=1}^{k} \pi^-_i v(x_i) + \sum_{i=k+1}^{n} \pi^-_i v(x_i)
\]

(8)

\( \pi^+_i \) and \( \pi^-_i \) are the decision weights given to each outcome.

Thus in evaluating a prospect using CPT, the transformation of each outcomes depends not only on the utility to be gained from that outcome (which is in turn dependent on whether the outcome is framed as a gain or a loss) and its probability of arising, but also on the ranking of that outcome compared to the other possible outcomes in the prospect. This RDU form is the only non-linear transformation of probabilities that is simultaneously able to satisfy stochastic dominance (Quiggin 1982).

CPT therefore incorporates a number of observed choice behaviours into the model. The characterisation of outcome as gains and
losses relative to a reference point, and the concept of loss aversion were incorporated into the asset allocation model of Hwang and Satchell. The further components of rank-dependence and nonlinear weighting of probabilities, however, are essential in explaining patterns of observed choice in risky prospects. This paper will include all these elements into the model, thus utilising the full CPT framework.

3.1 The Shape of the Weighting Function

A utility function that describes choice as being, for example, risk averse in gains and risk seeking in losses cannot capture the full complexity of observed choice patterns. Rather, the degree of risk aversion appears to be dependent on the probability and ranking of the outcome as well as the value of the outcome. In particular, Tversky and Kahneman (1992) discuss the “fourfold pattern of risk aversion” for binary gambles which shows a) risk aversion over gains of moderate to high probability, b) risk seeking over gains of small probability, c) risk aversion over losses of small probability and d) risk seeking over losses of moderate to large probability.

To account for this pattern, the shape of the functions $w^+$ and $w^-$ becomes fundamental in describing the actual choices made. Experimental observation has revealed a number of characteristics that any formulation of these functions must satisfy: Over/Underweighting, Decreasing Relative Sensitivity, and Increasing Absolute Sensitivity near $p=0$, and $p=1$. Luce (2000) is a good source for a fairly recent discussion of this evidence. Prelec (1998) aligns these three properties through a single axiom of compound invariance in the sense that, “given the axiom, the ordering of weighting functions by over/underweighting, subproportionality, and subadditivity coin-
cides”. This implies that the weighting functions are of the form.

\[ w^+(P_i) = \exp\{-\gamma^+(- \ln P_i)^\varphi\} \]
\[ w^-(P_i) = \exp\{-\gamma^-(\ln P_i)^\varphi\}, \]

where, \( P_i = \begin{cases} \sum_{i=1}^{n} p_j, & \text{if } x_i > 0 \\ \sum_{j=1}^{i} p_j, & \text{if } x_i \leq 0 \end{cases} \)

That is, \( P_i \) is the decumulative probability associated with outcome \( x_i \), (i.e., the probability of getting on outcome at least as good as \( v_i \)) where the outcomes are gains, and the cumulative probability associated with \( x_i \) for losses. With \( \varphi < 1 \), this results in the inverse-S shaped weighting function that is required for the fourfold pattern to be produced in simple prospects. Note that, with the axiom of compound invariance the resulting form has three parameters: \( \gamma \) is different for gains and losses, but \( \varphi \) is necessarily the same in both domains.

A number of alternative formulations have been proposed, for example that of Gonzalez and Wu (1999) who take a linear transformation of the log odds scale,

\[ w(p) = \frac{\delta p^{\gamma}}{\delta p^{\gamma} + (1 - p)^\gamma} \]

Although the axiomatic bases on which they are based differ, this function is extremely similar to Prelec’s function in its predictions. Indeed, so much so that extent that Gonzalez and Wu were unable to discriminate between them on their experimental data.

Prelec also offers a single parameter version of his function where both \( \gamma^+ \) and \( \gamma^- \) are equal to 1 enabling the function to be simplified to

\[ w^+(P_i) = w^-(P_i) = \exp\{-(- \ln P_i)^\varphi\} \]

Whilst Prelec does provide both an informal and an axiomatic argument for this further assumption, it seems clear from the experimental work of Gonzalez and Wu that the function does not

\footnote{We employ \( \varphi \) and \( \gamma \) throughout in place of Prelec’s \( \alpha \) and \( \beta \) so as to reduce the possibility of confusion with the parameters of the Weibull distribution which are commonly denoted \( \alpha \) and \( \beta \).}
provide a good description of individual differences in observed choice, which vary considerably in both curvature and elevation, whilst the single parameter only allows curvature to be varied. However, they fail to reject the one-parameter model when applied to the median data. Indeed, they state that the “one-parameter weighting functions proposed by Tversky and Kahneman and by Prelec fit the median data almost as well as the two-parameter, linear in log odds weighting function.”

When combined with the power value function described above and with $\varphi < 1$, Prelec’s one-parameter weighting function can lead to the fourfold pattern of risk attitudes - indeed he demonstrates that the weighting function will always produce an initial region of risk seeking (risk aversion) for sufficiently small values of gains (losses), regardless of whether the value function is concave or convex (Prelec 1998). Thus, it would seem that as long as we are concerned only with predicting aggregate behaviour, and not attempting to model individual differences, then Prelec’s axiomatised function provides us with an accurate yet parsimonious formulation with which to proceed. Note that Prelec’s single parameter form implies a single parameter $\varphi$ for both gains and losses. As we are concerned with median behaviour we employ this restriction here, although there do appear to be significant differences in the shape of the function for gains and that for losses when considering individual data.

4 Extension to Complex Prospects

Thus far, the framework of CPT has been fairly successful at explaining observed choice behaviour in experimental circumstances where the prospects are limited to simple and binary outcomes. Although CPT itself provides a framework that can be extended to more complex prospects with a large number of outcomes, the complexity of the experimental design that would be required to adequately test such models has meant that all such work has been performed with simpler prospects.

When trying to determine the implications of CPT for portfo-
lio choice, however, the decision is not a choice between simple prospects, but rather an optimisation of the proportions of different prospects, where the prospects themselves are continuous distributions over outcomes.

In Prelec’s functional form of the weighting function, the axiom of compound invariance and the proof based on this axiom are defined only for simple prospects. Nevertheless it is instructive to see what behaviour results if we assume Prelec’s function to be extendable into the domain of complex prospects.

If CPT is to be useful in determining how humans perform the task of choosing one index rather than another, or one source of (continuously distributed) risk rather than another then it becomes important how a prospect is evaluated when the number of possible outcomes in the prospect gets very large, or infinite in the continuous case.

The move from discrete to continuous CPT is an analogue of the same procedure for standard EUT. The discrete version of EUT is:

\[
V = \sum_{i=1}^{n} v(x_i) p_i \\
= \sum_{i=1}^{n} v(x_i) \left( \sum_{j=i}^{n} p_j - \sum_{j=i+1}^{n} p_j \right) \tag{12}
\]

In continuous form this becomes:

\[
V = \int v(x) dF(x) \tag{13}
\]

Recall that the discrete formulation for CPT may be given as:

\[
V = \sum_{i=1}^{k} v(x_i) \left( w^- \left( \sum_{j=1}^{i} p_j \right) - w^- \left( \sum_{j=1}^{i-1} p_j \right) \right) \\
+ \sum_{i=k+1}^{n} v(x_i) \left( w^+ \left( \sum_{j=i}^{n} p_j \right) - w^+ \left( \sum_{j=i+1}^{n} p_j \right) \right) \tag{14}
\]
The continuous version of this is:

\[
V = \int_{0}^{w^-(F(0))} v(x) \, dw^- (F(x)) + \int_{0}^{w^+(1-F(0))} v(x) \, dw^+(1 - F(x))
\]  

(15)

\(w^-(F(x))\) and \(w^+(1 - F(x))\) are transformations of the conditional continuous cumulative and decumulative functions respectively. From Eq. (6) we have \(F(0) = 1 - q\) and we may notate these transformed cumulative and decumulative distributions as

\[
\Pi^+(x) \equiv w^+(1 - F(x)) = w^+ \left(q \left(1 - F^+(x)\right)\right)
\]

and

\[
\Pi^-(x) \equiv w^-(F(x)) = w^- \left((1 - q) \left(1 - F^-(x)\right)\right)
\]  

(16)

It is important to note that, whilst in gains-only or loss-only prospects these transformations will result in functions that are themselves complete cumulative distribution functions, in mixed lotteries the combination of the gains transformation of the upper portion of the decumulative distribution and the loss transformation of the lower portion of the cumulative distribution will result in functions that are analogous to decumulative and cumulative distributions respectively. The analogy is not complete, however, because the total transformed probability under the transformed distributions will not, in general, equal 1.

Using this notation the evaluation may be written:

\[
V = \int_{0}^{w^-(1-q)} v(x) \, d\Pi^- (x) + \int_{0}^{w^+(q)} v(x) \, d\Pi^+ (x)
\]  

(17)

Furthermore we may enquire what transformation of the original pdf, \(f(x)\), is implied by this transformation of the cumulative and decumulative distributions. Denoting the transformed density functions as \(\pi^- (x)\) and \(\pi^+ (x)\) (which are analogous to the pdf’s of the transformed cumulative and decumulative distributions, \(\Pi^- (x)\) and \(\Pi^+ (x)\)), and using Eq. (5) we find that

\[
\pi^+(x) = w'^+ (F(x)) \, f(x) = w'^+ \left(q \left(1 - F^+(x)\right)\right) \, qf^+(x)
\]  

(18)
and

\[ \pi^{-}(x) = w^{-}(F(x)) f(x) = w^{-} ((1 - q) (1 - F^{-}(-x))) (1 - q) f^{-}(-x) \]  

(19)

We shall refer to these as the imputed distributions\(^\text{7}\). These demonstrate that to convert a continuous loss distribution using CPT, one multiplies the original distribution by the slope of the weighting function. We denote this multiplier for losses by \( m^{-}(x) \equiv w^{-} ((1 - q) (1 - F^{-}(-x))) \). For gains: \( m^{+}(x) \equiv w^{+} (q (1 - F^{+}(x))) \). Using this notation:

\[
\begin{align*}
V &= \int_{-\infty}^{0} v(x) \pi^{-}(x) \, dx + \int_{0}^{\infty} v(x) \pi^{+}(x) \, dx \\
&= \int_{-\infty}^{0} v(x) m^{-}(x) f(x) \, dx + \int_{0}^{\infty} v(x) m^{+}(x) f(x) \, dx
\end{align*}
\]  

(20)

(21)

Thus, the instantaneous degree of transformation of the probability distribution at any point is given by multiplying the objective pdf by \( m^{+} \) and \( m^{-} \). Where the slope of \( w \) is equal to 1, the multiplier is 1.

To obtain the pdf multiplier we take the first derivative of Prelec’s function (Eq. 11)

\[
w'(P) = \frac{\varphi}{P} (-\ln P)^{\varphi - 1} \exp\{-(-\ln P)^{\varphi}\}
\]  

(22)

The derivative is with respect to the conditional cumulative distribution for losses:

\[
m^{-}(x) = \frac{\varphi}{F(x)} (-\ln F(x))^{\varphi - 1} e^{-(-\ln F(x))^{\varphi}} \text{ for } x < 0
\]  

(23)

\[
m^{-}(x) = w^{-} ((1 - q) (1 - F^{-}(-x)))
\]  

(24)

Similarly for gains, we replace the conditional cumulative function

\(^{7}\)See appendix for detailed derivations.
by the conditional decumulative:

\[
m^+ (x) = \frac{\varphi}{1 - F(x)} (-\ln (1 - F(x)))^{\varphi-1} e^{-(-\ln(1-F(x)))^\varphi} \quad \text{for } x > 0
\]

\[
= w^+ (q (1 - F^+(x)))
\]  \hspace{1cm} (25) \hspace{1cm} (26)

It is worth exploring briefly the attributes of this multiplier. The value of the parameter \( \varphi \) is required to be less than 1 if the function is going to assume the shape required to produce the fourfold pattern of risk attributes for simple prospects. Assuming this to be the case, the loss multiplier function will display:

1. Convexity

2. Values below 1 away from the tails of the distribution and a minimum at \( F(x) = 1 - 1/e \approx 0.732 \).

3. Very steep slopes near \( F(x) = 0 \), and \( F(x) = 1 \), tending to infinity at the endpoints.

If we assume the value for Prelec’s function estimated by Wu and Gonzalez of \( \varphi = 0.74 \), then \( m^- (x) \) and \( m^+ (x) \) take the forms shown in Figure 1 and Figure 2. Since here we have assumed that the \( w \) function is identical for gains and losses, the multiplier function for a loss only prospect is a reflections about \( F(x) = 0.5 \) of that for a gains only prospect. As can be seen from this figure, the multiplier is highly non-linear, resulting in dramatic increases in the pdf at the extreme edges of the distributions.

The loss multiplier function given above shows the multiplier for the entire distribution. However, for mixed distributions, the loss portion of the KST pdf applies only to a portion of the whole distribution. Thus, only the leftmost portion of the multiplier given above will be used, up to the point where the \( F(x) \) is equal to \( 1 - q \) (the probability of getting a loss). Above this point the gains multiplier function is used. This generally results in a discontinuous jump in the pdf multiplier at the transition from losses to gains, except at the point where the graphs cross, (in this example at
$F(x) = 0.5$). One implication of this is that the total area under the new KST pdf will not generally equal 1 in a mixed distribution. In effect, this is equivalent to an implicit transformation of the values of $q$ and $(1 - q)$ as well as changing the shape of the loss and gain pdf’s. Figure 3 shows the effect of such a splicing of the two functions, assuming that $q = 80\%$.

### 4.1 CPT Asset Allocation Model

In the asset allocation notation, $u^+$ becomes

$$
\begin{align*}
    u^+ &= qE_B (x^g | x > 0) \\
    &= \int_0^\infty x^g m^+ (x) qf^+ (x) \, dx \\
    &= \int_0^\infty x^g \pi^+ (x) \, dx
\end{align*}
$$

(27)

where $E_B$ is the behavioural expected value.

This formulation requires the cumulative probability function which is not explicitly solvable for the Gamma function, and we instead use the Weibull function in this analysis. The use of the Weibull to describe asset returns has received considerable support recently as an accurate way of fitting the “fat tails” associated with empirically observed return distributions (see Laherrere and Sornette 1998; Sornette and Simonetti et al. 2000, and Malevergue and Sornette 2001) and indeed, as will be shown below, this provides a better fit for our data. The distribution is characterised by two positive parameters, $\alpha$ and $\beta$. For gains:

$$
qf^+ (x; \alpha_1, \beta_1) = q \frac{\alpha_1}{\beta_1^{\alpha_1}} x^{\alpha_1-1} e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}} , \quad 0 < x < \infty
$$

(28)

$$
q (1 - F^+ (x; \alpha_1, \beta_1)) = q e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}} , \quad 0 < x < \infty
$$

(29)

and for losses:
\begin{align}
(1 - q) f^-(x; \alpha_2, \beta_2) &= (1 - q) \frac{\alpha_2}{\beta_2^{\alpha_2}} (-x)^{\alpha_2 - 1} e^{-\left(\frac{-x}{\beta_2}\right)^{\alpha_2}}, \quad -\infty < x < 0 \\
(1 - q) (1 - F^- (-x; \alpha_2, \beta_2)) &= (1 - q) e^{-\left(\frac{-x}{\beta_2}\right)^{\alpha_2}}, \quad -\infty < x < 0
\end{align}

The subscripts 1 and 2 refer to the Weibull-KST distribution parameters for the gain and loss portions respectively.

Using this distribution we can calculate the effect of the cumulative probability transformation by examining the pdf of the imputed distributions \( \pi^+ (x) = m^+ (x) q f^+ (x; \alpha_1, \beta_1) \), and \( \pi^- (x) = m^- (x) (1 - q) f^- (-x; \alpha_2, \beta_2) \). As we demonstrate in the Appendix, if the original excess returns are distributed according to a Weibull \( f^+ (x; \alpha_1, \beta_1) \), then the imputed distribution is

\[ \pi^+ (x) = \frac{\varphi \alpha_1}{\beta_1^{\alpha_1}} x^{\alpha_1 - 1} \left( \left( \frac{x}{\beta_1} \right)^{\alpha_1} - \ln q \right) \varphi^{-1} e^{-\left( \left( \frac{x}{\beta_1} \right)^{\alpha_1} - \ln q \right)^\varphi} \]

with a similarly complex looking function for losses. Note that the value of \( q \) is intricately entwined with the pdfs for gains and losses so that the probability of a gain cannot be changed independently of the shape of the distribution for gains.

However, in the special case where the distribution ranges entirely over gains (i.e., \( q = 1 \)) or entirely over losses (\( q = 0 \)) then this simplifies dramatically to \( \pi (x) = f (x; \varphi \alpha, \beta) \). So evaluating a gains-only Weibull distribution (with parameters \( \alpha \) and \( \beta \)) where cumulative probability is assessed nonlinearly using Prelec’s function with parameter \( \varphi \), is equivalent to the evaluation, with no probability distortion, of the imputed Weibull distribution with parameters \( \alpha \varphi \) and \( \beta \). That this does not hold generally for mixed distributions is a result of the discontinuous jump in the multiplier at the reference point. For gains only and loss only distributions the area under the imputed distribution remains equal to 1 and the imputed distribution is therefore a probability density function itself. As mentioned above, this is not the case for mixed distributions and each portion of the
imputed density cannot be represented by simply transforming the parameters of the original distribution.

Using this to form \( u^+ \) (full derivations given in the Appendix) gives:

\[
\begin{aligned}
u^+ &= \beta_1^g \int_{(\ln q)^\varphi}^{\infty} \left( \frac{1}{y^{\varphi}} + \ln q \right)^{\frac{q}{\alpha_1}} e^{-y} dy \\
\end{aligned}
\]  

(33)

where \( y = \left( \left( \frac{x}{\beta_1} \right)^{\alpha_1} - \ln q \right)^\varphi \)

The value of \( u^- \) is similar:

\[
\begin{aligned}
u^- &= \beta_2^l \int_{(\ln(1-q))^\varphi}^{\infty} \left( \frac{1}{z^{\varphi}} + \ln (1 - q) \right)^{\frac{l}{\alpha_2}} e^{-z} dz \\
\end{aligned}
\]

(34)

with \( z = \left( \left( \frac{-x}{\beta_2} \right)^{\alpha_2} - \ln (1 - q) \right)^\varphi \)

Finally, for this distribution the utility function with initial wealth normalised to 1 is:

\[
U_{LA} = \theta^g u^+ - \lambda \theta^l u^- 
\]

(35)

and the optimal equity proportion is:

\[
\begin{aligned}
\theta &= \left( \frac{gu^+}{\lambda l u^-} \right)^{\frac{1}{l - g}} \\
&= \left( \frac{g \beta_1^g \int_{(\ln q)^\varphi}^{\infty} \left( \frac{1}{y^{\varphi}} + \ln q \right)^{\frac{q}{\alpha_1}} e^{-y} dy}{\lambda l \beta_2^l \int_{(\ln(1-q))^\varphi}^{\infty} \left( \frac{1}{z^{\varphi}} + \ln (1 - q) \right)^{\frac{l}{\alpha_2}} e^{-z} dz} \right)^{\frac{1}{l - g}} \\
\end{aligned}
\]

(36)
5 Empirical Investigation

5.1 Data

In what follows we use the same US financial data set as Hwang and Satchell to investigate the viable parameter space for the full CPT framework using this model. That is, we examine the parameter combinations that result in plausible combinations of risky and risk-free assets for a typical US investor, and compare these combinations relative to existing parameter estimates from experimental data.

For the risk-free asset we use the three month treasury bill, whilst for the risky asset we use the S&P 500 for the US market. We use 240 monthly observations from October 1982 to September 2002. Hwang and Satchell report the statistical properties of these data, showing that the monthly excess returns are non-normal.

What proportions constitute a reasonable allocation between risky and risk-free assets for individual investors? A complete characterisation of the risk profile of the portfolio of an individual will comprise assets across a number of classes\(^8\). Stripping the problem down from one of considering several asset classes to the simplified version of two asset classes in our model allows us to investigate the implications of CPT portfolio allocation in a simplified setting. As noted earlier the average proportion in domestic and foreign equities of large pension funds in 1993 was 83% in the UK and 43% in the USA so we could use these to signify reasonable proportions for institutional investors. It is possible that individuals may borrow in order to invest greater than 100% of their wealth into risky assets. Certainly if property is characterised as a risky asset in our two asset class world then large mortgage obligations will push the proportion of wealth in risky asset above 100% for many individuals. As will be seen, the model can certainly give these more extreme results, but our interests here are in ensuring that the model can predict median behaviour. That is, can the CPT model generate the range of observed market behavior using parameters that are consistent with

\(^8\)Friend and Blume (1975) investigate proportions of household investments in various asset classes based on household survey data. They show that the allocation across asset classes is highly variable across different household types.
our empirical and theoretical experience? In general, we rule out as unreasonable any parameter combinations that result in allocation into risky assets of greater than 100%.

5.2 Estimates of Weibull-KST Distribution

We fit the double Weibull distribution to the data using maximum likelihood estimators for gains and losses separately. We have used the entire data set for the current analysis as Hwang and Satchell found no significant differences between sub-periods, though selectively choosing periods representative of bull and bear markets are shown to have implications for the loss aversion parameter. Table 1 shows the results of this fitting procedure. These parameter values result in bimodality of the density function, but have larger ML values than the corresponding ML values if normality is assumed which indicates that the Weibull-KST distribution is a better fit to the underlying data. In addition the ML values are also substantially larger than those of the Gamma-KST distribution used by Hwang and Satchell which is further cause to believe that the underlying data are adequately represented by this distribution.

5.3 Previous CPT Parameter Estimates

A number of studies have attempted to estimate the parameters of CPT using experimental choice data. We are particularly interested in whether the use of similar estimates in our current model will yield reasonable results for the optimal amount of risky assets in individual portfolios. In addition, because of the potential difficulties in transporting empirical estimates from experimental data to real world choice scenarios, particularly where the experimental payoffs were purely hypothetical (Holt and Laury 2002), we will be interested in the sensitivity of the results to changes in the parameters, and in the restrictions that we can place on parameters through theoretical considerations. Previous estimates of the curvature of the power utility function from experimental data range from 0.32 (Camerer and Ho 1994) to 0.88 (Tversky and Kahneman 1992), although
many of these estimates used alternative specifications of the weighting function, either the form originally proposed by Tversky and Kahneman, or that of Gonzalez and Wu. Abdellaoui provides estimates using a methodology for eliciting standard sequences of outcomes which are independent of the weighting function. When his data are fitted separately to losses and gains using power utility the median estimates are $g = 0.89$ and $l = 0.92$. So the most common case expected from empirical observation is utility that is concave for gains, convex for losses, and more linear for losses than for gains (see also Heath, Huddart, and Lang 1999; Fennema and van Assen 1998).

Wu and Gonzalez provide an estimate in combination with Prelec’s weighting function of $g = 0.48$ and $\varphi = 0.74$. Loss aversion implies that $\lambda$ is greater than 1, and previous estimates seem to suggest that losses are treated around twice as sensitively as gains.

Our model has four parameters that interact in a number of complex ways and cannot be uniquely determined by asset allocations. We proceed by examining the restrictions that need to be placed on these parameters to result in reasonable prediction for $\theta$. The results are summarised in the text as numbered observations. The results of Hwang and Satchell hold in this model as well, so our focus here is on the effect of the novel element of our model, the introduction of nonlinear probability weightings.

5.4 Restrictions on $g$ and $l$

To examine the plausibility of the model for various parameter combinations, we first examine the optimal values of $\theta$, given various combinations of $g$ and $l$. The fourfold pattern of risk attitudes imposes no restrictions on the values of $g$ and $l$ if Prelec’s inverse S-shaped probability weighting function is used, though it influences the cut-off between risk-seeking and risk-aversion regions. In the current model, however, we have the constraint $g \neq l$, whereas much previous work has assumed the curvature in the loss and gain domains to be equal. Theoretical considerations of diminishing sensitivity from the reference point lead us to anticipate that both
curvature parameters are less than 1.

In Table 2, $\lambda = 2$ and the probability weighting function parameter ($\varphi$) has been set to 1, which removes the non-linearity in decision weights. The results thus mimic Hwang and Satchell’s model with two modifications: a) the use of the double Weibull distribution, rather than the double Gamma, and b) a slight modification of their utility function which has been performed in order to make the parameters comparable to previous studies. These results corroborate two of their findings: that the value for losses ($l$) should be larger than that for gains ($g$), and that the two parameters are close to each other.

It is evident from the tables above that the optimal proportion of equity approaches 0 as the difference between $l$ and $g$ approaches zero where $l$ is the greater, whilst $\theta$ approaches infinity as the difference between $g$ and $l$ approaches zero when $l$ is the smaller value. This implies that, at least for the data examined here, the model can always produce values for $\theta$ between 0 and 1 by making the difference between the curvature parameters sufficiently small with the correct sign. Equity allocations of below 100% can also be achieved if $l$ is vastly larger than $g$. However, in this extreme case the evaluation depends on the gains side only and loss aversion becomes irrelevant and we do not investigate this case further.

With $l > g$ the proportion of equity rises as the curvature parameters are driven apart so they are required to be relatively close to obtain values for $\theta$ below 100%. Over the ranges of variables displayed in the table the most appropriate results occur where the two variables are about 0.1 to 0.15 apart. This result is in agreement with other empirical assessments of the curvature of the gain and loss utility functions (Abdellaoui; Fennema and van Assen; Heath, Huddart and Lang). Figure 4 examines effect of the difference between the two curvature parameters for a broader range of these parameters (still with $\lambda = 2$). It shows that for low values of the gains curvature parameter, $g$, reasonable equity proportion may still be reached in situations where the loss parameter is very much greater than that for gains, although in this case gains become almost
irrelevant to the evaluation of the prospect and we do not examine it further. In addition, as the parameters rise substantially above 1, the difference between them is no longer constrained to be extremely close together. In what follows we restrict our attention to values of \( g \) and \( l \) that are close to 1, as would be expected from previous empirical analyses. Here the difference between them is restricted as discussed above. Figure 5 shows that a difference between \( l \) and \( g \) of 0.1 to 0.15 provides reasonable estimates of \( \theta \) for \( g \) between 0.4 and 2, and that \( \theta \) rises quickly above 100% if the difference is increased.

If we examine the solution for \( \theta \), it is possible to discern both why \( g \) and \( l \) are required to be close together to produce reasonable allocations of equity, and under which circumstances it will be the case that \( l \) is greater. Recall that the optimal equity proportion is given by Equation (4) (where \( g \neq l \)). Assuming \( l > g \), \( \theta \) will approach 0 in the limit as the difference \( l - g \) tends to 0 if and only if at the limit \( \frac{u^+}{\lambda u^-} < 1 \). If this inequality does not hold then \( \theta \) will approach infinity. For \( l < g \) the opposite results are true.

Thus, for given values of the other parameters, as the two curvature parameters get closer together the optimal equity proportion will always approach zero from one side and infinity from the other. \( \frac{u^+}{\lambda u^-} = 1 \) with \( g = l \) is the point of changeover where reasonable equity proportions are produced by \( l < g \) rather than \( l > g \). If this ratio is less than 1, then \( \theta \) tends to zero as the difference between the curvature parameters decreases; if the ratio is greater than 1, \( \theta \) tends to infinity. The change in the optimal proportion of equity as we increase the difference between \( g \) and \( l \) is positive on one side of the discontinuity at \( g = l \) (as \( \theta \) increases from 0) and negative on the other (as \( \theta \) decreases from infinity). That is:

\[
\frac{d\theta}{d(l-g)} > 0 \text{ for } l - g > 0 \tag{37}
\]

\[
\frac{d\theta}{d(l-g)} < 0 \text{ for } l - g < 0 \tag{38}
\]

For US excess returns and the parameter ranges used above \( \frac{u^+}{\lambda u^-} < 1 \)
in the limit as $g = l$ and therefore the equity proportion increases from 0 as we increase $l - g$. This result implies that the model results are not as sensitive to the degree of curvature on the utility functions per se, but rather to differences in curvature between gains and losses. Indeed, the model can always predict values of $\theta$ between 0 and 1 regardless of the curvature, by making the difference between the curvature parameters sufficiently small with the correct sign, as shown in Figure 5.

**Remark 1** $g \neq l$, but for values around 1, they must be sufficiently close together to produce appropriate equity allocations

**Remark 2** Reasonable equity allocations may be obtained with $l > g$ if
\[
\lim_{(l-g) \to 0} \left( \frac{u^+}{\lambda u^-} \right) < 1.
\]

### 5.5 Effect of Decision Weighting Function

These conclusions remain unchanged with the introduction of the nonlinearity of the decision function (Table 3). The parameter governing the curvature of probability weighting function has been decreased to the estimate of Wu and Gonzalez (1999) of $\varphi = 0.74$, resulting in an Inverse-S shaped decision weighting function. Here, $g$ still needs to be less than $l$, but two related effects may be observed. Firstly, the non-linearity causes the optimal proportion allocated to equity to decrease. Consequently, the difference between $l$ and $g$ needs to be greater to produce equivalent equity allocations. With $\lambda = 2$ this difference is about 0.25. Current empirical evidence as well as theoretical considerations (increased attention to extreme outcomes), suggest that $\varphi < 1$, at least for aggregate data and for the majority of individuals. However it is worth noting here that the implications for $g$ and $l$ hold over the whole range of values of $\varphi$, subject to a minimum bound for $\lambda$ which will be discussed below.

The effect of $\varphi$ may be seen in Figure 6 where $g$ is set to 0.7. With linear probability weights, $\theta$ rises quickly with $l - g$. As the probability weighting curve gets increasingly inverse-S shaped, the equity allocation decreases and the curvature difference is required
to be greater to maintain a reasonable allocation. At \( \varphi = 0.74 \), 
\( l - g \) needs to be around 0.25. As \( \varphi \) increases above 1, the opposite case is true. Essentially, an inverse-S shaped decision weighting function (\( \varphi < 1 \)) reduces the sensitivity of the equity proportion to the difference between the curvature parameters. 0.25 is a larger gap between \( l \) and \( g \), than is suggested by experimental estimates - one possible explanation for this is the relatively short return horizon of one month. Although increasing the gap is required to ensure the equity allocation is sufficiently large with the introduction of non-linear probability, Benartzi and Thaler (1995) show that this effect may also be achieved through lengthening the return evaluation horizon. Thus all the results here would be consistent with a smaller gap between \( l \) and \( g \), and a return horizon of one year rather than one month.

Figure 7 sets \( l - g = 0.25 \), and \( \lambda = 2 \). Over these ranges the equity allocation is relatively invariant to the absolute values of the curvature parameters, but strongly increasing in \( \varphi \). Focussing in on values for \( \varphi < 1 \) in Figure 8 we note that reasonable equity proportions are achieved for a relatively narrow band of the decision weighting parameter.

It is important to realise, however, that the parameters are by no means identifiable by asset allocation alone – a number of different combinations may lead to plausible proportions of risky assets. In particular, as we increase the LA parameter (\( \lambda \)) \( l \) and \( g \) are required to be further apart to retain the same proportions of risky and risk-free assets (Figure 9)\(^9\). This is intuitive – the greater the degree of loss aversion for a given utility function, the lower the proportion of equity we would expect to see. Thus, we see that determining the appropriate values for the parameters in this model will need to combine an analysis of which combinations are appropriate with empirical guidance on what the parameter ranges may be for specific individuals or groups.

\(^9\)Hwang and Satchell investigate the lower boundaries for loss aversion over bull and bear markets and conclude that the lower boundaries are considerably higher in bull markets. This would imply that \( g \) and \( l \) are further apart in bull markets.
Remark 3 The optimal equity allocation is increasing in $\varphi$. Specifically, nonlinearity of decision weights in the direction of an Inverse-S function ($\varphi < 1$) decreases the equity allocation.

Remark 4 The difference between $l$ and $g$ is about 0.25 with $\lambda = 2$ and $\varphi = 0.74$ for US equity returns. This difference increases with $\lambda$ and decreases with $\varphi$.

5.6 Lower Bounds on Loss Aversion ($\lambda$)

Given the results above, it is possible to restrict the range of values that may be taken by $\lambda$ if the $l > g$ assumption is to hold. Since the difference between $l$ and $g$ must decrease as the degree of loss aversion decreases to maintain a constant proportion of equity, there is a value of the Loss Aversion parameter below which $g$ is instead required to be greater than $l$. This point is given by $\lambda > \lim_{(l-g) \to 0} \left( \frac{u^+}{u^-} \right)$.

Figure 10 shows how this minimum value for $\lambda$ varies with values of $l = g$ and $\varphi$ for our US data set. This value is fairly stable for ranges of both the curvature parameters and Prelec’s decision weight parameter, and is around 1.2 to 1.5 if the decision weighting curve is inverse-S shaped. These values are below most experimental observations of this parameter which are around 2 to 2.5. We can conclude then, that as long as the individual is sufficiently loss averse it must be the case that the curvature parameter for losses is greater than that for gains. Note, however, that if $\varphi$ is sufficiently large then the minimum bound for $\lambda$ rises above 2. A corollary to the minimum bound for $\lambda$ therefore, is that for sufficiently high loss aversion, $\varphi$ may not be too large if the $l > g$ assumption is to give reasonable results. This means that any analysis that holds $\lambda$ constant (e.g., Figure 6) will show a discontinuity as $\varphi$ gets sufficiently large.

Remark 5 Reasonable equity allocations with $l > g$ as evidenced by the US excess returns data implies $\lambda > \lim_{(l-g) \to 0} \left( \frac{u^+}{u^-} \right)$.

Note that this result gives the minimum value for $\lambda$ in the limit as $(l - g) \to 0$. However, since the equity allocation is increasing
as risk aversion decreases, following Hwang and Satchell we may also investigate what minimum value of $\lambda$ is required to give equity allocations of below 100% away from this limit. That is, what is the minimum value of $\lambda$ required to give $\theta < 1$ for combinations of $\varphi$, $g$, and $l$:

\[
1 > \left( \frac{gu^+}{\lambda lu^-} \right) \frac{1}{l - g}
\]

\[
\lambda > \frac{gu^+}{lu^-} \tag{39}
\]

Figure 11 shows the resulting minimum values for $\lambda$ for ranges of $g$ and $\varphi$, given that $l - g = 0.25$. If we restrict attention to inverse S-shaped decision weighting functions with $\varphi < 1$ then a value of $\lambda$ around previous empirical estimates of 2 to 2.5 is always sufficient to produce equity allocations of below 100%.

**Remark 6** If $\frac{d\theta}{d(l - g)} > 0$ for $l - g$ close to 0 (so $\lim_{(l-g)\to 0} \left( \frac{u^+}{\lambda u^-} \right) < 1$), then $\theta < 100\%$ requires that $\lambda > \frac{gu^+}{lu^-}$ for given values of $g$, $l$ and $\varphi$.

### 5.7 Upper Restrictions on Loss Aversion ($\lambda$)

This minimum bound for given combinations of the other parameters is lower, the lower the difference between $l$ and $g$, reaching an absolute minimum at $g = l$. Figure 10 showed that this value is reasonably invariant to changes in both $v$ and $\varphi$. Note that a value of $\lambda$ above this minimum bound does not ensure that the optimal equity allocation is reasonable, merely that the assumption of $l > g$ can result in a reasonable percentage for certain parameter values. The closer the loss aversion parameter is to its lower bound, the higher the equity allocation — and this allocation may be significantly above 100%. To investigate which values of $\lambda$ are consistent with plausible allocations we proceed by setting $l - g = 0.25$ and examine the allocation for values of $\lambda$ above the minimum value.
Figure 12 shows the optimal values of $\theta$ for combinations of $\varphi$ and $\lambda$, when $l$ is set to 0.95 and $g = 0.7$. We see that, for a given value of $\varphi$, $\lambda$ may only vary over a relatively small range whilst retaining plausible model estimates for the proportion of risky assets. Consistent with the lower bound discussion above, this admissible loss aversion values increase with $\varphi$. We thus note that Loss Aversion and nonlinear probability weights are closely related. Setting $\varphi = 0.74$, $\lambda$ needs to be above 1.8 to give an equity proportion of below 100%. $\lambda = 2$ gives an allocation of 61% and for $\lambda > 2.6$ the equity proportion drops below 20%. With linear decision weights this band is both higher and wider, extending from about $\lambda = 2.4$ to 3.6.

**Remark 7** The optimal equity allocation is highly sensitive to loss aversion, requiring loss aversion parameters between about 2.4 and 3.6 for linear decision weights, $g = 0.7$ and $l - g = 0.25$. Loss aversion and nonlinear probability weights are intimately related. Decreasing $\varphi$ from 1 causes the appropriate band of $\lambda$ to narrow, and the values to decrease. For $\varphi = 0.74$ we have Loss Aversion in the approximate band of $(1.8, 2.6)$.\(^{10}\)

Most interestingly we see that the full model gives thoroughly plausible predictions of equity allocation when using as inputs parameter values which are consistent with those estimated from experimental data.

**Remark 8** Setting the parameter values to those indicated by previous empirical and theoretical work leads to reasonable equity allocations. In particular loss aversion, risk-aversion for gains, risk-seeking for losses, and an inverse S-shaped probability weighting function used together can generate reasonable equity allocations.

\(^{10}\)This is valid for the entire period of returns we are using here. These bands are higher in bull markets, and lower in bear markets (Hwang and Satchell 2003).
6 Discussion: CPT as a Framework for Portfolio Choice

6.1 Extension to Multiple Asset Classes

The current model extends only to two asset classes and has four free parameters. Whilst this is useful in providing us with some insight into the implications of CPT as a theory of choice over continuous distributions, even in this restricted model there are insufficient data to effectively estimate the parameters unless we employ time series data on asset allocations. In extending the model a number of issues may arise. In the first place there is the problem of defining the asset classes themselves: does one include home equity and human capital? A second problem is the mathematical problem of combining KST into multivariate distributions which adds a great deal of complexity to the analysis.

A deeper problem relates to the way individuals themselves perceive these asset classes. Evidence suggests that individuals separate their assets into different “mental accounts” thereby creating a psychological partition between different groups of assets (see Thaler 1999), and the analysis of Barberis and Huang (2001) suggests that the assumption that investors have narrowly framed mental accounts (i.e., at the level of individual stocks) enables us to explain stock returns better than assuming more broadly construed portfolio level mental accounts. These mental accounts may each contain a single asset class (e.g., equity investments treated as a distinct mental account), or accounts may combine assets from multiple asset classes. Our particular concern with mental accounting in the context of this model is that individuals may display very different risk characteristics (and therefore parameterisations) for assets in different mental accounts. The risk characteristics of an individual may actually be a composite of lower level risk characteristics associated with various mental accounts which may not be adequately incorporated into a "representative individual". This raises the possibility that separate parameters are required to model asset choice for each asset type within a single individual, or potentially even for the representative
6.2 Subjective Probability Assessments and Overconfidence

The model described above utilises the full framework of CPT to explain how people evaluate a continuous prospect. However, this is only done for known distributions. That is, the model above is a proposed mechanism for how individuals might transform a given probability distribution into decision weights. The given probability distribution is assumed to be an accurate representation of the underlying return distribution and the question of how individuals arrive at this distribution is left unasked.

This greatly simplifies the analysis as it means that an objectively observed and fitted distribution can be used as the basis for the model. Thus, the fitted KST distribution forms the inputs for $f(x)$ and $F(x)$, and the value of $q$ is determined simply by the observed probability of achieving the benchmark according to the underlying data. Once we introduce subjective assessment of the distribution on the part of the investor (i.e., without the chance to observe actual historic distributions over significant amounts of time), then $f(x)$ and $F(x)$ may themselves be generated by psychological biases, subjectivity and error, and could well be inaccurately calibrated relative to the reference point.

Certainly it seems that people would be unlikely to come to a subjective assessment of the underlying probability distribution that was perfectly accurate, and then reweight it according to psychophysical decision weights. Indeed Lichtenstein, Fischhoff et al. (1982) present evidence to suggest that people display considerable overconfidence when asked to provide a subjective assessment of a probability distribution. That is their subjective distribution is too tightly centered on their estimated mean. This adds a further layer of complexity to the problem of evaluating complex prospects as the decision transform needs to be applied to a subjectively assessed probability distribution. The implications of overconfidence are difficult to assess in the present context - the effect will be to narrow the distribution used to assess the equity returns. However, if
this narrowing is symmetrical around the estimated mean, then the current model would suggest that, both because of loss aversion and the fact that \( l > g \), the evaluation of the gains portion would improve relative to losses as the extreme tails that are no longer expected current more weight in losses than in gains. This would lead us to propose that overconfidence in the estimated distribution would also lead to increased risk taking, assuming that the estimated mean is still unbiased on average.

Fox and Tversky (1998) present a two-stage model in which individuals first assess the probability attached to the event, and thereafter apply the weighting function to the judged probability. Thus, \( \pi(A) \) is approximated by \( \pi[P(A)] \), where \( A \) is an uncertain event and \( P(A) \) is the subjective assessment of the probability of this event. The weights resulting from the combined stages are those applied in the decision context.

Investigating the decision weights that individuals apply to their own judged probabilities, Tversky and Fox note that the transformation, whilst displaying less order than that from given probabilities, is nearly identical. This provides a reason for the observation that deviations from expected utility theory are stronger under uncertainty than risk – the additional processing causes a compounding in the deviations from expected utility theory.

However, the current model could be a useful descriptive tool when applied to certain circumstances where the shape of the distributions is known reasonably well. In particular, this may be the case in professional portfolio management in fairly liquid markets where large amounts of information pertaining to historic returns is both available and utilised in order to assess risk. Of course it may also be the case that it is these environments where the degree of non-linearity is least pronounced due to a number or factors: the use of standardised risk assessment models may increase the objectivity of decision making; the use of benchmarks to select assets in some cases and the tendency for managers to display herding behaviour may mean that asset selection is driven more by social and group standards in the marketplace than by individual decision biases;
in addition, the imperfect link between portfolio returns and the personal gains and losses of the decision maker may distort a number of the psychophysical processes underlying the current model.

6.3 Psychological Validity of Extension to Continuous Distributions

A central question is whether individuals really do evaluate prospects with continuous (or nearly continuous) distributions in something approximating this manner. Indeed, there is an issue as to whether anybody is capable of simultaneously grasping the implications of a complete distribution of this nature, even when objectively given, and the experimental evidence regarding this topic relates to subjective, rather than objective assessments of probability distributions (Lichtenstein, Fischhoff et al. (1982) and Griffin and Tversky (1992)).

In practice, of course, the distribution of returns for an asset will normally not be objectively defined, unless the evaluation is being made by experts who know the precise nature of the underlying distribution. Overall, whilst CPT can be extended to continuous probability distributions, the situation is complicated by the subjective nature of assessments of these distributions. In addition, it is questionable whether the weightings provided by CPT truly reflect the process by which individuals evaluate such distributions. On one level the model implies that individuals proceed by somehow applying a complex weighting of outcome values across the entire distribution, rather than assessing only certain characteristics of the shape of the distribution (e.g., mean and standard deviation), or by referring to specific points on the distribution (for instance the probability of getting more than some specified aspiration level as discussed by Lopes and Oden (1999)) when making decisions.

This, however, is to mischaracterise the nature of the model at hand. CPT is unashamedly an “as if” model of human choice which incorporates observations of human choice into a descriptive model of the outcomes of the choice process, although it has the advantage
over many other "as if" models in that its parameters have natural psychological interpretations. It is not necessary that the model accurately models the decision process itself as long as it takes into account the observed regularities of the outputs of such a process. Even the apparently simpler models of probability distribution evaluation just mentioned above do not propose to model the actual decision process accurately, but are rather themselves “as if” models wherein the outcomes are modelled using a different description of the distribution input. Nevertheless, given that the choice regularities used in prospect theory were observed largely in much simpler choice scenarios, it is a legitimate concern to enquire whether they are still applicable when extended to the more cognitively complex world of continuous distributions.

In the case of individuals making portfolio choices from an extensive choice set involving differing risk profiles and frequently quite similar options which are presented in different frames, it is possible that choices modelled in the relatively “pure” psychophysical world described by CPT are swamped by alternative factors. The nature of the risk itself may be distorted by the framing of the problem. In addition, the choice has been shown to be strongly influenced by the perceived options available. Benartzi and Thaler (2001) provide evidence for the $1/n$ heuristic of choice in defined contribution saving plans where some investors tend to divide their investments evenly across the funds offered to them, regardless of the overall risk profile thus achieved. The existence of such naïve strategies could certainly indicate that many of the subtleties of choice exhibited by people in simple choice situations are easily masked by situational variables and alternative psychological considerations in complex environments.

7 Conclusions

In this article we have examined the implications of Cumulative Prospect Theory for individuals’ optimal asset allocation between risky and risk free assets. In doing so we extend the model of
Hwang and Satchell which examined optimal asset allocation in a world where utility is measured in terms of changes from a reference point, and where loss aversion leads investors to weight losses more than gains. These elements, however, only constitute part of the CPT framework: also required is the degree to which probability assessments are reweighted in a nonlinear way.

To incorporate nonlinear probability weighting into an asset selection problem, the CPT framework was extended to cover continuous distributions, rather than the discrete (and frequently very simple) gambles to which it had previously been applied and in which environments it had been tested. In the continuous form CPT with nonlinear decision weights is equivalent to multiplying the objective density function by the slope of the weighting function at the appropriate point on the objective cumulative distribution function for losses, and on the objective decumulative function for gains. This multiplier is convex; highly nonlinear; greater than 1 and tending to infinity for the extreme tails of the objective distribution; discontinuous at the reference point; and less than 1 away from the tails of the objective distribution.

We derived the continuous weighting function and then incorporated this into the Hwang/Satchell model, using the Weibull form of the KST distribution. Our choice of Weibull was largely based on its ability to accurately fit excess returns data, but the model could be extended any distribution. The resulting model was calibrated against US equity excess returns data to investigate the feasible parameter space in which reasonable results would occur. This process delivered the following broad conclusions:

1. The results are in agreement with Hwang and Satchell’s result using the restricted loss aversion world. Specifically the curvature parameter of the utility function for loss is greater than that for gains ($l > g$). This restriction is both demonstrated in the data and can be justified by the assumption that the equity allocation is an increasing function of the probability that equity outperforms cash. If both curvature parameters are restricted to be less than one so as to display diminishing sensitivity to
outputs away from the reference point, then this implies the loss utility function is more linear than that for gains. Also, the two curvature parameters are required to be fairly close together if the model is to produce reasonable values for the optimal proportion of equity.

2. \( l > g \) implies a lower bound on loss aversion for any given combinations of the other parameters. This lower bound decreases as \( l - g \) gets smaller, reaching an absolute minimum at the limit where \( l = g \) of \( \frac{u^+}{u^-} \).

3. The optimal equity proportions are highly sensitive to the loss aversion parameter. Reasonable results can only be obtained for the model with nonlinear decision weights if this parameter is restricted to a band approximately bounded by 1.8 and 2.6. This is consistent with empirical estimates of this parameter, which have tended to be around 2 to 2.25.

4. The model provides reasonable results for the optimal proportions of equity for parameter combinations that are in the range of previous empirical estimates of these parameters. However, the parameters are not identifiable solely from the resulting equity allocation and cannot be assessed or justified in isolation.

5. The proportion of the portfolio invested in equity increases with \( \varphi \). In particular, as the decision weighting function becomes more inverse-S shaped (as would be predicted from empirical and theoretical considerations) the proportion of equity held decreases. Thus, greater deviations from choice behaviour consistent with the expected utility model with regard to nonlinear probability assessment would appear to decrease risk-taking behaviour for US equity returns.
We apologise to readers for the small print in the Appendix. Larger print copies available upon request from the corresponding author.

8 Appendix

8.1 Derivation of Imputed pdfs

Where excess returns are distributed according to

\[ f(x) = \begin{cases} \frac{q}{2} f^+(x) & \text{for } x \geq 0 \\ (1 - q) f^-(-x) & \text{for } x < 0 \end{cases} \] (40)

for which the cumulative distribution is

\[ F(x) = \begin{cases} (1 - q) (1 - F^-(-x)) & \text{if } x < 0 \\ (1 - q) + q F^+(x) & \text{if } x \geq 0 \end{cases} \] (41)

the continuous form of the CPT is given by

\[ V = \int_0^{w^-(1-q)} v(x) d\Pi^- (x) + \int_0^{w^+(q)} v(x) d\Pi^+(x) \] (42)

where

\[ \Pi^+ (x) \equiv w^+ (1 - F(x)) = w^+ (q (1 - F^+(x))) \] (43)

and

\[ \Pi^- (x) \equiv w^- (F(x)) = w^- (1 - q) (1 - F^-(-x)) \] (44)

for \( x \) in the range of gains and losses respectively. Also \( \frac{dF(x)}{dx} = f(x) \)
Where the outcomes are distributed according to a Weibull distribution we have:

### 8.2 Transformation of Objective Distribution to Imputed Distribution

#### 8.2.1 Gains

Where the outcomes are distributed according to a Weibull distribution we have:

\[
 f^+ (x; \alpha_1, \beta_1) = \frac{\alpha_1}{\beta_1^{\alpha_1}} x^{\alpha_1 - 1} e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}, \quad 0 < x < \infty
\]

\[
 F^+ (x; \alpha_1, \beta_1) = 1 - e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}, \quad 0 < x < \infty
\]
The imputed gains distribution $\pi^+(x)$ is arrived at by multiplying the objective distribution $f^+(x; \alpha_1, \beta_1)$ by the multiplier $m^+(x) = w^+(q(1 - F^+(x)))$ which, when using Prelec's weighting function is

$$m^+(x) = \frac{\varphi}{q(1 - F^+(x))} \left(- \ln q \left(1 - F^+(x)\right)\right)^{\varphi - 1} e^{-\left(- \ln q(1 - F^+(x))\right)^\varphi}$$

(55)

$$= \frac{\varphi}{qe \left(\frac{x}{\beta_1}\right)^{\alpha_1}} \left(- \ln q \left(\frac{x}{\beta_1}\right)^{\alpha_1} \right)^{\varphi - 1} e^{-\left(- \ln q \left(\frac{x}{\beta_1}\right)^{\alpha_1}\right)^\varphi}$$

$$= \frac{\varphi}{q} e \left(\frac{x}{\beta_1}\right)^{\alpha_1} \left(- \ln q \right)^{\varphi - 1} e^{-\left(- \ln q \left(\frac{x}{\beta_1}\right)^{\alpha_1}\right)^\varphi}$$

(56)

Therefore the imputed distribution is:

$$\pi^+(x) = m^+(x) q f^+(x; \alpha_1, \beta_1)$$

(57)

$$= \frac{\varphi}{q} \left(\frac{x}{\beta_1}\right)^{\alpha_1} \left(- \ln q \right)^{\varphi - 1} e^{-\left(- \ln q \left(\frac{x}{\beta_1}\right)^{\alpha_1}\right)^\varphi} q \left(\frac{x}{\beta_1}\right)^{\alpha_1} - \left(- \ln q \left(\frac{x}{\beta_1}\right)^{\alpha_1}\right)^\varphi$$

(58)

Where the prospect in question has only gains outcomes (i.e., $q = 1$) this may be simplified greatly:

$$\pi^+(x) = \frac{\varphi \alpha_1}{\beta_1^{\alpha_1}} x^{\alpha_1 - 1} \left(\frac{x}{\beta_1}\right)^{\alpha_1} \left(- \ln q \right)^{\varphi - 1} e^{-\left(- \ln q \left(\frac{x}{\beta_1}\right)^{\alpha_1}\right)^\varphi}$$

(59)

This implies that the valuation of a gains only lottery may be given by:

$$V = \int_0^\infty v(x) f^+(x; \varphi \alpha_1, \beta_1) \, dx$$

$$= \int_0^\infty x^g \frac{\varphi \alpha_1}{\beta_1^{\alpha_1 + \varphi}} x^{\alpha_1 - 1} e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}} \, dx$$

(60)
Let \( y = \left( \frac{x}{\beta_1} \right)^{\varphi_{\alpha_1}} \),
then \( x = \beta_1 y^{\varphi_{\alpha_1}} \)
and \( dx = \frac{\varphi_{\alpha_1}}{\beta_1} y^{\varphi_{\alpha_1} - 1} dy \)
Substituting,

\[
V = \frac{\varphi_{\alpha_1}}{\beta_1^{\varphi_{\alpha_1}}} \int_0^{\infty} (\beta_1 y^{\varphi_{\alpha_1}})^{\varphi_{\alpha_1} - 1} e^{-y} \beta_1 y^{\varphi_{\alpha_1} - 1} dy
= \beta_1^\varphi \int_0^{\infty} y^{\varphi_{\alpha_1}} e^{-y} dy
= \beta_1^\varphi \Gamma \left( \frac{\varphi_{\alpha_1}}{\beta_1} + 1 \right)
\]

\[8.2.2 \text{ Losses}\]
For losses the objective pdf is:

\[
f^- (-x; \alpha_2, \beta_2) = \frac{\alpha_2}{\beta_2^\varphi} (-x)^{-1} e^{-\left( \frac{-x}{\beta_2} \right)^{\alpha_2}}, \quad -\infty < x < 0
\]

and the associated cdf is:

\[
F^- (-x; \alpha_2, \beta_2) = 1 - e^{-\left( \frac{-x}{\beta_2} \right)^{\alpha_2}}, \quad -\infty < x < 0
\]

The multiplier function is:

\[
m^- (x) = w^- \left( (1-q) (1-F^- (-x)) \right)
= \frac{\varphi}{(1-q) (1-F^- (-x))} (-\ln (1-q) (1-F^- (-x)))^{\varphi-1} e^{-(-\ln(1-q)(1-F^- (-x)))^\varphi}
\]

\[
= \frac{\varphi}{(1-q)(e^{-\left( \frac{-x}{\beta_2} \right)^{\alpha_2}})} \left( -\ln (1-q) e^{-\left( \frac{-x}{\beta_2} \right)^{\alpha_2}} \right)^{\varphi-1} \left( e^{-\left( \frac{-x}{\beta_2} \right)^{\alpha_2}} \right)^\varphi
\]

\[
= \frac{\varphi}{(1-q)} \left( \left( \frac{-x}{\beta_2} \right)^{\alpha_2} - \ln (1-q) \right)^{\varphi-1} \left( \frac{-x}{\beta_2} \right)^{\alpha_2} \left( \frac{-x}{\beta_2} \right)^{\alpha_2} \left( -\ln(1-q) \right)^\varphi
\]

Combining these as before to form the imputed distribution \( \pi^- (x) \):

\[
\pi^- (x) = m^- (x) (1-q) f^- (-x; \alpha_2, \beta_2)
= \frac{\varphi}{(1-q)} \left( \left( \frac{-x}{\beta_2} \right)^{\alpha_2} - \ln (1-q) \right)^{\varphi-1} e^{-\left( \frac{-x}{\beta_2} \right)^{\alpha_2}} \left( \frac{-x}{\beta_2} \right)^{\alpha_2} \left( -\ln(1-q) \right)^\varphi
\]

\[
= \frac{\varphi_{\alpha_2}}{\beta_2^\varphi} (-x)^{\alpha_2-1} \left( \left( \frac{-x}{\beta_2} \right)^{\alpha_2} - \ln (1-q) \right)^{\varphi-1} e^{-\left( \frac{-x}{\beta_2} \right)^{\alpha_2}} \left( -\ln(1-q) \right)^\varphi
\]
And for loss only distributions \((1 - q = 1)\):

\[
\pi^-(x) = m^-(x) (1 - q) f^-(x; \alpha_2, \beta_2)
\]
\[
= \frac{\varphi_2}{\beta_2^2} (-x)^{\alpha_2-1} \left( \left( \frac{-x}{\beta_2} \right)^{\alpha_2} - \ln (1 - q) \right)^{\varphi-1} e^{-\left( \left( \frac{-x}{\beta_2} \right)^{\alpha_2} - \ln (1 - q) \right)^{\varphi}}
\]
\[
= \frac{\varphi_2}{\beta_2^2} (-x)^{\varphi-1} e^{-\left( \frac{-x}{\beta_2} \right)^{\varphi}}
\]
\[
= m^-(x) f^-(x; \varphi_2, \beta_2)
\]

(68)

Which means the evaluation of a loss only function follows as before:

\[
V = \lambda \beta_2 \Gamma \left( \frac{l}{\varphi_2} + 1 \right)
\]

(69)

**8.3 Derivation of \(u^+\) for Mixed Functions**

\(u^+\) is

\[
u^+ = qE_B (x^g| x > 0)
\]
\[
= \int_0^\infty x^g m^+ (x) qf^+ (x) \, dx
\]

(70)

where \(E_B\) is the behavioural expected value.

Combining the Weibull-KST distribution and the multiplication function to form \(u^+\):

\[
u^+ = \int_0^\infty x^g f^+ (x) qm^+ (x) \, dx
\]

(71)

\[
= \int_0^\infty \frac{x^g \varphi_1 x^{\alpha_1-1}}{\beta_1^2} \left( \left( \frac{x}{\beta_1} \right)^{\alpha_1} - \ln q \right)^{\varphi-1} e^{-\left( \left( \frac{x}{\beta_1} \right)^{\alpha_1} - \ln q \right)^{\varphi}} \, dx
\]
\[
= \frac{\varphi_1}{\beta_1^2} \int_0^\infty \frac{x^{\alpha_1+\alpha_1-1}}{\beta_1} \left( \left( \frac{x}{\beta_1} \right)^{\alpha_1} - \ln q \right)^{\varphi-1} e^{-\left( \left( \frac{x}{\beta_1} \right)^{\alpha_1} - \ln q \right)^{\varphi}} \, dx
\]

(72)

Let \(y = \left( \left( \frac{x}{\beta_1} \right)^{\alpha_1} - \ln q \right)^{\varphi}\),

then \(x = \beta_1 \left( y^{\frac{1}{\varphi}} + \ln q \right)^{\frac{1}{\alpha_1}}\)

and \(dx = \frac{\beta_1}{\varphi_1} \left( y^{\frac{1}{\varphi}} + \ln q \right)^{\frac{1}{\alpha_1}-1} y^{\frac{1}{\varphi}-1} dy\)

Substituting,

\[
u^+ = \frac{\varphi_1}{\beta_1^2} \int_{(-\ln q)^{\varphi}}^\infty \left( \beta_1 \left( y^{\frac{1}{\varphi}} + \ln q \right)^{\frac{1}{\alpha_1}} \right)^{\alpha+\alpha_1-1} y^{1-\frac{1}{\varphi}} e^{-\beta_1 \left( y^{\frac{1}{\varphi}} + \ln q \right)^{\frac{1}{\alpha_1}}-1} y^{\frac{1}{\varphi}-1} dy
\]

(73)

\[
= \beta_1 \int_{(-\ln q)^{\varphi}}^\infty \left( y^{\frac{1}{\varphi}} + \ln q \right)^{\frac{1}{\alpha_1}} e^{-y} dy
\]

(74)
8.4 Derivation of \( u^- \) for Mixed Functions

To solve over the loss domain, \( u^- = \int_{-\infty}^{0} (-x)^{1} (1-q) f^- (-x) m^- (x) \, dx \), the value of \( u^- \) follows as before:

\[
\begin{align*}
    u^- &= \int_{-\infty}^{0} (-x)^{1} (1-q) f^- (-x) m^- (x) \, dx \\
    &= \int_{-\infty}^{0} (-x)^{1} \pi^- (x) \, dx \\
    &= \int_{-\infty}^{0} (-x)^{1} \frac{\varphi \alpha_2}{\beta_2^2} (-x)^{\alpha_2-1} \left( \frac{-x}{\beta_2^2} - \ln (1-q) \right)^{\varphi-1} e^{-\left( \frac{-x}{\beta_2} \right)^{\alpha_2-\ln(1-q)} \varphi} dx \\
    &= \frac{\varphi \alpha_2}{\beta_2^2} \int_{-\infty}^{0} (-x)^{1+\alpha_2-1} \left( \frac{-x}{\beta_2} \right)^{\alpha_2-\ln (1-q)} \left( \frac{-x}{\beta_2^2} \right)^{\alpha_2-\ln(1-q)} \varphi dx
\end{align*}
\]

Let \( z = \left( \frac{-x}{\beta_2^2} - \ln (1-q) \right)^{\varphi} \), then \( x = -\beta_2 \left( z^{\frac{1}{\varphi}} + \ln (1-q) \right)^{\frac{1}{\alpha_2}} \) and \( dx = -\beta_2 \frac{\varphi \alpha_2}{\beta_2^2} \left( z^{\frac{1}{\varphi}} + \ln (1-q) \right)^{\frac{1}{\alpha_2} - 1} z^{\frac{1}{\varphi} - 1} \, dz \)

\[
\begin{align*}
    u^- &= \frac{\varphi \alpha_2}{\beta_2^2} \int_{-\infty}^{\infty} \left( \left( z^{\frac{1}{\varphi}} + \ln (1-q) \right)^{\frac{1}{\alpha_2}} \right)^{1+\alpha_2-1} z^{1-\frac{1}{\alpha_2}} e^{-z} \left( \frac{-x}{\beta_2^2} \right)^{\alpha_2-\ln(1-q)} \varphi \, dz \\
    &= -\beta_2 \int_{-\infty}^{\infty} \left( z^{\frac{1}{\varphi}} + \ln (1-q) \right)^{\frac{1}{\alpha_2}} e^{-z} \, dz \\
    &= \beta_2 \int_{-\infty}^{\infty} \left( z^{\frac{1}{\varphi}} + \ln (1-q) \right)^{\frac{1}{\alpha_2}} e^{-z} \, dz
\end{align*}
\]

8.5 Total Evaluation of Mixed Functions and Solution for \( \theta \)

The total evaluation of the prospect is

\[
\begin{align*}
    V &= -\lambda u^- + u^+ \\
    &= -\lambda \beta_2 \int_{-\infty}^{\infty} \left( z^{\frac{1}{\varphi}} + \ln (1-q) \right)^{\frac{1}{\alpha_2}} e^{-z} \, dz + \beta_1 \int_{-\infty}^{\infty} \left( y^{\frac{1}{\varphi}} + \ln q \right)^{\frac{1}{\alpha_2}} e^{-y} \, dy
\end{align*}
\]

and the optimal equity proportion \( \theta \), is given by
\[ \theta = \left( \frac{gu^+}{\lambda u^-} \right) \frac{1}{l - g} \]  

\[ \left( \frac{g/\beta_1}{\lambda/\beta_2} \right) \frac{1}{l - g} \left( \frac{g/y + \ln q}{z + \ln (1 - q)} \right) \]  

(77)  

(78)
References


### Table 1: Estimates of the Weibull-KST parameters for US and UK excess market returns

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Table 2: Proportions of wealth allocated to equity for combinations of $g$ and $l$, with linear decision weights ($\phi = 1$) and $\lambda = 2$
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Table 3: Proportions of wealth allocated to equity for combinations of $g$ and $l$, with nonlinear decision weights ($\phi = 0.74$) and $\lambda = 2$
Figure 1: PDF Multiplier, $m^-(x)$, for Losses with $\varphi = 0.74$

Figure 2: PDF Multiplier, $m^+(x)$, for Gains with $\varphi = 0.74$
Figure 3: Spliced Multiplier Function (with $\varphi = 0.74$ and $q = 0.8$)

Figure 4: Proportion of equity allocation ($\theta$) with $\lambda = 2$, and linear decision weights ($\varphi = 1$)
Figure 5: Proportion of equity allocation ($\theta$) with $\lambda = 2$, and linear decision weights ($\varphi = 1$) - detail for near linear utility functions

Figure 6: Proportion of equity allocation ($\theta$) with $\lambda = 2$, and $g = 0.7$
Figure 7: Proportion of equity allocation ($\theta$) with $\lambda = 2$, and $l - g = 0.25$

Figure 8: Proportion of equity allocation ($\theta$) with $\lambda = 2$, and $l - g = 0.25$ ($\varphi < 1$)
Figure 9: Proportion of equity allocation ($\theta$) with $\varphi = 0.74$, and $g = 0.7$

Figure 10: Minimum values for $\lambda$ if reasonable equity proportions are to be produced with $l - g > 0$
Figure 11: Minimum values for $\lambda$ for $\theta < 100\%$ with $l - g = 0.25$

Figure 12: Equity allocations with $g = 0.7$ and $l = 0.95$