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FALLING REAL INTEREST RATES, HOUSE PRICES, AND THE INTRODUCTION OF THE PILL

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Abstract

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1 Introduction

The fall in real interest rates experienced in the past several decades has been an area of intense academic debate in recent years. Real interest rates have been negative in the United States and the Euro-zone since 2008. This has important implications for returns on investment, financial stability, and also the operation of monetary policy subject to the zero lower bound, see Teulings and Baldwin (2015) for an overview of the Secular Stagnation debate. Simultaneous to the fall in real interest rates, house prices have grown substantially in real terms, see Knoll, Schularick, and Steger (2014). This house price growth has led to housing becoming the largest single asset on a household's balance sheet, see Piketty (2014). In this paper we show that changes in demographic structure due to the introduction of the pill can explain simultaneously the fall in real interest rates and the rise in house prices over the past decades. The cohort born just before the introduction of the pill are far greater in size than the cohorts born before and after. Hence, the age composition of the population deviates strongly from the composition in the balanced growth path either before or after the introduction of the pill. This leads to a disruption in the life-cycle saving patterns of the overlapping generations, which can explain the fall in real interest rates and the rise in house prices.

The pill was first approved by the US Food and Drug Administration in 1960, however it took until the late sixties before its use was widely spread among the young female population, see Goldin and Katz (2002). It led to massive and sharp decline in fertility rates across most developed countries, by about 30% or more roughly around the year 1970 (see our discussion in section 2). The fall in fertility had corresponding effects on cohort size. Some 20 to 25 years later, the event had an echo effect, when the first cohort of mothers born after the introduction of the pill were supposed to give birth to the next generation. Since these post-pill cohorts were 30% smaller than their pre-pill predecessors, cohort size dropped again by some 30%. As a consequence, cohorts born after 1990 were about half the size of the large cohort born before 1970, although the exact numbers differ somewhat between countries depending on the exact pattern in fertility, the age of giving birth and migration. Since the population was growing rather rapidly in most countries in the decades before, the large cohort born between 1964 and 1968 are far greater than cohorts born before and after. The current composition of the population deviates therefore strongly from the distribution in any balanced growth path, whether a pyramid (in the case of a growing population) or an inverted pyramid (in the case of a shrinking population).

Because cohorts seek to smooth consumption across their life-cycles, a cohort saves during the years it is active on the labor market, first to repay the (implicit) debt build up during

early stage of life for the accumulation of human capital, and then to accumulate assets to finance their consumption after retirement. The desired asset-holding by a generation is at its peak around the age of retirement (around 65 in most countries). In a balanced growth path, this life-cycle pattern in the asset accumulation of any cohort is smoothly accommodated by the other cohorts which are in a different stages of their life-cycles. The large cohort currently aged between 45 and 49 disrupts this accommodating process. These cohorts are accumulating assets to finance their future consumption, while the usual absorbers of these savings, the retirees and the youngsters, are in short supply. Our analysis shows that the saving of this large cohort can explain the pattern of falling real interest rates. Our model predicts negative real interest rates in 2016, as is observed empirically. In fact, our model suggests that this downward trend will continue for another decade and half, overshooting the new balanced growth path level until hitting a trough around the year 2035. After 2035 the large cohort begins to retire and deplete their savings, thus reversing the trend of high savings, and eventually the real interest rate tends towards the new balanced growth path level. Note that Japan has an earlier large cohort, those born in the baby boom immediately after the Second World War. This cohort is so numerous that they dominate Japan's age distribution even in the year 2016. For this reason, Japan's age distribution leads that of the western world by about 15 years, and likewise our analysis when applied to Japan will occur 15 years earlier. In this sense, Japan's experience over the past two decades provides a good laboratory for countries whose large cohorts are 15 years younger, for example Germany.

Samuelson (1958) was the first to derive the balanced growth path relation between the growth rate g and the return to capital r in an economy with overlapping generations of workers and retirees. Under some assumptions (see section 4), the rate of return to capital is equal to the growth rate: $r = g$. For our purpose, nothing is lost in considering a world without technological progress. In such a world, g is the growth of the workforce, which is equal to the growth rate of the population in a balanced growth path. The introduction of the pill reduced the long-run constant rate of population growth g, which shall eventually lead to a lower return to capital r due to the $r = g$ relation. The transition to the new balanced growth path equilibrium is however non-trivial. The cohort size distribution carries memory and generates persistence; the enormous size of the large cohort born before the pill makes our demography of population more biased towards saving than any balanced growth path. Some 60 years after the introduction of the pill, this large cohort holds a large stock of assets to finance their future consumption. Hence, the supply of capital exceeds its demand during this episode, resulting in a fall in the real interest rate that overshoots the new balanced growth path level. Although this overshooting transitory, it is of first order importance as it lasts for several decades. The lower the elasticities of intertemporal

substitution in consumption and of substitution between capital and labor, the larger this effect. The empirical literature suggests these elasticities to be less than unity, in the order of magnitude of a half (see Chirinko (2004) and Havránek (2015)). Using these values, our calculations show that after an initial increase at the arrival of the shock, r falls gradually to a minimum that is about 2% below the new steady-value g. The trough in real interest rates occurs around 2035, some 65 years after the introduction of the pill.

An $r = g$ balanced growth path always exists, moreover this growth path will maximize the utility of the representative generation. However although the resource availability constraint is met every period, this outcome cannot be implemented easily as a market equilibrium. The critical condition that separates whether the market can generate the $r = g$ equilibrium asks whether the total net asset demand (across all cohorts) can be met by the size of the capital stock. When the size of the capital stock is insufficient to meet the demand of the saving cohorts, there is a problem of asset shortage and the economy is dynamically inefficient (with $r < g$). There are several solutions to this problem, some of them enforced by the government, i.e. either sovereign debt or a Pay-As-You-Go (PAYG) pension system, or we can have trade in bubbly assets, which is implemented by the market. Bubbly assets do not necessarily generate a real dividend but carry a positive price only because buyers expect these assets to carry a positive price in the future, see Tirole (1985). The possibility of trade in bubbly asset provides a market solution to the problem of dynamic efficiency. A bubble of the right size can provide sufficient assets to solve the asset shortage problem, thereby returning to the $r = g$ equilibrium.

We find that due the savings of the large cohort, there is a heightened period of acute asset shortage. This acute asset shortage requires a larger bubble to accommodate, and hence the growth of a bubble during this transition period is an efficient and perhaps unsurprising response to the introduction of the pill. The consumption smoothing of the large cohort comes as a cost. Due to the growth of the bubble, the large cohort buys the bubbly asset at an inflated price from the previous generation. This results in the previous generation earning a windfall profit in return for providing the supply of the bubbly asset. We may think of this windfall profit as a consumption smoothing tax that the large cohort pay for the service of using the bubble, however the large cohort is nonetheless better off as without this bubble they suffer even due to the lack of store of value. Although a larger bubble during this transition period may indeed be an efficient outcome that is sustainable under rational expectations (see our section 3), the appreciation of such a bubble is a feature that cannot be easily explained in rational framework. In particular rational bubbles without uncertainty appreciate at the real interest rate, however in our experience, and in our model in section 3, we see a fall in the real interest rate simultaneous to an acceleration of growth for the

bubble. For the purpose of our analysis, we content ourselves to offer an explanation for the fundamental driving forces behind the growth of bubbles. There are plausible explanations for the growth puzzle, for example speculative expectations or a process of learning, however it remains an open question that we remain agnostic towards from the perspective of this paper.

In practice, housing, in particular the land on which it is built, is an excellent candidate to serve as a bubbly asset. The supply of land is largely fixed, which is a favorable feature for a bubbly asset. Then, our analysis demonstrating the crucial role of bubbly assets during the transition phase has important policy implications. Policy makers and economists alike frequently voice worries about the increase house prices that have occurred in almost all countries since 1970, see Shiller (2007). They reason that these price increases are a bubble, usually interpreted as an irrational phenomenon that will lead to future disaster when the bubble bursts. However, the bubble in the value of housing may not be so irrational, serving the role of a store of value during the transition phase. Indeed, rational bubbles (without uncertainty) do not burst, although they may deflate gradually, contrary to a sudden collapse that characterizes irrational bubbles. This fits our general experience of long-standing house price appreciation over the past several decades. During episodes of low returns to regular investment, the purchase of bubbly assets is an efficient way to transact the intertemporal transfer of consumption between generations. In many definitions, Secular Stagnation is just that, a prolonged period of low returns to investment. Trying to avoid bubbles to pop up will just block the opportunity for cohorts to smooth their consumption over the life-cycle. Cohorts will seek alternative means to do so, potentially at higher cost. Hence in our view, an increase in the value of a bubbly asset such as housing is just a natural phenomenon during the transition to a lower rate of population growth.

Several past studies have looked to investigate the effect of demographics on real interest rates and house prices. Teulings and Gottfries (2015) argue that the increase in life expectancy not offset by an increase in the retirement age has raised savings supply. If this additional savings supply was directed to housing, it could explain the increase in house prices over the past decades. Rachel and Smith (2015) find that changes in demography has resulted in changes to the support ratio, the ratio of workers to the overall population. Taking an estimate for the elasticity of savings supply with respect to the support ratio, they estimate that demographic factors can lead to an increase in savings supply accounting for a 0.9% fall of real interest rates. Carvalho, Ferrero and Nechio (2016) make a similar argument and find that they can account for a reduction in the natural rate of interest of 1.5%. Goodhart and Erfurth (2014) consider future support ratios and project that support ratios will fall in the future, thereby reversing the period of high savings and low interest

rates. They predict that by 2025 real interest rates shall return to their historical norm of 2.5-3%. Mankiw and Weil (1989) consider life-cycle variation in the demand for housing and thus the implications of demographic change on house prices. They project that house price growth will slow from the 1990s onwards. Bakshi and Chen (1994) extend their analysis to consider the effect of demographics on the capital market as a whole and its implications for real interest rates. While the above studies vary in their approach, their analyses all have in common the limitation of being static or steady state in some fashion. Consider an analysis based on the level of the support ratio, which argues that within a steady state equilibrium high support ratios correspond to low real interest rates. Although this steady state argument broadly captures the direction of the demographic change, it misses the magnitude and the rich dynamics of the demographic transition. Contrast this with our model, where the main results are derived precisely from the dynamic transition outside of steady states. Specifically, embedded within the changes of the support ratio is the presence of a very large cohort born before the introduction of the pill. Rather than a smooth transition through many steady state equilibria, the size of this particular single cohort means that the transition is instead characterized by the rapid accumulation of an enormous quantity of assets. The implication is a much larger response for real interest rates, which would otherwise be missed without a full model of the transition dynamics.

The structure of the remainder of this paper is as follows. Section 2 presents a discussion on the demographic structure of the world's four biggest economies, the United States, China, Japan, and Germany. We propose a model of demographics and demonstrate that it can explain accurately the evolution of demography since 1970. In section 3, we present a simple model to explain the effect of the large cohort's savings on real interest rates and the size of the bubble. In section 4 we set out a general overlapping generations model of real interest rates. We calibrate this model and simulate the resulting real interest rate transition path. We conclude in section 5.

2 Demographic transition since the introduction of the pill

Figure 1 provides the age pyramids for the world's four largest economies, the United States, China, Japan, and Germany, in 2014.[1]

[1] Figure 1 is taken from http://www.indexmundi.com/germany/age_structure.html

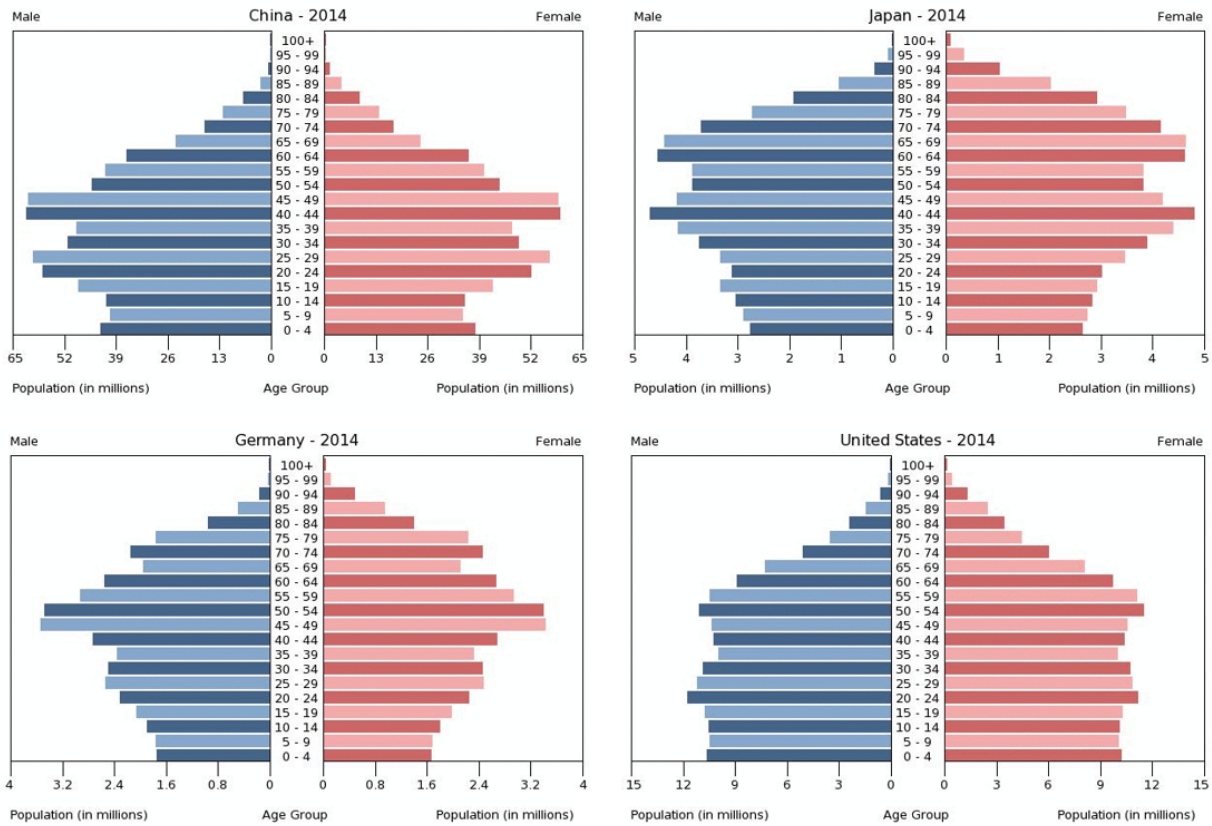

Figure 1: Age pyramids of the world's four largest economies.

All four graphs reveal a clear difference compared to the traditional graphs that applied 50 years ago and that do still apply for many developing countries: the pyramid has been replaced by a pillar (but with more mass in the middle section). However, at a more detailed level, the graphs also reveal clear differences between countries. For the sake of our discussion, we ignore the top of the pillar above the age of 65, which largely reflects the effect of a declining fraction of a cohort surviving up until the respective ages. Instead, we focus on the size of cohorts at lower ages, which largely reflects differences in the number of births over time, and in some cases subsequent migration flows. First, we may identify the relative size of the large cohort born before the introduction of the pill, who are aged around 45 in 2014. In each country we see a fall in cohort size as we move ahead in time, although the depth of this fall varies from country to country. The graph for the United States shows that they experience the smallest drop in cohort size following the introduction of the pill; their age pyramid comes closest to the ideal type of a pillar. We see that the biggest cohorts (age 20-24) are only about 10-15 % larger than the smallest (age 30-39). Moreover, cohort size does not reveal a clear systematic pattern over time, fluctuations being more or less erratic. The fall in fertility is offset by an inflow of new migrants, and hence the shape of the

American age pyramid is therefore largely consistent with a constant growth rate pattern associated with a stable population.

This is not the case in the other three countries, where roughly speaking, the bottom of the traditional age pyramid has been replaced not by a pillar but by an inverted pyramid, resulting in a diamond shape. However, there are clear difference even between these three countries. Japan stands out as its oldest large cohort (age 60-69) lead the largest cohorts in China (age 40-49) and Germany (age 45-54) by 20 and 15 years respectively. For Japan, their large cohort is those who were born in the baby boom immediately after the Second World War. In all three countries, there is an echo effect of the elderly large cohort some 20 to 25 years later, when the large cohort of fertile women gave rise to a boom in newborn babies. In Japan today, this echo cohort (in particular age 40-44) is even larger than the cohort of their parents due to mortality at the higher ages.

Germany stands out as the country among the four with the most clear diamond shape. After the introduction of the pill in the 1960s, fertility rates dropped by some 30%.[2] Again there is an echo effect some 20 to 25 years later, when the cohort of mothers declines by 30% due to the drop in fertility two decades before, leading to further 30% decline in cohort size. From then on, cohort size keeps shrinking steadily, since the fertility rate is below the reproduction rate. Looking in 2014, the smallest cohort is less than 50% of the largest cohort. We claim that the demographic transition experienced in each of these countries, particularly so for Germany, can be explained by a simple model of demographics where a country suffers a fertility shock around the time of the introduction of the pill.

2.1 A model of demographic transition

Suppose that households live for a total of J periods. Define the size of the cohort born at period t to be N_t. The total population of all those alive at period t is P_t, which is given by

$$P_t = \sum_{i=0}^{J-1} N_{t-i}. \tag{1}$$

Those between the ages of $\underline{F}$ and $\overline{F}$ are fertile. They determine the size of the newborn cohort according to

$$N_t = b_t \sum_{i=\underline{F}}^{\overline{F}-1} N_{t-1-i}, \tag{2}$$

[2]The actual fall in fertility in Germany was even larger, from 2.5 in 1967 to 1.4 1971. However, the fertility rate was exceptionally high for a short period between 1963 and 1967, being closer to 2 in the years before. See https://embryology.med.unsw.edu.au/embryology/index.php/Germany_Statistics

where b_t is the birth-rate at time t.

From these equations, we may derive the constant rate of population (and cohort size) growth for a given fixed birth-rate b. By equation 1, we see that if population grows at a constant rate of g_P and cohort size grows at a constant rate of g_N, then it must be the case that $g_P = g_N$. From here we shall refer to this common growth rate as simply g. We may rewrite equation 2 to

$$b = \frac{1}{\sum_{i=\underline{F}}^{\overline{F}-1}(1+g)^{-i}}, \tag{3}$$

which gives us a monotonically increasing relationship between the birth-rate b and the constant rate of population growth g.

Now we consider the effect of the pill. Supposing that the population is initially growing at the constant growth rate associated with $b = b^H$, given by equation 3, we model the effect of the pill as the following fertility shock

$$b_t = \begin{cases} b^H, & t < t^* \\ b^L & t \geq t^* \end{cases}, where\ b^H > b^L. \tag{4}$$

Using equation 2, and noting that the initial demographic state was that of constant growth, we may generate the demographic transition as a result of the introduction of the pill. In particular we find that in our simulations, the growth rate of the population tends towards the new constant growth rate given by equation 3 with $b = b^L$. Hence the effect of the pill on demography is a transition from the initial state of constant high population growth, to the eventual state of low constant growth.

2.1.1 Calibration to Germany

We apply our demographic model of the pill to compare the Germany in 2014. Taking the fertile sub-population to be those between the ages of 18 to 28, we begin under the assumption that the country was initially growing at the constant rate consistent with a Total Fertility Rate (TFR) of 2.5. We model the fertility shock as a sudden drop in TFR from 2.5 to 1.4 in the year of 1970 (see figure 2). Figures 2 and 3 below show the fit of our demographic model.

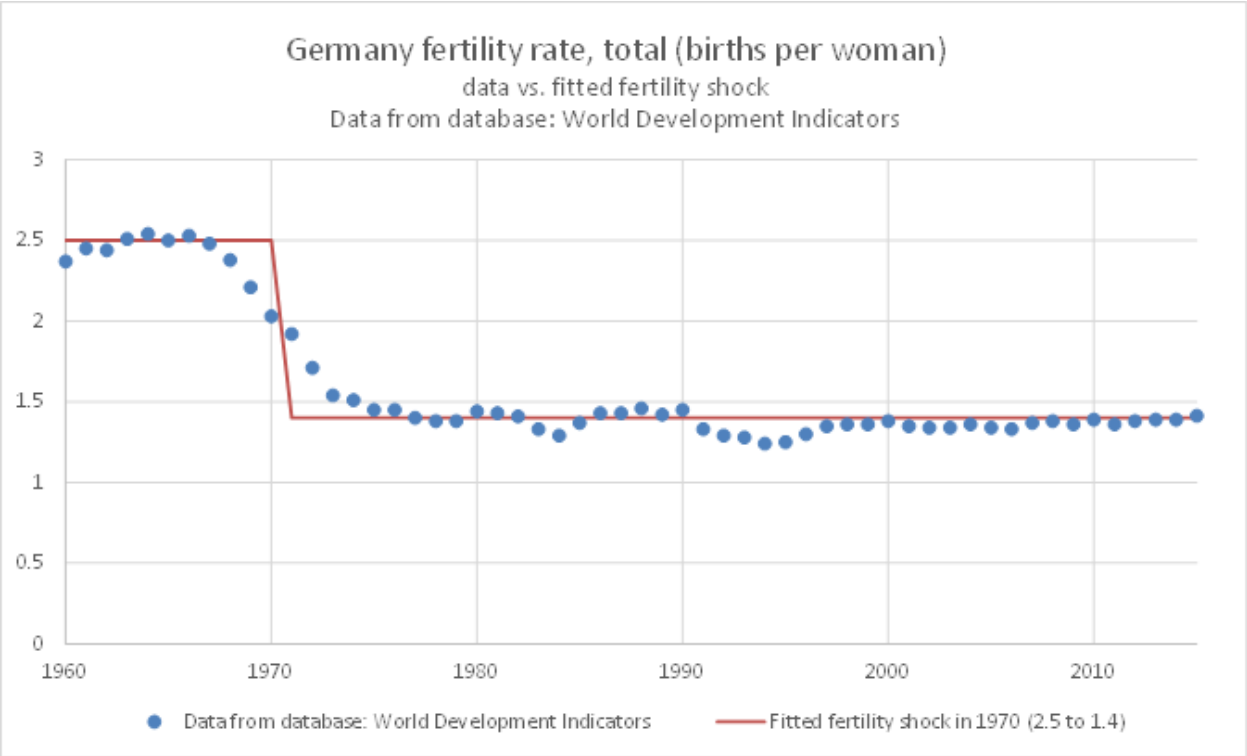

Figure 2: Germany's TFR and the impact of the pill.

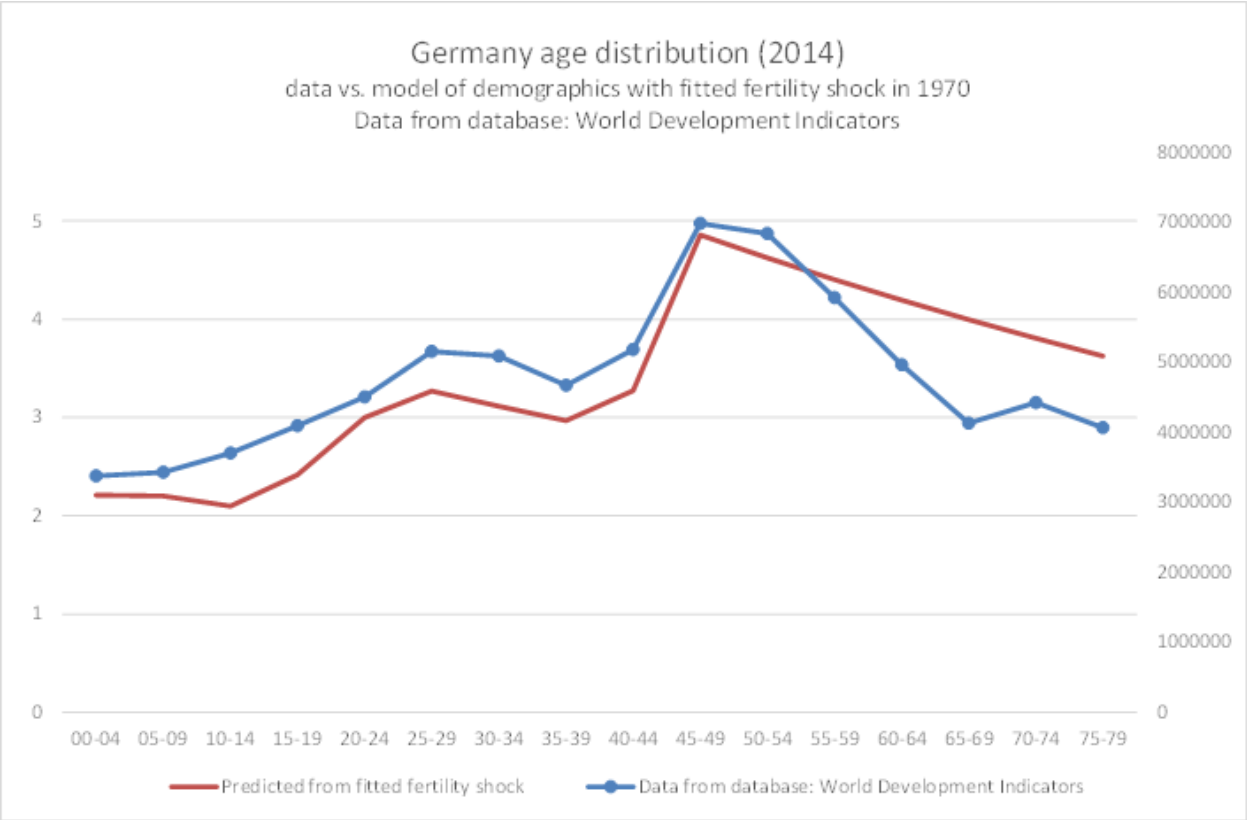

Figure 3: Germany's age distribution in 2014.

Looking at figure 2, we see that our assumption of a sudden fertility matches closely the data of Germany's TFR. In addition, from figure 3 we see that this simple fact can accurately explain the German age distribution in 2014, within the framework of our demographic model.

The important point to note is that the current age structure, in terms of its ratio between those approaching retirement to those who are young, is yet more extreme than the long-run state of constant rate population decay. Figure 4 contrasts the eventual constant rate of population decay, with our model's age distribution for Germany in 2014. We see in this comparison just how far off the current age distribution is, relative to even the case of constant population decay - there is considerably less mass in the young cohorts' part of the Germany's age distribution. This highlights the relative size of the large cohort born before the pill, and the relative scarcity of the absorbers of their savings (the younger cohorts). The implication is that our current demographic structure is intensely biased towards saving. Our hypothesis in this paper is that this transitional phenomenon has profound macroeconomic implications.

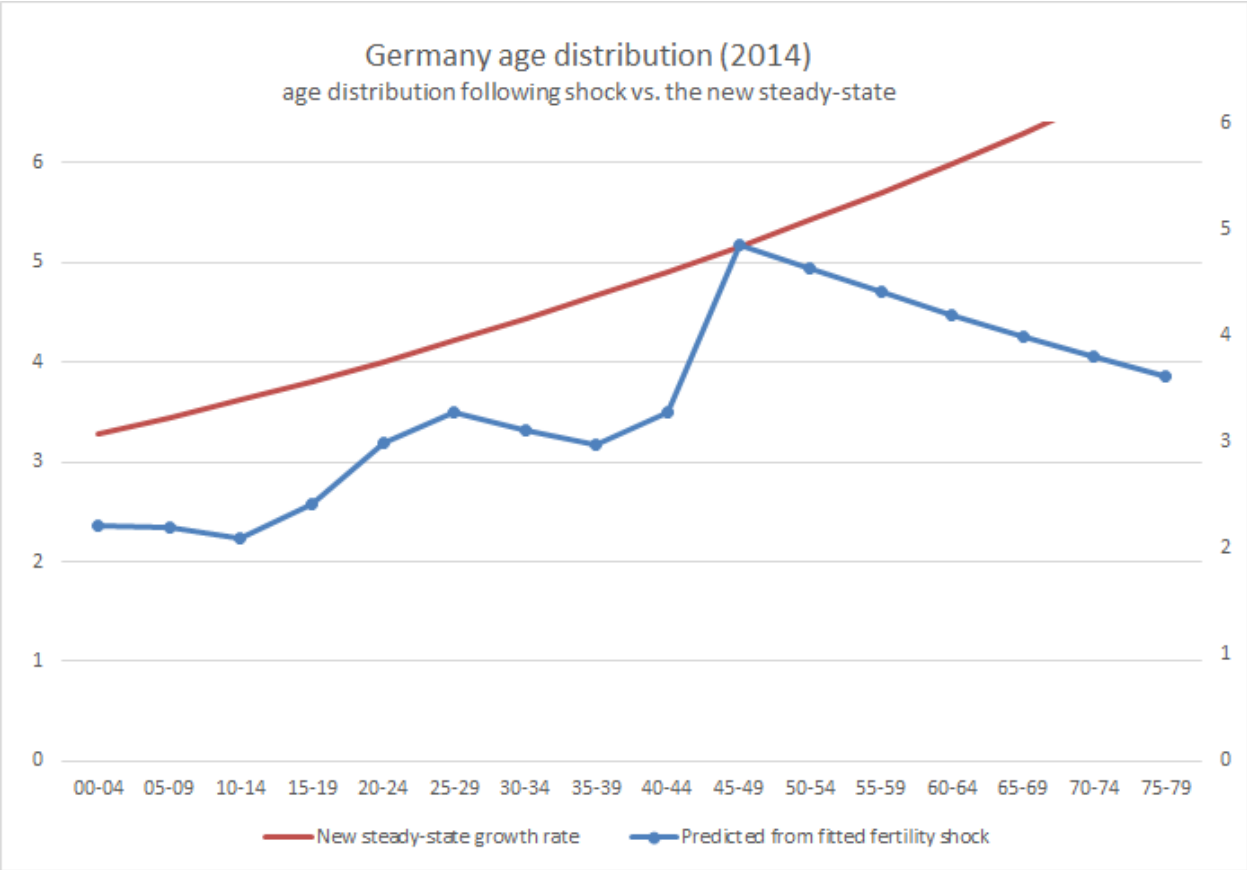

Figure 4: Germany's age distribution in 2014, contrasting with the eventual state of population decay.

This paper investigates the implications of this transitional age structure for the evolution of the real interest rate and house prices. We take this stylized model for Germany as a point of departure. Germany is not the only country in the Euro-zone facing this age structure. The graphs for Spain, Italy, The Netherlands, and Belgium look similar, somewhat more extreme for Spain and Italy, somewhat less extreme for The Netherlands and Belgium. Only France has a substantially different structure, similar to the pillar shape observed for the United States. To the extent that the Euro-zone has a closed capital market, our analysis therefore applies to the Euro-zone as a whole. The previous discussion suggests that this demographic pattern holds for Japan and China too, but with two differences, one in timing and another in magnitude. Regarding the timing: Japan runs 15 years ahead of Germany (and the rest of Euro-zone, except for France). Hence, if our analysis of the impact of demography covers an important driver of the macroeconomy, the evolution of Japan provides a nice laboratory for the future of the Euro-zone. China, for that matter, lags 5 years behind the Euro-zone. Regarding the magnitude: the magnitude of the distortion to age structure, in Germany and

the Euro-zone, dwarfs that of Japan. Hence, if Japan is a good a laboratory for the future of the Euro-zone, the effects will be much larger there in the Euro-zone than in Japan.

3 A simple model of transition dynamics

In this section we present the most simple model that demonstrates still the main mechanisms of our paper, namely that the savings of the large cohort leads to a period of acute asset shortage, which results in both a fall in the real interest rate and the growth of a bubble. Our model considers households that live for two periods. In the first period of their lives the households work and earn an income, some of which they must look to save in order to finance consumption in the second period (in retirement). The fertility shock leads to a disproportionately large cohort born just before the introduction of the pill, which we model in our two-generation economy as a one-period positive shock to cohort size. We show that the saving of the large cohort leads to a fall in real interest rates and an increase in the price of the bubble.

A household born in period τ has a CES utility function

$$U_\tau = \begin{cases} \frac{{c_{\tau,0}}^{1-\theta}}{1-\theta} + \beta \frac{{c_{\tau,1}}^{1-\theta}}{1-\theta} & for\ \theta \neq 1 \\ log(c_{\tau,0}) + \beta log(c_{\tau,1}) & for\ \theta = 1 \end{cases},$$

where $c_{\tau,i}$ denotes the consumption of the cohort born in period τ at age i. This leads to the following Euler equation

$$c_{\tau,1} = \beta^{1/\theta}(1 + r_\tau)^{1/\theta} c_{\tau,0}. \tag{5}$$

Capital fully depreciates between each period, hence

$$K_{t+1} = I_t,$$

where I_t is the level of capital investment in period t.

Production is Cobb-Douglas with respect to capital and labor

$$Y_t = K_t^\alpha L_t^{1-\alpha},$$

and perfect competition in the capital market gives

$$k_t = \left(\frac{\alpha}{r_t + 1}\right)^{\frac{1}{1-\alpha}}, \tag{6}$$

where k_t is the level of capital per worker.

Perfect competition in the labor market gives wages as a function of capital per worker

$$w_t = \alpha k_t^{\alpha-1}, \tag{7}$$

which is implicitly a function of r_{t-1} according to equation 6.

As a benchmark we begin by considering the case of a constant population. In this example, $g = 0$, and hence the dynamically efficient real interest rate is $r = 0$. Note that although this $r = g$ equilibrium always exists, it is not the only possible equilibrium. In a situation of asset shortage, r is smaller than g, the economy is dynamically inefficient, and integenerational transfers from the young to the old in each period, or alternatively trade in bubbles, increases the welfare of all future cohorts. In a situation of asset surplus, r is greater than g, and this situation is less easily resolved as you cannot improve all cohorts' welfare by saving more (as there is a welfare loss for the first cohort who saves more). Nonetheless the $r = g$ equilibrium is attractive as it maximizes the welfare of the representative household. In the case of our simple model, the only concern is the possibility of asset shortage. Whether or not a bubble is required to generate this equilibrium depends on if capital provides a sufficient store of value for the young cohort's savings. By solving for the supply of saving at $r = 0$ using equation 5, and comparing with the capital stock at $r = 0$, we arrive at the following condition for when there is a sufficient store of value (without a bubble)

$$\frac{1-\alpha}{\alpha}\frac{\beta}{1+\beta} \leq 1. \tag{8}$$

Our condition is exactly the same as (3.A22) in Barro and Sala-i-Martin (1999), when letting $g = 0$ and $\delta = 1$. The condition is more likely to hold for a high capital share α, and for a low β. Note as $g = 0$ in this example, the elasticity of intertemporal substitution, $1/\theta$, does not affect this condition. Specifically for $\alpha = 0.3$ and $\beta = 1$, this condition does not hold and hence a bubble is required for generating the dynamically efficient outcome.

A rational bubble, with fundamental value zero, provides the remaining store of value when equation 8 does not hold. Taking the supply of the bubbly asset to be fixed at 1, the size of the bubble is equal to its price p_t. Rational bubbles appreciate at the rate of the real interest rate, hence

$$p_{t+1} = (1 + r_t)p_t. \tag{9}$$

As in Samuelson (1958), trade in the bubble can generate the efficient $r = g$ outcome despite the bubble having no fundamental value. We suppose that in our case of a constant population, the size of the bubble is such that we have the dynamically efficient $r = 0$

equilibrium.

Next we look to analyze the effect of the introduction of the pill within the framework of our simple model. In our simple model it is not sufficient to consider a sudden drop in the birth-rate. Consider for example the following demographic model in our simple two generations model: $N_t = b_t N_{t-1}$. Because households live only for two periods, a drop in the birth-rate leads to the immediate adjustment of cohort size growth to the new lower level. This misses the entirety of the demographic transition to the new constant growth rate, namely the presence of the large cohort born before the shock. That is not to say that the cohort born before the shock is not "larger": assuming b_t fell from $b^H > 1$ to $b^L < 1$ the cohort born before the shock is still larger than those born before or after. However they are not "large" in the sense that this cohort is exactly the size you would expect them to be, given the size of the next cohort and the rate of population decay. The presence of the large cohort is important as otherwise the transition to the new growth rate would be a simple gradual convergence process. It is exactly the perverse demographic transition that drives the magnitude of the response for the real interest rate, resulting in a real interest rate fall that overshoots the new BGP level (see section 4). For this reason we consider the following demographic shock to analyze the effect of the pill within our simple model.

Suppose initially that cohort size is at a constant level of N^L which we normalize to 1. At time t^* there in a one-period increase in cohort size to $N^H = 1.5$. After t^* cohort size returns to the original constant level of N^L. This is summarized by

$$N_t = \begin{cases} N^H = 1.5 & t = t^* \\ N^L = 1 & t \neq t^*. \end{cases} \tag{10}$$

The cohort of size N^H is the large cohort born before the pill. The cohorts born before or after are smaller by a factor of 1/3, which matches approximately the relative sizes of cohorts we see in many countries. For simplicity, we consider constant population both before and after the shock. This corresponds to a population growth rate of 0, which means that the BGP both before and after has a constant size of the bubble (without exponential growth or decay), with the same dynamically efficient real interest rate of $r = 0$.

Because the increase in cohort size is known to be transitory to the large cohort, they are aware that the supply of savings they generate is large relative to what the smaller future generations can absorb. We shall show that the savings supply of the large cohort has two consequences. Firstly, the increase in savings leads to a fall in its return. Hence real interest rates fall as the large cohort accumulates assets for retirement. Secondly, the increase in savings makes the shortage of assets problem even more acute. Hence the bubble needs to

appreciate if it is to facilitate the savings of the large cohort.

With the arrival of new information, we allow the size of the bubble to jump at time t^*. This can be interpreted as coordination of the large cohort to raise the size of the bubble to a new level. Thereafter the size of the bubble evolves according to equation 18. To analyze the effect of this shock, we look to find the size of the bubble required at time t^* so that eventually the transition path brings economy back to the $r = 0$ equilibrium. While this transition path, associated with this particular size of the bubble, is just one of the many that are feasible, it has several attractive features that make it insightful to study. Firstly, this is the only transition path that eventually tends to the pre-shock dynamically efficient steady state equilibrium (with $r = 0$). Secondly, this transition path dominates, in terms of efficiency, all other feasible transition paths. To see this, first note that a larger bubble must eventually lead to a real interest rate of greater than 0. Following the argument in Tirole (1985), whenever r_t exceeds $g = 0$ it must be the case that the bubble is unstable, violating the resource constraint in some future period. Hence all other feasible transition paths have a smaller size of the bubble at time t^*. With a smaller bubble at time t^*, we therefore have a path for real interest rates that is lower at every time period. For any individual cohort, the loss from the lower return on savings dominates the gain from higher income when working. Hence the fall in real interest rates constitutes a net utility loss for each cohort.

3.1 Transition path simulations

Figure 5 shows the transition path in our simple model, under our baseline parameter specification of $(\alpha, \beta, \theta) = (0.3, 1, 2)$. With the arrival of the shock at time $t^* = 0$, the savings of the large cohort pushes down the return to capital and hence the real interest rate. The acute asset shortage in this period also manifests itself by an increase in the size of the bubble. The greater size of the bubble acts to accommodate the savings of the large cohort by generating larger young-to-old transfers. With the passage of time, the effect of the shock dies out and real interest rates gradually recover to the pre-shock steady state level. Likewise the size of the bubble evolves according to equation 18 and returns to the pre-shock steady state level.

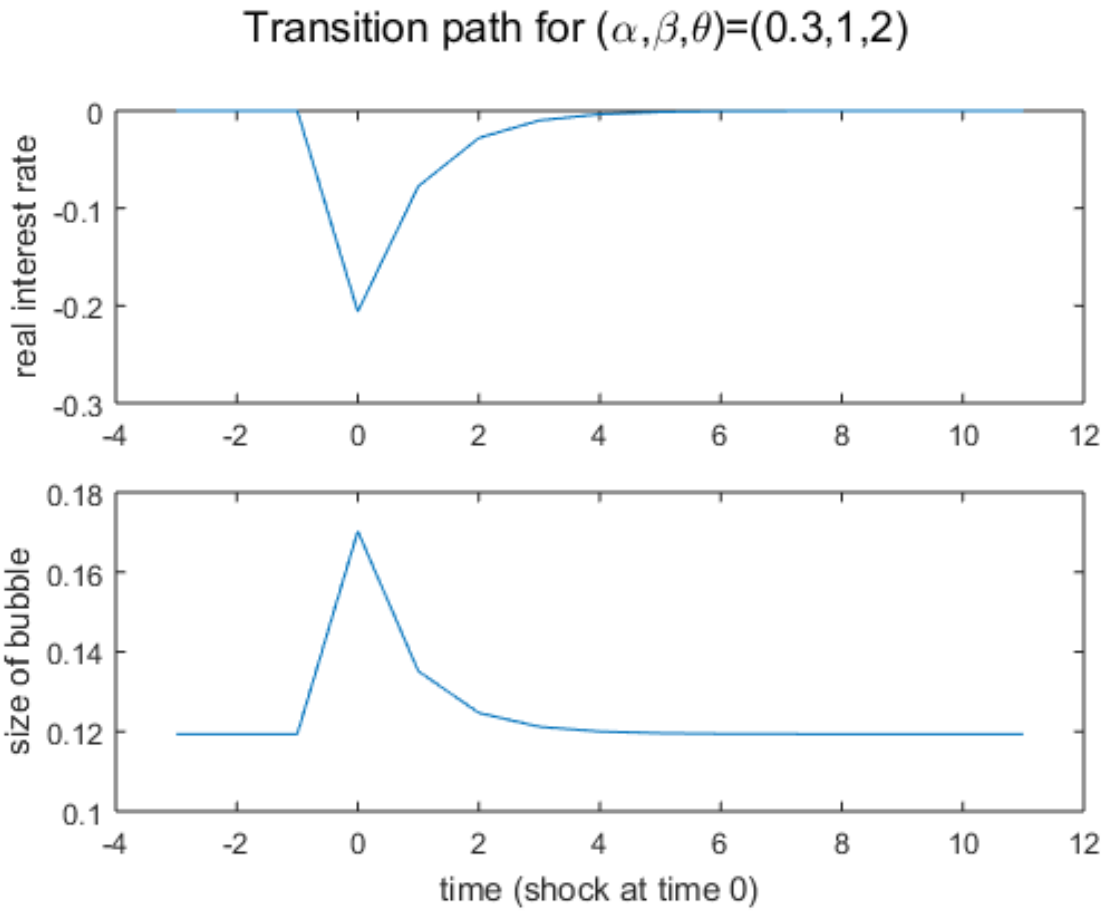

Figure 5: Real interest rate and size of bubble transition for $(\alpha, \beta, \theta) = (0.3, 1, 2)$.

The effect of the bubble is to reduce the fall in real interest rates in response to the shock from the pill. As discussed earlier, this makes retirement consumption cheaper relative to without the bubble, and this effect dominates the increase in wages due to a higher level of capital. Hence the bubble results in a net gain in welfare for all future cohorts. The increase in the size of the bubble means that the large cohort buys the assets from the previous generation at an inflated price. Nonetheless they are willing to do this, even despite the fact that the return on the bubble is negative, as they benefit by spreading the cost of their consumption shortage in retirement among all future cohorts. The windfall profit obtained by the last generation before the shock can be understood as a consumption smoothing tax on the large cohort, which the large cohort gladly pays in return for using the bubbly asset for consumption smoothing. In this light our bubble generates a smooth redistribution of wealth as the result of general equilibrium effects, and not as a result of government enforced transfers.

Regarding timing, the sharp response of real interest rates and the size of the bubble in

period t^* corresponds to the years that the large cohort is active on the labor market and saving for retirement. As the large cohort is born just before 1970, period t^* maps to a decrease in real interest rates and an increase in the price of the bubble over the period from 1985 to 2035. This fits broadly our experience over the past several decades, and specifically the exact transition of real interest rates will be examined closely in the next section.

While the households in this simple model are fully rational, the *growth* of the bubble in period t^* is not a feature that is explained by rational expectations. As previously discussed in the introduction, a rational bubble grows at the real interest rate. Hence a growing bubble is inconsistent with falling, and especially negative, real interest rates. In our simple model we abstract from this issue because in period t^* the size of the bubble is allowed to jump with the arrival of new information. Therefore the growth of the bubble in this period is not constrained to follow the rational expectations condition. In a full model with a finer discretization of time, this would not be so. Furthermore, the same critique would apply to any rational model that seeks to accompany falling real interest rates and a bubble that grows at an accelerating pace. In order to account for this phenomenon, one would need to introduce some kind of bounded rationality in the form of limited information. For example this could come from speculative expectations, or a process gradual learning. For our analysis we remain agnostic towards what explains the growth. Nonetheless we offer an explanation for the fundamental causes behind the growth of bubbles during the transition period, namely that in order to facilitate the savings of the large cohort a transitory period of greater intergenerational transfers is required. A period of greater intergenerational transfers is exactly what is delivered by the growth of a bubble.

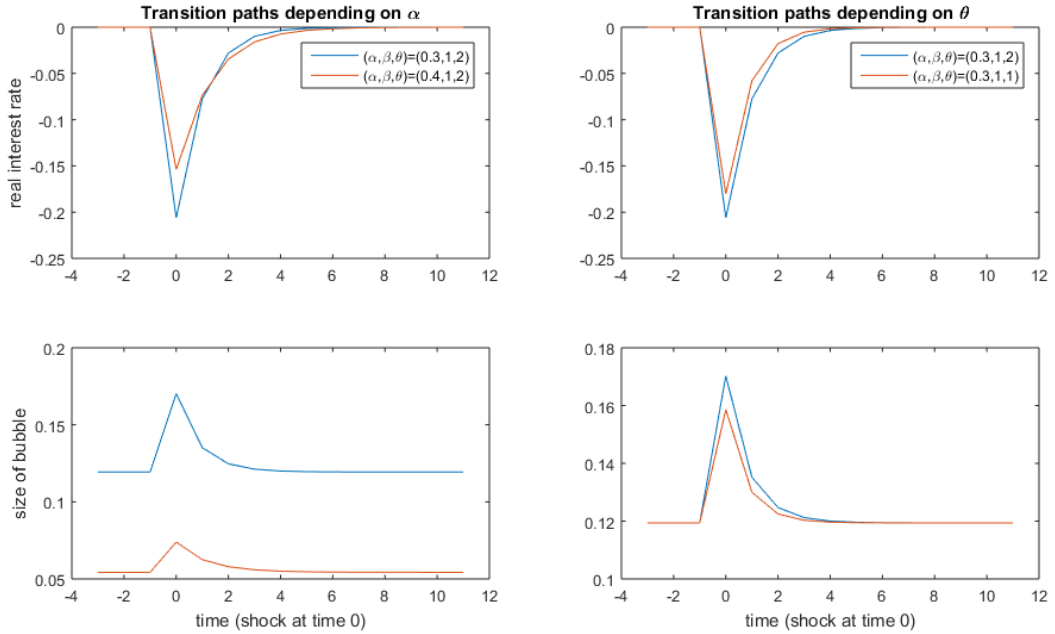

Figure 6: The effect of α and θ on transition dynamics.

Figure 6 shows how the transition dynamics are affected if we change our baseline parameter specification. The blue line remains always the transition path of our baseline specification with $(\alpha, \beta, \theta) = (0.3, 1, 2)$. We look to analyze the effect of deviations in the α and θ parameters. An increase in α increases the capital share of income, and hence increases the productivity of capital. This makes capital a better store of value, and we see this in the $(\alpha, \beta, \theta) = (0.4, 1, 2)$ simulation as the size of the bubble is smaller both in the steady states and in the transition periods.[3] Furthermore in this transition the real interest rate fall is also smaller. The elasticity of intertemporal substitution, $1/\theta$, measures the degree to which households are willing to substitute consumption across time (in response to intertemporal price changes). Although empirical estimates agree on $1/\theta$ being at around $1/2$, the widely used Cobb-Douglas utility function corresponds to an elasticity of $1/\theta = 1$. The implication of this assumption is that the income and substitution effects of real interest rate changes cancel out for savings supply; that is to say when the returns to retirement savings falls, households do not substitute towards saving more. Figure 6 shows the consequences of this assumption. The responses of both real interest rates and the size of the bubble is larger for the more realistic parameter specification with $\theta = 2$. Note however that θ does not affect the size of the bubble in the steady state equilibrium, unlike for parameter α.

[3]For $(\alpha, \beta, \theta) = (0.4, 1, 2)$, condition 8 is not met and hence a bubble (albeit smaller) is still required to generate the $r = g$ equilibrium. For $(\alpha, \beta, \theta) = (0.5, 1, 2)$, condition 8 is now met and there is no bubble in either the steady states nor the transition.

4 Real interest rate numerical simulations

We begin by laying the foundations of a general overlapping generations model. With this framework we first calibrate this model for Germany, and then use it to generate a detailed simulation for the path of the market-clearing real interest rates during the transition period of interest.

Firms produce output subject to a constant elasticity of substitution (CES) production function. Production takes two inputs, capital and labor, and exhibits constant returns to scale (CRTS):

$$Y_t = \begin{cases} \left(\alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha) L_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} & for\ \sigma \neq 1 \\ K_t^{\alpha} L_t^{1-\alpha} & for\ \sigma = 1 \end{cases} \tag{11}$$

In the CES production function, the elasticity of substitution between capital and labor is σ, where the limiting case of σ equal to 1 gives the Cobb-Douglas production function. Furthermore the limiting case of α equal to 0 gives the endowment income special case.

Capital accumulates according to

$$K_{t+1} = I_t + (1-\delta) K_t,$$

where $\delta \in (0, 1]$ is the depreciation rate of capital, and I_t is capital investment in period t.

The first order condition (FOC) of equation 11 with respect to capital, under perfect competition in the capital rental and capital investment market gives

$$r_{t-1} + \delta = \begin{cases} \alpha k_t^{-\frac{1}{\sigma}} \left[\alpha k_t^{\frac{\sigma-1}{\sigma}} + 1 - \alpha\right]^{\frac{1}{\sigma-1}} & for\ \sigma \neq 1 \\ \alpha k_t^{\alpha-1} & for\ \sigma = 1 \end{cases}, \tag{12}$$

and the FOC with respect to labor, under perfect competition in the labor market gives

$$w_t = \begin{cases} (1-\alpha) \left[\alpha k_t^{\frac{\sigma-1}{\sigma}} + 1 - \alpha\right]^{\frac{1}{\sigma-1}} & for\ \sigma \neq 1 \\ (1-\alpha) k_t^{\alpha} & for\ \sigma = 1 \end{cases} \tag{13}$$

where k_t denotes the level of capital per worker.

Next, we take our demographic model in section 2 to generate the transition of cohort sizes following the introduction of the pill. Households live for a total of J periods; a household

born in period τ has the following utility function

$$U_\tau = \begin{cases} \sum_{i=0}^{J-1} \beta^i \frac{c_{\tau,i}^{1-\theta}}{1-\theta} & for\ \theta \neq 1 \\ \sum_{i=0}^{J-1} \beta^i log c_{\tau,i} & for\ \theta = 1 \end{cases},$$

where again $c_{\tau,i}$ denotes the consumption of the cohort born in period τ at age i. We assume that households have access to a complete and perfect capital market, where they have the option of intertemporal trade through the purchase or selling of a one-period bond at the market real interest rate.[4] This utility function generates the following Euler equation

$$c_{\tau,i+1} = \beta^{1/\theta}(1 + r_{\tau+i})^{1/\theta} c_{\tau,i}, \tag{14}$$

which differs from the Euler equation in section 3 only due to the fact that a household's age, i, can now be greater than 1.

Each household receives a unitary labor endowment each period between entering the labor market at age χ, and leaving the labor market in retirement at age ψ. Households derive no utility from leisure, and hence they inelastically supply their labor to firms at the going market wage. Each household therefore earns a labor income for a total of $\psi - \chi$ periods, with life-time income for a household born at period τ being given by

$$W_\tau = \sum_{i=\chi}^{\psi-1} P_{\tau,i} w_{\tau+i},$$

where wage at time t is given by equation 13, and $P_{s,i}$ is the intertemporal price of period $s + i$ consumption in terms of period s income:

$$P_{s,i} = \prod_{j=0}^{i-1} (1 + r_{s+j})^{-1}.$$

The life-time budget constraint of a household born at period τ is given by

$$\sum_{i=0}^{J-1} c_{\tau,i} P_{\tau,i} \leq W_\tau. \tag{15}$$

[4]The assumption of trade through only one-period bonds simplifies our analysis at the exact moment of the fertility shock. During normal periods this assumption has no effect as each household can replicate any allocation achievable through general n-period bonds. However at the moment of the shock our assumption means that the household wealth effect of the shock is simply the change in the household's future discounted labor income. From this we are able to derive the exact consumption profile of each household in response to the surprise shock.

Finally the solution of the household's problem is given by the combination of the budget constraint in equation 15, and the Euler equation(s) given by equation 5.

Theorem 1. *Supposing that the population is growing at a constant rate of g. Assuming that $g > -\delta$, then there always exists a balanced growth path equilibrium characterized by a real interest rate of $r = g$.*

Theorem 1 states that when the size of the population grows at a constant rate of g, there always exists a balanced growth path (BGP) equilibrium where the real interest rate exactly equals the population growth rate. The proof of existence for the $r = g$ equilibrium in our general framework is given in our appendix; it follows approximately the argument given by Samuelson (1958), although he considers only the case of a two generation economy. Now we apply this to our demographic model of Germany from section 2, where those between the ages of 18 and 28 are fertile and each household lives for a total of 75 years. We begin with the $r = g$ BGP equilibrium for Germany which runs until the fertility shock which we take to occur in 1970. In 1970, the TFR suddenly falls from 2.5 to 1.4, and cohort size begins its transition towards the new lower (and negative) population growth rate. Our objective now is to calculate the real interest rate transition from the year 1970, as the population and the economy transitions towards the new BGP equilibrium.

The calculation of the transition path towards the new BGP equilibrium is highly non-trivial. Our baseline specification considers households that live for a total of 75 periods, and because each household is perfectly forward looking their consumption in each time period depends on the entire future path for real interest rates. We design an iterative adjustment algorithm that exploits the shared Euler equation of each household to narrow in on the equilibrium real interest rate path. The details of our algorithm is given in our appendix.

Below we show our results for the baseline specification of Germany. For the total 75 periods (years), a household enters the labor market at age 20 and retires at age 65. We take $\beta = 1$, $\delta = 0.075$, the elasticity of capital labor substitution to be $\sigma = 0.4$, and the elasticity of intertemporal substitution to be 0.5 (i.e. $\theta = 2$).[5]

[5]See Chirinko (2004), Havránek (2015) and Nadiri (1996) for estimates of the baseline specification parameters.

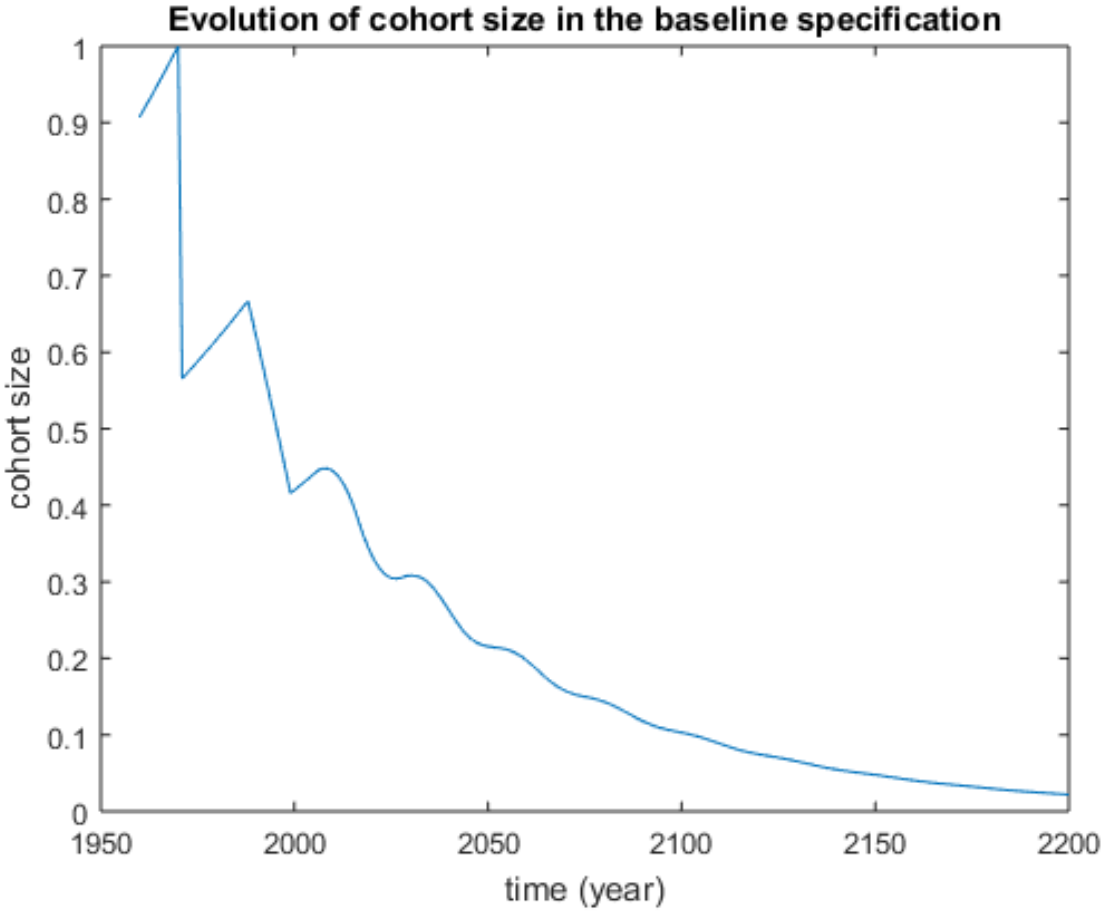

Figure 7: The demographic effect of the pill over a long horizon.

Figure 7 shows the evolution of cohort size after the introduction of the pill in 1970, over a very long horizon. Before the pill, population was growing at a constant rate of 1%. After the pill, the growth rate shall eventually converge to a constant decline of 1.5% per annum. However, due to persistence of the cohort distribution, it takes a significant amount of time before the economy converges to that new growth rate. After the introduction of the pill, there is an immediate steep decline, followed by a gradual increase because the size of the cohorts of mothers is still increasing. However after another 20 years or so, smaller cohorts enter the group of fertile women and hence cohort size starts declining again.

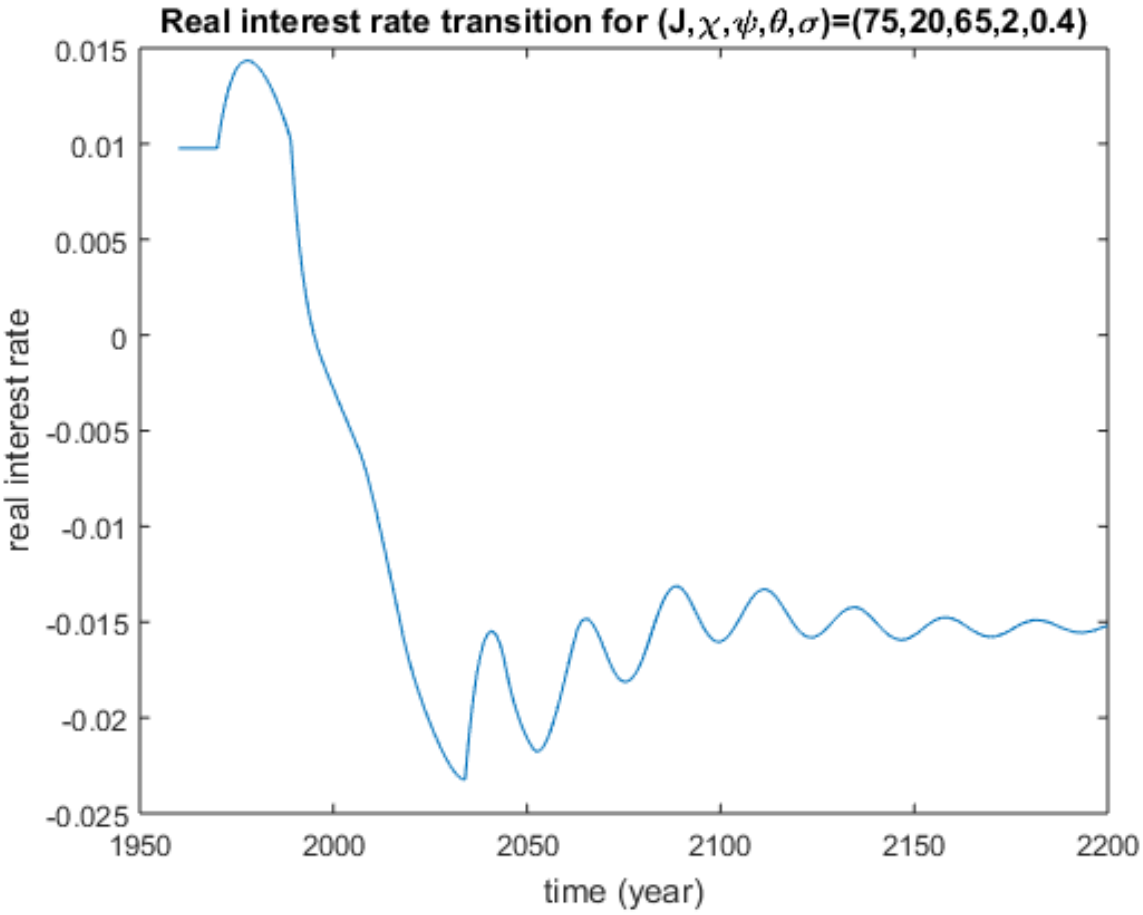

Figure 8: The evolution of the real interest rate over a long horizon.

Figure 8 shows the trajectory of the real interest rate for the baseline parameter specification. Initially, the real interest rate moves up. The composition of the population is hardly affected at that time since the number of smaller cohorts is still low, however the older cohorts feel richer because they realize that in subsequent periods the size of the cohorts borrowing for their education goes down, relaxing the tension on the goods market and making their claim of future output more valuable. Hence, they increase consumption, which drives up the real interest rate. From about 1985 onwards, the interest rate starts declining steeply, from 1.5% to -2% in 2035. What is striking is the length of the period for which the real interest rate declines. The downward pressure continues well into the 2030s when the large cohort begins to retire, and the depth of the trough significantly overshoots below the new BGP level.

After 2035, there is some recovery up to the level of the new BGP level, but this is only temporary as this uptick is just part of some residual cyclicality from the initial demographic shock. Each time when the original small cohort born just after the shock gives rise to a

new generation, there is another small fall in cohort size which is reflected in a fall in real interest rates. The length of one cycle is in our case is 23 years, which is the average age of a mother and hence the length of one generation. The shock's effect dampens quickly as the cohort size distribution tends to the new constant growth rate. This process is more-or-less complete by the year 2150.

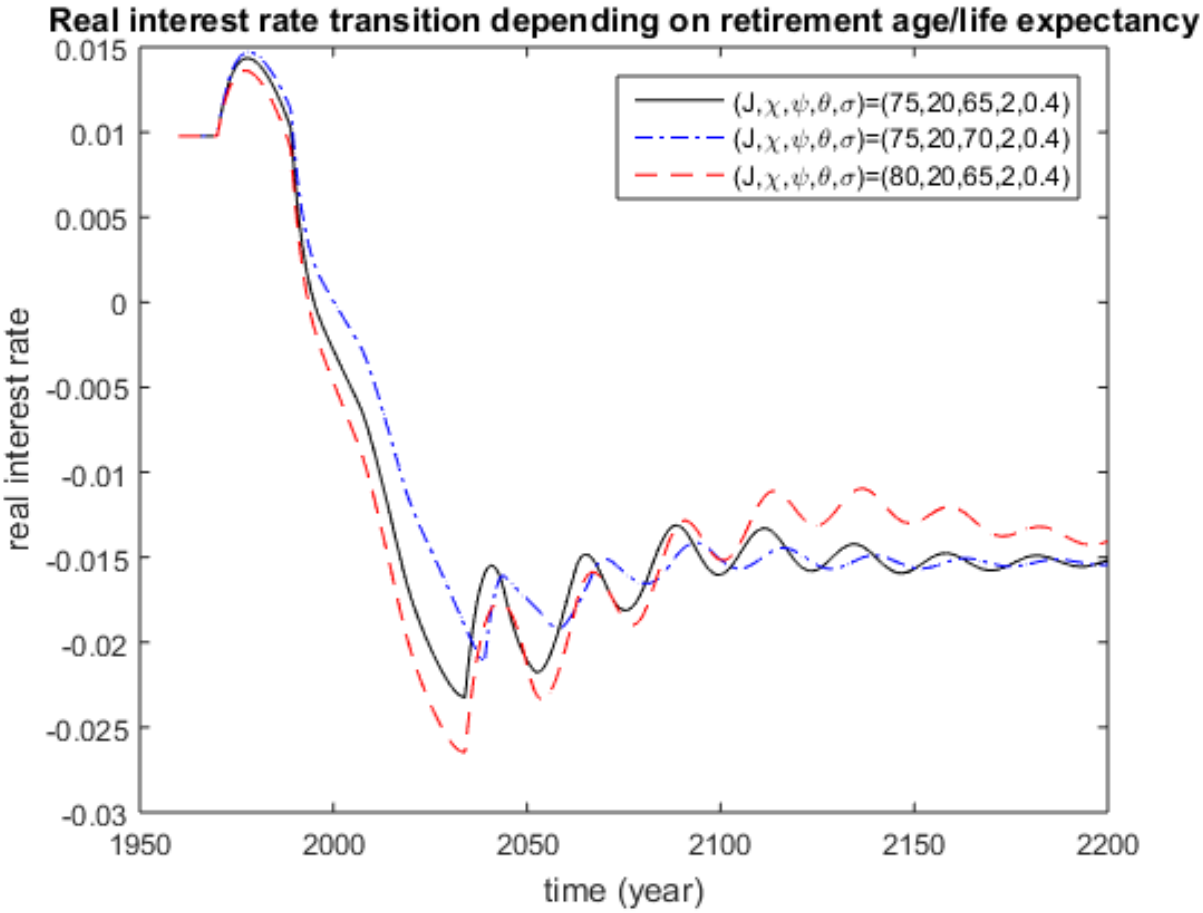

Figure 9: The effect of retirement age and longevity on the real itnerest rate transition.

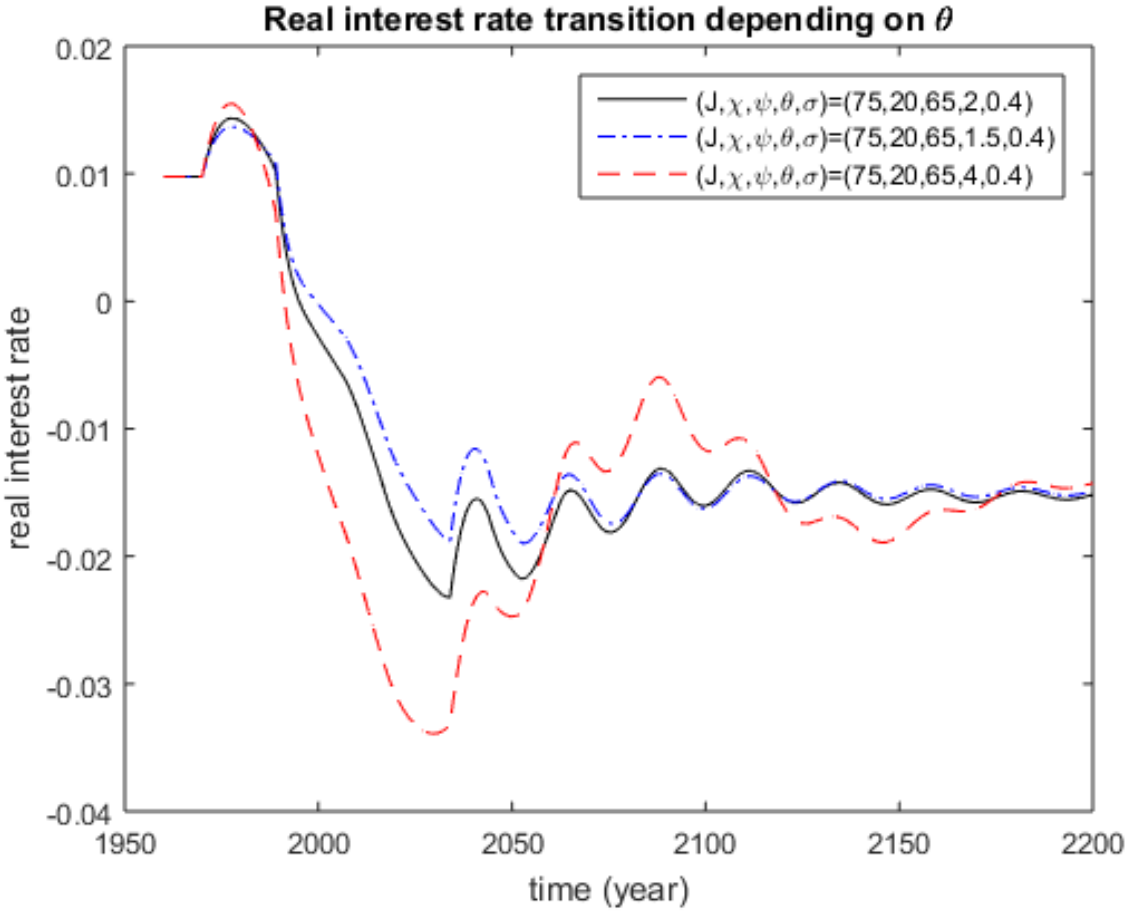

Figure 10: The effect of θ on the real interest rate transition.

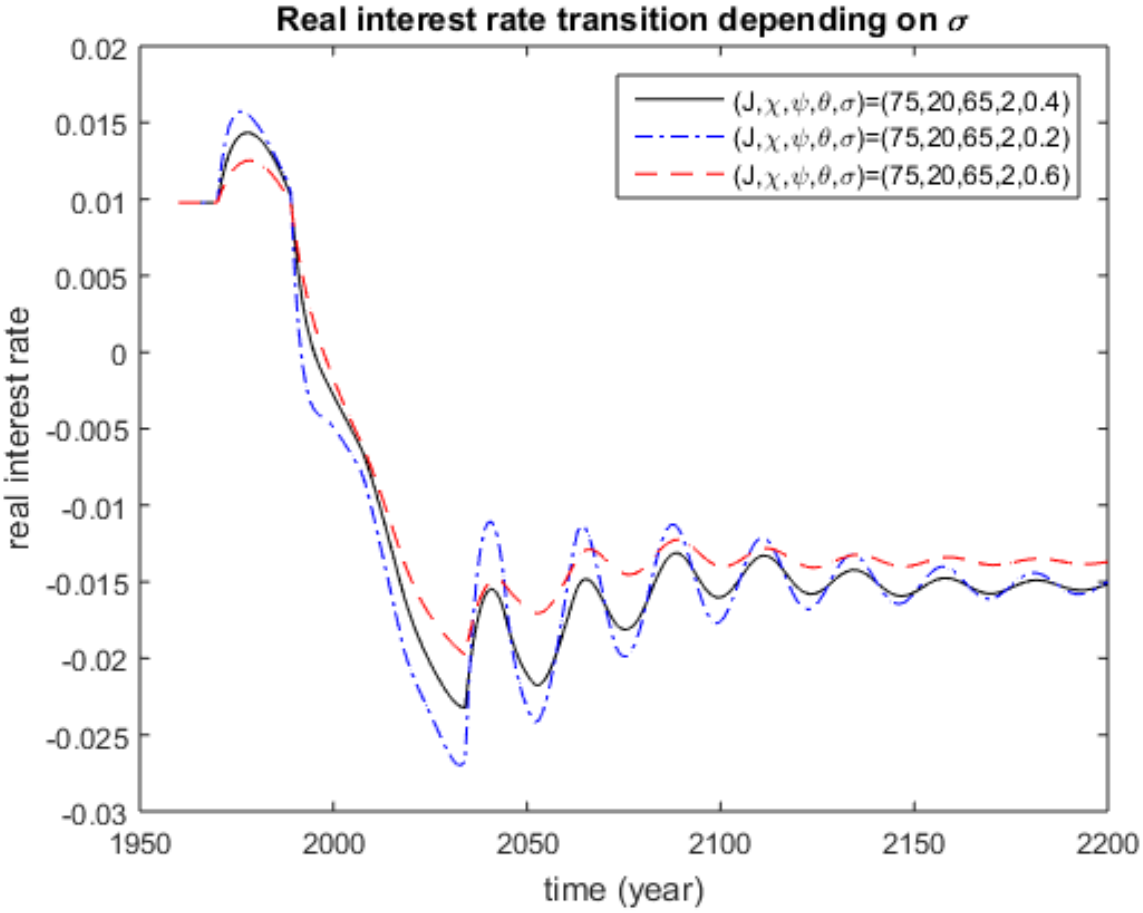

Figure 11: The effect of σ on the real interest rate transition.

A comparison of the trajectories for alternative parameter specifications provides an insight to the causes of the real interest rate fall and its depth. The trough in real interest rates is deeper for a longer longevity (holding retirement age constant), a lower retirement age (holding life expectancy constant), a lower elasticity of capital labor substitution, and for a lower elasticity of intertemporal substitution. None of these results come as a surprise. The first two facts are due to the resulting change to the number of years in retirement (which households need to save for). More years of retirement means a deeper fall in the real interest rate. In addition a higher retirement age postpones the timing of the trough since the large cohort continues the accumulate financial assets for a longer period. The effect of the elasticity of capital labor substitution matters, but not very much unless one really moves towards a Cobb-Douglas production function with the elasticity of substitution equal to unity. The cyclicality of the path becomes more pronounced for low values of the elasticity of substitution, as in these cases physical capital is less able to act as an external dampener to the variations in saving. Variations in the elasticity of intertemporal substitution matter

much more. A low elasticity (and therefore a high θ) means that households are less able to tolerate variation in consumption - hence this results in a deeper trough as households continue to save even when the return is strongly negative. For example for $\theta = 4$, the trough is much deeper and the shock takes much longer to fade out.

5 Conclusion

In this paper, we show that the fall in real interest rates and the growth of bubbles, in particular in housing, can both be explained by demographic changes as a result of the introduction of the pill. These demographic patterns are to varying degrees present in all high-income countries, and also China. This points to the conclusion that without some reversal of fertility rates, or a dramatic technological breakthrough, low interest rates are here to stay until at least around 2035. Specifically, Japan's demographic profile leads that of the western world by about 15 years, and we see that Japan has not escaped from its situation of low interest rates since it entered this period in the early 1990s.

Our analysis finds that the growth of a bubble during this transition period to lower growth may well be welfare-improving. By providing a much needed store of value in a period of acute asset-shortage, they help accommodate for the savings of the large cohort born before the pill. Although we find a case for the efficiency of a larger bubble during the transition, the actual process of growth appears to be inconsistent with rational expectations. Some form of bounded rationality is required to generate the growth of a bubble simultaneous to a fall in the real interest rate. Likewise psychological factors were probably at play. Further research on what could explain the mechanism of growth would be insightful for academic interest and policy alike. Nonetheless, policy that stifle the bubble in this transition process without providing an alternative only acts to prevent households from smoothing consumption. This will result in a deeper fall of real interest rates. In this light, there is a need for future study to find which policies are appropriate in this transition period of high savings, so that they generate the desirable effects of trade in bubbles but without the associated financial instability.

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Appendix

Proof of theorem 1

Proof. We look to first prove theorem 1 for the special case of an endowment economy with $\alpha = 0$. Using this result we shall generalize to general well-behaved CRTS production economies, which completes our proof.

We begin in our endowment income special case. First suppose that households are given the option to smooth their consumption across their life-times at the real interest rate g. This gives a set of desired intertemporal transfers for the representative household (representative as we consider a BGP). Suppose we start initially with no intertemporal transfers, so that for certain aggregate consumption equals aggregate income. We may introduce one desired intertemporal transfer at a time. For example suppose that each household wants to borrow one unit of consumption in the first period to be repaid by $1+g$ units of consumption in the second period. This transfer would lead to an expansion of the age one cohort's consumption by 1, and a decrease of the age two cohort's consumption by $1+g$. Because cohort sizes grow at rate g, this transfer does not break market-clearing. As each individual transfer does not break market-clearing, implementing the entire set of desired intertemporal transfers still maintain market-clearing. At this final allocation, each household is maximizing its utility subject to the real interest rate g, and we still have market-clearing, hence $r = g$ generates a BGP equilibrium.

Next we consider a CRTS production economy. We begin by supposing the BGP equilibrium real interest rate was g. We need to verify firstly market-clearing, and secondly the maximization of utility for the representative household.

$r = g$ implies a constant level for the level of capital per worker, and hence the equilibrium wage w. This means that the households labor income is equivalent to receiving a fixed income endowment of w in each period. We know from the endowment economy special case that aggregate consumption demand from all households, in a BGP with $r = g$, gives market-clearing in the endowment economy. This means that aggregate consumption must equal to aggregate labor income, which is w times the labor force.

Next, we use the fact that CRTS production functions allows us to decompose aggregate income exactly into aggregate labor income and aggregate capital income. We know that aggregate consumption equals to aggregate labor income. Therefore to show market-clearing, it suffices to show that aggregate capital income equals to aggregate investment.

Aggregate investment in the BGP must maintain the constant level of capital per worker implied by the real interest rate of g. Therefore we may write aggregate investment as the following function of the aggregate capital stock:

$$I_t = (g + \delta)K_t. \tag{16}$$

Likewise from the no-arbitrage condition on capital investment, it follows that

$$r + \delta = \partial Y_t / \partial K_t.$$

However under our initial assumption, $r = g$, therefore it follows that $\partial Y_t / \partial K_t = g + \delta$. Substituting $\partial Y_t / \partial K_t$ for $g + \delta$ in equation 16 shows that aggregate capital income (RHS) equals to aggregate investment (LHS). This verifies that we have market-clearing, and hence we conclude that $r = g$ generates a BGP equilibrium in the production economy.

□

Analytical solution for the simple model with C-D production

We consider an alternative specification of the simple two generation model, where we assume that the news of the demographic shock arrives before the consumption decisions at time $t = -2$ are made. This seems an unnatural assumption but it has the advantage that we can separate between the effect of the news and the effect of the adjustment itself, because at $t = -2$, the demographic shocks has no direct impact on the interest rate other than its effect of future interest rates. How will the economy deal with this shock?

For the sake of the exposition, we focus on the simplest case with Cobb Douglas preferences and technology $\theta = \sigma = 1$ and full depreciation $\delta = 1$. In that case

$$y = \alpha^{\alpha/(1-\alpha)}, k = \alpha^{1/(1-\alpha)}.$$

Furthermore, we focus on the case of asset shortage and hence trade in bubbly assets at the GGR path

$$\frac{\alpha}{1-\alpha} < \frac{\beta}{1+\beta},$$

see equation 8. The size of the bubble satisfies

$$p = \left(\frac{\beta}{\alpha} \frac{1-\alpha}{1+\beta} - 1 \right) k. \tag{17}$$

During the transition path, a rationale bubbles appreciates at the rate of the real interest rate

$$p_{t+1} = (1 + r_t)p_t. \tag{18}$$

Hence, p_{t+1} is the price retirees receive at period t for the bubbly assets and which they can

use for consumption at time t.

An equilibrium is a path for r_t for all $t \geq -1$ and the resale value of the bubble at $t = -2$, p_{-1}, that satisfies[6]

$$\begin{aligned}
1 &= \frac{\alpha}{\gamma}\left(1+\frac{p_{-1}}{k}\right)+\frac{\alpha}{\gamma}\left(1+r_{-1}\right)^{1/(\alpha-1)}, \qquad (19)\\
\left(1+r_{-1}\right)^{\alpha/(\alpha-1)} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}\left(1+r_{-1}\right)+\frac{\alpha}{\gamma}\left(1+r_0\right)^{1/(\alpha-1)}N_H,\\
\left(1+r_0\right)^{\alpha/(\alpha-1)} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}\left(1+r_0\right)\left(1+r_{-1}\right)^{\alpha/(\alpha-1)}N_H^{-1}+\frac{\alpha}{\gamma}\left(1+r_1\right)^{1/(\alpha-1)}N_H^{-1},\\
\left(1+r_1\right)^{\alpha/(\alpha-1)} &= \frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}\left(1+r_1\right)\left(1+r_0\right)^{\alpha/(\alpha-1)}N_H+\frac{\alpha}{\gamma}\left(1+r_2\right)^{1/(\alpha-1)},
\end{aligned}$$

and

$$\left(1+r_t\right)^{\alpha/(\alpha-1)}=\frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}\left(1+r_t\right)\left(1+r_{t-1}\right)^{\alpha/(\alpha-1)}+\frac{\alpha}{\gamma}\left(1+r_{t+1}\right)^{1/(\alpha-1)}, \text{ for } t>1, \qquad (20)$$

where $\gamma \equiv 1-\frac{1-\alpha}{1+\beta}$. All these equations have the same structure:.the left hand side is current output minus consumption of the current cohort of workers, the first term on the right hand side is consumption of retirees, and the second term on the right hand side is investment in next period's capital stock.

In the second line of 19, the demographic shock raises investment demand (resource shortage); in the third line, the demographic shock reduces consumption of retirees and investment relative to the current size of the workforce (resource abundance); in the third

[6]Since $k_{t+1} = I_t/N_t$ and

$$\begin{aligned}
c_{t,0} &= \frac{1}{\beta+1}w_t,\\
c_{t-1,1} &= \frac{\beta}{\beta+1}\left(1+r_t\right)w_{t-1},
\end{aligned}$$

equation **??**, **??**, and **??** can be written as

$$\begin{aligned}
k_t^{\alpha} &= c_{t,0}+c_{t-1,1}\frac{N_{t-1}}{N_t}+k_{t+1}\frac{N_{t+1}}{N_t},\\
k_t &= k\left(1+r_t\right)^{1/(\alpha-1)},\\
w_t &= \frac{1-\alpha}{\alpha}\left(1+r_t\right)k_t=\left(1-\alpha\right)y\left(1+r_t\right)^{\alpha/(\alpha-1)}.
\end{aligned}$$

Defining $\gamma = 1-\frac{1-\alpha}{1+\beta}$ and substitution of the last two equations in the first yields

$$\left(1+r_t\right)^{\alpha/(\alpha-1)}=\frac{\beta}{\gamma}\frac{1-\alpha}{1+\beta}\left(1+r_t\right)\left(1+r_{t-1}\right)^{\alpha/(\alpha-1)}\frac{N_{t-1}}{N_t}+\frac{\alpha}{\gamma}\left(1+r_{t+1}\right)^{1/(\alpha-1)}\frac{N_{t+1}}{N_t}.$$

line the demographic shock raises consumption of retirees (resource shortage). The first line of 19 deviates for two reasons. First, the capital labour ratio is at its steady state value, $k_{-2} = k$ since it was decided on before the information on the demographic shock arrived. Hence the reward for the factors of production is at its GGR value, so that $r_{-2} = 0$. Second, consumption of retirees is equal to the return to the capital stock, $(1 + r_{-2})\, k_{-2} = k$, plus the resale value of the bubbly assets bought last period, p_{-1}.

Equation 20 is a second order difference equation. It has an analytical solution[7]

$$1 + r_t = C^{\alpha^t},$$

where C is a constant of integration. For any α in the unit interval, $1 + r_t$ converges to unit for $t \to \infty$ and hence $r_\infty = 0$, which is the GGR equilibrium. Hence, equation 18 can be rewritten as

$$\begin{aligned} \frac{p}{p_{-1}} &= {}_{t=2}^{\infty}C_{t=-1}^{\alpha^t}{}^{1}\,(1 + r_t) = C_{t=-1}^{A}{}^{1}\,(1 + r_t)\,, \\ A &\equiv \frac{\alpha^2}{1-\alpha}. \end{aligned}$$

Substitution of 17 yields

$$\frac{p_{-1}}{k} = \left(\frac{\beta}{\alpha}\frac{1-\alpha}{1+\beta} - 1\right) C_{t=-1}^{-A}{}^{1}\,(1 + r_t)^{-1}\,. \tag{21}$$

The system 19 and 21 is a system of five equation, which can be solved for its five unknowns: $r_{-1,}..., r_1, p_{-1}$ and C, using $1 + r_2 = C^{\alpha^2}$.

Endowment economy algorithm

First we describe the special endowment income case, where $\alpha = 0$. As an analytic solution is unavailable, instead we look to numerically compute the transitional equilibrium. To do this, we exploit the Euler equation on an aggregate level to design an iterative algorithm,

[7]

$$\begin{aligned} v_{t+1}^{1+a} &= b v_{t+1} v_t^{1+a} + c v_{t+2}^{a} \\ v_t &= A h^{m^t} \\ A^{(1+a)} h^{m(1+a)m^t} &= b A h^{mm^t} A^{(1+a)} h^{(1+a)m^t} + c A^a h^{m^2 a m^t} \\ m &= \frac{1+a}{a} = \alpha \\ A &= bA^2 + c \Rightarrow A = 1 \end{aligned}$$

which we find converges to the transition path for real interest rates. The key idea is that each household has consumption that grows between any two adjacent periods by a factor of $\beta^{1/\theta}(1+r_t)^{1/\theta}$. If supposing we had market-clearing in any one period, then the real interest rate that guarantees market-clearing in the next period must be approximately the r_t such that $Y_{t+1} = Y_t\beta(1+r_t)$. This relation is not exact as in the next period, one cohort exits the population and another cohort enters the population. To derive the exact relation, note that aggregate consumption must match aggregate income Y_t. Therefore it follows that the equilibrium real interest rate in period t satisfies:

$$Y_{t+1} - N_{t+1}c_{t+1,0} = \beta^{1/\theta}(1+r_t)^{1/\theta}(Y_t - N_{t+1-J}c_{t+1-J,J-1}) \tag{22}$$

$Y_t - N_{t+1-J}c_{t+1-J,J-1}$ is the total consumption of all households in period t that will continue to live to the next period, and $Y_{t+1} - N_{t+1}c_{t+1,0}$ is the consumption of all households in period $t+1$ who were also alive the last period. This removes the part of aggregate income in period t that is consumed by the exiting cohort, and the part of aggregate income in period $t+1$ that is consumed by the entering cohort. Therefore $\beta^{1/\theta}(1+r_t)^{1/\theta}$ now gives the exact growth rate of the non-exiting consumption.

Our algorithm, for the endowment case, successively calculates the real interest rate transition using equation 22, within the limits of a transition window. We set real interest rates before the transition window to be the initial BGP level of g_H, and the real interest rates after the window to be the terminal BGP level g_L. Between the two BGP's we allow for a transition window of tJ periods, where t is an integer parameter we specify. The goal is to set t large enough so that there are sufficiently many periods for the transition to play out in full, but small enough so that the problem does not become too intensive computationally.

Our algorithm updates its calculation for the transition path every iteration. For any given iteration, and for each transition period within the transition window, we firstly need to calculate $Y_t - N_{t+1-J}c_{t+1,0}$ and $Y_{t+1} - N_{t+1}c_{t+1,0}$. While Y_t and Y_{t+1} do not change with respect to the real interest rate path, $N_{t+1}c_{t+1,0}$ and $N_{t+1-J}c_{t+1-J,J-1}$ is a function of the real interest rate. We therefore need to solve the household's problem for both the exiting cohort and the entering cohort to find these quantities, but we do not yet know the true real interest rate (path). To get around this problem, we calculate these quantities under the assumption that the interest rate transition is whatever we calculated from the previous iteration. Now we may calculate r_t using equation 22, which we take as the new iteration's real interest rate transition. Our algorithm computes additional iteration until the interest rate path converges. Furthermore we confirm our results by verifying that market-clearing holds in each period.

Production economy algorithm

The design of our algorithm for the production economy requires some adjustments from the endowment economy version. Following from equation 22, we now have

$$C_{t+1}(r_t, r_{t+1}) - N_{t+1}c_{t+1,0} = \beta^{1/\theta}(1+r_t)^{1/\theta}(C_t(r_{t-1}, r_t) - N_{t+1-J}c_{t+1-J,J-1}). \tag{23}$$

Equation 23 differs from equation 22 only in terms of the change from Y to C. C is aggregate consumption, which replaces Y as there is now the possibility of investment with capital. C depends on r_t as r_t determines the capital stock tomorrow and hence investment today, and C depends on r_{t-1} as r_{t-1} gives the capital stock today and hence determines output today.

While this is a small change in notation, it makes the problem drastically harder to solve. For example as r_t affects both the slope term, $\beta^{1/\theta}(1+r_t)^{1/\theta}$, and output both today and tomorrow, if you simply take output next period as given when calculating r_t using equation 23, you will end up over-adjusting. This is a big problem as over-adjustment leads to the algorithm's calculations to quickly explode. The above example is not the only source of co-dependence from the different terms in equation 23. For each period t which we look to calculate r_t, we can only take r_{t-1} and r_{t+1} as given, at the level of whatever the past iteration's path gives. But this leads to strange dynamics in the algorithm as once we calculate correctly the new r_t based on this past iteration's r_{t-1} and r_{t+1}, the next iteration's r_{t-1} and r_{t+1} will now be different. Hence we see that the goal posts, i.e. r_{t-1} and r_{t+1}, changes each iteration, and there is no guarantee that successive r_t calculations will lead to small adjustments towards the true level - in fact the result again is often explosive. We give a final example, although there are still more cases: in the production economy the $N_{t+1}c_{t+1,0}$ and $N_{t+1-J}c_{t+1-J,J-1}$ terms depend on the interest rate path not just through its effect on intertemporal prices, they also depend on interest rates through wages. This can also lead to perverse dynamics across iterations, where changes in the households' income leads to changes in C, which feeds back upon itself to create explosive behavior in the algorithm. Altogether all of the above issues compound together to make this calculation much more difficult.

Through much experimentation, we find one way of updating the transition interest rate path using equation 23 such that the algorithm is stable and converges, and furthermore converges to the true path (which we verify by checking market-clearing). The algorithm has two stages. In stage one: for each iteration and each transition period, we firstly calculate $N_{t+1}c_{t+1,0}$ and $N_{t+1-J}c_{t+1-J,J-1}$ by explicitly solving the household problem for the entering and leaving cohorts. Then we calculate C_t as a function of r_t by finding $Y_t - I_t$. This

internalizes the effect of r_t on C_t from the supply side perspective. For C_{t+1}, we calculate aggregate consumption demand in period $t+1$ by solving each alive cohort's household's problem in period t, and then aggregating. This serves to anchor C_{t+1} to a market-clearing level, which helps makes the adjustment in r_t less volatile. With all of the above terms, we may now solve equation 23 for r_t. Doing this for all periods in the transition window completes one iteration. We keep running iterations in this stage until it no longer converges further. Next we move to stage two. Stage two calculates the above terms the same way except for C_{t+1}. We now find C_{t+1} also in terms of aggregate consumption supply, i.e. $Y_{t+1} - I_{t+1}$, and we again internalize the effect of r_t on both Y_{t+1} and I_{t+1}. This gives us a different equation 23 as a function of r_t, which we solve. We keep running iterations in this final stage until the path converges a second time. This completes our algorithm, and we indeed find that the resulting interest rate path satisfies market-clearing in all transition periods.

Note that in stage one of the adjustment algorithm, we take C_{t+1} to be aggregate consumption demand in period $t+1$. This requires solving the household problem for households of all different ages, and then aggregating, and this is done for each period in the transition window. Furthermore this is repeated in each iteration, which can easily exceed 1000. The transition window scales with the size of the model, i.e. J, and likewise the number of differently aged households also scales with J. Finally, it appears that the number of iterations required also scales with J. This implies that the computational size of the task is approximately of order J^3, which quickly becomes enormous once we start to look calculate a full overlapping generations model with $J = 75$. For this reason, the algorithm for the production economy is very slow to converge.