UNIVERSITY OF CAMBRIDGE
Faculty of Economics

Cambridge Working Paper Economics

Cambridge Working Paper Economics: 1674

Long-Run Debt Ratios with Fiscal Fatigue

Donald Robertson

Demosthenes N. Tambakis

6 October 2016

We investigate the implications of fiscal fatigue – governments' declining ability to increase primary fiscal balances with rising public debt - utilising the cubic policy rule estimated by Ghosh et al. (2013). We characterize its equilibrium debt-output ratios and fiscal space, and analyze its dynamic stability in the deterministic (long-run) case. There may be up to three equilibria, of which the intermediate one will typically require a stability criterion stricter than fiscal solvency. We illustrate numerically for six developed economies.

Long-Run Debt Ratios with Fiscal Fatigue

Donald Robertson

Demosthenes N. Tambakis*

Pembroke College, Cambridge & Faculty of Economics

October 6, 2016

Abstract

We investigate the implications of fiscal fatigue – governments' declining ability to increase primary fiscal balances with rising public debt – utilising the cubic policy rule estimated by Ghosh et al. (2013). We characterize its equilibrium debt-output ratios and fiscal space, and analyze its dynamic stability in the deterministic (long-run) case. There may be up to three equilibria, of which the intermediate one will typically require a stability criterion stricter than fiscal solvency. We illustrate numerically for six developed economies.

Keywords: Debt sustainability; Debt-output ratio; Fiscal policy rules.

JEL classification codes: E6, H0, H6

---

*We thank participants at EEA-ESEM 2016 for helpful comments. The usual disclaimer applies. E-mail address for correspondence: dnt22@cam.ac.uk.

1 Introduction

How does "fiscal fatigue" — governments' declining ability to increase their primary (non-interest) surplus in response to rising debt levels — affect expected debt-output ratios and debt sustainability? In this paper we solve for the multiple long-run equilibrium debt ratios under the fiscal fatigue definition of Ghosh, Kim, Mendoza, Ostry and Qureshi (2013) and characterize their dynamic stability. Fiscal policy rules where the primary budget balance responds in non-linear fashion to debt accumulation have multiple turning points, resulting in potentially multiple equilibrium expected debt ratios.[1] In turn, each equilibrium implies a different measure of long-run "fiscal space", *i.e.* the distance between the expected debt ratio and the endogenous debt limit beyond which default becomes unavoidable. We chart the resulting challenge for sustaining a stable public debt ratio.

For the United States, the persistent fiscal deterioration in the aftermath of the Great Recession is an unprecedented response to historical debt buildup events; see D'Erasmo et al. (2015). Long term fiscal prospects are also worrying for the euro area following the sovereign debt crisis in periphery countries. Eichengreen and Panizza (2016) find that the magnitude and persistence of primary surpluses required for the single currency block to meet its 60 percent debt ratio target (*Fiscal Compact 2030*) is very rare, particularly when output growth is weak.

[1] Fiscal reaction functions mapping the lagged debt ratio to the primary surplus have attracted attention well before the Great Recession and ensuing public debt overhang. Monetary and fiscal policy reacting to debt shocks can be traced to Leeper (1991), who defined *passive* policy as being constrained by private and public optimization, while *active* policy is unconstrained.

The long run impact of worsening primary balances matters in two ways. First, insofar as a positive primary surplus response to debt accumulation is sufficient for sustainable debt dynamics under linear fiscal policy rules. This "fiscal responsibility" condition, due to Bohn (1995, 1998, 2008), has facilitated model-based tests of debt sustainability across countries and over time.[2] Of course, a fiscally irresponsible government may yet bring fiscal policy back on track at some future point, while being fiscally responsible may not prevent ever-increasing debt ratios and required primary surpluses exceeding GDP. Second, at high debt ratios governments find raising taxes or cutting primary expenditure increasingly difficult, and "fiscal fatigue" symptoms are amplified if output growth falters.[3] Contributing factors include government complacence because of cheap borrowing rates, i.e. the opposite of developing countries feeling market pressure; low tax compliance during episodes of weak growth resulting in a procyclical tax base, a feature which Talvi and Végh (2005) had identified for developing countries; as well as society's willingness to live with high debt, effectively discounting the risk of financial crises when a *public* backstop becomes essential; see Ostry, Ghosh and Espinoza (2015).

Against this background, Ghosh et al. (2013) formalized the fiscal fatigue notion by introducing a cubic debt rule in which the fiscal stance eventually deteriorates at an increasing rate as the debt ratio grows, counteracting any fiscal

[2]D'Erasmo, Mendoza and Zhang (2015) and Mendoza and Ostry (2008) survey model-based tests for developed countries and emerging market economies.

[3]See Alesina et al. (2014), Arrelano and Bai (2016) and the references therein. We review the empirical evidence on fiscal response coefficients in Section 4.

responsibility operating through the rule's conditional linear response. The government then faces an endogenous debt limit beyond which it is unable to adjust the primary balance to rising debt and has to default. Assuming such a description of fiscal fatigue is empirically relevant, the response of the primary balance to lagged debt changes sign twice. Denoting the debt-GDP ratio as d, Fig. 1 below presents three possible cases where the primary balance $s = f(d)$ intersects the growth-adjusted debt repayment schedule $(r - g)d$, where r is the real interest rate and g real output growth:

FIGURE 1 HERE

Equilibrium Debt Ratios with Fiscal Fatigue

In principle there can be up to three equilibrium debt ratios. However, the fiscal behavior implied by $f(d)$ to the left of its first turning point seems largely an artefact of the cubic assumption: it implies ever-growing budget surpluses with declining debt ratios. We thus consider Case I with a unique intersection in this low debt region to be unrealistic. Of the remaining, Case II includes an intersection in this region which we may again ignore, while in Case III the unique intersection occurs in the high debt region which we will argue is unstable. Thus we focus on the upward- and subsequent downward-sloping sections of $f(d)$. Fig. 2 zooms in on Case II:

FIGURE 2 HERE

Endogenous Debt Limit and Fiscal Space

The government is fiscally responsible in the intermediate region extending between turning points $d_{\min}$ and $d_{\max}$, and fiscal fatigue sets in to the right of $d_{\max}$. In the high debt region between $d_{\max}$ and the rightmost intersection, $\overline{d}$, growing debt is no longer offset by increasing surpluses. Any threshold debt-output ratio triggering output decline is likely located within this region.[4] Lastly, $\overline{d}$ represents the debt limit specific to fiscal policy rule $f(d)$. Beyond that, on average the primary balance cannot roll over accruing public debt and default becomes unavoidable.

Our main findings can be summarized as follows. First, there are up to three long-run equilibria, corresponding to the real solutions of a cubic polynomial in the unconditionally expected debt ratio. The equilibrium magnitudes depend non-linearly on the linear and higher order unconditional comovement between real interest rates and debt. Second, fiscal responsibility ($f' > 0$) is a weak criterion of debt sustainability if fiscal adjustment is a non-linear function of the debt ratio. Accordingly, we show that only the equilibrium in the intermediate debt region is dynamically stable while the two extreme ones are unstable. Our stability criterion, adopted also by Ghosh et al. (2013), is that the expected debt ratio should converge to a finite proportion of output in expectation. This requires that the fiscal reaction function slope exceeds the growth-adjusted real interest rate at each equilibrium point: $f' > r - g$. Assuming $r > g$, dynamic stability is more restrictive than fiscal responsibility as the latter cannot rule out an explosive debt-output

[4] Lo and Rogoff (2015) review the available rationales for sluggish post-crisis global growth. On the theoretical and empirical debate linking public debt levels and output growth see Ostry, Ghosh and Espinoza (2015) and Reinhart, Reinhart and Rogoff (2015).

ratio under fiscal fatigue. Therefore, a sustainable fiscal policy must on average react more aggressively to debt buildups than under a linear fiscal rule.

The framework sheds light on fiscal policy's short term potential to destabilize the debt ratio. Referring to Fig. 2 above, for a given slope of $(r-g)d$ and starting with an intersection in debt region $[d_{\min}, d_{\max}]$, consider a *short-term* shock to the debt ratio, *e.g.* because of a financial crisis-triggered recession. The government then needs to run a bigger surplus; thus fiscal policy following the rule is procyclical, amplifying the downturn and countering conventional wisdom on the role of automatic stabilizers.[5] Positive comovement between debt and real interest rates will further worsen the recessionary impact, while negative comovement will mitigate it. By contrast, a *permanent* shift up in $r-g$ (capturing secular stagnation and/or sovereign default concerns) requires a more aggressive fiscal stance if the government wishes to maintain the same debt ratio as before. At the current level, interest payments exceed the mandated surplus so debt starts to increase until it hits a new intersection above d^*. Such long-term shifts then lead to countercyclical fiscal policy, consistent with the consensus view.[6]

The expected debt limit specific to the fiscal rule and the fiscal space available to the government follow as a corollary. While Ghosh et al. (2013) work out *actual*

[5] On the potential for self-reinforcing austerity measures during the eurozone debt crisis see Alesina et al. (2014), Collignon (2012), Eichengreen and Panizza (2016) and Ghosh, Ostry and Qureshi (2013).

[6] See Lane (2003). However, the consensus whereby fiscal policy tends to be countercyclical (procyclical) in developed (developing) economies may be shifting since the financial crisis to encompass the financial cycle; see Borio, Lombardi and Zampolli (2016).

fiscal space in a stochastic environment by simultaneously solving for the default probability, the market interest rate and the fiscal rule's endogenous debt limit, we focus on *expected* fiscal space, defined as the (non-negative) distance between the stable equilibrium debt ratio — if it exists — and the expected debt limit. If the only real-valued equilibrium is unstable, however, it coincides with the debt limit beyond which default is certain — at least in our deterministic long-run setting — hence expected fiscal space is zero.

We employ a cubic debt rule featuring the fiscal response coefficients of Ghosh et al. (2013) to evaluate the long run debt ratios and implied debt limit of 6 developed economies. The unconditional moments and linear and nonlinear co-movements between each country's debt-GDP and its 10-year government bond yields are computed for 1995-2015 under two exogenous growth scenarios: "potential" (3 percent) and "post-crisis" (0.5 percent) average output growth. The numerical exercise is meant to illustrate the analytical framework; in particular, our non-structural approach means that the equilibrium debt ratios and implied long-run fiscal space (or lack thereof) need not be optimal, or indeed socially desirable.[7] That said, three features stand out. *First*, with the exception of Japan and Italy the countries in question have three equilibria. The stable debt ratios range from near 40 percent (the United States, fast growth) to 101 percent (Italy, slow growth). Of these countries, expected fiscal space is greater for the U.S. than the

[7] A government's own welfare function (expected reelection probability) may differ from the social optimum. Collard et al. (2016) show that such a "reluctant defaulter" will opt for much higher debt ratios than the standard strategic cost-benefit comparison would imply.

euro area economies, with the U.K. in between. Japan stands out with a single unstable debt ratio exceeding 250 percent of GDP under both growth scenarios, suggesting its expected fiscal space is zero. *Second*, a deteriorating macroeconomic environment — higher real rates and/or slower growth — raises the stable debt ratio and lowers the debt limit, shrinking expected fiscal space on both counts. Conversely, a more benign macroeconomic environment widens the government's expected fiscal maneuver room. This qualitative feature sets our model apart from Collard (2016), whose reluctant defaulter's optimal debt ratio closely tracks the maximum sustainable debt limit . *Third*, introducing excess fiscal fatigue raises the stable debt ratio and lowers expected fiscal space across the board. Italy then also displays a single unstable expected debt ratio albeit near zero, unlike Japan's.

The paper is structured as follows: Section 2 reviews the literature; Section 3 contrasts linear and nonlinear debt rules; Section 4 derives the unconditionally expected debt ratio(s) consistent with the cubic specification of Ghosh et al. (2013); Section 5 illustrates for six advanced economies; and Section 6 concludes.

2 Literature review

To the best of our knowledge, Ghosh et al. (2013) and Ghosh, Ostry and Qureshi (2013) are the only previous studies explicitly addressing fiscal fatigue. An exception is Shiamptanis (2015), who also finds the required stability criterion is tighter than those proposed by Bohn (1998).

Our non-structural approach shares Ostry, Ghosh and Espinoza's (2015) focus on "green-zone" cases with ample fiscal space, as opposed to the yellow- or red-zone where fiscal space is narrow or has run out. Further, our large numerical debt ratios and implied expected fiscal space are consistent with theoretical work on the optimal amount of public debt. Following upon Ayagari and McGrattan (1998) and Holmstron and Tirole (1998), Angeletos et al. (2013) show that public debt alleviates a financial friction by increasing the aggregate amount of collateral in crisis times. In a similar vein, Kocherlakota (2015) has suggested that issuing more debt may yield a higher natural real interest rate if Ricardian equivalence fails, thus moving the policy rate away from its zero bound and contributing to financial stability. Collard et al. (2015, 2016) have also calibrated high calibrated debt limits and optimal debt ratios, respectively, by assuming a government will only default as a last resort if it cannot service accruing debt, i.e. if its primary surplus falls short of $(r - g)d_t$.

In non-structural models, a linear budget response to lagged debt implies a unique expected debt ratio; see D'Erasmo et al.'s (2015) review of Bohn's contributions. Typically, structural models of fiscal policy with endogenous default also yield a unique Markov-perfect equilibrium (Aguiar and Gopinath (2006), Arrelano (2008)).[8] If there are multiple long run debt ratios consistent with a fiscal rule then identifying the stable one(s) becomes an important issue for debt management policy (Wyplosz (2013)). Equilibrium selection matters also for closed-economy DSGE models whose determinacy requires the equilibrium debt ratio

[8] An exception is Pergallini (2014); however, he assumes a conditionally increasing marginal fiscal response to rising debt, which rules out fiscal fatigue.

to log-linearize around its steady state (Linnemann (2006)), and open-economy models in which exogenous deviations of the debt ratio around its steady-state(s) determine the currency risk premium (Schmitt-Grihe and Uribe (2003)). There is also research where debt ceilings arise endogenously through state-dependent Laffer curves limiting governments' ability to raise taxes (Bi (2012), Arrelano and Bai (2016)). Their debt limits are time-varying with productivity, government spending and transfer shocks and have state-dependent distributions, whereas our measure is unconditionally expected.

Structural infinite-horizon models are better suited to analyze cases where sovereign default is imminent. For the euro area, Nerlich and Reuter (2015) report that procyclicality is stronger if countries have more fiscal space. Deficit bias is stronger if a government believes that the likelihood of default is remote, all else equal. In turn, its fiscal maneuver room tends to be higher with fiscal rules than without. Bi (2012) also finds that longer term fiscal reforms (if credible) have a better chance of reducing debt than short-term austerity measures.

Lastly, we assume that output growth is stochastic in principle, but independent of everything and has constant expectation, and we allow the unconditional comovement between real interest rates and debt ratios to take either sign.[9]

[9] By assuming the default probability is always increasing in the debt ratio, Ghosh et al. (2013) only allow a positive covariance between the levels, which excludes financial safe havens. For the U.S., Laubach (2009) finds that linear correlations are positive. For the eurozone, Ghosh, Ostry and Qureshi (2013) find an offsetting effect through investor expectations of a financial bailout.

3 Linear and non-linear debt rules

Denoting the period-t debt-output ratio as d_t, the ratio of the gross return on public debt to output growth from t to $t+1$ is $1 + r_t - g_t$, where $1 + r_t$ is the 1-period gross real interest rate contracted in period t and g_t is real output growth. The debt ratio evolves as

$$d_{t+1} = (1 + r_t - g_t)\, d_t - s_{t+1} \tag{1}$$

where s_{t+1} is the primary fiscal balance in period $t+1$, *i.e.* tax receipts less government spending in percent of output. We assume r_t is stochastic with constant unconditional expectation $E(r_t) = r$ and variance $var(r_t) = \sigma_r^2$ while g_t is time-varying but deterministic. The unconditional variance of d_t is $var(d_t) = \sigma_d^2$, and we define the unconditional comovement of the levels, squares and cubes of $1 + r_t$ and d_t as $\Theta = cov\{1 + r_t, d_t\}$, $\Lambda = cov\{(1 + r_t)^2, d_t^2\}$ and $\Gamma = cov\{(1 + r_t)^3, d_t^3\}$ respectively. Note that to obtain Θ, Λ, $\Gamma \neq 0$ it suffices that $\{d_t\}$ is unconditionally correlated with the real interest rate process $\{r_t\}$.

The benchmark linear debt rule for determining s_{t+1} is just

$$\begin{aligned} s_{t+1} &= f(d_t) + \mu_{t+1} \\ &= \rho d_t + \mu_{t+1} \qquad (2) \\ \mu_t &= \alpha Z_t + \varepsilon_t\ , \end{aligned}$$

where $\varepsilon \sim (0, \sigma_\varepsilon)$ is an i.i.d. shock to the primary balance and μ_{t+1} captures all determinants of s_{t+1} other than lagged debt, including proxies for temporary and cyclical fluctuations in output and government spending (Z_{t+1}).[10]

Bohn (1998) showed that $f'(d) = \rho > 0$ is sufficient for the debt ratio to be sustainable over time, *i.e.* for the infinite sequence of fiscal policies to meet the government's intertemporal budget and no-Ponzi constraints:

$$d_{t-1} = s_t + \sum_{j=1}^{\infty} E_t \left[{R_{jt}}^{-1} \cdot \ s_{t+j} \right] \tag{3}$$

$$0 = \lim_{n \to \infty} E_t [{R_{jt}}^{-1} \ \cdot d_{t+n}] \tag{4}$$

where ${R_{jt}}^{-1} = (1 + r_{jt})^{-1} = \beta^j E_t \left[\frac{u'(c_{t+j})}{u'(c_t)} \right]$ is the gross return on period-t debt maturing at $t+j$ and $\beta \in (0,1)$ is the discount factor. Under plausible assumptions about $\{r_t\}$, Bohn's proof only requires that μ_t and the present value of output are finite.

Mendoza and Ostry (2008) show that a fiscal authority committed to a linear debt rule as in (2) delivers the expected debt ratio:

$$Ed_t \equiv d^* = \frac{-\mu + (1-\rho)\Theta}{\rho(1+\overline{r}) - \overline{r}} \tag{5}$$

Eq. (5) follows from setting the linear coefficient in the debt rule as $\rho(1 + r_t - g_t)$, and $\overline{r}$ and μ are the unconditional means of the growth-adjusted real interest rate and temporary government spending, respectively. Thus, provided $\rho < 1$ a worse macroeconomic environment results in lower d^*, and vice versa. Further, assuming $\Theta = 0$, countries with higher ρ (more "fiscally responsible") will tend to have lower expected debt ratios than those with lower ρ. This counterfactual prediction arises

[10] The specific probability density function of ε_t is not required for our long-run purposes. It is critical, however, for short-run dynamic stability; see Section 4.2 and Ghosh et al. (2013).

because developing countries with procyclical fiscal policy (countercyclical primary balances) tend to have higher ρ in response to their greater macro-financial risk ($\Theta > 0$), the latter independently raising d^* given ρ. By contrast, an important reason why developed countries have historically been characterized by more countercyclical fiscal policy is that they are perceived to be less risky, and may even function as financial *safe havens*.[11] Lastly, as the default probability can only be zero, if the government if fiscally responsible, or one if it is not, the economy's actual fiscal space — the gap between d^* and the debt limit — is *either* infinite ($\rho > 0$) *or* zero ($\rho \leq 0$).

Based on the premise that the primary balance's response to debt accumulation is likely globally non-linear — i.e. over the whole debt range — in the rest of this paper we study the cubic functional form of Ghosh et al. (2013), who specify $f(d_t)$ as a continuously differentiable cubic polynomial. The non-linear debt rule in terms of d_t is:

$$\begin{aligned} s_{t+1} &= f(d_t) + \mu_{t+1} \\ &= \rho' d_t + \phi' d_t^2 + \psi' d_t^3 + \mu_{t+1} \\ \mu_{t+1} &= \alpha Z_{t+1} + \eta_{t+1} \end{aligned} \tag{6}$$

where $\eta \sim (0, \sigma_\eta)$ is a primary balance disturbance similar to ε above. We choose

[11] There is strong evidence for $\Theta > 0$ in developing countries; see Aguiar and Gopinath (2006) and Arrelano (2008). Safe haven status ($\Theta < 0$) is usually reserved for the United States and Japan; see respectively Prasad (2014) and Rogoff and Tashiro (2015).

to specify f in terms of the interest-adjusted debt ratio $\widetilde{d}_t = (1 + r_t)d_t$ and write:

$$\begin{aligned} s_{t+1} &= f(\widetilde{d}_t) + \mu_{t+1} \\ &= \rho\widetilde{d}_t + \phi\widetilde{d}_t^2 + \psi\widetilde{d}_t^3 + \mu_{t+1} \end{aligned} \quad (7)$$

$$\begin{aligned} \mu_{t+1} &= \alpha Z_{t+1} + \eta_{t+1} \\ \rho &\equiv \frac{\rho'}{1 + r_t}\ ,\ \phi \equiv \frac{\phi'}{(1 + r_t)^2}\ ,\ \psi \equiv \frac{\psi'}{(1 + r_t)^3} \end{aligned} \quad (8)$$

The $\widetilde{d}_t$ measure is also implicit in Mendoza and Ostry's (2008) unconditional derivation. The transformation simplifies the analytics and is without loss of generality as r_t is known at time t. Empirically, substituting $r_t = r$ guarantees the fiscal response coefficients are not stochastic. Differentiating eq. (7) with respect to $\widetilde{d}_t$ yields $d_{\min}, d_{\max} = \frac{-\phi \pm \sqrt{\phi^2 - 3\rho\psi}}{3\psi}$. Real-valued turning points require $\psi \leq \frac{\phi^2}{3\rho}$, which is always satisfied for $\rho > 0$ and $\psi < 0$.

We are interested in debt rules featuring $\psi < 0$. Referring to Fig. 2, for debt ratios in the range $[d_{\min}, d_{\max}]$, *i.e.* where f intersects the roll-over payment schedule from below, fiscal policy responds to growing debt by setting a bigger surplus. For example, a negative period-t shock to the surplus drops the economy vertically below the d^* intersection, so in period $t + 1$ the debt ratio rises above d^*. In that period, assuming no further shocks, the debt rule forces s_{t+1} to exceed $(r - g)d_{t+1}$ so debt is reduced and the economy returns towards d^*.

4 Equilibrium debt ratios

4.1 Existence

We derive the unconditionally expected debt ratios obtaining under cubic debt rule (7) and investigate their stability. Henceforth we set $g_t = g$, all t, so $r_t - g$ follows r_t up to a constant. Applying eq. (7) and $\widetilde{d}_t = (1 + r_t)d_t$ into eq. (1), the debt ratio evolves as

$$\begin{aligned} d_{t+1} &= (1 + r_t - g)d_t - \rho(1 + r_t)d_t \\ &\quad -\phi(1 + r_t)^2 d_t^2 - \psi(1 + r_t)^3 d_t^3 - \mu_{t+1} \end{aligned} \tag{9}$$

In Appendix A we establish that eq. (9) yields the following cubic polynomial in the long-run expected debt ratio d_i^*:

$$\Upsilon(d^*) \equiv d^{*3} + a_1 d^{*2} + a_2 d^* + a_3 = 0 \tag{10}$$

In general $\Upsilon(d^*)$ has up to three real-valued solutions, denoted d_i^*, $i \in \{1, 2, 3\}$. Its coefficients are given by:

$$\begin{aligned} a_1 &= \phi(\sigma_r^2 + (1 + r)^2)e \\ a_2 &= (\rho(1 + r) - (r - g))e - 3\sigma_d^2 \\ a_3 &= [\mu - (1 - \rho)\Theta + \phi\Lambda + \phi(\sigma_r^2 + (1 + r)^2)\sigma_d^2 + \psi\Gamma]e - \gamma_d\sigma_d^3 \end{aligned} \tag{11}$$

with $e = -\psi^{-1}[\gamma_r\sigma_r^3 + 3\sigma_r^2(1 + r) - (1 + r)^3]^{-1}$.

The behavior of $\Upsilon(d^*)$ is determined by fiscal response coefficients ρ, ϕ and ψ; the three unconditional moments of r_t and d_t; their unconditional covariances Θ,

Λ, Γ; the average growth-adjusted real interest rate $r - g$; and μ. The number of real solutions is a function of $D = Q^3 + R^2$, where

$$Q = \frac{3a_2 - a_1^2}{9} \;,\; R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54} \tag{12}$$

$D > 0$ implies one real and two complex roots; $D = 0$ implies all real roots and at least two equal; and $D < 0$ implies three real and unequal roots. Therefore, the likelihood of a single equilibrium in the "high debt" region — Case III in Fig. 1 — cannot be ruled out in principle. This point will become salient in Section 5.

4.2 Stability

We characterize the dynamic stability of expected debt ratio solutions in the deterministic case, and briefly discuss the implications of r_t being stochastic. When $r_t = r$, the debt ratio in eq. (9) follows the first-order difference equation:

$$\Delta d_{t+1} = (r - g)d_t - f(d_t) - \mu_{t+1} \tag{13}$$

Bohn's (1998) sufficient condition for fiscal solvency ($f'(d) > 0$) is a weak criterion of debt sustainability. For example, it would allow the debt ratio to grow without limit as long as the accompanying surpluses grow at some positive rate. Although the expected present value of debt would then be finite and the economy solvent, this might require primary surpluses exceeding GDP. But with fiscal fatigue such large surpluses become impossible; a stronger condition which keeps the debt ratio around some stable level is required.

In Appendix B we show that equilibrium debt ratio d_i^*, $i \in \{1, 2, 3\}$ is stable to small perturbations η_t (shocks to the primary balance) if:

$$f'(d_i^*) > r - g \tag{14}$$

Inequality (14) coincides with Ghosh et al.'s (2013) stability condition in the deterministic case. Thus, d_i^* is stable if at that debt ratio f intersects the growth-adjusted interest rate schedule $r - g$ from below, corresponding to Fig. 1, Case II where only the intermediate real solution (d_2^*) of $\Upsilon(d^*)$ is stable. Thus, minimum solution d_1^* and debt limit d_3^* are both unstable because $f' < 0$ unless $r < g$, which is ruled out by a modified golden rule commanding broad theoretical and empirical support in the long-run (Blanchard and Fischer (1989)).

Setting $\Delta d_{t+1} = 0$ in (13), differentiating eq. (7) with respect to d_i^* and plugging in inequality (14) yields:

$$3\psi(1+r)^2 d_i^{*2} + 2\phi(1+r)d_i^* + \rho - \frac{r-g}{1+r} > 0 \tag{15}$$

We denote the LHS quadratic polynomial roots by $\underline{d}$ and $\overline{d}$ and assume these are real and distinct. Given $\psi < 0$, inequality (15) holds for $d_i^* \in [\underline{d}, \overline{d}]$, where:

$$[\underline{d}, \overline{d}] = \left[\frac{1}{3\psi(1+r)}\left(-\phi \pm \sqrt{\phi^2 - 3\psi\left(\rho - \frac{r-g}{1+r}\right)}\right)\right] \tag{16}$$

Expected debt ratios are dynamically stable in the region defined by this closed interval. A sufficient condition for real-valued $\underline{d}$ and $\overline{d}$ then is:

$$\phi \geq \sqrt{3\psi\left(\rho - \frac{r-g}{1+r}\right)} \tag{17}$$

In turn, assuming $\phi > 0$ and $\psi < 0$, inequality (17) is satisfied if

$$\rho < \frac{r-g}{1+r} \tag{18}$$

The tighter stability condition under non-linear fiscal rules imposes an upper bound for the fiscal responsibility coefficient (ρ). Effectively, violating inequality (18) means the economy is in the single intersection Cases I and III of Fig. 1.

For stochastic r_t, difference equation (9) becomes stochastic as the interest-adjusted fiscal response coefficients ρ, ϕ and ψ in eq. (7) are random variables correlated with the debt process $\{d_t\}$. The stability of the solutions to (9) will depend on the magnitude of these correlations and the probability density function of primary balance shocks (η_{t+1}) to μ_{t+1}. In the short run, a shock to d_t rotates the $(r_t - g)d_t$ line as well as $f(d_t)$, the latter working through the changes in ρ, ϕ, ψ. It is then likely that $f' > r_t - g$ is not strong enough to force a finite expected debt ratio *unless* $Corr\{r_t, d_t\}$ is very negative. Intuitively, stability may be easier to attain for safe haven countries as their rollover interest payments go down with debt accumulation, at least in the short term. Conversely, $Corr(r_t, d_t) > 0$ makes stability less likely, all else equal. In that case, an extra "fiscal discipline" condition such as $f'' > 0$ may be required to make d_i^* stable to disturbances in the primary balance.[12]

4.3 Expected fiscal space

The expected fiscal maneuver room (in output terms) available to a government implementing a cubic debt rule such as (7) is the distance between the intermediate

[12] A full stability analysis for difference equations with stochastic parameters (response coefficients) would require imposing strong restrictions on the underlying processes $\{r_t\}$ and $\{d_t\}$. These lie beyond the scope of this paper.

(stable) and the maximum of the three real-valued solutions of polynomial $\Upsilon(d^*)$ in eq. (10), if they exist. The largest solution, d_3^*, corresponds to the maximum intersection point between $f(d)$ and $(r-g)d$ in Fig. 2. It measures the finite expected debt limit, denoted $\overline{d}$, beyond which the debt stock cannot be rolled over. The situation is irreversible, at least in expectation, because beyond $\overline{d}$ the government is unable to raise taxes and/or cut spending in line with rising debt and the primary balance worsens at an increasing rate. Alternatively, if $\Upsilon(d^*)=0$ has a single real solution, that is unstable and coincides with the debt limit. Expected fiscal space is then zero and public debt dynamics are unsustainable.[13]

Therefore, expected fiscal space S is *either* zero *or* the positive distance between the endogenous deterministic debt limit and stable expected debt ratio d_2^*:

$$\begin{aligned} \overline{d} &\equiv \arg\max_{i=\{1,2,3\}}[d_i^*] \Rightarrow && (19) \\ S &= d_3^* - d_2^* \text{ if } D \leq 0 \text{ , } \overline{d} \equiv d_3^* && (20) \\ S &= 0 \text{ if } D > 0 \text{ , } \overline{d} \equiv d^* \end{aligned}$$

Compared to the linear case where declining growth-adjusted real interest rates unambigously lower d^* (from Section 2 recall that S is infinite provided $\rho > 0$), there are now two reinforcing effects: higher g and/or lower r lowers expected debt ratio d_2^* in the stable (upward-sloping) debt region. At the same time, in the fiscal fatigue (downward-sloping) region, faster growth leads to higher $\overline{d}$ as it alleviates the debt burden; this is what Ostry et al. (2015) call the *organic* approach to

[13] In practice, the government is likely to lose market access and default well before the expected debt limit is reached, at least on the external component of public debt.

debt reduction. The net impact of $\Delta(r-g) < 0$ when $f(d)$ is a cubic function then is to increase expected fiscal space. Conversely, a secular deterioration in the macroeconomic environment implies less long-run fiscal space.

5 Long-run debt ratios: numerical illustration

We evaluate the expected public debt ratios, debt limit and fiscal space of France, Germany, Italy, Japan, the U.K. and the United States in three steps. *First*, solving polynomial $\Upsilon(d^*)$ requires the interest-adjusted fiscal response coefficients ρ, ϕ, ψ. Table 1 reviews the fiscal response estimates for cubic debt rule (6):

Table 1. Fiscal Response Coefficient Estimates[14]

	Sample	Linear ρ'	Quadratic ϕ'	Cubic ψ'
Bohn (1998)	1916-1995 U.S.	$0.028^{***}, 0.054^{***}$	0.106^{***}	-0.012
Bohn (2008)	1793-2003	$0.028^{***}, 0.147^{***}$		–
MO	1990-2005	$0.022^{***}, 0.038^{**}$	–	–
DMZ	1791-2014 U.S.	$0.078^{*}, 0.105^{***}$	0.003	–
	1951-2013	$0.028^{***}, 0.069^{***}$	–	
Ghosh et al.	1970-2007	$\mathbf{-0.2249}^{***}$	$\mathbf{0.0034}^{***}$	$\mathbf{-0.00001}^{***}$
	1985-2005	-0.0864	0.0017^{*}	-0.00001^{**}

[14] All estimates are for developed countries. MO and DMZ refer to Mendoza and Ostry (2008) and D'Erasmo, Mendoza and Zhang (2015). The non-linear coefficients often measure the conditional fiscal impact of deviations from a unique steady-state proxied by the average debt ratio.

While responsible fiscal behavior emerges over long time spans, to the best of our knowledge Ghosh et al.'s (2013) dynamic panel of 23 developed countries is the only study reporting negative ρ estimates. As discussed in Section 1, this is in line with a possible structural shift in the debt rule occuring post-2008 (D'Erasmo et al. (2015)). The positive ϕ' estimates are consistent with inequality (17): a positive quadratic response is required for stability. The cubic coefficient ψ' is small but significantly negative, consistent with the "increasing fatigue" assumption at high debt ratios. Accordingly, we adopt these authors' significant coefficients from 1970-2007.[15] The empirical fit of these coefficients is shown in Fig. 3:

FIGURE 3 HERE

Estimated Fiscal Reaction Functions

To highlight the sensitivity of the long run to fiscal fatigue, the green line shows the fiscal reaction function if the cubic response coefficient is raised by 20 percent: from the Ghosh et al. (2013) estimate to $\psi = -0.000012$. The two schedules largely overlap through debt ratios around 80 percent, after which they progressively diverge. The primary balance deteriorates rapidly when debt exceeds 150 percent of GDP, suggesting that long-term consolidation is highly sensitive to the cubic fiscal response as debt mounts. We return to this "excess fatigue" scenario below.

[15] The control variables affecting the primary balance in the preferred specification include the output gap, inflation, trade openness and the price of crude oil.

In the *second* step, the first three (unconditional) moments and the linear and nonlinear covariances between debt ratios and real bond yields required for the polynomial coefficients of $\Upsilon(d^*)$ are computed using annual general government gross debt ratios and monthly real 10-year government bond yields for each country j over the period 1995-2015. The descriptive statistics are in Table 2:

Table 2. Descriptive Statistics: 1995-2015[16]

10-year yields	*FRA*	*GER*	*ITA*	*JAP*	*U.K.*	*U.S.*
Mean r (%)	2.46	2.17	2.82	1.19	2.65	1.48
Std.dev. σ_r (%)	1.34	1.48	1.56	1.15	1.94	1.33
Skewness γ_r	0.36	-0.14	1.63	-1.09	-0.04	0.13
Debt ratios						
Mean d (%)	70.8	66.2	111.7	196.0	55.6	92.7
Std.dev. σ_d (%)	13.4	8.0	10.4	36.0	19.9	24.9
Skewness γ_d	0.77	0.49	0.62	0.08	0.73	0.23
$Corr\{1 + r_t, d_t\}$	-0.76	-0.86	0.11	-0.70	-0.82	-0.46
$\{\Delta(1 + r_t), \Delta d_t\}$	0.10	0.04	0.25	0.25	-0.15	0.12

[16] For consistency with the interest-adjusted debt transformation in eq. (9), the linear and non-linear co-movements are computed by matching the average r_t in January of a given year with the debt ratio of the previous year. We show the correlation coefficients of both levels and first differences and do not report the unscaled covariance measures (Θ, Λ, Γ); they are available upon request. Data sources: ECB, Eurostat and Federal Reserve Bank of St Louis (FRED).

The average 10-year real bond yields lie below their historical averages as sovereign term structures have flattened substantially since 2008.[17] With the exception of Italy, the negative co-movement between real yields and debt ratio levels would seem to indicate these sovereign debt markets function as safe havens. That is misleading, however, as the negative correlation simply captures the strong positive (negative) trend in debt ratios (real bond yields) over the period, and particularly since the global financial crisis: the correlation coefficients of the first-differenced data are all mildly positive except the U.K. We then adjust Ghosh et al.'s (2013) fiscal response coefficients ρ', ϕ', ψ' by country j's average real government bond yield (r^i), by eq. (8), and obtain interest-adjusted coefficients ρ^j, ϕ^j, ψ^j, $j = \{1, ..., 6\}$ reflecting country-specific macroecomic environments. However, we uniformly apply two exogenous average growth rates: 3 percent (corresponding to *pre-crisis* potential output) or 0.5 percent (*post-crisis*), in line with lower potential growth; see IMF (2015).[18] Lastly, we set average temporary government spending to $\mu = 0.022$, from Bohn (1998).

[17] The average 10-year real yield of the six economies is 2.13 percent per annum, against a steady-state annual real interest rate of 3.8 percent calibrated by D'Erasmo et al. (2015) for the commonly used deep parameter values (β, γ, σ). Also note the pronounced asymmetry in Japanese government bond returns, likely reflecting investors' one-sided expectations of future bond yield increases; see Fujiwara et al. (2011).

[18] Mendoza and Ostry (2008) employ 5 and 2.5 percent, D'Erasmo et al. (2015) impose zero growth-adjusted real rates ($r = g$), while Ghosh et al. (2013) use the 5-year country-specific average of the IMF's projected real output growth. In their 2013-2019 fiscal projections, Eichengreen and Panizza (2016) use *negative* growth-adjusted real interest rates for Japan, the U.K. and the U.S.

In the *third* step, the interest-adjusted response coefficients are applied to polynomial (10) to compute up to three equilibrium debt ratios d_i^*, $i = \{1, ..., 3\}$ for each economy. From eq. (19) recall that, if a country-specific expected debt limit $\overline{d}$ exists, it is uniquely determined as the maximum positive real solution of $Y(d^*)$. Hence, subtracting either d_2^* or the only real solution, as the case may be, from $\overline{d}$ yields each economy's expected fiscal space. The results are in Table 3:

Table 3. Long-run Debt Ratios and Fiscal Space[19]

Debt ratios (%)	*FRA*	*GER*	*ITA*	*JAP*	*U.K.*	*U.S.*
Actual	96	71	133	243	89	109.7
d_1^* $g = 0.03$	3.9	1.5	1.7	–	12.8	19.6
$g = 0.005$	3.1	1.2	1.4	–	9.2	12.0
excess fatigue	3.2	1.2	1.4	–	9.5	12.6
d_2^* $g = 0.03$	67.4	70.9	73.1	–	51.1	38.7
$g = 0.005$	94.2	97.7	101.1	–	78.2	68.5
excess fatigue	114.0	123.5	–	–	86.2	72.5
$\overline{d}$ $g = 0.03$	269.0	268.0	265.4	295.9	276.4	281.9
$g = 0.005$	242.9	241.4	237.9	277.1	253.0	259.8
excess fatigue	166.3	158.9	–	223.3	217.8	198.4
S $g = 0.03$	**201.6**	**197.1**	**192.3**	**0**	**225.3**	**243.2**
$g = 0.005$	**148.7**	**143.7**	**136.8**	**0**	**187.9**	**191.3**
excess fatigue	**52.3**	**35.4**	**0**	**0**	**101.7**	**125.9**

[19] Excess fatigue combines the slow growth scenario ($g = 0.005$) with $\psi' = -0.000012$, so $\psi^j = \frac{\psi'}{(1+r^j)^3}$, $j = \{1, ..., 6\}$.

We note that France, Germany, U.K. and the U.S. display three long-run debt ratios, of which the intermediate one (d_2^*) is stable. By contrast, Japan's fiscal record over the period yields a single unstable equilibrium in all three scenarios under consideration; Italy is also in this category, but only if there is excess fiscal fatigue. *Prima facie*, these two countries then have zero long-run fiscal space.[20]

Of the countries possessing a stable equilibrium, the smallest unstable solution (d_1^*) is near zero, somewhat higher for the U.S. Then, if average output growth is at potential, the stable debt ratio coincides with its actual (end-2015) value for Germany but is far smaller for the other four countries. The eurozone members' long-run debt ratios lie between 65 and 75 percent; they are lower for the U.K and the U.S. However, if average growth slows to 0.5 percent, the stable debt ratio rises and its gap with the actual figures drops by about 30 percent of GDP. In that "post-crisis" scenario, the eurozone members' long-run debt ratio is around 100 percent of GDP, while for Germany it is *above* its actual value. As slower growth also lowers the deterministic debt limit, the loss of long-run fiscal space is greater.

These magnitudes are broadly consistent with recent research. Our stable debt ratios are lower than the 76 and 60 percent steady-state calibrated by D'Erasmo et al. (2015) for the U.S. and EU-15 countries, respectively, while the U.S. historical figure imputed by Ghosh et al. (2013) is 78.7 percent. Further, with potential

[20]We note that, as of May 2014, the estimates of Moody's Analytics based on the methodology of Ghosh et al. (2013) and Ostry et al. (2015) also indicated zero actual fiscal space for Japan, Italy and Greece.

(*slow*) output growth, the average long-run debt ratio for the three eurozone members is 70 (*98*) percent. The EU's Fiscal Compact target debt ratio of 60 percent for 2030 then appears unrealistic unless growth picks up; see the discussion in Eichengreen and Panizza (2016). Lastly, Collard et al. (2016) calibrate the optimal debt ratio to 82 percent, far above that obtaining in standard strategic default models.

In terms of sensitivity analysis, we experiment with the larger fiscal fatigue coefficient (ψ) discussed in Fig. 3 above. The stable debt ratio rises marginally from its "post-crisis" level, relatively more so for France and Germany. At the same time, expected debt limits decline substantially for all economies (except Italy and Japan) so long-run fiscal space shrinks on both counts. With reference to Fig. 2, more fiscal fatigue tends to "compress" $f(d)$ so its last intersection with $(r-g)d$ occurs at a lower debt limit $\overline{d}$. Of course, actual default may occur before $\overline{d}$ and past $d_{\max}$, the turning point at which fiscal fatigue sets in, i.e. somewhere in the range where the threshold debt ratio beyond is detrimental to growth is located.

To summarize, our numerical exercise indicates a need for fiscal retrenchment in the potential growth scenario for all countries except Germany, to counter the public debt buildup.[21] In the slow growth scenario, only Italy and the U.S. require fiscal consolidation while Germany needs to *expand* fiscally to attain its higher long-run debt ratio. We emphasize these policy implication should be treated with

[21]Other than deliberate fiscal retrenchment, reducing public debt can be accomplished organically, through growth, or opportunistically if/as less distortionary revenue sources become available; see Ostry et al. (2015).

caution on several counts. Firstly, our simple numerical exercise primarily serves to illustrate the analytics. Secondly, our non-structural approach does not claim that the equilibrium debt ratio underlying the expected fiscal space (or lack thereof) is necessarily optimal or, indeed, desirable. In practice, governments might choose to behave optimally, or they may be self-interested and reluctant to default (Collard et al. (2016)), thus targeting a higher debt ratio than if they strategically evaluated costs and benefits. That said, a long-term objective of reducing the debt burden is consistent with the desire to create "fiscal room" against future contingencies, as well as to not risk sacrificing output growth beyond some debt ratio threshold, which we argued lies within the $[d_{\max}, \overline{d}]$ range in Fig. 2.

6 Concluding remarks

Motivated by post-crisis evidence of fiscal fatigue in developed economies, in this paper we studied the long-term implications of the non-linear fiscal reaction function (debt rule) proposed and estimated by Ghosh et al. (2013). We found that fiscal solvency is satisfied by up to three expected debt ratios, with their magnitude a function of the unconditional (linear and non-linear) comovement of real interest rates and of the two fundamental variables' own moments.

We analyzed dynamic stability in the deterministic (long-run) case, showing that only the intermediate equilibrium is stable and the stability criterion required under fiscal fatigue is stricter than fiscal solvency. Further, the expected debt limit beyond which default is unavoidable coincides with the unstable equilibrium in the high debt region, and the non-negative distance between that limit and the stable equilibrium measures the long-run fiscal space available to the government. We emphasized that cubic debt rules need not be optimal, or even desirable. Rather, they offer a useful gauge of long-term public debt sustainability insofar as they describe governments' average fiscal track record.

More generally, identifying the potential for multiplicity is arguably important in order "*... for policymakers to be aware of the full range of options they can eventually choose from...*" (Reinhart et al. (2015), p.S52). In that connection, while our framework cannot inform on the appropriate *speed* of fiscal adjustment, it may serve as input to medium-term fiscal consolidation and budgetary framework design. Classifying stability in the stochastic (short-run) case is an ambitious anaytical extension which we leave for future research. A novel feature of our unconditional approach is the expected debt ratios' sensitivity to the degree of (linear and non-linear) comovement between debt and real interest rates. On average, a country risk premium tends to make attaining stability harder, while financial safe haven status renders it easier. Quantifying the long-run impact of such considerations and that of skewness in real bond yield distributions seem useful empirical extensions.

References

[1] Aguiar, M., Gopinath, G., 2006. Defaultable debt, interest rates and the current account. Journal of International Economics 69: 64-83.

[2] Alesina, A., Barbiero, O., Favero, C., Giavazzi, F., Paradisi, M., 2014. Austerity in 2009-2013. Harvard University, unpublished.

[3] Angeletos, G.-M., Collard, F., Dellas, H., Diba, B., 2013. Optimal debt management and liquidity provision. NBER Working Paper 18800, February.

[4] Arrelano, C., 2008. Default risk and income fluctuations in emerging economies. American Economic Review 98: 690-712.

[5] Arrelano, C., Bai., Y., 2016. Fiscal austerity during debt crises. Federal Reserve Bank of Minneapolis, Staff Report 225, January.

[6] Ayagari, R., McGrattan, E., 1998. The optimum quantity of debt. Journal of Monetary Economics 42: 44-69.

[7] Blanchard, O., Fischer, S., 1989. *Lectures on Macroeconomics*, MIT Press.

[8] Bi, H., 2012. Sovereign default risk premia, fiscal limits, and fiscal policy, European Economic Review 56: 389-410.

[9] Bohn, H., 1995. The sustainability of budget deficits in a stochastic economy. Journal of Money, Credit and Banking 27: 257-271.

[10] Bohn, H., 1998. The behavior of U.S. public debt and deficits. Quarterly Journal of Economics 113: 949-963.

[11] Bohn, H., 2008. The sustainability of fiscal policy in the United States, in *Sustainability of Public Debt*, Neck, R., Sturm, J.-E. (eds.), MIT Press.

[12] Borio, C., Lombardi, M., Zampolli, F., 2016. Fiscal sustainability and the financial cycle. BIS Working Paper 552, March.

[13] Collard, F., Habib, M., Rochet, J.-C., 2015. Sovereign debt sustainability in advanced economies. Journal of the European Economic Association 13: 381-420.

[14] Collard, F., Habib, M., Rochet, J.-C., 2016. The reluctant defaulter: a tale of high government debt. Working paper, May.

[15] Collignon, S., 2012. Fiscal policy rules and the sustainability of public debt in Europe. International Economic Review 53: 539-567.

[16] D'Erasmo, Mendoza, E., Zhang, J., 2015. What is a sustainable public debt? *Handbook of Macroeconomics* Vol. II, Taylor, J., Uhlig, H. (eds.). North-Holland.

[17] Eichengreen, B., Panizza, U., 2016. A surplus of ambition: can Europe rely on large primary surpluses to solve its debt problem? Economic Policy 85: 5-49.

[18] Fujiwara, I., Körber, L., Nagakura, D., 2011. How much asymmetry is there in bond returns and exchange rates? Bank of Japan working paper No. 11-E-10, September.

[19] Ghosh, A., Kim, J., Mendoza, E., Ostry, J., Qureshi, M., 2013. Fiscal fatigue, fiscal space and debt sustainability in advanced economies. Economic Journal, vol. 0, pages F4-F30, 02.

[20] Ghosh, A., Ostry, J., Qureshi, M., 2013. Fiscal space and sovereign risk pricing in a currency union. Journal of International Money and Finance 34: 131-163.

[21] Holmgren, R., 1996. *A First Course in Discrete Dynamical Systems*, Springer.

[22] Holmstrom, B., Tirole, J., 1998. Private and public supply of liquidity. Journal of Political Economy 106: 1-40.

[23] IMF World Economic Outlook 2015, chapter 3. *Where Are we Headed? Perspectives on Potential Output*, International Monetary Fund, Washington, D.C., April.

[24] Kocherlakota, N., 2015. Public debt and the long-run neutral real interest rate. Speech at the Bank of Korea Conference, August 20.

[25] Lane, P., 2003. The cyclical behavior of fiscal policy: evidence from the OECD. Journal of Public Economics 87: 2661-2675.

[26] Leeper. E., 1991. Equilibria under “active” and “passive” monetary and fiscal policies. Journal of Monetary Economics 27: 129-147.

[27] Linnemann, L., 2006. Interest rate policy, debt, and indeterminacy with distortionary taxation. Journal of Economic Dynamics and Control 30: 487-510.

[28] Laubach, T., 2009. New evidence on the interest rate effects of budget deficits and debt. Journal of the European Economic Association 7: 858-885.

[29] Lo, S., Rogoff, K., 2015. Secular stagnation, debt overhang and other rationales for sluggish growth, six years on. BIS Working Paper 482, January.

[30] Mendoza, E., Ostry, J., 2008. International evidence on fiscal solvency: is fiscal policy "responsible"? Journal of Monetary Economics 55: 1081-1093.

[31] Nerlich, C., Reuter, W., 2013. The design of national fiscal frameworks and their budgetary impact. European Central Bank Working Paper 1588, September.

[32] Nerlich, C., Reuter, W., 2015. Fiscal rules, fiscal space and procyclical fiscal policy. ECB Working Paper 1872, December.

[33] Ostry, J., Ghosh, A., Espinoza, R., 2015. When should public debt be reduced? IMF Staff Discussion Note SDN/15/10, June.

[34] Pergallini, A., 2014. A note on nonlinear fiscal regimes and interest rate policy. Macroeconomic Dynamics 1-13.

[35] Prasad, E., 2014. *The Dollar Trap: How the U.S. Dollar Tightened its Grip on Global Finance*, Princeton University Press.

[36] Reinhart, C., Rogoff, K., 2010. Growth in a time of debt. American Economic Review 100: 573-578.

[37] Reinhart, C., Reinhart, V., Rogoff, K., 2015. Dealing with debt. Journal of International Economics 96 (Supplement 1): S43-S55.

[38] Rogoff, K., Tashiro, T., 2015. Japan's exorbitant privilege. Journal of the Japanese and International Economies 35: 43-61.

[39] Schmitt-Grohe, S., Uribe, M., 2003. Closing small open economy models. Journal of International Economics 61: 163-185.

[40] Shiamptanis, C., 2015. Risk assessment under a nonlinear fiscal policy rule. Economic Inquiry 53: 1539-1555.

[41] Talvi, E., Végh, C., 2005. Tax base variability and procyclical fiscal policy in developing countries. Journal of Development Economics 78: 156-190.

[42] Wyplosz, C., 2013. Fiscal rules: Theoretical issues and historical experiences, in *Fiscal Policy after the Financial Crisis*, Alesina, A., Giavazzi, F., (eds.), University of Chicago Press.

Appendix A. Expected debt ratio existence: proof of eqs. (10)-(11)

Applying the debt rule in eq. (7) to the debt evolution in eq. (9):

$$\begin{aligned} d_{t+1} &= (1+r_t-g)d_t - \rho(1+r_t)d_t \\ &\quad -\phi(1+r_t)^2 d_t^2 - \psi(1+r_t)^3 d_t^3 - \mu_{t+1} \end{aligned} \tag{A.1}$$

and taking unconditional expectations yields:

$$\begin{aligned} Ed_{t+1} &= (1-\rho)E\{(1+r_t)d_t\} - \phi E\{(1+r_t)^2 d_t^2\} \\ &\quad -\psi E\{(1+r_t)^3 d_t^3\} - gEd_t - \mu \end{aligned} \tag{A.2}$$

Recall the long-run covariances are $\Theta \equiv cov(1+r_t, d_t)$, $\Lambda = cov\{(1+r_t)^2, d_t^2\}$ and $\Gamma = cov\{(1+r_t)^3, d_t^3\}$. Expression (A.2) is then written as

$$\begin{aligned} Ed_{t+1} &= (1-\rho)E(1+r_t)Ed_t + (1-\rho)\Theta \\ &\quad -\phi cov\{(1+r_t)^2, d_t^2\} - \phi E(1+r_t)^2 E(d_t^2) \\ &\quad -\psi E\{(1+r_t)^3 d_t^3\} - gEd_t - \mu \end{aligned} \tag{A.3}$$

Applying the steady state definition $Ed_{t+1} = Ed_t = d^*$ to expression (A.3) yields:

$$\begin{aligned} d^* &= (1-\rho)(1+r)d^* + (1-\rho)\Theta \\ &\quad -\phi(\sigma_r^2 + (1+r)^2)(\sigma_d^2 + d^{*2}) \\ &\quad -\phi\Lambda - \psi E((1+r_t)^3 d_t^3) - gd^* - \mu \end{aligned} \tag{A.4}$$

where $\sigma_d^2 = E(d_t^2) - d^{*2}$ and $\sigma_r^2 = E(1+r_t)^2 - (1+r)^2$ are both unconditional variances and $\Lambda \equiv cov\{(1+r_t)^2, d_t^2\}$. Rearranging expression (A.4):

$$\begin{aligned} & \psi E[(1+r_t)^3 d_t^3] + \phi(\sigma_r^2 + (1+r)^2)\ d^{*2} + [\rho(1+r) - (r-g)]\ d^* \\ & +\phi(\sigma_r^2 + (1+r)^2)\sigma_d^2 - (1-\rho)\Theta + \phi\Lambda + \mu \\ = \ & 0 \end{aligned} \tag{A.5}$$

The first term in eq. (A.5) equals

$$\begin{aligned} & E[(1+r_t)^3 d_t^3] \\ = \ & cov\{(1+r_t)^3, d_t^3\} + E(1+r_t)^3 E(d_t)^3 \\ = \ & \Gamma + (\gamma_r \sigma_r^3 + 3(1+r)\sigma_r^2 - (1+r)^3) \\ & \cdot(\gamma_d \sigma_d^3 + 3d^* \sigma_d^2 - d^{*3}) \end{aligned} \tag{A.6}$$

where $\Gamma \equiv cov\{(1+r_t)^3, d_t^3\}$ and γ_r and γ_d are the third central moments of $1+r_t$ and d_t around their respective means. To expand the second term in (A.6) we apply $Ex^3 = \gamma_x \sigma_x^3 + 3(Ex)\sigma_x^2 - (Ex)^3$, where $\gamma_x = E\left(\frac{x-Ex}{\sigma_x}\right)^3$ and x is either $1+r_t$ or d_t. Substituting (A.6) into (A.5) and dividing through by ψ yields a cubic polynomial in d^*:

$$\Upsilon(d^*) \equiv d^{*3} + a_1 d^{*2} + a_2 d^* + a_3 = 0 \tag{A.7}$$

with coefficients

$$\begin{aligned} a_1 &= \phi(\sigma_r^2 + (1+r)^2)e \\ a_2 &= (\rho(1+r) - (r-g))e - 3\sigma_d^2 \\ a_3 &= [\mu - (1-\rho)\Theta + \phi\Lambda + \phi(\sigma_r^2 + (1+r)^2)\sigma_d^2 + \psi\Gamma]e \\ & \quad -\gamma_d\ \sigma_d^3 \end{aligned} \tag{A.8}$$

where

$$e = -\psi^{-1}[\gamma_r \sigma_r^3 + 3\sigma_r^2(1+r) - (1+r)^3]^{-1} \tag{A.9}$$

Expression (A.7) is polynomial $\Upsilon(d^*)$ in eq. (10) and its coefficients (a_1, a_2, a_3) are given in eq. (11). The Mendoza and Ostry (2008) solution in eq. (5) follows as a special case when all non-linear reaction terms in the debt rule are set to zero and the linear coefficient is modified to $\rho(1 + r_t - g)$, rather than $\rho(1 + r_t)$ as in (A.1).

Appendix B. Expected debt ratio stability: proof of inequality (14)

Expressing the debt evolution in discrete time, define G as

$$\begin{aligned} d_{t+1} &= d_t(1+r-g) - f(d_t) \\ &= G(d_t) \end{aligned} \tag{B.1}$$

where G is a continuous differentiable function with d^* a fixed point. Hence in steady state $G(d^*) = (1+r-g)d^* - f(d^*)$, where f is the cubic debt rule in eq. (6). That implies:

$$G'(d^*) = 1 + r - g - f'(d^*) \tag{B.2}$$

We now apply theorem 6.5 from Holmgren (1996), stated here without proof. If

$$\mid G'(d^*) \mid < 1 \tag{B.3}$$

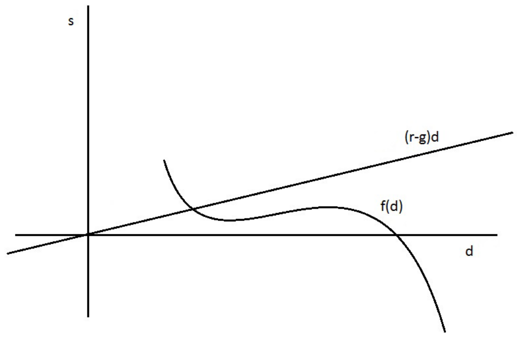
then there exists an open interval D containing d^* such that $G^n(d)$ converges to d^* for all $d \in D$ and $n \in \mathbb{Z}$. Conversely, if $\mid G'(d^*) \mid > 1$ then there exists an open interval containing d^* such that all points in the interval that are not equal to d^* must leave the interval under iteration of G. When G is the debt evolution equation whose fixed point is d^*, (B.3) becomes

$$-1 < 1 + r - g - f'(d^*) < 1 \tag{B.4}$$

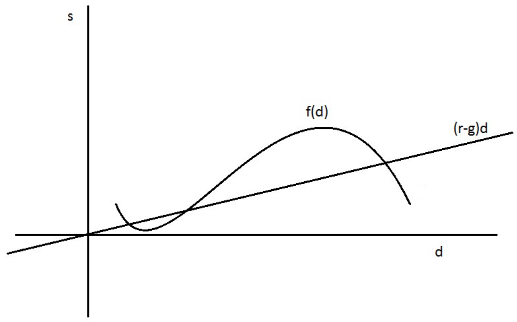
The lower bound in (B.4) is unrealistic as it implies an infinite net credit-output ratio. We are then left with

$$f'(d^*) > r - g \tag{B.5}$$

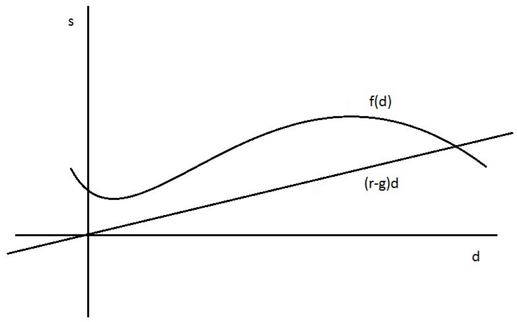
which is the stability criterion in inequality (14). □



Case I



Case II



Case III

Figure 1. Equilibrium Debt Ratios with Fiscal Fatigue

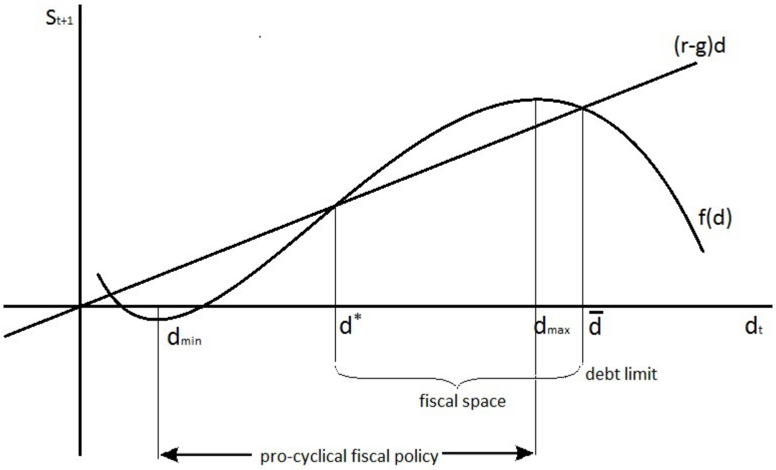


Figure 2. Endogenous Debt Limit and Fiscal Space

Figure 3. Estimated Fiscal Reaction Functions