This paper develops a theoretical framework for the study of war and conquest. The analysis highlights the role of three factors - the technology of war, resources, and contiguity network - in shaping the dynamics of appropriation and the formation of empires. The theory illuminates important patterns in imperial history.
The Strategy of Conquest

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Abstract

This paper develops a theoretical framework for the study of war and conquest. The analysis highlights the role of three factors – the technology of war, resources, and contiguity network – in shaping the dynamics of appropriation and the formation of empires. The theory illuminates important patterns in imperial history.

Keywords Balance of power, buffer state, empire, contest success functions, contiguity network, preemptive war, resources.

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1 Introduction

The history of the world .... is an imperial history, the history of empires. Empires were systems of influence or rule where ethnic, cultural or ecological boundaries were overlapped or ignored. Their ubiquitous presence arose from the fact that .... the endowments needed to build strong states were very unequally distributed. Against the cultural attraction, or physical force, of an imperial state, resistance was hard, unless reinforced by geographical remoteness or unusual cohesion. (Darwin [2007]; page 491)

For who is so indifferent or indolent as not to wish to know by what means and under what system of polity the Romans in less than fifty three years (220-167) succeeded in subjecting nearly the whole inhabited world to their sole government – a thing unique in history? (Polybius [2010] Volume 1; page 5)

In the study of history, a recurring theme is that the presence of small kingdoms is accompanied by bloody conflict; rulers fight each other incessantly, small parcels of land are exchanged, treasures are plundered, and capture of human beings is common. However, once a ruler acquires a large advantage relative to his neighbours he then quickly goes on to take them over, one after the other, and to create an empire. These empirical patterns lead us to ask: What are the circumstances under which rulers will choose to fight? What is the optimal timing of attack – now or later? When will the resource advantage of a ruler translate rapidly into domination over neighbours? What are the limits to the size of the empire? The goal of this paper is to develop a theoretical framework that can address these questions.

We consider a set of ‘kingdoms’. Every kingdom is endowed with resources and controlled by a ruler. Rulers desire to expand territory and acquire more resources. The ruler can wage a war on neighboring kingdoms. The winner of a war takes control of the loser’s resources and his kingdom; the loser is eliminated. The probability of winning a war depends on the resources of the combatants and on the technology of war that is defined by a contest success function. As the winning ruler expands his domain, he may be able to access and attack new
kingdoms. The neighborhood structure between kingdoms is reflected in a *contiguity network*. We model the interaction between rulers as a dynamic game and study its (Markov Perfect) equilibria.

Theorem 1 shows that there exists a pure strategy Markov Perfect equilibrium and the equilibrium payoffs are unique. This sets the stage for a study of how the main parameters – resources, the contiguity network, and the contest function – affect the dynamics of war and peace.

We start with the incentives to fight. Consider two rulers $A$ and $B$, with resources $x_A$ and $x_B$. When they fight, the expected payoff of $A$ is given by $(x_A + x_B)p(x_A, x_B)$, where $p(x_A, x_B)$ is the contest success function that defines the probability of winning for ruler $A$. Following Skaperdas [1996], a contest success function may be written as $p(x_A, x_B) = \frac{f(x_A)}{f(x_A) + f(x_B)}$, where $f$ is a positive and increasing function. Suppose ruler $A$ has more resources than ruler $B$. The contest success function is said to be *rich rewarding* if fighting is profitable for $A$ (and unprofitable for $B$), i.e., $(x_A + x_B)p(x_A, x_B) > x_A$. Conversely, the technology is said to be *poor rewarding* if $(x_A + x_B)p(x_A, x_B) < x_A$. We also examine the timing of optimal attack: should a ruler attack two rulers in a sequence rather than wait to fight the merged kingdom. Theorem 2 shows that a contest success function is rich rewarding if and only if $f$ exhibits increasing returns to scale and it is poor rewarding if and only if $f$ exhibits decreasing returns to scale. Moreover, a rich rewarding contest success function implies that attacking the two rivals in sequence is preferable; with a poor rewarding contest success function, attacking the larger kingdom formed after two rivals have fought is optimal.

Equipped with this result, we turn to the study of equilibrium dynamics.

First, we take up the case of a rich rewarding contest success function. Theorem 3 shows that in any configuration with three or more kingdoms, all rulers find it optimal to attack a neighbour. Thus, we are in a world with incessant warfare, the violence only stops when all opposition is eliminated. When the network is connected, all opposition is eliminated only with the hegemony of a single ruler.\(^2\) The arguments underlying this result are fairly general. We start by defining a *strong* ruler: this is a ruler who has a ‘full attacking sequence’ (involving

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\(^2\) Classical writers on war and more recent research both point to the decisive role of the army size and financial resources in securing victory, see e.g., Lewis [2010], Tzu [2008], Clausewitz [1993] and Howard [2009]. This is consistent with a rich rewarding contest success function. Some writers have drawn on the American experience in Vietnam to suggest that guerrilla warfare may give rise to a poor rewarding contest success function, see e.g., Hirshleifer [1995b]. For a detailed discussion on the empirical relevance of these concepts, see Section 6.

\(^3\) A network is connected if there is a path between any two kingdoms.
all other opponents), such that at each point he is stronger than the opponent. Clearly, at any point in time, the richest ruler is a strong ruler. It follows from the rich rewarding property that, if everyone else is peaceful, then such a strong ruler has a strict incentive to fight every other ruler. Next consider the case when other rulers may also wish to attack: does the strong ruler still have an incentive to implement a fully attacking sequence? The key observation here is a no-waiting property: when the contest success function is rich rewarding, it is always better to attack two opponents right away rather than to wait for them to fight each other (Theorem 2). It follows that the strong ruler has a dominant strategy to implement the full attacking sequence. So, there is always at least one ruler who wishes to fight to the finish. Anticipating this, and given the no-waiting property, every ruler, no matter how poor, has an incentive to fight a neighbour. Thus, in a connected network, in equilibrium, eventually there will be only one ruler left.

Turning to the role of resources and networks in shaping the prospects of individual rulers, for ease of exposition, in this part, we restrict attention to the well known Tullock Contest Function: the probability of ruler $A$ winning is $p(x_A, x_B) = x_A^\gamma / (x_A^\gamma + x_B^\gamma)$, for some $\gamma \in \mathbb{R}_+$. It can be shown that the function is rich rewarding if $\gamma > 1$ and poor rewarding if $\gamma < 1$ (and rulers are indifferent between war and peace if $\gamma = 1$). Proposition 1 highlights the role of strong and weak rulers: when $\gamma$ is large, the probability of a weak ruler becoming a hegemon becomes negligible. Within the set of strong rulers, those who have ‘exclusive’ access to weak kingdoms that have a significantly greater probability of becoming the hegemon (relative to their strong rivals).

Second, we take up poor rewarding contest success functions. Observe that, by definition, a poor ruler gains from fighting a rich rival. However, in this setting, waiting is better: so the poorer ruler would prefer to wait and allow for opponents to become large before engaging in a fight (Theorem 2). This gives rise to the prospect of peace. To make progress we divide the analysis into two parts. To start, consider resource distributions with a single rich ruler: if this ruler is sufficiently rich then his kingdom becomes an ‘irresistible’ prize; all other rulers have a strict incentive to fight to acquire the rich kingdom. So peace cannot be sustained and the outcome is hegemony. Next, consider the case where no ruler is very rich. Here we show that perpetual peace and a phase of war followed by peace may be sustained in equilibrium. The key to sustaining peace is the threat of imminent war. The equilibrium has the following structure: no ruler wishes to fight a single fight because, once this fight is undertaken, all rulers have an incentives to fight till the finish. It is this latter phase of war that makes war today unattractive. These arguments are summarized in Proposition 2. We illustrate...
through examples that the role of inequality is more general: across a range of networks and contest success functions, peace is more likely when resources are more similar. And, we illustrate through examples, that the dynamics of appropriation in the poor rewarding setting are ‘equalizing’. This is most clearly seen when Tullock parameter $\gamma$ is close to 0: across a range of networks, the equilibrium payoffs of all rulers are then more or less equal.

In the basic model, a conflict always yields a winner and a loser, all the resources are taken over by the winner, and the winner can implement a full attacking sequence (while all other rulers remain passive). Section 5 shows how our methods of analysis can be applied after we relax these features. The dynamics are now considerably richer and they yield new insights. The introduction of ties and gradual capture of parts of an empire leads to dynamics in which conflict among rulers can be protracted and involves the exchange of small parcels of territories. However, once a ruler succeeds in expanding his war making capacity, the rich rewarding property reinforces the strength and this rapidly speeds up the emergence of a hegemon. An extension of the model is presented in which a ruler can choose a single attack only: this accommodates the idea that rival rulers can become active once a ruler begins an attack sequence. We show that the incentives to wage war remain strong in this setting: hegemony is still the norm (Proposition 3). These three extensions suggest ways in which our central results on expansion and hegemony continue to obtain in more realistic models.

The interest turns next to the possibility of rivals forming an alliance to resist the aggression of an active ruler. When a ruler is picked to fight, the other rulers can form an alliance: this alliance puts together resources of all members to defend attack against any of them. The study of such defensive alliances delineates the circumstances under which a ‘balance of power’ can help restrain aggression and limit hegemony. The final extension involves (proportional) losses in war. We show that this means that a richer ruler will only wish to attack a poorer ruler if the resource differences are neither too small nor too large. This creates the possibility of ‘buffer’ states: poor kingdoms that are located in between large powerful neighbors and prevent the progression of conflict. These two extensions highlight the ways in which the framework can be enriched in ways that accommodate forces that limit the scope of hegemony.

We then relate our theory to history. Our theoretical framework has three key ingredients – resource seeking rulers, attack sequences constrained by the contiguity network, and contest

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4In the basic model, rulers allocate all resources to fighting; we consider a variant of the model with the guns vs butter trade-off and show that it provides a foundation for rich/poor rewarding contest success function. We also study resurrection: with a positive probability a defeated ruler can revive. In the basic model, we consider symmetric contest success functions; it is fairly straightforward to incorporate asymmetric rulers. Due to space constraints, these extensions are presented in an Online Appendix.
success functions. Section 6 begins by providing empirical background on them. Then the
section takes up three major historical episodes of expansion and empire formation – First
Chinese Empire, the Roman Empire, and the Spanish and Portuguese Empire in the New
World. In particular, the first two case studies draw attention the role of resource advantage
and rich rewarding contest success function in understanding the reinforcing expansion that
characterized the process. In these case studies, and extent of the empire was restricted by the
connectivity of the network. Our third case study highlights the malleability of the contiguity
network: the discovery of the New World and the changes in technology – a powerful navy,
gunpowder technology and seige forts – led to a reconfiguration of the contiguity network.
Spain and Portugal (and the other European powers) could access, and eventually conquer,
the New world. This ushered in the age of global empires. Finally, we take up a case study
where order among kingdoms arises out of a balance of power: this is Europe in the period
1650 AD - 1950 AD. We show how shifting alliances were effective in constraining a sequence
of dominant powers and thereby prevented a hegemony.

We now place our paper in the context of the literature and clarify its contributions. Our
paper studies the dynamics of war and peace and the formation of empires; related work
includes Hirshleifer [1995a], Jordan [2006], Krainin and Wiseman [2016], Levine and Modica
[2013], [2016], and Piccione and Rubinstein [2007]. A number of aspects of our framework
set it apart from existing work: we develop a non-cooperative and dynamic game with far-
sighted players, we consider general contest functions, and there is a network structure which
shapes the sequence of attack strategies and the scale of empires. Our analysis introduces
new concepts – rich/poor rewarding contest success functions and strong/weak rulers. They
enable us to address a range of very different questions, such as the timing and monotonicity
of optimal attack strategies, and how the prospects of individual rulers depend on the network
and on the nature of the contest success function. Finally, we provide a careful mapping from
our results onto the history of empires. Along all these dimensions we go beyond the existing
work.

The theoretical framework combines elements from the literature on contests, on resource

\footnote{Our framework also permits an exploration of major theories in the modern theory of international relations. A central tension in the literature concerns the contrasting prescriptions of ‘offensive’ and ‘defensive’ realism, see e.g., [Betts] 2013, [Mearsheimer] 2001 and [Waltz] 1979. Roughly speaking, ‘offensive’ realism advocates a strategy of persistent combativeness and aggression, while ‘defensive’ realism favors a strategy of restraint. Our paper reconciles these theories and locates their rationale in observable parameters such as the contiguity network, resources, and contest success functions. These points are discussed in the Online Appendix.}
wars, and on networks. We now discuss the relationship between our paper and these literatures.

There is a large literature on contests, for surveys see Konrad [2009] and Garfinkel and Skaperdas [2012]. We consider a general model of multi-player contests inspired by the axiomatic work of Skaperdas [1996]. In recent work, Konrad and Kovenock [2009], Groh, Moldovanu, Sela, and Sunde [2012], and Anbarci, Cingiz, and Ismail [2018] study multi-player sequential contests. In these papers the contest takes the form of an all-pay auction. The interest is in how individual heterogeneity and the sequential contest structure determine aggregate efforts and winning probabilities. By contrast, in our model, we abstract away from effort so that we can study the dynamics of conflict with general contest success functions and networks. To the best of our knowledge, the results on rich/poor rewarding contest success functions and strong/weak rulers, and the mapping from these results to imperial history, are novel in the context of this literature.

The role of resources in shaping violent conflict is an active field of study, see e.g., Acemoglu, Golosov, Tsyvinski, and Yared [2012], Caselli, Morelli, and Rohner [2013], and Novta [2016]. This literature provides evidence for appropriation of resources as a major motivation for war. The theoretical work is mostly limited to two players or to symmetric models; for an overview of the theory, see Baliga and Sjöström [2012]. Our paper contributes to this literature by studying the cumulative dynamics of appropriation and the expansion of territory within a contiguity network, and by linking these dynamics to major episodes of world history.

Finally, our paper is a contribution to the recent literature on conflict and networks, see e.g., Franke and Öztürk [2015], Hiller [2017], Kovenock and Roberson [2012], Huremović [2015], Jackson and Nei [2015], and König, Rohner, Thoenig, and Zilibotti [2017]. For an overview see Dziubiński, Goyal, and Vigier [2016]. Our paper advances this literature on two fronts: one, the dynamics of appropriation in inter-connected conflict and two, how these dynamics are decisively shaped by the contiguity network, the resources, and the contest success function.

The rest of the paper is organized as follows. Section 2 presents the basic model. Section 3 studies the incentives to fight and the optimal timing of attack. Section 4 presents the results on equilibrium dynamics and Section 5 discusses extensions of the basic model. Section 6

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6 For an early study of optimal strategy of attack in a three player game, see Shubik [1954]. Olszewski and Siegel [2016] study static contests with a large numbers of players.

7 In our model, a rich rewarding contest success function provides a rationale for waging a sequence of wars due to the compounding of spoils of war. This bears some resemblance to the earlier work of Garfinkel and Skaperdas [2000] and McBride and Skaperdas [2014] who study incentives for war in settings where rewards extend through time. In their model, war today is attractive as it facilitates expansion tomorrow.
provides a detailed mapping of the model onto imperial history – here we discuss the relation between aspects of history and the key ingredients of the model. We demonstrate how the model offers useful lens through which to view a number of major developments in imperial history. Section 7 concludes. The Appendix contains the proofs of the main results, while the Online Appendix covers a number of supplementary points.

2 The Model

We study a dynamic game in which rulers seek to maximize the resources they control by waging war and capturing new territories. There are three building blocks in our model: the interconnected ‘kingdoms’, the resource endowment for every kingdom, and the contest success function.

Let $V = \{1, 2, \ldots, n\}$, where $n \geq 2$ is the set of vertices. Every vertex $v \in V$ is endowed with resources, $r_v \in \mathbb{R}^{++}$. The vertices are connected in a network, represented by an undirected graph $G = \langle V, E \rangle$, where $E = \{uv : u, v \in V, u \neq v\}$ is the set of edges (or links) in $G$. A network $G$ is said to be connected if there is a path between any two vertices. For expositional simplicity, we restrict attention to (undirected) connected networks. Our insights extend in a natural way to directed networks.

A link between two vertices signifies ‘access’. Access may reflect physical contiguity. But, in principle, it goes beyond geography: we do not restrict attention to planar graphs. So our model allows for ‘virtual’ links, i.e., links made possible by advances in military and transport technology.

 Every vertex $v \in V$ is owned by one ruler. At the beginning, there are $N = \{1, 2, \ldots, n\}$ rulers. Let $\omega : V \rightarrow N$ denote the ownership function. The resources of ruler $i \in N$ under $\omega$, are given by

$$R_i(\omega) = \sum_{v \in \omega^{-1}(i)} r_v$$

(1)

The network together with the ownership configuration induces a neighbor relation between the rulers: two rulers $i, j \in N$ are neighbors in network $G = \langle V, E \rangle$ if there exists $u \in V$, owned by $i$, and $v \in V$, owned by $j$, such that $uv \in E$. Figure 1 illustrates vertices, resource endowments, and connections; vertices controlled by the same ruler share a common colour. The light line between vertices represents the interconnections, the dotted lines encircling

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8A graph is planar if it can be embedded in a plane, i.e. drawn in a plane in such a way that the edges intersect at their endpoints only. An example of a graph that is not planar is a clique with 5 nodes.
vertices owned by the same ruler indicate the ownership configuration, and the thick lines between vertices reflect the induced neighborhood relation between rulers.

![Figure 1: Neighboring Rulers](image)

When two rulers fight, the probability of winning is specified by a contest success function. Following Skaperdas [1996], we consider symmetric contest success functions with no ties. Given two rulers, A and B, with resources $x_A \in \mathbb{R}_{++}$ and $x_B \in \mathbb{R}_{++}$, respectively, $p(x_A, x_B)$ is the probability that A wins the conflict and $p(x_B, x_A)$ is the probability that B wins the conflict. Section 5 extends our model to allow for ties. Asymmetric contest success functions are presented in the Online Appendix.

The game takes place in discrete time: rounds are numbered $t = 1, 2, 3, \ldots$. At the start of a round, each of the rulers is picked with equal probability. The chosen ruler, (say) $i$, chooses either to be peaceful or to attack one of his neighbors. If a ruler attacks a rival, he does so with all his current resources. If he chooses peace, one of the remaining rulers is picked, and asked to choose between war and peace, and so forth. If no ruler chooses war, the game ends. If the attacker loses, the round ends. Otherwise, the attacker is allowed to attack neighbors.
until he loses, chooses to stop, or there are no neighbors left to attack. When two rulers $i$ and $j$ fight, the winner takes over the entire kingdom of the loser (and also inherits the boundaries, and hence the connections). This dynamic is illustrated in Figure 1: the orange kingdom wins the war with the red kingdom and expands. This expansion brings it in contact with new neighbors, the light and dark green kingdoms. The game ends when all rulers choose to be peaceful (the case of a single surviving ruler is a special case, as there is no opponent left to attack). Observe that, given these rules, the game ends after at most $n - 1$ rounds. It may of course end earlier: this happens if all the rulers choose peace at a round.

The configuration of kingdoms and rulers – who is a neighbor of whom – is (potentially) evolving over time. Given a set of vertices $U \subseteq V$, $G[U] = \langle U, \{vu \in E : v, u \in U\}\rangle$ is the subgraph of $G$ restricted to vertices in $U$ and links between them. The set of valid ownership configurations, given graph $G$, is denoted by

$$\mathcal{O} = \{o \in N^V : \text{for all } i \in N, G[o \circlearrowright i(i)] \text{ is connected}\}.$$  

As the graph is fixed, for simplicity, we omit it as an argument.

A state is a pair $(o, P)$, where $P \subseteq N$, is the set of rulers who were picked prior to $i$ and chose peace at $o$. Ruler $i$, picked at state $(o, P) \in \mathcal{O} \times 2^{N \setminus \{i\}}$, chooses a sequence of rulers to attack. A sequence $\sigma$ is feasible at $o$ in graph $G$ if either $\sigma$ is empty, or if $\sigma = j_1, \ldots, j_k$ and for all $1 \leq l < k$, $j_l \notin \{i, j_1, \ldots, j_{l-1}\}$ and $j_l$ is a neighbor of one of the rulers from $\{i, j_1, \ldots, j_{l-1}\}$ under $o$ in $G$. A sequence $\sigma$ is attacking if it is non-empty. Let $N^*$ denote the set of all finite sequences over $N$ (including the empty sequence). A strategy of ruler $i$ is a function $s_i : \mathcal{O} \times 2^{N \setminus \{i\}} \rightarrow N^*$ such that for every ownership configuration, $o \in \mathcal{O}$, and every set of rulers, $P \subseteq N \setminus \{i\}$, $s_i(o, P)$ is feasible at $o$ in $G$. Given ruler $i \in N$ and graph $G$, the set of strategies of $i$ is denoted by $S_i$; $S = \prod_{i \in N} S_i$ denotes the set of strategy profiles.

The probability that ruler 1 with resources $R_1$ wins a sequence of conflicts with rulers with resources $R_2, \ldots, R_m$, accumulating the resources of the losing opponents at each step of the sequence is

De Jong, Ghiglino, and Goyal [2014] introduced a model of conflict with resources and a network: the key difference is that conflict is imposed exogenously. Links are picked at random and rulers must fight. By contrast, in the present paper, the choice of waging a war or being at peace is the central object of study.

Observe that the only feasible sequence for rulers who do not own any vertices, and for the ruler who owns all vertices, is the empty sequence.
\[
p_{\text{seq}}(R_1, \ldots, R_m) = \prod_{k=2}^{m} p \left( \sum_{j=1}^{k-1} R_j, R_k \right).
\]  

Given \(o\), a set of rulers, \(P\), and a strategy profile \(s = (s_1, s_2, \ldots, s_n) \in S\), the probability that the game ends at \(o'\), is given by \(F(o' \mid s, o, P)\). We shall sometimes refer to a final ownership configuration as an outcome. The expected payoff to ruler \(i\) from strategy profile \(s \in S\) at state \((o, P)\) is:

\[
\Pi_i(s \mid o, P) = \sum_{o' \in O} F(o' \mid s, o, P) R_i(o').
\]

Every ruler seeks to maximize his expected payoff. The goals of rulers have been studied extensively; for classical discussions see [Hobbes 1651], [Machiavelli 1992], and for more recent work see [Jackson and Morelli 2007]. In Section 6, we provide an empirical basis for our assumption that rulers seek to maximize the resources they control.\(^{11}\)

A strategy profile \(s \in S\) is a Markov perfect equilibrium of the game if and only if, for every ruler \(i \in N\), every strategy \(s'_i \in S_i\), and every state, \((o, P) \in O \times 2^{N \setminus \{i\}}\), \(\Pi_i(s \mid o, P) \geq \Pi_i((s'_i, s_{-i}) \mid o, P)\). Standard arguments can be employed to establish:

**Theorem 1.** Fix a connected graph \(G\). For any symmetric contest success function, \(p\), and any resource endowment, \(r \in \mathbb{R}^V_{++}\), there exists an equilibrium and all equilibria are payoff equivalent.

The proof is presented in the Appendix.

## 3 The Incentives to Fight

This section introduces a general class of contest success functions and presents general results on incentives to fight for the two and three ruler setting. The notions of rich and poor rewarding contest success functions are introduced and a characterization is presented in

\(^{11}\)We assume that ruler’s utility is linear in resources. Risk-averse and risk-loving preferences can easily be accommodated. Suppose utility is given by \(u(x)\), with \(u(0) = 0\), \(u' > 0\) and \(u'' < 0\). This means that \(u(x + y) < u(x) + u(y)\). Expected payoff to \(x\) vs \(y\) can be written as:

\[
p(x, y)u(x+y) = p(x, y)(u(x) + u(y))(1 - d(x, y))
\]

where \(d(x, y) = 1 - u(x+y)/(u(x) + u(y))\). So \(0 < d(x, y) < 1\): in other words, risk-aversion creates a ‘cost’ of conflict. Section [ develops a model of losses in war.
terms of standard properties such as increasing and decreasing returns. The interest then turns to the timing and order of optimal attacks: conditions on the contest success functions are obtained under which rulers prefer to wait/not wait to attack.

In general, a contest success function is function $q : \mathbb{R}_{++} \rightarrow [0,1]^2$. Following Skaperdas [1996], we consider three axioms for contest success functions:

A1 For all $(x_1, x_2) \in \mathbb{R}_{++}^2$, $q_1(x_1, x_2) + q_2(x_1, x_2) = 1$.

A2 For all $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, $q_i(x_i, x_j)$ is increasing in $x_i$ and decreasing in $x_j$.

A3 For all $(x_1, x_2) \in \mathbb{R}_{++}^2$, $q_1(x_1, x_2) = q_2(x_2, x_1)$.

By axiom A3, the contest success function is symmetric and can be represented by function $p : \mathbb{R}_{++}^2 \rightarrow [0,1]$, where $q_1(x_1, x_2) = p(x_1, x_2)$ and $q_2(x_1, x_2) = p(x_2, x_1)$. Skaperdas [1996] shows that a contest success function satisfying axioms A1-3 takes the form

$$p(x, y) = \frac{f(x)}{f(x) + f(y)}.$$  

with an increasing, positive, function $f : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$. The study of contests remains a very active field of study; see Fu and Pan [2015] for a recent contribution and for references to the literature.

Recall that $(x + y)p(x, y)$ is the expected payoff of a ruler with resources $x$ who fights an opponent with resources $y$. We shall say that the contest success function, $p$, is rich rewarding if for all $x, y \in \mathbb{R}_{++}$ with $x > y$,

$$(x + y)p(x, y) > x$$  

Similarly, we shall say that $p$ is poor rewarding if for all $x, y \in \mathbb{R}_{++}$ with $x < y$,

$$(x + y)p(x, y) > x$$

A rich rewarding contest success function gives the richer side an incentive to fight, while poor rewarding one gives the poorer side an incentive to fight. We characterize rich and poor rewarding contest success functions in terms of standard properties of the function $f$. We also examine the timing of optimal attack: whether to attack now or to wait and attack later. A

\[12\] In addition, $f$ is unique up to positive multiplicative transformations.
contest success function, $p$, is said to have the no-waiting property if for all $x,y,z \in \mathbb{R}^+$, $p(x,y)p(x+y,z) > p(x,y+z)$. It is said to have the waiting property if for all $x,y,z \in \mathbb{R}^+$, $p(x,y)p(x+y,z) < p(x,y+z)$. With contest success functions having the no-waiting property, it is profitable for a ruler to attack the other two rulers in a sequence rather than wait to fight the merged kingdom. The converse is true in the case of contest success functions that exhibit the waiting property. Rich/poor rewarding and the timing of attacks are intimately related.

**Theorem 2.** Consider a contest success function, $p$, that satisfies (6).

- The function $p$ is rich rewarding if and only if $f$ exhibits increasing returns to scale; it is poor rewarding if and only if $f$ exhibits decreasing returns to scale.
- If $p$ is rich rewarding then it has the no-waiting property, while if $p$ is poor rewarding then it has the waiting property.

The proof is presented in the Appendix. The argument underlying this result proceeds as follows. Suppose that $x > y$. If $f$ exhibits increasing returns then $f(x)/(f(x)+f(y)) > x/(x+y)$. Multiplying both sides by $x+y$ now yields the desired implication. On the other hand, if the stronger side gains in expectation, then it must be that $(x+y)f(x)/(f(x)+f(y)) > x$. Rewriting and rearranging this gives us the inequality $f(x)/(f(x)+f(y)) > x/(x+y)$, which requires that $f$ exhibits increasing returns. A similar line of reasoning applies to the poor rewarding case. With regard to the timing, we begin by showing that the no-waiting property is equivalent to $f$ being super-additive. The next step demonstrates that super-additivity is a weaker property than increasing returns to scale, and that concludes the proof.

We illustrate the scope of these results through a consideration of the widely studied Tullock contest success function.

$$p(x,y) = \frac{x^\gamma}{x^\gamma + y^\gamma},$$

where $\gamma > 0$. Hence, $f(x) = x^\gamma$. If $\gamma > 1$ then $f$ has increasing returns to scale. From Theorem 2 it follows that the contest success function is rich rewarding and has the no-waiting property. On the other hand, if $\gamma < 1$, then $f$ exhibits diminishing returns to scale. It is therefore poor rewarding and the ruler would prefer to wait. Finally, observe that $(x+y)p(x,y) = x$, for all $x,y \in \mathbb{R}^+$ if $\gamma = 1$. So the contest success function is reward neutral; it is also timing neutral (as for all $x,y,z \in \mathbb{R}^+$, $p(x,y)p(x+y,z) = p(x+y,z)$).

To summarize:

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13The Online Appendix presents a discussion of the Hirshleifer Difference Contest Function.
Corollary 1. The Tullock contest success function is rich rewarding and has the no-waiting property if \( \gamma > 1 \); it is poor rewarding and has the waiting property if \( \gamma < 1 \). It is reward and timing neutral if \( \gamma = 1 \).

Theorem 2 presents a characterization of incentives to fight for the two ruler problem and it also provides an analysis of the optimal timing for the fully connected three rulers problem. We have also studied the question of optimal order of attack: should a ruler attack a poorer and then a richer opponent or is the converse order more profitable? We develop a general condition under which optimal attacking sequence is monotonically increasing (decreasing) in wealth. This is presented in the Online Appendix. In the case of the Tullock Contest Function this condition yields a clean implication: if \( \gamma > 1 \) then the optimal attack strategy prescribes attacking rivals in increasing order of resources; the converse holds if \( \gamma < 1 \). These results set the stage for the study of \( n \geq 3 \) rulers located in a connected network.

4 Conquest and Empire

This section studies equilibrium dynamics of war and peace and the formation of empires. The analysis for the rich rewarding case is reasonably complete: we show that equilibrium is characterized by incessant warfare and that the outcome is hegemony. The connectivity of the network defines the limits of the hegemony. The concepts of strong and weak rulers – that reflect resources and network architecture – play a key role in this analysis. The analysis of poor rewarding contest functions is more partial because the dynamics are considerably more complicated: we show that perpetual peace, perpetual war (and hegemony), and a phase of war followed by peace can all arise in equilibrium. Greater equality in initial resources makes peace more likely.

Given ownership configuration \( o \), the set of active rulers at \( o \) is

\[
\text{Act}(o) = \{ i \in N : \emptyset \subsetneq o^{-1}(i) \subsetneq V \}.
\]

An ordering of the elements of the set \( \text{Act}(o) \setminus \{i\} \), \( \sigma \), such that the sequence \( \sigma \) is feasible for \( i \) in \( G \) under \( o \) is called a full attacking sequence (or f.a.s). Figure 2 illustrates such a sequence (for the orange kingdom).

We are now ready to state our first main result on equilibrium dynamics.

Theorem 3. Consider a rich rewarding contest success function that satisfies (6). Suppose \( G \) is a connected network and let \( r \in \mathbb{R}_{++}^V \) be a generic resource profile. In equilibrium, every
active ruler chooses to attack a neighbor if $|A(o)| \geq 3$, and at least one of the active rulers attacks his opponent if $|A(o)| = 2$. The outcome is hegemony and the probability of becoming a hegemon is unique for every ruler.

The proof is presented in the Appendix. The result offers an account of the dynamics of conflict in a rich rewarding setting when rulers are driven by a desire to maximize resources under their control. It predicts incessant fighting, preemptive attacks, and long attacking sequences. It is worth drawing attention to the generality of this result: it holds for all rich rewarding contest functions, for any connected network, and for generic resources.

We discuss the arguments underlying the theorem. A ruler is said to be strong if he has an attacking sequence $\sigma = i_1, \ldots, i_k$, where for all $l \in \{1, \ldots, k\}$,

$$\sum_{j=0}^{l-1} R_{ij}(o) > R_{ik}(o).$$

In other words, at every step in the attacking sequence, the ruler has more resources than the next opponent. The set of strong rulers at ownership configuration $o$ is

$$S(o) = \{i \in \text{Act}(o) : i \text{ has a strong f.a.s. } \sigma \text{ at } o\}.$$

A ruler who is not strong is said to be weak. Note that (generically) in any state, the ruler with the most resources is strong, while the ruler with the least resources is weak. Thus both sets are non-empty in every network and for (generic) resource profiles.

The first step is to show that, assuming that all other rulers choose peace in all states,
it is optimal for a strong ruler to choose a full attacking sequence. This is true because the contest success function is rich rewarding and so a strong ruler has a full attacking sequence that increases his resources in expectation, at every step, along the sequence. The second step extends the argument to cover opponents who choose war. If opponents are active then the no-waiting property (from Theorem 2) tells us that it is even more attractive to not give them an opportunity to move. For a strong ruler it is therefore a dominant strategy to use an optimal full attacking sequence. The final step in the proof covers non-strong rulers to establish that with 3 or more active rulers, it is optimal for every ruler to choose a full attacking sequence. Observe that we have already shown that every non-strong ruler knows that he will be facing an attack sooner or later. This means that waiting can only mean that the opposition will become (larger and) richer. The no-waiting property then tells us that every ruler must attack as soon as possible. If there are only two active rulers then the richer ruler has a strict incentive to attack the poorer opponent (this follows from the definition of the rich rewarding contest function).

We now examine the role of the contiguity network and resources more closely. For expositional simplicity, we focus on the Tullock contest success function. Notice that, due to timing and order neutrality, there are no interesting network effects when \( \gamma = 1 \): equilibrium expected resources of any ruler remain equal to his initial resources. When \( \gamma \) is large it is never optimal to attack a richer ruler if other options are available. The optimal strategy for a strong ruler must involve attacking a poorer ruler at every stage in the attack sequence. Such a sequence is clearly not available for a weak ruler: the probability of a weak ruler becoming a hegemon converges to zero, as \( \gamma \) grows.

Given the initial ownership configuration \( o_0 \), a \( \gamma \), and resources \( r \), let \( \text{Prob}_i(r, \gamma | o_0) \) be the equilibrium probability of ruler \( i \) becoming the hegemon. Define

\[
\text{Prob}_i^*(r | o_0) = \text{Prob}_i(r, \lim_{\gamma \to +\infty} \gamma | o_0).
\]

**Proposition 1.** Suppose the contest success function is Tullock, the network \( G \) is connected, and the resources \( r \in \mathbb{R}^n_{++} \) are generic. The probability of a weak player becoming a hegemon becomes negligible as \( \gamma \) grows. Specifically,

\[
\text{Prob}_i^*(r | o_0) \begin{cases} 
\geq \frac{1}{|\text{Act}(o_0)|}, & \text{if } i \in S(o_0) \\
= 0, & \text{otherwise.}
\end{cases}
\]

The proof is presented in the Appendix.
Whether a ruler is strong or weak depends both on the distribution of resources and on the position of the ruler in the contiguity network. In Figure 3 we represent strong rulers in red and weak rulers in yellow. It is helpful to define the boundary of a set of vertices $U \subseteq V$ in $G$ is

$$B_G(U) = \{ v \in V \setminus U : \text{there exists } u \in U \text{ s.t. } uv \in E \}$$

A set of vertices, $U$, is weak if $G[U]$ is connected, $B_G(U) \neq \emptyset$, and for all $v \in B_G(U)$, $r_v > \sum_{u \in U} r_u$. A weak set of nodes is surrounded by a boundary, constituted of nodes, each of whom is endowed with more resources than the sum of resources of vertices within the set. Weak sets are illustrated in Figure 3. It is easy to see that, for any initial state $\phi$, a ruler is weak if his vertex belongs to a weak set and, otherwise, the ruler is strong.

![Figure 3: Weak rulers (surrounded by thick lines) and strong rulers](image)

Proposition 1 covers the case of large $\gamma$. We now turn to examples to show that the distinction between strong and weak rulers is central to the study of dynamics more generally, across rich rewarding $\gamma$. Consider three networks with 10 nodes: the clique network (with 45 links), a connected network with 27 links and a tree network (with 9 links). The resources endowments at the nodes are 2, 3, 6, 11, 13, 15, 16, 18, 21, and 23, respectively. The strong rulers are presented in purple, while the weak rulers are presented in yellow. These networks and resource endowments are presented in Figure 4. Observe that as we delete links from clique to obtain the network with 27 links, the number of weak rulers increases strictly (from 2 to 3) and the same happens as we go move from network with 27 links to the network with 9 links (the number goes up from 3 to 4).

We compute the equilibrium in these examples; the results are summarized in Figure 5.
The key point to note is that, even for $\gamma = 8$, the long run prospects of a ruler are essentially determined by whether he is strong or weak. Further study of examples that span a range of different values of $\gamma$ reveal that this pattern is reinforced when we increase $\gamma$.

![Figure 4: Examples of Networks](image)

Given the importance of strong and weak rulers, we briefly comment on how changes in resources and links affect the set of strong and weak rulers. Given a resource profile, observe that adding links to a network offers all rulers potentially more sequences of attack. This means that a ruler who was weak may now have a strong sequence. Adding links (weakly) therefore expands the set of strong rulers. The number of strong rulers is maximized in the complete network and it is minimized when the strongest ruler is at the center of a star network. Given a network and a resource configuration, an increase in resources of a ruler either maintains his status or switches him from weak to strong. Observe that an increase in resources of a ruler may well lead to another ruler becoming weak. From Proposition 1 we can infer that additional resources for one ruler can make a big difference to his and others’ long term prospects.

The discussion so far has focused on the difference between strong and weak rulers. We now argue that the network structure also shapes the relative prospects of different strong rulers. Consider an example with two strong rulers. Suppose the two rulers are 1 and 2, and they own vertices $v_1$ and $v_2$, respectively. The set of the remaining vertices, $V \setminus \{v_1, v_2\}$, can be partitioned into three sets: the set of nodes reachable from $v_2$ via $v_1$ only, denoted by $U_1$, the set of nodes reachable from $v_1$ via $v_2$ only, denoted by $U_2$, and the remaining nodes, $U_{12}$ (c.f. Figure 6).

To see the effects of the networks structure easily, suppose that $\gamma$ is large and that $r_{v_1} + R_{U_1} > r_{v_2} + R_{U_2} + R_{U_{12}}$. This ensures that ruler 1 remains strong as long as he is active.
Ruler 2, on the other hand, becomes weak if ruler 1 accumulates enough resources from the set $U_1$. The probability of ruler 1 becoming a hegemon is approximately $1/2 + q$, where $q$ is the probability that ruler 2 is picked to move before ruler 1 is picked to move and he is weak when that happens. Thus $q$ is the probability that 1 conquers sufficiently many nodes before ruler 2 is picked to move. To fix ideas, suppose that 1 needs to acquire all the nodes in $U_1$ to become uniquely strong. Suppose $|U_1| = k$. If $G[U_1]$ is a fully disconnected network then $q$ is approximately equal to $k!/(k+2)! = 1/((k+1)(k+2))$. If, on the other hand, $G[U_1]$ is a clique with $k-1$ strong rulers then $q$ is approximately equal to $(k-1)(k+1)!/(2(k+2)!)= (k-1)/(2(k+2))$. As $k$ gets large, the probability that ruler 1 becomes the hegemon converges to $1/2$ in the former case, and to 1 in the latter case.

Figure 6: Partitioning of a Graph with Two Strong Rulers

4.1 Poor Rewarding Contest Success Functions

We begin by recalling that in the poor rewarding setting, every bilateral conflict is profitable to the poorer of the two opponents. However, the poor rewarding property also implies that
rulers have a preference to wait before they fight. These two considerations suggest that the 
dynamics can be complicated. We are especially interested in the possibility of peace.

We start with noting that in equilibrium, at every ownership configuration, there is either 
peace or fight, regardless of the order in which the rulers are picked to move. Formally, given 
a strategy profile, $s$, an ownership configuration $o \in \mathcal{O}$ is *peaceful* under $s$, if for all $i \in N$
and all $P \in 2^{N \setminus \{i\}}$, $s_i(o, P)$ is the empty sequence. An ownership configuration $o \in \mathcal{O}$ is *conflictual* under $s$ if for every sequence $i_1, \ldots, i_n$ of rulers from $N$ there exists $k \in \{1, \ldots, n\}$
such that $s_{i_k}(o, \{i_1, \ldots, i_{k-1}\})$ is not empty. In other words, regardless of the order in which
the rulers are picked to move at $o$, one of the rulers chooses an attacking sequence.

By the observation above, the possibility of peace means that, in equilibrium, there exist
ownership configurations, with two or more active rulers, at which all the rulers prefer staying
peaceful to choosing fight. To make progress we divide the analysis into two parts: first, we
characterize situations where peace is impossible, and second, we turn to situations where
peace may be sustainable. We say that there is perpetual peace in a given strategy profile, if
the initial state is peaceful. We say that there is war followed by peace in a given strategy
profile if the initial state is not peaceful and no equilibrium outcome is hegemony.

**Proposition 2.** Consider a generic poor rewarding contest success function that satisfies (6).

1. For any connected network, $G$, and any generic resource endowment, $r \in \mathbb{R}_+^V$, every

   ownership configuration $o \in \mathcal{O}$ is either peaceful or conflictual in equilibrium.

2. For any connected network, $G$, any node $v \in V$, and any resource endowment of the

   other nodes, $r_{-v}$, there exists a resource level $\bar{r}_v$ such that for all $r_v > \bar{r}_v$, there is fight

   till hegemony in equilibrium under resource endowment $(r_v, r_{-v})$.

3. For any $n \geq 4$, there exists a network and a generic resource profile such that there is

   perpetual peace in equilibrium. Similarly, there exists a network and a generic resource

   profile such that there is war followed by peace in equilibrium.

Due to space constraints, the proof of this result is presented in the Online Appendix.
The result should be seen as a possibility result: it illustrates the rich range of outcomes
possible under the poor rewarding contest success function. A comparison of Theorem 3 with
Proposition 2 reveals contrasting optimal strategies (full attacking sequence versus no fighting)
and outcomes (hegemony versus multiple kingdoms) and highlights the key role of the contest
success function in shaping conflict dynamics. The hegemony result relies on quite different
arguments than the hegemony result under rich rewarding contest success function. In the poor rewarding case, the existence of a sufficiently rich ruler motivates other rulers to fight. However, due to the waiting property, these rulers may choose to fight only if others do not. This is in contrast to the rich rewarding case, where each ruler chooses fight whenever he is given a chance. The peace and war followed by peace outcomes rely on the idea of fear of conflict escalation. We propose a network and a (generic) resource profile for which, whenever any ruler chooses to fight, there will be fight till hegemony in the following states and the ruler who started the conflict will be involved in all the following conflicts. The resource endowments are such that it is never profitable for any ruler to be involved in fight till hegemony starting from the initial state. The main challenge is to show that such a resource endowment exists, for general $n$.

We next examine how networks, resources, and the contest success function affect the prospects of peace. We consider Tullock contest success function with two values of $\gamma$: 0.05 (low) and 0.8 (high) and networks with 10 nodes (as in the rich rewarding case). In addition we consider eight ranges of resources: $[45, 55]$, $[40, 60]$, $[35, 65]$, $[30, 70]$, $[25, 75]$, $[20, 80]$, $[15, 85]$, $[10, 90]$. For each triple of $\gamma$, number of links, $k$, and resource range, $[a, b]$, we pick 1000 random samples of connected networks of $k$ links with resources drawn uniformly from the set (of 10,000 evenly spaced values from) $[a, b]$. Figure 7 presents the frequencies of samples exhibiting peace in the first round as a function of the resource range. It suggests that peace is more likely when resources are drawn from a smaller range: this is true for both high and low values of $\gamma$ and true also across a wide range of networks. Taking together, Proposition 2 and our examples show that resource equality is conducive for peace.

This section concludes with an observation on equilibrium payoffs. We take up the same
three networks as in the rich rewarding case (from Figure 4) and we fix the Tullock parameter $\gamma$ to be equal to 0.05. Figure 8 presents the equilibrium payoffs and the Lorenz curves for the three networks and the initial resources. It is clear that, when $\gamma$ is very small, the equilibrium dynamics are powerfully equalizing. A comparison with Figure 5 also reveals the big difference between the rich and poor rewarding setting: the poorer kingdoms gain significantly in the latter setting, and this is reflected at the aggregate level via the Lorenz curves.

Figure 8: Equilibrium Payoffs and Lorenz Curves: $\gamma = 0.05$.

5 Extensions

In the basic model, in a war there is always a winner and a loser, the loser is eliminated, and all his resources are taken over by the winner. Moreover, the victor can employ his augmented resources to execute a sequence of attacks against rivals. We relax these features of the model and show that our methods of analysis can be applied to more general settings. The dynamics are now richer and this allows us to develop new insights. A general point that emerges is that conflict among rulers with small resources can be protracted and involve the exchange of small parcels of territories. Once a ruler has significant resources, the rich rewarding property will reinforce their strength, through further victories, and that can lead rapidly to the emergence of a hegemon. We next allow rivals of a currently active ruler to have opportunities to individually retaliate, before he executes further attacks. The main finding is that the incentives to wage war remain strong: hegemony is still the norm. These three extensions suggest ways in which the central results on expansion and hegemony continue to obtain in more realistic models.
The interest then turns to the possibility of rivals forming an alliance to resist the aggression of an active ruler. The study of such defensive alliances delineates the circumstances under which a ‘balance of power’ can help restrain aggression and limit hegemony. Finally, we introduce losses in war: an implication is that a richer ruler will only wish to attack a poorer ruler if the resource differences are neither too small nor too large. This creates the possibility for a small kingdom to act as a ‘buffer state’ and again help limit the extent of hegemony. The last two extensions highlight the ways in which the framework can be enriched in ways that accommodate forces which limit the scope of hegemony.

5.1 Gradual Conquest

We consider a situation in which the victor in a war acquires a part of the losers kingdom. This gradual expansion also creates the possibility that the losing ruler is not eliminated in a single battle, but that he survives and can engage in another fight. We will show that key elements of the analysis such as the incentive to fight and the emergence of hegemony carry over. The new element is that the equilibrium dynamics will typically involve parcels of land exchanging hands over time.

For expositional simplicity, we develop these points with the help of an example with two rulers $a$ and $b$, whose kingdoms are located on a line with four nodes. So the initial situation is as depicted in state V of Figure 9: kingdom $a$ covers nodes 1 and 2, while kingdom $b$ covers nodes 3 and 4. As before, a strategy is a mapping from the current ownership configuration to a sequence of attacks. The dynamics of conflict define a Markov Chain on five states as described in the figure.

Consider the incentives to fight in state III. If any ruler decides to fight at III, there will be two absorbing state reachable from III: I and one of the states, V, IV, or II, depending on whether the rulers choose peace at V and IV or not. Suppose that V is absorbing. Let $Q$ be the probability of reaching I from III (if fight is chosen by one of the rulers at III). Then state V is reached with probability $1 - Q$. Suppose that ruler $a$ does not find it profitable to fight at III: this means

$$Q(r_1 + r_2 + r_3 + r_4) + (1 - Q)(r_1 + r_2) < r_1 + r_2 + r_3.$$
Adding \( r_4 \) to both sides and rearranging terms we get

\[(1 - Q)(r_3 + r_4) > r_4.\]

This implies that ruler \( b \) has a strict incentive to attack ruler \( a \). This argument is fairly general: it extends easily, regardless of whether the other absorbing state is II, IV, or V. Moreover, similar considerations imply that in every state III, IV and V, one of the rulers always has an incentive to attack the other ruler. So, with a rich rewarding contest success function, when conquest involves gradual expansion of kingdoms, there is always war. Standard results in Markov Chains then allow us to infer that the dynamics will converge to one of the two absorbing states, states I and II; in other words, the outcome will be hegemony.

The example we have considered is simple: it has only two rulers and a specific network. But its draws out some important new features of the dynamics like exchange of parcels of land and gradual expansion. Based on this analysis, we conjecture that our arguments, on incentives to wage war and hegemony, should carry over to a setting with multiple rulers and a general network, if the contest success function is strongly rich rewarding.

### 5.2 Ties

In the basic model, every war yields a winner and a loser. The historical record shows that wars often ended without a clear victor, see Section 6. The possibility of ties can be easily accommodated. Following Blavatskyy [2010], replace axiom \( A_1 \) with additivity axiom, \( A_1' \), and introduce the axiom of independence of irrelevant alternatives, \( A_4 \). Suppose that \( q_{12}(x_1, x_2) \)
is the probability of a tie, given the resources $x_1$ and $x_2$.

**A1'** For all $(x_1, x_2) \in \mathbb{R}^2_{++}$, \( q_1(x_1, x_2) + q_2(x_1, x_2) + q_{12}(x_1, x_2) = 1 \),

**A4** For all $i, j \in \{1, 2\}$ the probability that $i$ wins given that $j \neq i$ does not win depends only on $x_i$, [Blavatskyy 2010](#) shows that a contest success function satisfying A1' and A2-4 takes the form

\[
q_i(x, y) = \frac{f(x)}{f(x) + f(y) + 1}, \quad q_{12}(x, y) = \frac{1}{f(x) + f(y) + 1},
\]

where $i \in \{1, 2\}$, with an increasing, positive, function $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$. It is important to note that the probability of ties is decreasing in the total resources of the two opponents involved in the war.

We can generalize our analysis to cover the case of a tie. Suppose that in the event of a tie, the current active ruler remains the ‘active’ ruler. Then it is possible to show that the distinction between rich and poor rewarding contest success functions can be generalized. In particular, Theorem [3](#) on rich rewarding continues to hold. However, the new element is that if rulers are small and poor then there may be long ‘intervals’ in which no kingdom is expanding. However, once a ruler wins some wars and acquires resources and expands, the probability of tie will decline and the further expansion will be rapid.

### 5.3 Short Attack Sequences

In the basic model, a ruler is allowed to choose a full attacking sequence of attacks. In particular, all other rivals remain passive, while this ruler executes this sequence. In this extension, we allow for rivals to have more opportunity to react and the goal of this section is to examine if our results are robust to this generalization.

We consider a variant of our model where rulers, when picked to move, can either choose peace or choose a sequence of attack of length 1 only, and then a new mover is drawn. A strategy of a ruler $i$ is a function $s_i : \mathcal{O} \times 2^{N \setminus \{i\}} \to N \cup \{\varepsilon\}$ such that for every ownership configuration, $o \in \mathcal{O}$, and every set of rulers, $P \subseteq N \setminus \{i\}$, $s_i(o, P)$ is feasible at $o$ in $G$, that is either $s_i(o, P)$ is empty or $s_i(o, P)$ consists of a neighbor of $i$ under $o$ in $G$. As the problem is especially relevant under the no-waiting property, in this discussion, we restrict attention to rich rewarding contest success functions. Notice that the proof of Theorem [1](#) can be adjusted in a straightforward way and so Theorem [1](#) is valid for the short-attack variant of the model.
In particular, equilibrium existence and payoff equivalence of equilibria hold in this model as well.

First, we take up the setting with a unique strong ruler. This situation arises naturally if one ruler controls more than half of the resources. But the condition is significantly more general. Given any network, $G$, recall that a maximal set of nodes such that any two distinct nodes in the set are reachable from each other by a path in $G$ is called a component in $G$. The set of all components of $G$ is denoted by $\mathcal{C}(G)$. In addition, given a set of nodes, $U \subseteq V$, $G - U = G[V \setminus U]$ denotes the graph obtained by removing the nodes in $U$ and all their links from $G$. A connected graph $G$ with resource endowment $\mathbf{r}$ has a unique strong node if and only if there exists a node $v \in V$ such that for every component $C \in \mathcal{C}(G - \{v\})$, $r_v > R_C$.

**Proposition 3.** Consider a rich rewarding contest success function that satisfies (6). Suppose the network $G$ is connected and a (generic) resource profile $\mathbf{r} \in \mathbb{R}^n_{++}$ is such that there is exactly one strong node. In equilibrium, at every ownership configuration, $\omega$, at least one ruler attacks his neighbor. So the outcome is hegemony and the probability of becoming a hegemon is unique for every ruler.

The proof is presented in the Online Appendix. The first observation is that if there is a unique strong ruler under some ownership configuration, then in every ownership configuration that follows in the course of the game, there is also a unique strong ruler. This is because no weak ruler can become strong, unless he fights and beats a strong ruler (in which case he becomes the unique strong ruler). Given this observation, we now show that at any state, for any strategy profile of the other rulers, the unique strong ruler increases his resources in expectation using the ‘optimal attacking’ strategy. We proceed by induction. For two rulers, which is the base step, the claim clearly holds. Assume now that the claim holds for $k$ rulers. We show that the result holds for $k+1$ active rulers. This is because, due to the rich-rewarding contest success function, any fight between the strong ruler and any other ruler increases his resources in expectation and then, by the induction hypothesis, the expected resources at the end of the game are even higher. If any other two rulers fight, then the resources of the strong ruler remain unchanged in any following state and then, by the induction hypothesis, they increase. Thus, in any equilibrium there cannot be peace, because, at any ownership configuration, the strong ruler prefers to attack one of his neighbors over remaining peaceful.

We now examine the case of multiple strong rulers, with the help of examples. Consider the same three networks (with corresponding resources) as in the basic model (c.f. Figure [4]) and assume that $\gamma = 8$. Equilibrium payoffs and Lorenz curves for the results are presented
in Figure 10. By way of illustration, the figure contains also the corresponding outcomes for the basic model. In all the examples every ruler chooses to fight in every state and so the outcome is hegemony. As in the case of the basic model, the expected resources of the weak rulers are close to 0. There is, however, much greater variation across the strong rulers. Their equilibrium payoffs are more affected by the initial resource distribution, as compared to the basic model. The ‘richest’ ruler gains most from the dynamics and has much higher expected payoffs. The Lorenz curves confirm this point: the one-step dynamics lead to greater inequality than the dynamics in the basic model.

Next, we study the frequency of peace. Suppose again that $n = 10$ nodes. We run calculations for $\gamma \in \{2, 4, 8, 16, 32\}$, number of links, $k \in \{9, 18, 27, 36, 45\}$, and resource ranges $[45, 55]$, $[40, 60]$, $[35, 65]$, $[30, 70]$, $[25, 75]$, $[20, 80]$, $[15, 85]$, and $[10, 90]$. For each combination of the three parameters we have drawn 1000 random samples. In each case we observe that there is fight till hegemony in equilibrium.
Taken together, Proposition 3 and these examples suggest that incessant warfare and the emergence of hegemony are robust features of the dynamics of appropriation in the rich rewarding setting.

5.4 Alliances

In the basic model an individual ruler chooses to attack a sequence of other rulers, one at a time. The background assumption is that the rulers that are being attacked must confront the attacker on their own. In the face of a powerful attacker, it would be reasonable for a ruler to form an alliance with other rulers who may eventually have to also face the same attacker. Starting with the classical account of Thucydides, alliances are a recurring theme in the history of warfare and empires. They have also been the subject of recent work, see Bloch [2012], König et al. [2017] and Jackson and Nei [2015]. An analysis of alliances is important but it is outside the scope of the present paper. In this section, our goal is more limited: we wish to elaborate on some considerations – relating to resources and networks – that arise when rulers can form such defensive alliances. We leave a more systematic and general analysis to future work.

We consider the following slight variation on our model: in a round, once a ruler has been picked, all the other active rulers have an opportunity to create alliances. The function of an alliance is limited: it puts together the resources of all its members and these resources can be deployed to defend any member of the alliance against an attack. To develop some intuition for how this can affect the dynamics of war suppose that the contest function is Tullock and that $\gamma$ is large. It follows then that a ruler $i$ will not want to attack a neighbor $j$ who is part of an alliance that has more resources.

An important question that arises at this point is who can form alliances with whom. In the simplest case, suppose there is no restriction. In other words, if $i$ is picked all other active rulers can contemplate an alliance. In this case a ruler $i$ will attack only if he has more resources than the sum of all the resources of the other rulers. This leads to our first observation: when alliances are unrestricted, hegemony will arise if and only if there is a ruler who controls more than half of all resources.

Next, we consider a simple restriction on alliance membership: suppose that members of an alliance must constitute a connected sub-graph of the residual contiguity network involving all rulers other than the ruler currently picked. Observe that in the case of a complete network all alliances are feasible and this formulation then corresponds to the simpler unrestricted
setting considered above. However, in more general networks, even a relatively poor ruler can
attack neighbors and become a hegemon. As an example consider a line network with an odd
number of rulers: the central node has \((n + 1)/2\) units of resources, while each of the other
\(n - 1\) nodes has exactly 1 unit of resources. The central node controls roughly one third of
all resources but there is no defensive alliance that can successfully protect the neighbors of
the central node. For \(\gamma\) large, it is optimal for the central node to implement a fully attacking
sequence. To rule out such situations we develop a sufficient condition for absence of war (and
consequently the lack of hegemony). Consider an ownership configuration with three or more
active rulers. There is no war in this configuration if for every ruler, \(i\), it is the case that each
of his neighbors is part of an alliance whose resources add up to more than the resources of
ruler \(i\).

These observations provide a theoretical basis for the notion of ‘balance of power’; for an
early discussion of this concept, see Hume [2006]; for more recent explorations of the idea, see
Kissinger [2015] and Betts [2013]. ‘Balance of power’ offers an alternative foundation to order
and stands in contrast to a theory in which order arises out of hegemony.

5.5 Losses in War

In the basic model, the winner retains all his resources and captures the entire resources of the
loser. Wars entail destruction of infrastructure and loss of lives; these losses can be especially
high in case the rivals use nuclear bombs. When losses are small, our earlier arguments
continue to hold, while if losses are very large then no ruler has an incentive to fight. The
interesting case is therefore one where the losses take on an intermediate value. We will show
that this intermediate range can give rise to the phenomenon of buffer states.\(^\text{14}\)

Let \(p\) be a rich rewarding contest success function. Suppose that a conflict between two
rulers entails a loss of a fixed fraction \(\delta \in (0, 1)\) of total resources. The expected payoff to a
ruler with \(x\) resources from a conflict with a ruler with \(y\) resources is:

\[
\Pi(x, y) = p(x, y)(x + y)(1 - \delta).
\]

We start by noting that the arguments in the proof of Theorem 1 can be extended to
establish equilibrium existence and uniqueness of equilibrium payoffs.

\(^{14}\)In our basic model, for simplicity, we assumed that kingdoms have no trade ties. Such ties may have a
bearing on losses from a war. But the relation between trade and incentives to fight is not straightforward;
for recent work on this question, see Jackson and Nei [2015] and Martin et al. [2008].
When $\delta = 0$, there are no losses in war: this is the benchmark model. More generally, regardless of the value of $\delta$, with a rich rewarding contest success function, expected resources of the poorer ruler are always smaller than his initial resources. Similarly, with a poor rewarding contest success function, expected resources of the richer ruler are always smaller than his initial resources. The principal impact of losses in war is to discourage war: so, in what follows, we focus on the rich rewarding case, as this is the case where all rulers have an incentive to fight at all points in the basic model.

If $\delta > 1/2$ then each ruler loses half of his resources in war: so it is clearly not profitable for the richer ruler to fight. Hence we restrict attention to the case where $\delta < 1/2$. Losses in war have an interesting implication for the incentives to fight. To see this suppose $x > y$. If $(1 - \delta)y < \delta x$ then the potential gain from the war is smaller than the loss to the richer ruler, and so attacking very small opponents is not profitable for the rich ruler. On the other hand, for a given $\delta < 1/2$, the richer ruler has no incentive to attack the poorer ruler if $y$ is sufficiently close to $x$: the total resources from such an attack are less than $2x$ while the probability of winning is less than $1/(2(1 - \delta))$ if $y$ is sufficiently close to $x$. Thus losses in war create an interesting structure of incentives: the richer ruler wishes to attack the poorer ruler only if the ruler is neither too poor but not too rich.

This structure of incentives gives rise to the possibility of buffer states. A buffer state is a ruler, lying between greater powers, and preventing them from attacking each other. The following example brings out this possibility.

Example 1 (Buffer State). Consider a line with three vertices (and rulers): $a$, $b$ and $c$ (in that order). Suppose that $R_a = 10$ and $R_c = 9$, $\gamma = 16$ and the cost of conflict is $\delta = 0.2$. Let us look at incentives to wage war as we vary the resources of ruler $b$. If $R_b \in (0, 1.79)$, ruler $a$ does not find it profitable to attack $b$ and then $c$, nor attack $b$ only. Similarly, $c$ does not find attacking $b$ or attacking $b$ and then $a$ profitable. Therefore peace in the initial state is the equilibrium outcome. Next consider $R_b \in (1.79, 10)$: one of the rulers now has an incentive to attack, and the outcome is either two rulers or a single hegemon. Finally, observe that if there was a link between $a$ and $c$, then ruler $a$ would definitely attack $c$. So the poor kingdom must offer the only path between the two large rulers.

More generally, a ruler $i$ is a buffer state at ownership configuration $\phi$, if $G - \phi^{-1}(i)$ is not connected and for any neighbour of $i$, $R_i(\phi) < R_j(\phi)$. In other words, the vertices owned by $i$ fragment the network and the resources of the buffer state are smaller than the resources of each of the surrounding rulers. The key step is to show that attacking a small ruler becomes
unprofitable not only in bilateral fights but also in longer sequences of fights. We show this under a mild assumption that \( f'(0) = 0 \). With this assumption the expected payoff from a sequence of fights is increasing in resources of an intermediate opponent in the sequence when these resources are sufficiently small and so, decreasing resource in this range decreases the payoffs which, with sufficiently large losses in war, makes including the poor ruler in the sequence unprofitable. These observations are formally stated and the proof is provided in the Online Appendix. The notion of buffer states is discussed in section 6.

We conclude by noting that losses in war are a different force driving peace as compared to the purely strategic considerations that arise under poor rewarding contest success function. These two forces may actually work against each other. Higher losses in war may prevent conflict escalation and this may encourage rulers to attack their neighbors. The Online Appendix illustrates this possibility.

6 Theory and History

Our theoretical framework has three key ingredients – resource seeking rulers, attack sequences constrained by the contiguity network, and the nature of the contest success functions. This section begins by providing empirical and historical background on them. We then take up four major historical episodes of empire formation and show how the developments are in line with our model’s predictions. This suggests that our model captures essential elements of imperial history in a parsimonious way.

6.1 Ingredients of the Approach:

We discuss three building blocks of the model: resource seeking rulers, the contiguity network, and rich/poor rewarding contest success functions.

Resource seeking Rulers: According to Thucydides [1989], there are three motives for war: greed, fear and honor. Elaborating on this, Hobbes [1886] says:

So that in the nature of man, we find three principal causes of quarrel. First, competition; secondly, diffidence; thirdly, glory. The first maketh men invade for gain; the second, for safety; and the third, for reputation. The first use violence, to make themselves masters of other mens persons, wives, children, and cattle; the second, to defend them; the third, for trifles. (Hobbes [1886]; page 64)
These observations are consistent with historical evidence. From ancient times onward conquering rulers have captured land, taken slaves, plundered treasure and sought glory. This was true in ancient and medieval times, and continues to hold in the modern era. We provide three examples.

The Roman Empire was founded on a series of hard fought campaigns. During the 2nd and 1st century BC, Roman generals waged ever more extensive wars and campaigns. Victory yielded land for the expanding Roman population, large number of slaves, and huge quantities of booty: in 50 years from 200-150 BC, the equivalent of 30 metric tonnes of gold was seized. In 62 BC, the victorious Pompey returned from the East with booty worth nearly 70 tonnes of gold (Kelly [2006]). Equally important was the high esteem in which successful generals were held. The highest honour for a general in Rome was a “Triumph”: a march of the general with his army through the city of Rome.

The second example concerns European global empires:

... the arch-characteristic of European imperialism was expropriation. Land was expropriated to meet the needs of plantations and mines engaged in long-distance commerce. Slave labour was acquired and carried thousands of miles to serve the same purpose. Native peoples were displaced, and their rights nullified, on the grounds that they had failed to make proper use of their land. Both native peoples and slaves (by different forms of displacement) suffered the effective expropriation of their cultures and identities. (Darwin [2007]; page 24)

Finally, control over resources remains a major motivation for wars in the modern world. For instance, the presence of large oil reserves has been suggested as a potential explanation for conflict in the Middle-East. The historical and political science literatures have suggested a potential role for natural resources in many cases of wars. Motivated by this descriptive literature, and the relatively large number of changes in boundaries between countries in the twentieth century, Caselli, Morelli, and Rohner [2015] present evidence that the presence and location of oil resources has significant and quantitatively important effects on inter-State conflicts in the period after World War 2.

The Contiguity Network: The contiguity network is a key ingredient in our model: it presents a general way to think about who can access or conquer whom. And as conflict proceeds through contiguity, the network naturally suggests a process through which a kingdom can grow via local expansion. To illustrate the value of the formulation, we present two examples.
Roman expansion over more than 450 years, from 500 BC to 30 BC: Figures 13 and 14 focus on moments of growth and presents the time line of expansion. The first observation is that through this period Roman expansion respects physical contiguity very tightly. The second point to note is that towards the end of this period, Rome developed a navy – something that can be thought of as a change in technology – and this allowed Rome to access Sicily. Thus the links in the network are shaped both by physical contiguity and by the state of technology.

Our second example is European global empires. From the 16th century onward, the military revolution and development of powerful navies in Europe enabled it to access the Americas. In our model, this is reflected in additional links in the physical contiguity network. The new links allowed imperial expansion to proceed along a network defined by access via water as well as the older links of physical contiguity on land, Parker [1988] and Hoffman [2015].

Rich/poor rewarding contest success functions: The contest success function maps the resources of the contestants onto the probability of victory and the probability of ties. In particular, the concepts of rich and poor rewarding play a central role in shaping the dynamics of conflict.

The key issue here is the relation between relative resources of the opponents and the probability of victory. We focus on early modern Europe; similar observations may be made for other times and for other parts of the world. Clausewitz [1993], drawing inspiration from the Napoleonic wars in Europe, writes:

In tactics, as in strategy, superiority in number is the most common element in victory... Bonaparte commanded 120,000 men at Dresden against 220,000 – not quite half. At Kolin, Frederick the Great’s 30,000 men could not defeat 50,000 Austrians; similarly, victory eluded Bonaparte at the desperate battle of Leipzig, though with is 160,000 men against 280,000, his opponent was far from being twice as strong.

These examples may show how in modern Europe even the most talented general will find it difficult to defeat an opponent twice his strength. .... with this discussion, we believe we have shown how significant superiority of numbers really is. It must be regarded as fundamental – to be achieved in every case and to the fullest extent possible. (Clausewitz [1993]: pages 228-232)

He goes on to add:
With the sole exception of Dresden in 1813, Bonaparte, the greatest general of modern times, always managed to assemble a numerically superior, or at least not markedly inferior, army for all the major battles in which he was victorious; and where he failed to do so – as at Leipzig, Brienne, Laon, and Belle-Alliance – he lost. ([Clausewitz [1993]; pages 335-336)

To further substantiate this point, we present evidence from the European experience during the nineteenth century. ([Howard [2009]; pages 100-101). Very seldom did eighteenth-century commanders operate with armies in excess of 80,000 men. These bounds were transcended, as we have seen, by the French armies of the revolutionary era, which supplemented their regular sources by organized or unorganized pillage; but the disaster which overtook the armies some 600,000 strong which Napoleon led into Russia in 1812 showed that even this ruthless improvisation had its limits. With the introduction of the railways these limits disappeared. Once the administrative complexities of moving armies by rail were mastered, as they were mastered by the Prussian General Staff in the 1860’s, the only restrictions on size were the numbers of men of military age in the community, the political and economic constraints on their conscription, and the administrative capacity to train, equip, and mobilize them. In 1870 the North German Confederation deployed against France exactly twice the number Napoleon had led into Russia – 1,200,000. .... (In 1870/71) Prussia had taken only a few weeks to destroy the armies first of the Austrian Empire and then of France and, in the latter case, to occupy the enemy capital in true Napoleonic fashion and dictate her own terms to a completely helpless foe. ([Howard [2009]; pages 100-101).

This discussion brings out the critical role of the effective army size – that rests on population size, the fiscal capacity of the state, and the military and transport technology – in shaping outcomes of war.

We map this discussion on to the issue of rich rewarding versus poor rewarding contest success functions. For concreteness, consider the Tullock Contest Function and suppose one army is twice the size of the other army. With an exponent $\gamma = 2$, the probability of winning for the larger army is 0.8, and with an exponent $\gamma = 4$ it is (approx) 0.95. On the other hand, with $\gamma = 0.5$ the probability of winning would be 0.6, and with $\gamma = 0$ the probability would be equal to 0.5. This suggests that an empirical record where an army twice the size of the
opponent wins the war with probability close to 1, is consistent with a rich rewarding contest success function.

Some authors have drawn attention to the ‘paradox of power’ in matters of conflict. They have suggested that the American experience in Vietnam is more in line with a poor rewarding contest success function, see e.g., Hirshleifer [1995b]. Broadly, one may argue that in some environments, due to physical environment or powerful cultural factors, the relationship between resources and winning is weak. In these circumstances, poor rewarding may be an appropriate model.

6.2 Implications

The analysis suggests that, when the contest success function has the rich rewarding property, the dynamics will exhibit incessant warfare. With small armies, war may be indecisive; however, once a ruler becomes dominant relative to his neighbours – either due to superior resource endowments or due to institutional or technological innovations – he will more easily expand his territory. Subsequent wars become decisive, and the speed of the expansion gathers pace. The size of empires is limited by the connectivity of the network. Moreover, if defensive alliances among rulers are possible – due to physical or cultural contiguity – dominant rulers may be contained and hegemony averted.

The predictions on hegemony are in line with three major episodes in world history – the first Chinese empire, the Roman Empire, and the Spanish Empire in the new World.\textsuperscript{15}

First Chinese Empire: We start with a discussion of one of the turning points in world history: the emergence of the first empire in China in 221 BC. The discussion draws heavily on Lewis [2010] and Overy [2010]. In China, the years between 475 BC and 221 BC were characterized by almost uninterrupted warfare between seven major states which is referred to as the \textit{Warring States Period}. The seven major kingdoms were Qin (located in the far west) the three Jins (located in the center on the Shanxi plateau; Han south along the Yellow River, Wei located in the middle, Zhao the most northernmost of the three), Qi (centred on the Shandong Peninsula), Chu (with its core territory around the valleys of the Han River),

\textsuperscript{15}There are other well known episodes that broadly conform to the pattern of an attack sequence leading to the formation of an empire, e.g., Cyrus forming the first Persian Empire, Alexander’s campaigns leading to the Greek Empire in Asia, Chandragupta setting up the Mauryan Empire in India, and the creation of the first Arab Empire. We have opted to focus on these three episodes as they provide broad coverage across space and time and at the same time we can present the empirical patterns in reasonable detail and systematically discuss them in relation to the theory.
and Yan (centered on modern-day Beijing). Initially, wars led to changes in the power of the different dynasties but all the kingdoms survived. However, from 320 BC to 221 BC, there was a major consolidation and by 221 BC, the Qin defeated all the other kingdoms and unified the entire area under one ruler, Qin Shi Huang. Figure 11 illustrates the dynamics and Figure 12 summarizes it.

We now relate, in some detail, our theoretical model the process of the emergence of this first empire. The first observation is that over a period stretching several hundred years, there was incessant warfare. The second observation is that in the first part of this period, from 475 BC to 360 BC, the armies were relatively small and the wars did not lead to the elimination of the major rulers. The third observation concerns the changes from 360 BC onward. The period after 360 BC witnessed major reforms of the Qin minister, Shuang Yang. After these reforms and the accompanying technological developments, the scale and violence in a war changed dramatically: now elimination of the losing ruler and conquest of his kingdom became much more likely, especially in a war between the Qin and one of the other warring states.

... the rise of Qin to dominance and its ultimate success in creating a unified empire depended on two major developments. First, under Shang Yang it achieved the most systematic version of the reforms that characterized the Warring States. These reforms entailed the registration and mobilization of all adult males for military service and the payment of taxes. While all Warring States were organized for war, Qin was unique in its extension of this pattern to every level of society, and in the manner in which every aspect of administration was devoted to mobilizing and provisioning its forces for conquest. (Lewis, 2010; page 38-39).

These reforms taken together meant that the ruler had the resources – both in terms of army size and in terms of tax revenue – to wage large scale wars. Equipped with such a large army the Qin ruler was able to implement a long attacking sequence: in 230 BC, Qin conquered Han, the weakest of the Seven Warring States. In 225 BC, Qin conquered Wei, followed in 223 BC by the conquest of Chu. Qin conquered Zhao and Yan in 222 BC. Finally, in 221 BC, Qin turned its attention to the last surviving Warring State opponent: the Qi. In the face of the great threat Qi surrendered.

16The size of the army was crucial in this contest: the first Qin invasion was a failure, when 200,000 Qin troops were defeated by a much larger Chu army with around 500,000 troops. The following year, Qin mounted a second invasion with 600,000 men and they defeated the Chu state. At their peak, the combined armies of Chu and Qin are estimated to have been in excess of a million soldiers.
Figure 11: The First Chinese Empire: Dynamics

Our final observation concerns the frontiers of the empire: the Qin empire was bounded by forests in the South, deserts and the Tibetan Plateau on the West, wasteland in the North and the Pacific Ocean in the East. These physical features, especially in the South, the West and the East, presented a physical constraint on further expansion. It is then possible to interpret China as a distinct ‘component’ of the world network, somewhat isolated from other parts of the world. The first Chinese Empire was a hegemon that was ‘limited’ by the connectivity of the physical contiguity network.

The Roman Empire: The Roman Empire has had a profound impact on the history of the Mediterranean area (and more broadly across Europe), over the past two and a half thousand years. Our discussion draws on Kelly [2006] and Polybius [2010]; the historical record is taken from the Encyclopedia Britannica, Scarre [1995], and Wittke et al. [2010]. Figures 13 and 14 summarize the expansion of Roman empire over the period 500 BC – 30 BC. In these Figures,
we distinguish physical contiguity from sea-based contiguity that was made possible after the
development of a Roman navy; the latter are represented with ‘dashed’ lines.

We begin with the Early Roman Empire and describe the period 500 BC to 272 BC. Rome’s first major war against an organized state was fought with Fidenae (437–426 BC), a
town located just upstream from Rome. Rome next fought a long and difficult war against Veii, an important Etruscan city not far from Fidenae. The conquest of Veii opened southern Etruria to further Roman expansion. Rome then proceeded to found colonies at Nepet and Sutrium and forced the towns of Falerii and Capena to become its allies. During the period 348–295 BC, Rome rapidly rose to a position of hegemony in Italy south of the Po valley. A key moment was the Third Samnite War (298–290 BC): Samnites persuaded the Etruscans, Umbrians, and Gauls to join them. Rome emerged victorious at the battle of Sentinum in 295 BC. The next major event was the Pyrrhic War, 280–275 BC. The conflict between Rome and Pyrrhus lasted 5 years and ended in a final Roman victory in 275 BC at Beneventum.

The period from 272 BC to 30 BC witnessed a massive expansion of the Roman empire
across the Mediterranean Sea and most of modern western Europe. Rome first began to make
Figure 14: Expansion of the Roman Republic 217 BC – 30 BC

war outside the Italian peninsula during the Punic wars against Carthage (in North Africa) around 264 AD. By 146 AD, Rome had defeated Carthage and taken over direct control over large parts of North Africa, and through its conflict with Carthage it also expanded its influence in Iberia. The wars with Macedonia led to control over Greece by 148 BC, and the defeat of the Selucid Emperor in 188 BC led to control over Asia Minor. Further conquests over the next hundred years would result in Rome’s conquest of large parts of modern Spain and most of modern France (Kelly [2006], Polybius [2010]). Figure 14 illustrates this growing hegemony.

Our theoretical analysis draws attention to four features in this process. The first point is the frequency of war: Rome was a warrior state and its vast territory had been acquired through a long series of hard fought campaigns over a period of over 500 years.

The second point pertains to the pace of expansion: over the period 500 BC – 272 BC, the expansion was limited to the Italian peninsula. This relatively slow pace of expansion is in line with the theoretical prediction of gradual acquisition of territory and slow movement of boundaries.

Three, once Rome had taken over the Italian peninsula, further expansion was rapid. Polybius [2010] presents a detailed discussion of the expansion during the period from 220 BC to 167 BC: a period that saw Rome take over parts of North Africa, Greece and Asia Minor. Later on, the period until 30 BC, saw a further massive expansion of Roman rule that led to rule over almost the entire coast around the Mediterranean Sea and over much of modern Western Europe. This pattern of slow followed by rapid expansion is consistent with a rich rewarding contest success function: once an empire attains a certain size advantage relative to its neighbours, it almost always wins subsequent battles and, due to the size of its army, also takes over territory more quickly.

Our final observation concerns the limits of the empire: the boundaries came to be defined by the Atlantic Ocean in the West, the Rhine and the Danube River in the North, the Sahara Desert in the South, and the Euphrates River in the East. Over the subsequent four hundred years, these boundaries would be contested but they would describe the limits of the empire broadly: these boundaries are consistent with the idea that the size of the empire is limited by the ‘connectivity’ of the contiguity network.

The Spanish Empire in the New World: European imperial expansion starting from around 1500 AD reshaped the medieval world and gave rise to the age of global empires. The expansion of Spanish domains in the newly discovered continent of America illustrates this moment in
Our discussion draws on Elliott [2006] and the historical record in taken from the Encyclopedia Britannica. Spanish conquest in the Americas started with the first voyage of Columbus in 1492 AD. This voyage created a new link in the contiguity network as it made a new island ‘accessible’. Equipped with the superior technology from Europe the Spanish quickly captured this island. The indigenous population was almost entirely annihilated, and the island became part of the Spanish domain. Moving onto Central America, the Spanish conquistador Hernan Cortes defeated the Aztecs in Mexico City by 1521 AD, and the Aztec Empire was largely conquered by 1532 AD. Continuing, on land and by sea, Spanish conquest had reached Cartagena by 1532 AD, and Caracas had been captured by 1567 AD. Further south, Francisco Pizzaro defeated the Inca ruler in 1532, and Spain set up the Viceroyalty of Peru in 1542, a vast area that included most parts of South America (other than the Portuguese Empire and Venezuela). The Mayans were defeated and the area of southern Mexico, Belize, Guatemala, and Honduras. El Salvador was eventually entirely taken over by 1697 AD.

European military technology played a central role in the dramatic speed and scale of these conquests. For instance, in 1532 AD, Pizarro captured the Inca emperor with 167 men fighting an imperial Inca army of between 5000-10,000 men. In 1536, 190 conquistadors held out for a year against an Inca army of over 100,000 men (Hoffman [2015]). In this context, the Spanish conquistadors were able to overwhelm opponents with significantly larger armies.

Our theoretical analysis draws attention to three aspects of this development. The first is the developments in Europe involving the successes of the Castilian kingdom through the 15th century. These successes set the stage for even further expansion across the world. The second point is the incessant fighting between the Spanish and the native kingdoms during this period. The third, and key, point is the reconfiguration of the contiguity network. This was made possible by the discovery of new sea routes to different parts of the world – the Caribbean Islands (and eventually America) by Columbus in 1492 AD. This discovery happened alongside a major change in the technology of war – the advancement of gunpowder and corresponding advances in the design of fortresses and the navy; for a systematic study of military revolution, see Rogers [1995] and Parker [1988]. Taken together these developments significantly altered the configuration of the contiguity network: previously ‘unknown’ parts of the world now

\[17\] In addition to military superiority, the Europeans were also helped by the vulnerability of indigenous populations in America to diseases such as small pox and measles. Almost 95% of the Aztec population died due to diseases introduced by the Spanish over the period. The indigenous population under the Incas was similarly greatly reduced due to epidemics of diseases brought by the Spanish. So the indigenous populations and its leadership could not present a real resistance to the conquistadors.
became ‘accessible’ and open to conquest. Imperial expansion now proceeded along this new network and gave rise to a truly global Spanish empire spanning three continents Europe, North America and South America.

We conclude this discussion by noting that the discovery by Vasco da Gama of a sea route to India in 1498 AD (and subsequent sea voyages of other explorers) had similarly far reaching implications: a combination of changes in the contiguity network and the military technology (working through the contest function and enhancing the prospects of victory) enabled European powers to conquer/control Australasia and eventually also control large parts of Asia (Hoffman [2015]).

Lack of Hegemony in Early Modern Europe, 1650 AD–1950 AD: We now consider a notable instance where in spite of regular warfare, hegemony failed to materialize: early modern Europe. The discussion draws heavily on Kissinger [2015]. He argues that the historical experience of Europe culminating in the formulation of the Peace of Westphalia helped lay the outlines of a global system of order with several sovereign states. Alliances play an important role in his account of the past but also in his perspective on the way ahead.

Our theoretical results on the role of defensive alliances delineate the circumstances – pertaining to resources and networks – under which alliances can constrain aggressive powers and thereby sustain peace among warring rulers/nations. At a time when networks constrain large movements of military, our theoretical result suggests that local alliances among powerful rulers will be key in shaping the extent of hegemony. We illustrate this point through a discussion of Louis XIV’s attempts to expand French dominance.

Louis XIV took control of the French crown in 1661 AD. He centralized the functioning of the state dramatically: the bureaucracy was greatly expanded and strengthened. Men from humble origins could be appointed senior ministers: the source of power of these men was the state and Louis XIV himself. The aristocracy and nobility moved along with Louis XIV to Versailles in 1682 AD. These changes led to a unified kingdom run by a skilled bureaucracy, and possessing a military that surpassed that of all the neighbours of France. Louis XIV then had France engage in almost continuous warfare for the rest of his long reign. Initially, French armies were victorious almost everywhere and French dominance over Europe loomed. But slowly, the tide turned. In the words of Henry Kissinger,

In the end . . . each new conquest galvanized an opposing coalition of nations. At first Louis’s generals won battles almost everywhere ultimately, they were defeated and checked everywhere. (Kissinger [2015]: page 34)
Eventually, England Holland and Austria formed the Grant Alliance in 1689 AD. This resulted in battles between very large armies, in some cases with over 100,000 men and most of the revenues of the major powers. The ensuing conflict with France is termed The Nine Years War. The war was financially crippling for participants; the average army size increased from 25,000 in 1648 to over 100,000 by 1697 AD. Between 1689–96, 80% of British government revenues were spent on the military. As war went on, both sides recognised decisive victory was no longer possible. In August 1696 AD, France and Savoy agreed a separate peace in the Treaty of Turin. The Treaty of Ryswick was only finalised once France agreed to return Luxembourg to Spain and Louis XIV recognised William as King. [Kissinger 2015] concludes as follows:

Louis sought what amounted to hegemony in the glory of France. He was defeated by a Europe that sought its order in diversity. (Kissinger 2015: page 35)

A similar pattern of changing defensive alliances prevented the rise of any subsequent dominant ruler, such as Napoleon Bonaparte and Adolf Hitler, to European hegemony. Throughout this period, England (and later Great Britain) played a central balancing role. It would ally with one or the other power guided largely by the consideration of balancing the power.

After the Second World War, leading powers could move their military resources more easily across different parts of the world. This suggests that a model in which network restrictions play a limited role will become more relevant over time. The analysis of that model suggests that locally powerful rulers will find it difficult to expand their territories as their neighbours could successfully call upon powerful allies from around the world to defend them. An aggressive ruler can expand his territories only in the special situation where he controls more than one half of all resources. This theoretical prediction is in line with empirical observation: after the end of the Second World War, the two major superpowers, United States and Russia, have intervened – in Korea, Afghanistan, and Kuwait – and successfully prevented the expansion of any significant single country at the expense of their neighbours.

6.3 Preemptive War

Theorem 3 says that in a rich rewarding setting, every ruler, even very poor rulers, will wish to fight. This result provides a rationale for poor rulers to fight that is consistent with arguments in moral philosophy:
“...a manifest to injure, a degree of active preparation that makes that intent a positive danger, and a general situation in which waiting, or doing anything other than fighting, greatly magnifies the risk.” [Walzer 1977]

Perhaps the best known instance of preemptive war in ancient history is the Peloponnesian War. In his account of the war, Thucydides argues that Sparta initiated war because it feared growing Athenian power. At the start of the Second World War, German occupation of the Netherlands, which was a neutral country, is widely regarded as an instance of preemptive war. This occupation was done to prevent use of the Netherlands by the British military, in due course. Finally, the Israel air attack on Egypt, Jordan and Syria in 1967 is considered as another instance of a preemptive war.

6.4 Buffer States

We next discuss the concept of ‘buffer state’. There is a large literature on buffer states; for a survey and a list of examples of such states, see Chay and Ross [1986]. This literature attributes three characteristics to such states.

‘...they are small countries, in both area and population; they are adjacent to two larger rival powers; and they are geographically located between these opposing powers’. [Chay and Ross 1986]

The example in the losses to war section brings out the key role of small resources and the intermediate physical location for a country or a kingdom to act as a ‘buffer state’.

In the late 19th century, Afghanistan was considered a buffer state between the British Empire and Russian Empire. In more recent times, Ukraine has been described as a buffer state between Russia and the NATO bloc (Mearsheimer March 2014, Walt 2014).

7 Concluding Remarks

This paper develops a theoretical framework for the study of the incentives to wage war to conquer territory and resources. Our innovation is that we locate the dynamics of appropriation within a contiguity network. The analysis develops a number of results on the interplay between the technology of war, the resources of rulers, and contiguity, that illuminate the process of the formation of empires. In a setting where the contest functions are rich rewarding,
starting from a situation with multiple kingdoms, the dynamics are characterized by incessant fighting. After an initial phase of uncertain and gradual growth, the pace of expansion of a ‘kingdom’ speeds up, and it grows rapidly through contiguous expansion. This expansion, and consequently the size of the empire, is limited by the connectivity of the network. The analysis also draws attention to the role of poor rewarding contest success functions, losses in war, and defensive alliances, in discouraging war and thereby limiting the scope of hegemony.

These results provide a parsimonious account of two major examples of hegemonies – the First Chinese Empire and the Roman Empire. Our work draws attention to the role of network connectivity in defining the limits of empire. The advance of technology – a powerful navy, gunpowder technology and seige forts – the contiguity network is redefined, and this paves the way for the age of global empires. This theoretical point is corroborated by the rise of Spanish Empire in the New World, during the period 1500 AD – 1750 AD. Finally, these results point to a set of considerations – involving defensive alliances and the balance of power — that account for the limited role of hegemony in Europe during the period 1650 AD-1950 AD.

The paper highlights the importance of network connections and the contest success functions. Rulers can alter both through strategic investments. In future work it would be interesting to incorporate this choice within the framework. The model examines external constraints to the extent of the empire: it is clear that institutional arrangements will also play a role in defining the limits of empire. This offers another avenue for further work.

References


$^{18}$For an early discussion of the role of institutions in empire building, see Polybius [2010].


APPENDIX: PROOFS

EQUILIBRIUM EXISTENCE AND PAYOFF UNIQUENESS

Proof of Theorem 1. We start with introducing a natural partial order of precedence on the set of ownership configurations, $\mathcal{O}$, and on the set of states, $\mathcal{O} \times 2^N$. Given any two ownership configurations, $\sigma \in \mathcal{O}$ and $\sigma' \in \mathcal{O}$, $\sigma \sqsubseteq \sigma'$ if and only if for all $v \in V$, either $\sigma(v) = \sigma'(v)$ or $\sigma(v) \neq \sigma'(u)$, for all $u \in V$. Informally, if $\sigma$ and $\sigma'$ are ownership configurations such that $\sigma'$ is obtained from $\sigma$ by some rulers expanding their territories, then $\sigma \sqsubseteq \sigma'$. Given any two states, $(\sigma, P) \in \mathcal{O} \times 2^N$ and $(\sigma', P') \in \mathcal{O} \times 2^N$, $(\sigma, P) \preceq (\sigma', P')$ if and only if either $\sigma \sqsubseteq \sigma'$ or $\sigma = \sigma'$ and $P \subseteq P'$. Informally, if state $(\sigma, P)$ precedes state $(\sigma', P')$ in the course of the game, then $(\sigma, P) \preceq (\sigma', P')$. We will also use $\sqsubset$ and $\prec$ to denote the strict orders associated with the respective partial orders, defined above. Given an ownership configuration, $\sigma \in \mathcal{O}$, let $\text{Succ}(\sigma) = \{\sigma' \in \mathcal{O} : \sigma \sqsubseteq \sigma'\}$ be the set of all ownership configurations that $\sigma$
precedes. Let \( \overline{\text{Succ}}(o) = \text{Succ}(o) \cup \{o\} \). Similarly, given a state \((o, P) \in \Omega \times 2^N\), let 
\( \text{Succ}(o, P) = \{(\alpha', P') \in \Omega \times 2^N : (o, P) \prec (\alpha', P')\} \) be the set of all states that \((o, P)\) precedes, and let \( \overline{\text{Succ}}(o, P) = \text{Succ}(\text{own}, P) \cup \{(\text{own}, P)\} \).

Since \( \Omega \) and \( \Omega \times 2^N \) are finite, there exist maximal elements of \( \succeq \) and \( \preceq \). Take any strategy profile, \( s \), defined recursively on \( \Omega \times 2^N \) starting from the maximal elements of \( \preceq \) as follows. If \((o, P)\) is such that \( o \) is maximal according to \( \succeq \) (i.e. there is only one active ruler at \( o \)) then, for all \( i \in N \), \( s_i(o, P) = \varepsilon \) (the unique feasible sequence of \( i \) at \( o \)). Otherwise, let \( s_i(o, P) \) be any sequence that maximises \( i \)'s expected payoff given the continuation payoff determined by \( s \) defined on the states in \( \text{Succ}(o, P) \). Clearly any such strategy profile is well defined and is a Markov perfect equilibrium of the game. Moreover, given the Markov perfection requirement and since at each state there are no simultaneous moves (only one player is picked to make a choice), every Markov equilibrium is a strategy profile of the form defined above.

We now turn to showing payoff equivalence of equilibria. Take any two Markov perfect equilibria of the game, \( s \) and \( s' \), and suppose that they are not payoff equivalent. Let \((o, P) \in \Omega \times 2^N\) be a maximal state, according to \( \preceq \), such that there exists a ruler \( i \in N \setminus P \) with 
\( \Pi_i(s | o, P) \neq \Pi_i(s' | o, P) \). Suppose that \( \Pi_i(s | o, P) > \Pi_i(s' | o, P) \) (the arguments for the inverse inequality are symmetric and omitted). Then \( i \) could strictly improve his payoff under \( s' \) by choosing a strategy \( s''_i \) different to \( s'_i \) at state \((o, P)\) only: \( s''_i(o, P) = s_i(o, P) \). Since \((o, P)\) is a maximal state, according to \( \preceq \), for which \( \Pi_i(s | o, P) \neq \Pi_i(s' | o, P) \), so for all states in \( \text{Succ}(o, P) \), \( s \) and \( s' \) yield the same payoff to \( i \) and the payoff to \( i \) at \((o, P)\) depends on his resources at \((o, P)\) and on his payoff at these states only. Thus \( \Pi_i((s'_{-i}, s''_i) | o, P) = \Pi_i(s | o, P) > \Pi_i(s' | o, P) \), a contradiction with the assumption that \( s' \) is a Markov perfect equilibrium of the game. Hence for all \( i \in N \) and \((o, P) \in \Omega \times 2^N \setminus \{i\} \), \( s \) and \( s' \) must yield the same payoff to \( i \).

\[ \square \]

INCENTIVES TO FIGHT

Proof of Theorem 2 We start with the rich rewarding case. The proof for the poor rewarding case is similar and omitted. Let \( p \) be a contest success function satisfying (6). Suppose that \( p \) is rich rewarding and take any \( x, y \in \mathbb{R}_{++} \) such that \( x > y \). Rewriting \( (x + y)p(x, y) > x \), it is equivalent to \( \frac{1}{1 + \frac{y}{x}} > \frac{1}{1 + \frac{x}{y}} \). Further, this is equivalent to \( f(x)/x > f(y)/y \). Hence
rich rewarding property is equivalent to \( f(x)/x \) being strictly on \( \mathbb{R}_{++} \), that is to \( f \) exhibiting increasing returns to scale.

We now turn to the timing result. Suppose that \( p \) has the no-waiting property. Then, for any \( x, y, z \in \mathbb{R}_{++} \),

\[
\frac{f(x)}{f(x) + f(y)} \frac{f(x + y)}{f(x + y) + f(z)} > \frac{f(x)}{f(x) + f(y + z)}.
\]

In particular, the inequality holds for \( z = x \) so

\[
\frac{f(x)}{f(x) + f(y)} \frac{f(x + y)}{f(x + y) + f(x)} > \frac{f(x)}{f(x) + f(x + y)}.
\]

holds for any \( x, y \in \mathbb{R}_{++} \). Dividing both sides by \( f(x) \) and multiplying them by the denominators we get

\[
f(x + y)(f(x) + f(x + y)) > (f(x) + f(y))(f(x + y) + f(x)).
\]

Dividing both sides by \( f(x) + f(x + y) \) yields

\[
f(x + y) > f(x) + f(y).
\]

Thus \( f \) is super-additive.

Next suppose that \( f \) is super-additive. Then, for any \( y, z \in \mathbb{R}_{++} \),

\[
f(y + z) > f(y) + f(z).
\]

Multiplying both sides of the inequality above by \( f(x + y) \), for any \( x, y, z \in \mathbb{R}_{++} \),

\[
f(x + y)f(y + z) > f(x + y)(f(y) + f(z)).
\]

Moreover,

\[
f(x + y)f(y + z) > f(x + y)f(y) + f(x + y)f(z) > f(x + y)f(y) + (f(x) + f(y))f(z).
\]

Adding \( f(x)f(x + y) \) to both sides we get

\[
f(x + y)(f(x) + f(y + z)) > (f(x) + f(y))(f(x + y) + f(z)).
\]
This can be rewritten as
\[
\frac{1}{f(x) + f(y)} \frac{f(x + y)}{f(x + y) + f(z)} > \frac{1}{f(x) + f(y + z)}.
\]

Multiplying both sides by \(f(x)\) we get
\[
\frac{f(x)}{f(x) + f(y)} \frac{f(x + y)}{f(x + y) + f(z)} > \frac{f(x)}{f(x) + f(y + z)}.
\]

To complete the proof, we show that increasing returns to scale imply super-additivity and that decreasing returns to scale imply sub-additivity. Suppose that \(f\) has increasing returns to scale. So \(f(x)/x\) is strictly increasing on \(\mathbb{R}_{++}\). For any \(x, y \in \mathbb{R}_{++}\),
\[
xf(x + y) > (x + y)f(x) \quad \text{and} \quad yf(x + y) > (x + y)f(y).
\]
Adding the two inequalities and dividing both sides by \(x + y\) we get \(f(x + y) > f(x) + f(y)\), that is \(f\) is strictly super-additive. The arguments for decreasing returns are similar and omitted.

By what was shown above, rich rewarding property of \(p\) implies that \(f\) exhibits increasing returns to scale which, in turn, implies that \(f\) is super-additive and, further, that \(p\) has the no-waiting property. By similar argument, poor rewarding property implies the waiting property.

\[\square\]

CONQUEST AND EMPIRE

We start by noting that the waiting and no-waiting properties extend to sequences of arbitrary length. Formally, let \(m \geq 3\), \(x_1, \ldots, x_m \in \mathbb{R}_{++}\), and \(1 \leq i < j \leq m\) such that \(i \neq 1\) or \(j \neq m\). If \(p\) has the no-waiting property, then
\[
p_{\text{seq}}(x_1, \ldots, x_{i-1}, x_i, \ldots, x_j, x_{j+1}, \ldots, x_m) > p_{\text{seq}} \left( x_1, \ldots, x_{i-1}, \sum_{l=i}^{j} x_l, x_{j+1}, \ldots, x_m \right)
\]

Proof of Theorem 3. The proof proceeds in three steps.

**Step 1:** Fix some state \(o\) with \(|\text{Act}(o)| \geq 2\). For a strong ruler \(i\), the optimal full attacking sequence maximizes his payoffs across all attacking sequences. Moreover, in generic case, it is a unique maximizer.
Let $o$ be a state with $|\text{Act}(o)| = m \geq 2$. Take an active ruler $j_0 \in \text{Act}(o)$ with maximal amount of resources $R_{j_0}(o)$. For generic resource values, such a ruler is unique. Pick a full attacking sequence $j_1, \ldots, j_{m-1}$ consisting of rulers in $\text{Act}(o) \setminus \{j_0\}$ that is feasible for $j_0$ in $G$ under $o$ (clearly such a sequence exists because $G$ is connected). Since $j_0$ has maximal amount of resources so, for all $1 \leq k \leq m - 1$, we have

$$\sum_{l=0}^{k-1} R_{j_l}(o) \geq R_{j_k}(o).$$

(11)

The expected payoff to ruler $j_0$ from the attacking sequence is

$$\pi_{j_0}(o \mid j_1, \ldots, j_{m-1}) = \left( \sum_{l=0}^{m-1} R_{j_l}(o) \right) \prod_{k=1}^{m-1} p \left( \sum_{l=0}^{k-1} R_{j_l}(o), R_{j_k}(o) \right)$$

$$= R_{j_0}(o) \prod_{k=1}^{m-1} p \left( \sum_{l=0}^{k-1} R_{j_l}(o), R_{j_k}(o) \right) \left( \frac{\sum_{l=0}^{k-1} R_{j_l}(o)}{\sum_{l=0}^{k-1} R_{j_l}(o)} \right).$$

(12)

Since $p$ is rich rewarding, so

$$p \left( \sum_{l=0}^{k-1} R_{j_l}(o), R_{j_k}(o) \right) \left( \frac{\sum_{l=0}^{k-1} R_{j_l}(o)}{\sum_{l=0}^{k-1} R_{j_l}(o)} \right) \geq 1,$$

(13)

with equality only if $k = 1$ and $R_{j_0}(o) = R_{j_1}(o)$.

At every step in the sequence, the expected resources are growing. So, for generic resource values, there is a full attacking sequence that dominates any partial attacking sequence. By definition, the optimal full attacking sequence maximizes payoffs across all attack sequences.

The first step has a powerful implication: in any state with 2 or more active rulers there is at least one ruler who has a strict incentive to attack, given that other rulers do not attack. Hence, in equilibrium, there must exist a hegemon.

In the dynamic game, in principle, a strong ruler may prefer to wait and allow others to move and then attack later. The next step shows that an optimal full attacking sequence dominates all such waiting strategies.

**Step 2:** Fix some state $o$ with $|\text{Act}(o)| \geq 2$ and a set or rulers, $P$. For any ruler $i \in N \setminus P$ strong at $o$, an optimal full attacking sequence is a dominant choice at $(o, P)$. Moreover, the choice is strictly dominant if $|\text{Act}(o)| \geq 3$.

Fix some state $o$. Let $\sigma_i(o)$ be the optimal sequence of ruler $i$ at $o$, assuming that the
game ends after \( i \) executes the sequence (successfully or not). In other words, \( \sigma_i(o) \) is the myopic optimal sequence of ruler \( i \) at \( o \). Notice that this sequence is independent of the set of rulers who chose peace prior to \( i \)'s move at a round at the state \( o \). Let \( \bar{\pi}_i(o) = \pi_i(o \mid \sigma_i(o)) \) denote the optimal myopic payoff ruler \( i \) can attain at \( o \).

**Claim.** The optimal myopic payoff is the highest that ruler \( i \) can hope to attain, i.e., for any state, \( o \), and any set of rulers, \( P \subseteq N \setminus \{i\} \), \( \bar{\pi}_i(o) \geq \Pi_i(s \mid o, P) \) for any feasible strategy profile \( s \). Moreover, if \( i \) is strong and there are at least three active rulers, then the inequality is strict.

The proof is by induction on the number of active rulers. For the induction basis, we show that the claim holds for 2 active rulers. If \( i \) is the richer ruler then, from the rich rewarding property, his myopic optimal strategy is to attack. It is also clear that attacking yields strictly higher payoffs if the other ruler does not attack, and weakly higher payoffs if the other ruler does attack. If \( i \) is the poorer ruler then not attacking is the optimal myopic strategy. In case the richer ruler attacks, the expected payoff to \( i \) is less due to the rich rewarding property. That completes the argument for 2 active rulers.

For the induction step, suppose that the claim holds for all \( y \leq X \), where \( X \geq 2 \), active rulers: we will show that it also holds for \( X + 1 \) active rulers. Given state \( o' \), set of rulers, \( P' \), and strategy profile, \( s' \), we will use \( \text{Atck}(s', o', P') \) to denote the set of rulers choosing attack at \( (o', P') \) under \( s' \), i.e. \( \text{Atck}(s', o', P') = \{ j \in N \setminus P' : s'_j(o', P') \neq \varepsilon \} \).

Fix some state \( o \) with \( X + 1 \) active rulers and a set of rulers, \( P \) such that \( \text{Act}(o) \setminus P \neq \emptyset \). Take an active ruler \( i \in \text{Act}(o) \setminus P \) and any strategy profile \( s \). If for all \( P' \subseteq N \) such that \( P \subseteq P' \), \( \text{Atck}(s, o, P') = \emptyset \), i.e. all players choose peace following \( P \) at \( o \), then the claim follows, because \( \sigma_i(o) \) is at least as good as the empty sequence at \( o \):

\[
\bar{\pi}_i(o) \geq \pi_i(o \mid \sigma_i(o)) \geq \pi_i(o \mid \varepsilon) = \Pi_i(s \mid o, P). \tag{14}
\]

Moreover, by Step 1, the inequality is strict if \( i \) is strong.

For the remaining part of the argument assume that there exists \( P' \subseteq N \) with \( P \subseteq P' \) such that \( \text{Atck}(s, o, P') \neq \emptyset \). We will establish that \( \bar{\pi}_i(o) \geq \Pi_i(s \mid o, P) \). Given a set of rulers \( P' \) and ruler \( j_0 \in N \setminus P' \) such that \( s_{j_0}(o, P') \neq \varepsilon \), let \( \Pi_i(s \mid o, P', j_0) \) denote the expected payoff to ruler \( i \) from strategy profile \( s \) conditional on ruler \( j_0 \) being selected to move at \( (o, P') \) and

\[\text{Throughout the proofs we use the standard notation, } \varepsilon, \text{ to denote empty sequences.}\]
$q(j_0, P' \mid \varnothing, s)$ denote the probability that $j_0$ is picked after $P'$ at $\varnothing$ under $s$. Then

$$
\Pi_i(s \mid \varnothing, P) = \sum_{P' \subseteq N \text{ s.t. } P \subseteq P', j_0 \in \text{Atck}(s, \varnothing, P')} \sum q(j_0, P' \mid \varnothing, s) \Pi_i(s \mid \varnothing, P', j_0) + \left(1 - \sum_{P' \subseteq N \text{ s.t. } P \subseteq P', j_0 \in \text{Atck}(s, \varnothing, P')} \sum q(j_0, P' \mid \varnothing, s)\right) \pi_i(\varnothing \mid \varepsilon) \tag{15}
$$

As we established above, $\bar{\pi}_i(\varnothing) \geq \pi_i(\varnothing \mid \varepsilon)$, with strict inequality if $i$ is strong. Thus to show the claim, it is enough to show that

$$
\bar{\pi}_i(\varnothing) \geq \Pi_i(s \mid \varnothing, P', j_0), \tag{16}
$$

for each $P' \subseteq N$ with $P \subseteq P'$ and each attacking ruler $j_0 \in \text{Atck}(s, \varnothing, P')$, with strict inequality for at least one $P'$ and $j_0 \in \text{Atck}(s, \varnothing, P')$ in the case of $i$ being strong.

So take any set of rulers, $P' \subseteq N$ with $P \subseteq P'$ and any ruler $j_0 \in \text{Atck}(s, \varnothing, P')$. Three cases are possible:

(i). $j_0 \neq i$ and $i$ is not in the attacking sequence $s_{j_0}(\varnothing)$ of $j_0$,

(ii). $j_0 \neq i$ and $i$ is in the attacking sequence $s_{j_0}(\varnothing)$ of $j_0$,

(iii). $j_0 = i$.

*Case (i).* Ruler $j_0$ is different to $i$ and does not have $i$ in his attacking sequence $s_{j_0}(\varnothing)$. Let $F(\varnothing' \mid s, \varnothing, P', j_0)$ be the probability of reaching ownership state $\varnothing'$ in the next round from state $\varnothing$ under strategy profile $s$ when $j_0$ is selected to move after $P'$ (and executes attacking sequence $s_{j_0}(\varnothing, P')$). Then

$$
\Pi_i(s \mid \varnothing, P', j_0) = \sum_{\varnothing' \in \emptyset} F(\varnothing' \mid s, \varnothing, P', j_0) \Pi_i(s \mid \varnothing'). \tag{17}
$$

To show (16) it is enough to show that

$$
\bar{\pi}_i(\varnothing) \geq \Pi_i(s \mid \varnothing') = \Pi_i(s \mid \varnothing', \emptyset), \tag{18}
$$

for each state $\varnothing'$ that can be reached in the next round with positive probability from $\varnothing$ when $j_0$ plays the attacking sequence $s_{j_0}(\varnothing, P')$ after $P'$ at $\varnothing$. We will show that the inequality is strict when $i$ is strong.
Ownership state \(o'\) is reached after at least one fight and so has at most \(X\) active rulers. Hence, by the induction hypothesis, \(\pi_i(o') \geq \Pi_i(s \mid o', \emptyset)\), and so to show (18) it is enough to show that
\[
\pi_i(o) \geq \pi_i(o'). \tag{19}
\]
Take an optimal myopic sequence, \(\sigma_i(o')\), of \(i\) at \(o'\). There are two sub-cases to be considered.

(a) Sequence \(\sigma_i(o')\) does not contain the rulers in the sequence of fights that leads to \(o'\). This means, in particular, that \(\sigma_i(o')\) is not a full attacking sequence. Hence, by Step 1, \(i\) is not strong.

Since \(\sigma_i(o')\) does not contain the rulers in the sequence of fights that leads to \(o'\), it can be executed at state \(o\). By optimality of \(\sigma_i(o)\) at \(o\)
\[
\pi_i(o) = \pi_i(o \mid \sigma_i(o)) \geq \pi_i(o \mid \sigma_i(o')) = \pi_i(o' \mid \sigma_i(o')) = \pi_i(o'). \tag{20}
\]

(b) Sequence \(\sigma_i(o')\) contains at least one ruler in the sequence of fights that leads to \(o'\). This is true, in particular, when \(i\) is strong because, by Step 1, \(\sigma_i(o')\) must be a full attacking sequence then.

Since \(\sigma_i(o')\) contains at least one ruler in the sequence of fights that leads to \(o'\), \(\sigma_i(o)\) is an optimal myopic strategy for \(i\) at \(o\) and so payoff dominates \(\sigma'\) at \(o\). Hence
\[
\pi_i(o) = \pi_i(o \mid \sigma_i(o)) \geq \pi_i(o \mid \sigma') > \pi_i(o' \mid \sigma_i(o')) = \pi_i(o'). \tag{21}
\]
Hence (19) and, consequently, (18) hold with strict inequality.

Case (ii). Ruler \(j_0\) is different to \(i\) and has \(i\) in his attacking sequence \(s_{j_0}(o)\). Let \(s_{j_0}(o) = j_1, \ldots, j_m\) be the sequence selected by \(j_0\) at \(o\) under strategy \(s_{j_0}\). Then \(i = j_k\) for some \(1 \leq k \leq m\). Given \(l \in \{1, \ldots, m\}\), let \(o^l\) be the state reached after \(j_0\) looses the \(l\)th fight in the sequence. The expected payoff to \(i\) from \(s\) at \(o\) given that \(j_0\) is selected to move after set
$P'$ or rulers is equal to

$$
\Pi_i(s \mid o, P', j_0) = \sum_{l=1}^{k-1} F(o^l \mid s, o, P', j_0) \Pi_i(s \mid o^l) + \\
\left(1 - \sum_{l=1}^{k-1} F(o^l \mid s, o, P', j_0)\right) p(r_i(o), \sum_{l=0}^{k-1} r_{j_l}(o)) \Pi_i(s \mid o^k),
$$

(22)

where $j_1, \ldots, j_{k-1}$ are the rulers attacked by $j_0$ prior to attacking $i$.

Hence to show (16) it is enough to show that (18) holds for all $o' = o^l$, $l \in \{1, \ldots, k-1\}$, reachable after a sequence of fights of $j_0$ in which $j_0$ looses before facing $i$, and that

$$
\bar{\pi}_i(o) \geq p(r_i(o), \sum_{l=0}^{k-1} r_{j_l}(o)) \Pi_i(s \mid o^k).
$$

(23)

holds for $o^k$, reachable by a sequence of fights of $j_0$ in which $i$ is attacked by $j_0$ and wins. (18) is shown by the same arguments as in point (ii) above. In particular, the inequality in (18) is strict when $i$ is strong. For (23), let $\tau$ be a sequence of rulers $\{j_0, \ldots, j_{k-1}\}$ feasible to $i$ at $o$ (clearly such a sequence exists). Then sequence $\sigma' = \tau \sigma_i(o^k)$, consisting of $\tau$ and an optimal myopic sequence of $i$ at $o^k$, is feasible for $i$ at $o$. By the no-waiting property and its generalization, (10), $\tau$ yields at least the same payoff to $i$ as the sequence of fights that leads to $o'$ (the inequality is strict, unless $k = 1$). Combining this with the induction hypothesis we get

$$
\bar{\pi}_i(o) \geq \pi_i(o \mid \tau \sigma_i(o^k)) \geq p(R_i(o), \sum_{l=0}^{k-1} R_{j_l}(o)) \pi_i(o^k \mid \sigma_i(o^k))
$$

$$
\geq p(R_i(o), \sum_{l=0}^{k-1} R_{j_l}(o)) \Pi_i(s \mid o^k),
$$

(24)

with strict inequality, unless $k = 1$.

Case (iii). Ruler $i$ is picked to move at $o$ after $P'$. The strategy chosen by $i$ under strategy profile $s$ at $(o, P')$ is $s_i(o, P')$. Let $o'$ be the state that is reached if $i$ wins all the attacks in sequence $s_i(o, P')$. Then sequence $\sigma' = s_i(o, P') \sigma_i(o')$, consisting of $s_i(o, P')$ and an optimal myopic sequence of $i$ at $o'$, is feasible for $i$ at $o$. State $o'$ is reached after at least one fight and has at most $X$ active rulers. By the induction hypothesis, $\bar{\pi}_i(o') \geq \Pi_i(s \mid o')$ and it follows
that
\[ \pi_i(\sigma) \geq \pi_i(\sigma | s_i(\sigma), \sigma') \geq \pi_i(\sigma' | \sigma'(\sigma')) = \pi_i(\sigma') \geq \Pi_i(s | \sigma'). \]  
(25)

The inequality is strict unless the sequence \( s_i(\sigma)\sigma_i(\sigma') \) is the same as the optimal myopic sequence of \( i \) at \( \sigma \).

To complete the proof of the claim, we argue that \( \pi_i(\sigma) > \Pi_i(s | \sigma, P) \) if \( i \) is strong and there are at least 3 active rulers at \( \sigma \). As we established above, if \( i \) is strong then \[16\] holds with equality in two cases only: \( j_0 = i \) and \( s_i(\sigma, P) \) is the optimal myopic sequence of \( i \) at \( \sigma \), or \( j_0 = j \neq i \), \( j_0 \) attacks \( i \) first under \( s_{j0}(\sigma, P) \) and \( j_0 \) is the first ruler to be attacked by \( i \) under his optimal myopic sequence of attacks. Generically the second case is possible for at most one ruler other then \( i \). Hence with at least three active rulers there is at least one for which the inequality in \[16\] is strict. This completes the proof of the claim.

From Step 1, we know that in any state \( \sigma \), there exists a strong ruler for whom the full attacking sequence is the optimal stand alone strategy and it is optimal for him to choose it after any set of rulers \( P \) at \( \sigma \). It now follows from the claim above that for this strong ruler the optimal full attacking sequence dominates all other strategies, and the domination is strict if there are at least three active rulers at \( \sigma \). The final step in the proof takes up non-strong rulers. We show that faced with rulers such that at every state at least one of them attacks, every ruler will find it profitable to choose an optimal full attacking sequence.

**Step 3:** Let \( i \in N \) be a ruler, \( \tilde{s} \) be a strategy profile such that for every state \( \sigma \) and for every permutation of \( N, j_1, \ldots, j_n \), there exists \( k \in \{1, \ldots, n\} \) such that \( j_k \neq i \) and \( \tilde{s}_{jk}(\sigma, \{j_1, \ldots, j_{k-1}\}) \neq \varepsilon \). Let \( s_i \) be a best response of \( i \) to \( \tilde{s}_{-i} \). Then for every state \( \sigma \) such that \( i \in \text{Act}(\sigma) \) and \( |\text{Act}(\sigma)| \geq 3 \), and for every set of rulers, \( P \subseteq N \setminus \{i\} \) such that \( \text{Atck}(\tilde{s}, \sigma, P) \setminus \{i\} \neq \emptyset \), \( s_i(\sigma, P) \) is an optimal full attacking sequence of \( i \) at \( \sigma \).

Let \( i \in N \) be a ruler and let \( \tilde{s}_{-i} \) be a strategy profile of the other rulers, as stated above. The assumption means that at every state \( \sigma \), for any draw of rulers, with probability 1 a ruler other than \( i \) would choose attack if \( i \) would not. Let \( s_i \) be a strategy such that at every state \( \sigma \) where ruler \( i \) is active and there are at least three active rulers, and for every \( P \subseteq N \setminus \{i\} \) with \( \text{Atck}(\tilde{s}, \sigma, P) \setminus \{i\} \neq \emptyset \), \( s_i(\sigma, P) \) is an optimal full attacking sequence for \( i \). We show that for any other strategy, \( s_i' \), of ruler \( i \), every state \( \sigma \in \emptyset \) with \( |\text{Act}(\sigma)| \geq 2 \), and every set of rulers \( P \subseteq N \setminus \{i\} \) such that \( \text{Atck}(\tilde{s}, \sigma, P) \setminus \{i\} \neq \emptyset \),

\[ \Pi_i((s_i, \tilde{s}_{-i}) | \sigma, P) > \Pi_i((s_i', \tilde{s}_{-i}) | \sigma, P), \]  
(26)
with strict inequality when $|\text{Act}(o)| \geq 3$. Notice that, by Step 2, the claim holds if $i$ is strong at $o$. For the remaining part of the proof we will consider rulers who are not strong at the given states.

The argument is by induction on the number of active rulers. As it proceeds along lines similar to Step 2, it is omitted.

Proof of Proposition 1. A sequence $\sigma \in \mathbb{R}^*$ is strong if either $\sigma = \varepsilon$ or $\sigma = x_0, \ldots, x_m$ and for all $k \in \{1, \ldots, m\}$, $\sum_{j=0}^{k-1} x_j > x_k$. A sequence $\sigma \in \mathbb{R}^*$ is weak if it is not strong.

Let $p(x, y \mid \gamma) = \frac{x^\gamma y^\gamma}{x^\gamma + y^\gamma}$. Since \[
\frac{\partial p}{\partial \gamma} = \left( \frac{x^\gamma y^\gamma}{x^\gamma + y^\gamma} \right) (\ln(x) - \ln(y))
\]
and \[
\lim_{\gamma \to +\infty} \frac{x^\gamma}{x^\gamma + y^\gamma} = \lim_{\gamma \to +\infty} \frac{1}{1 + (\frac{y}{x})^\gamma} = \begin{cases} 
1, & \text{if } x > y \\
0, & \text{if } x < y.
\end{cases}
\]
so for $x > y$, $p(x, y \mid \gamma)$ is increasing and converges to 1 when $\gamma \to +\infty$, and for $x < y$, $p(x, y \mid \gamma)$ is decreasing and converges to 0 when $\gamma \to +\infty$. In addition, for any strong sequence $\sigma$, $p_{\text{seq}}(\sigma \mid \gamma)$ is increasing when $\gamma$ is increasing. This is because for all $k \in \{1, \ldots, m\}$, $\sum_{j=0}^{k-1} x_j > x_k$, and so $\lim_{\gamma \to +\infty} \prod_{k=1}^{m} p\left( \sum_{j=0}^{k-1} x_j, x_k \mid \gamma \right) = 1$ and $\prod_{k=1}^{m} p\left( \sum_{j=0}^{k-1} x_j, x_k \mid \gamma \right)$ is increasing when $\gamma$ is increasing. On the other hand, for any weak $\sigma = x_0, \ldots, x_m$, $\lim_{\gamma \to +\infty} p_{\text{seq}}(\sigma \mid \gamma) = 0$. This is because there exists $k \in \{1, \ldots, m\}$ such that $\sum_{j=0}^{k-1} x_j < x_k$ and for any such $k$, $\lim_{\gamma \to +\infty} p\left( \sum_{j=0}^{k-1} x_j, x_k \mid \gamma \right) = 0$. Since for all other $k \in \{1, \ldots, m\}$, $p\left( \sum_{j=0}^{k-1} x_j, x_k \mid \gamma \right) \leq 1$ so $\lim_{\gamma \to +\infty} \prod_{k=1}^{m} p\left( \sum_{j=0}^{k-1} x_j, x_k \mid \gamma \right) = 0$. Consequently, for any non-empty sequence $\sigma = x_0, \ldots, x_m$,

$$
\lim_{\gamma \to +\infty} p_{\text{seq}}(\sigma \mid \gamma) = \begin{cases} 
1, & \text{if } \sigma \text{ is strong} \\
0, & \text{if } \sigma \text{ is weak}.
\end{cases}
$$

The claim on probability of hegemony for strong rulers now follows. \qed
ONLINE APPENDIX

PROOFS AND ADDITIONAL RESULTS

PROOFS I: POOR REWARDING CONTEST SUCCESS FUNCTIONS

Proof of Proposition 2. Part 1: The argument presented here is true for general contest functions. Take any ownership configuration, $o \in \mathcal{O}$, and any active ruler, $i \in \text{Act}(o)$. Given a state $(o, P) \in \mathcal{O} \times 2^{N \setminus \{i\}}$, an attacking sequence, $\sigma$, is an optimal attacking sequence if it maximises the payoff of $i$ at $(o, P)$ across all attacking sequences that are feasible to $i$ at $o$ and given the continuation payoffs determined by $s$ on the states in $\text{Succ}(o, P)$. Notice that if a sequence is an optimal attacking sequence for $i$ at $(o, P)$, then it is an optimal attacking sequence of $i$ at $(o, P')$, for any $P' \subseteq N$. Thus its optimality depends on the ownership configuration and the expected payoff determined by $s$ on ownership configurations $o' \in \text{Succ}(o)$, only. Clearly, at every state $(o, P) \in \mathcal{O} \times 2^{N \setminus \{i\}}$ an expected payoff maximising ruler chooses between the empty sequence (peace) and an optimal attacking sequence at $o$.

Given ownership configuration $o$, let $E(s, o)$ be the set of rulers, active at $o$, for whom an optimal attacking sequence at $o$ yields higher payoff than the empty sequence. It is easy to see that if $E(s, o) = \emptyset$ and $s$ is an equilibrium, then $s_i(o, P) = \varepsilon$, for all $i \in N$ and $P \in 2^{N \setminus \{i\}}$. On the other hand, suppose that $E(s, o) \neq \emptyset$ and take any sequence $i_1, \ldots, i_n$ of rulers from $N$. Let $i_k$ be the last ruler from $E(s, o)$ in the sequence. Generically, no ruler is indifferent between peace and an optimal attacking sequence. Hence, if $s$ is an equilibrium then, for every $l > k$, $s_{i_l}(\{i_1, \ldots, i_{l-1}\})$ is the empty sequence and, consequently, $s_{i_k}(\{i_1, \ldots, i_{k-1}\})$ is an optimal attacking sequence of $i_k$ at $o$ under the continuation of $s$. Hence if $E(s, o) \neq \emptyset$ then $o$ is conflictual under $s$.

Part 2: Let $p$ be a poor rewarding contest success function satisfying (6). Then $p(x, y) = f(x)/(f(x) + f(y))$ and, by Theorem 2, $f(x)/x$ is decreasing. Since $f(x)/x$ is decreasing and positive on $\mathbb{R}_{++}$ so $\lim_{x \to +\infty} f(x)/x$ exists and is finite. Let $\lim_{x \to +\infty} f(x)/x = L$.

Consider a sequence of fights where a ruler with $x \in \mathbb{R}_{++}$ resources first fights a ruler with $y \in \mathbb{R}_{++}$ resources and then fights with $m \geq 1$ rulers with resources $z_1, \ldots, z_m \in \mathbb{R}_{++}$. The
expected payoff to the rulers with \( x \) resources from such a sequence of fights is equal to

\[
\pi(x, y, z_1, \ldots, z_m) = p_{\text{seq}}(x, y, z_1, \ldots, z_m)(x + y + z_1 + \ldots + z_m)
\]

\[
= x \cdot \frac{f(x)}{f(x) + f(y)} \cdot \frac{x + y}{x} \cdot \prod_{i=1}^{m} \left( \frac{f(x + y + \sum_{j=1}^{i-1} z_j)}{f(x + y + \sum_{j=1}^{i-1} z_j) + f(z_i)} \cdot \frac{x + y + \sum_{j=1}^{i} z_j}{x + y + \sum_{j=1}^{i} z_j} \right).
\]

We will show that for sufficiently large \( y \), \( \pi(x, y, z_1, \ldots, z_m) > x \). We consider two cases separately: \( L > 0 \) and \( L = 0 \).

Suppose first that \( L > 0 \). Notice that

\[
\lim_{y \to +\infty} \frac{f(x)}{f(x) + f(y)} \cdot \frac{x + y}{x} = \lim_{y \to +\infty} \frac{f(x)}{f(x) + f(y)} \cdot \left( \frac{x}{y} + 1 \right) = \frac{f(x)}{x} \cdot \frac{x}{L} > 1.
\]

Similarly

\[
\lim_{y \to +\infty} \frac{f(x + y + \sum_{j=1}^{i-1} z_j)}{f(x + y + \sum_{j=1}^{i-1} z_j) + f(z_i)} \cdot \frac{x + y + \sum_{j=1}^{i} z_j}{x + y + \sum_{j=1}^{i} z_j} = \frac{L}{L} = 1.
\]

Hence \( \lim_{y \to +\infty} \pi(x, y, z_1, \ldots, z_m) = t > x \) and so for sufficiently large \( y \), \( \pi(x, y, z_1, \ldots, z_m) > x \).

Second, suppose that \( L = 0 \). After winning the conflict with the ruler with \( y \) resources, in every subsequent conflict in the sequence the starting ruler has higher resources than his opponent. Hence the probability of winning each of these conflicts is more than 1/2. In the event of winning all the conflicts in the sequence, the starting ruler owns at least \( x + y + \sum_{j=1}^{m} z_j \) resources. By these observations \( \pi(x, y, z_1, \ldots, z_m) \geq \left( \frac{1}{2^m} \right) \left( \frac{f(x)}{f(x) + f(y)} \right) (x + y) \). On the other hand, since \( L = 0 \) so, for sufficiently large \( y \),

\[
\frac{f(y)}{y} + \left( 1 - \frac{1}{2^m} \right) \frac{f(x)}{y} < \frac{1}{2^m} \frac{f(x)}{x}.
\]

Multiplying both sides by \( y/f(x) \) and reorganizing, this is equivalent to

\[
\frac{f(y)}{f(x)} + 1 < \frac{1}{2^m} \left( 1 + \frac{y}{x} \right).
\]

Taking the inverses of both sides and then multiplying both sides by \((x + y)/2^m\), this is
equivalent to
\[
\left( \frac{1}{2^m} \right) \left( \frac{f(x)}{f(x) + f(y)} \right) (x + y) > x.
\]
Hence, for sufficiently large \( y \), \( \pi(x, y, z_1, \ldots, z_m) > x \).

Now, let \( G \) be a connected network over the set of nodes, \( V \), and let \( r \in \mathbb{R}_{++} \) be a resource endowment. Fix any vertex \( v \in V \). Take any ownership configuration \( o \in \mathcal{O} \). If there is a ruler who owns all the vertices under \( o \) then we are done. Assume otherwise. There are at least two active rulers under \( o \), \( |\text{Act}(o)| \geq 2 \). Let \( i \) be the ruler owning vertex \( v \), \( o(v) = i \), and let \( j \in \text{Act}(o) \) be any active neighbor of \( i \) under \( o \). Let \( \sigma \) be a permutation of \( \text{Act}(o) \setminus \{j\} \) starting with \( i \). Sequence \( \sigma \) is a full attacking sequence of \( j \) at \( o \). By what we have shown above, if \( r_v \) is sufficiently large, then \( \Pi(j, o; \sigma) > R_j(o) \) and so by choosing \( \sigma \) ruler \( j \) strictly increases his expected payoff. Since at every ownership configuration \( o \) with at least two active rulers there exists a ruler who can increase his expected resources by choosing attack, so every equilibrium outcome is hegemony.

**Part 3** Let \( v \in V \) be a vertex and let \( G \) be a star network with centre \( v \). Let \( p \) be a Tullock contest success function with \( \gamma \in (0, 1) \). Take any \( y > 0 \). Let the resource vector \( r \) be such that \( r_u = y \), for each spoke \( u \in V \setminus \{v\} \), and \( r_v = x \), for the centre. We will show that there exists (a range of values of) \( x \) such that there is an equilibrium where each ruler chooses peace in the initial ownership configuration. Similarly, we will show that there exists (a range of values of) \( x \) such that there is an equilibrium where each ruler at a spoke chooses a sequence of fights that leads to a ownership configuration with peace (so we have war followed by peace in equilibrium).

The expected payoff from a full attacking sequence of \( m \) fights to a ruler owning a spoke in a star over at least \( m + 1 \) vertices, when each spoke is endowed with \( y \) resources and the centre is endowed with \( x \) resources, is
\[
\varphi(x, y, m) = (x + my) p(y, x) \prod_{i=1}^{m-1} p(x + iy, y) = (x + my) \left( \frac{y^\gamma}{x^\gamma + y^\gamma} \right) \prod_{i=1}^{m-1} \left( \frac{(x + iy)^\gamma}{(x + iy)^\gamma + y^\gamma} \right)
\]

The key to the constructions of resource endowments enabling equilibria described above is the following claim:

**Claim.** For all \( m \geq 2 \), \( \gamma \in [0, 1) \), and \( y > 0 \), there exists a unique \( x^*_m = x^*_m(y, \gamma) > y \), such that
that
\[
\varphi(x, y, m) \begin{cases} 
<y & \text{if } x \in (y, x_m^*), \\
= y & \text{if } x = x_m^*, \\
> y & \text{if } x > x_m^*.
\end{cases}
\] (27)

Moreover, \(x_{m+1}^*(y, \gamma) > x_m^*(y, \gamma)\).

Before proving the claim, we provide the construction of resource endowments. Taking any \(x \in (\max(y, x_{n-2}^* - y), x_{n-1}^*)\) guarantees that no ruler has incentives to engage in a full attacking sequence (and the interval is non-empty, as \(x_{n-2}^* > y\), for \(n \geq 4\)). Moreover, after at least one fight, every ruler at a spoke has incentives to fight if no other ruler fights, as a full attacking sequence yields him expected payoff higher than \(y\). Thus any ruler deviating from peaceful strategy profile leads to fight till hegemony, which is not profitable for the deviating ruler. Therefore there is an equilibrium where all rulers choose peace in the initial ownership configuration. Similarly, taking any \(x \in (\max(0, x_{n-3}^*-2y), x_{n-2}^*-y)\) guarantees that after one fight by a spoke, an ownership configuration with resources at the centre as described above is reached. Moreover, at such a state, no ruler has incentives to engage in a full attacking sequence. Thus there is an equilibrium where (1) in the initial state each ruler owning a spoke chooses to attack the centre and the ruler owning the centre chooses peace, (2) in the state with \(n-1\) vertices every vertex chooses peace, and (3) in any state with at most \(n-2\) at least one vertex chooses attack. In this equilibrium there is one conflict followed by peace.

Notice that the two constructions given above are generic: analogous argument could be conducted if spokes were endowed with resource sufficiently close to each other and the centre was endowed with resources within a range close to the range given in the construction above.

We now provide the proof of the claim. To this end, we establish four properties of function \(\varphi\), from which the claim follows. Fix any \(\gamma \in [0, 1)\).

First, we show that, for all \(x, y \in \mathbb{R}_+\) and \(m \geq 3\), \(\varphi(x, y, m) < \varphi(x, y, m-1)\). Notice that, \(y^{1-\gamma} \leq (x+(m-1)y)^{1-\gamma}\). Multiplying both sides by \(y^\gamma (x+(m-1)y)^{\gamma} \) we get \(y(x+(m-1)y)^{\gamma} < y^\gamma (x+(m-1)y)\). Reorganizing, we obtain \((x+my)(x+(m-1)y)^{\gamma} < ((x+(m-1)y)^{\gamma} + y^\gamma)(x+(m-1)y)\). Dividing both sides by the RHS we get \(\frac{x+my}{x+(m-1)y} \left( \frac{(x+(m-1)y)^{\gamma}}{(x+(m-1)y)^{\gamma} + y^\gamma} \right) < 1\). This, together with the fact that \(\varphi(x, y, m) = \varphi(x, y, m-1)\left( \frac{(x+(m-1)y)^{\gamma}}{(x+(m-1)y)^{\gamma} + y^\gamma} \right) \left( \frac{x+my}{x+(m-1)y} \right)\) yields \(\varphi(x, y, m) < \varphi(x, y, m-1)\).

Second, we show that \(\varphi\) is strictly increasing in \(x\) for \(x > y\). First derivative of \(\varphi\) with
respect to \( x \) is

\[
\frac{\partial \varphi}{\partial x} = \left( \frac{\gamma y^\gamma}{x^\gamma + y^\gamma} \right) (x + my) \prod_{i=1}^{m-1} \left( \frac{(x + iy)^\gamma}{(x + iy)^\gamma + y^\gamma} \right) - \left( \frac{x^{\gamma-1}}{x^\gamma + y^\gamma} \right) + \sum_{j=1}^{m-1} \frac{y^\gamma}{(x + jy)((x + jy)^\gamma + y^\gamma)}.
\]

(28)

Since \( \gamma \in [0, 1) \) so \((1 - \gamma)(x + y) > 0\). Reorganizing we get \( x + y + (m - 1)\gamma y > \gamma(x + my) \). Dividing both sides by \( \gamma(x + y)(x + my) \) we get \( \frac{1}{\gamma(x + my)} + \frac{(m - 1)y}{(x + y)(x + my)} > \frac{x^{\gamma-1}}{x^\gamma + y^\gamma} \). Since \( x > y \) and \( \gamma \in [0, 1) \) so \((x/y)^{1-\gamma} > 1\) and so \( \frac{1}{x+y} > \frac{1}{x+y(x+y)^{1-\gamma}} = \frac{x^{\gamma-1}}{x^\gamma + y^\gamma} \). Hence

\[
\frac{1}{\gamma(x + my)} + \frac{(m - 1)y}{(x + y)(x + my)} > \frac{x^{\gamma-1}}{x^\gamma + y^\gamma}.
\]

(29)

Notice that

\[
\frac{(m - 1)y}{(x + y)(x + my)} = \left( \frac{1}{x + y} \right) - \left( \frac{1}{x + my} \right) = \sum_{i=1}^{m-1} \left( \frac{1}{x + iy} \right) - \sum_{i=2}^{m} \left( \frac{1}{x + iy} \right) = \sum_{i=1}^{m-1} \left( \frac{1}{x + iy} - \frac{1}{x + (i + 1)y} \right) = \sum_{i=1}^{m-1} \left( \frac{y}{(x + iy)((x + iy) + y)} \right)
\]

Moreover, for \( \gamma \in [0, 1) \), \( x > y \), and \( i \geq 1 \),

\[
\frac{y}{(x + iy)((x + iy) + y)} = \frac{1}{(x + iy) \left( \left( \frac{x}{y} + i \right) + 1 \right)} < \frac{1}{(x + iy) \left( \left( \frac{x}{y} + i \right)^\gamma + 1 \right)} = \frac{y^\gamma}{(x + iy)((x + iy)^\gamma + y^\gamma)}
\]

Thus

\[
\frac{(m - 1)y}{(x + y)(x + my)} < \sum_{i=1}^{m-1} \left( \frac{y^\gamma}{(x + iy)((x + iy)^\gamma + y^\gamma)} \right)
\]

which, together with (29), implies

\[
\frac{1}{\gamma(x + my)} + \sum_{i=1}^{m-1} \left( \frac{y^\gamma}{(x + iy)((x + iy)^\gamma + y^\gamma)} \right) > \frac{x^{\gamma-1}}{x^\gamma + y^\gamma}.
\]
Therefore, by that and (28), \( \partial \varphi / \partial x > 0 \) for all \( x > y \) and so \( \varphi \) is increasing in \( x \) on \( (y, +\infty) \).

Third, we show that for all \( y \in \mathbb{R}_{++} \) and \( m \geq 3 \), \( \lim_{x \to +\infty} \varphi(x, y, m) = +\infty \). To see that notice that \( \lim_{x \to +\infty} \prod_{i=1}^{m-1} p(x+iy, y) = 1 \) and \( \lim_{x \to +\infty} p(y, x)(x+my) = \left( \frac{y^\gamma}{1+(\frac{y}{x})^\gamma} \right) (x^{1-\gamma} + m \left( \frac{y}{x^\gamma} \right)) = +\infty \), and so the property follows.

Fourth, we show that \( \varphi(y, y, m) < y \). To see that we start with

\[
\varphi(y, y, m) = \left( \frac{1}{2} \right) (m+1)y \prod_{i=1}^{m-1} \left( \frac{(i+1)^\gamma}{(i+1)^\gamma + 1} \right) = \left( \frac{1}{2} \right) (m+1)y \prod_{i=2}^{m} \left( \frac{i^\gamma}{i^\gamma + 1} \right).
\]

Since \( \frac{i^\gamma}{i^\gamma + 1} = 1 - \left( \frac{1}{i^\gamma + 1} \right) \), \( \gamma \in [0,1) \), \( i \geq 1 \), so \( \frac{i^\gamma}{(i^\gamma + 1)} \) is increasing in \( \gamma \). Hence \( \varphi(y, y, n) < \left( \frac{1}{2} \right) (m+1)y \prod_{i=2}^{m} \left( \frac{i^\gamma}{i^\gamma + 1} \right) = \left( \frac{1}{2} \right) y \left( \frac{2^n}{2^n} \right) 2 = y \).

By the four properties of \( \varphi \), established above, for all \( m \geq 2 \), \( \gamma \in [0,1) \), and \( y > 0 \), there exists a unique \( x^*_m = x^*_m(y, \gamma) > y \), such that (27) holds. Moreover, since for all \( x, y \in \mathbb{R}_{++} \) and \( m \geq 3 \), \( \varphi(x, y, m) < \varphi(x, y, m-1) \), and since \( \varphi \) is increasing in \( x \) for \( x > y \), \( x^*_m(y, \gamma) > x^*_m(y, \gamma) \). This completes the proof. \( \square \)

PROOFS II: SHORT ATTACK SEQUENCES

Proof of Proposition 3. Throughout the proof we use the precedence relations on ownership configurations and states, as well as the sets \( \text{Succ} \) and \( \overline{\text{Succ}} \) introduced in proof of Theorem 1.

Notice that if \( i \in \text{Act(}o\text{)} \) is the unique strong ruler at \( o \), then for all \( o' \in \text{Succ}(o) \) there is exactly one strong ruler in \( \text{Act}(o') \) and if \( i \in \text{Act}(o') \) then \( i \) is strong. This is because no weak ruler has a strong full attacking sequence and therefore no such ruler can become strong, unless he wins a conflict with a strong ruler (in which case he replaces the unique strong ruler in the subsequent state).

Given a ruler \( i \in N \) and an ownership configuration, \( o \), a strategy \( s_i \) of \( i \) is an attacking strategy at \( o \) if, for every ownership configuration \( o' \in \overline{\text{Succ}}(o) \) such that \( i \in \text{Act}(o) \) and \( |\text{Act}(o)| \geq 2 \), and every set of rulers \( P \in 2^{N \setminus \{i\}} \), \( s_i(o', P) \neq \varepsilon \). Thus, at state \( o \) and at any state following \( o \) in the course of the game, \( i \) never chooses to stay peaceful under \( s_i \), unless he is not active or is the unique active ruler.

Given a ruler \( i \in N \), an ownership configuration, \( o \), and a strategy profile of the other ruler, \( s_{-i} \), we define an attacking strategy \( s_i \) that is a best attacking response of \( i \) to \( s_{-i} \) at \( o \). The strategy is defined recursively on the set of states \( \text{Succ}(o, \overline{\emptyset}) \), starting from the maximal elements under \( \leq \). If \( (o', P) \) is such that \( o' \) is maximal according to \( \leq \) in \( \overline{\text{Succ}}(o) \) then, for
all \( i \in N, s_i(o', P) = \varepsilon \) (the unique feasible choice of \( i \) at \( o \)). Otherwise, let \( s_i(o, P) \) be any neighboring ruler attacking whom maximises \( i \)'s expected payoff across all neighbors of \( i \) at \( o \), given the continuation payoff determined by \( s = (s_i, s_{-i}) \) defined on states in \( \text{Succ}(o, P) \).

Notice that if \( j \) is such a ruler at \((o, P)\) then, for any \( P \in 2^{N\setminus\{i\}} \), attacking \( j \) maximises \( i \)'s expected payoff across all neighbors of \( i \). Moreover, generically, such a neighbor is unique.

Now we are ready to give main part of the proof. First we show, for any strategy profile, \( s \), any ownership configuration, \( o \in \mathcal{O} \), any ruler \( i \in N \), and any set of rulers \( P \subseteq N \setminus \{i\} \), that if \( i \) is the unique strong ruler at \( o \) then any best attacking response, \( s_i^* \), of \( i \) to \( s_{-i} \) at \( o \) yields \( i \) an expected payoff greater than \( R_i(o) \).

The proof is by induction on the number of active rulers at \( o \). For the induction basis, suppose that \( |\text{Act}(o)| = 2 \) and that \( i \) is the single strong ruler at \( o \). Let \( j \) be the other active ruler. Since \( p \) is rich rewarding and \( i \) is strong, the other active ruler is weak and attacking him increases \( i \)'s payoff in expectation. Thus the claim holds.

For the induction step, take any \( 2 < m \leq n \) suppose that the claim holds for any ownership configuration \( o \) with \( |\text{Act}(o)| < m \) active rulers. Take any ownership configuration, \( o \in \mathcal{O} \), with a unique strong ruler, \( i \in \text{Act}(o) \). Notice that since \( s_i^* \) is an attacking strategy, so \( s_i^*(o, P) \neq \varepsilon \), for all \( P \in 2^{N\setminus\{i\}} \). Hence, with probability 1, a ruler choosing attack will be selected at \( o \). Thus the strategy profile \( \tilde{s} = (s_i^*, s) \) determines a probability distribution \( Q(\cdot | \tilde{s}, o) \) on the set \( A(o) = \{(j, k) \in \text{Act}(o) \times \text{Act}(o) : j \neq k\} \) where, given \((j, k) \in A(o)\), \( Q(j, k | \tilde{s}, o) \) is the probability that ruler \( j \) attacks ruler \( k \) at \( o \). Given two rulers, \( j, k \in \text{Act}(o) \), active at \( o \) let \( o[j \to k] \) denote the ownership configuration resulting from \( j \) winning a conflict with \( k \). The expected payoff to \( i \) at \( o \), \( \Pi_i(\tilde{s} | o) \), is equal to

\[
\Pi_i(\tilde{s} | o) = \sum_{(j, k) \in A(o)} Q(j, k | \tilde{s}, o) \left( p(R_j(o), R_k(o))\Pi_i(\tilde{s} | o[j \to k]) + p(R_k(o), R_j(o))\Pi_i(\tilde{s} | o[k \to j]) \right) + \\
\sum_{(j, i) \in A(o)} Q(j, i | \tilde{s}, o) p(R_i(o), R_j(o))\Pi_i(\tilde{s} | o[i \to j]) + \\
Q(i, s_i^*(o, P) | \tilde{s}, o) p(R_i(o), R_{s_i^*(o,P)}(o))\Pi_i(\tilde{s} | o[i \to s_i^*(o,P)])
\]

By the observation at the beginning of the proof, \( i \) remains a unique strong ruler at each ownership configuration \( o[j, k] \) with \((j, k) \in A(o)\) such that \( j \neq i \) and \( k \neq i \). Similarly, \( i \)
remains a unique strong ruler at each ownership configuration $o[i, j]$ with $j \in A(o)$. Thus, by the induction hypothesis, for all $(j, k) \in A(o)$, $\Pi_{i}(s \mid o[j \rightarrow k]) > R_{i}(o[j \rightarrow k])$. In the case of $j \neq i$ and $k \neq i$, $R_{i}(o[j \rightarrow k]) = R_{i}(o)$. In the case of $k = i$, $p(R_{i}(o), R_{j}(o)) \Pi_{i}(s \mid o[i \rightarrow j]) > p(R_{i}(o), R_{j}(o)) R_{i}(o[j \rightarrow k]) = p(R_{i}(o), R_{j}(o))(R_{i}(o) + R_{j}(o)) > R_{i}(o)$, as $R_{i}(o) > R_{j}(o)$ and $i$ is rich rewarding. Hence $\Pi_{i}(s \mid o) > R_{i}(o)$.

Now, suppose that there is a unique strong node in $G$ under resource endowment $r$. Then there is a unique strong ruler at the ownership configuration. Take any equilibrium $s$ of the game. By the observation above, there is a unique strong ruler at every ownership configuration $o \in O$. In addition, point 1 of Proposition 2 extends immediately to the short sequence of attack (the proof does not make any assumptions about the sequences that the rulers choose). Hence in the short sequence model, like in the basic model, every ownership configuration is either peaceful or conflictful under $s$. Take any peaceful ownership configuration $o$. It must be that there is a unique active ruler at $o$ as otherwise, by what was shown above, if no other active ruler attacks his neighbor, the unique strong ruler attacks one of his neighbors. Hence there is fight till hegemony under $s$. By generic uniqueness of equilibrium payoffs, the probability of becoming a hegemon is generically unique.

**PROOFS III: LOSSES IN WAR**

**Proposition 4.** Fix a connected network $G$, resource endowment $r \in \mathbb{R}^{V+}$, and a rich rewarding contest success function, $p$, satisfying (6) with continuous and continuously differentiable function $f$. Then there exist threshold values of the losses in war $0 < \delta_{1} < \delta_{2} \leq 1/2$ such that

1. If $\delta < \delta_{1}$ then Theorem 3 holds.

2. If $\delta > \delta_{2}$ then the equilibrium outcome is perpetual peace.

3. If $f'(0) = 0$ then buffer states can help prevent a hegemony.

**Proof of Proposition 4** Given set of vertices $U \subseteq V$ and resource endowment $r$ let $r_{U} = \sum_{v \in U} r_{v}$. Also, given a resource endowment $r$ over a set of vertices $V$ let $Z(r) = \{(r_{U}, r_{U'}) : U, U' \in 2^{V} \setminus \{\emptyset\}, U \cap U' = \emptyset\}$.

**Point 1** Fix the set of vertices $V$ and resource endowment $r$. Assume first that $p$ is rich rewarding. Take any $x, y \in \mathbb{R}_{++}$. Since $p(x, y)(x + y) > x$ so $\delta_{1}^{x, y} = 1 - x/((x + y)p(x, y)) > 0$ and for any $\delta \in (0, \delta_{1}^{x, y}), p(x, y)(1 - \delta)(x + y) > x$. Next, assume that $p$ has the no-waiting
property. Take any \( x, y, z \in \mathbb{R}_+^+ \). By monotonicity and continuity of \( p \), \( p(x, y)p((1 - \delta)(x + y), z)((1 - \delta)(x + y) + z) \) is continuous and decreasing in \( \delta \). Moreover, by the no-waiting property, \( p(x, y)p(x, y + z)(x + y + z) > p(x, y + z)(x + y + z) \). Thus \( \delta_{x,y}^2 = \sup \{0 < \delta \leq 1 : p(x, y)p((1 - \delta)(x + y), z)((1 - \delta)(x + y) + z) > p(x, y + z)(x + y + z)\} \) is well defined and for all \( \delta \in (0, \delta_{x,y}^2) \), \( p(x, y)p((1 - \delta)(x + y), z)((1 - \delta)(x + y) + z) > p(x, y + z)(1 - \delta)(x + y + z) \).

Let \( \delta^1 \) be the point \( \min_{x,y \in \mathbb{Z}_+} \delta_{x,y}^1 \), \( \delta^2 \) be the point \( \min_{x,y \in \mathbb{Z}_+} \delta_{x,y}^2 \), and \( \delta \) be the minimum of \( \delta^1 \) and \( \delta^2 \). Since \( Z(r) \) is finite and non-empty so \( \delta \) is well defined and positive. For any \( \delta \in (0, \delta \) and amounts of resources from \( Z(r) \), expected payoff from attacking a poorer side is higher than current resource holding and expected payoff from attacking two opponents in a sequence is higher than payoff from letting them fight and attacking them afterwards. Hence the argument in proof of Theorem 3 works. This proves point 1.

Point 2: Let \( Z_i = (1 - \delta)^i x_0 + \sum_{j=1}^i (1 - \delta)^{i-j+1} x_j \). The expected payoff to a ruler with \( x_0 \) resources from a sequence of conflicts with \( m \) rulers with resources \( x_1, \ldots, x_m \) is given by

\[
\Pi_{\text{seq}}(x_0, x_1, \ldots, x_m) = Z_m \prod_{i=1}^{m} p(Z_{i-1}, x_i) = x_0 \prod_{i=1}^{m} p(Z_{i-1}, x_i) \frac{Z_i}{Z_{i-1}}.
\]

Recall that if \( \delta \geq 1/2 \) then, for any \( x, y \in \mathbb{R}_+^+ \), \( (x+y)(1-\delta)p(x, y) < x \). Hence \( p(Z_{i-1}, x_i)Z_i/Z_{i-1} = p(Z_{i-1}, x_i)(1-\delta)(Z_{i-1}+x_i)/Z_{i-1} < 1 \), for all \( i \in \{1, \ldots, m\} \) and, consequently, \( \Pi_{\text{seq}}(x_0, x_1, \ldots, x_m) < x_0 \). Hence no non-empty sequence of fights is profitable if \( \delta \geq 1/2 \) and so there is peace in equilibrium on any network and for any resource endowment \( r \in \mathbb{R}_+^Y \). This completes the proof of point 2.

Point 3: To prove the point, we will show first that with sufficiently large value of \( \delta \) any sequence of fights containing a fight with sufficiently small ruler yields lower expected payoff than some of its subsequences ending before the fight with the small ruler. This observation allows us to support the idea of buffer states in network settings.

Take any vector of resources \( (x_0, \ldots, x_m) \in \mathbb{R}_+^m \) and fix some \( k \in \{0, \ldots, m\} \). Assume that \( x_i > 0 \), for all \( i \neq k \). We will compare the expected payoffs from the sequence of fights involving the sequence of resources \( x_0, \ldots, x_m \) with sequences of fight involving the sequence of resources \( x_0, \ldots, x_l \), for \( l \in \{0, \ldots, k-1\} \).

The inequality \( \Pi_{\text{seq}}(x_0, \ldots, x_m) < \Pi_{\text{seq}}(x_0, \ldots, x_l) \) is equivalent to

\[
Z_m \prod_{i=l+1}^{m} p(Z_{i-1}, x_i) < Z_l.
\]
We will show first, that the inequality (30) is satisfied if \( \delta \) is sufficiently close to 1/2. Notice that \( LHS = \Pi_{seq}(Z_{l-1}, x_1, \ldots, x_m) \) and, by what we observed earlier, this is less than \( Z_{l-1} \), if \( \delta \geq 1/2 \). Since the expected payoff is continuous in \( \delta \) so, for any \( l \in \{0, \ldots, k-1\} \) and \( x_k \in \mathbb{R}_+ \), there exists \( \delta(l, x_k) \in [0, 1/2] \) such that (30) is satisfied for all \( \delta \in [0, \delta(l, x_k)) \). Let \( \delta^*(x_k) = \min_{l \in \{0, \ldots, k-1\}} \delta(l, x_k) \). For any \( \delta \in (\delta^*(x_k), 1/2) \) there exists \( l \in \{0, \ldots, k-1\} \) such that the sequence of fights with resources \( x_0, \ldots, x_l \) yields higher expected payoff than the sequence of fights \( x_0, \ldots, x_m \).

Second, we show that the LHS is strictly increasing in \( x_k \) on the interval \((0, x)\) with \( x \) sufficiently close to 0. Notice that

\[
\frac{\partial LHS}{\partial x_k} = (1 - \delta)^{-l} \left( \prod_{i=l+1}^m p(Z_{i-1}, x_i) \right) \left( (1 - \delta)^{m-k+1} \right.
- Z_m \left( \frac{f'(x_k)}{f(x_k)} p(x_k, Z_{k-1}) \right.
- \left. \sum_{i=k+1}^m \frac{f'(Z_{i-1})}{f(Z_{i-1})} p(x_i, Z_{i-1})(1 - \delta)^{i-k} \right).
\]

Since \( f'(0) = 0 \) so

\[
\left. \frac{f'(x_k)}{f(x_k)} p(x_k, Z_{k-1}) \right|_{x_k=0} = \left. \frac{f'(x_k)}{f(x_k) + f(Z_{k-1})} \right|_{x_k=0} = 0, \text{ while }
\]

\[
\left. \sum_{i=k+1}^m \frac{f'(Z_{i-1})}{f(Z_{i-1})} p(x_i, Z_{i-1})(1 - \delta)^{i-k} \right|_{x_k=0} > 0.
\]

Since \( f' \) is continuous around 0 so, for any \( \delta \in [0, 1) \), there exists \( \bar{x}(\delta) > 0 \) such that \( \frac{\partial LHS}{\partial x_k} > 0 \) on \((0, \bar{x}(\delta)) \) and, consequently, \( LHS \) is increasing in \( x_k \) on \((0, \bar{x}(\delta)) \). Let \( x^* = \inf_{\delta \in [0, 1/2]} \bar{x}(\delta) \) for any \( \delta \in [0, 1/2) \), \( LHS \) is increasing in \( x_k \) on \((0, x^*) \).

The two observations above allow us to conclude the following: any sequence of fights, \( x_1, \ldots, x_m \) with \( k \in \{0, \ldots, m\} \) yields lower expected payoff than one of its sequences, that ends after \( l \leq k - 1 \) fights if \( x_k \in (0, x^*) \) and \( \delta \in (\delta^*(x^*), 1/2) \). Moreover, if \( \delta^* > 0 \), then the sequence of length \( m \) yields weakly higher payoff than any of its subsequences of length \( l \leq k - 1 \) and yields exactly the same payoff as at least one of these subsequences. In this case, increasing \( x_k \) within the interval \((0, x^*) \) will make the sequence of length \( m \) yield strictly higher payoff than any of the subsequences of length \( l \leq k - 1 \).

Now suppose that \( G \) has a node, \( v \) such that \( G - \{v\} \) is disconnected. By the conclusion above, there exists a range \((\delta^{**}, 1/2)\) of losses in war and a range \((0, x^{**})\) of amounts of
resources such that in equilibrium no sequence of fights involves a fight with the ruler owning \( v \) if resources at \( v, r_v \in (0, x^*) \). It is important to note that the requirement on the resources at \( v \) is necessary. With \( \delta = 0 \), for any value of \( r_v \in \mathbb{R}_{++} \), there exists at least one strong ruler whose full attacking sequence is better than any other sequence. By the analysis above, there exists \( \delta \) and \( r \) such that this sequence (or a subsequence of it) involves \( v \), if \( r_v > r \) and does not involve \( v \) if \( r_v < r \).

This implies that buffer states prevent spread of conflict, if their resources are sufficiently low and losses in war are sufficiently high.

\[ \square \]

![Figure 15: Network where increasing cost of conflict leads to war.](image)

Example 2 (Greater losses lead to more war). Consider a star network over 4 vertices, as presented in Figure 15. Assume \( \gamma = 0.5 \). Every spoke is endowed with \( y \) resources and the centre is endowed with \( x \) resources. Let \( y = 1.0 \) and \( x \in (2.1, 2.9) \).

Suppose that cost of conflict \( \delta = 0 \). The expected payoff to a spoke ruler with 1.0 resources from executing an attacking sequence of length \( m \leq 2 \) when the centre ruler has \( z \in (3.1, 3.9) \) resources is \( \varphi(z, 1.0, m) \geq \varphi(z, 1.0, 2) \in (1.23, 1.37) \) (recall function \( \varphi \) as defined in proof of Proposition 2). Hence after any spoke ruler attacks the centre at the initial ownership configuration, there will be fight till hegemony in any equilibrium. Payoff to the spoke ruler from executing an attacking sequence of length 3 at the initial state is \( \varphi(x, 1.0, 3) \in (0.889, 0.999) \). Thus it is not profitable for a spoke ruler to attack the centre at the initial state. Since \( \gamma < 1 \) and \( x < y \) so it is not profitable for the centre ruler to attack as well. Hence there is an equilibrium with peace at the initial state.

Suppose now that cost of conflict \( \delta = 0.2 \). The expected payoff to a spoke ruler with \( y \) resources from executing an attacking sequence of length \( m \) when the centre ruler has \( x \)
resources is
\[
\psi(x, y, m \mid \delta) = \left( x(1-\delta)^m + \sum_{j=1}^{m} (1-\delta)^j y \right) p(y, x) \prod_{i=1}^{m-1} p \left( (1-\delta)^i x + \sum_{j=1}^{i} (1-\delta)^j y, y \right). \tag{31}
\]

Consider the ownership configuration resulting from two attacks by spoke on a centre: there are two active rulers, one with 1.0 resources and another one with \( z = 0.8(0.8(1.0 + x) + 1.0) \in (2.784, 3.296) \) resources. Expected payoff to the poorer ruler from attacking the richer ruler is \( \psi(z, 1.0, 1 \mid 2.0) \in (1.13, 1.23) \). Hence the poorer ruler finds it profitable to attack the richer one. Consider now the ownership configuration resulting from one attack by a spoke ruler on the centre. There are two spokes, each endowed with 1.0 resources and the centre endowed with \( z = 0.8(1.0 + x) \in (2.48, 3.12) \) resources. Any attacker anticipates two fights after an attack. Expected payoff to a spoke from two fights is \( \psi(z, 1.0, 2 \mid 2.0) \in (0.73, 0.81) \). Thus a spoke ruler does not want to attack and (with \( \gamma < 1 \) and \( z > y \)) the centre ruler does not want to attack as well. Lastly, consider the initial ownership configuration. Payoff to a spoke from attacking the centre is \( \psi(x, 1.0, 1 \mid 2.0) \in (1.01, 1.16) \). This leads to an ownership configuration with peace. Hence every spoke finds it profitable to attack the centre and so there is no peace at the initial ownership configuration. The increase in the cost of conflict prevents conflict escalation at states with less than four rulers. This in turn raises incentives for rulers to attack in the initial state. \( \Delta \)

ADDITIONAL RESULTS I: OPTIMAL ORDER OF ATTACK

The contest success function has the poor-first property if the expected payoffs of attacking the poor ruler followed by the rich ruler are larger, i.e., for all \( x, y, z \in \mathbb{R}_{++} \), with \( y < z \), \( p(x, y)p(x+y, z) > p(x, z)p(x+z, y) \). Similarly, a technology has the rich-first property if the converse it true. Define:
\[
h(s, t) = \frac{f(t)f(s+t)}{f(s+t)-f(s)-f(t)}.
\]

We can now state:
Proposition 5. Let \( p \) be a contest success function that satisfies (6). The function \( p \) has poor-first (rich-first) property if and only if \( h(s,t) \) is strictly increasing (decreasing) in \( t \in \mathbb{R}_{++} \) for all \( s \in \mathbb{R}_{++} \).

Proof of Proposition 5. We provide the proof for the poor-first case. The arguments for the rich-first case are similar are omitted. Let \( x, y, z \in \mathbb{R}_{++} \) with \( y > z \) and suppose that \( p(x, y)p(x + y, z) > p(x, z)p(x + z, y) \). This may be rewritten as:

\[
\frac{f(x)}{f(x) + f(y)} \frac{f(x + y)}{f(x + y) + f(z)} > \frac{f(x)}{f(x) + f(z)} \frac{f(x + z)}{f(x + z) + f(y)}
\]

Dividing both sides by \( f(x) \) and multiplying them by the denominators, we get

\[
f(x + y)(f(x) + f(z))(f(x + z) + f(y)) > f(x + z)(f(x) + f(y))(f(x + y) + f(z))
\]

This is equivalent to

\[
f(y)f(x + y)(f(x) + f(z)) + f(x)f(x + z)f(x + y) + f(z)f(x + z)f(x + y) > f(x)f(x + z)(f(x) + f(y)) + f(x)f(x + y)f(x + z) + f(y)f(x + y)f(x + z)
\]

Subtracting \( f(x)f(x + z)f(x + y) + f(z)f(x + z)f(x + y) + f(y)f(x + y)f(x + z) \) from both sides this is equivalent to

\[
f(y)f(x + y)(f(x) + f(z)) - f(y)f(x + y)f(x + z) > f(z)f(x + z)(f(x) + f(y)) - f(z)f(x + z)f(x + y)
\]

Reorganizing and multiplying both sides by \(-1\), this is equivalent to

\[
f(z)f(x + z)(f(x + y) - (f(x) + f(y))) > f(y)f(x + y)(f(x + z) - (f(x) + f(z))).
\]

Dividing both sides by \((f(x + y) - (f(x) + f(y)))(f(x + z) - (f(x) + f(z)))\), this is equivalent to

\[
\frac{f(z)f(x + z)}{f(x + z) - f(x) - f(z)} > \frac{f(y)f(x + y)}{f(x + y) - f(x) - f(y)}.
\]

This completes the proof.

In the case of the Tullock Contest Function, \( h(s,t) \) is increasing (decreasing) in \( t \) for all \( s \).
if \( \gamma > 1 \) (\( \gamma < 1 \)); \( h(s, t) \) remains constant in \( t \) for all \( s \), if \( \gamma = 1 \).

This result can be generalized to cover optimal order of attack with \( n \) opponents: if all opponents are accessible then the order of attack is monotonically strictly increasing (decreasing) if \( h(x, y) \) is increasing (decreasing) in \( y \) for all \( x \). The qualification ‘if all opponents are neighbours’ is important. If some opponents are not neighbours then it may be optimal to attack a richer neighbor in preference to a poor neighbour, so as to reach other poorer opponents first. Here is an example. Suppose \( G \) is a line network with 4 rulers, \( a, b, c, \) and \( d \), each controlling one vertex (in that order). Suppose that resources of ruler \( a \) are \( x \in (0, 2) \). The resources of \( b, c \) and \( d \) are respectively 2, 2 and 1. If \( x < 1.83 \) then the optimal full attacking sequence of ruler \( b \) is \((a, c, d)\): so it prescribes attacking the weakest neighbour first. On the other hand, if \( x > 1.84 \) then the optimal full fighting sequence is \((c, d, a)\): it is better to attack a stronger neighbour, \( c \), first to get access to weak \( d \), and then attack \( a \).

ADDITIONAL RESULTS II: HIRSHLEIFER’S CONTEST SUCCESS FUNCTION

Another widely used contest success function, along the Tullock contest success function, is the so called difference form proposed by Hirshleifer [1989]:

\[
p(x, y) = \frac{\exp(\gamma x)}{\exp(\gamma x) + \exp(\gamma y)},
\]

where \( \gamma > 0 \). Thus \( f(x) = \exp(\gamma x) \) and it is easy to check that \( f(x)/x \) is increasing on interval \((0, 1/\gamma)\) and decreasing on \((1/\gamma, +\infty)\). Thus the function maintains the poor rewarding and, consequently, the waiting properties on the interval \((0, 1/\gamma)\) and maintains the rich rewarding and, consequently, the no-waiting properties on the interval \((1/\gamma, +\infty)\). Hence if the minimal resources in the network at the initial ownership configuration are greater than \( 1/\gamma \), all the results obtained for the rich rewarding case would hold for this contest success function as well and if the total resources in the network are less than \( 1/\gamma \), the results for the poor rewarding case apply.

For the order of fights properties of Hirshleifer’s contest success function, notice that

\[
h(s, t) = \frac{\exp(\gamma t) \exp(\gamma(s + t))}{\exp(\gamma(s + t) - \exp(\gamma s) - \exp(\gamma t))} = \frac{1}{\exp(-\gamma t) - \exp(-2\gamma t) - \exp(-\gamma(s + t))}.
\]

Taking the derivative with respect to \( t \) and comparing it to 0 we can see that \( h(s, t) \) is decreasing in \( t \) when \( \exp(-\gamma t) < 1/2 - \exp(-\gamma s)/2 \) and is increasing in \( t \) when the inequality
is reversed. The LHS of the inequality is decreasing in \( t \) while the RHS is increasing in \( s \). Moreover, the functions \( \exp(-\gamma x) \) and \( 1/2 - \exp(-\gamma x)/2 \) intersect at \( x = \ln(3)/\gamma > 1/\gamma \). Thus on the interval \((0, 1/\gamma)\) the contest success function maintains the poor rewarding and the rich first properties and on interval \((\ln(3)/\gamma, +\infty)\) it maintains the rich rewarding and the poor first property.

**FURTHER EXTENSION I: COSTS OF FIGHTING**

In the basic model, if there is a war then it is assumed that both rulers always allocate all their resources to fighting. In the literature on war and contests a key concern has been the trade-off between guns and butter. The losses in war is one way of modeling the costs of war. But it is clear that, as we move toward a more general theory of war, we will need to introduce the opportunity costs of resources allocated to war more directly. This section proposes a simple way of introducing costs of war and shows how it can be used to develop a theory of incentives to fight that is analogous to that of rich/poor rewarding contest success functions.

We will present a simple example with two rulers \( A \) and \( B \). Rulers have resources \( r_A \) and \( r_B \). The prize of the war is \( r_A + r_B \). The rulers allocate resources to combat, \( z_i \leq r_i \), for \( i \in \{A, B\} \). Suppose the cost of resources is \( c_i(x) \), with \( c_i(0) = 0 \), \( c_i'(\cdot) \geq 0 \) and \( c_i'' \geq 0 \).

The probability of win for ruler \( i \) is given by the Tullock contest function, where \( \gamma = 1 \):

\[
\frac{z_i}{z_i + z_j}
\]

The expected payoff to rulers from the war are:

\[
(r_A + r_B) \frac{z_A}{z_A + z_B} - c_A(z_A) \ & \ (r_A + r_B) \frac{z_B}{z_B + z_A} - c_B(z_B).
\]

Suppose for simplicity that the richer country has a lower marginal cost of war. Let \( r_A > r_B \); so \( c'_A(x) < c'_B(x) \) for all \( x > 0 \).

Let us focus attention on interior equilibrium, \( z_i \in (0, R_i) \). In an interior equilibrium resource commitments must satisfy:

\[
\frac{z_B}{(z_A + z_B)^2} = c'_A(z_A) \ & \ \frac{z_A}{(z_A + z_B)^2} = c'_B(z_B).
\]
It follows that
\[ \frac{z_B}{c_A'(z_A)} = \frac{z_A}{c_B'(z_B)}. \]

Ruler A allocates more effort to war and has a higher probability of winning: this is intuitive: the prize is the same, but the costs of war are lower for Ruler A. Let us now compare the ex-post distribution of resources. For computational simplicity, consider the following functional form for the cost function:
\[ c_i(x) = \frac{x^2}{r_i^\gamma}. \]

Then the interior equilibrium resource allocation is:
\[ \frac{z_A}{z_B} = \left[ \frac{r_A}{r_B} \right]^{\frac{1}{\gamma}}. \]

The richer ruler gains resources from a war if
\[ (r_A + r_B) \frac{z_A}{z_A + z_B} > r_A. \]

After substituting equilibrium allocations and rearranging terms, this is equivalent to
\[ \frac{r_B}{r_A} > \left[ \frac{r_B}{r_A} \right]^{\frac{1}{\gamma}}. \]

So the richer ruler gains in resources if and only if \( \gamma > 2 \).

Finally, in the spirit of Hirshleifer [1995a], consider a multi-period version of this game with myopic rulers. If the conditions for interior equilibrium remain satisfied over time, then the larger ruler will gradually expand his domain and will conquer the poorer kingdom, if and only if \( \gamma > 2 \). If \( \gamma \in (0, 2) \) then the poorer ruler will catch up with the richer ruler and the two kingdoms will remain in eternal conflict. So we may think of \( \gamma = 2 \) as the threshold corresponding to the rich/poor rewarding distinction in the basic model.

FURTHER EXTENSION II: RESURRECTION

In the basic model, a losing ruler is eliminated. The empirical record suggests that losing rulers may withdraw and sometime come back to fight the winner. The gradual expansion model, in the main text of the paper, provides one way to model this. We now consider an alternative in which the loser is eliminated and his entire kingdom taken over, but he may be
resurrected at some point in the future with only his initial node as the endowment.

The model now works as follows: the active rulers are picked to make their choices at odd number rounds \( t = 1, 3, \ldots \) and these rounds proceed like in the original model. At even numbered rounds \( t = 2, 4, \ldots \), each removed ruler is resurrected with probability \( \rho \in (0, 1) \). The resurrected ruler gets the vertex that was owned by him at the initial round. This vertex is removed from the set of vertices owned by his previous owner.

The game now has an infinite horizon so we adapt the payoffs as follows. Given a state \( o \), a strategy profile, \( s \), and the probability of resurrection, \( \rho \), let \( H(o \mid s, \rho) \) be the stationary probability of being at state \( o \) of the Markov chain determined by \( s \) and \( \rho \). We write the expected payoff to ruler \( i \) from strategy profile \( s \) is:

\[
\Pi_i(s) = \sum_{o \in O} H(o \mid s, \rho) R_i(o).
\]

We consider a simple example with two rulers and two nodes, with resources \( r_1 \) and \( r_2 \), respectively. Suppose that \( r_1 > r_2 \). Let the resurrection probability be \( \rho \in (0, 1] \). The expected payoff to the stronger ruler from choosing attack is:

\[
\frac{\rho}{1 + \rho} r_1 + \left( \frac{p(r_1, r_2)}{1 + \rho} \right) (r_1 + r_2)
\]

which is greater than \( r_1 \) if \( p \) is strong rewarding. Hence the stronger ruler should always choose attack (regardless of the value of \( \rho \)). This suggests the following conjecture: there will always be rulers show will choose strong attacking sequence at every state, regardless of the choices of other players.

FURTHER EXTENSION III: ASYMMETRIC RULERS

We next take up the issue of symmetry. In the basic model, we assume that all rulers have the same technology and this is reflected in a common function \( f \). This symmetry is plausible at some points in time and in some parts of the world, as in medieval and modern Europe (see Clausewitz [1993] and Howard [2009]). But, in other contexts, it is less so: for instance, it has been argued that the superiority of European military technology was a key factor in explaining the success of European nations in establishing empires in the Americas and in

\[\text{[20]}\] We have a three-state Markov chain here: the initial state, the one were player 1 is the hegemon and the one were player 2 is the hegemon. From this it is elementary to get this expected payoff.
Asia (Hoffman 2015). This motivates a generalization of contest success function, beyond the symmetric case.

Following Rai and Sarin 2009 let us retain axioms A1 and A2, but drop the symmetry axiom, A3. This yields

\[ q_i(x, y) = \frac{f_i(x)}{f_1(x) + f_2(y)}, \] (33)

where \( i \in \{1, 2\} \), with increasing and positive functions \( f_1, f_2 : \mathbb{R}_{++} \to \mathbb{R}_{++} \).

The idea that a ruler has superior technology can be represented in terms of domination. The function \( f_1 : \mathbb{R}_{++} \to \mathbb{R}_{++} \) everywhere dominates function \( f_2 : \mathbb{R}_{++} \to \mathbb{R}_{++} \) on \( \mathbb{R}_{++} \) if and only if, for all \( x \in \mathbb{R}_{++} \), \( f_1(x) \geq f_2(x) \). Clearly, if \( f_1 \) everywhere dominates \( f_2 \) on \( \mathbb{R}_{++} \) then for all \( x \in \mathbb{R}_{++} \), \( q_1(x, x) \geq q_2(x, x) \). Ruler 1 then has a more powerful technology as compared to ruler \( j \).

Our analysis of the basic model extends in a straightforward manner if the richer rulers also have the superior technology. If however the poorer ruler has the more powerful technology then a consideration of the incentives to fight would involve a comparison of both resources and technology in the optimal sequence of attack. We leave a general analysis of this problem to future work.

**THEORY OF INTERNATIONAL RELATIONS**

This section discusses the relation between our theoretical model and the theory of international relations. A central tension in the theory of international relations concerns the contrasting prescriptions of ‘offensive’ and ‘defensive’ realism, see e.g., Betts 2013, Mearsheimer 2001 and Waltz 1979. Roughly speaking, ‘offensive’ realism advocates a strategy of persistent combativeness and aggression, while ‘defensive’ realism favors a strategy of restraint. Our paper reconciles these theories, and locates their rationale in observable parameters such as the contiguity network, resources and technology.

A comparison of the dynamics under rich rewarding and poor rewarding reveals contrasting optimal strategies (full attacking sequence versus no war) and outcomes (hegemony versus multiple kingdoms). It highlights the role of the contest success function and offers one possible resolution to a key tension in the modern literature on international relations: whether nations should be offensive or defensive. The optimality of full attacking sequence – both for rich and poor rulers – echoes the central arguments for ‘offensive’ realism:

\[ \text{Clark and Riis 1998} \] study asymmetric contest success function satisfying, in addition, the axiom of homogeneity, and this leads to a generalized Tullock Contest Success Function.
“Given the difficulty of determining how much power is enough for today and tomorrow, great powers recognize that the best way to ensure their security is to achieve hegemony now, thus eliminating any possibility of a challenge by another great power. Only a misguided state would pass up an opportunity to be the hegemon in the system because it thought it already had sufficient power to survive.”

Mearsheimer [2001]

By contrast, our analysis of the dynamics of conflict under a poor rewarding contest success function offers a potential foundation for the thesis of ‘defensive realism’. This thesis argues that

“...the first concern of states is not to maximize power but to maintain their position in the system.” Waltz [1979]

Indeed, the arguments in the peace result rest on the desire of rulers to retain their current status: such an engagement may be myopically attractive but is ultimately a losing proposition as it encourages other rulers to engage in war. The considerations underlying the peace outcome in our theoretical model are in line with the traditional arguments for ‘defensive realism’.