We introduce the commands twexp and twgravity that implement the estimators developed in Jochmans (2017) for exponential regression models with two-way fixed effects. twexp is applicable to generic $n \times m$ panel data. twgravity is written for the special case where the data is a cross-section on dyadic interactions between $n$ agents. A prime example of the latter is cross-sectional bilateral trade data, where the model of interest is a gravity equation with importer and exporter effects. Both twexp and twgravity can deal with data where $n$ and $m$ are large, that is, the case of many fixed effects. They make use of Mata and are very fast to execute.
twexp and twgravity: Estimating exponential regression models with two-way fixed effects

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Abstract. We introduce the commands twexp and twgravity that implement the estimators developed in Jochmans (2017) for exponential regression models with two-way fixed effects. twexp is applicable to generic $n \times m$ panel data. twgravity is written for the special case where the data is a cross-section on dyadic interactions between $n$ agents. A prime example of the latter is cross-sectional bilateral trade data, where the model of interest is a gravity equation with importer and exporter effects. Both twexp and twgravity can deal with data where $n$ and $m$ are large, that is, the case of many fixed effects. They make use of Mata and are very fast to execute.

Keywords: exponential regression, gravity model, panel data, two-way fixed effects.

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1 Introduction

The exponential-regression model finds wide application in the analysis of non-negative outcomes such as count data. It has also shown itself to be an attractive alternative to the log-linearized regression model. Indeed, following Santos Silva and Tenreyro (2006), constant-elasticity models are now routinely estimated from data in levels rather than logarithms. This paper presents two Stata routines to estimate exponential regressions with two-way fixed effects.

We consider double-indexed data on a non-negative outcome, $y_{ij}$, and a $p$-vector of regressors, $x_{ij}$. The routine twexp is designed to estimate the slope vector $\gamma$ in the $n \times m$ panel model

$$y_{ij} = e^{(\alpha_i + \beta_j + x_{ij}^T \gamma)} \varepsilon_{ij}, \quad \mathbb{E}(\varepsilon_{ij} | x_{11}, \ldots, x_{nm}) = 1,$$

where $i = 1, \ldots, n$ and $j = 1, \ldots, m$ and we let $e(a) := \exp(a)$. Here, $\alpha_i$ and $\beta_j$ are fixed effects and $\varepsilon_{ij}$ is a latent disturbance. A slight variation to this is a cross-sectional data set in which we observe outcomes and regressors for the $n \times (n - 1)$ pairwise interactions between agent $i = 1, \ldots, n$ and $j \neq i$. This is different from the panel-data case as,
here, we do not observe \( y_{ii} \) and \( x_{ii} \). The routine \texttt{twgravity} is designed to handle this case. Its name is derived from the leading example of such an application being the estimation of a gravity equation from a cross-section of bilateral trade flows. Here, the outcome is the directed trade flow from \( i \) to \( j \), the regressors are measures of distance or (dis-)similarity between \( i \) and \( j \), and \( \alpha_i \) and \( \beta_j \) are exporter and importer effects, respectively.

The most popular estimator of (1) is the pseudo maximum-likelihood estimator (PMLE) that arises from treating the \( y_{ij} \) as conditionally-independent Poisson variates. If we introduce the shorthand

\[
u_{ij}(\alpha_i, \beta_j, \gamma) := y_{ij} - e^{(\alpha_i + \beta_j + x_{ij}^\top \gamma)},
\]

the PMLE solves the \( p \) first-order conditions for \( \gamma \),

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} u_{ij}(\alpha_i, \beta_j, \gamma) = 0,
\]

jointly with the \( n + m \) first-order conditions for the effects \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_m \),

\[
\sum_{j=1}^{m} u_{ij}(\alpha_i, \beta_j, \gamma) = 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} u_{ij}(\alpha_i, \beta_j, \gamma) = 0, \quad j = 1, \ldots, m,
\]

subject to a suitable normalization on the fixed effects, such as \( \sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{m} \beta_j \), for example. In spite of the presence of the growing number of nuisance parameters the estimator of \( \gamma \) is consistent and has a correctly-centered limit distribution when either \( n \) is large and \( m \) is small or when both \( n \) and \( m \) are large (and of a similar magnitude). Details on the theoretical properties are available in \text{Wooldridge (1999)} and \text{Fernández-Val and Weidner (2016)}.

The pseudo-Poisson approach suffers from two drawbacks. The first is a numerical one. Indeed, the large amount of fixed effects implies that a simple approach that combines, say, \texttt{poisson} with \( n + m \) dummy variables will be infeasible in many data sets. The routines \texttt{poi2hdfe} (Guimarães 2016) or \texttt{ppmlhdfe} (Correia et al. 2019) are designed especially to deal with this problem and are useful alternatives here. The second drawback is that the plug-in estimator of the covariance matrix of the above moment conditions is severely biased. The origin of the problem is again the estimation of the incidental parameters. Indeed, calculating the covariance matrix requires estimating terms involving

\[
u_{ij}(\alpha_i, \beta_j, \gamma)^2
\]
which requires estimates of the fixed effects. The latter are both numerous and estimated with low precision, creating an incidental-parameter bias in the estimated covariance matrix. The bias can be severe, as evidenced by the simulation results in Egger and Staub (2016), Jochmans (2017), and Pfaffermayer (2019). The practical implication of this is that the standard errors will often not be an accurate reflection of the statistical precision of the parameter estimates. Often they will be too small. Consequently, reported confidence interval will be too narrow and test procedures will overreject under the null.

Equation (1) is an important member of the class of multiplicative-error models. For such models moment conditions have been derived that are free of fixed effects (Charbonneau 2013, Jochmans 2017). They allow inference on \( \gamma \) to be separated from estimation of \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_m \). twexp and twgravity implement estimators based on these moments. Both routines are designed to be computationally efficient and are very fast to implement. Hence, our routines should be a useful addition to the toolbox of empirical researchers working with count data and trade data. Furthermore, as the whole problem is free of nuisance parameters the standard errors do not suffer from an incidental-parameter bias.

2 Moment conditions and estimators

Consider (1) under the assumption that the errors are mutually independent. Then, using that

\[
E \left( \frac{y_{ij}}{e(x_{ij}^\gamma)} \bigg| \mathbf{x}_{11}, \ldots, \mathbf{x}_{nm} \right) = e(\alpha_i + \beta_j)
\]

for all \((i, j)\), we have

\[
E \left( \frac{y_{ij}}{e(x_{ij}^\gamma)} \frac{y'_{i'j'}}{c(x_{i'j'}^\gamma)} - \frac{y_{ij'}}{e(x_{ij'}^\gamma)} \frac{y_{i'j}}{c(x_{i'j}^\gamma)} \bigg| \mathbf{x}_{11}, \ldots, \mathbf{x}_{nm} \right) = 0
\]

(2)

for all \(i, i'\) and \(j, j'\). This (conditional) moment condition for \( \gamma \) is free of incidental parameters. Equation (2) implies unconditional moment conditions that can form the basis of a method-of-moment (MM) estimator of \( \gamma \). Our Stata routines implement two of these estimators.

The first estimator, which we dub GMM1 below, uses the levels of the covariates,
where we define the variance of the moments is estimated by

\[ V \]  

The use of the summands that make up \( s \) the estimator is

\[ \hat{\gamma}_1 := \arg \min_{\gamma} s_1(\gamma)^T s_1(\gamma). \]

Under suitable regularity conditions \( \hat{\gamma}_1 \) is consistent and asymptotically normal. Its asymptotic variance has a sandwich form and can be estimated as \( Q_1^{-1} V_1 Q_1^{-T} \), where

\[ Q_1 := -\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} \sum_{j'=1}^{m} x_{ij} \left\{ y_{ij} y_{i'j'} (x_{ij} + x_{i'j'})^T \right\} e(x_{ij}, \hat{\gamma}_1) e(x_{i'j'}, \hat{\gamma}_1) e(x_{ij}, \hat{\gamma}_1) e(x_{i'j'}, \hat{\gamma}_1) \],

is the Jacobian of the empirical moments evaluated at the point estimator and the variance of the moments is estimated by

\[ V_1 := \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} v_{ij}^T, \]

where we define the \( p \)-vector \( v_{ij} \) as

\[ 4 \sum_{i' \neq i \neq j} \sum (x_{ij} - x_{i'j'}) (x_{i'j} - x_{i'j'}) \left\{ \frac{y_{ij}}{e(x_{ij}, \hat{\gamma}_1)} \frac{y_{i'j'}}{e(x_{i'j'}, \hat{\gamma}_1)} - \frac{y_{ij'}}{e(x_{ij}, \hat{\gamma}_1)} \frac{y_{i'j'}}{e(x_{i'j'}, \hat{\gamma}_1)} \right\}. \]

The use of \( V_1 \) is needed to handle the fact that each observation appears in many of the summands that make up \( s_1(\gamma) \).

The second estimator we implement, GMM2, is

\[ \hat{\gamma}_2 := \arg \min_{\gamma} s_2(\gamma)^T s_2(\gamma), \]

which is of the same form as \( \hat{\gamma}_1 \) but solves the empirical moment equations

\[ s_2(\gamma) := \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j'=1}^{m} \sum_{j'=1}^{m} x_{ij} \left\{ \frac{y_{ij}}{e(-x_{ij}, \gamma)} \frac{y_{i'j'}}{e(-x_{i'j'}, \gamma)} - \frac{y_{ij'}}{e(-x_{ij}, \gamma)} \frac{y_{i'j'}}{e(-x_{i'j'}, \gamma)} \right\} = 0. \]

1. As written here the moment equations of GMM1 can be set arbitrarily close to zero when the regressors are all non-negative by setting one of the elements of \( \gamma \) arbitrarily large. This can be resolved by transforming all regressors into deviations from their overall mean. Doing so does not alter the roots of the original estimating equation. Both of our Stata routines perform this normalization by default.
The large-sample behavior of this estimator parallels that of \( \hat{\gamma}_1 \). The matrices \( Q_2 \) and \( V_2 \) needed to estimate the variance of the limit distribution are readily obtained. We omit further details here for brevity. There is an array of other possible estimators that can be derived from the conditional moment conditions above. Motivations for the estimators considered here are given in the supplementary material to Jochmans (2017).

The choice between the two estimators depends on the application at hand. The simulation results in Jochmans (2017) show that GMM2 tends to be more efficient than GMM1 in designs where the conditional variance increases with the conditional mean while GMM1 is relatively more precise in the other situations. In extensive numerical work we have found that GMM1 is extremely stable, making it very reliable. When the linear index \( x_{ij}^\top \gamma \) can take on very large values the objective function of GMM2 can have multiple local maxima and regions over which it is fairly flat. This can be understood by noting that \( s_2(\gamma) \) can be obtained from \( s_1(\gamma) \) by multiplying through the latter’s summand with \( c((x_{ij} + x_{i'j'} + x_{i'j} + x_{ij'})^\top \gamma) \). This complicates numerical optimization using gradient-based methods such as the Newton algorithm that we use. Our code checks whether a global optimum has been reached by verifying whether the empirical moments are (up to tolerance) equal to zero at the solution and gives a warning if not. If this happens we suggest to experiment with different starting values or to switch to GMM1 in stead.

The large number of terms in \( s_1(\gamma) \) and \( s_2(\gamma) \) may suggest that evaluation of the objective function is time consuming, making estimation and inference based on them infeasible in large data sets (see, for example, the discussion in Egger and Staub 2016). This is not the case. Careful inspection and subsequent re-arrangement of terms reveals that evaluation of these equations is immediate in any matrix-based language (here, Mata). Additional details on this are provided in the appendix. The same is true for the Jacobian matrices \( Q_1 \) and \( Q_2 \) as well as for the variance estimators \( V_1 \) and \( V_2 \). `twexp` and `twgravity` are written for balanced data sets. The implementation of our efficient computations would require adjustment to deal with gaps in the data. The exact form of the adjustment depends on the pattern of missingness of the data and is, therefore, not easily programmed in a generic manner. We note that merely dropping observations for which information is missing is not sufficient. This is because of the structure of the empirical moments, where each summand depends on quadruples of observations. One may, of course, decide to resort to brute-force evaluation of the criterion in such cases.
3 Stata commands

3.1 Command: twexp

The command `twexp` is designed for (balanced) $n \times m$ panel data sets.

Syntax

`twexp depvar [indepvars], indn(varname) indm(varname) model(option) init(vec)`

Here,

- `indn(varname)` declares the cross-sectional dimension of the panel.
- `indm(varname)` declares the time-series dimension of the panel.
- `model(option)` determines whether GMM1 or GMM2 is implemented.
- `init(vec)` specifies the starting value for the numerical optimization.

A table in standard layout reports point estimates, standard errors, $z$-statistics and $p$-values for the null that the coefficient in question is equal to zero, and 95% confidence intervals for each of the coefficients. The vector of point estimates and their estimated covariance matrix can be recovered by typing `matrix list e(b)` and `matrix list e(V)`, respectively.

3.2 Command: twgravity

The command `twgravity` is designed for a cross-section on dyadic interactions between $n$ agents. Agents do not interact with themselves, so $y_{ii}$ and $x_{ii}$ are not defined. This is like a panel model with $m = n - 1$. In the vectors and matrices defined in Section 2 this only requires modifying the range over which the sums go. To evaluate the criterion function efficiently, however, additional intervention is needed (see the discussion on gaps in the previous section). Therefore, a different Stata command is provided to deal with this case.

Syntax

`twgravity` has the same syntax as `twexp`:
\textit{Jochmans and Verardi}

\texttt{twgravity \textit{depvar} [ \textit{indepvars} ]}, \texttt{\textit{indn}(\textit{varname}) \textit{indm}(\textit{varname}) \textit{model}(\textit{option})}

\texttt{init(vec)}

Here, again, \texttt{\textit{indn}(\textit{varname})} identifies the first agent in the dyad.

\texttt{\textit{indm}(\textit{varname})} identifies the second agent in the dyad.

\texttt{\textit{model}(\textit{option})} determines whether GMM1 or GMM2 is implemented.

\texttt{\textit{init(vec)}} specifies the starting value for the numerical optimization.

The screen output has the same form as before.

\section{Example}

We use the model and data of \textit{Santos Silva and Tenreyro (2006)} to illustrate the use of \texttt{twgravity}. The data set can be downloaded from \url{http://personal.lse.ac.uk/tenreyro/lgw.html}.

The data is a cross-section on bilateral trade flows between 136 countries. The outcome variable is bilateral trade, measured in 1,000 U. S. dollars (\texttt{trade}). The regressors are all measures of distances between the importing and exporting country. They are (the logarithm of) geographical distance (\texttt{ldist}) and a set of dummies that aim to capture other factors of relatedness. These dummies indicate whether or not countries $i$ and $j$ share a common border (\texttt{border}), speak the same language (\texttt{comlang}), have a colonial history (\texttt{colony}), and take part in a common free-trade agreement (\texttt{comfrt_wto}). For each observation the variables \texttt{s1_im} and \texttt{s2_ex} identify the importer and exporter, respectively.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Obs</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>trade</td>
<td>18,360</td>
<td>172129.5</td>
<td>1829058</td>
<td>0</td>
<td>1.01e+08</td>
</tr>
<tr>
<td>ldist</td>
<td>18,360</td>
<td>8.785508</td>
<td>.7415775</td>
<td>4.876723</td>
<td>9.898691</td>
</tr>
<tr>
<td>border</td>
<td>18,360</td>
<td>.0196078</td>
<td>.1386522</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>comlang</td>
<td>18,360</td>
<td>.209695</td>
<td>.407102</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>colony</td>
<td>18,360</td>
<td>.1704793</td>
<td>.3760636</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>comfrt_wto</td>
<td>18,360</td>
<td>.0250546</td>
<td>.1562948</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Estimating this model by GMM1 is done by typing
and completes in .81 seconds (using Stata/MP 15.1 on a MacBook 1.4HGz Intel Core i7 with 16GB RAM). The following output is reported.

| trade     | Coef.  | Std. Err. | z     | P>|z|  | [95% Conf. Interval] |
|-----------|--------|-----------|-------|------|----------------------|
| ldist     | -.8165761 | .188396 | -4.33 | 0.000 | -1.185825 - .4473268 |
| border    | .4873677  | .2339165 | 2.08  | 0.037 | .0288999 .9458356    |
| comlang   | .2594789  | .2119004 | 1.22  | 0.221 | -.0288999 .5480433   |
| colony    | .1648687  | .1955009 | 0.84  | 0.399 | - .2183059 .5480433  |
| comfrt_wto| .3064196  | .217326  | 1.41  | 0.159 | -.1195316 .7323707   |

Changing the estimator used to GMM2 is done by typing

twgravity trade ldist border comlang colony comfrt_wto, indn(s2_ex)  
    indm(s1_im) model(GMM2)

which terminates after 1.85 seconds with the following output.

| trade     | Coef.  | Std. Err. | z     | P>|z|  | [95% Conf. Interval] |
|-----------|--------|-----------|-------|------|----------------------|
| ldist     | -.7509313 | .0567805 | -13.23| 0.000 | -.8622191 -.6396436 |
| border    | .1490604  | .0771748 | 1.93  | 0.053 | -.0021994 .3003202   |
| comlang   | .4909294  | .0929732 | 5.28  | 0.000 | .3087052 .6731536    |
| colony    | .2128996  | .1212684 | 1.76  | 0.079 | -.0247821 .4505813   |
| comfrt_wto| .3298556  | .1249293 | 2.64  | 0.008 | .0849987 .5747126    |

These results correspond to those reported in Table 5 of Jochmans (2017). To appreciate the computational speed, estimation by PMLE takes just under 16 seconds when using poisson with dummies, 3.87 second when using poi2hdfe, and 1.65 seconds when using ppmlhdfe.

5 Simulation

We use simulated data to further illustrate twgravity. The simulation design has two binary regressors. They are independent and take on the value one with probability .05 and .50, respectively. This makes the first regressor sparse. The coefficient on each
The regressor is set to unity. All fixed effects are set to zero and errors are drawn from a log-normal distribution such that their logs follow a standard-normal distribution. The regressors are drawn once and held fixed across the 5,000 Monte Carlo replications. The errors are redrawn in each replication. The sample size was set to \( n = 25 \), yielding \( 25 \times 24 = 600 \) observations at the dyad level. Simulation results for a variety of other designs and different sample sizes are reported in Jochmans (2017).

The first table below contains summary statistics for the three point estimators considered. BGMM11 refers to the GMM1 point estimator of the first coefficient and BGMM12 refers to the GMM1 point estimator of the second coefficient. This naming convention is also used for GMM2. BPPML1 and BPPML2 refer to the PMLE point estimates.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Obs</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>BGMM11</td>
<td>5,000</td>
<td>.9542519</td>
<td>.3584049</td>
<td>-.2982407</td>
<td>2.925951</td>
</tr>
<tr>
<td>BGMM12</td>
<td>5,000</td>
<td>1.002699</td>
<td>.109982</td>
<td>.5822876</td>
<td>1.549646</td>
</tr>
<tr>
<td>BGMM21</td>
<td>5,000</td>
<td>.9396433</td>
<td>.3955723</td>
<td>-.3508639</td>
<td>3.290722</td>
</tr>
<tr>
<td>BGMM22</td>
<td>5,000</td>
<td>.9997944</td>
<td>.1121814</td>
<td>.5487244</td>
<td>1.508134</td>
</tr>
<tr>
<td>BPPML1</td>
<td>5,000</td>
<td>.940787</td>
<td>.3754578</td>
<td>-.3382381</td>
<td>2.783273</td>
</tr>
<tr>
<td>BPPML2</td>
<td>5,000</td>
<td>1.002575</td>
<td>.1124283</td>
<td>.5688691</td>
<td>1.547408</td>
</tr>
</tbody>
</table>

GMM1 does best in terms of both bias and standard deviation but all estimators perform quite well. The average computational effort for GMM1, GMM2, and PMLE (each starting at a vector of zeros) was .1414 seconds, .1435 seconds, and .1780 seconds, respectively.

The next table provides corresponding summary statistics for the estimated standard errors for each estimator.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Obs</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>SEGMM11</td>
<td>5,000</td>
<td>.310138</td>
<td>.0805269</td>
<td>.1471121</td>
<td>.7484761</td>
</tr>
<tr>
<td>SEGMM12</td>
<td>5,000</td>
<td>.1115835</td>
<td>.0143905</td>
<td>.0828566</td>
<td>.2427083</td>
</tr>
<tr>
<td>SEGMM21</td>
<td>5,000</td>
<td>.3345286</td>
<td>.0903741</td>
<td>.1373527</td>
<td>.8006971</td>
</tr>
<tr>
<td>SEGMM22</td>
<td>5,000</td>
<td>.1157641</td>
<td>.0168373</td>
<td>.0827409</td>
<td>.4340205</td>
</tr>
<tr>
<td>SEPPML1</td>
<td>5,000</td>
<td>.2538752</td>
<td>.0547559</td>
<td>.1251421</td>
<td>.5346598</td>
</tr>
<tr>
<td>SEPPML2</td>
<td>5,000</td>
<td>.1025859</td>
<td>.0128109</td>
<td>.0756624</td>
<td>.2152773</td>
</tr>
</tbody>
</table>

It is of interest to compare the Monte Carlo standard deviation (in the previous table) to the average standard error (in the current table). The ratio of the latter to the former
is .8654 and 1.0145 for GMM1, .8457 and 1.0319 for GMM2, and .67612 and .9125 for PMLE. Thus, the standard errors for pseudo-Poisson estimator are quite a bit too low, on average.

6 Conclusion

We have introduced the Stata routines `twexp` and `twgravity` for exponential-regression models with two-way fixed effects. These estimators are based on Jochmans (2017). They are fast to compute, even in large data sets, and yield reliable standard errors for inference.

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8 Appendix

**Additional computational details for GMM1**

Fix the value of $\gamma$ and introduce the shorthands $e_{ij} := e(x_{ij}^T \gamma)$ and $u_{ij} := y_{ij}/e_{ij}$. First consider the pure panel data case. The (symmetrized) moment conditions for GMM1 are

$$s_1(\gamma) = \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} x_{ij} \left\{ u_{ij} u_{i'j'} - u_{ij} u_{i'j'} \right\}.$$  

Note that

$$\sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} x_{ij} u_{ij} u_{i'j'} = \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} x_{ij} u_{ij} u_{i'j'} = \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} x_{ij} (u_{ij} \bar{u}),$$

where $\bar{u} := \sum_{i=1}^{n} \sum_{i'=1}^{n} u_{ij}'$ is the grand mean of the $u_{ij}$. Likewise,

$$\sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} x_{ij} u_{ij} u_{i'j'} = \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} u_{ij} u_{i'j'} = \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} x_{ij} (\bar{u}, \bar{u}_j),$$

where $\bar{u}_i := \sum_{j=1}^{m} u_{ij}$ and $\bar{u}_j := \sum_{i=1}^{n} u_{ij}$ are the means taken with respect to each of the two dimensions of the data. Consequently,

$$s_1(\gamma) = \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{m} \sum_{j'=1}^{m} x_{ij} \left\{ u_{ij} \bar{u} - \bar{u}_i \bar{u}_j \right\},$$

which is fast to evaluate in any matrix-based language. Expressions for the Jacobian matrix $Q_1$ and for $v_{ij}$ follow in the same manner. All these expressions are used in the implementation of *twexp*. 
In *twgravity* self-links are ruled out, i.e., the observations $y_{ii}, x_{ii}$ are not in the data. In this case the empirical moments for GMM1 become

$$s_1(\gamma) = \sum_{i=1}^{n} \sum_{i' \neq i} \sum_{j' \neq j} \sum_{j' \neq j} x_{ij} \left\{ u_{ij} u_{i'j'} - u_{ij} u_{i'j'} \right\};$$

note the change in the range of the sums. It is convenient to define $y_{ii} = 0$ and $x_{ii} = 0$. Then, in the same way as before,

$$\sum_{i=1}^{n} \sum_{i' \neq i} \sum_{j' \neq j} \sum_{j' \neq j} x_{ij} u_{ij} u_{i'j'} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} u_{ij} (\bar{u}_i - \bar{u}_j + u_{ji})$$

and

$$\sum_{i=1}^{n} \sum_{i' \neq i} \sum_{j' \neq j} \sum_{j' \neq j} x_{ij} u_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} (\bar{u}_i \bar{u}_j - \bar{u}_{ij})$$

where $\bar{u}_{ij} := \sum_{i' \neq i} u_{ij} u_{i'j}$. Consequently, in this case we have

$$s_1(\gamma) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \left\{ u_{ij} \bar{u} - \bar{u}_i \bar{u}_j \right\} - \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \left\{ u_{ij} (\bar{u}_i + \bar{u}_j - u_{ji}) - u_{ij} \right\}$$

The additional term on the right-hand side compared to the corresponding expression above is a correction term for the absence of self links in the data. The Jacobian matrix and the covariance matrix of the moment conditions can again be obtained in a similar manner.

**Additional computational details for GMM2**

Fix the value of $\gamma$ and introduce the shorthand $e_{ij} := e(x_{ij}^{\top} \gamma)$ First consider the pure panel data case. The (symmetrized) moment conditions for GMM2 are

$$s_2(\gamma) = \sum_{i=1}^{n} \sum_{i' \neq i} \sum_{j=1}^{m} \sum_{j' = 1}^{m} x_{ij} \left\{ y_{ij} y_{i'j'} e_{ij} e_{i'j'} - y_{ij} y_{i'j'} e_{ij} e_{i'j'} \right\}.$$

Here, defining the $n \times m$ matrices $(Y)_{ij} := y_{ij}$ and $(E)_{ij} := e_{ij}$ we can compactly write

$$x_{ij} y_{ij} \sum_{i'=1}^{n} \sum_{j'=1}^{m} \varphi_{ij} \varphi_{i'j'} = x_{ij} y_{ij} (EY^{\top} E)_{ij},$$

$$x_{ij} e_{ij} \sum_{i'=1}^{n} \sum_{j'=1}^{m} y_{i'j'} e_{i'j'} = x_{ij} e_{ij} (YE^{\top} Y)_{ij};$$

note that the terms on the right-hand side here are quadratic forms in $E$ and $Y$. Hence,

$$s_2(\gamma) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \left\{ y_{ij} (EY^{\top} E)_{ij} - e_{ij} (YE^{\top} Y)_{ij} \right\},$$

which is again immediate to compute in any matrix-based language. When self-links are ruled out—again defining $y_{ii} = 0$, $x_{ii} = 0$, and now also setting $e_{ii} = 0$, no further modification is needed for GMM2.