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Abstract

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Abstract For any game, we provide a justification for why in the long-run outcomes are mostly both efficient and egalitarian/symmetric in the Rawlsian sense. We do this by constructing an adaptive dynamic framework with four features. First, agents select rules to implement actions. Second, rule selection satisfies some minimal payoff monotonicity: rules that do best are chosen with a positive probability. Third, in choosing rules agents are subject to "small" random mutation. Fourth mutation is payoff-dependent with agents mutating more when they do badly than when they do well. Our main result is: if the set of feasible rules R is sufficiently rich then outcomes that survive maximise the payoff of the player that does least well. We also show that if R is restricted to those that do best-reply on uniform histories then outcomes that survive are efficient and egalitarian amongst the set of minimum weak CURB sets. Finally, we consider long-run outcomes assuming mutation is payoff-independent; in contrast to our strong selection result above, in this case we show indeterminacy: any outcome can survive if R is sufficiently rich.

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1 Introduction

An important objective of the literature on evolutionary dynamics has been to provide insights into equilibrium selection. Which equilibrium, or more generally outcome, will be selected/survive in the long-run? Our aim in this paper is very much in line with this objective. Specifically, we provide a fairly general dynamic framework to explain why in the long-run only outcomes that are both *efficient* and *egalitarian* (symmetric) in the Rawlsian sense are mostly observed.

While rest points of most adaptive evolutionary models with deterministic dynamics are closely related to the concept of Nash equilibrium, these models have not been

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successful in addressing the question "which equilibrium" because they are typically non-ergodic and/or may have long-run behaviour that do not correspond to a Nash equilibrium. One approach to "which equilibrium" question is to introduce "small" amount of perpetual random mutation/shocks into adaptive evolutionary models and consider outcomes that will be "mostly" observed in the long-run. These outcomes are called stochastic stable (or SS for short); formally they correspond to the support of the limit of the invariant distribution of the dynamics as the mutation becomes arbitrarily small. The SS approach was pioneered by Foster and Young (1990), Kandori et al. (1993) and Young (1993). It has been successful in selecting between strict Nash equilibria in specific applications.[1]

Our framework is similar to the SS approach; however we differ from most of the literature on two issues: rules and mutation. The SS approach typically assumes that the agents in the population follow some fixed adaptive rule (such as best reply or imitation). However, it turns out that the specific equilibria selected in these models depend very much on the adaptive rules allowed. For example, in 2×2 coordination games with two Nash equilibria, best response dynamics (choosing an action that is a best reply to frequency distribution of past actions) selects the risk dominant equilibrium (Kandori et al. (1993) and Young (1993)), whereas imitation dynamics (choosing an action that performed best on average in the past) selects the efficient equilibrium (Robson and Vega-Redondo (1996)). Therefore, to answer the "which equilibrium" question, we need to explore the dynamics in which multiple rules coexist. The possibility of heterogeneous rules, however, is not just a matter of equilibrium selection. It is simply not plausible that a single behavioural rule can capture all important properties of human behaviour.

Once one allows for the possibility of multiple rules, it is important to allow for selection and competition between rules.[2] While rules do not change very often, agents do revise these policies intermittently; thus, in playing a game, players not only change their actions, but also revise rules that dictate the actions they choose.

Our first departure from the typical work on SS is that evolution in this paper is at the level of rules; thus we allow for multiple rules and consider their selection and evolution. Specifically, our evolutionary model, consisting of a single population of

[1] Sone surveys of deterministic and stochastic evolutionary dynamics are Weibull (1995), Samuelson (1997), Young (1998a), Fudenberg and Levine (1998) and Wallace and Young (2014).

[2] Kaniovski et al. (2000), Juang (2002), Josephson (2009) and Schipper (2009) discuss models with 2 or 3 rules. All the references except the second do not allow players to revise their rules.

agents randomly matched to play an N-player normal form game repeatedly, has the following features: (i) there is a set of feasible rules, where a rule is a mapping from the (finite) set of histories to the action set, (ii) agents adopt rules to play the role of each player and the actions chosen by agents in each role at each date is dictated by the rule she follows, (iii) at the beginning of each date every agent has the opportunity to revise her rule according to some evolutionary selection criterion that satisfies some minimal monotonicity condition that says a rule is chosen with a positive probability if it has done best in the past (henceforth, we use wpp to refer to "with a positive probability") and (iv) selection of rules at any date allows for the possibility of mutation. Our framework is very general as we impose minimal restrictions on the underlying game, the set of feasible rules and the rule selection criterion.[3]

Most of the literature on SS also assumes exogenous mutation. One exception is Bergin and Lipman (1996) who show that, for any adaptive dynamics with finite state space, every long-run prediction of the system without mutation is SS if every arbitrary state-dependent mutation is allowed. Thus, it follows from Bergin and Lipman (1996) that selection results such as those by Kandori et al. (1993) and Young (1993) do not necessarily hold with state-dependent mutation.

But what does mutations supposed to represent? One common answer, experimentation, is difficult to reconcile with the constant rate of mutation across histories/states. Surely agents experiment less in a state with high payoff than in a state with low payoff.[4] Our second departure from the SS literature is payoff-dependent mutation.[5] Specifically, we assume mutation is "less" likely (has a lower order of magnitude) in states in which the agent is doing well than in states that the agent is doing badly (our mutation assumption is similar to the perturbation assumption in Myerson's notion of Proper equilibrium).[6]

[3]Alternatively, we can think of our set-up as an evolutionary framework applied to repeated games, where rules represent strategies in the repeated game and the evolutionary selection criterion describes how agents choose between different repeated game strategies at each date.

[4]There is also strong evidence that bacteria mutate when they are under stress, i.e. when they are poorly adapted to their environment (Bjedov et al. (2003)). Such mutations, known as stress-induced mutagenesis, have also been discovered in other organisms, including yeast, algae, nematodes, flies, human cancer cells and plants (Ram and Hadany (2014), Fitzgerald et al. (2017) and references therein).

[5]Maruta (2002), Blume (2003) and Bilancini and Boncinelli (2020) consider payoff related mutation and selection issue in symmetric 2×2 games. van Damme and Weibull (2002) model mutation rates as mistake probabilities that are endogenously determined by effort levels that are set by agents and show that selection of risk-dominant outcome in 2×2 games holds in their framework.

[6]There is some experimental support for the assumption that mutation is payoff-dependent. Lim

Given the above features, we obtain very sharp selection results for any arbitrary game. Our main result is that, for any normal form game, if the set of feasible rules is sufficiently rich then any SS outcome must maximise the payoff of the player that does least well - henceforth, we shall refer to any such outcome as MaxMin norm.[7] Thus, any SS must be both efficient and egalitarian in Rawlsian sense. Therefore, in common interest games, our result implies that the strictly Pareto-dominant outcome will emerge in the long-run. In Nash bargaining games, the SS allocation maximises the share of the player that receives the least payoff in the allocation; thus if the underlying bargaining game is symmetric, then equal shares are the solution. Our result does not necessarily select an equilibrium, as MaxMin norms may not be Nash. In the Prisoners' Dilemma game, for example, our framework selects cooperation (a non-Nash outcome), as it is the unique MaxMin norm.

The selection of MaxMin norm is established by showing that if the set of feasible rules is sufficiently rich, then MaxMin outcomes are more difficult to escape from and no harder to reach than non-MaxMin outcomes. Very informally, there are two reasons why this is the case in our framework. First, our payoff-dependent mutation implies that the likelihood of a single mutation out of a MaxMin norm is less than that out of any non-MaxMin norm, as the payoff of any agent that does least well in the latter is less than the payoff of every agent in the former. Second, given that selection is at the level of rules and the set of feasible rules is sufficiently rich, reaching a state in which a MaxMin norm $\overline{a}$ is played from any state is no harder than reaching any other state. This we show by constructing a rule profile $r = (r_1, .., r_N)$, where r_n is the rule that specifies how to play the game in the role of player n, such that wpp (a) for each n, starting at any state, with at most *one* mutation, all agents adopt rule r_n and (b) once all agents adopt the profile r, with at most *one* mutation all agents will end up playing a MaxMin norm $\overline{a}$. If r is feasible then reaching a state in which $\overline{a}$ is played is no harder than reaching any state in which $\overline{a}$ is not chosen.

The property in (a) above effectively describes an invading ability as it says that,

and Neary (2016) and Mäs and Nax (2016) consider deviations (mutations) from myopic best-response in repeated coordination games in experimental setting. They find that the probability of such mutation is higher the lower the payoff from not mutating.

[7]The concept of MaxMin norms was introduced into stochastic evolutionary dynamics by Young (1998b). In that paper, he describes contract games and shows that, with state-independent mutation, a SS conventional contract is efficient and approximates MaxMin norms. In our framework, our result selects MaxMin norms exactly in general games under payoff-dependent mutation.

with a single mutation in each role, r can invade the population and that in (b) describes a triggering attribute as it says if all agents use r then with a single mutation it can be triggered to play $\overline{a}$ eventually. Given our selection criterion that at each date a rule is chosen wpp if it has done best in the past, the invading property is ensured if r_n does best response at some point after invasion. The triggering property of starting with r and eventually playing MaxMin $\overline{a}$ after a single mutation has the flavour of "a secret handshake" - the single mutation effectively triggers r to eventually play $\overline{a}$ forever.[8] Since a rule is a mapping from histories to actions, it is possible for a rule to satisfy both properties by playing best response at some point after invasion and $\overline{a}$ after the secret handshake triggering mutation. In fact, there is a large set of rules that satisfy both the invading and triggering properties (see Section 4). For our selection result, all that is needed is that the set of feasible rules contains some such rules.[9]

As mentioned above, our selection result may select a MaxMin norm $\overline{a}$ that is not Nash. Since this means playing $\overline{a}$ repeatedly in a SS state, the rule(s) that are used in such a state must play $\overline{a}$ if $\overline{a}$ has been played by all in the past. However, by appealing to some rationality arguments, it may be argued that at any history in which the same action profile has been played repeatedly, agents will choose a (myopic) best response to that action profile as they extrapolate that the same action profile will be played in the future; thus, $\overline{a}$ cannot persist indefinitely if $\overline{a}$ is not Nash. In the paper, we also consider the case when the set of feasible rules is restricted to playing best response when the history of actions has been uniform. Although such a restriction is inconsistent with every SS inducing a non-Nash MaxMin norm, we are still able to obtain a selection result that has the flavour of MaxMin norm. Very roughly, with this restriction we show that any SS outcome maximises the payoff of the player that does least well amongst all outcomes that are Nash - more precisely the selection is amongst the set of minimum weak CURB sets (a set-theoretic generalisation of Nash equilibrium).[10]

[8]In the literature (e.g. Robson (1990), Fudenberg and Maskin (1990), and Binmore and Samuelson (1992)) secret handshake allows a group of mutants to recognise each other, and thereby cooperate amongst themselves and punish those outside the group, and invade. Here, the idea is somewhat different: all agents are using the same rule profile r and a single mutation to a secret handshake triggers all agents using r to change behaviour so that $\overline{a}$ will happen eventually.

[9]As mentioned in footnote 3, our framework can be thought of as applying evolutionary arguments to selecting strategies in repeated games. However, in contrast to the repeated game literature, our analysis for why efficiency and egalitarianism may evolve does not involve constructing rules that incentivise players to choose a norm on the basis of future punishment and rewards; the two features of being invading and triggering that we need some feasible rules to have are not necessarily about behaviour that induces appropriate intertemporal incentives.

[10]In the extension, we also provide some conditions on birth of new rules that ensure selection is

Our selection results are driven by the assumption that mutation rate by any agent depends inversely on the payoffs the agent obtains in the past. To bring out the importance of this assumption we also consider the case in which mutation rate is history-independent (and thus independent of past payoffs), as in most of the literature on SS. Under this alternative mutation assumption, we obtain the opposite of the strong selection stated above: any outcome is SS if the set of feasible rules is sufficiently rich. Such indeterminacy result with history-independent mutation arises because, for any outcome a, we can construct rules that are invading and triggering towards a. Effectively, the existence of such rules implies that drift between any pair of outcomes can be achieved through a minimal (and equal) number of mutations; but then since with history-independent mutation the likelihood of moving from one outcome to another in the long-run depends only on the number of mutations, all outcomes are SS and can survive in the long-run.

Given this indeterminacy claim, we conclude that the selection results obtained in the literature on evolutionary models with small random history-independent mutation depend critically on limiting the set of feasible rules (e.g. restricting the set to best-reponse or imitation rules), as indeterminacy and a "folk theorem type result" seem inevitable if the set of feasible rules is sufficiently rich. Excluding some rules of behaviour from the outset may of course be reasonable if there is good a priori reasoning for restricting agents' behaviour. For example, for reasons mentioned above, we could restrict the set of feasible rules to those that best reply at uniform histories. We show that under this restriction any Nash equilibrium (in fact any minimum weak CURB set) is SS if the set of feasible rules is sufficiently rich and mutation is history-independent. This implies that, under history-independent mutation, to obtain any selection between Nash equilibria (or minimum weak CURB sets), we need to impose more restrictions on the set of feasible rules than best reply at uniform histories.

Finally, we like to briefly discuss an important feature of our results regarding speed of convergence.[11] The literature on SS is sometimes criticised on the grounds that convergence to a SS state can be too slow. For example, in models in which all agents are assumed to follow best response rules (e.g. Kandori et al. (1993) and Young (1993)), the number of mutations needed for the system to switch from one equilibrium

amongst Nash equilibria.

[11]We like to thank Larry Samuelson for emphasising the importance of this feature to us.

to another has to be increasing in the population size. When the latter size is large and the mutation rate is small, the chance that such a switch occurs is extremely small. This implies that the time it takes for a system to converge to any SS outcome could be unreasonably long.[12] Such a criticism does not apply to our results as the number of mutations needed in our framework to reach a state in which a particular outcome (e.g. MaxMin norm) is played is most minimal (at most $N+1$: one for each of the N roles plus an additional one to induce triggering) and independent of the size of the population. Effectively, since both invading in any role and the secret handshake require at most one mutation, the time taken to reach a long-run outcome does not need to increase with the number of agents.

The plan of the paper is as follows. In Section 2 we describe the model. An overview of the main results is given in Section 3; we also provide a sketch of some of the arguments in this section. In Section 4 we provide detailed results. Section 5 describes how our results can be extended when some of our assumptions are relaxed. The proofs of all the claims in Sections 4 and 5 are in the Appendix and Online Appendix.

2 Underlying Game and Evolutionary Dynamics

One-shot game. The stage game is a normal form game denoted by $G = \{A_n, \pi_n\}_{n=1}^N$, where N is the number of players (with some abuse of notation, we also use N to denote the set of players) and, for any player n, A_n is the set of finite (pure) actions and $\pi_n : A \to \mathbb{R}$ is the payoff function, where $A \equiv A_1 \times ... \times A_N$. For any N-tuple $x = (x_1, .., x_N)$, x_n and x_{-n} respectively refer to the n-th component and all components other than n. For any n and any $a \in A$, player n's best reply to a_{-n} is $B_n(a_{-n}) \equiv \arg\max_{a'_n \in A_n} \pi_n(a'_n, a_{-n})$; we also use the notation $B_n(a)$ to denote $B_n(a_{-n})$. Let $B(a) = \times_n B_n(a)$. For simplicity, we assume, for all n, $\pi_n(a) \neq \pi_n(a')$ if $a \neq a'$. Denote the set of (pure) Nash equilibria in G by E, with e a typical element of E (the set E could be empty).

Repeated matching game. There is a single population consisting of I agents (we also use I to denote the set of agents). At any discrete date $t = 1, 2, ...$, each member of the population is randomly matched with $N-1$ other agents to play the stage game G in the role of some player. For simplicity, assume that at each date all agents are

[12] Convergence can be fast if the interaction between agents is through a network and agents respond only to the choices of their neighbours (see Young (1998a)). Similar kind of fast convergence can also be obtained without local interactions if payoff gains from switching between equilibria and mutation rates are sufficiently high (e.g. Kreindler and Young (2013)).

randomly matched to play the game, i.e. $I = MN$ for some integer M. Hence, there are exactly M matches in each period (we also use M to denote the set of matches). We assume $M > 1$. Our results also hold for the case when $M = 1$; however, the analysis is different, somewhat simpler and less interesting given some of the other assumptions we make below; we discuss this case in Online Appendix. To simplify, assume also that the probability of being assigned to any match or to any role is the same for all agents.

Our set up is standard except that in order to handle a general game with different players we allow agents to take the role of different players. Thus, at each date every agent cares about both the action profile in the match she participates in and the role she plays in that match. Thus, the preference of each agent is over the set $A \times N$. Given that all agents are assumed to have the same payoff function $\pi_n(.)$ in each role n, it follows that all agents are assumed to have identical preferences (even though the payoffs function for different roles/players may not be the same).[13]

History and Rules. For any finite set X, denote the set of probability distributions over X by ΔX and the set distributions over X with full support by $Int(\Delta X)$**.**

We assume agents have finite period memory T: so they only recall the outcome (action profiles) in the previous $T \geq 1$ periods (we use T to describe both the memory and the set $\{1, .., T\}$). For any period and any $m \in M$, $a^m \in A$ denotes the outcome in match m, and $\phi = (a^1, .., a^M) \in A^M$ refers to the outcome of all matches. At any date, denote the outcomes in the previous T periods, henceforth called history, by $\theta = \{a^{m,t}\}_{m\in M, t\in T}$, where $a^{m,t} \in A$ refers to the outcome $T - t + 1$ periods ago in match m. Let $\Theta \equiv A^{M\times T}$ be the set of histories. For any $\theta = \{a^{m,t}\}_{m\in M,t\in T} \in \Theta$, define the set of action profiles played in θ by $A(\theta) = \{a \in A \mid a = a^{m,t}$ for some $m \in M$ and $t \in T\}$ and the action played in role n of match m and in the t-th period of θ by $\theta_n^{m,t} = a_n^{m,t}$. Also, let $\theta^{m,t} = \{\theta_n^{m,t}\}_{n\in N}$ and $\theta^t = \{\theta^{m,t}\}_{m\in M}$. For any $a \in A$, let $\phi(a) \in A^M$ be the 1-period outcome in which a is played in every match, and $\theta(a) \in \Theta$ be the history in which $\phi(a)$ is played in every period. Denote $\Theta^u = \cup_{a\in A}\theta(a)$ as the set of uniform histories. A history $\theta(a) \in \Theta^u$ is called an equilibrium history if $a \in E$. A history θ is called stationary if $\theta^t = \theta^{t'}$ for all t and t'. Let Θ^s be the set of stationary histories.

When choosing an action in any role at any date, each agent adopts a rule to imple-

[13] Only ordinal ranking over $A \times N$ implied by $(\pi_1, .., \pi_N)$ is important for our results. Hence, the results in this paper also hold if agents have different payoff functions in each role as long as they all have a common preference ordering over the set $A \times N$.

ment her actions. A rule on how to play the game at each date in any role $n \in N$ is a mapping $r_n : \Theta \to \Delta A_n$ from the set of histories to the set of probability distributions over A_n.[14] For any $\theta \in \Theta$ and $Q_n \subseteq A_n$, $r_n(\theta)[Q_n]$ refers to the probability that rule r_n attaches to Q_n when θ is observed. We denote the profile of rules by $r = (r_1, ..., r_N)$ and the probability that r chooses any $a \in A$ by $r(\theta)[a] = \times_{n \in N} r_n(\theta)[a_n]$. Also, let $\overline{R}_n$ be the set of all rules in role $n \in N$ and $\overline{R} = \times_{n \in N} \overline{R}_n$ be the set of all rule profiles.

At any date, assume that the set of rules agents may adopt in any role $n \in N$ is some fixed non-empty subset R_n, henceforth called the set of feasible rules in role n, of $\overline{R}_n$. Let $R = \times_{n \in N} R_n \subseteq \overline{R}$ be the set of feasible rule profiles. When the meaning is clear we drop the word profile from both rule profile and action profile.

Next we ask if there should be some restriction on the set R, and if so, what. One restriction would be to require feasible rules to be deterministic (pure) i.e. $R \subset \overline{R}^{pure}$, where $\overline{R}^{pure} = \times_{n \in N} \overline{R}_n^{pure}$ and $\overline{R}_n^{pure} = \left\{ r_n \in \overline{R}_n \,|\forall\, \theta \in \Theta,\, r_n(\theta)[a_n] = 1 \text{ for some } a_n \in A_n \right\}$.

By appealing to bounded rationality type reasoning, another natural restriction might be to restrict rules to simple ones. The simplest rules are history-independent ones that always take the same action. Denote the set of such rules, henceforth called simple rules, in role n by $S_n = \left\{ r_n \in \overline{R}_n^{pure} \,|r_n(\theta) = r_n(\theta'), \forall \theta \text{ and } \theta' \in \Theta \right\}$. Let $S = \times_n S_n$. Also, define the set of totally mixed simple rule profiles by $S^{mixed} = \left\{ r \in \overline{R} \,|r(\theta) = r(\theta') \in Int(\Delta A), \forall \theta \text{ and } \theta' \in \Theta \right\}$. We could also consider rules with higher order complexity than simple ones. For example, rule profiles with "complexity" one level higher than S may be those that do one action at some histories and another at every other history. Denote the set of such rules, henceforth called 2-complexity rules, by $S^2 = \left\{ r \in \overline{R}^{pure} \,|\exists a, b \in A \text{ s.t. } r(\theta)[\{a, b\}] = 1, \forall \theta \right\}$.

A most common type of rule used in the literature is (myopic) best reply. Other common rules are better reply, fictitious play, imitation and no regret learning. Some of these rules are motivated by certain rationality considerations; others are not. A most minimal rationality restriction on the set of rules is concerned with behaviour at equilibrium histories. In such histories *every* agent in any role is taking the same action and is obtaining the maximum payoff given the choice of the others in every match of any period; hence, it is very reasonable to assume that the rules prescribe

[14] In the above definition, rules obey anonymity with respect to individual agent's experiences as the behaviour depends only on the history of actions. A more general type of rule might also depend on the identity of the agents that adopt different actions in the previous periods. We adopt such anonymity to simplify the analysis.

the same equilibrium behaviour in the next period. This behaviour is satisfied by most rules in the literature. Such rules, henceforth called equilibrium rules, is denoted by $R^E = \{r \in \overline{R}\,|r(\theta(e))[e] = 1 \text{ for all } e \in E\}$.

Being an equilibrium rule imposes rationality restrictions only at equilibrium histories. Many typical rules used in the literature also impose some rationality restrictions on non-equilibrium histories. For example, a weak restriction on the set of rules, henceforth called uniform best reply, is to play best reply at all uniform histories. Denote this set by $R^u = \{r \in \overline{R}\,|r(\theta(a))[B(a)] = 1 \text{ for all } a \in A\}$. A justification for restricting rules to R^u is: if in the past the same action profile is played in every match at every date then agents will assume that the same behaviour would be followed in the future (particularly if T and/or M are large) and hence, play a best reply.

A stricter restriction than uniform best reply is to require any rule r_n in role n to best reply at any history in which the same action is played in all matches of any period in all roles other than n, i.e. $r_n(\theta)[B_n(a_{-n})] = 1$ for any $a_{-n} \in A_{-n}$ and $\theta \in \Theta^{-n}(a_{-n})$, where $\Theta^{-n}(a_{-n}) \equiv \{\theta \in \Theta \mid \theta^{m,t}_{-n} = a_{-n} \text{ for all } m \text{ and } t\}$.

For histories in which other players do not play the same action, we could also require rules in any role to play a best reply to some distribution over the set of actions observed in the history. An examples of such a behaviour in any role is a rule that samples one of the matches in θ and best replies to it wpp. We call such rules sampling best reply and define the set of such rules by $R^s = \times_{n \in N} R^s_n$, where for each n

$$R^s_n = \left\{ r_n \in \overline{R}_n \;\middle|\; \begin{array}{c} \text{(a) } r_n(\theta)[a_n] > 0 \Rightarrow a_n = B_n(\theta^{m',t'}) \text{ for some } m' \text{ and } t' \text{ and} \\ \text{(b) } \exists\, m \text{ and } t \text{ s.t. } \forall \theta \in \Theta,\, r_n(\theta)[B_n(\theta^{m,t})] > 0. \end{array} \right\} \tag{1}$$

One could also argue that rules should not depend on payoff-irrelevant features of history; for example, they should be match-neutral in the sense that rules should be sensitive only to past actions and not to the identity of matches in which the actions took place. Formally, we say a rule r is *match-neutral* if, for any $\theta \in \Theta$ and any permutation of matches $k : M \to M$, $r(\{\theta^{m,t}\}_{m,t}) = r(\{\theta^{k(m),t}\}_{m,t})$.

While many of the above restrictions (as well as several others) on the set R may be reasonable, to widen the applicability of our results, we assume only the following:

Assumption on R: *(i) R is finite and (ii) either $S \subset R$ or $R \cap S^{mixed} \neq \emptyset$.*

Restriction (i) above is assumed to ensure that the dynamics has a *finite* state space. It imposes no restriction if the set of rules involve pure strategies only. Restriction (ii) is assumed to ensure that all actions can be played wpp at least through mutation.[15]

States. At any date, for any $n \in N$, $m \in M$ and $t \in T$, let $r_n^{m,t} \in R_n$ refer to the rule adopted $T - t + 1$ periods ago by the agent in role n, in the mth match and $r^{m,t} = (r_1^{m,t}, ..., r_N^{m,t})$ be the rule profile adopted $T - t + 1$ periods ago by the agents in the mth match. Also, let $r^t = (r^{1,t}, ..., r^{M,t}) \in R^M$ be the rules adopted $T - t + 1$ periods ago in all the M matches. The set of T-period rule-histories is R^{MT}. We then define $\Omega \equiv \Theta \times R^{MT}$ as the set of all states with typical element $\omega = \{a_n^{m,t}, r_n^{m,t}\}_{n \in N, m \in M, t \in T}$. Since the number of players, actions and rules, and players' memory are all finite, the state space Ω is finite.

For any state $\omega = \{a_n^{m,t}, r_n^{m,t}\}_{n \in N, m \in M, t \in T}$, let $\theta[\omega] = \{a_n^{m,t}\}_{n \in N, m \in M, t \in T}$ be the history of actions in ω, $A(\omega) \equiv \{a \in A \mid a \in A(\theta[\omega])\}$ be the set of actions played in ω, $R[\omega] = \{r_n^{m,t}\}_{n \in N, m \in M, t \in T}$ be the history of rules in ω, and $R_n(\omega) \equiv \{r_n \in R_n \mid r_n = r_n^{m,t}$ for some $m \in M$ and $t \in T\}$ be the set of rules adopted in role n in ω. Let $R(\omega) = \times_{n \in N} R_n(\omega)$. A state $\omega \in \Omega$ is called uniform if $\theta[\omega] \in \Theta^u$. A uniform state ω such that $A(\omega) = a$ and $R(\omega) = r$ for some $a \in A$ and $r \in R$ is denoted by $(\theta(a), r)$.

Full history. At any date, the roles and the matches of each agent i in the previous T periods are defined as follows. For any t, $n^t(i) \in N$ and $m^t(i) \in M$ respectively refer to the role assigned to i and the match participated in by i, $T - t + 1$ periods ago. We refer to $\upsilon^t(i) = (n^t(i), m^t(i))$ as the assignment of i, $T - t + 1$ periods ago and $\upsilon = \{\upsilon^t(i)\}_{i \in I, t \in T}$ as the assignments over T periods. Let Υ be the set of assignments over T periods. At any date if $\omega \in \Omega$ is the state and $\upsilon \in \Upsilon$ is the assignment, we call $h = (\omega, \upsilon)$ the full history of the previous T periods. Define the set of full histories by H and the set of full histories associated with any $\omega \in \Omega$ by $H(\omega) = \{h \in H \mid h = (\omega, \upsilon)$ for some $\upsilon \in \Upsilon\}$.

Rule Selection and Inertia. Since agents have T-period memory recall, the selection criterion used by any agent in any role is a mapping from the set of full histories H to some distribution over feasible rules. For expositional simplicity, we assume that the criterion for each agent is anonymous (i.e. the choice of the rule at each date depends only on the state) and all agents in the same role use a common criterion to revise rules.

Definition 1 *For any $n \in N$, a rule selection criterion $\rho_n : \Omega \to \Delta R_n$ is a mapping which assigns a probability distribution over R_n for any $\omega \in \Omega$. Let $\rho \equiv (\rho_1, ..., \rho_N)$.*

[15] The second restriction is not needed for some of the results. Also, we could replace it by the assumption that there is action mutation (there is noise in the implementation of rules).

To ensure asynchronisation of behaviour across different agents, we introduce inertia both in selection of rules and in implementation of actions. Specifically, rule inertia stipulates that at any date, if an agent is assigned the same role as in the previous period then with probability $p^r \in (0,1)$ she faces rule inertia and chooses the same rule as she did in the previous period, and with probability $1-p^r$ there is no rule inertia and she chooses her rule according to the selection criterion $\rho_n(.)$ for all n. Action inertia stipulates that at any date, if an agent is assigned the same role and using the same rule as in the previous period then with probability $p^a \in (0,1)$ she faces action inertia and takes the same action as she did in the previous period, and with probability $1-p^a$ there is no action inertia and she chooses her action according to her rule. When an agent is assigned a different role from that in the previous period or is using a new rule, we assume the agent is not subject to action inertia. The sequence of events at each date is thus such that first agents are assigned matches and roles. Second, those subject to rule inertia choose the same rule as in the previous period and agents not subject to rule inertia revise their rules according to ρ_n for each role n. Third, agents subject to action inertia take the same actions as in the previous period and agents not subject to action inertia take actions according to their rules.

There are a number of points to note regarding the inertia described above. First, we have assumed that rule inertia applies to agents only if they play the same role in the previous period, and action inertia applies to agents only if they play the same role and use the same rule as in the previous period. This is not the only way of introducing inertia. We have adopted this approach because it seems more natural, even though the analysis turns out to be significantly more complicated as a result of our specific inertia assumptions. In any case the results in this paper do not depend on this specification of inertia. Second, in the above we assumed that at each date roles and rules are decided before action inertia. Again nothing hinges on such a specification. Third, inertia of any kind is not necessarily needed to obtain many of the results (see Section 5).

Dynamics without mutation. The above describes a dynamical system that can be represented by a Markov chain with state space Ω.[16] The evolutionary process

[16]Since at any date the rule (action) an agent chooses when subject to rule (action) inertia depends on the rule (action) chosen by that agent in the previous period, the dynamics of the system at any date depends on the assignments $\upsilon \in \Upsilon$ each agent has had in the previous T periods, in addition to the state $\omega \in \Omega$. However, since the roles and the matches agents are assigned to are independent of the identity of the players, the state $\omega \in \Omega$ is still sufficient to describe the transition dynamics.

starts at some initial state $\omega^0 \in \Omega$. Given $\{G, R, \rho, p^r, p^a, \omega^0\}$, the system then evolves with states changing from one period to the next. Hence, the dynamics of the system $\Lambda : \Omega \to \Delta\Omega$ is a mapping which assigns to any state a probability distribution over the set of states. Since the state space Ω is finite, the dynamics will eventually settle on one of its recurrent classes within finite time. We denote a recurrent class of Λ by $C \subseteq \Omega$ and the set of all recurrent classes by Γ. Also, let $A(C) = \cup_{\omega \in C} A(\omega)$ and $R(C) = \cup_{\omega \in C} R(\omega)$ be, respectively the set of actions and rules that can occur in recurrent class $C \in \Gamma$.

Restrictions on the selection criterion. To preserve the generality of our results, we like to impose minimal restriction on the rule selection criterion. One common reasonable assumption is to assume that agents choose a rule from the existing ones:

Assumption (No-Birth): *For any $n \in N$ and $\omega \in \Omega$, $\rho_n(\omega)[r_n] > 0$ only if $r_n \in R_n(\omega)$.*

The above No-Birth assumption has a "must see" feature in that it excludes the possibility that agents select a rule that has not been played in the previous T-periods of history. One consequence of the No-Birth assumption is that if at any state all agents choose the same rule in a given role then that rule will continue being chosen in that role, as there is no possibility of adopting a new rule (so with No-Birth, new rules are possible only through mutation). The No-Birth assumption also implies that every recurrent class $C \in \Gamma$ must contain a unique rule profile (i.e. $R(C)$ is singleton) because once a rule is not chosen for T periods that rule will disappear forever.

Another restriction is to apply some rationality to ρ. A minimal notion of rationality is that rules that receive the highest payoffs in the last T periods would be selected wpp:

Assumption (Monotonicity): *Fix any $n \in N$, $\omega \in \Omega$ and $r_n \in R_n(\omega)$. Denote the set of all matches and periods in which r_n has been adopted by agents in role n in ω* by $D_n(r_n, \omega) \equiv \{(m, t) \in M \times T \mid R[\omega]_n^{m,t} = r_n\}$. Then $\rho_n(\omega)[r_n] > 0$ *if*

$$\pi_n(\theta[\omega]^{m,t}) \geq \pi_n(\theta[\omega]^{m',t'}),\ \forall (m,t) \in D_n(r_n, \omega) \text{ and } \forall (m', t'). \tag{2}$$

Monotonicity assumption stipulates that $r_n \in R_n(\omega)$ is chosen wpp at any state ω if the payoff of any agent who adopted rule r_n in role n in ω is no less than that obtained by any other agent in role n in ω. This condition is weak because it requires a rule to be selected wpp if it has done best in *all* of the periods it was used in the history.

We assume both No-Birth and Monotonicity. There are four remarks concerning these assumptions. First, both are "backward looking" as they require agents to recall the past T periods and, in any role n, (a) choose one of the existing rules and (b) if there exists a rule that has done best in the last T periods then choose that rule wpp.

Second, No-Birth is a kind of imitation assumption in the sense that it assumes that agents choose amongst rules that have been observed; this may not always be true. In Section 5 we consider some alternatives to No-Birth.

Third, Monotonicity attributes a payoff to rule r_n even for periods in which the agents adopting r_n were subject to action inertia. It may be more reasonable to apply Monotonicity to matches in which r_n has not been subject to action inertia, and the actions implemented, and hence payoffs obtained, were the results of implementing r_n. Our results hold irrespective of whether the selection criterion applies when agents are subject to action inertia or not: in our proofs, whenever we appeal to Monotonicity to show that some rule r_n is selected, we only consider states in which all agents in role n are not subject to action inertia (and thereby implement their rules); see footnotes 43, 44 and 45.

Fourth, while Monotonicity does not require agents to know the details of the rules adopted in all roles, matches and dates in the past T periods, it does require agents to know *wpp* the rule(s) that did best in the history. Having some knowledge of the actual rules adopted in the past and their performance may not be such a strong requirement. After all individuals do have some ideas of how others (or at least those close to them) react to past events; especially, information about successful rule of behaviour is often publicly discussed and advertised. Nevertheless, it may be argued that such information may not always be available. In Section 5, we consider the case in which agents *only* know the past actions (and not the rules that actually adopted in each match in the past) and show that our main selection results also extend in this setting.

Mutation. Next, we add mutation to the dynamical system Λ. Specifically, assume that at each date, in selecting their rules, agents mutate with a small probability and choose randomly a feasible rule. Mutation may come from players' intentional experimentation, involuntary trembles or mistakes, pure fantasy, or other factors. As we mentioned in the introduction, the probability of mutation by any agent at any date may depend on how well the agent has done in recent past. Formally, for any agent i and any full history $h \in H$, we denote the probability of mutation (or mutation rate) by i at h by $\epsilon^{f(i,h)}$ for some function $f : I \times H \to (0, \infty)$ and the random choice that

i makes at h, conditional on mutating and being assigned role n, by some probability distribution over feasible rules $\chi_n(i,h) \in Int(\Delta R_n)$. Thus, any agent i, conditional on being assigned role n, with probability $1-\epsilon^{f(i,h)}$ chooses the rule prescribed by the rule selection criteria ρ_n, while with probability $\epsilon^{f(i,h)}$, she randomly picks a feasible rule according to $\chi_n(i,h)$. Effectively, $f(i,h)$ measures the resistance of i to mutation at history h. Our mutation assumption implies that at any date every $r_n \in R_n$ is chosen with a probability no less than $\min_i \epsilon^{f(i,h)}\chi_n(i,h)[r_n] > 0$, for all n.

The precise sequence of events at each date for the dynamics with mutation is:

(i) Mutation: Each agent i at any full history $h \in H$ mutates with probability $\epsilon^{f(i,h)}$ or does not mutate with probability $1-\epsilon^{f(i,h)}$.

(ii) Matching and Role assignment: Agents are matched and assigned roles.

(iii) Rules: If an agent has mutated and is assigned role n, she randomly picks a rule in R_n according to $\chi_n(i,h)$. If an agent has not mutated and is assigned some role n different from that she used in the previous period then she chooses her rule according to the selection criterion $\rho_n(.)$. If an agent has not mutated and is assigned the same role n as in the previous period then with probability p^r she is subject to rule inertia, in which case she chooses her previous rule (in that role), and with probability $1-p^r$ she is not subject to rule inertia, in which case she chooses her rule according to $\rho_n(.)$.

(iv) Actions: If an agent is either assigned a different role from the previous period or has chosen a different rule from the previous period then she implements her rule by choosing the action prescribed by the rule. If an agent is assigned the same role and has chosen the same rule as in the previous period then with probability p^a she is subject to action inertia, in which case she chooses her previous period action (in that role), and with probability $1-p^a$ she is not subject to action inertia, in which case she implements her rule by choosing the action prescribed by the rule.

The precise sequence of events described above is not important for the results of this paper. We could change the order of mutation, rule selection, matching and role assignment, and implementation of actions, and the same conclusions will follow.

To simplify the exposition we make the dynamics with mutation anonymous by assuming throughout that f is anonymous with respect to the identity of the agents: for any permutation $k: I \to I$,

$$f(i,(\omega,\{\upsilon^t(i)\}_t) = f(k(i),(\omega,\{\hat{\upsilon}^t(k(i))\}_t)) \text{ for all } (\omega,\{\upsilon^t(i)\}_t) \in \Omega \times \Upsilon, \quad (3)$$

where $\hat{\upsilon}^t$ is the assignment at t that satisfies $\upsilon^t(i) = \hat{\upsilon}^t(k(i))$ for all t.

As we argued in the introduction, the likelihood of mutating by an agent may depend on her personal experience; in particular, it is higher when agents are doing badly than when they do well. Formally, for any $h = \{a_n^{m,t}, r_n^{m,t}, n^t(i), m^t(i)\}_{n\in N, m\in M, t\in T, i\in I}$, denote the payoff of each agent i in period t of full history h by $\pi^t(i,h) = \pi_{n^t(i)}(a^{m^t(i),t})$. Then a most minimal restriction on mutation rates that reflects the idea that each agent's mutation rate is negatively correlated with individual's recent payoff history is:

$$\text{for all } h,\, \overline{h} \in H,\ f(i,h) > f(i,\overline{h}) \text{ if and only if } \pi^T(i,h) > \pi^T(i,\overline{h}). \tag{4}$$

We say agents are subject to history dependent mutation (henceforth, HDM) if f satisfies (4).[17] Note that HDM is a restriction on mutation rates for a given agent i across different full histories. There is no restriction on mutation rates across different agents.

In this paper, we also compare our selection results under HDM with the more standard set-up in which the mutation rates at any date are independent of the past. We say agents are subject to history independent mutation (henceforth, HIM) if

$$\text{there exists } \eta \in (0,\infty) \text{ s.t. } f(i,h) = \eta \text{ for all } (i,h) \in I \times H. \tag{6}$$

The dynamics with mutation with arbitrary function $f : I \times H \to (0,\infty]$ is a Markov chain with state space equal to full histories H. However, under either HDM or HIM, we can also describe the dynamics with mutation as a Markov chain on the set of states Ω (i.e. to define the transition dynamics, it is not necessary to describe the past assignments). Clearly, this is the case with HIM as past assignments do not influence mutation rates and hence the evolution of future states. With HDM, we have

$$\Sigma_{\overline{\upsilon}\in\Upsilon} \Pr((\overline{\omega},\overline{\upsilon}) \mid (\omega,\upsilon)) = \Sigma_{\overline{\upsilon}\in\Upsilon} \Pr((\overline{\omega},\overline{\upsilon}) \mid (\omega,\hat{\upsilon})),\ \forall\omega,\overline{\omega}\in\Omega \text{ and } \upsilon,\hat{\upsilon}\in\Upsilon, \tag{7}$$

as f satisfies (3) and (4).[18] But then by the lumpability result of Kemeny and Snell

[17] In (4) mutation rates at any date depend only on how well each agent has done in the previous period. Our results under HDM also hold if mutation rates depend negatively on how each agent has done in the entire T-period histories; in particular, they remain valid if we replace (4) by:

$$\begin{aligned} &\text{(a) } \forall h, \overline{h} \in H,\ f(i,h) > f(i,\overline{h}) \text{ if } \pi^t(i,h) > \pi^{t'}(i,\overline{h}), \text{ for all } t,\, t' \in T \\ &\text{(b) } \forall h, \overline{h} \in H,\ f(i,h) > f(i,\overline{h}) \text{ only if } \pi^t(i,h) > \pi^{t'}(i,\overline{h}) \text{ for some } t \text{ and } t'. \end{aligned} \tag{5}$$

Condition (5) is the most minimal condition consistent with the idea that mutation rates are negatively correlated with payoffs in the T-period history. In the analysis below we adopt the stronger condition (4) to simplify the analysis (see Footnote 19).

[18] The argument for (7) is as follows: For some permutation $k : I \to I$, $\hat{\upsilon}^T(i) = \upsilon^T(k(i))$ for all i. Let $\tilde{\upsilon} \in \Upsilon$ be such that $\tilde{\upsilon}^t(i) = \upsilon^t(k(i))$ for all i and t. For all i, since $\hat{\upsilon}^T(k(i)) = \tilde{\upsilon}^T(k(i))$, by (4), $f(k(i),(\omega,\hat{\upsilon})) = f(k(i),(\omega,\tilde{\upsilon}))$; also, by (3), $f(k(i),(\omega,\tilde{\upsilon})) = f(i,(\omega,\upsilon))$. Hence, $f(k(i),(\omega,\hat{\upsilon})) = f(i,(\omega,\upsilon))$ for all i. Since $k(.)$ is a permutation and the rest of the dynamics does not depend on past

(1976) the dynamics over elements of Ω can be described by a Markov chain.[19]

The dynamics with mutation is ergodic and thus has a unique invariant distribution that summarises the "long-run" behaviour of the process from any initial state. Fixing $f(.,.)$ and $\chi_n(.,.)$ for all n, we parametrise the dynamics with mutation and its invariant distribution by ϵ and denote them respectively by $\Lambda^\epsilon : \Omega \to \Delta\Omega$ and $\mu^\epsilon \in \Delta\Omega$.

We are interested in cases where mutation is arbitrarily small. Fixing $f(.,.)$ and $\chi_n(.,.)$ for all n, as ϵ approaches zero the invariant distribution μ^ϵ converges to an invariant distribution μ of the unperturbed dynamics Λ. The states that μ attaches a positive probability are SS states and are denoted by $\Omega^* \equiv \{\omega \mid \mu(\omega) > 0\}$. Any SS state $\omega \in \Omega^*$ is observed in the long-run when ϵ is arbitrarily small; hence any $\omega \in \Omega^*$ must belong to a recurrent class $C \in \Gamma$ of the unperturbed dynamics Λ.

One way of characterising Ω^* involves locating the recurrent class(es) of Λ with the least stochastic potential (see Young (1993)). Define the resistance to mutation by any set of agents $I' \subseteq I$ at $h \in H$ by $\sum_{i \in I'} f(i,h)$. Also, define the resistance between two full histories $h = (\omega, \upsilon)$ to $h' = (\omega', \upsilon')$, $\overline{res}(h,h')$, as the minimum resistance to moving from h to h' in one period: Formally, let $I(h,h') \equiv \{I' \subseteq I \mid$the system transits from h to h' in one period wpp with mutation by some subset of $I'\}$; then $\overline{res}(h,h') = \min_{I' \in I(h,h')} \sum_{i \in I'} f(i,h)$ if $I(h,h') \neq \emptyset$ and $\overline{res}(h,h') = \infty$, otherwise. We define the resistance between two states ω and ω', $\widehat{res}(\omega,\omega')$, as the least resistance to moving the system from ω to ω' in one period: $\widehat{res}(\omega,\omega') = \min_{h' \in H(\omega')} \overline{res}(h,h')$ for any $h \in H(\omega)$.[20]

For any pair C and $C' \in \Gamma$, let $res(C,C')$ be the least amount of resistance for the system to switch from C to C' within finite periods.[21] Also, let Ψ be the directed graph in which each vertex corresponds uniquely to some $C \in \Gamma$, and for any C' and $C'' \in \Gamma$ with $C' \neq C''$ there exists a directed edge, denoted by (C', C''), connecting C' to C''. For any $C \in \Gamma$, a *C-tree*, denoted by τ_C, is a spanning tree belonging to

assignments, (7) must hold.

[19]Describing the dynamics as a Markov chain over Ω is however not feasible under HDM if replace (4) by the weaker condition (5) mentioned in Footnote 17. This is because (3) and (5) cannot guarantee (7) when $T > 1$. So to prove the main results under HDM when (4) is replaced by (5), the analysis has to be conducted assuming that the dynamics is a Markov Chain over H. This is feasible; however, the exposition is more involved.

[20]By (3), $\min_{h' \in H(\omega')} \overline{res}(h,h')$ is the same for all $h \in H(\omega)$.

[21]Formally, for any distinct pair C and $C' \in \Gamma$, a path is a sequence of states $\zeta = (\omega_1, ..., \omega_k)$ for some positive integer k with $\omega_1 \in C$ and $\omega_k \in C'$. The resistance of ζ is the sum of the resistances of all the transitions: $\widetilde{res}(\zeta) = \sum_{\ell=1}^{k-1} \widehat{res}(\omega_\ell, \omega_{\ell+1})$. We define $res(C,C') = \min_{\zeta \in F(C,C')} \widetilde{res}(\zeta)$, where $F(C,C')$ is the set of all such paths from C to C'.

the graph Ψ, such that from every vertex $C' \neq C$, there is a unique path directed from C' to C. Let T_C be the set of all such C-trees. The resistance of a C-tree $\tau \in T_C$, denoted by $res(\tau)$, is defined by the sum of the resistances of all its edges: $res(\tau) = \Sigma_{(C',C'')\in\tau} res(C', C'')$. For any $C \in \Gamma$, the stochastic potential $\gamma(C)$ is the least resistance among all C-trees: $\gamma(C) = \min_{\tau\in T_C} res(\tau)$. We can then characterise the set of SS states by $\Omega^* = \{\omega \in \Omega \mid \omega \in C \text{ for some } C \in \arg\min_{C'\in\Gamma} \gamma(C')\}$.[22] We also say a recurrent class $C^* \in \Gamma$ is SS if $C^* \in \arg\min_{C'\in\Gamma} \gamma(C')$ and denote the set of all SS recurrent classes by $\Gamma^* = \arg\min_{C\in\Gamma} \gamma(C)$.

3 Overview of the Results

In this section, we state the main results of the paper assuming HDM, and then the indeterminacy results under HIM in order to contrast them with our main results. We also provide sketches for some of our claims.

First, some additional notation. A set theoretic generalisation of strict Nash equilibrium is the concept of minimum CURB set (Basu and Weibull 1990). Here, we define another generalisation - minimum weak CURB set (Klimm and Weibull 2009). Fix any n and $Y_n \subseteq A_n$ and let $Y = \times_n Y_n$. Then $\widetilde{B}_n(Y) = \{a_n \in A_n \mid a_n = B_n(a') \text{ for some } a' \in Y\}$ is the set of best replies to Y for player n; let $\widetilde{B}(Y) = [\widetilde{B}_1(Y) \times ... \times \widetilde{B}_N(Y)]$. Then Y is a weak CURB set (WCURB) if $\widetilde{B}(Y) \subseteq Y$. A set is a minimum WCURB (MWCURB) if it is WCURB and contains no smaller WCURB sets. Let W be the set of all the MWCURB sets in G. Clearly, a MWCURB set always exists and any such set is singleton only if it is a (strict) Nash equilibrium.[23]

For any $Y \subseteq A$ let $u(Y) = \min_{n\in N \text{ and } a\in Y} \pi_n(a)$. We call any $\overline{a} \in A$ that satisfies $\overline{a} \in \arg\max_{a\in A} u(a)$ as a MaxMin norm and $\overline{u} = \max_{a\in A} u(a)$ as the MaxMin norm payoff. Similarly, we define any equilibrium $\overline{e}$ and any MWCURB $\overline{Q}$ that satisfy $\overline{e} \in \arg\max_{e\in E} u(e)$ and $\overline{Q} \in \arg\max_{Q\in W} u(Q)$ as a MaxMin equilibrium norm and MaxMin MWCURB norm, respectively. Also, define $\overline{u}_E = \max_{e\in E} u(e)$ and $\overline{u}_W = \max_{Q\in W} u(Q)$ as the MaxMin equilibrium norm payoff and the MaxMin MWCURB norm payoff, respectively. These MaxMin terms can be trivially shown to satisfy the following.

Remark: *(i) Any MaxMin norm $\overline{a}$ is efficient. Furthermore, if $a \in A$ is efficient and*

[22]This follows from Theorem 4 in Young (1993) and that, by the definition of $\widehat{res}(.,.)$, , $\widehat{res}(\omega,\omega') \geq 0$ and $0 < \lim_{\epsilon\to 0} \frac{\Lambda^\epsilon(\omega)[\omega']}{\epsilon^{\widehat{res}(\omega,\omega')}} < \infty$, for any two states ω and $\omega' \in \Omega$.

[23]MWCURB set is a refinement of the concept of minimum CURB set, as any minimum CURB set contains a MWCURB set.

$\pi_n(a) = \pi_{n'}(a)$ *for all* n, $n' \in N$ *then* a *is the unique MaxMin norm.*

(ii) Any MaxMin equilibrium norm $\overline{e}$ *is efficient amongst the set of equilibria. Furthermore, if* $e \in E$ *is efficient amongst the set of Nash equilibria and* $\pi_n(e) = \pi_{n'}(e)$ *for all* n, $n' \in N$ *then* e *is the unique MaxMin equilibrium norm.*

For any set of states $\Omega' \subseteq \Omega$, with some abuse of notation, we define the minimum payoff any player can obtain in all elements of Ω' by $u(\Omega') = \min_{\omega \in \Omega'} u(A(\omega))$.

Main Results - HDM. Our results depend on what rules are feasible. As argued above, it may be reasonable to assume that only deterministic rules are feasible. Or it may be argued, on the basis of some rationality reasoning, that at some histories rules must best reply; e.g. requiring rules to best reply at all equilibrium histories may be a reasonable assumption. Alternatively, bounded cognitive reasoning suggests that agents choose rules that are not too complex.

Our first main result shows that under HDM if R is sufficiently rich to include some pure equilibrium rule or some set of low complexity rules then the minimum payoff any agent can obtain in any SS recurrent class is the MaxMin norm payoff $\overline{u}$.

Theorem 2 *Assume HDM. There exists* $r \in R^E \cap \overline{R}^{pure}$ *and* $R' \subseteq S^2$ *such that if either (i)* $r \in R$ *or (ii)* $R' \subset R$ *then* $u(C) = \overline{u}$ *for all* $C \in \Gamma^*$.

As shown in Section 4, the choice r or R' in Theorem 2 is not unique; the only requirement for the result is that any one of them is feasible (R could of course contain any other rule including non-pure, non-equilibrium or non-2-complexity ones). For any $C \in \Gamma^*$, the claim in Theorem 2 that $u(C) = \overline{u}$ is equivalent to saying that every $a \in A(C)$ is a MaxMin norm. So the theorem is a very strong selection result as it predicts that in the long-run only MaxMin norms are selected. Hence, we have:

Corollary 3 *Assume HDM. If* R *includes either all pure equilibrium rules or all 2-complexity rules then every action played in any* $C \in \Gamma^*$ *must be a MaxMin norm.*[24]

It may be argued that neither all pure equilibrium rules nor all 2-complexity rules are reasonable because these rules do not always satisfy other properties such as match-neutrality or best reply at uniform histories. In Section 4, we show that if either $M > 2$ or $T > 1$ then r or R' in Theorem 2 can be chosen so that they satisfy match-neutrality.

The choice of r or R' in Theorem 2, however, may not satisfy uniform best reply property if MaxMin norms are not equilibria. For example, suppose MaxMin norm $\overline{a}$ is

[24] Note that both $R^E \cap \overline{R}^{pure}$ and S^2 consist of pure rules and hence are finite.

unique; then r and R' we construct in Theorem 2 stipulate playing $\overline{a}$ at $\theta(\overline{a})$; but this violates uniform best reply if $\overline{a} \notin E$. Our second main result provides a characterisation of SS under HDM that is consistent with limiting the set of feasible rules to uniform best reply ones. Specifically, we show that with HDM, under some weak conditions, the minimum payoff any agent can obtain in any $C \in \Gamma^*$ is bounded below by the MaxMin MWCURB norm payoff $\overline{u}_W$; also, this lower bound binds if every $r \in R$ best replies at every uniform history wpp.

Theorem 4 *Suppose HDM. (i) If $N = 2$ and $R \cap R^s \neq \emptyset$ then $u(C) \geq \overline{u}_W$, for all $C \in \Gamma^*$. (ii) There exists $r \in R^u \cap \overline{R}^{pure}$ such that if $r \in R$ then $u(C) \geq \overline{u}_W$, for all $C \in \Gamma^*$. (iii) If $R \subseteq \{r \in \overline{R} \mid r(\theta(a))[B(a)] > 0$, for all $a \in A\}$ then, in both (i) and (ii), $u(C) = \overline{u}_W$ for all $C \in \Gamma^*$.*[25]

The assumption needed to ensure that $u(C) \geq \overline{u}_W$ for all $C \in \Gamma^*$ in Theorem 4 is particularly minimal for the case with $N = 2$ as it only requires R to contain *at least one* sampling best reply rule profile. For arbitrary number of players the assumption needed is still weak because there are no restrictions on R other than feasibility of a specific uniform best response rule. For games in which every MWCURB norm is singleton the above characterisation will be in terms of MaxMin equilibrium norm.

Corollary 5 *Assume HDM and either $N = 2$ and $R \cap R^s \neq \emptyset$ or $R^u \cap \overline{R}^{pure} \subseteq R$. Then in any $C \in \Gamma^*$ the minimum payoff any agent receives is bounded below by $\overline{u}_W$, and the bound binds if $R \subseteq \{r \in \overline{R} \mid r(\theta(a))[B(a)] > 0$, for all $a \in A\}$. Also, if in addition every MWCURB set is singleton, then every action played in any $C \in \Gamma^*$ is a MaxMin equilibrium norm.*

Results with HIM. We next characterise the set Γ^* assuming HIM under similar conditions as in our two main results. Our first claim is the analogue of Theorem 2:

Theorem 6 *Assume HIM. Fix any $a \in A$. There exists $r \in R^E \cap \overline{R}^{pure}$ and $R' \subset S^2$ such that if either (i) $r \in R$ or (ii) $R' \subset R$ then $A(C) = a$ for some $C \in \Gamma^*$.*

Thus, with HIM we can show the following total indeterminacy result.

Corollary 7 *Assume HIM. Every action profile will be played in some SS state if R contains either all pure equilibrium rules or all 2-complexity rules.*

[25] We assumed that either $S \subseteq R$ or $S^{mixed} \cap R \neq \emptyset$; hence, condition $R \subseteq \{r \in \overline{R} \mid r(\theta(a))[B(a)] > 0$ for all $a \in A\}$ imply that the latter must hold.

One way of obtaining some selection under HIM would be to restrict the set of feasible rules. However, even if R is restricted to uniform or samply best reply rules, indeterminacy with respect to the set of MWCURB may still persist.

Theorem 8 *Assume HIM and fix any $Q \in W$. (i) If $N = 2$ and $R \cap R^s \neq \emptyset$ then there exists $C \in \Gamma^*$ such that $A(C) = Q$. (ii) There exists $r \in R^u$ such that if $r \in R$ then there exists $C \in \Gamma^*$ such that $A(C) = Q$.*

Corollary 9 *Assume HIM. If either $N = 2$ and $R \cap R^s \neq \emptyset$ or $R^u \cap \overline{R}^{pure} \subseteq R$, then every $Q \in W$ can be sustained in an SS recurrent class.*

Note the contrast between the results under HIM with those under HDM. In the former, the less restrictions we impose on the set of rules the larger the indeterminacy, whereas with HDM the less restrictions are imposed the stronger the selection result.

Sketches for some of the ideas of the proofs. First, some additional notation. For any $m \in M$, a' and $a'' \in A$, let $\overline{\phi}(m, a'' \mid a') \in A^M$ be the 1-period outcome that involves playing a'' in match m and playing a' in every other match and $\overline{\theta}(m, a'' \mid a') \in \Theta^s$ be the stationary history consisting of playing $\overline{\phi}(m, a'' \mid a')$ in each period. We refer to any $\overline{\theta}(m, a'' \mid a')$ such that a'' differs from a' in at most by one component (i.e. $a''_{-n} = a'_{-n}$ for some n) by stationary 1-deviation history. For any $m \in M$, $n \in N$ and $a'_{-n} \in A_{-n}$, let $\hat{\Theta}^{-n}(m \mid a'_{-n}) = \{\theta \in \Theta \mid \forall t, \exists\ b^t_n$ and $c^t_n \in A_n$ s.t. $\theta^t = \overline{\phi}(m, (b^t_n, a'_{-n}) \mid c^t_n, a'_{-n})\}$. Any $\theta \in \hat{\Theta}^{-n}(m \mid a'_{-n})$ is called a 1-deviation history. Clearly, $\hat{\Theta}^{-n}(m \mid a'_{-n}) \cap \Theta^s = \cup_{a'_n, a''_n} \overline{\theta}(m, (a''_n, a'_{-n}) \mid a'_n, a'_{-n})$.

Two properties of rules are critical for our results: (a) *1-mutation from $a \in A$* and (b) *invading.* The former refers to rules that if adopted by all agents then with at most one mutation the system reaches a recurrent class in which all play a in every match. The latter are rules that if introduced through one mutation in each role, they will eventually be adopted by all agents wpp. More formally, $r_n \in R_n$ is invading at $C' \in \Gamma$ if starting from C' with at most one mutation by any agent the system reaches wpp some $C \in \Gamma$ such that $R(C) = (r_n, R_{-n}(C'))$; also $r \in R$ is invading if r_n is invading at all $C' \in \Gamma$, for all $n \in N$. The selection results in Theorems 2 and 4 (for the case when every $Q \in W$ is singleton, i.e. $W = E$) is based on the following:

Claim1. Under HDM, for any $a \in A$, $u(C^0) \geq u(a)$ for all $C^0 \in \Gamma^*$ *if there exists $r \in R$ that is 1-mutation from a and invading. Hence, $u(C) = \overline{u}$* for all $C \in \Gamma^*$ if *there exists $r \in R$ that is 1-mutation from $\overline{a}$*, for some MaxMin norm $\overline{a}$, and invading.

Given that $u(\overline{a}) \geq u(a)$ for any MaxMin norm $\overline{a}$ and any $a \in A$, the second part of Claim 1 follows immediately from the first part. The assumption that there exists $r \in R$ that is 1-mutation from a and invading ensures that starting from any $C^0 \in \Gamma$ with at most $N+1$ sequential mutations (one for each role and one to play a), by agents that receive the lowest payoffs in the recurrent classes at which mutation takes place, the system reaches some $C \in \Gamma$ in which all agents play a. Given that mutation is payoff dependent, this implies that, for any C^0-tree, if $u(C^0) < u(a)$ then there exists a C-tree with a lower resistance than that C^0-tree; hence for all $C^0 \in \Gamma^*$ it must be that $u(C^0) \geq u(a)$. The C-tree is obtained by adding and deleting edges to/from the C^0-tree. Figure 1 illustrates the argument when $N = 2$ and $R_n(C^0) \neq r_n$ for all n.

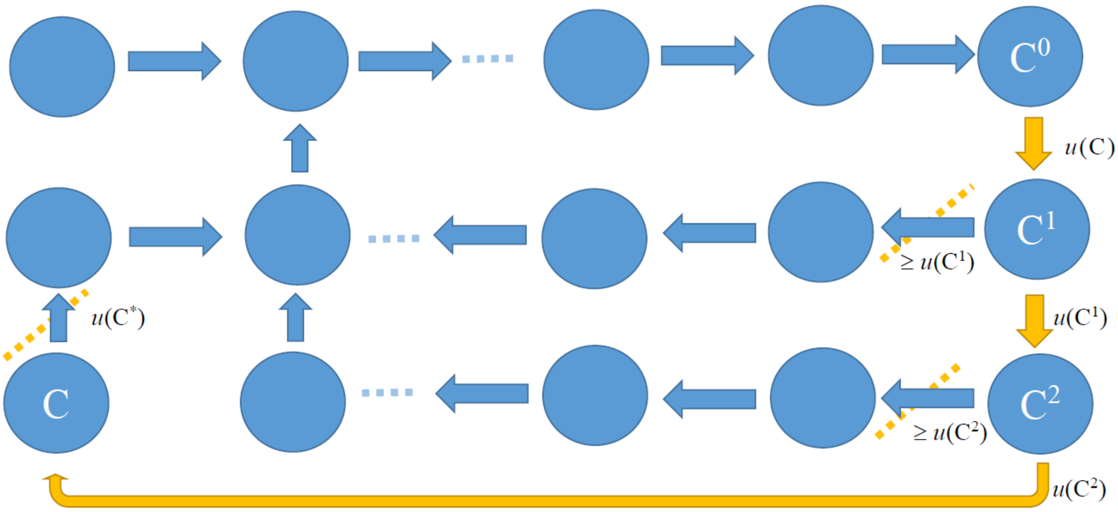

Figure 1. Suppose $C^0 \in \Gamma^*$ and $u(C^0) < u(a)$. The blue circles and the blue arrows represent any C^0-tree. Since r is invading there exists C^1 and $C^2 \in \Gamma$ such that $R(C^2) = r$ and, for each $\ell = 1,2$, one mutation by an agent who obtains $u(C^{\ell-1})$ at $C^{\ell-1}$ moves the system from $C^{\ell-1}$ to C^ℓ. Also, since r is 1-mutation from a, if $A(C^2) \neq a$ then one mutation by an agent that obtains $u(C^2)$ at C^2 ensures that system reaches some C s.t. $A(C) = a$ from C^2. Given that $u(C^0) < u(a)$, if one adds the three yellow edges (C^0, C^1), (C^1, C^2) and (C^2, C) to C^0-tree and delete the three edges in the C^0-tree starting from C^1, C^2 and C then one is left with a C-tree with a lower resistance than that of C^0 (as the three yellow edges (C^0, C^1), (C^1, C^2) and (C^2, C) respectively require one mutation by an agent that obtains a payoff $u(C^0)$, $u(C^1)$ and $u(C^2)$, whereas the three edges in the C^0-tree starting from C^1, C^2 and C respectively require at least one mutation by an agent with a payoff no less than $u(C^1)$, $u(C^2)$ and $u(a)$). But this is a contradiction.

Given Claim 1, to demonstrate the selection results with HDM we need to show the existence of a rule that is both 1-mutation from a and invading. First, consider rules that are 1-mutation from a. Effectively, these rules can be triggered to play a in finite time after receiving an appropriate signal through one mutation. There are many ways to construct such rules. For example, when $N = 2$ and a is an equilibrium, any sampling best reply rule $r \in R^s$ is 1-mutation from a. To see this suppose the system is at some $\omega \in \Omega$ at date t such that $R(\omega) = r$ and $A(\omega) \neq a$. Assume there is a mutation and the mutating agent is assigned role n and the mutation is to a simple rule that plays a_n. Then, at $t+1$ all agents in role $k \neq n$ wpp sample the mutant's last period action a_n

and play the best response a_k to it. If at $t+2$ all agents in role k are subject to action inertia and all in role n implement their rules, then a will be played by all wpp (all in role k play a_k by action inertia, the simple rule does a_n by assumption and r_n also does a_n as it is the best reply to a_k). But then, by $r \in R^s$ and $a \in B(a)$, wpp all will play a indefinitely.

When $a \notin E$ or $N > 2$, the sampling best reply rule $r \in R^s$ may not behave as above after mutation to a simple rule that plays a_n, and hence, it may not be 1-mutation from a. Nevertheless, any rule r that behaves in a similar fashion as that described above is. Specifically, for 1-mutation from a in the general case, we need (i) starting at any state ω in which all use r with (at most) one mutation the system reaches a *state* with history $\theta(a)$ and (ii) once $\theta(a)$ is reached, a will be played henceforth. The latter follows if

$$r(\theta(a))[a] = 1. \tag{8}$$

For the former, suppose that $r(\theta[\omega])(a') > 0$ for some a' such that $a'_n \neq a_n$ for some n. Then one mutation to a simple rule that plays a_n in role n at ω moves the system wpp to some state with history θ such that $\theta^T = \overline{\phi}(m, (a_n, a'_{-n}) \mid a')$ for some m. Then a sufficient condition for (i) is $r(\theta)[a] > 0$, as it ensures that after the mutation wpp all will play a and hence, by action inertia the system reaches a state with history $\theta(a)$. However, $r(\theta)[a] > 0$ at all θ with $\theta^T = \overline{\phi}(m, (a_n, a'_{-n}) \mid a')$ is not necessary for (i). In the proof we use a weaker condition. First, we require only that the triggering starts at stationary history in which $\overline{\phi}(m, (a_n, a'_{-n}) \mid a')$ happens at every date, i.e. at $\overline{\theta}(m, (a_n, a'_{-n}) \mid a')$. Second, we allow for the possibility that the triggering happens in two stages: first agents in roles other than n play a_{-n} and then agents in role n play a_n (this 2-stage triggering allows for richer behaviour such as the sampling best reply described above). One condition we use in the proof for triggering[26] is

$$\forall n \text{ and } a'_{-n} \in A_{-n}, \exists\, m \text{ s.t. for all } \theta \in \cup_{a'_n \in A_n} \overline{\theta}(m, (a_n, a'_{-n}) \mid a') \tag{9}$$
$$\text{(a) } r_{-n}(\theta)[a_{-n}] > 0, \text{ if } a'_{-n} \neq a_{-n} \text{ and } a'_n \neq a_n \text{ and (b) } r_n(\theta)[a_n] > 0, \text{ if } a'_{-n} = a_{-n}.$$

Rules that are invading must have some appropriate best reply properties. For example, a set of sufficient conditions for r to be invading is (8) and playing best reply

[26]This condition is not necessary for ensuring triggering. For example, instead suppose we assume the rule plays a at every non-stationary history; then starting from a stationary history, one mutation can start the triggering by inducing a non-stationary history and thereby resulting in all choosing a (see (15b) below).

at all $\theta \neq \theta(a)$ that are 1-deviation for some match:

$$\forall n \text{ and } a'_{-n} \in A_{-n},\ \exists\ m' \text{ s.t. } r_n(\theta)[B_n(a'_{-n})] > 0,\ \forall \theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n}) \backslash \theta(a) \tag{10}$$

The basic idea why this guarantees that r_n is invading involves three steps: First, starting from any state action inertia ensures that agents in roles other than n can play some action profile a'_{-n} persistently. Second, for any m', one mutation by any agent to r_n can induce some 1-deviation history $\theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n})$. Third, the best reply property at 1-deviation histories described in (10), together with (8), ensure that r_n does at least as well as $R_n(C')$ wpp (assuming that all agents persist with playing a'_{-n}), and thereby, using Monotonicity, results in r_n being adopted by all in role n.

From the above any r that satisfies (8), (9) and (10) is invading and 1-mutation from a. Hence, (i) of Theorem 2 follows immediately from Claim 1 if the set of rules that satisfy (8), (9) and (10) contains a pure equilibrium rule. This is trivially the case if $a \in E$. In fact, in this case every pure rule in the set is a uniform best reply and the set contains all sampling best reply rules when $N = 2$. Hence, the claim in (i) and (ii) of Theorem 4 that $u(C) \geq \overline{u}_W = \overline{u}_E$ for all $C \in \Gamma^*$ must hold when $W = E$.[27]

When $a \notin E$ there also exists a pure equilibrium rule that satisfy (8), (9) and (10); however, in this case, such a rule may not be match-neutral. Such match sensitivity is however not necessary for our selection result, as neither (9) is necessary for 1-mutation from a nor (10) is necessary for invading. We also show the existence of two alternative sets of 1-mutation from a and invading rules that are match-neutral. In one alternative (described in the next section) we preserve (10) and replace (9b) by one that requires the rules to play a at histories that are neither stationary nor 1-deviation. When $T > 1$, this modification ensures 1-mutation from a without excluding best reply behaviour at every 1-deviation history. A pure and match-neutral example of such modification in any role n is a rule that, for any a'_{-n}, play $B_n(a'_{-n})$ at any history θ other than $\theta(a)$ in which all agents in roles other than n play a'_{-n} (i.e. at every $\theta \in \cup_m \hat{\Theta}^{-n}(a'_{-n}) \backslash \theta(a)$) and play a_n otherwise; see (16) below.[28] Another approach (considered in Online Appendix) preserves triggering property (9b) and modifies (10).

To establish (ii) of Theorem 2, we extend Claim 1 to sets of rules; specifically,

[27] The inequality $u(C) \geq \overline{u}_W$ clearly binds if, for every $C \in \Gamma$, $A(C) \cap W \neq \emptyset$. We show that the latter holds if $R \subseteq \{r \in \overline{R} \mid r(\theta(a))[B(a)] > 0 \text{ for all } a \in A\}$, thereby establishing (iii) of Theorems 4.

[28] In the Prisoners' Dilemma game, this example says that for any n: (i) Defect if there has been a deviation from Cooperation in the past and all agents in the role of player $k \neq n$ either always chose Defect or always chose Cooperate in the past, and (ii) Cooperate, otherwise.

$u(C) \geq u(a)$ for all $C \in \Gamma^*$ if there exists $R' \subseteq R$ such that each $r \in R'$ is 1-mutation from a and at every $C' \in \Gamma$ there exists $r \in R'$ that is invading at C'. An example of such R' is the set of all 2-complexity rules such that: (a) every $r \in R'$ satisfies (8) and (9) and (b) for each a', R' contains a rule that does $B(a')$ at every 1-deviation history $\theta \neq \theta(a)$; conditions (a) and (b) ensure respectively that each $r \in R'$ is 1-mutation from a and at each $C' \in \Gamma$ there exists $r \in R'$ that is invading at C'. The existence of such R' then establishes (ii) of Theorem 2.

The indeterminacy results under HIM in Theorems 6 and 8 (when every $Q \in W$ is singleton, i.e. $W = E$) are based on the following claim.

Claim 2. Under HIM, for any $a \in A$, there exists $C \in \Gamma^*$ *such that* $A(C) = a$ *if there exists* $r \in R$ *that is both 1-mutation from* a and *invading.*

The proof of Claim 2 is simpler than that of Claim 1: it involves showing that for any $C^0 \in \Gamma^*$ and for any C^0-tree, there exists $C \in \Gamma$, with $A(C) = a$, and a new tree with root C that has the same resistance as C^0-tree. The new tree is obtained in by adding and deleting equal number of edges to/from the C^0-tree - the existence of a feasible rule that is both 1-mutation from a and invading makes this exercise feasible.

Part (i) of Theorem 6 and Theorem 8 (when Q is singleton) then follow from Claim 2, given that every pure rule that satisfies (8), (9) and (10) is 1-mutation from and invading, and the set of such rules (a) contains an equilibrium rule and (b) is a subset of R^u and contains R^s if $a \in E$.[29] Part (ii) of Theorems 6 follows by extending Claim 2 to sets of rules.

4 Proofs of the Theorems 2, 4, 6 and 8

Proofs of Theorems 2 and 4. We start with two definitions and a critical lemma.

Definition 10 *Any* $R' \subseteq R$ *is 1-mutation from* $Q \subseteq A$ *if for any* $C' \in \Gamma$ *such that* $A(C') \subsetneq Q$ *and* $R(C') \in R'$*, and any* $\omega \in C'$*, one mutation by any agent at* ω *moves the system from* ω *wpp to some* $C \in \Gamma$ *such that* $A(C) \subseteq Q$ *and* $R(C) \in R'$*.*[30]

[29]The existence of $r \in R$ that is both 1-mutation from a and invading is more than what is needed to establish Claim 2. Hence, our results under HIM can be obtained under a weaker condition than the existence of a feasible rule that satisfies (8), (9) and (10); see the discussion at the end of the next section.

[30]Throughout, the statement at state $\omega \in \Omega$ refers to any situation in which the system is at some $h \in H(\omega)$. Similarly, the statement one mutation by an agent at ω moves the system from ω wpp to some set of states $\Omega' \subseteq \Omega$ refers to any situation in which the system is at some $h \in H(\omega)$ and then a single mutation by the agent moves the system from h wpp to some $h' \in H(\omega')$ for some $\omega' \in \Omega'$.

Definition 11 *Any $R'_n \subseteq R_n$ in role n is invading at $C' \in \Gamma$ if, whenever $R'_n \cap R_n(C') = \emptyset$, for any $\omega \in C'$ there exists $r_n \in R'_n$ such that one mutation by any agent at ω moves the system wpp from ω to some $C \in \Gamma$ with $R(C) \in (r_n, R_{-n}(C'))$. Any $R' \subseteq R$ is invading if R'_n is invading at all $C' \in \Gamma$, for all n.*

Lemma 12 *Fix any $Q \subseteq A$ and $C^0 \in \Gamma^*$. Suppose there exists $R' \subseteq R$ such that R' is 1-mutation from Q and invading. Then $u(C^0) \geq u(Q)$.*[31]

Next, we define three properties that ensure that a rule is 1-mutation from Q and invading. To do so, for any $n \in N$ and $a'_{-n} \in A_{-n}$, define $B_n(a'_{-n} \mid Q) = \arg\max_{a_n \in Q_n} \pi_n(a_n, a'_{-n})$ as the Q-constrained best reply to a'_{-n}. Denote the set of uniform histories with actions in Q by $\Theta^u(Q) \equiv \{\theta' \mid \theta' = \theta(a) \text{ for some } a \in Q\}$. The first property, henceforth called Q-constrained property, requires playing actions in Q if actions observed belong to Q:

$$r(\theta)[Q] = 1 \;\forall \theta \text{ s.t. } A(\theta) \subseteq Q. \tag{11}$$

Condition (11) reduces to (8) when Q is equal to some $a \in A$. The second property is:

$$\begin{gathered}\forall n \text{ and } a'_{-n} \in A_{-n},\ \exists\, m \text{ s.t. for all } \theta \in \cup_{a_n \in Q_n, a'_n \in A_n} \overline{\theta}(m, (a_n, a'_{-n}) \mid a'),\\ \text{(a) } r_{-n}(\theta)[Q_{-n}] > 0, \text{ if } a'_{-n} \notin Q_{-n} \text{ and } a'_n \notin Q_n\text{; and}\\ \text{(b) } r_n(\theta)[B_n(a'_{-n} \mid Q)] > 0, \text{ if } a'_{-n} \in Q_{-n}.\end{gathered} \tag{12}$$

This property, henceforth called Q-triggering, says that, for any n and a'_{-n}, there exists m such that at any stationary 1-deviation history $\theta \in \cup_{a_n \in Q_n, a'_n \in A_n} \overline{\theta}(m, (a_n, a'_{-n}) \mid a')$ the rule does with wpp (a) some action in Q_{-n} in roles other than n if $a'_{-n} \notin Q_{-n}$ and $a'_n \notin Q_n$ and (b) the Q-constrained best reply to a'_{-n} in role n if $a'_{-n} \in Q_{-n}$.[32] Condition (12) reduces to (9) when Q is equal to some $a \in A$. The third property, henceforth called Q-best, requires playing best reply at all $\theta \notin \Theta^u(Q)$ that are 1-deviation for some match:

$$\forall n \text{ and } a'_{-n} \in A_{-n},\ \exists\, m' \text{ s.t. } r_n(\theta)[B_n(a'_{-n})] > 0,\ \forall \theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n}) \backslash \Theta^u(Q). \tag{13}$$

Condition (13) reduces to (10) when Q is equal to some $a \in A$.

[31] The concept of a set $R'_n \subseteq R$ being invading at $C' \in \Gamma$ requires one mutation by *any* agent to result in invasion in the sense that wpp it result in all agents in role n to adopt some $r_n \in R'_n$. Lemma 12 also holds with a weaker concept of invasion that requires one mutation by any agent that receives the lowest payoff in C' to induce invasion. In the paper, we adopt the stronger concept of invading to simplify the exposition. In Subsection 2.2 of the Online Appendix, we use the weaker concept to broaden the scope of our selection of MaxMin norm result.

[32] The definition of Q-triggering in (12) can be weakend somewhat by changing the order of the quantifiers. To simplify the exposition, we adopt the definition in (12).

Lemma 13 *Fix any $R' \subseteq R$ and $Q \subseteq A$. Suppose every $r \in R'$ satisfies (11) and (12). Then R' is 1-mutation from Q.*

Lemma 14 *Fix any Q and $r \in R$ such that (11) and (13) hold. Then r is invading if either (i) $Q \in W$ and r satisfies (12b) or (ii) Q is singleton.*

We refer to any r that satisfies (11), (12) and (13) by a Q-rule and denote the set of Q-rules by $R_Q = \{r \in \overline{R} \,|\, r \text{ satisfies (11), (12) and (13)}\}$. Then, by Lemmas 13 and 14, any $r \in R_Q$ is both 1-mutation from Q and invading if either $Q \in W$ or Q is singleton. Hence, by Lemma 12 (setting R' to equal r) we have:

Proposition 15 *Assume HDM. Fix any Q such that either $Q \in W$ or Q is singleton. Suppose $R \cap R_Q \neq \emptyset$. Then $u(C) \geq u(Q)$ for any $C \in \Gamma^*$.*

Assuming $R \cap R_Q \neq \emptyset$ is not too restrictive. First, the three conditions (11), (12) and (13) are restrictions at 1-deviation histories and at histories in which only actions in Q has been played; they do not put any restrictions at any other type of histories.

Second, consider the case when $Q \in W$. Then $B(a') \in Q$ for any $a' \in Q$. Hence, in this case, $r \in R_Q$ if and only if it satisfies (11), (12a) and

$$\forall n \text{ and } a'_{-n} \in A_{-n},\ \exists\ m' \text{ s.t. } r_n(\theta)[B_n(a'_{-n})] > 0,\ \forall \theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n}) \tag{14}$$

(this is because when $Q \in W$, (14) is equivalent to (13) and (12b)). Clearly, (11), (12a) and (14) allow for pure rules and are consistent with each other.[33] Also, (14) implies best reply wpp at any $\theta \in \Theta^u$; hence, $R_Q \cap \overline{R}^{pure}$ is a non-empty subset of R^u when $Q \in W$. Also, with two players R_Q includes all sampling best reply rules if $Q \in W$.

Lemma 16 *Assume $N = 2$. If $Q \in W$ then $R^s \subset R_Q$.*

The claims in (i) and (ii) of Theorem 4 that $u(C) \geq \overline{u}_W$ for any $C \in \Gamma^*$ follows from Lemma 16, Proposition 15 and $R_Q \cap \overline{R}^{pure}$ being a non-empty subset of R^u when $Q \in W$. The claim in (iii) of Theorem 4 that, for any $C \in \Gamma^*$, $u(C) = \overline{u}_W$ if $R \subseteq \{r \in \overline{R} \mid r(\theta(a))[B(a)] > 0, \text{ for all } a \in A\}$ then follows from the next lemma.

Lemma 17 *Fix any $C \in \Gamma$. Assume $R(C)(\theta(a))[B(a)] > 0$ for all $a \in A$. Then there exists $Q \in W$ such that $Q \subseteq A(C)$; hence $u(C) \leq \overline{u}_W$.*

[33] A match-neutral examples of the set $R_Q \cap \overline{R}^{pure}$ when $Q \in W$ is a rule r such that, for any n, $r_n(\theta)[Q_n] = 1$ if $\theta \in \cup_{m,k \neq n, a', a_k \neq a'_k} \overline{\theta}(m, (a_k, a'_{-k}) \mid a')$ and $r_n(\theta)[B_n(\sigma)] = 1$ for some $\sigma \in \Delta(\cup_{m,t} \theta^{m,t})$ otherwise, where $B_n(\sigma)$ is the best reply to σ. Another example, is rule r such that, for any n, $r_n(\theta)[B_n(a'_{-n})] = 1$ if $\theta \in \cup_m \hat{\Theta}(m \mid a'_{-n})$ for all a'_{-n} and $r_n(\theta)[Q_n] = 1$ otherwise.

Third, consider the case when Q is singleton and equal to some arbitrary $a \in A$. Then (8), (9) and (10) respectively are equivalent to (11), (12) and (13), and hence describe a-constrained, a-triggering and a-best properties. Thus, the set of a-rules is given by $R_a = \{r \in \overline{R} \,|\, r \text{ satisfies (8), (9) and (10)}\}$. Note that any pure $r \in R_a$, by (8), plays $B(a)$ at $\theta(a)$ if and only if $a \in E$ and, by (13), plays e at $\theta(e)$ if $e \in E \backslash a$. Hence, although $R_a \cap \overline{R}^{pure}$ does not belong to R^u if $a \notin E$, $R_a \cap \overline{R}^{pure} \subset R^E$ for all $a \in A$.

Also, R_a is well-defined and contains pure rules. This is because (8), (9) and (10) allow pure rules and are consistent with each other. Specifically, (8), (9a) and (10) are consistent with each other because they apply at different histories. Condition (8) is consistent with (9b) because both allow a_n to be played wpp. Finally, consider (9b) and (10); the former requires, for some m, playing a_n at stationary 1-deviation history $\overline{\theta}(m, a \mid a'_n, a_{-n})$ whereas the latter requires, for some m', playing $B_n(a'_{-n})$ at any $\theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n}) \backslash \theta(a)$. Clearly, the two conditions are also consistent if (a) $a_{-n} \neq a'_{-n}$, (b) $a_n = B_n(a_{-n})$ or (c) m' and m are distinct. Given $M > 1$, (c) is always possible, and hence (9b) and (10) are consistent. However, when $a_n \neq B_n(a)$ and the rule is deterministic, such consistency is possible if and only if, for all a'_n, r_n plays different actions at $\overline{\theta}(m, a \mid a'_n, a_{-n})$ and $\overline{\theta}(m', a \mid a'_n, a_{-n})$ for some distinct pair m and m'; thus consistency with both (9b) and (10) may require match sensitive behaviour.[34]

Fix any $C \in \Gamma^*$ and MaxMin norm $\overline{a} \in A$. With HDM, by Proposition 15, $u(C) \geq u(\overline{a})$ if $R \cap R_{\overline{a}} \neq \emptyset$. Since $u(\overline{a}) \geq u(a)$ and $R_{\overline{a}} \cap \overline{R}^{pure}$ is a non-empty subset of R^E, there exists $r \in R^E$ such that if $r \in R$ then $u(C) = u(\overline{a})$. This establishes (i) of Theorem 2.

In Section 2 we mentioned that it may be appropriate to limit rules to match-neutral ones on the grounds that behaviour should not depend on payoff-irrelevant aspects of history. Although the set of pure a-rules includes match-neutral rules when $a \in E$, as we argued above, this may not be the case if $a \notin E$ because in this case (9b) can be inconsistent with (10). Our selection results, however, also hold with match-neutral rules. We can achieve this by modifying either (9) or (10) in the definition of R_a. Below, we consider modifying the former (as the analysis is simpler) and consider the latter in

[34]Such sensitivity of course is not necessary if $a_n = B_n(a)$ or if the rule is mixed and plays both a_n or $B_n(a)$ with a fixed positive probability at every 1-deviation history $\overline{\theta}(m', a \mid a'_n, a_{-n})$ for all m' and a'_n. An example of of a pure a-rule is any r that, for some m and any θ does the following : play a at if $\theta \in \theta(a) \cup \{\cup_{k, a'_k \neq a_k, a'_{-k}} \overline{\theta}(m, (a_k, a'_{-k}) \mid a')\}$, and play $B(\sigma)$ for some $\sigma \in \Delta(\cup_{m',t} \theta^{m',t})$, otherwise; the behaviour this example stipulates is match sensitive if $a \neq B(a)$.

Online Appendix. Specifically, here we replace a-triggering (9) by the following:

$$\text{(a) } \forall n,\ r_{-n}(\theta)[a_{-n}] > 0,\ \forall \theta \in \cup_{m,a',a''_n \neq a'_n} \overline{\theta}(m, (a''_n, a'_{-n}) \mid a') \text{ and}$$
$$\text{(b) } r(\theta)[a] > 0,\ \forall\ \theta \in \Theta \backslash \{\Theta^s \cup \Theta^-\}. \tag{15}$$

where $\Theta^- \equiv \cup_{n,a'_{-n}} \Theta^{-n}(a'_{-n})$ is the set of histories in which all agents in $N-1$ roles take the same action. Condition (15a) is an a-triggering condition similar to (9a) for roles other than n. Condition (15b) requires playing a wpp at any non-stationary history $\theta \notin \Theta^-$; so if such a history is reached this condition triggers a. We refer to any rule that satisfies (8), (10) and (15) by a-plus rule and denote the set of such rules by R_a^+.

Note the following. First, since Θ^s is empty when $T = 1$, (15b) imposes restrictions only if $T > 1$. Second, by (8) and (10), $R_a^+ \cap \overline{R}^{pure} \subset R^E$. Third, (15) admits pure rules and is consistent with (8) and (10) as the set of histories at which (15) imposes restrictions is different from those at which (8) and (10) apply restrictions to. Furthermore, R_a^+ in contrast to R_a always contains pure rules that are match-neutral. For example, R_a^+ includes the pure and match-neutral rule r defined by: for any n and θ

$$r_n(\theta)[B_n(a'_{-n})] = 1 \text{ if } \theta \in \Theta^{-n}(a'_{-n}) \backslash \theta(a) \text{ for some } a'_{-n} \text{ and } r_n(\theta)[a_n] = 1 \text{ otherwise.} \tag{16}$$

Next fix any $r \in R_a^+ \cap R$. In Online Appendix (Lemma 39), we show r is 1-mutation from a if $T > 1$. Also, given that r satisfies (11) and (13) for the case when $Q = a$, by Lemma 14 (setting Q equal to a), r is invading. Hence, by Lemma 12 (setting Q to equal a) we have:

Proposition 18 *Assume HDM. Fix any $a \in A$ and suppose $R \cap R_a^+ \neq \emptyset$. If $T > 1$ then $u(C) \geq u(a)$ for any $C \in \Gamma^*$.*

The inequality in Proposition 18 binds if a is set equal to some MaxMin norm $\overline{a}$. Hence, given that $R_{\overline{a}}^+ \cap \overline{R}^{pure}$ is a non-empty subset of R^E and includes match-neutral rules, there exists a pure match-neutral equilibrium rule $r \in R_{\overline{a}}^+$ such that if $r \in R$ and $T > 1$ then $u(C) = u(\overline{a})$ for any $C \in \Gamma^*$; thus, if $T > 1$ then the result in (i) of Theorem 2 holds even if R is restricted to match-neutral ones. However, neither $R \cap R_{\overline{a}}^+ \neq \emptyset$ nor $T > 1$ are necessary for selection of MaxMin norm result with match-neutral rules. As mentioned above, in Online Appendix, we construct another set of rules, called R_a^* (obtained by modifying (10) in the definition of R_a), that contain pure match-neutral equilibrium rules and is such that if $R \cap R_a^* \neq \emptyset$ then $u(C) \geq u(a)$ for any $C \in \Gamma^*$, with inequality binding if a is a MaxMin norm. These results are obtained for all values of T, including $T = 1$ (for the latter case M has to exceed 2).

Next we turn to the proof of part (ii) of Theorem 2. Fixing $a \in A$, we need R to include, for each $b \in A$, a 2-complexity rule $r \in S^2$ that satisfies

$$\forall n \text{ and } a'_{-n},\ \exists m \text{ s.t. } r_n(\theta)[b_n] = 1 \text{ for all } \theta \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \theta(a), \tag{17}$$

a-constrained (8) and a-triggering (9).[35] Formally, let

$$\mathcal{R}_a = \left\{ R' \subset S^2 \;\middle|\; \begin{array}{c} \forall r \in R',\ (8) \text{ and } (9) \text{ are satisfied and} \\ \forall b \in A,\ \exists\ r \in R' \text{ that satisfies (17).} \end{array} \right\} \tag{18}$$

In Online Appendix (Lemma 40), we show any $R' \in R_a$ such that $R' \subseteq R$ is 1-mutation from a and invading. Hence, by Lemma 12 (setting Q to equal a), we have:

Proposition 19 *Assume HDM. Fix any $a \in A$. Suppose there exists $R' \in \mathcal{R}_a$ such that $R' \subseteq R$. Then $u(C) \geq u(a)$ for every $C \in \Gamma^*$.*

Also, $\mathcal{R}_a$ is clearly well-defined.[36] Hence, Part (ii) of Theorem 2 follows from Proposition 19 for the case when a is set equal to a MaxMin norm.

As with the case of a-rules, the behaviour of rules belonging to any $R' \in R_a$ may also be match sensitive, as (17) and (9b) respectively require playing b_n at $\overline{\theta}(m, a \mid a'_n, a_{-n})$ and a_n at $\overline{\theta}(m', a \mid a'_n, a_{-n})$ for some m and m' and for all a'_n. Our selection result for 2-complexity rules, however, also holds with match-neutral rules. In Online Appendix (Proposition 44), we establish an equivalent result to Proposition 19, and thereby establish (ii) of Theorem 2, with match-neutral 2-complexity rules.

Proofs of Theorems 6 and 8. First, we define a weaker version of invading.

Definition 20 *Any $R'_n \subseteq R_n$ for role n is weakly invading (henceforth, w-invading) at $C' \in \Gamma$ if, whenever $R'_n \cap R_n(C') = \emptyset$, there exists $\omega \in C'$, $r_n \in R'_n$ and $i \in I$ such that one mutation by i at ω moves the system wpp from ω to some $C \in \Gamma$ with $R(C) \in (r_n, R_{-n}(C'))$. Any $R' \subseteq R$ is w-invading if R'_n is w-invading at each $C' \in \Gamma$ for all n.*

Lemma 21 *Assume HIM. Fix any $Q \subseteq A$. Suppose there exists $R' \subseteq R$ such that R' is 1-mutation from Q and w-invading. Then there exists $C \in \Gamma^*$ such that $A(C) \subseteq Q$.*

[35] Condition (17) is similar to (10) except that the latter requires playing a best response at all $\theta \in \Theta^{-n}(m \mid a'_{-n}) \backslash \theta(a)$, whereas (17) requires playing a fixed action b_n at all such histories.

[36] An example of a set of rules belonging to $\mathcal{R}_a$ is the set of rules $\cup_{b \in A} \tilde{r}^{a,b}$ where, for some fixed match m, $\tilde{r}^{a,b}(\theta)[b] = 1$ if $\theta \in \{\cup_{a'_{-n}} \hat{\Theta}^{-n}(m \mid a'_{-n})\} \backslash \theta(a)$ and $\tilde{r}(\theta)[a] = 1$ otherwise.

The above lemma enables us to state the following.

Proposition 22 *Assume HIM. (i) Fix any $a \in A$. Suppose either $R_a \cap R \neq \emptyset$ or there exists $R' \in \mathcal{R}_a$ such that $R' \subseteq R$. Then there exists $C \in \Gamma^*$ such that $A(C) = a$. (ii) Fix any $Q \in W$. Suppose $R_Q \cap R \neq \emptyset$. Then there exists $C \in \Gamma^*$ such that $A(C) = Q$.*

Given that any $r \in R_a$ and each $R' \in \mathcal{R}_a$ are 1-mutation from a and invading (and hence w-invading), part (i) of Proposition 22 follows from Lemma 21. Next, fix any $Q \in W$. Since any $r \in R_Q$ is 1-mutation from Q and invading (and hence w-invading), by Lemma 21, there exists $C \in \Gamma^*$ such that $A(C) \subseteq Q$. Also, by Lemma 17, there exists $Q' \in W$ such that $Q' \subseteq A(C)$; but since $A(C) \subseteq Q$ and MWCURB sets do not intersect, $Q = Q' = A(C)$. This establishes (ii) of Proposition 22.

Theorem 6 follows from (i) of Proposition 22, $R_a \cap \overline{R}^{pure}$ being a non-empty subset of R^E, $\mathcal{R}_a$ being well-defined and every $R' \in \mathcal{R}_a$ being a subset of S^2, for any $a \in A$. Theorem 8 follows from (ii) of Proposition 22, Lemma 16 and $R_Q \cap \overline{R}^{pure}$ being a non-empty subset of R^u, for any $Q \in W$.[37]

5 Extension

No Inertia. Action inertia allows for the possibility of persistence in actions in every role; this enables us to show that, for any role n, rules that do best reply at histories in which all in roles other than n take the same actions are invading. If there were no action inertia, agents in roles other than n do not necessarily take the same actions as in the past and hence rules that best reply to past behaviour of others may not be invading. However, action inertia may be dispensable if, for all n, R_n were sufficiently large so that, for each action profile that agents in roles other than n might take, there is a rule $r_n \in R_n$ that does a best reply to it (as then some member of R_n can invade).[38]

[37] As explained above, a pure a-rule is not match-neutral if $a \notin E$. Proposition 22, and hence Theorem 6, also hold if we assume R is pure and match-neutral; e.g. given that every feasible $r \in R_a^+ \cup R_a^*$ is 1-mutation from a and invading, the claim that in (i) of Proposition holds if $\{R_a^+ \cup R_a^*\} \cap R \neq \emptyset$.

Note also that since, for any Q, Lemma 21 requires the feasibility of some R' that is 1-mutation from Q and *w-invading* and Lemma 12 requires the feasibility of some R' that is 1-mutation from Q and *invading*, the scope of Proposition 22 is significantly wider than the claim in Proposition 15. For example, replace the a-best property (10) in the definition of R_a by the weaker condition of requiring best reply only at all $\theta \in \Theta^u \backslash \theta(a)$. This set contains R_a, includes pure match-neutral rules and can be shown to be 1-mutation from a and w-invading. Hence, by Lemma 21, if r contains one such rule then the claim in (i) of Proposition 22 holds. However, since this set is not necessarily invading, its feasibility is not sufficient to establish Proposition 15 for the case when $Q = a$.

[38] The argument is similar to the case of 2-complexity rules where we show that any $R' \in \mathcal{R}_a$ is invading.

Rule inertia allows for persistence in rules. Since at any date the history of previous T periods is relevant to Monotonicity selection criterion, rule inertia may be necessary for a single mutation to succeed if $T > 1$; without it a single mutation may die immediately after one period and hence our analysis would not carry if $T > 1$. However, rule inertia may be dispensable if $T = 1$ or more generally if we strengthen the Monotonicity selection criterion so that it depends only on the history of play in the last period.

Dispensing with action and rule inertia is indeed possible as we can show the following without assuming action inertia (with or without rule inertia): if R is sufficiently rich and the selection criterion is strengthened as suggested above then (a) with HDM only MaxMin norms survive and (b) with HIM any outcome survives. Specifically, we replace condition (2) in the Monotonicity Assumption by

$$\begin{cases} \forall n, \omega \in \Omega \text{ and } r_n \in R_n(\omega), \rho_n(\omega)[r_n] > 0 \text{ if } D_n^T(r_n, \omega) \neq \emptyset \text{ and} \\ \pi_n(\theta[\omega]^{m,T}) \geq \pi_n(\theta[\omega]^{m',T}) \; \forall m \in D_n^T(r_n, \omega) \text{ and } m' \in M, \end{cases} \tag{19}$$

where $D_n^T(r_n, \omega) = \{m \mid (m, T) \in D_n(r_n, \omega)\}$. For any role, (2) says any rule r_n is selected wpp at any date if the payoff it receives is highest in the previous T periods; in contrast, according to (19) r_n is selected wpp at any date if it performs best in the previous period. Note when $T = 1$, (19) is equivalent to (2). Next, fix any $a \in A$, let

$$\widetilde{\mathcal{R}}_a = \left\{ R' \subset \overline{R}^{pure} \;\middle|\; \begin{array}{l} \text{(a) } \forall r \in R' \text{ and } \theta,\, r(\theta)[a] = 1 \text{ if } \theta^T \in \phi(a) \cup \{\cup_{m,n,a' \text{s.t.} a'_n \neq a_n} \overline{\phi}(m, (a_n, a'_{-n}) \mid a')\} \\ \text{(b) } \forall b \in A,\, \exists r \in R' \text{ s.t. } \forall \theta,\, r(\theta)[b] = 1 \text{ if } \theta^T \in \cup_{a' \neq a} \phi(a') \end{array} \right\} \tag{20}$$

consist of sets of rules such that (a) every rule in this set plays a if the last period of the history is either $\phi(a)$ or $\overline{\phi}(m, (a_n, a'_{-n}) \mid a')$ for some m, n and a' such that $a'_n \neq a_n$, and (b) for each $b \in A$ there exist a rule in the set that plays b at all histories in which some uniform outcome $\phi(a')$, for some $a' \neq a$, is played in the last period of the history. The set $\widetilde{\mathcal{R}}_a$ is similar to $\mathcal{R}_a$ except that the behaviour prescribed by rules in $\mathcal{R}_a$ relates to histories that are uniform or 1-deviation, whereas for $\widetilde{\mathcal{R}}_a$ it relates to histories in which the outcome in the last period is uniform or 1-deviation.

Note that $\widetilde{\mathcal{R}}_a$ is well-defined as conditions (a) and (b) in (20) are restrictions at different sets of histories. In fact, $\widetilde{\mathcal{R}}_a$ includes well-defined elements that consist of match-neutral 2-complexity rules (an example is the set $\cup_{b \in A} \bar{r}^{a,b}$, where $\bar{r}^{a,b}(\theta)[a] = 1$ if $\theta^T \in \phi(a) \cup \{\cup_{m,n,a' \text{ s.t. } a'_n \neq a_n} \overline{\phi}(m, (a_n, a'_{-n}) \mid a')\}$ and $\bar{r}^{a,b}(\theta)[b] = 1$, otherwise).

Proposition 23 *Suppose no action inertia and Monotonicity satisfies (19). Fix any*

$a \in A$ and assume there exists $R' \in \widetilde{\mathcal{R}}_a$ such that $R' \subseteq R$. Then (i) under HDM $u(C) \geq u(a)$ for every $C \in \Gamma^*$ and (ii) under HIM there exists $C \in \Gamma^*$ such that $A(C) = a$.[39]

Parts (i) of and (ii) of Proposition 23 and their proofs are respectively similar to Proposition 19 and 2-complexity part in (i) of Proposition 22 and their proofs. It follows from (i) of Proposition 23 that if $R' \subseteq R$ for some $R' \in \widetilde{\mathcal{R}}_{\overline{a}}$ for some MaxMin norm $\overline{a}$ then $u(C) = \overline{u}$ for every $C \in \Gamma^*$.

Relaxing Monotonicity: Selection When Rules Are Not Observable. Monotonicity is a weak assumption except that it implicitly assumes that at any date agents observe wpp the identity of the rule that performed the best in the previous history. Here, we drop Monotonicity and extend our analyses to the case in which agents observe past actions but not the identity of rules that were used to implement them.

To do this, we assume here that agents first identify actions in the observed history that generated highest payoffs and then select existing rules that could have possibly generated these best actions in the history. Since at each date agents only recall T-period history, to identify rules that could have induced some specific outcome in the history, agents need to make assumptions regarding what could have happened before the history. We refer to any sequence of actions that could have happened before the observed history by a *pre-history,* and describe rules that could have generated the observed actions as *consistent* rules. Formally, $r_n \in R_n$ is said to be consistent with $\theta \in \Theta$ at $(m,t) \in M \times T$ given pre-history $\tilde{\theta} \in \Theta$ if $r_n(\tilde{\theta}^t, ..., \tilde{\theta}^T, \theta^1, ..., \theta^{t-1})[\theta_n^{m,t}] > 0$. For any n, r_n, θ, t and $\tilde{\theta}$, let $M_n(r_n, \theta, t \mid \tilde{\theta}) \equiv \{m \mid r_n$ is consistent with θ at (m,t) given $\tilde{\theta}\}$. Denote the set of matches at which the best payoff in θ at t in role n is achieved by $M_n^B(\theta, t) \equiv \{m \mid \pi_n(\theta^{m,t}) \geq \pi_n(\theta^{m',t'})$ for all $(m', t')\}$.

Definition 24 *Fix n. Rule r_n is Justifiable at $\omega \in \Omega$ if $r_n \in R_n(\omega)$ and there exists $\tilde{\theta} \in \Theta$ such that $M_n(r_n, \theta[\omega], t \mid \tilde{\theta}) = M_n^B(\theta[\omega], t)$ for all t. Rule selection $\rho_n(.)$ is Justifiable if $\rho_n(\omega)[r_n] > 0$ for any ω and r_n that is Justifiable at ω.*

Thus, for a rule to be Justifiable at ω (and hence be selected wpp) it is sufficient to demonstrate the existence of some pre-history $\tilde{\theta}$ such that every action in $\theta[\omega]$ that could

[39] Proposition 23 assumes no action inertia; however, it makes no assumption regarding rule inertia as it is consistent with both rule inertia and no rule inertia. The proposition however, does not cover the case when there is action inertia but no rule inertia; however, this case is somewhat uninteresting because it is more likely that agents revise their actions more often than they revise their rules.

have been induced by the rule given $\tilde{\theta}$ generates the highest payoff in $\theta[\omega]$. It may be argued that some pre-histories are more reasonable than others. For example, if in some role n all agents have taken the same action a'_n in state $\omega \in \Omega$, then it may be reasonable to argue that in the definition of Justifiability only pre-histories in which all in role n have taken action a'_n should be considered. An even weaker selection criterion would be to select any rule r_n wpp if, *for all* pre-histories $\tilde{\theta}$, $M_n(r_n, \theta[\omega], t \mid \tilde{\theta}) = M_n^B(\theta[\omega], t)$ for all t. Clearly, this property cannot hold at $t = 1$ if r_n is history-dependent.[40] However, our main results still hold if the selection criterion chooses Justifiable rules that, *for all* pre-histories, induce the best actions after date 1.

Definition 25 *Fix $n \in N$. Rule r_n is s-Justifiable at $\omega \in \Omega$ if it is Justifiable and $M_n(r_n, \theta[\omega], t \mid \tilde{\theta}) = M_n^B(\theta[\omega], t)$ for all $\tilde{\theta} \in \Theta$ and $t > 1$. Rule selection $\rho_n(.)$ is s-Justifiable if $\rho_n(\omega)[r_n] > 0$ for any ω and r_n that is s-Justifiable at ω.*

If $T = 1$, then s-Justifiable and Justifiable rules selection criteria are equivalent. However, when $T > 1$, s-Justifiable is a significantly weaker selection criterion. Below we sketch a brief intuition for why the main results of this paper hold if the selection criterion is s-Justifiable, instead of Monotonic (see Online Appendix for the details).

In the proofs of the main results we appeal to Monotonicity to show that, for any n, a rule r_n satisfying some properties (e.g. (11) and (13)) is selected wpp at any state ω in which all agents in roles other than n chose some a'_{-n} and r_n plays $b_n = B_n(a'_{-n})$ in each period (see for example Lemma 14). Thus, the role of Monotonicity in our proofs is to ensure that for any ω and $r_n \in R_n(\omega)$, $\rho_n(\omega)[r_n] > 0$ if

$$\theta[\omega] \in \Theta^{-n}(a'_{-n}) \text{ for some } a'_{-n}, \text{ and } \pi_n(b_n, a'_{-n}) \geq \pi_n(\theta[\omega]_n^{m',t'}, a'_{-n}) \ \forall (m', t'), \qquad (21)$$

where b_n is the action r_n takes at every period in state ω. But the knowledge that r_n was the rule that chose b_n in each period in ω is not necessary for our selection results. To see this, fix any ω satisfying (21). Let $\Theta^{-n}(a'_{-n} \mid T) = \{\theta \in \Theta \mid \theta_{-n}^{m',T} = a'_{-n} \text{ for all } m'\}$ be the set of histories in which all do a'_{-n} in roles other than n in the *last period.* Suppose $r_n(\tilde{\theta})[b_n] = 1$ for all $\tilde{\theta} \in \Theta^{-n}(a'_{-n} \mid T)$; thus the behaviour of r_n does not depend on the history of play before the last period if a'_{-n} was played in all matches in the last period. Then, for any $\tilde{\theta} \in \Theta^{-n}(a'_{-n} \mid T)$, both $M_n(r_n, \theta[\omega], 1 \mid \tilde{\theta})$ and $M_n^B(\theta[\omega], 1)$ are equal to $\{m' \mid \theta[\omega]_n^{m',1} = b_n\}$. Also, for all $\tilde{\theta} \in \Theta$ and $t > 1$, $M_n(r_n, \theta[\omega], t \mid \tilde{\theta})$ and $M_n^B(\theta[\omega], t)$

[40]If $\pi_n(\theta^{m,1}) \geq \pi_n(\theta^{m',t'})$ for some m and all (m', t'), then such a condition would require $r_n(\tilde{\theta})[\theta_n^{m,1}] > 0$ for every (pre-history) $\tilde{\theta}$, which is only possible when r_n is history-independent.

are equal to $\{m' \mid \theta[\omega]_n^{m',t} = b_n\}$, i.e. r_n chooses b_n at $(\tilde{\theta}^t, .., \tilde{\theta}^T, \theta[\omega]^1, .., \theta[\omega]^{t-1})$ for all $t > 1$ because $\theta[\omega]_{-n}^{m',t-1} = a'_{-n}$ for all m'. Hence, r_n is s-Justifiable at ω.

It therefore follows from the above that with s-Justifiable selection criterion we have that at any ω satisfying (21), $\rho_n(\omega)[r_n] > 0$ if $r_n(\tilde{\theta})[b_n] = 1$ for all $\tilde{\theta} \in \Theta^{-n}(a'_{-n} \mid T)$. The latter restriction is effectively a 1-period recall condition and our analysis could allow for it. More specifically, all our results, other than Proposition 18, hold if we replace Monotonicity assumption with s-Justifiable selection criterion because the critical sets such as R_Q and R_a allow for rules that have the following 1-period recall property: for all n, a'_{-n} and $\tilde{\theta} \in \Theta^{-n}(a'_{-n} \mid T)$, the rule does $B_n(a'_{-n})$ (or $B_n(a'_{-n} \mid Q_n)$) at $\tilde{\theta}$.[41]

Selection with Birth. Given No-Birth assumption, without mutation the system may become stuck in state(s) in which all agents adopt the same rule even if the rule does badly in this state. Although mutation may prevent such an outcome persisting, it may be argued that even with no mutation new rules may be adopted if incumbent rules induce non-reasonable outcomes. For example, if the same action $a \in A$ occurs in every match and in all of the last T periods, then agents might assume that a will be played again in the next period; hence, it may be reasonable to assume that at $\theta(a)$ agents choose $B(a)$. Thus, even if rules that do $B(a)$ at $\theta(a)$ are absent in the previous T periods, we might still expect agents to select such rules at $\theta(a)$. The following formalises this idea.

Assumption (u-Birth): *For any ω, n and $r_n \in R_n \backslash R_n(\omega)$, $\rho_n(\omega)[r_n] > 0$ if and only if $\theta[\omega] = \theta(a)$ for some $a \in A$, $r_n(\theta(a))[B_n(a)] > 0$ and $r'_n(\theta(a))[B_n(a)] = 0$ for all $r'_n \in R_n(\omega)$.*

If we replace No-Birth by u-Birth assumption then, by Lemma 67 (see Online Appendix), $A(C)$ contains a MWCURB set, for each $C \in \Gamma$. This has two implications. First, with u-Birth indeterminacy results under HIM are more limited as it would be restricted to MWCURB sets; i.e. while Theorem 8 still holds, Theorem 6 no longer holds if $a \notin E$. Second, with u-Birth under HDM we select a MaxMin MWCURB norm:

Proposition 26 *Suppose HDM and u-Birth. Assume $R \cap R^*_{\overline{Q}} \neq \emptyset$ for some MaxMin MWCURB norm $\overline{Q} \in W$. Then $u(C) = \overline{u}_W$ for any $C \in \Gamma^*$.*[42]

[41] The result in Proposition 18 regarding the set R_a^+ also holds if Monotonicity assumption is replaced by Justifiable criterion. However it does not hold with s-Justifiable criterion. This is because to apply s-Justifiability we need rules that have the 1-period recall property mentioned above, whereas when $T > 1$ (as is the case in Proposition 18) rules in the set R_a^+ may not allow for 1-period recall.

[42] Thus, with u-Birth, our selection of MaxMin MWCURB norm is similar to that in (iii) of Theorem

For any $a \in A$, u-Birth assumption allows for new rules to emerge at uniform state $\omega = (\theta(a), R[\omega])$ only if the existing rules are not best replying to a. We could modify u-Birth and remove this only if requirement.

Assumption (u*-Birth): *For any ω, n and $r_n \in R_n \backslash R_n(\omega)$, $\rho_n(\omega)[r_n] > 0$ if and only if $\theta[\omega] = \theta(a)$ for some $a \in A$ and $r_n(\theta(a'))[B_n(a')] = 1$ for any $a' \in A$.*

If we replace No-Birth by u*-Birth, then by Lemma 68 (see Online Appendix), $A(C) \in E$ for all $C \in \Gamma$ if $R^u \cap R^{pure} \subseteq R$. This has two implications. First, with u*-Birth and $R^u \cap R^{pure} \subseteq R$, our indeterminacy results under HIM are more limited as it would be restricted to equilibrium sets, i.e. Theorem 6 and 8 respectively hold only when $a \in E$ and $Q \in E$. Second, with u*-Birth under HDM we select a MaxMin equilibrium norm:

Proposition 27 *Suppose HDM and u*-Birth. Assume $R^u \cap R^{pure} \subseteq R$ and $R \cap R^*_{\overline{e}} \neq \emptyset$ for some MaxMin equilibrium norm $\overline{e} \in E$. Then $u(C) = \overline{u}_{\overline{e}}$ for any $C \in \Gamma^*$.*

6 References

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4. In the latter, the selection claim followed from the assumption that $R \subseteq \{r \in \overline{R} \mid r(\theta(a))[B(a)] > 0$ for all $a \in A\}$; here u-Birth plays the same role.

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7 Appendix: Proof of Lemmas 12, 13 and 14

Unless explicitly stated, all statements below hold independently of the mutation (HDM or HIM) assumption.

By anonymity of f described in (3), for any $C \in \Gamma$, $h \in C \times \Upsilon$, i and $i' \in I$, there exists $h' \in C \times \Upsilon$ such that $f(i,h) = f(i',h')$; thus

$$\min_{h \in C \times \Upsilon} f(i,h) = \min_{h' \in C \times \Upsilon} f(i',h') \text{ for all } C \in \Gamma,\ i \text{ and } i' \in I \tag{22}$$

Given (22), for any $C \in \Gamma$, define $f^{\min}(C) \equiv \min_{h \in C \times \Upsilon} f(i,h)$ for all i. Then

$$res(C, C') \geq f^{\min}(C) \text{ for any two distinct } C \text{ and } C' \in \Gamma. \tag{23}$$

Lemma 28 *Assume HDM. Fix any $C \in \Gamma$. For any $i \in I$ and $h \in C \times \Upsilon$, if $\pi^T(i,h) = u(C)$ then $f(i,h) = f^{\min}(C)$.*

Proof. Suppose not; then $\pi^T(i,h) = u(C)$ and $f(i,h) > f(i,h')$ for some $h' \in C \times \Upsilon$. Then by (4), $\pi^T(i,h) > \pi^T(i,h')$. But this implies $u(C) > \pi^T(i,h')$; a contradiction. ■

Lemma 29 *Assume HDM. Fix any $R' \subseteq R$ and $C^0 \in \Gamma$. Suppose R' is invading. Then there exists $\{C^1, .., C^N\}$ such that, for any $n = 1, .., N$, (i) $C^n \in \Gamma$; (ii) $R_n(C^n) \in R'_n$, and $R_{-n}(C^n) = R_{-n}(C^{n-1})$; (iii) $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$ if $R_n(C^{n-1}) \notin R'_n$ and 0 otherwise; and (iv) $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$.*

Proof. The proof consists of applying the next claim recursively starting from $n = 1$.

The Induction Claim for any $n \geq 1$. Assume that if $n > 1$ then there exist $\{C^1, .., C^{n-1}\}$ such that, for any $\ell = 1, .., n-1$, $C^\ell \in \Gamma$; $R_\ell(C^\ell) \in R'_\ell$ and $R_{-\ell}(C^\ell) = R_{-\ell}(C^{\ell-1})$; and $res(C^{\ell-1}, C^\ell) = f^{\min}(C^{\ell-1})$ if $R_\ell(C^{\ell-1}) \notin R'_\ell$ and 0 otherwise. Then there exists $C^n \in \Gamma$ such that $R_n(C^n) \in R'_n$ and $R_{-n}(C^n) = R_{-n}(C^{n-1})$; $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$ if $R_n(C^{n-1}) \notin R'_n$ and 0 otherwise; and $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$.

Proof of the Induction Claim. If $R_n(C^{n-1}) \in R'_n$ set $C^n = C^{n-1}$; then the claim holds trivially. So suppose $R_n(C^{n-1}) \notin R'_n$. By definition, there exists $\omega^{n-1} \in C^{n-1}$, $i \in I$ and $h \in H(\omega^{n-1})$ such that $\pi^T(i, h) = u(C^{n-1})$. Since R' is invading, there exists $r_n \in R'_n$ such that one mutation by i at ω^{n-1} moves the system wpp from ω^{n-1} to some $C^n \in \Gamma$ with $R(C^n) \in (r_n, R_{-n}(C^{n-1}))$. Hence, by Lemma 28, $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$.

Next fix any C^0-tree $\tau^*_{C^0} \in \arg\min_{\tau \in T_{C^0}} res(\tau)$ and define the following tree operations on $\tau^*_{C^0}$: For all $\ell = 1, .., n$ such that $R_\ell(C^{\ell-1}) \neq R_\ell(C^\ell)$, (i) construct the edge $C^{\ell-1} \to C^\ell$ (by the induction assumption and $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$, $res(C^{\ell-1}, C^\ell) = f^{\min}(C^{\ell-1})$) and (ii) delete the edge starting at C^ℓ in $\tau^*_{C^0}$, say $C^\ell \to \tilde{C}^\ell$ (by (23) $res(C^\ell, \tilde{C}^\ell) \geq f^{\min}(C^\ell)$). The tree operations in (i) and (ii) therefore induce a C^n-tree $\tau \in T_{C^n}$ such that $res(\tau) = \gamma(C^0) + \sum_{\ell=1}^{n} res(C^{\ell-1}, C^\ell) - \sum_{\ell=1}^{n} res(C^\ell, \tilde{C}^\ell) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$. The last part of the induction claim then follows from $\gamma(C^n) \leq res(\tau)$. ■

Lemma 30 *Fix any $C' \in \Gamma$ and $Q \subseteq A$. Assume R' is 1-mutation from Q and $R(C') \in R'$. Then there exist $C \in \Gamma$ such that $A(C) \subseteq Q$, $R(C) \in R'$ and $\gamma(C) + f^{\min}(C) \leq \gamma(C') + f^{\min}(C')$.*

Proof. If $A(C') \subseteq Q$ then the claim holds trivially (set $C = C'$). So suppose $A(C') \subsetneq Q$ and fix any agent i and any C'-tree $\tau^*_{C'} \in \arg\min_{\tau \in T_{C'}} res(\tau)$. Denote $R(C')$ by r. Since $A(C') \subsetneq Q$ and R' is 1-mutation from Q, one mutation by any agent at any $\omega' \in C'$ moves the system from ω' to some $C \in \Gamma$ such that $A(C) \subseteq Q$ and $R(C) \in R'$ wpp. Also, by definition, there exists $\omega \in C'$ and $i \in I$ such that i receives $u(C')$ at the last period of ω. Hence, by Lemma 28, $res(C', C) = f^{\min}(C')$.

Next, define the following tree operations on $\tau^*_{C'}$: (i) Construct the edge $C' \to C$ and (ii) delete the edge starting at C, say $C \to \tilde{C}$. Since $res(C', C) = f^{\min}(C')$ and, by (23), $res(C, \tilde{C}) \geq f^{\min}(C)$, the tree operations in (i) and (ii) induce a C-tree $\tau_C \in T_C$, such that $res(\tau_C) = \gamma(C') + res(C', C) - res(C, \tilde{C}) \leq \gamma(C') + f^{\min}(C') - f^{\min}(C)$. The claim in the lemma then follows from $\gamma(C) \leq res(\tau_C)$. ■

Lemma 31 *Fix any $C' \in \Gamma$ and i. There exists $h \in C' \times \Upsilon$ such that $\pi^T(i,h) = u(C')$.*

Proof. Given the definition of $u(C')$, there exists $\omega' \in C'$, $m \in M$ and $n \in N$ such that $\pi_n(\theta[\omega']^{m,T}) = u(C')$. This implies any $h = (\omega, \{v^t(i)\}_{t\in T, i\in I}) \in H(\omega)$ such that $v^T(i) = (m,n)$ satisfies the claim in the lemma. ■

Proof of Lemma 12. Suppose not; then $u(C^0) < u(Q)$. Fix any C^0-tree $\tau^*_{C^0} \in \arg\min_{\tau \in T_{C^0}} res(\tau)$. Since R' is invading, by Lemma 29, there exists $C^N \in \Gamma$ such that

$$\gamma(C^N) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^N) \tag{24}$$

and $R(C^N) \in R'$. Since R' is 1-mutation from Q, by Lemma 30, there exists $C \in \Gamma$ such that $A(C) \subseteq Q$, $R(C) \in R'$ and $\gamma(C) \leq \gamma(C^N) + f^{\min}(C^N) - f^{\min}(C)$. Then, by (24),

$$\gamma(C) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C). \tag{25}$$

By $A(C) \subseteq Q$, we have $u(Q) \leq u(C)$; therefore, $u(C^0) < u(C)$. Next, fix any $i \in I$. By Lemma 31, there exists $h^0 \in C^0 \times \Upsilon$ and $h \in C \times \Upsilon$ such that $\pi^T(i,h^0) = u(C^0)$ and $\pi^T(i,h) = u(C)$. Given $u(C^0) < u(C)$ and (4), we then have $f(i,h^0) < f(i,h)$. Thus, by Lemma 28, $f^{\min}(C^0) = f(i,h^0) < f(i,h) = f^{\min}(C)$. But then, by (25), $\gamma(C) < \gamma(C^0)$; this contradicts the supposition that $C^0 \in \Gamma^*$. ■

Lemma 32 *Fix any $\omega \in \Omega$ and $t' \geq 1$. Starting from ω wpp the system reaches $\omega' \in \Omega$ such that $\theta[\omega']^{m,t} = \theta[\omega]^{m,T}$ and $R[\omega']^{m,t} = R[\omega]^{m,T}$ for all m and $t \geq t'$.*

The proof of Lemma 32 follows trivially from rule and action inertia.

Henceforth we shall use the term *one mutation by an agent at ω to r_n in role n of match m* to refer to the situation in which an agent mutates at ω and then chooses r_n after being assigned role n in match m. Also, for any m, n, r' and r_n, denote the profile of rules in different matches in any period $\{r^{m'}\}_{m'\in M} \in \overline{R}^M$ such that $r^m = (r_n, r'_{-n})$ and $r^{m'} = r'$ for all $m' \neq m$ by $q^n(m, r_n \mid r')$.

Lemma 33 *Fix any $m \in M$, $n \in N$, $r_n \in R_n$, $r' \in R$ and $\omega \in \Omega$ such that $R(\omega) = r'$ and $r'_n \neq r_n$. Assume $r'(\theta[\omega])[a'] > 0$ and $r_n(\theta[\omega])[b_n] > 0$ for some $a' \in A$ and $b_n \in A_n$. Then one mutation by any agent i at ω to r_n in role n of match m moves the system wpp from ω to some state $\hat{\omega}$ such that $\theta[\hat{\omega}]^T = \overline{\phi}(m, (b_n, a'_{-n}) \mid a')$, $R[\hat{\omega}]^T = q^n(m, r_n \mid r')$ and, for all $t < T$, $\theta[\hat{\omega}]^t = \theta[\omega]^{t+1}$ and $R[\hat{\omega}]^t = R[\omega]^{t+1}$.*

Proof. Suppose that the system is in state ω at some date τ and the following happens: (i) agent i is the only agent that is in a mutating status, (ii) agent i is assigned role n in match m, (iii) other agents are randomly matched to other roles and matches (iv) agent i mutates to r_n and plays b_n and every other agent in any role k follows her rule r'_k and plays a'_k. Then at $\tau + 1$ the state $\hat{\omega}$ satisfying the claim in the lemma will be reached wpp with one mutation by i. ■

Lemma 34 *Fix any $m \in M$, $n \in N$, $r_n \in R_n$, $r' \in R$ and $\omega \in \Omega$ such that $R[\omega]^T = q^n(m, r_n \mid r')$. Assume $r'(\theta[\omega])[a'] > 0$ and $r_n(\theta[\omega])[b_n] > 0$ for some $a' \in A$ and $b_n \in A_n$. Fix any $\bar{b}_n \in \{b_n, \theta[\omega]_n^{m,T}\}$ and $c_k^{m'} \in \{a'_k, \theta[\omega]_k^{m',T}\}$ for any k and m' such that either $k \neq n$ or $m' \neq m$. Then from ω wpp the system reaches state $\hat{\omega}$ such that $\theta[\hat{\omega}]^{m,T} = (\bar{b}_n, c_{-n}^m)$, $\theta[\hat{\omega}]^{m',T} = (c_1^{m'}, .., c_N^{m'})$ for all $m' \neq m$, $R[\hat{\omega}]^T = q^n(m, r_n \mid r')$, and $\theta[\hat{\omega}]^t = \theta[\omega]^{t+1}$ and $R[\hat{\omega}]^t = R[\omega]^{t+1}$ for all $t < T$.*

Lemma 35 *Fix any $Q \subseteq A$, $\omega \in \Omega$, $r \in R$ such that $\theta[\omega]^{m,T} \in Q$ and $R[\omega]^{m,T} = r$ for all m, and (11) holds. Then from ω wpp the system reaches some $C \in \Gamma$ with $A(C) \subseteq Q$.*

The proof of Lemma 34 is similar to that of Lemma 33, and Lemma 35 follows trivially from Lemma 32 (see Online Appendix for the proofs of Lemmas 34 and 35).

Lemma 36 *Fix any $Q \subseteq A$, $n \in N$, $m \in M$, $a \in Q$, $\omega^1 \in \Omega$, $r \in R$ and $s_n \in S_n$ such that $\theta[\omega^1]_{-n}^{m',T} = a_{-n}$ for all m', $R[\omega^1]^T = q^n(m, s_n \mid r)$, $s_n(\theta')[a_n] > 0$ for all $\theta' \in \Theta$, r satisfies (11) and there exists $b_n \in B_n(a_{-n} \mid Q) \cup a_n$ such that*

$$r_n(\theta[\omega^1])[b_n] > 0 \text{ and } r_n(\overline{\theta}(m, a \mid b_n, a_{-n})[b_n] > 0. \tag{26}$$

Then from ω^1, the system reaches wpp some $C \in \Gamma$ such that $A(C) \subseteq Q$ and $R(C) = r$.

Proof. Suppose the system is at ω^1 at some date τ. It follows from $r_n(\theta[\omega^1])[b_n] > 0$, $\theta[\omega^1]_{-n}^{m',T} = a_{-n}$ for all m', $s_n(\theta[\omega^1])[a_n] > 0$ and Lemma 34, that the system reaches wpp at date $\tau + 1$ state $\omega^2 \in \Omega$, where $\theta[\omega^2]^T = \overline{\phi}(m, a \mid b_n, a_{-n})$, $\theta[\omega^2]^t = \theta[\omega^1]^{t+1}$ for all $t < T$ and $R[\omega^2]^T = q^n(m, s_n \mid r)$. But then by Lemma 32 the system reaches wpp at date $\tau + T$ state $\omega^3 \in \Omega$, where $\theta[\omega^3] = \overline{\theta}(m, a \mid b_n, a_{-n})$ and $R[\omega^3]^t = q^n(m, s_n \mid r)$ for all t. By (26), $r_n(\theta[\omega^3])[b_n] > 0$. Also, $s_n(\theta[\omega^3])[a_n] > 0$.

Thus, assuming that for the next T periods all agents in role n follow their rules and agents in roles other than n suffer from action inertia, we have (applying Lemma 34 T times) that the system reaches wpp at date $\tau + 2T$ state ω^4 such that $\theta[\omega^4] = \overline{\theta}(m, a \mid b_n, a_{-n})$, $R[\omega^4] = R[\omega^3]$. Since $b_n = B_n(a_{-n} \mid Q_n) \cup a_n$, it must be that r_n performs at least as well as s_n in state ω^4. Thus, at date $\tau + 2T$ all agents in role n, by the Monotonicity assumption, will adopt r_n wpp.[43] Then, by Lemma 32 the system reaches wpp some $C \in \Gamma$ such that $A(C) \subseteq Q$ and $R(C) = r$. ■

Proof of Lemma 13 . Fix any $C' \in \Gamma$, $\omega \in C'$, $r \in R'$, $a' \in A$ and $i \in I$ such that $A(C') \subsetneq Q$, $R(C') = r$ and $r(\theta[\omega])[a'] > 0$. Since r satisfies (11) and $A(C') \subsetneq Q$, we have $a' \notin Q$ (otherwise, by Lemma 35, $A(C') \subseteq Q$). Fix any n such that $a'_n \notin Q_n$, $a_n \in Q_n$ and $s_n \in S_n$ such that $s_n(\theta')[a_n] > 0$ for all $\theta' \in \Theta$. There are two cases.

Case A. $a'_{-n} \in Q_{-n}$. By (12b) and $a'_{-n} \in Q_{-n}$, there exists m such that

$$r_n(\overline{\theta}(m, (a_n, a'_{-n}) \mid a''_n, a'_{-n}))[B_n(a'_{-n} \mid Q)] > 0 \text{ for all } a''_n \in A_n. \tag{27}$$

Also, by $r(\theta[\omega])[a'] > 0$, $s_n(\theta[\omega])[a_n] > 0$, Lemma 33 and Lemma 32, starting from ω at some date τ, one mutation by any agent i at ω to s_n in role n of match m moves the system wpp T periods later to some state ω^1, where $\theta[\omega^1] = \overline{\theta}(m, (a_n, a'_{-n}) \mid a')$ and $R[\omega^1]^{,t} = q^n(m, s_n \mid r)$ for all t. Condition (27) implies that $r_n(\theta[\omega^1])[b_n] > 0$ and $r_n(\overline{\theta}(m, (a_n, a'_{-n}) \mid b_n, a'_{-n})[b_n] > 0$, where $b_n = B_n(a'_{-n} \mid Q)$. So, given that $a_n \in Q_n$ and $a'_{-n} \in Q_{-n}$ and r satisfies (11), at ω^1 all the assumptions in Lemma 36 hold; hence, it follows from Lemma 36 that from ω^1 the system reaches wpp some $C \in \Gamma$ such that $A(C) \subseteq Q$ and $R(C) = r \in R'$.

Case B. $a'_{-n} \notin Q_{-n}$. By (12a) and $a'_n \notin Q_n$, there exists m' such that $r_{-n}(\overline{\theta}(m', (a_n, a'_{-n}) \mid a'))[a_{-n}] > 0$ for some $a_{-n} \in Q_{-n}$. Also, by $r(\theta[\omega])[a'] > 0$, $s_n(\theta[\omega])[a_n] > 0$, Lemma 33 and Lemma 32, starting from ω at some date τ, one mutation by any agent i at ω to s_n in role n of match m' moves the system wpp to some state $\hat{\omega}$ at date $\tau + T$ such that $\theta[\hat{\omega}] = \overline{\theta}(m', (a_n, a'_{-n}) \mid a')$ and $R[\hat{\omega}]^t = q^n(m', s_n \mid r)$ for all t. Also, by (12b) and $a_{-n} \in Q_{-n}$, there exists m such that

$$r_n(\overline{\theta}(m, a \mid a''_n, a_{-n}))[B_n(a_{-n} \mid Q)] > 0 \text{ for all } a''_n \in A_n. \tag{28}$$

Then, by $r_{-n}(\overline{\theta}(m', (a_n, a'_{-n}) \mid a'))[a_{-n}] > 0$, Lemma 34 and Lemma 32, from $\hat{\omega}$ the system reaches T periods later wpp $\omega^1 \in \Omega$, where $\theta[\omega^1] = \overline{\theta}(m, a \mid a'_n, a_{-n})$, $R[\omega^1]^{m,t} = (s_n, r_{-n})$ and $R[\omega^1] = R[\hat{\omega}]$. Thus, by condition (28), $r_n(\theta[\omega^1])[b_n] > 0$ and $r_n(\overline{\theta}(m, a \mid$

[43] In this proof we appeal to Monotonicity to select r_n and eliminate s_n at date $\tau + 2T$ in state ω^4 with history $\theta[\omega^4] = \overline{\theta}(m', a \mid b_n, a_{-n})$. Given that $\theta[\omega^3]$ is also equal to $\overline{\theta}(m', a \mid b_n, a_{-n})$ we could appeal to Monotonicity and select r_n earlier at date $\tau + T$ in state ω^3. However, history $\theta[\omega^3]$ at date $\tau + T$ is obtained after r_n being subject to action inertia for $T - 1$ periods whereas history $\theta[\omega^4]$ at date $\tau + 2T$ is reached by r_n being active (not subject to action inertia) for the previous T periods. Given that we want to allow for the possibility that Monotonicity applies only after the rules in a particular role are all active, in the proof we appeal to Monotonicity to select r_n at date $\tau + 2T$ rather than earlier.

$b_n, a_{-n})[b_n] > 0$, where $b_n = B_n(a_{-n} \mid Q)$. So, given that $a_n \in Q_n$ and $a_{-n} \in Q_{-n}$ and r satisfies (11), at ω^1 all the assumptions in Lemma 36 hold; hence, by Lemma 36, from ω^1 the system reaches wpp some $C \in \Gamma$ such that $A(C) \subseteq Q$ and $R(C) = r \in R'$. ■

For any ω^0, $a'_{-n} \in A_{-n}$, m, n, r_n, r' and $\tau = 1, 2, .., $ let $K^\tau(m, n, r_n, r' \mid a'_{-n}, \omega^0) =$

$$\left\{ \{b_n^t, c_n^t, \omega^t\}_{t=1}^{\tau} \;\middle|\; \begin{array}{c} \text{for all } 1 \le t \le \tau \text{ and for all } m' \neq m,\ r_n(\theta[\omega^{t-1}])[b_n^t] > 0,\ r'_n(\theta[\omega^{t-1}])[c_n^t] > 0, \\ \theta[\omega^t] = (\theta[\omega^{t-1}]^2, .., \theta[\omega^{t-1}]^T, \overline{\phi}(m, (b_n^t, a'_{-n}) \mid c_n^t, a'_{-n}), \\ R[\omega^t]^{t'} = R[\omega^{t-1}]^{t'+1} \text{ for all } t' < T, \text{ and } R[\omega^t]^T = q^n(m, r_n \mid r'). \end{array} \right\}$$

be the set of sequences of actions in role n and states of length τ that can occur after ω^0 assuming that, in each period, all agents in roles other than n play a'_{-n}, the agent in role n of match m play according to r_n and all other agents in role n play according to r'_n. The set $K^\tau(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ is non-empty as it can be defined recursively.

Lemma 37 *Fix any $a \in A$, $\omega^0 \in \Omega$, $a'_{-n} \in A_{-n}$, $m \in M$, $n \in N$, $r_n \in R_n$ and $r' \in R$ such that $\theta[\omega^0] \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \theta(a)$, $R[\omega^0]^T = q^n(m, r_n \mid r')$ and*

$$r_n(\theta)[B_n(a'_{-n})] > 0 \text{ for all } \theta \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \theta(a) \text{ and } r_n(\theta(a))[B_n(a_{-n}) \cup a_n] > 0. \tag{29}$$

Then from ω^0 the system reaches wpp some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$.

Proof. First, we establish the following induction claim.

The Induction Claim for any $t \geq 1$. Assume that if $t > 1$ then there exists $\{b_n^{t'}, c_n^{t'}, \omega^{t'}\}_{t'=1}^{t-1} \in K^{t-1}(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ such that $b_n^{t'} = B_n(a'_{-n})$ for all $t' < t$ and starting from ω^0 the system reaches wpp ω^{t-1}. Then there exists $\{b_n^t, c_n^t, \omega^t\}$ such that $b_n^t = B_n(a'_{-n})$, $r_n(\theta[\omega^{t-1}])[b_n^t] > 0$, $r'_n(\theta[\omega^{t-1}])[c_n^t] > 0$, $\theta[\omega^t]^T = \overline{\phi}(m, (b_n^t, a'_{-n}) \mid c_n^t, a'_{-n})$, $R[\omega^t]^T = q^n(m, r_n \mid r')$, $\theta[\omega^t]^{t'} = \theta[\omega^{t-1}]^{t'+1}$ and $R[\omega^t]^{t'} = R[\omega^{t-1}]^{t'+1}$ for all $t' < T$, and starting from ω^{t-1} the system reaches wpp ω^t at the next date.

Proof of the Induction Claim. Fix any c_n^t such that $r'_n(\theta[\omega^{t-1}])[c_n^t] > 0$. Then the proof follows from applying Lemma 34 to ω^{t-1} if it can be shown that $r_n(\theta[\omega^{t-1}])[B_n(a'_{-n})] > 0$. If $t = 1$ or $a_n = B_n(a_{-n})$ then this follows from (29), given that $\theta[\omega^{t-1}] \in \hat{\Theta}^{-n}(m \mid a'_{-n})$ and $\theta[\omega^0] \neq \theta(a)$. If $t > 1$ and $a_n \neq B_n(a_{-n})$ then, given that $b_n^{t-1} = B_n(a'_{-n})$, we have $\theta[\omega^{t-1}] \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \theta(a)$; but then, by (29), $r_n(\theta[\omega^{t-1}])[B_n(a'_{-n})] > 0$.

By applying the claim recursively starting from $t = 1$, there exists $\{b_n^t, c_n^t, \omega^t\}_{t=1}^T \in K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ such that $b_n^t = B_n(a'_{-n})$ for every $t \in T$ and starting from ω^0 the system reaches wpp ω^T. Since by definition $\omega^T \in \hat{\Theta}^{-n}(m \mid a'_{-n})$ and $R[\omega^T]^t = q^n(m, r_n \mid r')$ and $b_n^t = B_n(a'_{-n})$ for all t, r_n performs at least as well as r'_n at every period in ω^T. Thus, by Monotonicity, r_n will be adopted by all agents in role n wpp at ω^T.[44] Hence, by Lemma 32, the system reaches some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$. ■

Lemma 38 *Fix any $a \in A$, $\omega^0 \in \Omega$, $a'_{-n} \in A_{-n}$, $m \in M$, $n \in N$, $r_n \in R_n$ and $r' \in R$ such that $R(\omega^0) = r'$, $r'_{-n}(\theta[\omega^0])[a'_{-n}] > 0$ and r_n satisfies (29). Then one mutation by any agent at ω^0 to r_n in role n of match m moves the system wpp from ω^0 to some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$.*

Proof. Fix any $\{b_n^t, c_n^t, \omega^t\}_{t=1}^T \in K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ such that if $r_n(\theta[\omega^0])([B_n(a'_{-n})] > 0$ then $b_n^1 = B_n(a'_{-n})$. Clearly, $K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ exists. The proof has two steps.

Step 1. One mutation by any agent at ω^0 to r_n in role n of match m moves the system wpp from ω^0 to state ω^T. To show this note that, by $R(\omega^0) = r'$, $r'_{-n}(\theta[\omega^0])[a'_{-n}] > 0$, $r_n(\theta[\omega^0])[b_n^1] > 0$, $r'_n(\theta[\omega^0])[c_n^1] > 0$ and Lemma 33, one mutation by any agent at ω^0 to r_n in role n of match m moves the system wpp from ω^0 to state ω^1. Since $r_n(\theta[\omega^{t-1}])[b_n^t] > 0$ and $r'_n(\theta[\omega^{t-1}])[c_n^t] > 0$ for every $t > 1$, by applying Lemma 34 $(T-1)$ times, we have that starting from ω^1 the system reaches wpp ω^T.

Step 2. Starting from ω^T the system will reach some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$. To show this there are two cases to consider.

Case 1. $\theta[\omega^T] = \theta(a)$: By the definition of ω^T, $R[\omega^T]^t = q^n(m, r_n \mid r')$ for all t. Also, since $\theta[\omega^T] = \theta(a)$, r_n performs at least as well as r'_n in ω^T. Thus, by Monotonicity, r_n will be adopted by all agents in role n wpp at ω^T.[45] Hence, by Lemma 32, from ω^T the system reaches wpp T periods later some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$.

Case 2. $\theta[\omega^T] \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \theta(a)$: Given that $R[\omega^T]^T = q^n(m, r_n \mid r')$ and r_n satisfies (29), by Lemma 37, from ω^T the system reaches some $C \in \Gamma$ s.t. $R_n(C) = (r_n, r'_{-n})$. ■

Proof of Lemma 14. Fix any $C' \in \Gamma$, r' and $n \in N$, such that $r' = R(C')$. If $r_n = r'_n$ then r_n is invading at C'. So suppose that $r'_n \neq r_n$. Fix any $\omega \in C'$ and a'_{-n} such that $r'_{-n}(\theta[\omega])[a'_{-n}] > 0$. If $Q \in W$, then as r satisfies (12b) and (13), there exists m such that $r_n(\theta)[B_n(a'_{-n})] > 0$ for all $\theta \in \hat{\Theta}^{-n}(m \mid a'_{-n})$. Also, if Q is singleton and equal to some $a \in A$, then as r satisfies (11) and (13), $r_n(\theta(a)[a_n] = 1$ and $r_n(\theta)[B_n(a'_{-n})] > 0$ for all $\theta \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \theta(a)$. Thus, in both cases r_n satisfies (29). Hence, by Lemma

[44] Note that in state ω^T all agents in role n are active. Therefore, this result holds if Monotonicity assumption applied only in roles where agents do not suffer from action inertia.

[45] Note that in state ω^T all agents in role n are active. Therefore, this result holds if Monotonicity assumption applied only in roles where agents do not suffer from action inertia.

38, one mutation by *any* agent at ω to r_n in role n of match m moves the system from ω to some $C \in \Gamma$ satisfying $R(C) = (r_n, R_{-n}(\omega))$ wpp. Hence, r_n is invading at C'. ■

Online Appendix to "Rules and Mutation - A Theory of How Efficiency and Rawlsian Egalitarianism/Symmetry May Emerge"

In this Online Appendix, we complete the remaining discussions and proofs (according to the sections) in the paper. All references are to statements in the paper.

1 Section 2 of the Paper: The Case of $M = 1$

All our results also extend to the case when $M = 1$. The analysis in this case is somewhat different; in particular, with rule inertia and No-Birth it is much simpler as evolutionary forces has effectively no bite on rule selection under these assumptions.

When $M = 1$, there is exactly one agent in each role and thus $I = N$. Then at any state, for any n, whenever one agent mutates to some rule $r_n \in R_n$, given rule inertia assumption, the system reaches a state ω' in which *all agents* in role n play r_n (i.e. $R_n(\omega') = r_n$) wpp. By No-Birth, this means that all agents will use r_n henceforth. This implies that all rules are invading. Therefore, all rule profiles can be adopted with at most N mutation (one for each role).

The above conclusion implies that, to obtain any of our results under the assumptions of rule inertia and No-Birth, it suffices for the set of feasible rules to contain rules that have only 1-mutation from property. For example consider the set of equilibrium rules $R^E_Q = \{r \in R^E \mid r(\theta)[Q] = 1 \text{ for all } \theta \in \Theta\backslash \cup_e \theta(e)\}$ that always does an action in Q at non-equilibrium histories. Any $r \in R^E_Q$ is 1-mutation from Q. To see this, fix any C such that $R(C) = r$ and $A(C) \subsetneq Q$, and any $\omega \in C$. Then $\theta[\omega] = \theta(e)$ for some $e \in E\backslash Q$. Next, fix any m, n, $a_n \neq e_n$ and s_n such that $s_n(\theta)[a_n] > 0$ for all $\theta \in \Theta$, and consider a mutation at ω by any agent to s_n in role n of match m. Then from ω the system reaches some $\omega' \in \Omega$ such that $\theta[\omega'] = \overline{\theta}(m, (a_n, e_{-n}) \mid e)$ and $R[\omega'] = q^n(m, s_n \mid r)$. At this history r_n does better than s_n and thus, by Monotonicity, wpp s_n is eliminated at ω' and all choose r_n. Furthermore since $\theta[\omega'] \notin \cup_e\theta(e)$ we have $r(\theta[\omega'])[Q] = 1$. Therefore, all will play an action in Q henceforth.

Given that every $r \in R_Q^E$ is invading and 1-mutation from Q, it follows that under HDM if $R_Q^E \cap R$ is non-empty then the claim in Proposition 15 that $u(C) \geq u(Q)$ for all $C \in \Gamma^*$ holds. The claim in (i) of Theorem 2 then follows by setting Q equal to a MaxMin norm.

The other selection results under HDM and the results under HIM also extend by similar type of reasoning.

2 Section 4 and the Appendix in the Paper

2.1 Outstanding Proofs from Section 4 and the Appendix

Proof of Lemma 16. Fix $r \in R^s$ and n. Denote the player other than n by k. First, consider any $\theta \in \Theta$ such that $A(\theta) \subseteq Q$. Then by (1a) and $Q \in W$, $r_n(\theta)[a_n] > 0$ implies that $a_n = B_n(\theta^{m',t'}) \in Q_n$ for some m' and t'. Thus, r_n satisfies (11). Second, fix any $a'_n \notin Q_n$, $a'_k \notin Q_k$ and $a_n \in Q_n$. By (1b), there exists m such that $r_k(\overline{\theta}(m, (a_n, a'_k) \mid a'))[B_k(a_n)] > 0$. But given that $Q \in W$, $B_k(a_n) \in Q_k$. Hence, r satisfies (12a). Third, fix any $a'_k \in Q_k$, $a'_n \in A_n$ and $a_n \in Q_n$. By (1b), there exists m such that $r_n(\overline{\theta}(m, (a_n, a'_k) \mid a'))[B_n(a'_k)] > 0$. But given that $Q \in W$, $B_n(a'_k) \in Q_n$. Hence, r satisfies (12b). Finally, r trivially satisfies (13) because for any $n \in N$, $a' \in A$, $a''_n \in A_n$, by (1b), $r_n(\overline{\theta}(m, (a''_n, a'_k) \mid a'))[B_n(a'_k)] > 0$ for any m. ■

For any $C \in \Gamma$, define $A^u(C) \equiv \{a \in A \mid (\theta(a), r) \in C\}$.

Proof of Lemma 17. For all n and $a \in A$, let $F_n(a) = \widetilde{B}_n(a) \cup a_n$ and $F(a) = \times_n F_n(a)$. Define recursively the sequence $Y^1(a), Y^2(a), ...$ as follows: $Y^1(a) = \{a\}$ and, for any integer $k \geq 1$, $Y^{k+1}(a) = \cup_{a' \in Y^k(a)} F(a')$. Let $Y(a) = \{a' \in A \mid a' \in Y^k(a) \text{ for some } k\}$ and $Y = \cup_{a \in A^u(C)} Y(a)$.

For all $a \in A^u(C)$, by $R(C)(\theta(a))[B_n(a)] > 0$ and Lemma 34 (setting $r' = R(C)$), $F(a) \subseteq A^u(C)$. Hence, by recursion for every $a \in A^u(C)$ and $k \geq 1$ we have $Y^{k+1}(a) \subseteq A^u(C)$. This implies that $Y \subseteq A^u(C)$.

Next we show Y is a WCURB: Fix any $a \in \widetilde{B}(Y)$. Then there exists $a^1 \in Y$ such that $a \in \widetilde{B}(a^1)$; thus, $a \in Y^2(a^1)$. But by $a^1 \in Y \subseteq A^u(C)$ we have $Y^2(a^1) \subseteq Y$. Hence $a \in Y$. Therefore, Y is a WCURB set. But this together with $Y \subseteq A^u(C)$ implies that there exists $Q \in W$ such that $Q \subseteq A(C)$. Since $R(C)(\theta(a))[B(a)] > 0$ for all $a \in A$, this implies that $u(C) \leq u(Q) \leq \overline{u}_W$. ■

Lemma 39 *Suppose $T > 1$. Fix any a. Every $r \in R_a^+$ is 1-mutation from a.*

Proof. Fix any $i \in I$, $C' \in \Gamma$, $\omega \in C'$, $r \in R_a^+$ and a' such that $R(C') = r$, $A(C') \neq a$ and $r(\theta[\omega])[a'] > 0$. Clearly, $a' \neq a$; otherwise, by Lemma (35), $A(C') = a$. Let n be such that $a'_n \neq a_n$. There are two cases.

Case A: $a'_{-n} \neq a_{-n}$. Fix any m. Suppose the system is at ω at some date τ and agent i mutates, is assigned role n in match m and mutation is to some $s_n \in S_n$ such that $s_n(\theta)[a_n] > 0$ for all $\theta \in \Theta$. Then by Lemmas 33 and 32 the system reaches wpp at date $\tau + T$ state $\hat{\omega}$, where $\theta[\hat{\omega}] = \overline{\theta}(m, (a_n, a'_{-n}) \mid a')$, $R[\hat{\omega}]^t = q^n(m, s_n \mid r)$ for all t. Since $a'_n \neq a_n$ and $a'_{-n} \neq a_{-n}$, by (15a), $r_{-n}(\theta[\hat{\omega}])[a_{-n}] > 0$. Thus, by Lemma 34, the system reaches at date $\tau + T + 1$ wpp state ω^1 with $\theta[\omega^1]^T = \overline{\phi}(m, (a_n, a_{-n}) \mid a'_n, a_{-n})$, $\theta[\omega^1]^t = \theta[\hat{\omega}]^{t+1}$ for all $t < T$ and $R[\omega^1] = R[\hat{\omega}]$. Since $T > 1$, $a'_n \neq a_n$ and $a'_{-n} \neq a_{-n}$, we have $\theta[\omega^1] \in \Theta \backslash \{\Theta^s \cup \Theta^-\}$. By (15b), this implies $r_n(\theta[\omega^1])[a_n] > 0$. Also, by (8), $r_n(\theta(a))[a_n] > 0$. Since $s_n(\theta')[a_n] > 0$ for all $\theta' \in \Theta$, we have that at ω^1 all the assumptions in Lemma 36 hold for the case in which $Q = a$ and $b_n = a_n$; hence, it then follows from that lemma that starting at state ω^1 the system reaches wpp some $C \in \Gamma$ such that $A(C) = a$ and $R(C) = r \in R'$.

Case B: $a'_{-n} = a_{-n}$. Fix any $k \neq n$. Since $a'_{-k} \neq a_{-k}$, by (10), there exists m such that $r_k(\theta')[B_k(a'_{-k})] > 0$, for all $\theta' \in \hat{\Theta}^{-k}(m \mid a'_{-k})$. Suppose the system is at ω and agent i mutates, is assigned role k of match m and mutation is to $s_k \in S_k$ such that, for some $a''_k \neq B_k(a'_{-k})$, $s_k(\theta')]\ [a''_k] > 0$ for all $\theta' \in \Theta$. Then, by Lemmas 33 and 32**,** the system reaches wpp some $\omega' \in \Omega$ with $\theta[\omega'] = \theta(m, (a''_k, a'_{-k}) \mid a')$, $R[\omega']^t = q^k(m, s_k \mid r)$ for all t. Since $r_k(\theta')][B_k(a'_{-k})] > 0$, for all $\theta' \in \hat{\Theta}^{-k}(m \mid a'_{-k})$, by applying Lemma 34 T times, starting from ω' the system reaches wpp some $\hat{\omega} \in \Omega$ with $\theta[\hat{\omega}] = \overline{\theta}(m, (a''_k, a'_{-k}) \mid B_k(a'_{-k}), a'_{-k})$, $R[\hat{\omega}]^t = q^k(m, s_k \mid r)$ for all t. Given that $\pi_k(B_k(a'_{-k}), a'_{-k}) > \pi_k(a''_k, a'_{-k})$, r_k performs better than s_k in state $\hat{\omega}$ and next date all agents in role k will adopt r_k wpp. Since $\theta[\hat{\omega}] \in \hat{\Theta}^{-k}(m \mid a'_{-k})$, $r_k(\theta[\hat{\omega}])[B_k(a'_{-k})] > 0$; furthermore, since $a''_k \neq B_k(a'_{-k})$ and (15a) holds, $r_n(\theta[\hat{\omega}])[a_n] > 0$. Hence, by Lemma 34, starting from $\hat{\omega}$ next date the system reaches wpp some $\tilde{\omega} \in \Omega$ with $\theta[\tilde{\omega}]^T = \overline{\phi}(m, (a_n, B_k(a'_{-k}), a'_{-n,k}) \mid a'_n, B_k(a'_{-k}), a'_{-n,k})$, $\theta[\tilde{\omega}]^{T-1} = \overline{\phi}(m, (a'_n, a''_k, a'_{-n,k}) \mid a'_n, B_k(a_{-k}), a'_{-n,k})$ and $\theta[\tilde{\omega}]^t = \theta[\hat{\omega}]^{t+1}$ for $t < T - 1$, $R[\tilde{\omega}]^T = r$ and $R[\tilde{\omega}]^t = R[\hat{\omega}]^{t+1}$ for all $t < T$. Since $a_n \neq a'_n$ and $a''_k \neq B_k(a'_{-k})$, by (15b), $r(\theta[\tilde{\omega}])[a] > 0$. Hence, starting from from $\tilde{\omega}$ next date the system reaches wpp some $\tilde{\omega}'$ with $\theta[\tilde{\omega}']^T = \phi(a)$, $\theta[\tilde{\omega}']^t = \theta[\tilde{\omega}]^{t+1}$ for $t < T$, $R[\tilde{\omega}']^T = r$ and $R[\tilde{\omega}']^t = R[\tilde{\omega}]^{t+1}$ for all $t < T$. Hence, by Lemma (35), the system reaches wpp some C such that $A(C) = a$

and $R(C) = r$. ■

Lemma 40 *Fix any a. Any $R' \in \mathcal{R}_a$ is 1-mutation from a and invading if $R' \subseteq R$.*

Proof. Fix any $R' \in \mathcal{R}_a$ such that $R' \subseteq R$. Since for every $r \in R'$, (8) and (9) hold, by Lemma 13, R' is 1-mutation from a.

Next, fix any $n \in N$, $C' \in \Gamma$ and r' such that $r' = R(C')$. If $r'_n \in R'_n$ then R'_n is invading at C'. So assume $r'_n \notin R'_n$. Fix any $\omega \in C'$ and a'_{-n} such that $r'_{-n}(\theta[\omega])[a'_{-n}] > 0$. Let $b_n = B_n(a'_{-n})$. Given the definition of R_a in (18), there exists $r_n \in R'_n$ and m such that $r_n(\theta)[b_n] = 1$ for all $\theta \in \hat{\Theta}^{-n}(m \mid a') \backslash \theta(a)$; but then by Lemma 38 and $b_n = B_n(a'_{-n})$, one mutation by *any* agent at ω to r_n in role n of match m moves the system from ω to some $C \in \Gamma$ satisfying $R(C) = (r_n, R_{-n}(C'))$ wpp. Thus, R'_n is invading at C'. ■

Lemma 41 *Assume HIM. Suppose $R' \subseteq R$ is w-invading. For any $C^0 \in \Gamma$, there exists a sequence $\{C^1, .., C^N\}$ such that, for any $n = 1, .., N$, (i) $C^n \in \Gamma$; (ii) $R_n(C^n) \in R'_n$ and $R_{-n}(C^n) = R_{-n}(C^{n-1})$; and (iii) $res(C^{n-1}, C^n) = \eta$ if $C^{n-1} \neq C^n$ and 0, otherwise.*

Proof. The proof consists of applying the next claim recursively starting from $n = 1$.

The Induction Claim for any $n \geq 1$. Assume if $n > 1$ then there exists $\{C^1, .., C^{n-1}\}$ such that, for any $\ell = 1, .., n-1$, (i) $C^\ell \in \Gamma$, (ii) $R_\ell(C^\ell) \in R'_\ell$ and $R_{-\ell}(C^\ell) = R_{-\ell}(C^{\ell-1})$, and (iii) $res(C^{\ell-1}, C^\ell) = \eta$ if $R_\ell(C^{\ell-1}) \notin R'_\ell$ and 0, otherwise. Then there exists $C^n \in \Gamma$ such that $R_n(C^n) \in R'_n$ and $R_{-n}(C^n) = R_{-n}(C^{n-1})$, and $res(C^{n-1}, C^n) = \eta$ if $R_n(C^{n-1}) \notin R'_n$ and 0, otherwise.

Proof of the claim. There are two possibilities. If $R_n(C^{n-1}) \in R'_n$ then the claim follows by setting $C^n = C^{n-1}$. If $R_n(C^{n-1}) \notin R'_n$, then the claim holds because, given that R'_n is w-invading at C^{n-1}, there exists $r_n \in R'_n$, $i \in I$ and $\omega \in C^{n-1}$ such that one mutation by i at ω moves the system wpp from ω to some $C^n \in \Gamma$ satisfying $R(C^n) = (r_n, R_{-n}(C^{n-1}))$. ■

Proof of Lemma 21. Fix any $C^0 \in \Gamma^*$ and any C^0-tree $\tau^*_{C^0} \in \arg\min_{\tau \in T_{C^0}} res(\tau)$. By assumption $R' \subseteq R$ is 1-mutation from Q and w-invading. The latter implies that there exists a sequence $\{C^1, .., C^N\}$ that satisfies, for any $n = 1, .., N$, the claims (i) -(iii) in Lemma 41. But then there must exists $C^{N+1} \in \Gamma$ such that (a) $A(C^{N+1}) \subseteq Q$ and $R(C^{N+1}) \in R'$ and (b) $res(C^N, C^{N+1}) = \eta$ if $C^N \neq C^{N+1}$ and 0, otherwise (if $A(C^N) \subsetneq Q$ this follows from $R(C^N) \in R'$ and R' being 1-mutation from Q, and if $A(C^N) \subseteq Q$ then set $C^{N+1} = C^N$).

Next, perform the following tree operations on $\tau^*_{C^0}$: For any $n = 1, .., N+1$ such that $C^{n-1} \neq C^n$, (i) construct the edge $C^{n-1} \to C^n$ (by the previous argument $res(C^{n-1}, C^n) = \eta$) and (ii) delete the edge starting at C^n, say $C^n \to \tilde{C}^n$ (clearly, $res(C^n, \tilde{C}^n) \geq \eta$).

The tree operations (i) and (ii) induce a C^{N+1}-tree without increasing the total resistance; hence $\gamma(C^{N+1}) \leq \gamma(C^0)$. But since $C^0 \in \Gamma^*$ we have $C^{N+1} \in \Gamma^*$; this together with $A(C^{N+1}) \subseteq Q$ and $R(C^{N+1}) \in R'$ completes the proof. ■

Lemma 42 *Fix any $C \in \Gamma$, $\omega \in C$, $r \in R$ and $a \in A$ such that $\rho(\omega)[r] > 0$ and $r(\theta[\omega])[a] > 0$. Then there exists $\omega' \in C$ such that $\theta[\omega'] = \theta(a)$ and $R(\omega') = r$.*

Proof. Suppose at ω all agents revise their rule to r and choose a. Since this can happen wpp, there exists $\hat{\omega} \in C$ such that $\theta[\hat{\omega}]^{m,T} = a$ and $R[\hat{\omega}]^{m,T} = r$ for all m. The claim in the lemma then follows from Lemma 32. ■

Proof of Lemma 34. Suppose that the system is in state ω at some date τ and the following happens: (i) all agents are assigned the same roles and the same matches as at date $\tau - 1$, (ii) all agents are subject to rule inertia, (iii) the agent in role n of match m plays $\bar{b}_n$ (this is feasible because if $\bar{b}_n = b_n$ then the agent follows her rule r_n and plays b_n and if $\bar{b}_n = \theta[\omega]^{m,T}_n$ then the agent suffers action inertia and plays $\theta[\omega]^{m,T}_n$) and (iv) every other agent in any role k in any match m' plays $c^{m'}_k$ (this is feasible because if $c^{m'}_k = a'_k$ then the agent follows her rule r'_k and plays a'_k and if $c^{m'}_k = \theta[\omega]^{m',T}_k$ then the agent suffers action inertia and plays $\theta[\omega]^{m',T}_k$). Then at $\tau + 1$ state $\hat{\omega}$ satisfying the properties claimed in the lemma will be reached wpp. ■

Proof of Lemma 35. Starting from ω, by Lemma 32, the system eventually reaches wpp a state $\omega' \in \Omega$ such that $\theta[\omega']^{m,t} = \theta[\omega]^{m,T} \in Q$ and $R[\omega']^{m,t} = r$ for all m and t. The claim then follows from (11). ■

2.2 An Alternative Approach to Demonstrating Selection of MaxMin Norm with Match-Neutral Rules

In Proposition 15, we established the claim that under HDM, for any $a \in A$, $u(C) \geq u(a)$ for all $C \in \Gamma^*$ if $R \cap R_a \neq \emptyset$ (with the inequality binding when a is a MaxMin norm). The set of pure rules belonging to R_a, however, are not match-neutral if $a \notin E$. In the paper, we stated that there are different ways of extending the same claim to match-neutral rules. One alternative to R_a is R^+_a; we showed that the latter contains pure match-neutral rules and that, for the case of $T > 1$, $u(C) \geq u(a)$ for all $C \in \Gamma^*$ if

$R \cap R_a^+ \neq \emptyset$. Here, we consider an alternative approach to constructing match-neutral rules that establish the same claim (without restricting the analysis to $T > 1$).

Fix any a. Consider the following alternative to (10):

$$\forall n,\ a'_{-n} \text{ and } \theta \in \cup_m \hat{\Theta}^{-n}(m \mid a'_{-n}),\ \text{(a) } r_n(\theta)[B_n(a'_{-n})] > 0 \text{ if } \theta \notin \cup_{a'_n} \overline{\theta}(m, a \mid a'_n, a_{-n})$$
$$\text{and (b) } r_n(\theta)[B_n(a_{-n}) \cup a_n] > 0, \text{ otherwise.} \tag{30}$$

For any m, n, a'_{-n} and $\theta \in \hat{\Theta}^{-n}(m \mid a'_{-n})$, (30) requires playing wpp a best reply if $\theta \notin \cup_{a'_n} \overline{\theta}(m, a \mid a'_n, a_{-n})$ and either a best response or a_n, otherwise.

We shall refer to the set of rules that satisfy a-constrained (8), a-triggering (9), and (30) by R_a^* and call elements of this set a-star rules. We will consider a-star rules that satisfy one of the following conditions:

$$\text{if } a_n \neq B_n(a_{-n}) \text{ then } r_n(\theta)[a_n] = 0\ \forall \theta \in \Theta^s \backslash \overline{\Theta}(a) \tag{31}$$

or

$$\text{if } a_n \neq B_n(a_{-n}) \text{ then } r_n(\theta)[a_n] = 0\ \forall \theta \in \Theta \backslash \{\Theta^s \cup \Theta^-\}, \tag{32}$$

where $\overline{\Theta}(a) = \cup_{m',k,a'} \overline{\theta}(m', (a_k, a'_{-k}) \mid a')$ is the set of stationary 1-deviation histories from a. Conditions (31) and (32) require that whenever a_n is not a best reply to a_{-n}, either (i) a_n is played at a stationary history θ only if $\theta \in \overline{\Theta}(a)$ or (ii) a_n is played at a non-stationary history θ only if $\theta \in \Theta^-$ respectively. Denote the set of a-star rules that satisfy (31) by $R_a^*(1)$ and the set of a-star rules that satisfy (32) by $R_a^*(2)$.

Note the following points. First, (30) implies that $r_n(\theta(a'))[B_n(a'_{-n})] > 0$ for all $a' \neq a$; hence, $R_a^*(j) \cap \overline{R}^{pure} \subset R^E$ for any $j = 1, 2$. Second, $R_a^*(j)$ is non-empty for any $j = 1, 2$: Clearly, (30), (31) and (32) are well defined, and are consistent with each other; furthermore, they are also are consistent with (8) and (9) because (30), (31) and (32) allow for the possibility of playing a_n at every stationary 1-deviation histories $\overline{\Theta}(a)$. Third, $R_a^*(j)$ contains pure rules that are match-neutral for any $j = 1, 2$. An example of such a rule is the following: in any role n, play (i) a_n at every stationary 1-deviation history $\theta \in \{\cup_{m,a'_n} \overline{\theta}(m, a \mid a'_n, a_{-n})\} \cup \{\cup_{m,k \neq n, a'_k \neq a_k} \overline{\theta}(m, (a_k, a'_{-k}) \mid a')\}$, (ii) $B_n(a'_{-n})$ at every 1-deviation history $\theta \in \cup_{m,a'_n} \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \overline{\theta}(m, a \mid a'_n, a_{-n})$ for all a'_{-n} and (iii) $B_n(a_{-n})$ at every other history. Fourth, when $a_n = B_n(a)$, (31) and (32) are trivially satisfied; hence, if $a \in E$ then $R_a^* = R_a^*(1) = R_a^*(2)$.

We next demonstrate the claim that, for every $C \in \Gamma^*$, $u(C)$ is bounded below by $u(a)$ if a-star rules are feasible.

Proposition 43 *Assume HDM. Fix any $a \in A$. Suppose either (a) $M > 2$ and $R \cap R^*_a(1) \neq \emptyset$ or (b) $T > 1$ and $R \cap R^*_a(2) \neq \emptyset$. Then $u(C) \geq u(a)$ for any $C \in \Gamma^*$.*[1]

Since Proposition 43 holds for any a and, for any $j = 1, 2$, $R^*_a(j) \cap \overline{R}^{pure}$ is a non-empty subset of R^E and includes match-neutral rules, it follows that if either $M > 2$ or $T > 1$ then there exists a match-neutral $r \in R^E \cap \overline{R}^{pure}$ such that if r is feasible then $u(C) = u(\overline{a})$ for any $C \in \Gamma^*$. Thus, Part (i) of Theorem 2 holds even if R is restricted to match-neutral ones as long as either $M > 2$ or $T > 1$.

Our results for 2-complexity rules also extends to match-neutral rules. For this we weaken (17) to: for all n

$$r_n(\theta)[b_n] = 1 \text{ if } \theta \in \cup_{m,a'} \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \overline{\theta}(m, a \mid a'_n, a_{-n}) \tag{33}$$

and require (8), (9) and either (31) or (32):[2] Let

$$\mathcal{R}^*_a(1) = \left\{ R' \subset S^2 \;\middle|\; \begin{array}{c} \forall r \in R' \text{ satisfies (8), (9) and} \\ \forall b \in A,\ \exists\, r \in R' \text{ that satisfies (33) and (31) } \forall n \end{array} \right\} \tag{34}$$

$$\mathcal{R}^*_a(2) = \left\{ R' \subset S^2 \;\middle|\; \begin{array}{c} \forall r \in R' \text{ satisfies (8), (9) and} \\ \forall b \in A,\ \exists\, r \in R' \text{ that satisfies (33) and (32) } \forall n \end{array} \right\} \tag{35}$$

Clearly, $\mathcal{R}^*_a(j)$ is non-empty and includes pure match-neutral rules for all $j = 1, 2$. We can now state our result for match-neutral 2-complexity rules.

Proposition 44 *Assume HDM. Fix any $a \in A$. Suppose there exists $R' \subseteq R$ such that either (a) $M > 2$ and $R' \in \mathcal{R}^*_a(1)$ or (b) $T > 1$ and $R' \in \mathcal{R}^*_a(2)$. Then $u(C) \geq u(a)$ for every $C \in \Gamma^*$.*

Proposition 44 when a is a MaxMin norm demonstrates that if either $M > 2$ or $T > 1$ then part (ii) of Theorem 2 holds even if R is restricted to match-neutral ones.

Proof of Propositions 43 and 44

In the rest of this subsection $a \in A$ is fixed throughout. Also, for any $m \in M$ and $\omega \in \Omega$, we use the term m-agent of ω to denote any agent that plays in match m in the last period of ω. Next, we introduce the concept of payoff invading (henceforth, p-invading) rules.

[1] Note that the above results excludes the possibility of $T = 1$ and $M = 2$. In this case we conjecture that a similar result may not be possible as there may not be enough leeway to construct a pure match-neutral rule that is both 1-mutation from a and has appropriate invading property.

[2] Condition (33) is similar to (30a) except that the latter requires playing a best response at such histories whereas (33) requires playing a fixed action b_n at all such histories.

Definition 45 *(i) Any* $R_n' \subseteq R_n$ *for role* n *is p-invading at* $C' \in \Gamma$ *if, whenever* $R_n' \cap R_n(C') = \emptyset$*, for any* $\omega \in C'$ *and* $m \in M$ *such that* $u(\theta[\omega]^{m,T}) = u(C')$*, there exists* $r_n \in R_n'$ *such that one mutation by any* m*-agent of* ω *at* ω *moves the system wpp from* ω *to some* $C \in \Gamma$ *with* $R(C) \in (r_n, R_{-n}(C'))$*.*

(ii) Any $R' \subseteq R$ *is p-invading at* $v \in \mathbb{R}$ *if* R_n' *is p-invading at all* $C' \in \Gamma$ *such that* $u(C') \leq v$*, for all* n*.*

Lemma 46 *Fix any* $C' \in \Gamma$ *and* $r' \in R$ *such that* $R(C') = r'$*. There exists* $\omega \in C'$*,* $m \in M$*,* $d \in A$ *and* $a'' \in A$ *such that* $\theta[\omega] = \overline{\theta}(m, a'' \mid d)$*,* $u(\omega) = u(a'') = u(C')$ *and either* $d \neq a$ *or* $r'(\theta[\omega])[a] = 1$*. If in addition* $u(C') < u(a)$ *then* $a'' \neq a$*.*

Proof. Given the definition of $u(C')$, there exists $a'' \in A(C')$ and $\omega' \in C'$ such that $u(a'') = u(C')$ and $\theta[\omega']^{m,T} = a''$ for some m. Fix any $d \in A$ such that $r'(\theta[\omega'])[d] > 0$. By Lemmas 34 (setting $r_n = r_n'$) and 32, starting from ω' wpp the system reaches ω such that $\theta[\omega] = \overline{\theta}(m, a'' \mid d)$ and $R(\omega) = r'$. Clearly, $u(\omega) = u(a'') = u(C')$ and $\omega \in C'$. This implies the first claim in the lemma if $r'(\theta[\omega])[a] = 1$. So suppose otherwise; then exists $d' \neq a$ such that $r'(\theta[\omega])[d'] > 0$. But then, by Lemmas 34 (setting $r_n = r_n'$) and 32, starting from ω wpp the system reaches state $\hat{\omega}$ such that $\theta[\hat{\omega}] = \overline{\theta}(m, a'' \mid d')$. But then the first claim in the lemma must also hold in this case as $\hat{\omega} \in C'$ and $d' \neq a$.

If $u(C') < u(a)$ then, by $u(C') = u(a'')$, we also have $a'' \neq a$. ■

Lemma 47 *Fix any* $R' \subseteq R$ *and* $C^0 \in \Gamma^*$*. Suppose* R' *is p-invading at* $u(C^0)$*. Then there exists* $\{C^1, .., C^N\}$ *such that, for any* $n = 1, .., N$*, (i)* $C^n \in \Gamma$*; (ii)* $R_n(C^n) \in R_n'$*, and* $R_{-n}(C^n) = R_{-n}(C^{n-1}))$*; (iii)* $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$ *if* $R_n(C^{n-1}) \neq R_n(C^n)$ *and* 0*, otherwise; (iv)* $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$*; and (v)* $u(C^0) \geq u(C^n)$*.*

Proof. The proof consists of applying the following claim recursively starting from $n = 1$.

The Induction Claim for any $n \geq 1$. Assume that if $n > 1$ then there exist $\{C^1, .., C^{n-1}\}$ such that, for any $\ell = 1, .., n-1$: $C^\ell \in \Gamma$; $R_\ell(C^\ell) \in R_\ell'$ and $R_{-\ell}(C^\ell) = R_{-\ell}(C^{\ell-1}))$; $res(C^{\ell-1}, C^\ell) = f^{\min}(C^{\ell-1})$ if $R_\ell(C^{\ell-1}) \notin R_\ell'$ and 0, otherwise; and $u(C^0) \geq u(C^\ell)$. Then there exists $C^n \in \Gamma$ such that $R_n(C^n) \in R_n'$ and $R_{-n}(C^n) = R_{-n}(C^{n-1}))$; $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$ if $R_n(C^{n-1}) \notin R_n'$ and 0, otherwise; $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$; and $u(C^0) \geq u(C^n)$.

Proof of the Induction Claim. Since $u(C^0) \geq u(C^\ell)$ for $\ell = 1, .., n-1$, the assumption that R' is p-invading at $u(C^0)$ implies that R_n' is p-invading at C^{n-1}. If $R_n(C^{n-1}) \in R_n'$

set $C^n = C^{n-1}$; then the claim holds trivially. So suppose $R_n(C^{n-1}) \notin R'_n$. By the definition of $u(C^{n-1})$, there exists $\omega^{n-1} \in C^{n-1}$ and m such that $u(\theta[\omega^{n-1}]^{m,T}) = u(C^{n-1})$. Since R' is p-invading at C^{n-1}, there exists $r_n \in R'_n$ such that one mutation by any m-agent of ω^{n-1} at ω^{n-1} to r_n moves the system wpp from ω^{n-1} to some $C^n \in \Gamma$ with $R(C^n) \in (r_n, R_{-n}(C^{n-1}))$. Given that one such m-agent of ω^{n-1} receives $u(C^{n-1})$ at the last period of ω^{n-1}, by Lemma 28, $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$. Furthermore, by the same reasoning as in the proof of The Induction Claim in the proof of Lemma 29 $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$. Finally, we need to show $u(C^0) \geq u(C^n)$. To establish this, suppose otherwise. Fix any $i \in I$. Then, by Lemma 31, there exists $h^0 \in C^0 \times \Upsilon$ and $h^n \in C^n \times \Upsilon$ such that $\pi^T(i, h^0) = u(C^0) < u(C^n) = \pi^T(i, h^n)$. Hence, by (4), $f(i, h^0) < f(i, h^n)$ and, by Lemma 28, $f^{\min}(C^0) = f(i, h^0) < f(i, h^n) = f^{\min}(C^n)$. This, together with $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$, implies $\gamma(C^n) < \gamma(C^0)$, thereby contradicting $C^0 \in \Gamma^*$. ■

Lemma 48 *Fix any $C^0 \in \Gamma^*$. Suppose there exists $R' \subseteq R$ such that R' is p-invading at $u(C^0)$ and 1-mutation from a. Then $u(C^0) \geq u(a)$.*

Proof. Suppose not; then $u(C^0) < u(a)$. Fix any C^0-tree $\tau^*_{C^0} \in \arg\min_{\tau \in T_{C^0}} res(\tau)$. Since R' is p-invading at $u(C^0)$, by Lemma 47, there exists $C^N \in \Gamma$ such that

$$\gamma(C^N) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^N) \tag{36}$$

and $R(C^N) \in R'$. Since R' is 1-mutation from a, by Lemma 30, there exists $C \in \Gamma$ such that $A(C) = a$, $R(C) \in R'$ and $\gamma(C) \leq \gamma(C^N) + f^{\min}(C^N) - f^{\min}(C)$. But then by (36) we have

$$\gamma(C) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C). \tag{37}$$

Since $A(C) = a$, we have $u(C^0) < u(C)$. Fix any i. Then, by Lemma 31, there exists $h^0 \in H(C^0)$ and $h \in H(C)$ such that $\pi^T(i, h^0) = u(C^0)$ and $\pi^T(i, h) = u(C)$. Given $u(C^0) < u(C)$ and (4), we have $f(i, h^0) < f(i, h)$. Thus, by Lemma 28 we have $f^{\min}(C^0) = f(i, h^0) < f(i, h) = f^{\min}(C)$. But then, by (37), $\gamma(C) < \gamma(C^0)$; this contradicts the supposition that $C^0 \in \Gamma^*$. ■

Lemma 49 *Fix any $\omega^0 \in \Omega$, $a'_{-n} \in A_{-n}$, $m \in M$, $n \in N$, $r_n \in R_n$ and $r' \in R$ such that $\theta[\omega^0] \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \cup_{m', a'_n} \overline{\theta}(m', a \mid a'_n, a_{-n})$, $r_n(\theta')[B_n(a'_{-n})] > 0$ for all $\theta' \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \cup_{m', a'_n} \overline{\theta}(m', a \mid a'_n, a_{-n})$ and $R[\omega^0]^T = q^n(m, r_n \mid r')$. Then from ω^0 the system reaches some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$.*

Proof. First, we establish the following claim.

Claim A. There exists some $\tau \in T$ and $\{b_n^t, c_n^t, \omega^t\}_{t=1}^{\tau} \in K^{\tau}(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ such that $\theta[\omega^{\tau}]_n^{m,t} = B_n(a'_{-n})$ for every $t \in T$ and starting from ω^0 the system reaches wpp state ω^{τ}.

We prove this claim by recursion in two steps.

Step 1. There exists $\{b_n^1, c_n^1, \omega^1\} \in K^1(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ such that $b_n^1 = B_n(a'_{-n})$ and starting from ω^0 the system reaches wpp state ω^1.

By assumption, $r(\theta[\omega^0])[B_n(a'_{-n})] > 0$. Hence, the claim in this step follows from Lemma 34.

Step2. Fix any $0 < t < T$. If there exists $\{b_n^{t'}, c_n^{t'}, \omega^{t'}\}_{t'=1}^{t} \in K^t(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ such that $b_n^t = B_n(a'_{-n}) \neq \theta[\omega^t]_n^{m,t'}$ for some $t' < T$, then there exists $\{b_n^{t+1}, c_n^{t+1}, \omega^{t+1}\} \in K^1(m, n, r_n, r' \mid a'_{-n}, \omega^t)$ such that $b_n^{t+1} = B_n(a'_{-n})$ and starting from ω^t the system reaches wpp ω^{t+1}.

Since $b_n^t = B_n(a'_{-n}) \neq \theta[\omega^t]_n^{m,t'}$ for some $t' < T$, it follows that $\theta[\omega^t] \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \setminus \cup_{m',a'_n} \overline{\theta}(m', a \mid a'_n, a_{-n})$. But then by assumption $r_n(\theta[\omega^t])[B_n(a'_{-n})] > 0$. Then the claim in this step follows from Lemma 34.

The statement in Claim A follows from Step 1 if $T = 1$ or $\theta[\omega^0]_n^{m,t} = B_n(a'_{-n})$ for every $t > 1$ (in these case $\tau = 1$); otherwise, it follows from Step 1 and by applying Step 2 recursively.

By Claim A, $\theta[\omega^{\tau}]_n^{m,t} = B_n(a'_{-n})$ for every $t \in T$. Hence, at every period in state ω^{τ}, r_n performs at least as well as rule r'_n. Thus, by Monotonicity, r_n will be adopted by all agents in role n wpp at ω^{τ}.[3] Hence, by Lemma 32, the system will reach some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$. ■

Lemma 50 *Fix any $\omega^0 \in \Omega$, $a'_{-n} \in A_{-n}$, $m \in M$, $n \in N$, $r_n \in R_n$ and $r' \in R$ such that $\theta[\omega^0] = \overline{\theta}(m, a'' \mid d)$ for some a'' and $d \in A$, $r'_{-n}(\theta[\omega^0])[a'_{-n}] > 0$ and r_n satisfies (30). Suppose if $r_n(\theta)[B_n(a_{-n})] = 0$ for some $\theta \in \cup_{m',a'_n \neq a_n} \overline{\theta}(m', a \mid a'_n, a_{-n})$ then one of the following holds: (i) $r'_{-n}(\theta[\omega^0])[a_{-n}] = 0$, (ii) $r'_n(\theta[\omega^0])[a_n] = 1$, (iii) $r_n(\theta[\omega^0])[a_n] = 0$, (iv) $\theta[\omega^0] \in \cup_{c_n \neq a_n} \theta(c_n, a_{-n})$ or (v) $a'' \neq a$, $d \neq a$ and (32) holds. Assume also that either (a) $R[\omega^0] = r'$ and there is one mutation by any agent at ω^0 to r_n in role n of match m or (b) $R[\omega^0]^T = q^n(m, r_n \mid r')$. Then starting from ω^0 the system reaches some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$ wpp.*

[3]Note that in state ω^{τ} all agents in role n are active. Therefore, this result holds if Monotonicity assumption applied only in roles where agents do not suffer from action inertia.

Proof. Fix any $\{b_n^t, c_n^t, \omega^t\}_{t=1}^T \in K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ such that if $r_n(\theta[\omega^0])([B_n(a'_{-n})] > 0$ then $b_n^1 = B_n(a'_{-n})$. Given the definition of $K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$, such a sequence exists. Furthermore, under the assumptions of the lemma the system reaches ω^T from ω^0 wpp (when (a) in the lemma holds this follows from applying Lemma 33 followed by Lemma 34 $T-1$ times, and when (b) in the lemma holds this follows from applying Lemma 34 T times). Now if

$$\pi_n(b_n^{t'}, a'_{-n}) \geq \pi_n(c_n^{t''}, a'_{-n}) \text{ for all } t' \text{ and } t'' \in T \tag{38}$$

then r_n performs at least as well as r'_n in every period of ω^T; hence, by Monotonicity, wpp r_n will be adopted by all agents in role n in the next period, and thus the claim follows from Lemma 32. So suppose otherwise. To complete the proof of the lemma it is then sufficient to show the following:

$$\text{from } \omega^T \text{ the system reaches some } C \in \Gamma \text{ s.t. } R_n(C) = (r_n, r'_{-n}). \tag{39}$$

Since (38) does not hold, it follows from the definition of ω^T that $\theta[\omega^T] \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \theta(a)$ and $R[\omega^T]^t = q^n(m, r_n \mid r')$ for all t. Then, given that r_n satisfies (30), by Lemmas 37 and 49**,** (39) holds if either $r_n(\theta')[B_n(a_{-n})] > 0$ for all $\theta' \in \cup_{m',a'_n} \overline{\theta}(m', a \mid a'_n, a_{-n}) \backslash \theta(a)$ or $\theta[\omega^T] \notin \cup_{m',a'_n} \overline{\theta}(m', a \mid a'_n, a_{-n})$. So assume that neither of these two conditions hold. Then, by assumption, $a_n \neq B_n(a_{-n})$, $\theta[\omega^T] \in \cup_{m',a'_n \neq a_n} \overline{\theta}(m', a \mid a'_n, a_{-n})$ and one of the conditions (i)-(v) in the lemma holds.

Next we show that $\theta[\omega^T] \notin \cup_{a'_n \neq a_n} \overline{\theta}(m, a \mid a'_n, a_{-n})$. Suppose not; then $a'_{-n} = a_{-n}$, $c_n^t \neq a_n$ and $b_n^t = a_n$ for all $t \in T$. But these contradict (i)-(iii) in the lemma. So either (iv) or (v) must hold. If the former holds then $\theta[\omega^0] \in \hat{\Theta}^{-n}(m \mid a_{-n}) \backslash \cup_{m',a'_n} \overline{\theta}(m', a \mid a'_n, a_{-n})$ and thus, by construction of b_n^1, $b_n^1 = B_n(a_{-n})$; but this contradicts $b_n^1 = a_n \neq B_n(a_{-n})$. So assume (v) holds. Then $a'' \neq a$ and $d \neq a$; this together with $b_n^1 = a_n \neq B_n(a_{-n})$ implies that $\theta[\omega^0]_{-n}^{m',T} \neq a_{-n}$ *for some* m'. Since we also have assumed $\theta[\omega^T] \in \overline{\theta}(m, a \mid a'_n, a_{-n})$ for some $a'_n \neq a_n$, it follows that $\theta[\omega^1]_{-n}^{m',T} = a_{-n}$; but then given that r_n satisfies (32), we have $r_n(\theta[\omega^1])[a_n] = 0$. But this contradicts $b_n^2 = a_n$.

Given the above, to complete the proof we only need to consider the case in which $M = 2$ and $\theta[\omega^T] = \overline{\theta}(m', a \mid a'_n, a_{-n})$ for some $a'_n \neq a_n$ and $m' \neq m$. This means $b_n^t = a'_n$ for all $t \in T$ and therefore, given that we have assumed (38) is not the case, it must also be that $a'_n \neq B_n(a_{-n})$. Also, since $\theta[\omega^T] = \overline{\theta}(m', a \mid a'_n, a_{-n})$, by (30), there exists $b_n^{T+1} \in \{a_n \cup B_n(a_{-n})\}$ such that $r_n(\theta[\omega^T])[b_n^{T+1}] > 0$. Therefore, $b_n^{T+1} \neq b_n^t$ for all $t \leq T$. Next, fix any c_n^{T+1} such that $r_n'(\theta[\omega^T])[c_n^{T+1}] > 0$ and consider the two different possibilities.

Case A. Either $c_n^{T+1} \neq a_n$ or $T > 1$. For any $j \in \{0, 1\}$, by Lemma 34, starting from ω^T the system reaches wpp state $\omega(j) \in \Omega$ such that $\theta[\omega(j)]^T = \phi(m, (b_n^{T+j}, a_{-n}) \mid c_n^{T+1}, a_{-n}))$, $\theta[\omega(j)]^t = \theta[\omega^T]^{t+1}$ for all $t < T$ and $R[\omega(j)] = R[\omega^T]$. If $c_n^{T+1} \neq a_n$ then, by $b_n^T = a_n' \neq a_n$, $\theta[\omega(0)] \in \hat{\Theta}^{-n}(m \mid a_{-n}) \backslash \cup_{m', d_n'} \overline{\theta}(m', a \mid d_n', a_{-n})$ and if $T > 1$ then, by $b_n^{T+1} \neq b_n^T$, $\theta[\omega(1)] \in \hat{\Theta}^{-n}(m \mid a_{-n}) \backslash \cup_{m', d_n'} \overline{\theta}(m', a \mid d_n', a_{-n})$. Since starting from ω^T the system reaches wpp both $\omega(0)$ and $\omega(1)$, by Lemma 49 in both cases the system reaches some $C \in \Gamma$ s.t. $R_n(C) = (r_n, r_{-n}')$ from ω^T. Hence, we have (39).

Case B. $c_n^{T+1} = a_n$ and $T = 1$. By Lemma 34, starting from ω^T the system reaches wpp $\hat{\omega}$ such that $\theta[\hat{\omega}] = \phi(m, (b^{T+1}, a_{-n}) \mid (c_n^{T+1}, a_{-n}))$ and $R[\hat{\omega}] = R[\omega^T]$. Given that $b_n^{T+1} \in \{a_n \cup B_n(a_{-n})\}$, $\pi_n(b_n^{T+1}, a_{-n}) \geq \pi_n(c_n^{T+1}, a_{-n})$. Hence, r_n performs at least as well as rule r_n'. Thus, by Monotonicity, wpp r_n will be adopted by all agents in role n and Lemma 32 the system reaches some $C \in \Gamma$ such that $R_n(C) = (r_n, r_{-n}')$ from $\hat{\omega}$. Hence, we have (39). ■

Lemma 51 *Fix any $m \in M$, $n \in N$, $r_n \in R_n$, $r' \in R$ and $\omega \in \Omega$ such that $R(\omega) = r'$ and $r_n' \neq r_n$. Assume $r'(\theta[\omega])[a'] > 0$ and $r_n(\theta[\omega])[b_n] > 0$ for some $a' \in A$ and $b_n \in A_n$. Fix any $c_k^{m'} \in \{a_k', \theta_k^{m',T}\}$ for any $k \neq n$ and $m' \neq m$. Then one mutation by any m-agent of ω at ω to r_n in role n of match m moves the system wpp from ω to some state $\hat{\omega}$ such that $\theta[\hat{\omega}]^t = \theta[\omega]^{t+1}$ for all $t < T$, $\theta[\hat{\omega}]^{m,T} = (b_n, a_{-n}')$ and $\theta[\hat{\omega}]^{m',T} = (a_n', c_{-n}^{m'})$ for all $m' \neq m$, $R[\hat{\omega}]^T = q^n(m, r_n \mid r')$ and $R[\hat{\omega}]^{m'',t} = r'$ for all $(m'', t) \neq (m, T)$.*

Proof. Suppose that the system is in state ω at some date τ and consider any agent i that plays in match m in the last period of ω. Next assume that at τ (i) i is the only agent that is in a mutating status, (ii) i is assigned role n in match m (iii) every agent other than i that played in match m at date $\tau - 1$ is randomly assigned a role other than n in match m, (iv) all other agents are assigned the same matches and the same roles as at date $\tau - 1$, (v) agent i mutates to r_n and plays b_n and all other agents in match m follow their rule r_{-n}' and play a_{-n}', and (vi) in any match $m' \neq m$, the agent in role n follow her rule r_n' and plays a_n' and, for all $k \neq n$, the agent in role k plays $c_k^{m'}$ (this is feasible because if $c_k^{m'} = a_k'$ then the agent follows his rule r_k' and plays a_k' and if $c_k^{m'} = \theta_k^{m',T}$ then the agent suffers action inertia and plays $\theta_k^{m',T}$). Hence, at $\tau + 1$ the state $\hat{\omega}$ satisfying the claim in the lemma will be reached. ■

Lemma 52 *Fix any* $\omega^0 \in \Omega$, $a'_{-n} \in A_{-n}$, $m \in M$, $n \in N$, $r_n \in R_n$ *and* $r' \in R$ *such that* $\theta[\omega^0] = \overline{\theta}(m, a'' \mid d)$ *for some* $a'' \neq a$ *and* $d \in A$, $R[\omega^0] = r'$ *and* $r'_{-n}(\theta[\omega^0])[a'_{-n}] > 0$. *Assume either* $d \neq a$ *or* $r'(\theta[\omega^0])[a] = 1$. *Suppose* r_n *satisfies*

$$r_n(\theta)[B_n(a'_{-n})] = 1 \text{ for all } \theta \in \cup_{m,a'_n} \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \overline{\theta}(m, a \mid a'_n, a_{-n}), \tag{40}$$

and either (a) $M > 2$ *and (31) or (b)* $T > 1$ *and (32). Then one mutation by any* m*-agent of* ω^0 *at* ω^0 *to* r_n *in role* n *of match* m *moves the system from* ω^0 *wpp to some* $C \in \Gamma$ *such that* $R_n(C) = (r_n, r'_{-n})$.

Proof. The claim in the lemma follows from Lemma 50 if either $r_n(\theta)[B_n(a_{-n})] > 0$ for all $\theta \in \cup_{m', a'_n \neq a_n} \overline{\theta}(m', a \mid a'_n, a_{-n})$ or any of the conditions (i)-(v) in Lemma 50 holds. So assume otherwise; then $a_n \neq B_n(a)$ (this follows from (30)), $r'_{-n}(\theta[\omega^0])[a_{-n}] > 0$, $r'_n(\theta[\omega^0])[c^1_n] > 0$ for some $c^1_n \neq a_n$, $r_n(\theta[\omega^0])[a_n] > 0$, either $d \neq a''$ or $d = a''$ with $d_\ell \neq a_\ell$ for some $\ell \neq n$ and (v) in Lemma 50 does not hold. But by assumption $r'_n(\theta[\omega^0])[c^1_n] > 0$ for some $c^1_n \neq a_n$ implies that $d \neq a$. Since either $M > 2$ and (31) or $T > 1$ and (32), it then follows from $a'' \neq a$, $d \neq a$ and (v) in Lemma 50 not holding that $M > 2$ and $r_n(\theta)[a_n] = 0$ for all $\theta \in \Theta^s \backslash \overline{\Theta}(a)$.

Next we show that there exists $\ell \neq n$ such that $d_\ell \neq a_\ell$. Suppose not; then it must be that $d \neq a''$. Since $a_n \neq B_n(a)$, $r_n(\theta[\omega^0])[a_n] > 0$, $\theta[\omega^0] \in \Theta^s$, $M > 2$ and $r_n(\theta)[a_n] = 0$ for all $\theta \in \Theta^s \backslash \overline{\Theta}(a)$, it must then be that $\overline{\theta}(m, a'' \mid d) = \overline{\theta}(m, (a_k, d_{-k}) \mid d)$ for some k such that $d_k \neq a_k$. If $k \neq n$ then the claim follows. If $k = n$ then $a'' = (a_n, d_{-n})$. But given that $a'' \neq a$, it then follows that there exists $\ell \neq n$ such that $d_\ell \neq a_\ell$; a contradiction.

Since $r_n(\theta[\omega^0])[a_n] > 0$, $r'_n(\theta[\omega^0])[c^1_n] > 0$, $r'_{-n}(\theta[\omega^0])[a_{-n}] > 0$ and $\theta[\omega^0]^{m',T}_\ell = d_\ell$ for all $m' \neq m$, by applying Lemma 51 once and Lemma 32 $T-1$ times, one mutation by any m-agent of ω^0 at ω^0 to r_n in role n of match m moves the system wpp from ω^0 to state $\tilde{\omega}$ such that $\theta[\tilde{\omega}] = \overline{\theta}(m, a \mid (c^1_n, d_\ell, a_{-n,\ell}))$, $R^t[\tilde{\omega}] = q^n(m, r_n \mid r')$ for all t. Since $c^1_n \neq a_n$ and $d_\ell \neq a_\ell$, $\theta[\tilde{\omega}] \in \Theta^s \backslash \overline{\Theta}(a)$; therefore $r_n(\theta[\tilde{\omega}])[a_n] = 0$. The claim in the lemma then holds because, by Lemma 50 (see (iii) and (b) in Lemma 50), from $\tilde{\omega}$ the system reaches some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$ wpp. ■

Lemma 53 *Fix any* $r \in R^*_a$ *and* $v < u(a)$. *Then* r *is p-invading at* v *if either* $M > 2$ *and (31) or* $T > 1$ *and (32)*

Proof. Fix any $n \in N$, $C' \in \Gamma$ and r' such that $r' = R(C')$ and $u(C') \leq v$. If $r_n = r'_n$ then r_n is p-invading at C'. So suppose that $r'_n \neq r_n$. Since $u(C') \leq v < u(a)$, by Lemma

46, there exists $\omega \in C'$, $m \in M$, $d \in A$ and $a'' \neq a$ such that $\theta[\omega] = (\overline{\theta}(m, a'' \mid d)$, $u(C') = u(a'') = u(\omega)$ and either $d \neq a$ or $r'(\theta[\omega])[a] = 1$. Let a'_{-n} be such that $r'_{-n}(\theta[\omega])[a'_{-n}] > 0$. Then, given that r satisfies (30) and either $M > 2$ and (31) or $T > 1$ and (32) for all n , it follows from Lemma 52 that one mutation by any m-agent of ω at ω to r_n in role n of match m moves the system from ω to some $C \in \Gamma$ satisfying $R(C) = (r_n, R_{-n}(\omega))$ wpp. Thus, r_n is p-invading at C'. ■

Proof of Proposition 43. Suppose not; then there exists $C' \in \Gamma^*$ such that $u(C') < u(a)$. By assumption and Lemma 53, there exist $r \in R \cap R^*_a$ that is p-invading at $u(C')$. Also, given that r satisfies (11) and (12) for the case when $Q = a$, by Lemma 13 (setting $R' = r$), r is 1-mutation from a. Thus, it follows from Lemma 48 (by setting R' to equal r) that $u(C') \geq u(a)$; but this is a contradiction. ■

Lemma 54 *Fix any $R' \subseteq R$ such that either (a) $M > 2$ and $R' \in \mathcal{R}^*_a(1)$ or (b) $T > 1$ and $R' \in \mathcal{R}^*_a(2)$. Assume $v < u(a)$. Then R' is p-invading at v.*

Proof. Fix any $n \in N$, $C' \in \Gamma$ and r' such that $r' = R(C')$ and $u(C') \leq v$. If $r'_n \in R'_n$ then R'_n is p-invading at C'. So assume $r'_n \notin R'_n$. Since $u(C') \leq v < u(a)$, by Lemma 46, there exists $\omega \in C'$, $m \in M$, $d \in A$ and $a'' \neq a$ such that $\theta[\omega] = (\overline{\theta}(m, a'' \mid d)$, $u(C') = u(a'') = u(\omega)$ and either $d \neq a$ or $r'(\theta[\omega])[a] = 1$.

Next, fix any a'_{-n} such that $r'_{-n}(\theta[\omega])[a'_{-n}] > 0$. By assumption there exists $r_n \in R'_n$ that satisfies (40) and and either $M > 2$ and (31) or $T > 1$ and (32) for all n. But then, given that $r'_{-n}(\theta[\omega])[a'_{-n}] > 0$, it follows from Lemma 52 that one mutation by any m-agent of ω at ω to r_n in role n of match m moves the system from ω to some $C \in \Gamma$ satisfying $R(C) = (r_n, R_{-n}(\omega))$ wpp. Therefore, r_n is p-invading at C'. ■

Proof of Proposition 44. Suppose not; then there exists $C' \in \Gamma^*$ such that $u(C') < u(a)$. By assumption and Lemma 54, there exists $R' \subseteq R$ such that $R' \in \mathcal{R}^*_a(j)$ for some j and R' is p-invading at $u(C')$. Also, since for every $r \in R'$, (8) and (9) hold, by Lemma 13 R' is 1-mutation from a. Then it follows from Lemma 48 that $u(C') \geq u(a)$; but this is a contradiction. ■

3 Section 5 of the Paper

3.1 Proof of the Claims with No Inertia

First we introduce a concept of invasion based on last period actions.

Definition 55 *(i) Any* $R'_n \subseteq R_n$ *for role* n *is T-invading at* $C' \in \Gamma$ *if, whenever* $R'_n \cap R_n(C') = \emptyset$ *, there exists* $a' \in A(C')$*,* $\omega \in C'$ *and* $r_n \in R'_n$ *such that* $\theta[\omega]^T = \phi(a')$ *and* $u(a') = u(\omega) = u(C')$*, and one mutation by any agent at* ω *to* r_n *moves the system wpp from* ω *to some* $C \in \Gamma$ *with* $R(C) \in (r_n, R_{-n}(C'))$*.*

(ii) Any $R' \subseteq R$ *is T-invading at* $v \in \mathbb{R}$ *if* R'_n *is T-invading at all* $C' \in \Gamma$ *such that* $u(C') \leq v$*, for all* n*.*

Lemma 56 *For any* $C \in \Gamma$*,* $R(C)$ *is unique and for any* $a' \in A(C)$ *there exists* ω' *such that* $\theta[\omega']^T = \phi(a')$*. Also, if* $R(C) \in \overline{R}^{pure}$ *then, for each* $\omega \in C$*,* $\theta[\omega]^t \in \cup_{a' \in A} \phi(a')$ *for all* t*.*

Proof. Fix any $\omega \in C$. Let $r \in R(\omega)$ such that $\rho(\omega)[r] > 0$. Then starting from ω the system moves next date wpp to some ω' such that $R[\omega']^{m,T} = r$ for all m. By (19) after $T-1$ periods the system reaches wpp some state ω'' such that $R[\omega'']^{m,t} = r$ for all m and t. By No-Birth assumption it follows that $R(C) = r$ and thus $R(C)$ is unique.

Next, fix any $a' \in A(C)$. As agents are not subject to action inertia, it must be that there exists $\omega'' \in C$ such that $R(C)(\omega'')[a'] > 0$. Then starting from ω'' the system moves next date wpp to some ω' such that $\theta[\omega']^T = \phi(a')$.

Finally, the last claim in the lemma follows trivially from $R(C)$ being unique. ■

Lemma 57 *Fix any* $R' \subseteq R$ *and* $C^0 \in \Gamma^*$*. Suppose* R' *is T-invading at* $u(C^0)$*. Then there exists* $\{C^1, .., C^N\}$ *such that, for any* $n = 1, .., N$*, (i)* $C^n \in \Gamma$*; (ii)* $R_n(C^n) \in R'_n$*, and* $R_{-n}(C^n) = R_{-n}(C^{n-1}))$*; (iii)* $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$ *if* $R_n(C^{n-1}) \neq R_n(C^n)$ *and* 0*, otherwise; (iv)* $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$*; and (v)* $u(C^0) \geq u(C^n)$*.*

Proof. The proof consists of applying the following claim recursively starting from $n = 1$.

The Induction Claim for any $n \geq 1$*.* Assume that if $n > 1$ then there exist $\{C^1, .., C^{n-1}\}$ such that, for any $\ell = 1, .., n-1$: $C^\ell \in \Gamma$; $R_\ell(C^\ell) \in R'_\ell$ and $R_{-\ell}(C^\ell) = R_{-\ell}(C^{\ell-1}))$; $res(C^{\ell-1}, C^\ell) = f^{\min}(C^{\ell-1})$ if $R_\ell(C^{\ell-1}) \notin R'_\ell$ and 0, otherwise; and $u(C^0) \geq u(C^\ell)$. Then there exists $C^n \in \Gamma$ such that $R_n(C^n) \in R'_n$ and $R_{-n}(C^n) = R_{-n}(C^{n-1}))$; $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$ if $R_n(C^{n-1}) \notin R'_n$ and 0, otherwise; $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$; and $u(C^0) \geq u(C^n)$.

Proof of the Induction Claim. Since $u(C^0) \geq u(C^\ell)$ for $\ell = 1, .., n-1$, the assumption that R' is T-invading at $u(C^0)$ implies that R'_n is T-invading at C^{n-1}. If $R_n(C^{n-1}) \in R'_n$ set $C^n = C^{n-1}$; then the claim holds trivially. So suppose $R_n(C^{n-1}) \notin R'_n$. By the definition of $u(C^{n-1})$, there exists $a^{n-1} \in A(C^{n-1})$, $\omega^{n-1} \in C^{n-1}$ and $r_n \in R'_n$ such that $\theta[\omega^{n-1}]^T = \phi(a^{n-1})$ and $u(a^{n-1}) = u(\omega^{n-1}) = u(C^{n-1})$, and one mutation by any agent at ω^{n-1} to r_n moves the system wpp from ω^{n-1} to some $C^n \in \Gamma$ with $R(C^n) \in (r_n, R_{-n}(C^{n-1}))$. Given that there exists at least one agent who receives $u(C^{n-1})$ at the last period of ω^{n-1}, by Lemma 28, $res(C^{n-1}, C^n) = f^{\min}(C^{n-1})$. Furthermore, by the same reasoning as in the proof of The Induction Claim in the proof of Lemma 47, $\gamma(C^n) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^n)$ and $u(C^0) \geq u(C^n)$. ■

Lemma 58 *Fix any $C^0 \in \Gamma^*$. Suppose there exists $R' \subseteq R$ such that R' is T-invading at $u(C^0)$ and 1-mutation from a. Then $u(C^0) \geq u(a)$.*

Proof. Suppose not; then $u(C^0) < u(a)$. Fix any C^0-tree $\tau^*_{C^0} \in \arg\min_{\tau \in T_{C^0}} res(\tau)$. Since R' is T-invading at $u(C^0)$, by Lemma 57, there exists $C^N \in \Gamma$ such that $\gamma(C^N) \leq \gamma(C^0) + f^{\min}(C^0) - f^{\min}(C^N)$ and $R(C^N) \in R'$. The rest of the proof is identical to that in Lemma 48. ■

Next we show that for any a, $R' \in \widetilde{\mathcal{R}}_a$ is both invading and T-invading at any $v < u(a)$ and 1-mutation from a.

Lemma 59 *Fix any a, $R' \subseteq R$ and $v < u(a)$. Suppose $R' \in \widetilde{\mathcal{R}}_a$. Then R' is both w-invading and T-invading at v.*

Proof. Fix any $n \in N$, $C' \in \Gamma$ and r' such that $r' = R(C')$. If $r'_n \in R'_n$ then R'_n is both w-invading and T-invading at C'. So assume $r'_n \notin R'_n$. Next, we establishing two claims.

Claim 1. Assume $A(C') = a$. Then R'_n is w-invading at C'.

Since $A(C') = a$, C' is singleton and equal to some ω such that $\theta[\omega] = \theta(a)$ and $r'(\theta[\omega])[a] = 1$. Fix any $r_n \in R'_n$. By (20a), $r_n(\theta[\omega])[a_n] = 1$. By Lemma 33, one mutation by any agent at ω to r_n in role n of match m moves the system wpp from ω to some state ω' such that $\theta[\omega']^{m'',T} = a$ for all m'', $R[\omega']^{m,T} = (r_n, r'_{-n})$ and $R[\omega']^{m',t} = r'$ for all $(m', t) \neq (m, T)$. As r_n performs equally well as r'_n does at the last period of ω', by Monotonicity assumption (19), wpp r_n will be chosen by all agents in role n for the next period. Applying (19) for $T - 1$ more periods, the system reaches wpp some state ω'' such that $R[\omega'']^{m',t} = (r_n, r'_{-n})$ for all m' and t. This establishes the claim.

Claim 2. Suppose there exists $\omega \in C'$ and $a'' \neq a$ such that $\theta[\omega]^T = \phi(a'')$. Then one mutation by any agent at ω to r_n moves the system wpp from ω to some $C \in \Gamma$ with $R(C) \in (r_n, R_{-n}(C'))$.

Fix any a' such that $r'(\theta[\omega])[a'] > 0$. Let $b_n = B_n(a'_{-n})$. As $R' \in \widetilde{\mathcal{R}}_a$, by (20b), $\theta[\omega]^T = \phi(a'')$ and $a'' \neq a$, there exists $r_n \in R'_n$ such that $r_n(\theta[\omega])[b_n] = 1$. Therefore, by Lemma 33, one mutation by any agent at ω to r_n in role n of match m moves the system wpp from ω to some state ω' such that $\theta[\omega']^{m,T} = (b_n, a'_{-n})$, $\theta[\omega']^{m',T} = a'$ for all $m' \neq m$, $\theta[\omega']^t = \theta[\omega]^{t+1}$ for all $t < T$, $R[\omega']^{m,T} = (r_n, r'_{-n})$ and $R[\omega']^{m',t} = r'$ for all $(m', t) \neq (m, T)$. As $b_n = B_n(a'_{-n})$, by Monotonicity assumption (19), wpp r_n will be chosen by all agents in role n for the next period. Applying (19) for $T-1$ more periods the system reaches wpp some state ω'' such that $R[\omega'']^{m',t} = (r_n, r'_{-n})$ for all m' and for all t. This establishes the claim.

Now if $A(C') \neq a$, by Lemma 56, there exists $\omega \in C'$ and $a'' \neq a$ such that $\theta[\omega]^T = \phi(a'')$. So it follows by Claims 1 and 2 that R_n is w-invading at C'. Furthermore, if $u(C') \leq v$ then by $v < u(a)$ we have $A(C') \neq a$. So, by Claim 2, R_n is T-invading at C' if $u(C') \leq v$. ■

Lemma 60 *Fix any $a \in A$, $R' \in \widetilde{\mathcal{R}}_a$ such that $R' \subseteq R$. Then R' is 1-mutation from a.*

Proof. Fix any $i \in I$, $C' \in \Gamma$, $\omega \in C'$, and $r \in R'$ such that $A(C') \neq a$ and $R(C') = r$. Since $r \in R' \subseteq \overline{R}^{pure}$, by Lemma 56, there exists $a'' \in A(C')$ such that $\theta[\omega]^T = \phi(a'')$. Let $a' \in A$ be such that $r(\theta[\omega])[a'] = 1$. Next note that $a'_n \neq a_n$ for some n. Otherwise, $r(\theta[\omega])[a] = 1$ and, by (20a), a will be chosen by all agents for all the following periods; but this implies $C' = (\theta(a), r)$, contradicting the assumption that $A(C') \neq a$. Furthermore, since $R' \in \widetilde{\mathcal{R}}_a$ and $\theta[\omega]^T = \phi(a'')$, by (20), there exists $r'_n \in R'_n$ such that $r'_n(\theta[\omega])[a_n] = 1$.

Next, suppose at date τ the system is at state ω. Fix any m. By Lemma 33, one mutation by any agent at ω to r'_n in role n of match m moves the system wpp from ω to some state ω' such that $\theta[\omega']^T = \overline{\phi}(m, (a_n, a'_{-n}) \mid a')$, $R[\omega']^{m,T} = (r'_n, r_{-n})$ and $R[\omega']^{m',t} = r$ for all $(m', t) \neq (m, T)$. Now, by (20a), $r'_n(\theta[\omega'])[a_n] = 1$ and $r(\theta[\omega'])[a] = 1$. Given that $\rho_n(\omega)[r_n, r'_n] = 1$, it then follows that wpp the system reaches a state ω'' such that $\theta[\omega'']^T = \phi(a)$, $R[\omega'']^{m',T} \in \{r, (r'_n, r_{-n})\} \subset R'$ for all m'. Then by applying (20a) and (19) for the next $T-1$ periods, the system reaches wpp a recurrent class $A(C) = a$ and $R(C) \in \{r, (r'_n, r_{-n})\}$ and we establish the claim in the lemma. ■

Proof of Proposition 23 . Suppose there exists $R' \subseteq R$ such that $R' \in \widetilde{\mathcal{R}}_a$. Then, by Lemmas 59 and 60, R' is w-invading, T-invading at $v < u(a)$ and 1-mutation from a. Then (i) under HDM, by Lemma 58, that $u(C) \geq u(a)$ for every $C \in \Gamma^*$ and (ii) under HIM, by Lemma 21, there exists $C \in \Gamma^*$ such that $A(C) = a$. ■

3.2 Proof of the Claims When Monotonicity and Observability of Rules Are Relaxed

In this subsection we shall demonstrate how the selection result in Proposition 15 and the indeterminacy result in Proposition 22, both regarding R_Q, can be extended when Monotonicity selection criterion is replaced by s-Justifiable criterion. The other results with the sets $\mathcal{R}_a$ and R_a^* (for the case when $M > 2$) also extend by similar type of reasoning.

Fix any $Q \subseteq A$. We first modify the Q-triggering property in (12) by

$$\forall n \text{ and } a'_{-n} \in A_{-n}, \exists\, m \text{ s.t. for all } a_n \in Q_n,\, a'_n \in A_n \text{ and } \theta \text{ s.t. } \theta^T = \overline{\phi}(m, (a_n, a'_{-n}) \mid a')$$
$$\text{(a) } r_{-n}(\theta)[Q_{-n}] > 0, \text{ if } a'_{-n} \notin Q_{-n} \text{ and } a'_n \notin Q_n \text{ and} \tag{41}$$
$$\text{(b) } r_n(\theta)[B_n(a'_{-n} \mid Q)] = 1, \text{ if } a'_{-n} \in Q_{-n}.$$

Condition (41) is the same restriction as (12) except that, for any n and $a'_{-n} \in A_{-n}$, the latter is a restriction at all stationary $\theta \in \cup_{a_n \in Q_n, a'_n \in A_n} \overline{\theta}(m, (a_n, a'_{-n}) \mid a')$ whereas the former applies at all θ such that $\theta^T \in \cup_{a_n \in Q_n, a'_n \in A_n} \overline{\phi}(m, (a_n, a'_{-n}) \mid a')$. Next, we modify the Q-star best property in (13) by

$$\forall n \text{ and } a'_{-n} \in A_{-n}, \exists\, m' \text{ s.t. } r_n(\theta)[B_n(a'_{-n})] = 1,\ \forall \theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n}; T) \backslash \Theta^u(Q; T), \tag{42}$$

where, with some abuse of notation, $\Theta^u(Q;T) = \left\{\theta \in \Theta \,\middle|\, \theta^T \in \cup_{a \in Q} \phi(a)\right\}$ and $\hat{\Theta}^{-n}(m' \mid a'_{-n}; T) = \left\{\theta \in \Theta \,\middle|\, \theta^T \in \cup_{a'_n, a''_n \in A_n} \overline{\phi}(m', (a''_n, a'_{-n}) \mid a'_n, a'_{-n})\right\}$. Condition (42) is the same as (13) except that, for any n and $a'_{-n} \in A_{-n}$, the latter is a restriction at all stationary $\theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n}) \backslash \Theta^u(Q)$ whereas the former applies at all θ such that $\theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n}; T) \backslash \Theta^u(Q; T)$. Now define the sets R_Q^T as follows.

$$R_Q^T = \left\{ r \in \overline{R} \,|\, r \text{ satisfies conditions (11), (41) and (42)} \right\}.$$

Note that if $Q \in W$ then $r \in R_Q^T$ if and only if it satisfies (11), (41a) and

$$\forall n \text{ and } a'_{-n} \in A_{-n}, \exists\, m' \text{ s.t. } r_n(\theta)[B_n(a'_{-n})] = 1,\ \forall \theta \in \hat{\Theta}^{-n}(m' \mid a'_{-n}; T). \tag{43}$$

(this is because when $Q \in W$ (43) is equivalent to (42) and (41b)).

The analogue of Propositions 15 for the set R_Q^T with s-Justifiable selection criterion is the following.

Proposition 61 *Assume HDM and rule selection criterion $\rho(.)$ is s-Justifiable. Fix any Q such that either $Q \in W$ or Q is singleton and suppose $R \cap R_Q^T \neq \emptyset$. Then*

$u(C) \geq u(Q)$ *for any* $C \in \Gamma^*$.

Proof of Proposition 61

Given Lemma 12, the proof of Proposition 61 involves showing that any $r \in R \cap R_Q^T$ is 1-mutation from Q and invading when $Q \in W$ or Q is singleton. In the Appendix to the paper, we showed that every $r' \in R \cap R_Q$ satisfies these properties if selection selection criteria is Monotonic. Here, we use a similar reasoning to show that with s-Justifiable selection criterion the same holds for any $r \in R \cap R_Q^T$.

For the rest of this subsection, assume rule selection criterion $\rho(.)$ is s-Justifiable. To show that any $r \in R \cap R_Q^T$ is 1-mutation from Q we need to first derive an analogue of Lemma 36 with s-Justifiable criterion.

Lemma 62 *Fix any* $Q \subseteq A$, $n \in N$, $m \in M$, $a \in Q$, $\tilde{\omega} \in \Omega$, $r \in R$ *and* $s_n \in S_n$ *such that* $\theta[\tilde{\omega}]_{-n}^{m',T} = a_{-n}$ *for all* m', $R[\tilde{\omega}]^T = q^n(m, s_n \mid r)$, $s_n(\theta')[a_n] > 0$ *for all* $\theta' \in \Theta$, r *satisfies (11) and*

$$r_n(\theta)[b_n] = 1, \forall \theta \text{ s.t. } \theta^T \in \cup_{a_n'' \in A_n} \overline{\phi}(m, a \mid a_n'', a_{-n}). \tag{44}$$

where $b_n = B_n(a_{-n} \mid Q)$. *Then from* $\tilde{\omega}$, *the system reaches wpp some* $C \in \Gamma$ *such that* $A(C) \subseteq Q$ *and* $R(C) = r$.

Proof. Fix any a_n' such that $r_n(\theta[\tilde{\omega}])[a_n'] > 0$. Since $\theta[\tilde{\omega}]_{-n}^{m',T} = a_{-n}$ for all m', and $s_n(\theta[\tilde{\omega}])[a_n] > 0$, by Lemma 34, starting at state $\tilde{\omega}$ the system reaches next date wpp state $\omega^0 \in \Omega$, where $\theta[\omega^0]^T = \overline{\phi}(m, a \mid a_n', a_{-n})$, $R[\omega^0]^T = q^n(m, s_n \mid r)$, $\theta[\omega^0]^t = \theta[\tilde{\omega}]^{t+1}$ and $R[\omega^0]^t = R[\tilde{\omega}]^{t+1}$ for all $t < T$. As $\theta[\omega^0]_{-n}^{m',T} = a_{-n}$ for all m', $s_n(\theta')[a_n] > 0$ for all $\theta' \in \Theta$ and (44) holds, it follows from applying Lemma 34 T times that starting at ω^0 the system reaches T periods later state $\omega^T \in \Omega$, where $\theta[\omega^T] = \overline{\theta}(m, a \mid b_n, a_{-n})$ and $R[\omega^T]^t = q^n(m, s_n \mid r)$ for all t. Since $b_n = B_n(a_{-n} \mid Q_n)$, $a_n \in Q_n$, $\theta[\omega^T] = \overline{\theta}(m, a \mid b_n, a_{-n})$ and r_n satisfies (44), we have $M_n(r_n, \theta[\omega^T], 1 \mid \widetilde{\theta}) = M_n^B(\theta[\omega^T], 1)$ for any pre-history $\widetilde{\theta} \in \Theta$ such that $\widetilde{\theta}^T \in \cup_{a_n'' \in A_n} \overline{\phi}(m, a \mid a_n'', a_{-n})$ and $M_n(r_n, \theta[\omega^T], t \mid \tilde{\theta}') = M_n^B(\theta[\omega^T], t)$ for any $\tilde{\theta}' \in \Theta$ and $t > 1$. Hence, r_n is s-Justifiable. As $\rho(.)$ is s-Justifiable, the next period all agents in role n will adopt r_n wpp. Then, by (11) and Lemma 32, the system reaches wpp a recurrent class C such that $A(C) \subseteq Q$ and $R(C) = r$. ■

Lemma 63 *Fix any* $R' \subseteq R$ *and* $Q \subseteq A$. *Suppose every* $r \in R'$ *satisfies (11), (41). Then* R' *is* 1*-mutation from* Q.

Proof. The proof of this lemma is identical to that of Lemma 13 except that here we appeal to Lemma 62 instead of Lemma 36. ■

To show that any $r \in R \cap R_Q^T$ is invading, we need to first derive analogue of Lemmas 37 and 38 with s-Justifiable criterion.

Lemma 64 *Fix any* $a \in A$, $\omega^0 \in \Omega$, $a'_{-n} \in A_{-n}$, $m \in M$, $n \in N$, $r_n \in R_n$ *and* $r' \in R$ *such that* $\theta[\omega^0] \in \hat{\Theta}^{-n}(m \mid a'_{-n})$, $R[\omega^0]^T = q^n(m, r_n \mid r')$ *and the following holds*

$$\begin{aligned} r_n(\theta)[B_n(a'_{-n})] &= 1 \ \forall \theta \in \hat{\Theta}^{-n}(m \mid a'_{-n}; T) \backslash \Theta^u(a; T) \text{ and} \qquad (45) \\ \text{either } r_n(\theta)[B_n(a'_{-n})] &= 1 \ \forall \theta \in \Theta^u(a; T) \text{ or } r_n(\theta)[a_n] = 1 \ \forall \theta \in \Theta^u(a; T). \end{aligned}$$

Suppose also that either $\theta[\omega^0] \notin \Theta^u(a;T)$ *or* $r_n(\theta)[B_n(a'_{-n})] = 1$ *for all* $\theta \in \Theta^u(a;T)$. *Then starting from from* ω^0 *the system reaches some* $C \in \Gamma$ *such that* $R(C) = (r_n, r'_{-n})$.

Proof. There are two steps to this proof.

Step 1. There exists $\{b_n^t, c_n^t, \omega^t\}_{t=1}^T \in K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ *such that* $b_n^t = B_n(a'_{-n})$ *for every* $t \in T$ and *starting from* ω^0 *the system reaches wpp* ω^T*:* The proof of this step consists of applying the following claim recursively starting from $t = 1$.

The Induction Claim for any $t \geq 1$. Assume that if $t > 1$ then there exists $\{b_n^{t'}, c_n^{t'}, \omega^{t'}\}_{t'=1}^{t-1} \in K^{t-1}(m, n, r_n, r' \mid a'_{-n}, \omega^0)$ such that $b_n^{t'} = B_n(a'_{-n})$ for all $t' < t$ and starting from ω^0 the system reaches wpp ω^{t-1}. Then there exists $\{b_n^t, c_n^t, \omega^t\}$ such that $b_n^t = B_n(a'_{-n})$, $r_n(\theta[\omega^{t-1}])[b_n^t] = 1$, $r'_n(\theta[\omega^{t-1}])[c_n^t] > 0$, $\theta[\omega^t]^T = \overline{\phi}(m, (b_n^t, a'_{-n}) \mid c_n^t, a'_{-n})$, $R[\omega^t]^T = q^n(m, r_n \mid r')$, $\theta[\omega^t]^{t'} = \theta[\omega^{t-1}]^{t'+1}$ and $R[\omega^t]^{t'} = R[\omega^{t-1}]^{t'+1}$ for all $t' < T$, and starting from ω^{t-1} the system reaches wpp ω^t at the next date.

Proof of the Induction Claim. Fix any c_n^t such that $r'_n(\theta[\omega^{t-1}])[c_n^t] > 0$. Then the proof follows from applying Lemma 34 to ω^{t-1} if it can be shown that $r_n(\theta[\omega^{t-1}])[B_n(a'_{-n})] = 1$. If $t = 1$ or $a_n = B_n(a_{-n})$ then this follows from $\theta[\omega^{t-1}] \in \hat{\Theta}^{-n}(m \mid a'_{-n})$, (45) and that either $\theta[\omega^0] \notin \Theta^u(a;T)$ or $r_n(\theta)[B_n(a'_{-n})] = 1$ for all $\theta \in \Theta^u(a;T)$. If $t > 1$ and $a_n \neq B_n(a_{-n})$ then, given that by the induction assumption $b_n^{t-1} = B_n(a'_{-n})$, we have $\theta[\omega^{t-1}] \in \hat{\Theta}^{-n}(m \mid a'_{-n}) \backslash \Theta^u(a;T)$; but then, by (45), $r_n(\theta[\omega^t])[B_n(a'_{-n})] = 1$.

Step 2. Starting from ω^T *the system reaches wpp some* $C \in \Gamma$ *such that* $R_n(C) = (r_n, r'_{-n})$*:* First, given that $\{b_n^t, c_n^t, \omega^t\}_{t=1}^T \in K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$, we have $\theta[\omega^T] \in \hat{\Theta}^{-n}(m \mid a'_{-n})$ and $R[\omega^T]^T = q^n(m, r_n \mid r')$. Second, it must be that

$$r_n(\theta)[B_n(a'_{-n})] = 1 \text{ for all } \theta \in \Theta \text{ s.t. } \theta^T \in \cup_{1 \leq t < T} \theta[\omega^T]^t \qquad (46)$$

To show this consider two cases. If $a_n = B_n(a'_{-n})$ then by (45) $r_n(\theta)[B_n(a'_{-n})] = 1 \ \forall \theta \in$

$\hat{\Theta}^{-n}(m \mid a'_{-n}; T)$; but then (46) follows from $\theta[\omega^T] \in \hat{\Theta}^{-n}(m \mid a'_{-n})$. If $a_n \neq B_n(a'_{-n})$ then $\theta[\omega^T]^t \neq \phi(a)$ for every $t \in T$; but then (46) follows from $\theta[\omega^T] \in \hat{\Theta}^{-n}(m \mid a'_{-n})$ and (45).

Now given that by the previous step $b_n^t = B_n(a'_{-n})$ for all t, it follows from (46) that $M_n(r_n, \theta[\omega^T], 1 \mid \tilde{\theta}) = M_n^B(\theta[\omega^T], 1)$ for all $\tilde{\theta}$ such that $\tilde{\theta}^T \in \cup_{1 \leq t' < T} \theta[\omega^T]^{t'}$ and, that $M_n(r_n, \theta[\omega^T], t' \mid \tilde{\theta}) = M_n^B(\theta[\omega^T], t')$ for all $t' > 1$ and $\tilde{\theta} \in \Theta$. Thus, r_n is s-Justifiable. As $\rho(.)$ is s-Justifiable, at state ω^T all agents in role n will adopt r_n wpp. Hence, by Lemma 32, the system reaches some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$. ■

Lemma 65 *Fix any $a \in A$, $\omega^0 \in \Omega$, $a'_{-n} \in A_{-n}$, $m \in M$, $n \in N$, $r_n \in R_n$ and $r' \in R$ such that $R[\omega^0] = r'$, $r'_{-n}(\theta[\omega^0])[a'_{-n}] > 0$ and r_n satisfies (45). Then one mutation by any agent at ω^0 to r_n in role n of match m moves the system wpp from ω^0 to some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$.*

Proof. Fix any $\{b_n^t, c_n^t, \omega^t\}_{t=1}^T \in K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$. Given the definition of $K^T(m, n, r_n, r' \mid a'_{-n}, \omega^0)$, such a sequence exists. The rest of the proof follows from the following two steps.

Step 1. One mutation by any agent at ω^0 to r_n in role n of match m moves the system wpp from ω^0 to state ω^T. This step is identical to Step 1 in the proof of Lemma 38; it follows by exactly the same reasoning.

Step 2. Starting from ω^T the system will reach some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$. Given the definition of ω^T, we have $\theta[\omega^T] \in \hat{\Theta}^{-n}(m \mid a'_{-n})$ and $R[\omega^T]^t = q^n(m, r_n \mid r')$ for all t. There are two cases.

Case 1. $\theta[\omega^T] \in \Theta^u(a; T)$, $B_n(a'_{-n}) \neq a_n$ and $r_n(\theta)[a_n] = 1$ for all $\theta \in \Theta^u(a; T)$. First we show that in this case $\theta[\omega^T] = \theta(a)$. Suppose not; then there exists $t \in T$ such that $\theta[\omega^{t-1}] \notin \Theta^u(a; T)$ and $\theta[\omega^t]^T = \phi(a)$. But $\theta[\omega^{t-1}] \notin \Theta^u(a; T)$, together with $\theta[\omega^{t-1}] \in \hat{\Theta}^{-n}(m \mid a'_{-n}; T)$ and (45), imply that $r_n(\theta[\omega^{t-1}])([B_n(a'_{-n})] = 1$; but this, together with $B_n(a'_{-n}) \neq a_n$, contradict $\theta[\omega^t]^T = \phi(a)$.

It then follows from $\theta[\omega^T] = \theta(a)$ and (45) that $r_n(\theta)[a_n] = 1$ for all $\theta \in \Theta^u(a; T)$. Hence, $M_n(r_n, \theta(a), 1 \mid \tilde{\theta}) = M_n^B(\theta(a), 1)$ for all $\tilde{\theta} \in \Theta^u(a; T)$ and that $M_n(r_n, \theta(a), t' \mid \tilde{\theta}) = M_n^B(\theta(a), t')$ for all $t' > 1$ and $\tilde{\theta} \in \Theta$. Thus, given that $\theta[\omega^T] = \theta(a)$, r_n is s-Justifiable at ω^T. As $\rho(.)$ is s-Justifiable, at state ω^T all agents in role n will adopt r_n wpp. So, by Lemma 32, the system reaches some $C \in \Gamma$ such that $R_n(C) = (r_n, r'_{-n})$.

Case 2. $\theta[\omega^T] \notin \Theta^u(a; T)$ or $r_n(\theta)[B_n(a'_{-n})] = 1$ for all $\theta \in \Theta^u(a; T)$. Since $\theta[\omega^T] \in$

$\hat{\Theta}^{-n}(m \mid a'_{-n})$, $R[\omega^T]^T = q^n(m, r_n \mid r')$, and r_n satisfies (45), it follows from Lemmas 64 that starting from ω^T the system reaches some $C \in \Gamma$ s.t. $R_n(C) = (r_n, r'_{-n})$. ■

Lemma 66 *Fix any Q and $r \in R \cap R^T_Q$. Then r is invading if either $Q \in W$ or Q is singleton.*

Proof. Fix any $C' \in \Gamma$, r' and $n \in N$, such that $r' = R(C')$. If $r_n = r'_n$ then r_n is invading at C'. So suppose that $r'_n \neq r_n$. By Lemma 46, there exists $\omega \in C'$ such that $\theta[\omega] \in \Theta^s$ and $u(C') = u(\omega)$. Fix any a'_{-n} such that $r_{-n}(\theta[\omega])[a'_{-n}] > 0$. There are two cases.

Case 1. $Q \in W$. Then as r satisfies (43), there exists m such that $r_n(\theta)[B_n(a'_{-n})] = 1$ for all $\theta \in \hat{\Theta}^{-n}(m \mid a'_{-n}; T)$.

Case 2. Q is singleton and equal to some $a \in A$. Then as r satisfies (41b) and (42), we have $r_n(\theta)[a_n] = 1$ for all $\theta \in \Theta^u(a; T)$ and $r_n(\theta)[B_n(a'_{-n})] = 1$ for all $\theta \in \hat{\Theta}^{-n}(m \mid a'_{-n}; T) \backslash \Theta^u(a; T)$.

Thus, in both cases r_n satisfies (45). Hence, by Lemma 65, one mutation by any agent at ω to r_n in role n of match m moves the system from ω to some $C \in \Gamma$ satisfying $R(C) = (r_n, R_{-n}(\omega))$ wpp. Therefore, r_n is invading at C'. ■

3.3 Proof of the Claims with Birth

Lemma 67 *Assume u-Birth. Fix any $C \in \Gamma$. Then $R(C)$ is unique, $R(C)(\theta(a))[B(a)] > 0$ for any $a \in A^u(C)$ and there exists $Q \in W$ such that $Q \subseteq A(C)$.*

Proof. First note that by u-Birth assumption, for any n, if $R_n(\omega)$ is unique and $R_n(\omega) \in R^u_n$ for some $\omega \in C$, then $R_n(C)$ is unique and equal to $R_n(\omega)$. Second, note that Lemma 42 holds with u-Birth assumption. Next we establish two claims.

Claim 1: Fix any n, $\omega \in C$ and $a \in A$ such that $\theta[\omega] = \theta(a)$ and $R(\omega)$ is singleton; then $R_n(\omega)(\theta(a))[B_n(a)] > 0$. Suppose not; then by u-Birth assumption, there exists $r_n \in R^u_n$ such that $\rho_n(\omega)[r_n] > 0$. By Lemma 42 this implies that there exists $\omega' \in C$ such that $R_n(\omega') = r_n$. Then by the argument in the previous paragraph $R_n(C) = r_n \in R^u_n$. But this contradicts the supposition.

Claim 2: For any $\omega \in C$ if $R(\omega)$ is singleton then $R(C) = R(\omega)$. To show this, it is sufficient to show that for any $\omega \in C$ such that $R(\omega) = r$, $\rho(\omega))[r] = 1$. For any such ω, either $\theta[\omega] \in \Theta^u$ in which case by Claim 1 and u-Birth assumption $\rho(\omega))[r] = 1$ or $\theta[\omega'] \notin \Theta^u$ in which case the same conclusion follows directly from u-Birth assumption.

Now by Lemma 42, there exists $\omega \in C$ such that $R(\omega)$ is singleton. Then it follows from Claims 2 and 1 that $R(C)$ is unique, $R(C)(\theta(a))[B(a)] > 0$ for any $a \in A^u(C)$.

Finally, by Lemma 17 and $R(C)(\theta(a))[B(a)] > 0$ for any $a \in A^u(C)$, there exists $Q \in W$ such that $Q \subseteq A(C)$. ■

Proof of Proposition 26. For any $C \in \Gamma$, by Lemma 67 , $A(C)$ contains some $Q \in W$; so $u(C) \leq \overline{u}_W$. Hence,the claim in the Proposition follows from Proposition 15. ■

Lemma 68 *Suppose u*-Birth. If* $R^u \cap R^{pure} \subseteq R$*, then* $A(C) \in E$ *for all* $C \in \Gamma$*.*

Proof. Fix any $C \in \Gamma$. First note that by u*-Birth assumption a new rule can be born at any uniform history only if the rule does a best reply to the action profile played at that history. Hence, it follows that for any $\omega \in C$ such that $\theta[\omega] = \theta(e)$ for some $e \in E$, $R(\omega)$ is singleton and $R(\omega)(\theta[\omega])[e] = 1$ it must be that $A(C) = e$.

Second, fix any $r \in R^u \cap R^{pure}$ and $e' \in E$ such that $r(\theta)[e'] = 1$ for all $\theta \notin \Theta^u$. Clearly such a rule exists and by assumption is feasible.

Third, by Lemma 42, there exists a uniform state $\omega' \in C$ such that $\theta[\omega'] \in \Theta^u$. Also, by u*-Birth Assumption, $\rho(\omega')[r] > 0$. Fix any a such that $r(\theta[\omega'])[a] = 1$. Then by Lemma 42, there exists $\omega'' \in C$ such that $\theta[\omega''] = \theta(a)$ and $R(\omega'') = r$. Now there are two cases.

Case 1. $a \in E$. Then by $r \in R^u \cap R^{pure}$ and the claim in the first paragraph of this proof $A(C) = a \in E$.

Case 2. $a \notin E$. Suppose the system is at state ω'' at some date and the following happens: (i) all agents are assigned the same roles and the same matches as in the previous period, (ii) all agents adopt the same rule r, (iii) all agents in some match m implement their rule r and play $B(a)$ and all other agents are subject to action inertia and play a. Then the system reaches next period state $\tilde{\omega}$ such that $\tilde{\theta}[\tilde{\omega}]^T = \phi(m, B(a) \mid a)$ and $R(\tilde{\omega}) = r$. As $a \notin E$, it follows that $B(a) \neq a$ and $\theta[\tilde{\omega}]$ is not uniform. But then by assumption $r(\theta[\tilde{\omega}])[e'] > 0$. Hence, by Lemma 42, there exists $\hat{\omega} \in C$ such that $\theta[\hat{\omega}] = \theta(e')$ and $R(\hat{\omega}) = r$. Then by $r \in R^u \cap R^{pure}$ and the claim in the first paragraph of this proof $A(C) = e' \in E$. ■

Proof of Proposition 27. For any $C \in \Gamma$, by Lemma 68 , $A(C) \in E$; hence $u(C) \leq \overline{u}_E$. So the claim in the Proposition follows from Proposition 15. ■