Panel Unit Root Tests in the Presence of a Multifactor Error Structure*

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February 10, 2012

Abstract

This paper extends the cross-sectionally augmented panel unit root test (CIPS) proposed by Pesaran (2007) to the case of a multifactor error structure, and proposes a new panel unit root test based on a simple average of cross-sectionally augmented Sargan-Bhargava statistics (CSB). The basic idea is to exploit information regarding the m unobserved factors that are shared by k observed time series in addition to the series under consideration. Initially, we develop the tests assuming that m^0, the true number of factors is known, and show that the limit distribution of the tests does not depend on any nuisance parameters, so long as $k \geq m^0 - 1$. Small sample properties of the tests are investigated by Monte Carlo experiments and shown to be satisfactory. Particularly, in contrast to other existing panel unit root tests with cross-sectional dependence, the proposed CIPS and CSB tests have the correct size for *all* combinations of the cross section (N) and time series (T) dimensions considered. The power of both tests rise with N and T, although the CSB test performs better than the CIPS test for smaller sample sizes. The proposed tests are also compared with a number of existing tests in the literature, although the comparison is complicated by the fact that most of the existing tests exhibit size distortions. The only test that stands out is the recent test proposed by Bai and Ng (2010). The various testing procedures are illustrated with empirical applications to real interest rates and real equity prices across countries.

JEL Classifications: C12, C15, C22, C23

Keywords: Panel Unit Root Tests, Cross Section Dependence, Multifactor Error Structure, Fisher Inflation Parity, Real Equity Prices.

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*We would like to thank Anindya Banerjee, Soren Johansen and Joachim Westerlund for useful comments. We are grateful to Roger Moon and Benoit Perron for helpful email correspondence with regard to the implementation of the tests in Moon et al. (2007). We would also like to thank Serena Ng for generously providing her matlab codes for computation of the test statistics proposed by Bai and Ng (2010). This version has also benefited greatly from constructive comments and suggestions by the Editor (Cheng Hsiao), an associate editor and three anonymous referees.

1 Introduction

There is now a sizeable literature on testing for unit roots in panels where both cross section (N) and time series (T) dimensions are relatively large. Reviews of this literature are provided in Banerjee (1999), Baltagi and Kao (2000), Choi (2004), and in Breitung and Pesaran (2008). The so called first generation panel unit root tests pioneered by Levin, Lin and Chu (2002) and Im, Pesaran and Shin (2003) focussed on panels where the idiosyncratic errors were cross-sectionally uncorrelated. More recently, to deal with a number of applications such as testing for purchasing power parity or cross country output convergence, the second generation panel unit root tests have focussed on the case where the errors are allowed to be cross-sectionally correlated.

Three main approaches have been proposed. The first, pioneered by Maddala and Wu (1999), and developed further by Chang (2004), Smith et al. (2004), Cerrato and Sarantis (2007), and Palm et al. (2011), applies bootstrap methods to panel unit root tests. The main idea of this approach is to approximate the distribution of the test statistic under cross section dependence by block bootstrap resampling to preserve the pattern of cross section dependence in the panel. This approach allows for general cross section dependence structures, however, it is mainly suited to panels with large T and relatively small N.

The second approach is due to Bai and Ng (2004, 2010) and proposes tests based on a decomposition of the observed series, y_{it}; $i = 1, 2, ..., N$, $t = 1, 2, ..., T$, into two unobserved components, common factors and idiosyncratic errors, and tests for unit roots in both of these components. It is also tested if the unobserved common factors are cointegrated. This is known as the PANIC (panel analysis of nonstationarity in idiosyncratic and common components) approach, and provides indirect tests of unit roots in the observed series. The factors are estimated from m^0 principal components (PC) of Δy_{it}. It is assumed that m^0, the true number of factors, is known or estimated from the observations. If it is found that the estimated factors contain unit roots and are not cointegrated it is then concluded that the N series are integrated of order 1. If the presence of a unit root in the factors is rejected, in the second stage the PANIC procedure applies panel unit root tests to the N idiosyncratic errors. Estimates of idiosyncratic errors are obtained as defactored observations, also known as PANIC residuals. Moon and Perron (2004) follow a similar approach in that they base their test on a principal components estimator of common factors. In particular, their test is based on defactored observations obtained by projecting the panel data onto the space orthogonal to the estimated factor loadings. Bai and Ng (2010) propose two panel unit root tests which are applied to the PANIC residuals. The first one is based on a pooled estimate of the autoregressive root fitted to the PANIC residuals, as in Moon and Perron (2004), and the second one employs a panel version of the modified Sargan-Bhargava test (PMSB).[1]

The third approach, proposed in Pesaran (2007), augments the individual Dickey-Fuller (DF) regressions of y_{it} with cross section averages, $\bar{y}_{t-1} = N^{-1}\Sigma_{j=1}^{N} y_{j,t-1}$ and $\Delta\bar{y}_t$, to take account of error cross section dependence. These cross-sectionally augmented DF regressions can be further augmented with lagged changes $\Delta y_{i,t-s}$, $\Delta\bar{y}_{t-s}$, for $s = 1, 2, ...$, to deal with possible serial correlation in the residuals. These doubly augmented DF regressions are referred to as CADF regressions. The panel unit root test statistic is then computed as the average of the CADF statistics. It is shown that the average statistic is free of nuisance parameters but, due to non-zero cross correlation of the individual, $CADF_i$, statistics, the average statistic has a

[1] Westerlund and Larsson (2009) provide further theoretical results on the asymptotic validity of the pooled versions of the PANIC procedure.

non-normal limit distribution as N and $T \to \infty$. Monte Carlo experiments show that Pesaran's test has desirable small sample properties in the presence of a single unobserved common factor but show size distortions if the number of common factors exceeds unity.[2] A small sample comparison of some of these tests is provided in Gengenbach, Palm and Urbain (2009).[3]

The data generating mechanisms underlying the PANIC approach differ in one important respect from the ones considered by Moon and Perron (2004) and Pesaran (2007). The latter studies assume that under the null of unit roots the common factor components have the same order of integration as the idiosyncratic components, whilst the PANIC approach allows the order of integration of the factors to differ from that of the idiosyncratic components. However, if the primary objective of the exercise is to test for unit roots in the observed series, y_{it}, the distinction between the common and idiosyncratic components of y_{it} is not essential and the panel unit root test can be implemented using the Moon-Perron or Pesaran's set up. The distinction will become relevant if the unit root null hypothesis is not rejected. In that case it would indeed be of interest to investigate further whether the source of the non-stationarity lies with the common factors, the idiosyncratic components, or both.

The present paper extends Pesaran's CIPS to the case of a multifactor error structure. This is a non-trivial yet important extension which is much more broadly applicable. It has also the advantage of being intuitive and simple to implement. Following Bai and Ng (2010) we also consider a panel unit root test based on simple averages of cross-sectionally augmented Sargan-Bhargava type statistics, which we denote by CSB**.** The presence of multiple unobserved factors poses a number of additional challenges. In order to deal with a multifactor structure, we propose to utilize the information contained in a number of k additional variables, $\mathbf{x}_{it}$, that together are assumed to share the common factors of the series of interest, y_{it}. The ADF regression for y_{it} is then augmented with cross-sectional averages of y_{it} and $\mathbf{x}_{it}$.[4]

The requirement of finding such additional variables seems quite plausible in the case of panel data sets from economics and finance where economic agents often face common economic environments. Most macroeconomic theories postulate the presence of the same unobserved common factors (such as shocks to technology, tastes and fiscal policy), and it is therefore natural to expect that many macroeconomic variables, such as interest rates, inflation and output share the same factors. If anything, it would be difficult to find macroeconomic time series that do not share one or more common factors. For example, in testing for unit roots in a panel of real outputs one would expect the unobserved common shocks to output (that originate from technology) to also manifest themselves in employment, consumption and investment. In the case of testing for unit roots in inflation across countries, one would expect the unobserved common factors that correlate inflation rates across countries to also affect short-term and long-term interest rates across markets and economies. The fundamental issue is to ascertain the nature of dependence and persistence that is observed across markets and over time. The

[2] The cross section augmentation procedure is also employed by Hadri and Kurozumi (2009) in their work on testing the null of stationarity in panels.

[3] Other panel unit root tests have also been proposed by Chang (2002), who employs a non-linear IV method, Choi and Chue (2007) who use a subsampling method to account for cross-section correlation, and Phillips and Sul (2003) who use an orthogonalisation procedure to deal with error cross-dependence in the case of a single common factor.

[4] The idea of augmenting ADF regressions with other covariates has been investigated in the unit root literature by Hansen (1995) and Elliott and Jansson (2003). These authors consider the additional covariates in order to gain power when testing the unit root hypothesis in the case of a single time series. In this paper we augment ADF regressions with cross section averages to eliminate the effects of unobserved common factors in the case of panel unit root tests.

present paper can, therefore, be viewed as a first step in the process of developing a coherent framework for the analysis of unit roots and multiple cointegration in large panels.

Initially we develop the tests assuming that m^0, the true number of factors is known, and show that the limit distribution of $CIPS$ and CSB tests does not depend on any nuisance parameters, so long as $k \geq m^0 - 1$. But, in practice m^0 is rarely known. Most existing methods of estimating m^0, such as the information criteria of Bai and Ng (2002), assume that the unobserved factors are strong, in the sense discussed in Chudik, Pesaran and Tosetti (2011). However, in many empirical applications we may not be sure that all the factors are strong. Bailey, Kapetanios and Pesaran (2012, BKP) show that the strength of the factors is determined by the nature of the factor loadings, and depends on the exponent of the cross-sectional dependence, α, defined as $\ln(n)/\ln(N)$, where n is the number of non-zero factor loadings. The value $\alpha = 1$ corresponds to the case of a strong factor, while $\alpha < 1$ gives rise to a large set of practically plausible values ranging from semi-strong to weaker factors. BKP find that for many macroeconomic and financial series of interest, the value of the exponent is less than one. This result casts some doubt on the practical justification of panel unit root tests based on estimated factors by principal components, which are discussed above. The solution offered in this paper deals with the uncertainty surrounding the true number of factors by assuming that there exists a sufficient number of k additional regressors that *together* share at least $m^0 - 1$ of the factors in the model that influence the variable under consideration. This approach does not require all the factors to be strong. This way, by selecting $k = m_{\max} - 1$, where $m_{\max}$ is the assumed maximum number of factors, the estimation of m^0 will not be needed.

Small sample properties of $CIPS$ and CSB tests are investigated by Monte Carlo experiments. These tests are shown to have the correct size for *all* combinations of N and T considered in a number of different experiments. This contrasts the results obtained for some of the prominent existing tests in the literature such as the pooled tests of Bai and Ng (2004, 2010) and Moon and Perron (2004), as well as the defactored versions of optimal tests of Ploberger and Phillips (2002) and Moon et al. (2007), that tend to be over-sized when T is small. The panel version of the modified Sargan-Bhargava test of Bai and Ng (2010) on the other hand appears to be undersized. The experimental results also show that the proposed CSB test has satisfactory power, which for some combinations of N and T tends to be higher than that of the $CIPS$ test. Power comparisons with other tests are complicated by the fact that many of the tests proposed in the literature are over-sized, in some cases substantially when T is relatively small. Empirical applications to Fisher's inflation parity and real equity prices across different economies illustrate how the proposed tests perform in practice.

The plan of the paper is as follows. Section 2 sets out the panel data model, formulates the $CIPS$ test and derives its asymptotic distribution. Section 3 presents the CSB test. Section 4 discusses the proposed tests in the presence of residual serial correlation. Section 5 describes the Monte Carlo experiments and reports the small sample results. Section 6 presents the empirical applications, and Section 7 provides some concluding remarks.

Notation: L denotes a lag operator such that $L^\ell \mathbf{x}_t = \mathbf{x}_{t-\ell}$, K denotes a finite positive constant such that $K < \infty$, $||\mathbf{A}|| = [tr(\mathbf{AA}')]^{1/2}$, $\mathbf{A}^+$ denotes the Moore-Penrose inverse of $\mathbf{A}$, $\mathbf{I}_q$ is a $q \times q$ identity matrix, $\boldsymbol{\tau}_q$ and $\mathbf{0}_q$ are $q \times 1$ vectors of ones and zeros, respectively, $\mathbf{0}_{q\times r}$ is a $q \times r$ null matrix, $\overset{N}{\Longrightarrow}$ $(\overset{N}{\rightarrow})$ denotes convergence in distribution (quadratic mean (q.m.) or mean square errors) with T fixed as $N \rightarrow \infty$, $\overset{T}{\Longrightarrow}$ $(\overset{T}{\rightarrow})$ denotes convergence in distribution (q.m.) with N fixed (or when there is no N-dependence) as $T \rightarrow \infty$, $\overset{N,T}{\Longrightarrow}$ denotes sequential convergence in distribution with $N \to \infty$ first followed by $T \to \infty$, $\overset{(N,T)_j}{\Longrightarrow}$ denotes joint convergence in distribution with N,$T \to \infty$ jointly with certain restrictions on the expansion rates of T and N to be specified, if any.

2 Panel Data Model and the $CIPS$ Test

Let y_{it} be the observation on the i^{th} cross section unit at time t, and suppose that it is generated as

$$\Delta y_{it} = \beta_i(y_{i,t-1} - \boldsymbol{\alpha}'_{iy}\mathbf{d}_{t-1}) + \boldsymbol{\alpha}'_{iy}\Delta\mathbf{d}_t + u_{it},\ i = 1, 2, ..., N;\ t = 1, 2, ..., T, \tag{1}$$

where $\beta_i \leq 0$, $\mathbf{d}_t$ is 2×1 vector consisting of an intercept and a linear trend so that $\mathbf{d}_t = (1, t)'$. Without loss of generality, it is assumed that $\mathbf{d}_0 \equiv \mathbf{0}$, and $\Delta\mathbf{d}_1 \equiv (0, 1)'$. Consider the following multifactor error structure

$$u_{it} = \boldsymbol{\gamma}'_{iy}\mathbf{f}_t + \varepsilon_{iyt}, \tag{2}$$

where $\mathbf{f}_t$ is an $m^0 \times 1$ vector of unobserved common effects, $\boldsymbol{\gamma}_{iy}$ is the associated vector of factor loadings, and ε_{iyt} is the idiosyncratic component. This set up generalises Pesaran's (2007) one factor error specification. We assume that these error processes satisfy the following assumptions:

Assumption 1 (idiosyncratic errors): The idiosyncratic shocks, ε_{iyt}, $i = 1, 2, ..., N$; $t = 1, 2, ..., T$, are independently distributed both across i and t, with zero means, variances, σ_i^2, $(0 < \sigma_i^2 \leq K)$, and finite fourth-order moments.

Remark 1 *This assumption, which implies that the idiosyncratic shocks are serially uncorrelated, will be relaxed in Section 4. It is also possible to relax the assumption that the idiosyncratic errors are cross-sectionally independent, and replace it by assuming that $\varepsilon'_{iyt}s$ are cross-sectionally weakly dependent in the sense of Chudik, Pesaran, and Tosetti (2011). However, such an extension will not be considered in this paper.*

Assumption 2 (factors): The $m^0 \times 1$ vector $\mathbf{f}_t$ follows a covariance stationary process, with absolute summable autocovariances, distributed independently of $\varepsilon_{iyt'}$ for all i, t and t'. Specifically, we assume that $\mathbf{f}_t = \boldsymbol{\Psi}(L)\mathbf{v}_t$ where $\mathbf{v}_t \sim IID(\mathbf{0}, \boldsymbol{\Omega}_m)$, which has finite fourth-order moments, $\boldsymbol{\Psi}(L) = \sum_{\ell=0}^{\infty}\boldsymbol{\Psi}_\ell L^\ell$, where $\{\ell\boldsymbol{\Psi}_\ell\}_{\ell=0}^{\infty}$ is absolute summable such that $\sum_{\ell=0}^{\infty}\ell|\psi_{rs}^{(\ell)}| < \infty$ for all r, s, with $\psi_{rs}^{(\ell)}$ being the $(r, s)^{th}$ element of $\boldsymbol{\Psi}_\ell$. Specifically, it is assumed that the inverse of $\boldsymbol{\Lambda}_f$ defined by

$$\boldsymbol{\Lambda}_f = \boldsymbol{\Psi}(1), \tag{3}$$

exists.

Remark 2 *Assumption 2 is quite general but rules out the possibility of the factors having unit roots. In our set up this makes sense since otherwise all series in the panel could be $I(1)$ irrespective of whether $\beta_i = 0$ or not. Also if $\boldsymbol{\gamma}'_{iy}\mathbf{f}_t$ is assumed to be $I(1)$ and cointegrated with y_{it}, then y_{it} will be $I(1)$ even if $\beta_i = 0$, and a test of $\beta_i = 0$ as a unit root test will not be meaningful, also noted by Hansen (1995, p. 1159) in a similar context.*

Combining (1) and (2) it follows that

$$\Delta y_{it} = \beta_i(y_{i,t-1} - \boldsymbol{\alpha}'_{iy}\mathbf{d}_{t-1}) + \boldsymbol{\alpha}'_{iy}\Delta\mathbf{d}_t + \boldsymbol{\gamma}'_{iy}\mathbf{f}_t + \varepsilon_{iyt}. \tag{4}$$

The hypothesis that all observed series, y_{it}, have *unit roots and are not cross unit cointegrated* can be expressed as

$$H_0 : \beta_i = 0 \text{ for all } i, \tag{5}$$

against the alternative

$$H_1 : \beta_i < 0 \text{ for } i = 1, 2, ..., N_1, \beta_i = 0 \text{ for } i = N_1 + 1, N_1 + 2, .., N,$$

where $N_1/N \to \kappa$ and $0 < \kappa \leq 1$ as $N \to \infty$.

Under the null hypothesis, (4) can be solved for y_{it} to yield

$$y_{it} = y_{i0} + \boldsymbol{\alpha}'_{iy}\mathbf{d}_t + \boldsymbol{\gamma}'_{iy}\mathbf{s}_{ft} + s_{iyt},\ i = 1, 2, ..., N;\ t = 1, 2, ..., T, \tag{6}$$

where

$$\mathbf{s}_{ft} = \mathbf{f}_1 + \mathbf{f}_2 + \cdots + \mathbf{f}_t, \text{ and } s_{iyt} = \varepsilon_{iy1} + \varepsilon_{iy2} + \cdots + \varepsilon_{iyt},$$

with y_{i0} being a given initial value. Therefore, under H_0 and Assumptions 1 and 2, y_{it} is composed of the initial value, y_{i0}, a common stochastic component, $\mathbf{s}_{ft} \sim I(1)$, and an idiosyncratic component, $s_{iyt} \sim I(1)$, so that while all units of the panel share the common stochastic trends, $\mathbf{s}_{ft}$, there is no cointegration among them. Under the alternative stationarity hypothesis, $\beta_i < 0$, we must have $y_{it} \sim I(0)$, and it is therefore *essential* that $\mathbf{f}_t$ is at most an $I(0)$ process.[5]

Remark 3 *Our primary objective is to test for the presence of a unit root in the y_{it} process, which is observed. In contrast, Bai and Ng (2004) consider whether the source of non-stationarity is due to the common factors and/or the idiosyncratic components, neither of which are observed directly. To see how our approach is related to the Bai and Ng (2004, p.1130-1) PANIC framework, consider their specification*

$$\begin{aligned} y_{it} &= \mu_i + \boldsymbol{\gamma}'_{iy}\mathbf{F}_t + e_{iyt}, \\ \Delta\mathbf{F}_t &= \mathbf{C}(L)\mathbf{u}_t, \\ (1-\rho_i L)e_{iyt} &= \varepsilon_{iyt}, \end{aligned} \tag{7}$$

where rank $(\mathbf{C}(1)) = r$, with $0 \leq r \leq m^0$ and r is the number of factors that are $I(1)$, and for simplicity let $\varepsilon_{iyt} \sim iid(0, \sigma^2_\varepsilon)$. Bai-Ng objective is "to determine r and test if $\rho_i = 1$ when neither $\mathbf{F}_t$ nor e_{iyt}, is observed." (Bai and Ng, 2004;p.1130). From (7) it readily follows that

$$\Delta y_{it} = \beta_i(y_{i,t-1} - \mu_i - \boldsymbol{\gamma}'_{iy}\mathbf{F}_{t-1}) + \boldsymbol{\gamma}'_{iy}\Delta\mathbf{F}_t + \varepsilon_{iyt}, \tag{8}$$

where $\beta_i = -(1-\rho_i)$. Under $H_0 : \beta_i = 0$, (8) becomes

$$\Delta y_{it} = \boldsymbol{\gamma}'_{iy}\Delta\mathbf{F}_t + \varepsilon_{iyt}, \text{ or } y_{it} = y_{i0} + \boldsymbol{\gamma}'_{iy}\mathbf{F}_t + s_{iyt},$$

and since within the Bai and Ng framework $\mathbf{F}_t$ and s_{iyt} are both $I(1)$ processes, then y_{it} must also be $I(1)$. Under the alternative hypothesis $H_1 : \beta_i < 0$, it follows from (8) that if $\mathbf{F}_t$ is $I(1)$ (and possibly cointegrated with y_{it}), y_{it} will be $I(1)$, unless $r = 0$ and there are no common stochastic trends. Therefore, it is meaningful to interpret a test of $\beta_i = 0$ as a panel unit root test only if $\mathbf{F}_t$ is assumed to be $I(0)$. See also Remark 2.

[5] One can test whether f_t is $I(0)$ by applying time series unit root tests to cross section averages, $\bar{y}_t = N^{-1}\sum_{i=1}^N y_{it}$, for $t = 1, 2, ..., T$. It can be shown that such tests are asymptotically valid as T and $N \to \infty$, so long as $T/N \to 0$. However, the power of such tests will depend on T and the cross section dimension is only relevant in ensuring that $\bar{y}_t$ is a good proxy for f_t.

In the case where $m^0 = 1$, Pesaran (2007) proposes a test of $\beta_i = 0$ jointly with $f_t \sim I(0)$, based on DF (or ADF) regressions augmented by current and lagged cross-sectional averages of y_{it} as proxies for the unobserved $\mathbf{f}_t$. He shows that the resultant test is asymptotically invariant to the factor loadings, $\boldsymbol{\gamma}_{iy}$. To deal with the case where $m^0 > 1$ we assume that in addition to y_{it}, there exist k additional observables, say $\mathbf{x}_{it}$, which depend on at least the same set of common factors, $\mathbf{s}_{ft}$, although with different factor loadings. For example, in the analysis of output convergence it is reasonable to argue that output, investment, consumption, real equity prices, and oil prices have the same set of factors in common. Similarly, short term and long term interest rates and inflation across countries are likely to have a number of factors in common.

More specifically, suppose the $k \times 1$ vector of additional regressors follow the general linear process

$$\Delta \mathbf{x}_{it} = \mathbf{A}_{ix} \Delta \mathbf{d}_t + \boldsymbol{\Gamma}_{ix} \mathbf{f}_t + \boldsymbol{\varepsilon}_{ixt},\ i = 1, 2, ..., N;\ t = 1, 2, ..., T, \tag{9}$$

where $\mathbf{x}_{it} = (x_{i1t}, x_{i2t}, ..., x_{ikt})'$, $\boldsymbol{\Gamma}_{ix} = (\boldsymbol{\gamma}_{ix1}, \boldsymbol{\gamma}_{ix2}, ..., \boldsymbol{\gamma}_{ixk})'$, $\mathbf{A}_{ix} = (\mathbf{a}_{ix1}, \mathbf{a}_{ix2}, ..., \mathbf{a}_{ixk})'$, and $\boldsymbol{\varepsilon}_{ixt}$ is the idiosyncratic component of $\mathbf{x}_{it}$ which is $I(0)$ and distributed independently of $\varepsilon_{iyt'}$ for all i, t and t'. Solving for $\mathbf{x}_{it}$ we have

$$\mathbf{x}_{it} = \mathbf{x}_{i0} + \mathbf{A}_{ix} \mathbf{d}_t + \boldsymbol{\Gamma}_{ix} \mathbf{s}_{ft} + \mathbf{s}_{ixt},\ i = 1, 2, ..., N;\ t = 1, 2, ..., T, \tag{10}$$

where $\mathbf{s}_{ixt} = \sum_{s=1}^{t} \boldsymbol{\varepsilon}_{ixs}$. Combining (6) and (10) we obtain

$$\mathbf{z}_{it} = \mathbf{z}_{i0} + \boldsymbol{\Gamma}_i \mathbf{s}_{ft} + \mathbf{A}_i \mathbf{d}_t + \mathbf{s}_{it}, \tag{11}$$

where $\mathbf{z}_{it} = (y_{it}, \mathbf{x}'_{it})'$, $\boldsymbol{\Gamma}_i = \left(\boldsymbol{\gamma}_{iy}, \boldsymbol{\Gamma}'_{ix}\right)'$, $\mathbf{A}_i = (\boldsymbol{\alpha}_{iy}, \mathbf{A}'_{ix})'$, and $\mathbf{s}_{it} = (s_{iyt}, \mathbf{s}'_{ixt})'$.

Assumption 3 (factor loadings): $\|\mathbf{A}_i\| \le K$ and $\|\boldsymbol{\Gamma}_i\| \le K$, for all i, with the factors normalized such that $E(\mathbf{f}_t \mathbf{f}'_t) \equiv \mathbf{I}_m$.

Assumption 4 (initial conditions): $E||\mathbf{s}_{f1}|| \le K$, $E||\mathbf{z}_{i0}|| \le K$, and $E||\mathbf{s}_{i1}|| \le K$, for all i.

Remark 4 *Assumption 3 imposes minimal conditions on the factor loadings. For example, it does not rule out possible dependence between the factor loadings and idiosyncratic errors Also the normalisation of* $\mathbf{f}_t$ *under Assumption 3 can be achieved by suitable transformations of* $\boldsymbol{\Gamma}_i$ *and* $\mathbf{f}_t$ *(also note that* $\boldsymbol{\Psi}_0$ *in Assumption 2 is unrestricted). Assumption 4 is also routine in the literature on unit roots.*

Averaging (11) across i we obtain

$$\bar{\mathbf{z}}_t = \bar{\mathbf{z}}_0 + \bar{\boldsymbol{\Gamma}} \mathbf{s}_{ft} + \bar{\mathbf{A}} \mathbf{d}_t + \bar{\mathbf{s}}_t, \tag{12}$$

where $\bar{\mathbf{z}}_t = N^{-1} \sum_{i=1}^{N} \mathbf{z}_{it}$, $\bar{\mathbf{A}} = N^{-1} \sum_{i=1}^{N} \mathbf{A}_i$, and $\bar{\mathbf{s}}_t = N^{-1} \sum_{i=1}^{N} \mathbf{s}_{it}$.[6] Writing (4), (11) and (12) in matrix notation, under the null for each i we have

$$\Delta \mathbf{y}_i = \mathbf{F} \boldsymbol{\gamma}_{iy} + \Delta \mathbf{D} \boldsymbol{\alpha}_{iy} + \boldsymbol{\varepsilon}_{iy}, \tag{13}$$

$$\Delta \mathbf{Z}_i = \mathbf{F} \boldsymbol{\Gamma}'_i + \Delta \mathbf{D} \mathbf{A}'_i + \mathbf{E}_i, \tag{14}$$

$$\Delta \bar{\mathbf{Z}} = \mathbf{F} \bar{\boldsymbol{\Gamma}}' + \Delta \mathbf{D} \bar{\mathbf{A}}' + \bar{\mathbf{E}}, \tag{15}$$

[6] Weighted cross section averages could also be used with appropriate granularity restrictions on the weights.

where $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_T)'$, $\Delta\mathbf{D} = (\Delta\mathbf{d}_1, \Delta\mathbf{d}_2, ..., \Delta\mathbf{d}_T)'$, with $\Delta\mathbf{d}_1 = (0,1)'$, $\boldsymbol{\varepsilon}_{iy} = (\varepsilon_{iy1}, \varepsilon_{iy2}, ..., \varepsilon_{iyT})'$, $\Delta\mathbf{Z}_i = (\Delta\mathbf{z}_{i1}, \Delta\mathbf{z}_{i2}, ..., \Delta\mathbf{z}_{iT})'$, $\mathbf{E}_i = (\boldsymbol{\varepsilon}_{i1}, \boldsymbol{\varepsilon}_{i2}, ..., \boldsymbol{\varepsilon}_{iT})'$ with $\boldsymbol{\varepsilon}_{it} = (\varepsilon_{iyt}, \boldsymbol{\varepsilon}'_{ixt})'$ $\Delta\bar{\mathbf{Z}} = (\Delta\bar{\mathbf{z}}_1, \Delta\bar{\mathbf{z}}_2, ..., \Delta\bar{\mathbf{z}}_T)'$ and $\bar{\mathbf{E}} = N^{-1}\sum_{i=1}^{N}\mathbf{E}_i$. From (15), if $\bar{\boldsymbol{\Gamma}}$ has full column rank m^0, it follows that

$$\mathbf{F} = \left(\Delta\bar{\mathbf{Z}} - \Delta\mathbf{D}\bar{\mathbf{A}}' - \bar{\mathbf{E}}\right)\bar{\boldsymbol{\Gamma}}\left(\bar{\boldsymbol{\Gamma}}'\bar{\boldsymbol{\Gamma}}\right)^{-1}. \quad (16)$$

However, as shown in Appendix A, $\bar{\mathbf{E}} \overset{N}{\to} \mathbf{0}$, and hence we obtain that

$$\mathbf{F} - \left(\Delta\bar{\mathbf{Z}} - \Delta\mathbf{D}\bar{\mathbf{A}}'\right)\bar{\boldsymbol{\Gamma}}\left(\bar{\boldsymbol{\Gamma}}'\bar{\boldsymbol{\Gamma}}\right)^{-1} \overset{N}{\to} \mathbf{0}. \quad (17)$$

This implies that under the null hypothesis linear combinations of $\Delta\bar{\mathbf{Z}}$ and $\Delta\mathbf{D}$ provide valid approximations of $\mathbf{F}$ for large N. This condition on the rank of the cross section average of the factor loadings is stated as an assumption below:

Assumption 5 (rank condition): The $(k+1) \times m^0$ matrix of factor loadings $\boldsymbol{\Gamma}_i$ is such that

$$rank(\bar{\boldsymbol{\Gamma}}) = m^0 \leq k+1, \text{ for any } N \text{ and as } N \to \infty, \quad (18)$$

where $\bar{\boldsymbol{\Gamma}} = N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i$, and $\bar{\boldsymbol{\Gamma}} \overset{N}{\to} \boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}$ is a fixed bounded matrix with rank m^0.

Remark 5 *It is not necessary that y_{it} and $(x_{i1t}, x_{i2t}, ..., x_{ikt})$ have the same cross-sectional dimensions. This is illustrated in Section 6. Also it is not necessary for the rank condition to hold for all cross section units individually, but it must hold on average. For example, the rank condition holds so long as a non-zero fraction of factor loadings, $\boldsymbol{\Gamma}_i$, are full rank as $N \to \infty$. Also, so long as Assumption 5 is satisfied, we do not necessarily require that $\lim_{N\to\infty} N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i'\boldsymbol{\Gamma}_i$ exists and is positive definite, which is typically assumed for the identification of factors. See, for example, Assumption A(ii) of Bai and Ng (2004) and Assumption 6 of Moon and Perron (2004). For example, under our framework, a factor can be weak in the equation for y_{it} and strong in the equations for $\mathbf{x}_{it}$, and vice versa. Such cases do not invalidate the rank condition.*

In view of the above we shall base our test of the panel unit root hypothesis on the t-ratio of the Ordinary Least Squares (OLS) estimator of b_i $(\hat{b}_i)$ in the following cross-sectionally augmented regression

$$\Delta y_{it} = b_i y_{it-1} + \mathbf{c}_i'\bar{\mathbf{z}}_{t-1} + \mathbf{h}_i'\Delta\bar{\mathbf{z}}_t + \mathbf{g}_i'\mathbf{d}_t + \epsilon_{it}. \quad (19)$$

The t-ratio of $\hat{b}_i$ is given by

$$t_i(N,T) = \frac{\Delta\mathbf{y}_i'\bar{\mathbf{M}}\mathbf{y}_{i,-1}}{\hat{\sigma}_i\left(\mathbf{y}_{i,-1}'\bar{\mathbf{M}}\mathbf{y}_{i,-1}\right)^{1/2}} = \frac{\sqrt{T-2k-5}\,\Delta\mathbf{y}_i'\bar{\mathbf{M}}\mathbf{y}_{i,-1}}{\left(\Delta\mathbf{y}_i'\bar{\mathbf{M}}_i\Delta\mathbf{y}_i\right)^{1/2}\left(\mathbf{y}_{i,-1}'\bar{\mathbf{M}}\mathbf{y}_{i,-1}\right)^{1/2}},$$

where $\Delta\mathbf{y}_i = (\Delta y_{i1}, \Delta y_{i2}, ..., \Delta y_{iT})'$, $\mathbf{y}_{i,-1} = (y_{i0}, y_{i1}, ..., y_{i,T-1})'$, $\bar{\mathbf{M}} = \mathbf{I}_T - \bar{\mathbf{W}}\left(\bar{\mathbf{W}}'\bar{\mathbf{W}}\right)^{-1}\bar{\mathbf{W}}'$, $\bar{\mathbf{W}} = (\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, ..., \bar{\mathbf{w}}_T)'$, $\bar{\mathbf{w}}_t = \left(\Delta\bar{\mathbf{z}}_t', \mathbf{d}_t', \bar{\mathbf{z}}_{t-1}'\right)'$, $\hat{\sigma}_i^2 = \Delta\mathbf{y}_i'\bar{\mathbf{M}}_i\Delta\mathbf{y}_i/(T-2k-5)$, and $\bar{\mathbf{M}}_i = \mathbf{I}_T - \bar{\mathbf{W}}_i\left(\bar{\mathbf{W}}_i'\bar{\mathbf{W}}_i\right)^{-1}\bar{\mathbf{W}}_i'$, with $\bar{\mathbf{W}}_i = \left(\bar{\mathbf{W}}, \mathbf{y}_{i,-1}\right)$. For the intercept only case the degrees of freedom adjustment for $\hat{\sigma}_i^2$ is $T-2k-4$. Using (16) in (13)

$$\Delta\mathbf{y}_i = \Delta\bar{\mathbf{Z}}\boldsymbol{\delta}_i + \Delta\mathbf{D}\boldsymbol{\alpha}_i + \sigma_i\boldsymbol{\upsilon}_i, \quad (20)$$

where $\boldsymbol{\delta}_i = \bar{\boldsymbol{\Gamma}}\left(\bar{\boldsymbol{\Gamma}}'\bar{\boldsymbol{\Gamma}}\right)^{-1}\boldsymbol{\gamma}_{iy}$, $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{iy} - \bar{\mathbf{A}}'\boldsymbol{\delta}_i$, $\boldsymbol{\upsilon}_i = \left(\boldsymbol{\varepsilon}_{iy} - \bar{\mathbf{E}}\boldsymbol{\delta}_i\right)/\sigma_i$, it is also easily seen that $E(\boldsymbol{\upsilon}_i\boldsymbol{\upsilon}_i') = \mathbf{I}_T + O(N^{-1})$. Therefore,

$$\bar{\mathbf{M}}\Delta\mathbf{y}_i = \sigma_i\bar{\mathbf{M}}\boldsymbol{\upsilon}_i. \tag{21}$$

From (14) and (15) we also have

$$\mathbf{Z}_{i,-1} = \boldsymbol{\tau}_T\mathbf{z}_{i0}' + \mathbf{S}_{f,-1}\boldsymbol{\Gamma}_i' + \mathbf{D}_{-1}\mathbf{A}_i' + \mathbf{S}_{i,-1}.$$

Taking cross-sectional averages gives

$$\bar{\mathbf{Z}}_{-1} = \boldsymbol{\tau}_T\bar{\mathbf{z}}_0' + \mathbf{S}_{f,-1}\bar{\boldsymbol{\Gamma}}' + \mathbf{D}_{-1}\bar{\mathbf{A}}' + \bar{\mathbf{S}}_{-1}, \tag{22}$$

where $\mathbf{S}_{f,-1} = (\mathbf{0}_{m^0}, \mathbf{s}_{f1}, ..., \mathbf{s}_{f,T-1})'$, $\mathbf{D}_{-1} = (\mathbf{0}_2, \mathbf{d}_1, ..., \mathbf{d}_{T-1})'$, $\mathbf{Z}_{i,-1} = (\mathbf{z}_{i0}, \mathbf{z}_{i1}, ..., \mathbf{z}_{iT-1})'$, $\mathbf{S}_{i,-1} = (\mathbf{0}_{k+1}, \mathbf{s}_{i1}, ..., \mathbf{s}_{i,T-1})'$, $\bar{\mathbf{Z}}_{-1} = (\bar{\mathbf{z}}_0, \bar{\mathbf{z}}_1, ..., \bar{\mathbf{z}}_{T-1})'$ and $\bar{\mathbf{S}}_{-1} = N^{-1}\sum_{i=1}^N \mathbf{S}_{i,-1}$.

Similarly from (20)

$$\mathbf{y}_{i,-1} = \mathring{y}_{i0}\boldsymbol{\tau}_T + \bar{\mathbf{Z}}_{-1}\boldsymbol{\delta}_i + \mathbf{D}_{-1}\boldsymbol{\alpha}_i + \sigma_i\mathring{\mathfrak{s}}_{i,-1}, \tag{23}$$

where

$$\mathring{\mathfrak{s}}_{i,-1} = (\mathbf{s}_{iy,-1} - \bar{\mathbf{S}}_{-1}\boldsymbol{\delta}_i)/\sigma_i, \tag{24}$$

$\mathbf{s}_{iy,-1} = (0, s_{iy1}, ..., s_{iy,T-1})'$, and $\mathring{y}_{i0} = y_{i0} - \bar{\mathbf{z}}_0'\boldsymbol{\delta}_i$. Therefore,

$$\bar{\mathbf{M}}\mathbf{y}_{i,-1} = \sigma_i\bar{\mathbf{M}}\mathring{\mathfrak{s}}_{i,-1}. \tag{25}$$

Using (21) and (25), $t_i(N,T)$ can be re-written as

$$t_i(N,T) = \frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}\mathring{\mathfrak{s}}_{i,-1}}{\left(\frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}_i\boldsymbol{\upsilon}_i}{T-2k-5}\right)^{1/2}\left(\mathring{\mathfrak{s}}_{i,-1}'\bar{\mathbf{M}}\mathring{\mathfrak{s}}_{i,-1}\right)^{1/2}}. \tag{26}$$

For fixed N and T, the distribution of $t_i(N,T)$ will depend on the nuisance parameters through their effects on $\bar{\mathbf{M}}_i$ and $\bar{\mathbf{M}}$. However, this dependence vanishes either as $N \to \infty$, for a fixed T, or as N and $T \to \infty$, jointly. In addition, under Assumption 4 the effect of the initial cross section mean, $\bar{\mathbf{z}}_0$, also vanishes asymptotically, either as $N \to \infty$ for a fixed T, or as N and $T \to \infty$, jointly.[7]

The main results concerning the asymptotic distribution of $t_i(N,T)$ are summarised in the theorem below. The proof is given in the Appendix for the case where $\mathbf{d}_t = (1,0)'$. The results for the case where $\mathbf{d}_t = (1,t)'$ can be derived in a similar manner and are provided in a Supplement available from the authors on request.

Theorem 2.1 *Suppose the series $\mathbf{z}_{it}$, for $i = 1,2,...,N$, $t = 1,2,...,T$, is generated under (5) according to (11) and $d_t = 1$. Then under Assumptions 1-5 and as N and $T \to \infty$, such that $\sqrt{T}/N \to 0$, $t_i(N,T)$ given by (26) has the same sequential ($N \to \infty, T \to \infty$) and joint $[(N,T)_j \to \infty]$ limit distribution, is free of nuisance parameters, and is given by*

$$CADF_i = \frac{\int_0^1 W_i(r)dW_i(r) - \boldsymbol{\omega}_{i\mathbf{v}}'\mathbf{G}_{\mathbf{v}}^{-1}\boldsymbol{\pi}_{i\mathbf{v}}}{\left(\int_0^1 W_i^2(r)dr - \boldsymbol{\pi}_{i\mathbf{v}}'\mathbf{G}_{\mathbf{v}}^{-1}\boldsymbol{\pi}_{i\mathbf{v}}\right)^{1/2}}, \tag{27}$$

[7] The importance of initial values for power properties of panel unit root tests is discussed in Moon et al. (2007), Breitung and Westerlund (2009), and Harris et al. (2010). A further investigation of this issue for the case where the errors are cross sectionally dependent is clearly worthwhile, but will not be pursued in this paper.

where

$$\boldsymbol{\omega}_{i\mathbf{v}} = \begin{pmatrix} W_i(1) \\ \int_0^1 [\mathbf{W_v}(r)]\, dW_i(r) \end{pmatrix}, \; \boldsymbol{\pi}_{i\mathbf{v}} = \begin{pmatrix} \int_0^1 W_i(r)dr \\ \int_0^1 [\mathbf{W_v}(r)]\, W_i(r)dr \end{pmatrix},$$

$$\mathbf{G_v} = \begin{pmatrix} 1 & \int_0^1 [\mathbf{W_v}(r)]'\, dr \\ \int_0^1 [\mathbf{W_v}(r)]\, dr & \int_0^1 [\mathbf{W_v}(r)]\, [\mathbf{W_v}(r)]'\, dr \end{pmatrix},$$

$W_i(r)$ is a scaler standard Brownian motion and $\mathbf{W_v}(r)$ is m^0-dimensional standard Brownian motion defined on [0,1], associated with ε_{iyt} and $\mathbf{v}_t$, respectively. $W_i(r)$ and $\mathbf{W_v}(r)$ are mutually independent.

See Appendix A for a proof.

Remark 6 *Since the random variables $CADF_i$ form an exchangeable sequence, conditional on $\mathbf{W_v}(r)$, $CADF_i$ and $CADF_j$ are independently distributed; see, for example, Theorem 1.2.2 in Taylor et al. (1985, p.13). Unconditionally, however, they are correlated with the same degree of dependence for all $i \neq j$.*

Remark 7 *When the factors are serially uncorrelated, namely $\mathbf{f}_t \equiv \mathbf{v}_t \sim IID(\mathbf{0}, \mathbf{I}_m)$, (see Assumptions 2 and 3), even for a finite T the limit distribution of $t_i(N,T)$ as $N \to \infty$, does not depend on the factor loadings and σ_i. In the case where the factors are serially correlated the limit distribution of $t_i(N,T)$ does depend on the serial correlation patterns of $\mathbf{f}_t$ when T is finite. However, as Theorem 2.1 states, the dependence of $t_i(N,T)$ on the autocovariances of $\mathbf{f}_t$ vanishes in the limit when $T \to \infty$ and $N \to \infty$, jointly.*

Remark 8 *When the y_{it} process does not contain a linear time trend but the additional regressors $\mathbf{x}_{it}$ (or some subset thereof) do, the augmented regression (19) must include a linear trend term in order to eliminate the effects of such a trend in $\bar{\mathbf{x}}_t$. Alternatively, in such a case, it can be shown that Theorem 2.1 holds when the additional regressors are replaced by a detrended version of $\bar{\mathbf{x}}_t$. See the empirical Section 6 for more details.*

The panel unit root test can now be based on the average of the t-ratios

$$CIPS_{NT} = N^{-1} \sum_{i=1}^{N} t_i(N,T), \tag{28}$$

which can be viewed as the cross-sectionally augmented version of the IPS test advanced in Im et al. (2003). As in Pesaran (2007), it is theoretically more convenient to work with a suitably truncated version of the $CIPS_{NT}$ test statistic defined by

$$CIPS_{NT}^{*} = N^{-1} \sum_{i=1}^{N} t_i^{*}(N,T). \tag{29}$$

where

$$t_i^{*}(N,T) = \begin{cases} t_i(N,T), & \text{if } -K_1 < t_i(N,T) < K_2, \\ -K_1, & \text{if } t_i(N,T) \le -K_1, \\ K_2, & \text{if } t_i(N,T) \ge K_2, \end{cases} \tag{30}$$

and the truncation points K_1 and K_2 are chosen using a normal approximation for $t_i(N,T)$. Specifically, they are set as $K_1 = -E(CADF_i) - \Phi^{-1}(\epsilon/2)\sqrt{Var(CADF_i)}$, and $K_2 = E(CADF_i) + \Phi^{-1}(1-\epsilon/2)\sqrt{Var(CADF_i)}$, where $\Phi^{-1}(\ .\ )$ is the inverse of the cumulative standard normal distribution function. K_1 and K_2 can be obtained using simulated values of $E(CADF_i)$ and $Var(CADF_i)$ with $\epsilon = 1\times 10^{-6}$ for $N=200$, and $T=200$. As with $t_i(N,T)$, the limiting distribution of $t_i^*(N,T)$, denoted by $CADF_i^*$, exists and will be free of nuisance parameters. We have

$$CADF_i^* = \begin{cases} CADF_i, & \text{if } -K_1 < CADF_i < K_2, \\ -K_1, & \text{if } CADF_i \leq -K_1, \\ K_2, & \text{if } CADF_i \geq K_2. \end{cases} \tag{31}$$

It is now straightforward to show that under the null hypothesis the asymptotic distribution of $CIPS^*$ exists and is free from nuisance parameters. To see this, let

$$\Delta_{NT}^* = N^{-1}\sum_{i=1}^{N}\left[t_i^*(N,T) - CADF_i^*\right], \tag{32}$$

and note that $CIPS_{NT}^* = \overline{CADF}^* + \Delta_{NT}^*$, where $\overline{CADF}^* = N^{-1}\sum_{i=1}^{N} CADF_i^*$. Also, by Theorem 2.1 and the relationships (30) and (31), for each i, $\Delta_i^* = t_i^*(N,T) - CADF_i^* \rightarrow_p 0$, as $(N,T)_j \to \infty$. Therefore, $\Delta_{NT}^* = CIPS_{NT}^* - \overline{CADF}^* \rightarrow_p 0$, as $(N,T)_j \to \infty$, since $E\left|t_i^*(N,T) - CADF_i^*\right| \leq E\left|t_i^*(N,T)\right| + E\left|CADF_i^*\right| < K < \infty$, given the truncated nature of the underlying random variables. Furthermore, since by construction, $E\left|CADF_i^*\right| < K < \infty$ for each i, then conditional on $\mathbf{W_v}$, (using Theorem 1.2.2 in Taylor et al. (1985, p.13)), we have

$$\begin{aligned} \overline{CADF}^* = N^{-1}\sum_{i=1}^{N} CADF_i^* \overset{a.s.}{\to} E\left[CADF_1^* | \mathbf{W_v}, -K_1 < CADF_1 < K_2\right] \\ +\pi_2 K_2 - \pi_1 K_1, \end{aligned} \tag{33}$$

where $\pi_1 = \Pr[CADF_i^* \leq -K_1|\mathbf{W_v}]$ and $\pi_2 = \Pr[CADF_i^* \geq K_2|\mathbf{W_v}]$. From (33) we have that $\overline{CADF}^*$ converges in distribution as $N \to \infty$. Hence, it also follows that conditional on $\mathbf{W_v}$, the truncated statistic, $CIPS_{NT}^*$, will converge to the same distribution as the limiting distribution of $\overline{CADF}^*$. But due to the dependence of $CADF_i^*$ over i, the limiting distribution is not normal and its critical values need to be computed by stochastic simulation. Further for suitably small choice of ϵ, the simulated critical values of the untruncated statistic, $\overline{CADF}$, is very close to those of $\overline{CADF}^*$. The reason for introducing the truncated version of $CIPS$ statistic is purely technical and is aimed at circumventing the difficult problem of establishing that the untruncated statistics, $t_i(N,T)$, have moments. The computation of the critical values of $\overline{CADF}$ is discussed in Section 4.2.

3 The CSB Test

The cross-sectional augmentation approach can also be exploited in the case of other unit root tests, such as the test proposed by Sargan and Bhargava (1983). In the single time series case, the Sargan-Bhargava statistic was modified by Stock (1999) to allow for serial correlation. This test has also been recently adopted by Bai and Ng (2010) in the panel context with good effects.

Recall that the data generating process for y_{it} under the null is given by

$$\Delta y_{it} = \boldsymbol{\alpha}_{iy}'\Delta\mathbf{d}_t + \boldsymbol{\gamma}_{iy}'\mathbf{f}_t + \varepsilon_{iyt}.$$

For each i, the cross-sectionally augmented Sargan-Bhargava statistic, is given by

$$CSB_i(N,T) = T^{-2}\sum_{t=1}^{T}\widehat{u}_{it}^2/\hat{\sigma}_i^2,$$

where

$$\widehat{u}_{it} = \sum_{j=1}^{t}\hat{\varepsilon}_{ij}, \text{ and } \hat{\sigma}_i^2 = \sum_{t=1}^{T}\hat{\varepsilon}_{it}^2/\left[T-(k+1)\right],$$

and $\hat{\varepsilon}_{it}$ are the OLS residuals from the regressions of Δy_{it} on $\Delta\bar{\mathbf{z}}_t$, in the case of models with an intercept only. If the underlying series are trended, $\hat{\varepsilon}_{it}$ must be calculated from a regression of Δy_{it} on an intercept and $\Delta\bar{\mathbf{z}}_t$, with $\hat{\sigma}_i^2$ computed as $\hat{\sigma}_i^2 = \sum_{t=1}^{T}\hat{\varepsilon}_{it}^2/\left[T-(k+2)\right]$. The use of cross-sectional augmentation as a way of dealing with the unobserved factors is justified using (17), which renders $\hat{\varepsilon}_{it}$ free of the nuisance parameters (namely the factor loadings). It is now easy to prove that for each i, $CSB_i(N,T)$ statistic converges to a functional of Brownian motions, which is independent of the factors as well as their loadings.[8] The CSB test is then based on the cross-sectional average of the $CSB_i(N,T)$ statistics, given by

$$CSB = N^{-1}\sum_{i=1}^{N}CSB_i(N,T). \tag{34}$$

Computation of the critical values for the CSB statistic using stochastic simulations is described in Section 4.2.

4 The Case of Residual Serial Correlation

In this section we relax Assumption 1, and consider the implications of residual serial correlation for our proposed tests. In error factor models, residual serial correlation can be modelled in a number of different ways, directly *via* the idiosyncratic components, through the factor(s), or a mixture of the two. We focus on the serial correlation in the idiosyncratic errors and model the residual serial correlation as

$$\zeta_{iyt} = \theta_i\zeta_{iy,t-1}+\eta_{iyt}, \ \ |\theta_i|<1, \text{ for } i=1,2,...,N; \ t=1,2,...,T, \tag{35}$$

where ζ_{iyt} is the idyosyncratic component of $u_{it} = \boldsymbol{\gamma}_{iy}'\mathbf{f}_t + \zeta_{iyt}$, and η_{iyt} is independently distributed across both i and t, with zero means and variances, $0<\sigma_{i\eta}^2<K<\infty$.

To keep the exposition simple we confine our analysis to the first order stationary processes, though the analysis readily extends to higher order processes. Under (35) we have

$$\Delta y_{it} = \beta_i(y_{i,t-1}-\boldsymbol{\alpha}_{iy}'\mathbf{d}_{t-1}) + \boldsymbol{\alpha}_{iy}'\Delta\mathbf{d}_t + \boldsymbol{\gamma}_{iy}'\mathbf{f}_t + \zeta_{iyt}(\theta_i), \tag{36}$$

where $\zeta_{iyt}(\theta_i) = (1-\theta_i L)^{-1}\eta_{iyt}$. We also assume the coefficients of the autoregressive process to be homogeneous across i, although this could be relaxed at the cost of more complex mathematical details. Under the null that $\beta_i = 0$, with $\theta_i = \theta$ and $\mathbf{d}_t = (1,0)'$, (36) reduces to

$$\Delta y_{it} = \boldsymbol{\gamma}_{iy}'\mathbf{f}_t + \zeta_{iyt}(\theta), \tag{37}$$

[8] A proof of this is provided in a Supplement, which is available from the authors on request.

and upon using (35) under the null hypothesis we have

$$\Delta y_{it} = \theta \Delta y_{i,t-1} + \boldsymbol{\gamma}_{iy}'(\mathbf{f}_t - \theta \mathbf{f}_{t-1}) + \eta_{iyt}. \tag{38}$$

The individual CADF regressions can be written as

$$\Delta \mathbf{y}_i = b_i \mathbf{y}_{i,-1} + \bar{\mathbf{W}}_{i1}\mathbf{h}_i + \epsilon_i, \text{ for } i = 1, 2, ..., N, \tag{39}$$

where $\bar{\mathbf{W}}_{i1} = (\Delta \mathbf{y}_{i,-1}, \Delta\bar{\mathbf{Z}}, \Delta\bar{\mathbf{Z}}_{-1}, \boldsymbol{\tau}_T, \bar{\mathbf{Z}}_{-1})$, which is a $T \times (3k+5)$ matrix. The t-ratio of $\hat{b}_i$ in regression (39) is given by

$$t_i(N,T) = \frac{\Delta \mathbf{y}_i' \bar{\mathbf{M}}_{i1} \mathbf{y}_{i,-1}}{\hat{\sigma}_i \left(\mathbf{y}_{i,-1}' \bar{\mathbf{M}}_{i1} \mathbf{y}_{i,-1}\right)^{1/2}} = \frac{\sqrt{T-(3k+6)}\Delta \mathbf{y}_i' \bar{\mathbf{M}}_{i1} \mathbf{y}_{i,-1}}{\left(\Delta \mathbf{y}_i' \bar{\mathbf{M}}_{i1,p} \Delta \mathbf{y}_i\right)^{1/2} \left(\mathbf{y}_{i,-1}' \bar{\mathbf{M}}_{i1} \mathbf{y}_{i,-1}\right)^{1/2}}, \tag{40}$$

where $\bar{\mathbf{M}}_{i1} = \mathbf{I}_T - \bar{\mathbf{W}}_{i1}(\bar{\mathbf{W}}_{i1}'\bar{\mathbf{W}}_{i1})^{-1}\bar{\mathbf{W}}_{i1}'$, $\hat{\sigma}_i^2 = [T-(3k+6)]^{-1}\Delta \mathbf{y}_i' \bar{\mathbf{M}}_{i1,p} \Delta \mathbf{y}_i$ and $\bar{\mathbf{M}}_{i1,p} = \mathbf{I}_T - \mathbf{P}_{i1}(\mathbf{P}_{i1}'\mathbf{P}_{i1})^{-1}\mathbf{P}_{i1}'$, $\mathbf{P}_{i1} = (\bar{\mathbf{W}}_{i1}, \mathbf{y}_{i,-1})$.

Combining (9) with (37), similarly to (14) we obtain

$$\Delta \mathbf{Z}_i = \mathbf{F}\boldsymbol{\Gamma}_i' + \mathbf{E}_i, \tag{41}$$

where $\mathbf{E}_i = (\boldsymbol{\zeta}_{iy}'(\theta), \mathbf{E}_{ix}')'$, with $\mathbf{E}_{ix} = (\boldsymbol{\varepsilon}_{ix1}, \boldsymbol{\varepsilon}_{ix2}, ..., \boldsymbol{\varepsilon}_{ixT})'$, and $\boldsymbol{\zeta}_{iy}(\theta) = \left(\zeta_{iy1}(\theta), \zeta_{iy2}(\theta), ..., \zeta_{iyT}(\theta)\right)'$, with the common factors $\mathbf{F}$, and factor loadings $\boldsymbol{\Gamma}_i$ defined as in the previous section. Taking cross section averages of (41) we obtain $\Delta\bar{\mathbf{Z}} = \mathbf{F}\bar{\boldsymbol{\Gamma}}' + \bar{\mathbf{E}}$, where as before $\bar{\mathbf{E}} = N^{-1}\sum_{i=1}^{N} \mathbf{E}$. Therefore, assuming that the rank condition, (18), holds

$$\mathbf{F} = \left(\Delta\bar{\mathbf{Z}} - \bar{\mathbf{E}}\right) \bar{\boldsymbol{\Gamma}} \left(\bar{\boldsymbol{\Gamma}}'\bar{\boldsymbol{\Gamma}}\right)^{-1}. \tag{42}$$

Writing (38) in matrix notation and using (42) we have

$$\Delta \mathbf{y}_i = \theta \Delta \mathbf{y}_{i,-1} + (\Delta\bar{\mathbf{Z}} - \theta\Delta\bar{\mathbf{Z}}_{-1})\boldsymbol{\delta}_i + \sigma_{i\eta}\boldsymbol{\upsilon}_i, \tag{43}$$

with

$$\boldsymbol{\upsilon}_i = [\boldsymbol{\eta}_{iy} - (\bar{\mathbf{E}} - \theta\bar{\mathbf{E}}_{-1})\boldsymbol{\delta}_i]/\sigma_{i\eta},$$

and $E(\boldsymbol{\upsilon}_i\boldsymbol{\upsilon}_i') = \mathbf{I}_T + O(N^{-1})$. Further from (37) using (42) it follows that

$$\mathbf{y}_{i,-1} = \alpha_{iy}\boldsymbol{\tau}_T + \mathring{y}_{i0}\boldsymbol{\tau}_T + \bar{\mathbf{Z}}_{-1}\boldsymbol{\delta}_i + \sigma_{i\eta}\mathring{\mathfrak{s}}_{i\zeta,-1},$$

where

$$\mathring{\mathfrak{s}}_{i\zeta,-1} = \left(\mathbf{s}_{i\zeta,-1} - \bar{\mathbf{S}}_{-1}\boldsymbol{\delta}_i\right)/\sigma_{i\eta},$$

$\mathbf{s}_{i\zeta,-1} = (0, s_{i\zeta1}, ..., s_{i\zeta,T-1})'$ with $s_{i\zeta t} = \sum_{s=1}^{t} \zeta_{iys}(\theta)$, $\bar{\mathbf{S}}_{-1} = (\bar{\mathbf{s}}_{\zeta,-1}, \bar{\mathbf{S}}_{x,-1})$ with $\bar{\mathbf{s}}_{\zeta,-1} = N^{-1}\sum_{i=1}^{N} \mathbf{s}_{i\zeta,-1}$ and $\mathring{y}_{i0} = y_{i0} - \bar{\mathbf{z}}_0'\boldsymbol{\delta}_i$.

The test statistic (40) then becomes

$$t_i(N,T) = \frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}_{i1}\mathring{\mathfrak{s}}_{i\zeta,-1}}{\left(\frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}_{i1,p}\boldsymbol{\upsilon}_i}{T-3k-6}\right)^{1/2}\left(\mathring{\mathfrak{s}}_{i\zeta,-1}'\bar{\mathbf{M}}_{i1}\mathring{\mathfrak{s}}_{i\zeta,-1}\right)^{1/2}}. \tag{44}$$

Theorem 4.1 *Suppose the series* $\mathbf{z}_{it}$*, for* $i = 1, 2, ..., N$*,* $t = 1, 2, ..., T$*, is generated under (5) according to (41) and* $|\theta| < 1$*. Then under Assumptions 1-5 and as* N *and* $T \to \infty$*,* $t_i(N, T)$ *in (44) has the same sequential* $(N \to \infty, T \to \infty)$ *and joint* $[(N, T)_j \to \infty]$ *limit distribution given by (27) obtained for* $\theta = 0$*.*

The proof is provided in a Supplement available from the authors on request.

For a general AR(p) error specification, the CADF regressions in (39) must also be augmented by further lagged changes. More specifically, in this case the $t_i(N, T)$ statistic should be computed as the OLS t-ratio of b_i in the following p^{th} order augmented regression:

$$\Delta \mathbf{y}_i = b_i \mathbf{y}_{i,-1} + \bar{\mathbf{W}}_{ip} \mathbf{h}_{ip} + \epsilon_i, \tag{45}$$

where $\bar{\mathbf{W}}_{ip} = (\Delta \mathbf{y}_{i,-1}, \Delta \mathbf{y}_{i,-2}, ..., \Delta \mathbf{y}_{i,-p}; \Delta \bar{\mathbf{Z}}, \Delta \bar{\mathbf{Z}}_{-1}, ..., \Delta \bar{\mathbf{Z}}_{-p}; \boldsymbol{\tau}_T; \bar{\mathbf{Z}}_{-1})$, which is a $T \times (k + 2)(p + 1)$ data matrix. In the case where $\mathbf{d}_t = (1, t)'$, (45) should include a linear trend term, with the degrees of freedom term associated with the error variance adjusted accordingly.

Similarly it can be shown that the $CSB_i(N, T)$ statistics have the same limiting distribution as for $\theta = 0$,[9] and from the above it follows that in the case of first order residual serial correlation, the cross section augmented regression should be augmented further with the term $\Delta \bar{\mathbf{z}}_{t-1}$, so that

$$\Delta y_{it} = b_i \Delta y_{i,t-1} + \mathbf{c}'_{i0} \Delta \bar{\mathbf{z}}_t + \mathbf{c}'_{i1} \Delta \bar{\mathbf{z}}_{t-1} + \epsilon_{it},$$

which for higher order serial correlation generalises to

$$\Delta y_{it} = \sum_{\ell=1}^{p} b_{i\ell} \Delta y_{it-\ell} + \sum_{\ell=0}^{p} \mathbf{c}'_{i\ell} \Delta \bar{\mathbf{z}}_{t-\ell} + \epsilon_{it}, \tag{46}$$

with

$$CSB_i(N, T) = T^{-2} \sum_{t=1}^{T} \hat{u}_{it}^2 / \hat{\sigma}_i^2,$$

where $\hat{u}_{it} = \sum_{j=1}^{t} \hat{\epsilon}_{ij}$, $\hat{\sigma}_i^2 = \sum_{t=1}^{T} \hat{\epsilon}_{it}^2 / [T - p - (p + 1)(k + 1)]$, and

$$\hat{\epsilon}_{it} = \Delta y_{it} - \sum_{\ell=1}^{p} \hat{b}_{i\ell} \Delta y_{it-\ell} - \sum_{\ell=0}^{p} \hat{\mathbf{c}}'_{i\ell} \Delta \bar{\mathbf{z}}_{t-\ell}.$$

In the case where $\mathbf{d}_t = (1, t)'$, (46) should include an intercept term, with the degrees of freedom term associated with the error variance adjusted accordingly.

4.1 Uncertainty Surrounding the Number of Factors

So far we have considered the case in which the true number of unobserved factors, m^0, is given. In practice m^0 is rarely known, although it is reasonable to assume that it is bounded by a finite integer value, $m_{\max}$. In the case of the proposed test there are two possible ways that one could proceed when m^0 is not known.

One approach would be to estimate m^0 using a suitable statistical technique such as the information criteria proposed by Bai and Ng (2002). Most existing methods of estimating m^0 assume that the unobserved factors are strong, in the sense discussed in Chudik, Pesaran and

[9] A proof is included in a Supplement available upon request from the authors

Tosetti (2011). However, in many empirical applications we may not be sure that all unobserved factors are strong. Bailey, Kapetanios and Pesaran (2012) propose measuring the strength of the factors by the exponent of the cross-section dependence, α, defined as $\ln(n)/\ln(N)$, where n is the number of non-zero factor loadings. The value $\alpha = 1$ corresponds to the case of a strong factor, while values of α in the range $(1/2, 1)$ correspond to factors that are semi-strong. Bailey et al. (2012) estimate the exponent of the cross-sectional dependence for macroeconomic and financial series of interest, and find that mostly it is less than one. This raises interesting technical issues concerning the determination of the number of factors and if they are strong, as assumed by the standard theory.

Alternatively, assuming that there exists a sufficient number of additional regressors that share at least $m^0 - 1$ of the factors included in the model for y_{it}, one could set $k = m_{\max} - 1$ (where $m^0 \leq m_{\max}$), and use the k additional regressors for augmenting the regressions when computing $CIPS$ and CSB statistics. This approach is likely to work in practice when $m_{\max}$ is relatively small (2 or 3), and does not require all the factors to be strong. However, when $m_{\max}$ is believed to be large, the $CIPS$ and CSB tests are likely to lose power due to loss of degrees of freedom. More importantly, it might be difficult to find a sufficient number of additional regressors to deal with the adverse effects of the unobserved factors on our proposed test statistics.[10]

4.2 Computation of Critical Values of CIPS and CSB Tests

Critical values for the $CIPS$ and CSB tests for different values of k, N, T, and lag-augmentation order, p, are obtained by stochastic simulation. We report p-specific critical values, as in Im et al. (2003) and Pesaran (2007), which results in better small sample properties of the tests. To compute the critical values, y_{it} is generated as

$$y_{it} = y_{it-1} + \varepsilon_{iyt},\ i = 1, 2, ..., N;\ t = 1, 2, ..., T, \tag{47}$$

where $\varepsilon_{iyt} \sim iidN(0, 1)$ with $y_{i0} = 0$. The j^{th} element of the $k \times 1$ vector of the additional regressors x_{it}, is generated as

$$x_{ijt} = x_{ij,t-1} + \varepsilon_{ixjt},\ i = 1, 2, ..., N;\ j = 1, 2, ..., k;\ t = 1, 2, ..., T, \tag{48}$$

with $\varepsilon_{ixjt} \sim iidN(0, 1)$ and $x_{ij0} = 0$.

For the CIPS test the individual $t_i(N, T)$ statistic is calculated as the t-ratio of the coefficient on $y_{i,t-1}$ of the CADF regression of Δy_{it} on an intercept, y_{it-1}, $\bar{\mathbf{z}}'_{t-1}$, $\Delta\bar{\mathbf{z}}'_t, \Delta\bar{\mathbf{z}}'_{t-1}, ..., \Delta\bar{\mathbf{z}}'_{t-p}$, and $\Delta y'_{i.t-1}, ..., \Delta y'_{i.t-p}$ under Case I where the model only contains an intercept, and Case II where the CADF regressions also include a linear time trend. The $CIPS$ statistic is then computed as $CIPS_{NT} = N^{-1} \sum_{i=1}^{N} t_i(N, T)$.

For the CSB test, the individual CSB_i statistic is computed as $CSB_i = T^{-2} \sum_{t=1}^{T} \hat{e}_{it}^2 / \hat{\sigma}_i^2$, with $\hat{u}_{it} = \sum_{j=1}^{t} \hat{e}_{ij}$ and $\hat{\sigma}_i^2 = \sum_{t=1}^{T} \hat{e}_{it}^2 / [T - p - (p+1)(k+1)]$, where $\hat{e}_{it}$ are the estimated residuals from the regression of Δy_{it} on $\Delta y_{i,t-1}, ..., \Delta y_{i,t-p}$ and $\Delta\bar{\mathbf{z}}'_t, \Delta\bar{\mathbf{z}}'_{t-1}, ..., \Delta\bar{\mathbf{z}}'_{t-p}$, under Case I. Under Case II, $\hat{\sigma}_i^2 = \sum_{t=1}^{T} \hat{e}_{it}^2 / [T - (p+1)(k+2)]$, where $\hat{e}_{it}$ are the estimated residuals from the regression of Δy_{it} on an intercept, $\Delta y_{i,t-1}, ..., \Delta y_{i,t-p}$ and $\Delta\bar{\mathbf{z}}'_t, \Delta\bar{\mathbf{z}}'_{t-1}, ..., \Delta\bar{\mathbf{z}}'_{t-p}$. The CSB statistic is computed as $CSB_{NT} = N^{-1} \sum_{i=1}^{N} CSB_i$.

[10] In the presence of uncertainty regarding the integration and/or the cointegration properties of the additional regressors, one could employ the bounds testing approach proposed by Pesaran et al. (2001). A detailed discussion of such an approach is outside the scope of the present paper.

The $100\alpha\%$ critical values of the $CIPS$ and CSB statistics are computed for $N, T = 20, 30, 50, 70, 100, 200$, $k = 0, 1, 2, 3$ and $p = 0, 1, ..., 4$, as their α quantiles for $\alpha = 0.01, 0.05, 0.1$ based on $10,000$ and $50,000$ replications, respectively.[11] To save space, the critical values are reported in a separate supplement ($CIPS$ and CSB Critical Value Tables) which is available from the authors upon request.

It is worth noting that the reported critical values of the $CIPS$ test statistics depend on k, and not on m^0. When $m^0 \leq k+1$ the asymptotic distribution of the $CIPS$ test depends on m^0, but the critical values depend on the number of additional regressors actually included when simulating the critical values. A similar situation also arises when critical values are computed for standard DF statistics by stochastic simulations. The critical values depend on whether linear trends are included in the DF regressions. Similarly, what matters in our analysis is the nature of the deterministic variables and the number of additional $I(1)$ regressors that are added to the DF and SB regressions when carrying out the stochastic simulations.

5 Small Sample Performance: Monte Carlo Evidence

In what follows we investigate by means of Monte Carlo simulations the small sample properties of the $CIPS$ and CSB tests defined by (28) and (34), respectively, and compare their performance to several tests proposed in the literature. Specifically, we consider the pooled test statistic $P_{\hat{e}}$ of Bai and Ng (2004) based on the PANIC residuals, a panel version of the modified Sargan–Bhargava test (denoted by $PMSB$) and a PANIC residual-based Moon and Perron (2004) type test (denoted by P_b), both of which are proposed by Bai and Ng (2010), the t_b^* statistic of Moon and Perron (2004) for the case of an intercept only,[12] a defactored version of the optimal invariant test of Ploberger and Phillips (2002), denoted by PP, for the case of an intercept and a linear trend, and the defactored version of the common point optimal test of Moon, Perron and Phillips (2007), denoted by CPO. The theory of the CPO test is developed by Moon et al. for the serially uncorrleated case, but it is claimed (see Section 6.4 in Moon et al. (2007, p. 436)), that replacing variances in their CPO statistic with long-run variances should result in a test with a correct size under quite general short memory error autocorrelations. However, our preliminary experiments suggested that this claim might not be valid. Upon communicating these results to the authors, Moon, Perron and Phillips have provided us with another modification of the CPO test that appropriately allows for residual serial correlation (see Moon, Perron and Phillips, 2011). In addition to replacing the variance of the errors by the long run variance, in this recent paper Moon et al. also adjust the centering of the statistic to accommodate for the second-order bias induced by the correlation between the error and lagged values of the dependent variable. In our Monte Carlo simulations we also include the modified CPO test, denoted by $\widetilde{CPO}$.

Details of the computation of the statistics $P_{\hat{e}}$, P_b, $PMSB$, PP, CPO, and $\widetilde{CPO}$ are provided in a Supplement, which is available from the authors on request. The $P_{\hat{e}}$ test is defined in Section 2.4 of Bai and Ng (2004, p.1140), the t_b^* test in Section 2.2.2 of Moon and Perron (2004, p.91), the P_b and $PMSB$ tests in Section 3, p.1094, eq. (9) and Section 3.1,

[11] It is also possible to simulate the critical values directly using (27) by replacing the integrals of the Brownian motions with their simulated counterparts. Our analysis suggests that the critical values obtained from this procedure closely match the ones tabulated in the supplement to the paper.

[12] The t_a^* test of Moon and Perron (2004) is not included since they summarise the experimental results saying "in almost all cases, the test based on the t_b^* statistic has better size properties." Similarly, the P_a test of Bai and Ng (2010) is not included.

p.1095, eq.(11), respectively of Bai and Ng (2010), the CPO and PP tests in Section 4.1, p.424; Section 5.1, p.427; and Section 5.3.1, p.429, eq. (20), respectively, in Moon et al. (2007), and the $\widetilde{CPO}$ test in Section 2.2, p.4; Section 2.3, p.5, of Moon et al. (2011). In computing the CPO and $\widetilde{CPO}$ test statistics we set the constant term (the 'c' term in Moon et al.) to unity. Also, following Moon and Perron (2004), the long-run variances for the $PMSB$, P_b, t_b^*, PP, CPO and $\widetilde{CPO}$ test statistics are estimated by means of the Andrews and Monahan (1992) method using the quadratic spectral kernel and prewhitening. See Moon and Perrorn (2004) for further details.

The details of the computation of the critical values for the $CIPS$ and CSB tests are set out in Section 4.2. Both the $CIPS$ and CSB tests reject the null when the value of the statistic is smaller than the relevant critical value, at the chosen level of significance. We do not report size adjusted results, since such results are likely to have limited value in empirical applications. See, for example, Horowitz and Savin (2000).

5.1 Monte Carlo Design

In their Monte Carlo experiments Bai and Ng (2010, Section 5) set $m^0 = 1$ and do not allow for serial correlation in the idiosyncratic errors. Here we consider a more general set up and allow for two factors ($m^0 = 2$), and also consider experiments where the idiosyncratic errors are serially correlated. Following Bailey, Kapetanios and Pesaran (2012) we generate one of the factors in the y_{it} equations as strong and the second factor as semi-strong. Accordingly, the data generating process (DGP) for the $\{y_{it}\}$ is given by

$$y_{it} = d_{iyt} + \rho_i y_{i,t-1} + \gamma_{iy1} f_{1t} + \gamma_{iy2} f_{2t} + \varepsilon_{iyt}, i = 1, 2, ..., N; t = -49, ..., T, \tag{49}$$

with $y_{i,-50} = 0$, where $\gamma_{iy1} \sim iidU[0, 2]$, for $i = 1, 2, ..., N$; $\gamma_{iy2} \sim iidU[0, 1]$ for $i = 1, ..., [N^\alpha]$, and $\gamma_{iy2} = 0$ for $i = [N^\alpha]+1, [N^\alpha]+2, ..., N$ (where $[\cdot]$ denotes the integer part); $f_{\ell t} \sim iidN(0, 1)$ for $\ell = 1, 2$, $\varepsilon_{iyt} \sim iidN(0, \sigma_i^2)$ with $\sigma_i^2 \sim iidU[0.5, 1.5]$. The exponent of cross-sectional dependence of the first (strong) factor is 1, and for the second (semi-strong) factor, it is set to 0.75, guided by the empirical results reported in Bailey et al. (2012). See, also Chudik et al. (2011).

At the stage of implementing the tests, we assume that $m_{\max} = 2$, and hence set $k = m_{\max} - 1 = 1$. The additional regressor, x_{it}, is generated as

$$\Delta x_{it} = d_{ix} + \gamma_{ix1} f_{1t} + \varepsilon_{ixt}, \tag{50}$$

where

$$\varepsilon_{ixt} = \rho_{ix}\varepsilon_{ixt-1} + \varpi_{ixt}, , \varpi_{ixt} \sim iidN(0, 1 - \rho_{ix}^2), \tag{51}$$

$i = 1, 2, ..., N; t = -49, ..., T$, with $\varepsilon_{ix,-50} = 0$, and $\rho_{ix} \sim iidU[0.2, 0.4]$. The factor loadings in (50) are generated as $\gamma_{ix1} \sim iidU[0, 2]$, so that

$$E(\mathbf{\Gamma}_i) = \begin{pmatrix} 1 & \frac{1}{2}N^{-0.25} \\ 1 & 0 \end{pmatrix}. \tag{52}$$

and hence the rank condition (18) is satisfied when N is finite, but fails when $N \to \infty$. In this way we also check the robustness of the CIPS and CSB tests to failure of the rank condition for sufficiently large N.

We considered two specifications for the deterministics in y_{it} and x_{it}. For the case of an intercept only, $d_{iyt} = (1 - \rho_i)\alpha_{iy}$ with $\alpha_{iy} \sim iidN(1, 1)$ and $d_{ix} = 0$; for the case of an intercept and a linear trend, $d_{iyt} = \mu_{iy} + (1 - \rho_i)\delta_i t$ with $\mu_{iy} \sim iidU[0.0, 0.02]$ and $\delta_i \sim iidU[0.0, 0.02]$, and $d_{ix} = \delta_{ix}$ with $\delta_{ix} \sim iidU[0.0, 0.02]$.

To examine the impact of the residual serial correlation on the proposed tests we consider the DGPs in which the idiosyncratic errors ε_{iyt} are generated as

$$\varepsilon_{iyt} = \rho_{iy\varepsilon}\varepsilon_{iyt-1} + (1 - \rho_{iy\varepsilon}^2)^{1/2}\eta_{iyt}, \text{ for } t = -49, -48, ..., 0, 1, ..., T, \tag{53}$$

with $\varepsilon_{iy,-50} = 0$, where $\eta_{iyt} \sim iidN(0, \sigma_i^2)$, and $\sigma_i^2 \sim iidU[0.5, 1.5]$. We considered a positively serially correlated case, $\rho_{iy\varepsilon} \sim iidU[0.2, 0.4]$, as well as a negatively serially correlated case, $\rho_{iy\varepsilon} \sim iidU[-0.4, -0.2]$. The first 50 observations are discarded.

The parameters α_{iy},δ_i,μ_{iy}, δ_{ix}, $\rho_{iy\varepsilon}$, γ_{iy1},γ_{iy2}, ρ_i, γ_{ix1}, ρ_{ix}, and σ_i are redrawn over each replication. The DGP is given by (49) with $\rho_i = \rho = 1$ for size, and $\rho_i \sim iidU[0.90, 0.99]$ for power. All tests are conducted at the 5% significance level. All combinations of $N, T = 20, 30, 50, 70, 100, 200$ are considered, and all experiments are based on 2,000 replications each.

In the case where the errors of y_{it} are serially correlated, lag augmentation is required for the asymptotic validity of the $CIPS$ and CSB tests as well as the pooled tests of Bai and Ng (2004). For these tests, in the Monte Carlo results that follow, lag augmentation is selected according to $\hat{p} = \left[4(T/100)^{1/4}\right]$ (where $[\cdot]$ denotes the integer part). For the other tests, the statistics are adjusted using a non-parametric estimator of the long run variance. In our Monte Carlo results we use the long run variance of Andrews and Monahan (1992). Also note that the asymptotic normality of the $PMSB$, $P_{\hat{e}}$, P_b, t_b^*, PP, CPO and $\widetilde{CPO}$ test statistics require $N/T \to 0$ as N and T go to infinity, while the asymptotic validity of the $CIPS$ and CSB tests only requires that $\sqrt{T}/N \to 0$, which allows N and T to expand at the same rate.

5.2 Results

Size and power of the tests are summarised in Tables 1 to 6. Table 1 provides the results for the panel with an intercept only, and with serially uncorrelated idiosyncratic errors. The size properties of the $P_{\hat{e}}$, t_b^*, and P_b tests are very similar: they tend to over-reject the null moderately across combinations of N and T, with the extent of over-rejection rising as N increases. These results are consistent with those reported in Gengenbach, Palm and Urbain (2009) and Bai and Ng (2010). The CPO test, and its modified version, $\widetilde{CPO}$, have good size properties when T is larger than N, but these tests begin to show serious size distortions as N increases relative to T, which is in line with the condition $N/T \to 0$ that underlies the theory of these tests. The $PMSB$ test of Bai and Ng (2010) tends to under-reject the null when T and N are small, which is in accordance with the results reported in Bai and Ng (2010, Table 1). For example, when $T = N = 20$, the estimated size is 0.65% at the 5% nominal level. In contrast, $CIPS$ and CSB tests have the correct size for all combinations of sample sizes, even when T is small relative to N. In terms of power, the CSB test has satisfactory power which is almost consistently higher than that of $CIPS$, though most of the other tests do tend to display higher power (which could partly be due to the over-sized nature of the other tests). An exception is the PMSB test for small values of T and N, which exhibits lower power than the CSB test.

The results for the case with a linear trend are summarised in Table 2. The tendency of the over-rejection of $P_{\hat{e}}$ for small T is more serious than for the case with an intercept only. For example, even when $T = 200$ and $N = 100$, the size of $P_{\hat{e}}$ is 8.4%. The size of the defactored version of the Ploberger and Phillips test, the PP test, which is only considered for the case with an intercept and a linear trend, is close to the nominal level only when T is much larger than N. The size distortion of the P_b test is similar to that for the case of an intercept only case, though somewhat less pronounced. The CPO test grossly over-rejects for all combinations of N and T. The over-rejection tendency of the $\widetilde{CPO}$ test is now even more pronounced as compared to the intercept only case. The $PMSB$ test is now even more under-sized. When $T = N = 20$, the size of the $PMSB$ test is 0.20%, and even when $N = T = 100$, the size of the $PMSB$ is 1.85% at the 5% nominal level. Again, the $CIPS$ and CSB tests have the correct size for all combinations of sample sizes and their power rise in N and T, as to be expected. Power discrepancies between the CSB and $CIPS$ tests are less pronounced in this case, with the former still showing higher power than the latter. The other tests have higher power than these two tests, but given their size distortions a straightforward power comparison would be problematic. The $PMSB$ test continues to be an exception for smaller values of T, where now the power of this test is almost negligible for $T = 20$, and for $T = 30$ the power ranges from 0.85 to 2.75 across different values of N. Even when $T = 70$, the CSB test has greater power that the PMSB test, for small N.

Tables 3 and 4 present the results for the case where ε_{iyt} are positively serially correlated for the intercept only and linear trend cases, respectively. The results for the case where ε_{iyt} are negatively serially correlated are summarised in Tables 5 and 6. The effect of allowing for residual serial correlation on the $P_{\hat{e}}$, P_b, PP and $\widetilde{CPO}$ tests is to accentuate the tendency of these tests to over-reject the null. Positive serial correlation in ε_{iyt} seems to be more problematic for the size of these tests as compared to negative serial correlation. As was noted earlier, the CPO test is not centered appropriately when there are residual serial correlation, and one needs to use the modified version, the $\widetilde{CPO}$ test in such cases. This is supported by the reported results. The size of the CPO test varies from one extreme (of zero) in the case of positive serial correlation, to another extreme of (unity) in the presence of negative serial correlation. But, as expected the $\widetilde{CPO}$ test has good size properties for values of $T > N$, although it continues to show significant size distortions when $N > T$. The $PMSB$ test, in the case of positive serial correlation, shows some tendency to over-reject for small T and large N. By contrast, the effect of negative serial correlation on the $PMSB$ test is relatively minor, but as in the serially uncorrelated case reported in Tables 1 and 2, the $PMSB$ test tend to under-reject. The size and power of the $CIPS$ and CSB tests are not much affected by residual serial correlation once the underlying regressions are augmented with lagged changes as in (45) and (46). As the results in Tables 1-6 show, the $CIPS$ and CSB tests do not display any size distortions for all values of N and T, irrespective of whether the idiosyncratic errors are serially correlated or not.

Overall, $CIPS$ and CSB tests perform well in most cases, always having the correct size. The evidence on power is mixed, with no one test dominating, and the outcomes difficult to compare due to the size distortion of some of the tests, and the fact that the power of the tests are differently affected by the number of factors and the choice of factor loadings.

6 Empirical Applications

As an illustration of the proposed tests we consider two applications. One to the real interest rates across $N = 32$ economies, and another to the real equity prices across $N = 26$ markets. For both applications we employ quarterly observations over the period $1979Q2 - 2009Q4$ (i.e. 123 data points). Under the Fisher parity hypothesis, real interest rates, defined as the difference between the nominal short-term interest rate and the inflation rate, are stationary. The second application is chosen as it is generally believed that real equity prices are $I(1)$, and it would be interesting to see if the outcomes of the tests considered in this paper are in line with this belief.

As noted in Section 4.1, and as with the other panel unit root tests that are based on principal components, we need to decide on $m_{\max}$. In the present application we set $m_{\max} = 4$. This choice is based on the recent literature that argues that 2 to 6 unobserved common factors are sufficient to explain variations in most macroeconomic variables. See, for example, Stock and Watson (2002) and Eickmeier (2009), among others. This suggests that at most three additional $I(1)$ regressors ($k = m_{\max} - 1 = 3$) are needed for the implementation of $CIPS$ and CSB tests. The set of regressors that are likely to share common factors with real interest rates, $r_{it}^S - \pi_{it}$, and real equity prices, eq_{it}, are as follows:

	y_{it}	Additional regressors ($\mathbf{x}_{it}$)
Real Interest Rates ($N = 32$)	$r_{it}^S - \pi_{it}$	$poil_t, r_{it}^L, eq_{it}, ep_{it}, gdp_{it}$
Real Equity Prices ($N = 26$)	eq_{it}	$poil_t, r_{it}^L, \pi_{it}, ep_{it}, gdp_{it}$

where

$$r_{it}^S = 0.25 * \ln(1 + R_{it}^S/100),\ \pi_{it} = p_{it} - p_{it-1} \text{ with } p_{it} = \ln(CPI_{it}),\ poil_t = \ln(POIL_t),$$
$$r_{it}^L = 0.25 * \ln(1 + R_{it}^L/100),\ ep_{it} = e_{it} - p_{it} \text{ with } e_{it} = \ln(E_{it}),\ eq_{it} = \ln(EQ_{it}/CPI_{it}),$$
$$gdp_{it} = \ln(GDP_{it}/CPI_{it}),$$

R_{it}^S is the short-term (three month) rate of interest, measured in per annum in per cent in country i at time t, CPI_{it} the consumer price index, $POIL_t$ the price of Brent Crude oil, R_{it}^L the long-term rate of interest per annum in per cent (typically the yield on ten year government bonds), E_{it} the nominal exchange rate of country i in terms of US dollars, EQ_{it} the nominal equity price index, and GDP_{it} the nominal Gross Domestic Product of country i during period t in domestic currency.[13]

When testing for unit roots in real interest rates, $\rho_{it}^S = r_{it}^S - \pi_{it}$, we consider ADF regressions without linear trends, and de-trend the possibly trended variables $poil_t$, eq_{it}, ep_{it} and gdp_{it}, before the ADF regressions are augmented with their cross-sectional averages. See, also Remark 8. These de-trended components are computed as residuals from the regressions of $poil_t$, eq_{it}, ep_{it} and gdp_{it} on a linear deterministic trend which does not affect the $I(1)$ properties of these variables.

The 32 countries considered are: Argentina, Australia, Austria, Belgium, Brazil, Canada, Chile, China, France, Finland, Germany, Indonesia, India, Italy, Japan, Korea, Malaysia, Mexico, Netherlands, New Zealand, Norway, Peru, Philippines, Spain, Sweden, Switzerland, Singapore, South Africa, Thailand, Turkey, UK, and the US. Note that not all components of $\mathbf{x}_{it}$ are available for all countries due to data limitations. In particular, there are 26 series for eq_{it}, 31 series for ep_{it}, and 18 series for r_{it}^L.

For $m_{\max} = 4$, we consider the application of the $CIPS$ and CSB tests allowing the number of factors, m^0, to take any value between 1 and 4, as a way of dealing with the sampling uncertainty associated with basing our tests on a particular choice of m^0.[14] To check the

[13] The data are publicly available at: http://www-cfap.jbs.cam.ac.uk/research/gvartoolbox/download.html. A detailed description of the data and sources can be found in the Appendix of the userguide of the gvartoolbox by Smith, L.V. and A. Galesi (2011), available at the same web adress.

[14] Setting $m_{\max} = 4$, and using the information criterion IC_1 proposed by Bai and Ng (2004), the number of factors selected was $\hat{m}^0 = 3$ for the real interest rates, and $\hat{m}^0 = 4$ for the real equity prices. The last result suggests that the number of factors in real equity prices could be even higher than 4, but we did not consider $m_{\max} > 4$ given the number of observations available and the tendency of IC_1 selection criteria to over-estimate the number of factors, particularly if some of the factors are not strong.

robustness of the test outcomes to the choice of the additional regressors used in augmentation, we present the results of these tests for all possible combinations of candidate regressors. For $m^0 = 1$ no additional regressors are required for augmentation apart from $\bar{y}_t$, for $m^0 = 2$ one additional regressor is required, and so on. We set the lag order to $\hat{p} = \left[4(T/100)^{1/4}\right]$, as discussed in the previous section.

The test results for the real interest rates are reported in Table 7. As can be seen, the $CIPS$ test strongly rejects the null hypothesis of the panel unit root at the 1% level, for all values of m^0, and for all combinations of candidate regressors. The test results based on the CSB test are very similar, although there are some exceptions. The CSB test does not reject when $m^0 = 3$ with $\mathbf{x}_{it} = (poil_t, \bar{r}_t^L)$, and when $m^0 = 4$ with $\mathbf{x}_{it} = (poil_t, \bar{r}_t^L, \overline{eq}_t)$, $\mathbf{x}_{it} = (poil_t, \bar{r}_t^L, \overline{ep}_t)$ and $\mathbf{x}_{it} = (poil_t, \bar{r}_t^L, \overline{gdp}_t)$, out of the twenty possible combinations. These results suggest that for a significant number of countries the Fisher parity holds. This is in line with recent findings reported in Westerlund (2008) using panel cointegration tests.

The results of panel unit root tests applied to real equity prices are summarised in Table 8. The test outcomes are generally as to be expected. The null hypothesis of a panel unit root in real equity prices cannot be rejected in most cases. When the $CIPS$ test is used, the null of the panel unit root is rejected once at the 1% level (out of 26 cases), and 6 times at the 5% level. There are fewer rejections when the CSB test used, namely 2 out of 26 cases. Overall, the test results are in line with the generally accepted view that real equity prices approximately follow random walks with a drift.

We also applied the other panel unit root tests considered in the MC experiments (except for the CPO test which is not valid when the errors are serially correlated) to our two datasets. The results are summarised in Table 9. For computation of the $P_{\hat{e}}$ statistic, the lag order $\hat{p} = \left[4(T/100)^{1/4}\right]$ was used. Application of these tests to real interest rates yield mixed results. The P_b, t_b^* and $\widetilde{CPO}$ tests strongly reject the null hypothesis at the 1%, for all values of m^0, which is in accordance with the results of the $CIPS$ and CSB tests. In contrast, the $P_{\hat{e}}$ test does not reject the null hypothesis, irrespective of the number of factors considered. Results for the $PMSB$ test are mixed. It rejects the null at the 5% level for $m^0 = 1$ and 4, but fails to reject the null when $m^0 = 2$ and 3; illustrating the sensitivity of this test to the assumed number of factors. For real equity prices almost all the tests considered show strong rejections of the null hypothesis for all values of m^0, which conflicts with the generally accepted view that real equity prices approximately follow random walks with a drift, the exceptions being the $P_{\hat{e}}$, the PP and $\widetilde{CPO}$ tests (the latter two only when $m^0 = 4$), and the $PMSB$ test only when $m^0 = 1$.

7 Concluding Remarks

This paper considers two simple panel unit root tests that are valid in the presence of cross-sectional dependence induced by m^0 stationary common factors. The first test, $CIPS$, is an extension of the test proposed in Pesaran (2007) and is based on the average of t-ratios from ADF regressions augmented by the cross section averages of the dependent variable as well as k additional regressors with similar common factor features. The second test, CSB, is based on averages of cross-sectionally augmented Sargan-Bhargava statistics. Initially we develop the tests assuming that m^0, the true number of factors is known, and show that the limit distributions of the tests do not depend on any nuisance parameters, so long as $k \geq m^0 - 1$. To deal with the uncertainty that surrounds the value of m^0 in practice, we propose to either choose the number of additional regressors as $k = m_{\max} - 1$, where $m_{\max} \leq m^0$, which avoids having to estimate m^0, or to estimate m^0 consistently using suitable selection criteria.

Small sample properties of the proposed tests are investigated by Monte Carlo experiments, which suggest that the proposed $CIPS$ and CSB tests have the correct size across all combinations of N and T considered. In contrast, the tests proposed by Bai and Ng (2004, 2010), Moon and Perron (2004), Moon et al. (2007), and Moon et al. (2011) display size distortions across combinations of N and T, with the extent of these distortions rising with N, in some cases substantially. In terms of power, the results are mixed. The power of the CSB and $CIPS$ tests rises in N and T and reaches quite acceptable levels when N and T are sufficiently large. For smaller T, the CSB test has higher power than that of $CIPS$, and should thus be preferred in such cases. However, both of these tests have lower power for smaller N and T than the alternative tests proposed in the literature, though due to the substantial size distortions of the other tests a straightforward power comparison of CSB and $CIPS$ and its competitors could be problematic. In empirical applications it is important that the tests being considered have the correct size, otherwise their use could lead to misleading conclusions.

The paper also applies the various panel unit root tests to real interest rates and real equity prices across countries. All tests, except the $P_{\hat{e}}$ test of Bai and Ng (2004) and the $PMSB$ test of Bai and Ng (2010) reject the null of a unit root in real interest rates, which is in line with panel cointegration tests of the Fisher parity equation. For real equity prices, only our proposed tests, $CIPS$ and CSB, and $P_{\hat{e}}$ test do not reject the null of panel unit roots in real equity prices across, which is in accordance with the generally accepted view that real equity prices approximately follow random walks with a drift.

The better small sample results reported for the $CIPS$ and CSB tests as compared to the other tests proposed in the literature comes at a cost, as the tests require the existence of additional $I(1)$ regressors that share the same common factors with y_{it}. We have argued that this might not be a problem when m^0, the true number of factors in y_{it}, is not too large. For example, if $m^0 \leq 2$, only one additional regressor is needed at most to apply the test, and this is unlikely to be a problem in practice, where most macro and finance series are often driven by a small number of common factors. For larger values of m^0 a more careful consideration of the testing problem is required. In such cases it seems more appropriate if the problem of panel unit root testing is considered as part of a more general problem, where robustness of the panel unit root test outcomes to alternative assumptions regarding the integration and cointegration properties of the additional regressors is considered and evaluated.

Appendix A Mathematical Proofs

Lemmas

Lemma A.1 *Under Assumptions 1-5*

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{iy}'\bar{\mathbf{E}}/T &= O_p\left(T^{-1/2}N^{-1/2}\right),\ \mathbf{s}_{iy,-1}'\bar{\mathbf{E}}/T = O_p\left(N^{-1/2}\right),\ \textit{uniformly over } i \\
\bar{\mathbf{S}}_{-1}'\boldsymbol{\varepsilon}_{iy}/T &= O_p\left(N^{-1/2}\right),\ \mathbf{s}_{iy,-1}'\bar{\mathbf{S}}_{-1}/T^2 = O_p\left(N^{-1/2}\right),\textit{uniformly over } i \\
\bar{\mathbf{E}}'\bar{\mathbf{E}}/T &= O_p\left(N^{-1}\right),\ \bar{\mathbf{S}}_{-1}'\bar{\mathbf{E}}/T = O_p\left(N^{-1}\right),\ \bar{\mathbf{S}}_{-1}'\bar{\mathbf{S}}_{-1}/T^2 = O_p\left(N^{-1}\right) \\
\mathbf{F}'\bar{\mathbf{E}}/T &= O_p\left(T^{-1/2}N^{-1/2}\right),\ \bar{\mathbf{S}}_{-1}'\mathbf{F}/T = O_p\left(N^{-1/2}\right) \\
\boldsymbol{\tau}_T'\bar{\mathbf{E}}/T &= O_p\left(T^{-1/2}N^{-1/2}\right),\ \bar{\mathbf{S}}_{-1}'\boldsymbol{\tau}_T/T = O_p\left(T/N\right) \\
\mathbf{S}_{f,-1}'\bar{\mathbf{E}}/T &= O_p\left(N^{-1/2}\right),\ \frac{\mathbf{S}_{f,-1}'\bar{\mathbf{S}}_{-1}}{T^2} = O_p\left(N^{-1/2}\right).
\end{aligned}
$$

Similar order results hold for the case of serially correlated errors.

Proof. See Appendix A.1 of Pesaran (2007). ■

Lemma A.2 *Consider a full column rank $m\times n$ matrix $\mathbf{A}$ ($m>n$) and an $n\times n$ non-singular symmetric matrix $\boldsymbol{\Omega}$. Then $\mathbf{A}'(\mathbf{A}\boldsymbol{\Omega}\mathbf{A}')^{+}\mathbf{A}=\boldsymbol{\Omega}^{-1}$, where $(\mathbf{A}\boldsymbol{\Omega}\mathbf{A}')^{+}$ is the Moore-Penrose inverse of $\mathbf{A}\boldsymbol{\Omega}\mathbf{A}'$.*

Proof. Using the fact that $\mathbf{A}^{+}\mathbf{A}=\mathbf{I}_n$ and $\mathbf{A}'\mathbf{A}'^{+}=\mathbf{I}_n$ and that $(\mathbf{A}\boldsymbol{\Omega}\mathbf{A}')^{+}=\mathbf{A}'^{+}\boldsymbol{\Omega}^{-1}\mathbf{A}^{+}$ (see Magnus and Neudecker 1999, p.34), $\mathbf{A}'(\mathbf{A}\boldsymbol{\Omega}\mathbf{A}')^{+}\mathbf{A}=\mathbf{A}'\mathbf{A}'^{+}\boldsymbol{\Omega}^{-1}\mathbf{A}^{+}\mathbf{A}=\boldsymbol{\Omega}^{-1}$, as required. ■

Proof of Theorem 2.1:

In the case where $\mathbf{d}_t=1$, using (26) we have

$$
t_i(N,T)=\frac{\frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}\mathring{\mathbf{s}}_{i,-1}}{T}}{\left(\frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}_i\boldsymbol{\upsilon}_i}{T-2k-4}\right)^{1/2}\left(\frac{\mathring{\mathbf{s}}_{i,-1}'\bar{\mathbf{M}}\mathring{\mathbf{s}}_{i,-1}}{T^2}\right)^{1/2}}, \tag{A.1}
$$

where $\boldsymbol{\upsilon}_i=(\boldsymbol{\varepsilon}_{iy}-\bar{\mathbf{E}}\boldsymbol{\delta}_i)/\sigma_i$, and $\mathring{\mathbf{s}}_{i,-1}=\left(\mathbf{s}_{iy,-1}'-\bar{\mathbf{S}}_{-1}\boldsymbol{\delta}_i\right)/\sigma_i$, $\bar{\mathbf{M}}=\mathbf{I}_T-\bar{\mathbf{W}}\left(\bar{\mathbf{W}}'\bar{\mathbf{W}}\right)^{-1}\bar{\mathbf{W}}'$, and $\bar{\mathbf{W}}=\left(\Delta\bar{\mathbf{Z}},\boldsymbol{\tau}_T,\bar{\mathbf{Z}}_{-1}\right)$. Also, since $d_t=1$, from (15) and (22) we have $\Delta\bar{\mathbf{Z}}=\mathbf{F}\bar{\boldsymbol{\Gamma}}'+\bar{\mathbf{E}}$ and $\bar{\mathbf{Z}}_{-1}=\boldsymbol{\tau}_T\bar{\mathbf{z}}_0'+\mathbf{S}_{f,-1}\bar{\boldsymbol{\Gamma}}'+\bar{\mathbf{S}}_{-1}$. Let $\mathbf{W}_f=(\mathbf{F},\boldsymbol{\tau}_T,\mathbf{S}_{f,-1})$ and $\bar{\boldsymbol{\Xi}}=\left(\bar{\mathbf{E}},\mathbf{0}_T,\bar{\mathbf{S}}_{-1}\right)$ so that

$$
\bar{\mathbf{W}}'=\mathbf{Q}_N\mathbf{W}_f'+\bar{\boldsymbol{\Xi}}',\ \text{where}\ \underset{(2k+3)\times(2m^0+1)}{\mathbf{Q}_N}=\begin{pmatrix}\bar{\boldsymbol{\Gamma}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & 1 & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{z}}_0 & \bar{\boldsymbol{\Gamma}}\end{pmatrix}. \tag{A.2}
$$

Consider the numerator of (A.1), and note that

$$
\frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}\mathring{\mathbf{s}}_{i,-1}}{T}=\frac{\boldsymbol{\upsilon}_i'\mathring{\mathbf{s}}_{i,-1}}{T}-\left(\boldsymbol{\upsilon}_i'\bar{\mathbf{W}}\mathbf{B}\right)\left(\mathbf{B}\bar{\mathbf{W}}'\bar{\mathbf{W}}\mathbf{B}\right)^{-1}\left(\frac{\mathbf{B}\bar{\mathbf{W}}'\mathring{\mathbf{s}}_{i,-1}}{T}\right), \tag{A.3}
$$

where $\underset{(2k+3)\times(2k+3)}{\mathbf{B}}=\begin{pmatrix}T^{-1/2}\mathbf{I}_{k+2} & \mathbf{0}\\ \mathbf{0} & T^{-1}\mathbf{I}_{k+1}\end{pmatrix}$. Using Lemma A.1 together with the results in Proposition 17.1 of Hamilton (1994; p.486) we have

$$
\frac{\mathring{\mathbf{s}}_{i,-1}'\boldsymbol{\upsilon}_i}{T^{3/2}}=\frac{\mathbf{s}_{iy,-1}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i^2T^{3/2}}+O_p\left(\frac{1}{\sqrt{NT}}\right)\overset{(N,T)_j}{\Longrightarrow}\int_0^1 W_i(r)dW_i(r). \tag{A.4}
$$

where $W_i(r)$ is a standard Brownian motion defined on [0,1], associated with ε_{iyt}. Using (A.2) it follows that

$$
\mathbf{B}\bar{\mathbf{W}}'\boldsymbol{\upsilon}_i=\mathbf{B}\mathbf{Q}_N\mathbf{W}_f'\boldsymbol{\upsilon}_i+\mathbf{B}\bar{\boldsymbol{\Xi}}'\boldsymbol{\upsilon}_i, \tag{A.5}
$$

$$\mathbf{B}\bar{\mathbf{W}}'\mathring{\mathbf{s}}_{i,-1}/T = \mathbf{B}\mathbf{Q}_N\mathbf{W}_f'\mathring{\mathbf{s}}_{i,-1}/T + \mathbf{B}\bar{\boldsymbol{\Xi}}'\mathring{\mathbf{s}}_{i,-1}/T, \tag{A.6}$$

$$\mathbf{B}\bar{\mathbf{W}}'\bar{\mathbf{W}}\mathbf{B} = \mathbf{B}\mathbf{Q}_N\mathbf{W}_f'\mathbf{W}_f\mathbf{Q}_N'\mathbf{B} + \mathbf{B}\mathbf{Q}_N\mathbf{W}_f'\bar{\boldsymbol{\Xi}}\mathbf{B} + \mathbf{B}\bar{\boldsymbol{\Xi}}'\mathbf{W}_f\mathbf{Q}_N'\mathbf{B} + \mathbf{B}\bar{\boldsymbol{\Xi}}'\bar{\boldsymbol{\Xi}}\mathbf{B}. \tag{A.7}$$

From Lemma A.1, it is easily seen that, as $(T,N) \xrightarrow{j} \infty$ with $\sqrt{T}/N \to 0$,

$$\mathbf{B}\bar{\boldsymbol{\Xi}}'\boldsymbol{\upsilon}_i \overset{(N,T)_j}{\rightarrow} \mathbf{0},\ \mathbf{B}\bar{\boldsymbol{\Xi}}'\mathring{\mathbf{s}}_{i,-1}/T \overset{(N,T)_j}{\rightarrow} \mathbf{0},\ \mathbf{B}\bar{\boldsymbol{\Xi}}'\bar{\boldsymbol{\Xi}}\mathbf{B} \overset{(N,T)_j}{\rightarrow} \mathbf{0}, \text{ and } \mathbf{B}\mathbf{Q}_N\mathbf{W}_f'\bar{\boldsymbol{\Xi}}\mathbf{B} \overset{(N,T)_j}{\rightarrow} \mathbf{0}. \tag{A.8}$$

Define $\underset{(2m^0+1)\times(2m^0+1)}{\mathbf{C}} = \begin{pmatrix} T^{-1/2}\mathbf{I}_{m^0+1} & \mathbf{0} \\ \mathbf{0} & T^{-1}\mathbf{I}_{m^0} \end{pmatrix}$, so that, using Lemma A.1 and the results in Proposition 17.1 and 18.1 of Hamilton (1994; p.486, p.547-8), as $(T,N) \xrightarrow{j} \infty$ with $\sqrt{T}/N \to 0$ we have

$$\mathbf{B}\mathbf{Q}_N\mathbf{W}_f'\boldsymbol{\upsilon}_i = \mathbf{Q}_N\mathbf{C}\mathbf{W}_f'\boldsymbol{\upsilon}_i \overset{(N,T)_j}{\Longrightarrow} \mathbf{Q}\boldsymbol{\vartheta}_{if}, \tag{A.9}$$

$$\mathbf{B}\mathbf{Q}_N\mathbf{W}_f'\mathring{\mathbf{s}}_{i,-1}/T = \mathbf{Q}_N\mathbf{C}\mathbf{W}_f'\mathring{\mathbf{s}}_{i,-1}/T \overset{(N,T)_j}{\Longrightarrow} \mathbf{Q}\boldsymbol{\kappa}_{if}, \tag{A.10}$$

$$\mathbf{B}\mathbf{Q}_N\mathbf{W}_f'\mathbf{W}_f\mathbf{Q}_N'\mathbf{B} = \mathbf{Q}_N\mathbf{C}\mathbf{W}_f'\mathbf{W}_f\mathbf{C}\mathbf{Q}_N' \overset{(N,T)_j}{\Longrightarrow} \mathbf{Q}\boldsymbol{\Upsilon}_f\mathbf{Q}', \tag{A.11}$$

where

$$\mathbf{Q} = \underset{N\to\infty}{\text{plim}}\,\mathbf{Q}_N,\ \boldsymbol{\vartheta}_{if} = \begin{pmatrix} \boldsymbol{\Lambda}_f\mathbf{W}_{\mathbf{v},i}(1) \\ \boldsymbol{\Lambda}_f^*\boldsymbol{\omega}_{i\mathbf{v}} \end{pmatrix},\ \boldsymbol{\kappa}_{if} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\Lambda}_f^*\boldsymbol{\pi}_{i\mathbf{v}} \end{pmatrix},\ \boldsymbol{\Upsilon}_f = \begin{pmatrix} \mathbf{I}_{m^0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_f^*\mathbf{G}_{\mathbf{v}}\boldsymbol{\Lambda}_f^{*\prime} \end{pmatrix},\ \boldsymbol{\Lambda}_f^* = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_f \end{pmatrix}, \tag{A.12}$$

$$\boldsymbol{\omega}_{i\mathbf{v}} = \begin{pmatrix} W_i(1) \\ \int_0^1 [\mathbf{W}_{\mathbf{v}}(r)]\,dW_i(r) \end{pmatrix},\ \boldsymbol{\pi}_{i\mathbf{v}} = \begin{pmatrix} \int_0^1 W_i(r)dr \\ \int_0^1 [\mathbf{W}_{\mathbf{v}}(r)]\,W_i(r)dr \end{pmatrix},\ \mathbf{G}_{\mathbf{v}} = \begin{pmatrix} 1 & \int_0^1 [\mathbf{W}_{\mathbf{v}}(r)]'\,dr \\ \int_0^1 [\mathbf{W}_{\mathbf{v}}(r)]\,dr & \int_0^1 [\mathbf{W}_{\mathbf{v}}(r)]\,[\mathbf{W}_{\mathbf{v}}(r)]'\,dr \end{pmatrix},$$

$\boldsymbol{\Lambda}_f$ is defined by (3), $\mathbf{W}_{\mathbf{v},i}(1)$ is defined such that $T^{-1/2}\sum_{t=1}^T \mathbf{v}_t\varepsilon_{iyt}/\sigma_i \overset{T}{\Longrightarrow} \mathbf{W}_{\mathbf{v},i}(1)$ with $\mathbf{v}_t$ defined as in Assumption 2, $\mathbf{W}_{\mathbf{v}}(r)$ is an m^0-dimensional standard Brownian motion associated with $\mathbf{v}_t$ defined on [0,1], and $W_i(r)$ is defined as above. These two groups of Brownian motions $(\mathbf{W}_{\mathbf{v}}(r), W_i(r))$ are independent of each other. Collecting the results from (A.5) to (A.11), as well as using Lemma A.2 (since $\mathbf{Q}$ has full column rank) we have

$$\left(\boldsymbol{\upsilon}_i'\bar{\mathbf{W}}\mathbf{B}\right)\left(\mathbf{B}\bar{\mathbf{W}}'\bar{\mathbf{W}}\mathbf{B}\right)^{-1}\left(T^{-1}\mathbf{B}\bar{\mathbf{W}}'\mathring{\mathbf{s}}_{i,-1}\right) \overset{(N,T)_j}{\Longrightarrow} \boldsymbol{\vartheta}_{if}'\mathbf{Q}'\left(\mathbf{Q}\boldsymbol{\Upsilon}_f\mathbf{Q}'\right)^{+}\mathbf{Q}\boldsymbol{\kappa}_{if} \tag{A.13}$$

$$= \boldsymbol{\vartheta}_{if}'\boldsymbol{\Upsilon}_f^{-1}\boldsymbol{\kappa}_{if} = \boldsymbol{\omega}_{i\mathbf{v}}'\boldsymbol{\Lambda}_f^{*\prime}\left(\boldsymbol{\Lambda}_f^*\mathbf{G}_{\mathbf{v}}\boldsymbol{\Lambda}_f^{*\prime}\right)^{-1}\boldsymbol{\Lambda}_f^*\boldsymbol{\pi}_{i\mathbf{v}} = \boldsymbol{\omega}_{i\mathbf{v}}'\mathbf{G}_{\mathbf{v}}^{-1}\boldsymbol{\pi}_{i\mathbf{v}}.$$

Therefore, together with (A.3), (A.4) and (A.13), as $(T,N) \xrightarrow{j} \infty$ with $\sqrt{T}/N \to 0$ we have

$$\frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}\mathring{\mathbf{s}}_{i,-1}}{T} \overset{(N,T)_j}{\Longrightarrow} \int_0^1 W_i(r)dW_i(r) - \boldsymbol{\omega}_{i\mathbf{v}}'\mathbf{G}_{\mathbf{v}}^{-1}\boldsymbol{\pi}_{i\mathbf{v}}. \tag{A.14}$$

In a similar manner, noting that as $(T,N) \xrightarrow{j} \infty$ with $\sqrt{T}/N \to 0$

$$\frac{\mathring{\mathbf{s}}_{i,-1}'\mathring{\mathbf{s}}_{i,-1}}{T^2} = \frac{\mathbf{s}_{iy,-1}'\mathbf{s}_{iy,-1}}{\sigma_i^2T^2} + O_p\left(\frac{1}{\sqrt{N}}\right) \overset{(N,T)_j}{\Longrightarrow} \int_0^1 W_i^2(r)dr, \tag{A.15}$$

it follows that

$$\frac{\mathring{\mathbf{s}}_{i,-1}'\bar{\mathbf{M}}\mathring{\mathbf{s}}_{i,-1}}{T} \overset{(N,T)_j}{\Longrightarrow} \int_0^1 W_i^2(r)dr - \boldsymbol{\pi}_{i\mathbf{v}}'\mathbf{G}_{\mathbf{v}}^{-1}\boldsymbol{\pi}_{i\mathbf{v}}. \tag{A.16}$$

Next, consider $\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}_i\boldsymbol{\upsilon}_i/(T-2k-4)$. Note that $\bar{\mathbf{M}}_i\boldsymbol{\upsilon}_i$ are the residuals from the regression of $\boldsymbol{\upsilon}_i$ on $\bar{\mathbf{W}}_i = (\bar{\mathbf{W}}, \mathbf{y}_{i,-1})$, but from equation (23) $\mathbf{y}_{i,-1}$ has components $(\bar{\mathbf{Z}}_{-1}, \boldsymbol{\tau}_T, \mathring{\mathbf{s}}_{i,-1})$. As $(\bar{\mathbf{Z}}_{-1}, \boldsymbol{\tau}_T) \subset \bar{\mathbf{W}}$, but $\mathring{\mathbf{s}}_{i,-1}$ is not contained in $\bar{\mathbf{W}}$, we have $\bar{\mathbf{M}}_i\boldsymbol{\upsilon}_i = \bar{\mathbf{M}}_i^*\boldsymbol{\upsilon}_i$, where $\bar{\mathbf{M}}_i^* = \mathbf{I}_T - \bar{\mathbf{H}}_i\left(\bar{\mathbf{H}}_i'\bar{\mathbf{H}}_i\right)^{-1}\bar{\mathbf{H}}_i'$ with $\bar{\mathbf{H}}_i = (\bar{\mathbf{W}}, \mathring{\mathbf{s}}_{i,-1})$. Thus

$$\frac{\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}_i^*\boldsymbol{\upsilon}_i}{T-2k-4} = \frac{\boldsymbol{\upsilon}_i'\boldsymbol{\upsilon}_i}{T-2k-4} - \frac{\left(\boldsymbol{\upsilon}_i'\bar{\mathbf{H}}_i\mathbf{B}_*\right)\left(\mathbf{B}_*\bar{\mathbf{H}}_i'\bar{\mathbf{H}}_i\mathbf{B}_*\right)^{-1}\left(\mathbf{B}_*\bar{\mathbf{H}}_i'\boldsymbol{\upsilon}_i\right)}{T-2k-4}, \tag{A.17}$$

where $\underset{(2k+4)\times(2k+4)}{\mathbf{B}_*} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & T^{-1} \end{pmatrix}$. First note that using Lemma A.1 we have

$$\boldsymbol{\upsilon}_i'\boldsymbol{\upsilon}_i/(T-2k-4) \overset{(N,T)_j}{\rightarrow} 1. \tag{A.18}$$

We also have that

$$\mathbf{B}_*\bar{\mathbf{H}}_i'\boldsymbol{\upsilon}_i = \begin{pmatrix} \mathbf{B}\bar{\mathbf{W}}'\boldsymbol{\upsilon}_i \\ \hat{\mathbf{s}}_{i,-1}'\boldsymbol{\upsilon}_i/T \end{pmatrix}, \; \mathbf{B}_*\bar{\mathbf{H}}_i'\bar{\mathbf{H}}_i\mathbf{B}_* = \begin{pmatrix} \mathbf{B}\bar{\mathbf{W}}'\bar{\mathbf{W}}\mathbf{B} & \mathbf{B}\bar{\mathbf{W}}'\hat{\mathbf{s}}_{i,-1}/T \\ \hat{\mathbf{s}}_{i,-1}'\bar{\mathbf{W}}\mathbf{B}/T & \hat{\mathbf{s}}_{i,-1}'\hat{\mathbf{s}}_{i,-1}/T^2 \end{pmatrix},$$

so then using (A.4), (A.15), and following the same line of analysis as for the results in (A.13), it can be seen that $\left(\boldsymbol{\upsilon}_i'\bar{\mathbf{H}}_i\mathbf{B}_*\right)\left(\mathbf{B}_*\bar{\mathbf{H}}_i'\bar{\mathbf{H}}_i\mathbf{B}_*\right)^{-1}\left(\mathbf{B}_*\bar{\mathbf{H}}_i'\boldsymbol{\upsilon}_i\right)$ in (A.17) will tend to a function of standard Brownian motions as $(T,N)\xrightarrow{j}\infty$ with $\sqrt{T}/N\to 0$. Thus, dividing by $T-2k-4$ makes the second term of (A.17) asymptotically negligible, and together with the results in (A.17) and (A.18) we have that $\frac{\boldsymbol{\upsilon}_i\bar{\mathbf{M}}_i^*\boldsymbol{\upsilon}_i}{T-2k-4}\overset{(N,T)_j}{\rightarrow} 1$. Thus, as $(T,N)\xrightarrow{j}\infty$ with $\sqrt{T}/N\to 0$,

$$\boldsymbol{\upsilon}_i'\bar{\mathbf{M}}_i\boldsymbol{\upsilon}_i/(T-2k-4)\overset{(N,T)_j}{\rightarrow} 1. \tag{A.19}$$

Finally, from the results in (A.1), (A.14), (A.16) and (A.19), we have, as $\sqrt{T}/N\to 0$,

$$t_i(N,T)\overset{(N,T)_j}{\Longrightarrow}\frac{\displaystyle\int_0^1 W_i(r)dW_i(r)-\boldsymbol{\omega}_{i\mathbf{v}}'\mathbf{G}_{\mathbf{v}}^{-1}\boldsymbol{\pi}_{i\mathbf{v}}}{\left(\displaystyle\int_0^1 W_i^2(r)dr-\boldsymbol{\pi}_{i\mathbf{v}}'\mathbf{G}_{\mathbf{v}}^{-1}\boldsymbol{\pi}_{i\mathbf{v}}\right)^{1/2}}, \tag{A.20}$$

as required. Condition $\sqrt{T}/N\to 0$ is satisfied so long as $T/N\to\delta$, as N and $T\to\infty$, where δ is a fixed finite non-zero positive constant. For sequential asymptotics, with $N\to\infty$, first, we note that for a fixed T and as $N\to\infty$, $\mathbf{Q}=\text{plim}_{N\to\infty}\mathbf{Q}_N$, and by Lemma A.1, (A.8) continues to hold (replacing '$\overset{(N,T)_j}{\rightarrow}$' by '$\overset{N}{\rightarrow}$'). Then, letting $T\to\infty$ yields (A.20).

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Table 1: Size and Power of Alternative Panel Unit Root Tests with Two Factors,
Factors and Idiosyncratic Errors are Serially Uncorrelated, $m^0 = 2$ Known, With an Intercept Only

	Size: $\rho_i = \rho = 1$						**Power**: $\rho_i \sim iidU[0.90, 0.99]$					
(T, N)	20	30	50	70	100	200	20	30	50	70	100	200
$CIPS(\hat{p}, k = 1)$												
20	5.75	6.40	5.10	5.50	5.50	6.10	7.80	10.70	10.85	13.15	11.95	14.85
30	5.40	6.60	5.35	5.70	5.85	6.15	11.40	13.65	17.10	17.10	18.55	21.85
50	5.00	5.60	5.90	6.10	4.80	5.90	17.35	22.10	27.10	27.50	32.05	38.40
70	5.45	4.85	4.60	5.70	5.35	5.25	27.95	33.40	40.75	47.45	50.00	56.35
100	5.65	7.05	6.10	4.95	5.75	5.45	44.65	54.45	67.10	68.20	78.60	82.15
200	4.95	4.55	5.60	5.65	4.85	4.80	97.40	99.50	99.95	99.95	100.00	100.00
$CSB(\hat{p}, k = 1)$												
20	6.35	6.10	5.60	4.95	5.80	6.10	14.25	15.80	18.50	23.45	24.80	31.20
30	5.70	5.85	5.20	5.60	5.55	4.10	20.50	24.80	31.70	36.80	40.50	46.95
50	6.35	6.00	5.80	5.85	5.55	5.55	39.20	47.75	62.20	70.30	77.25	87.70
70	5.70	5.80	6.35	6.15	5.75	5.60	61.40	75.40	89.55	94.30	98.00	99.50
100	4.55	5.20	5.95	6.10	5.40	6.60	79.05	89.65	97.95	98.70	99.60	99.95
200	6.50	4.75	6.15	5.15	6.20	5.85	94.85	97.80	99.45	99.90	99.95	100.00
$P_{\hat{e}}(\hat{p})$												
20	10.50	10.15	13.40	13.05	14.15	19.65	23.45	28.05	35.60	42.30	53.40	74.60
30	9.40	8.40	9.05	8.35	7.45	11.00	30.45	39.30	52.10	64.75	76.90	93.85
50	8.65	8.45	9.25	9.25	10.40	10.35	59.10	70.60	88.30	94.35	97.50	99.50
70	6.65	7.55	7.85	7.90	8.05	8.65	77.00	89.60	97.50	98.70	99.75	100.00
100	7.20	7.10	6.95	6.20	6.10	6.70	90.80	97.70	99.65	99.90	99.95	100.00
200	7.25	6.60	6.75	5.85	5.75	6.50	99.80	100.00	100.00	100.00	100.00	100.00
$PMSB$												
20	0.65	1.10	1.35	1.10	2.00	3.55	3.95	6.25	11.25	16.20	23.25	46.25
30	1.15	1.25	1.45	1.60	1.60	2.00	10.50	20.20	35.55	50.95	68.05	89.60
50	1.45	1.85	1.90	2.35	2.05	2.35	41.25	61.30	84.95	92.30	96.60	98.90
70	1.85	2.40	2.55	2.40	2.25	1.85	68.05	85.25	96.25	98.00	99.25	99.75
100	2.10	3.10	3.50	2.60	3.10	2.65	88.20	97.40	99.25	99.85	99.90	100.00
200	3.05	2.90	3.15	3.80	3.60	2.85	99.50	99.95	100.00	100.00	100.00	100.00
P_b												
20	8.65	8.65	9.50	9.40	11.65	19.35	28.95	35.45	51.60	63.00	76.30	93.20
30	7.35	7.70	7.55	8.10	8.60	12.70	47.80	60.95	78.55	86.65	94.30	98.70
50	7.55	6.95	7.60	6.05	7.80	8.95	77.90	88.55	96.05	98.00	98.85	99.60
70	7.05	7.50	6.95	7.00	7.25	5.95	90.45	95.75	99.20	99.20	99.85	100.00
100	7.25	6.60	7.15	6.70	6.00	7.20	96.80	99.45	99.80	99.95	100.00	100.00
200	8.30	6.75	6.45	6.15	5.55	5.65	99.95	100.00	100.00	100.00	100.00	100.00
t_b^*												
20	10.45	10.05	13.10	13.75	18.00	20.50	82.75	91.30	97.00	97.80	98.45	99.55
30	10.35	9.65	10.80	10.50	13.55	16.65	93.25	96.55	99.05	99.05	99.80	99.75
50	7.65	9.05	7.95	7.95	9.95	11.35	98.05	99.40	99.70	99.95	100.00	100.00
70	8.10	7.85	7.80	8.20	9.10	10.05	99.30	99.80	99.90	99.90	100.00	100.00
100	7.95	7.50	7.70	7.35	7.85	7.70	99.90	100.00	100.00	100.00	100.00	100.00
200	8.20	6.65	6.55	7.05	6.25	6.85	100.00	100.00	100.00	100.00	100.00	100.00
CPO												
20	7.00	8.05	10.40	14.50	17.50	28.90	32.75	46.85	66.15	76.60	84.95	93.65
30	6.60	7.60	9.65	11.65	13.15	23.25	50.80	67.20	83.50	91.00	95.70	97.80
50	5.90	5.85	7.90	9.25	11.65	16.15	78.20	89.75	96.30	98.15	99.25	99.35
70	5.35	6.45	7.40	8.40	10.50	13.10	90.30	96.45	99.00	99.15	99.70	99.95
100	5.35	6.15	7.40	7.65	8.90	12.95	96.60	99.50	99.85	99.90	99.90	100.00
200	5.90	5.45	6.50	6.70	7.35	10.15	99.90	100.00	100.00	100.00	100.00	100.00
$\widetilde{CPO}$												
20	7.80	10.15	14.45	18.60	23.50	39.80	32.50	46.45	65.00	75.30	83.95	94.35
30	7.85	8.10	11.65	13.60	16.65	26.95	48.20	64.20	82.00	88.70	94.75	97.30
50	6.45	5.35	8.25	9.45	11.85	16.45	74.65	87.30	95.75	97.65	99.05	99.15
70	6.10	6.20	7.85	8.55	10.35	13.05	87.80	95.45	98.65	98.95	99.60	99.95
100	5.45	6.10	7.50	7.80	9.15	12.90	95.95	99.15	99.75	99.90	99.85	100.00
200	5.90	5.25	6.70	6.40	7.20	10.75	99.85	99.95	100.00	100.00	100.00	100.00

Notes: y_{it} is generated as $y_{it} = d_{iyt} + \rho_i y_{i,t-1} + \gamma_{iy1} f_{1t} + \gamma_{iy2} f_{2t} + \varepsilon_{iyt}, i = 1, 2, ..., N; t = -49, 48, ...0, 1, ..., T$, with $y_{i,-50} = 0$, where $\gamma_{iy1} \sim iidU[0, 2]$, for $i = 1, 2, ..., N$; $\gamma_{iy2} \sim iidU[0, 1]$ for $i = 1, ..., [N^\alpha]$ and $\gamma_{iy2} = 0$ for $i = [N^\alpha] + 1, [N^\alpha] + 2, ..., N$ (where $[\cdot]$ denotes the integer part); $f_{\ell t} \sim iidN(0, 1)$ for $\ell = 1, 2$, $\varepsilon_{iyt} \sim iidN(0, \sigma_i^2)$ with $\sigma_i^2 \sim iidU[0.5, 1.5]$; $\Delta x_{it} = d_{ix} + \gamma_{ix1} f_{1t} + \varepsilon_{ixt}$, where, $d_{ix} = 0$, $\varepsilon_{ixt} = \rho_{ix}\varepsilon_{ixt-1} + \varpi_{ixt}$,, $\varpi_{ixt} \sim iidN(0, 1 - \rho_{ix}^2)$, $i = 1, 2, ..., N; t = -49, 48, ...0, 1, ..., T$, with $\varepsilon_{ix,-50} = 0$, and $\rho_{ix} \sim iidU[0.2, 0.4]$. The factor loadings in (50) are generated as $\gamma_{ix1} \sim iidU[0, 2]$; $d_{iyt} = (1 - \rho_i)\alpha_{iy}$ with $\alpha_{iy} \sim iidN(1, 1)$. The parameters α_{iy}, $\rho_{iy\varepsilon}$, γ_{iy1},γ_{iy2}, ρ_i, γ_{ix1}, ρ_{ix}, and σ_i are redrawn over each replication. The first 50 observations are discarded. The $CIPS(\hat{p})$ test and the $CSB(\hat{p})$ test are the proposed panel unit root test, defined by (28) and (34), respectively, based on cross section augmentation using y_{it} and x_{it} with lag-augmentation order selected according to $\hat{p} = \left[4(T/100)^{1/4}\right]$. $P_{\hat{e}}(\hat{p})$ is the test of Bai and Ng (2004) with lag-augmentation order $\hat{p} = \left[4(T/100)^{1/4}\right]$ and $PMSB$ and P_b are the pooled tests of Bai and Ng (2010), all of which are based on two extracted factors from y_{it},. The t_b^* test is the Moon and Perron (2004) test, the CPO test is the defactored constant point optimal test of Moon, Perron and Phillips (2007), and the $\widetilde{CPO}$ is the defactored point optimal test with serially correlated errors of Moon, Perron and Phillips (2011), based on two extracted factors from y_{it}. The $PMSB$, P_b, t_b^*, CPO and $\widetilde{CPO}$ tests use the automatic lag-order selection for the estimation of the long-run variances following Andrews and Monahan (1992). All tests are conducted at the 5% significance level, and the $CIPS(\hat{p})$ and $CSB(\hat{p})$ tests are based on the critical values for the corresponding $\hat{p} = \left[4(T/100)^{1/4}\right]$ and the number of additional regressors, k. All experiments are based on 2000 replications.

Table 2: Size and Power of Alternative Panel Unit Root Tests with Two Factors, Factors and Idiosyncratic Errors are Serially Uncorrelated, $m^0 = 2$ Known, With an Intercept and a Linear Trend

	Size: $\rho_i = \rho = 1$						**Power**: $\rho_i \sim iidU[0.90, 0.99]$					
(T, N)	20	30	50	70	100	200	20	30	50	70	100	200
$CIPS(\hat{p}, k = 1)$												
20	6.45	5.20	6.30	6.30	5.45	5.50	7.25	6.55	7.85	7.85	5.80	8.05
30	5.30	5.40	5.90	6.80	5.85	5.45	6.85	8.15	9.00	10.45	11.95	11.75
50	6.35	5.45	5.65	6.10	5.85	5.35	10.00	10.40	13.00	14.00	17.90	20.75
70	5.55	5.50	5.60	5.20	4.65	4.65	14.70	17.40	22.15	25.75	26.65	31.35
100	5.20	5.90	6.30	5.25	5.00	5.10	23.45	29.60	37.85	39.40	46.45	52.10
200	5.60	5.70	5.65	5.30	6.15	3.75	83.80	91.25	97.85	99.25	99.80	99.95
$CSB(\hat{p}, k = 1)$												
20	6.35	5.40	5.80	5.15	5.20	5.65	8.60	8.85	11.55	12.10	13.35	19.25
30	6.80	6.15	5.80	5.95	5.85	5.70	10.65	12.10	14.45	18.45	20.65	25.80
50	5.95	5.80	5.20	5.60	4.50	5.80	15.50	19.15	23.50	29.65	33.55	41.75
70	6.05	4.95	5.90	5.70	5.85	5.25	25.50	33.60	46.45	54.70	65.75	80.40
100	4.65	5.55	5.80	6.35	5.45	5.00	44.15	58.25	75.85	84.95	91.95	97.90
200	5.40	5.10	5.10	6.20	6.15	5.75	87.20	94.85	98.75	99.60	99.85	100.00
$P_{\hat{e}}(\hat{p})$												
20	15.25	18.00	21.45	21.65	29.05	36.30	17.40	19.25	25.10	26.35	32.30	43.50
30	12.25	11.95	12.65	14.75	14.80	19.90	15.75	17.25	19.50	24.00	25.85	40.25
50	10.80	10.95	12.75	10.95	13.40	17.70	20.95	25.50	34.55	39.05	47.80	71.90
70	8.85	9.20	10.35	11.40	12.70	12.95	30.00	39.35	52.50	64.80	75.65	92.85
100	7.60	7.45	8.00	7.75	7.35	6.50	45.75	58.55	76.50	85.70	91.40	98.70
200	8.40	7.45	7.25	8.20	8.40	7.75	94.20	98.45	99.80	99.90	100.00	100.00
$PMSB$												
20	0.20	0.25	0.25	0.45	0.30	0.75	0.40	0.20	0.15	0.55	0.35	0.75
30	0.35	0.50	0.35	0.75	0.95	0.55	0.85	1.40	1.70	2.10	2.80	2.75
50	1.45	1.30	1.35	1.00	0.85	0.90	7.05	9.10	14.65	19.20	26.20	48.00
70	1.55	1.55	1.25	1.40	1.65	0.90	16.20	24.85	42.00	54.10	68.70	88.20
100	2.30	2.60	2.55	2.30	1.85	1.65	41.20	58.55	80.10	89.30	92.10	97.95
200	3.45	2.90	2.35	3.10	3.20	2.60	90.50	96.60	98.90	99.45	99.80	99.90
P_b												
20	5.80	5.65	6.20	6.25	8.35	9.55	7.80	7.35	9.05	9.30	10.00	15.50
30	6.05	6.35	5.90	6.50	5.90	7.25	10.00	10.90	13.20	16.65	19.50	29.70
50	7.45	5.25	6.25	4.85	5.60	6.65	23.35	28.55	37.45	44.30	54.80	77.65
70	7.65	5.90	6.20	5.20	5.05	4.95	37.30	48.60	63.30	72.95	82.70	94.60
100	7.70	6.60	5.90	6.00	5.05	4.80	63.10	75.70	89.10	94.45	95.30	98.70
200	7.60	5.80	5.65	5.65	5.10	5.35	95.15	97.80	99.35	99.55	100.00	99.95
PP												
20	0.65	0.35	1.55	0.85	1.45	2.60	1.25	0.75	2.35	1.65	2.75	5.10
30	1.00	1.00	1.25	1.65	2.10	2.45	2.00	3.50	4.55	5.95	8.15	13.55
50	2.20	2.25	2.60	1.40	2.15	2.95	11.10	14.75	23.65	28.75	39.85	60.75
70	2.45	2.30	1.85	2.75	3.30	3.55	22.65	33.95	48.80	61.00	74.05	87.55
100	3.20	3.05	3.05	3.60	3.90	4.00	47.70	66.10	84.35	90.50	92.30	96.95
200	3.75	3.30	3.60	4.90	4.30	6.05	92.20	97.10	99.05	99.55	99.90	99.90
CPO												
20	45.45	48.15	51.90	55.40	56.15	59.30	50.35	53.55	58.35	63.50	63.60	68.65
30	44.15	50.75	54.10	58.35	62.65	67.15	53.15	61.60	65.80	71.35	75.35	81.30
50	42.75	49.05	56.55	61.40	67.20	75.35	62.55	70.20	79.85	83.65	89.25	93.40
70	39.55	49.20	54.95	63.20	66.25	73.65	67.40	79.90	87.50	92.05	94.30	97.45
100	37.95	43.80	51.50	58.40	62.55	71.90	77.20	87.45	94.20	96.65	97.70	99.10
200	32.00	33.80	42.65	48.15	54.55	63.50	94.95	97.85	99.55	99.80	100.00	100.00
$\widetilde{CPO}$												
20	12.80	18.40	32.60	41.45	51.20	74.00	15.15	24.65	41.55	51.80	62.65	82.35
30	8.15	11.80	16.45	24.65	32.60	52.80	14.15	21.80	33.05	46.50	58.25	79.20
50	5.65	7.45	11.50	12.75	16.85	29.45	22.90	32.10	50.15	62.25	74.50	87.55
70	4.40	5.80	7.00	9.50	13.15	20.75	32.65	49.65	67.65	78.65	88.40	94.55
100	4.45	4.65	6.75	8.10	9.85	15.40	54.50	73.20	89.40	94.45	95.10	98.30
200	3.85	3.75	5.05	7.05	6.85	10.25	92.35	97.40	99.20	99.75	99.90	99.95

Notes: y_{it} is generated as described in the note to Table 1, but $d_{iyt} = \mu_{iy} + (1 - \rho_i)\delta_i t$ with $\mu_{iy} \sim iidU[0.0, 0.02]$ and $\delta_i \sim iidU[0.0, 0.02]$, and $d_{ixt} = \delta_{ix}$ with and $\delta_{ix} \sim iidU[0.0, 0.02]$. The PP test is a defactored version of the optimal invariant test of Ploberger and Phillips (2002), based on two extracted factors from y_{it}. See also the notes to Table 1 for the specification of the rest of the parameters and the test statistics.

Table 3: Size and Power of Alternative Panel Unit Root Tests with Two Factors,
Factors are Serially Uncorrelated but Idiosyncratic Errors are Positively Serially Correlated, $m^0 = 2$ Known,
With an Intercept Only

	Size: $\rho_i = \rho = 1$						**Power**: $\rho_i \sim iidU[0.90, 0.99]$					
(T, N)	20	30	50	70	100	200	20	30	50	70	100	200
$CIPS(\hat{p}, k = 1)$												
20	5.00	5.65	4.05	4.30	3.80	4.15	7.40	8.65	8.50	10.70	9.25	11.65
30	4.40	5.45	3.85	4.20	4.15	4.55	9.65	11.45	14.55	14.80	16.00	18.75
50	4.30	5.30	5.25	4.70	3.90	5.15	16.55	20.40	24.05	24.65	28.75	34.60
70	4.90	5.00	4.45	5.00	4.35	4.30	26.10	30.55	37.55	44.40	45.15	51.50
100	5.45	6.20	5.60	4.10	5.55	4.95	41.95	51.10	62.85	62.65	74.60	78.25
200	4.75	4.45	5.05	5.55	4.65	4.55	96.45	99.10	99.80	100.00	100.00	100.00
$CSB(\hat{p}, k = 1)$												
20	6.85	6.40	6.15	6.30	6.75	6.20	13.40	15.30	17.70	23.25	23.65	30.55
30	5.40	6.25	5.65	5.65	6.00	4.45	18.65	22.95	28.75	33.85	37.70	43.90
50	5.90	5.60	5.95	5.65	5.65	6.10	36.65	43.55	58.60	67.40	74.25	86.10
70	5.15	6.15	5.50	5.85	5.20	5.80	60.45	74.85	90.65	95.70	99.05	99.85
100	4.35	4.80	5.75	5.75	5.15	6.30	80.35	90.90	98.60	99.50	99.90	100.00
200	6.35	4.40	5.40	5.10	5.65	5.45	97.25	99.10	99.75	100.00	100.00	100.00
$P_{\hat{e}}(\hat{p})$												
20	12.00	15.40	18.15	18.00	21.65	31.00	22.40	24.10	30.30	37.05	44.20	63.60
30	10.05	9.90	11.55	12.05	10.60	15.60	30.00	35.75	48.60	61.20	75.45	93.45
50	8.55	9.10	9.35	9.45	10.60	11.90	58.45	71.60	89.40	95.35	98.60	99.80
70	7.35	6.95	8.20	7.70	8.55	9.80	78.05	91.15	98.55	99.45	99.90	100.00
100	7.20	7.35	6.25	6.40	5.70	6.95	93.05	98.50	100.00	100.00	100.00	100.00
200	7.50	6.45	5.65	6.70	5.60	6.65	99.95	100.00	100.00	100.00	100.00	100.00
$PMSB$												
20	1.70	3.80	8.45	7.60	12.50	21.50	3.40	5.65	10.40	13.50	21.85	42.55
30	1.35	2.90	3.85	5.15	6.55	10.05	8.80	17.30	32.80	49.25	67.65	92.50
50	1.80	2.10	2.15	2.75	3.25	5.75	40.10	61.20	87.40	94.90	98.85	99.90
70	1.70	2.35	2.65	2.60	2.95	3.50	68.70	88.25	98.20	99.45	100.00	100.00
100	2.00	2.95	3.55	3.00	3.80	3.45	90.60	98.60	99.90	100.00	100.00	100.00
200	2.85	2.50	3.25	3.85	3.55	3.15	99.90	100.00	100.00	100.00	100.00	100.00
P_b												
20	9.65	12.60	16.65	17.80	21.95	32.85	21.00	24.60	39.15	46.30	60.05	83.95
30	7.15	8.35	10.80	11.45	13.30	19.85	38.95	52.40	72.90	83.55	94.60	99.45
50	7.00	6.30	7.20	6.95	8.25	11.40	76.70	88.10	97.55	99.10	99.65	99.90
70	6.25	6.50	7.05	7.05	7.75	6.85	90.95	97.25	99.80	99.85	99.95	100.00
100	6.65	6.75	6.55	5.95	5.70	7.50	97.95	99.85	100.00	100.00	100.00	100.00
200	8.65	6.85	6.00	5.80	5.60	5.75	99.95	100.00	100.00	100.00	100.00	100.00
t_b^*												
20	8.70	8.20	11.35	12.05	15.70	20.65	80.45	90.70	97.95	98.95	99.70	99.95
30	7.85	7.50	8.95	9.00	10.60	13.30	92.40	96.95	99.45	99.65	99.95	100.00
50	6.25	6.85	6.45	6.65	7.70	8.45	98.55	99.80	100.00	99.95	100.00	100.00
70	6.70	6.30	5.85	5.85	7.00	6.75	99.75	99.95	100.00	100.00	100.00	100.00
100	6.85	6.70	6.55	5.75	5.55	6.00	100.00	100.00	100.00	100.00	100.00	100.00
200	7.40	6.10	6.30	6.55	5.45	6.05	100.00	100.00	100.00	100.00	100.00	100.00
CPO												
20	0.05	0.05	0.10	0.00	0.05	0.10	0.60	0.40	0.60	0.60	1.15	1.40
30	0.00	0.00	0.00	0.00	0.00	0.00	0.85	1.20	1.60	2.65	4.55	10.90
50	0.00	0.00	0.00	0.00	0.00	0.00	3.10	6.35	15.15	27.40	43.40	75.30
70	0.00	0.00	0.00	0.00	0.00	0.00	7.85	19.55	44.70	66.10	83.25	96.10
100	0.00	0.00	0.00	0.00	0.00	0.00	20.50	49.10	82.80	93.60	97.40	99.75
200	0.00	0.00	0.00	0.00	0.00	0.00	75.25	95.30	99.80	100.00	100.00	100.00
$\widetilde{CPO}$												
20	8.00	11.30	16.35	23.85	29.45	51.20	32.80	46.20	68.15	81.35	89.20	97.60
30	6.75	8.25	12.20	13.90	18.55	33.55	48.85	67.15	85.25	93.15	97.75	99.45
50	5.60	5.20	7.80	8.55	11.70	16.70	78.05	90.70	98.05	99.05	99.75	99.90
70	5.25	6.15	7.35	8.30	9.85	11.30	89.80	97.60	99.60	99.75	99.95	100.00
100	4.95	5.70	5.85	6.80	7.50	9.90	97.05	99.90	100.00	100.00	100.00	100.00
200	5.85	5.05	5.15	5.40	5.60	7.60	99.95	100.00	100.00	100.00	100.00	100.00

Notes: y_{it} is generated as described in the notes to Table 1, except that $\varepsilon_{iyt} = \rho_{iy\varepsilon}\varepsilon_{iyt-1} + (1 - \rho_{iy\varepsilon}^2)^{1/2}\eta_{iyt}, \eta_{iyt} \sim iidN(0, \sigma_i^2)$, $\varepsilon_{iy,-50} = 0$, $\sigma_i^2 \sim iidU[0.5, 1.5]$, $\rho_{iy\varepsilon} \sim iidU[0.2, 0.4]$. See also the notes to Table 1 for the specification of the rest of the parameters and the test statistics.

Table 4: Size and Power of Alternative Panel Unit Root Tests with Two Factors,
Factors are Serially Uncorrelated but Idiosyncratic Errors are Positively Serially Correlated, $m^0 = 2$ Known,
With an Intercept and a Linear Trend

	Size: $\rho_i = \rho = 1$						**Power**: $\rho_i \sim iidU[0.90, 0.99]$					
(T, N)	20	30	50	70	100	200	20	30	50	70	100	200
$CIPS(\hat{p}, k = 1)$												
20	5.05	3.90	3.85	4.15	3.50	2.75	6.15	4.55	5.50	5.65	4.10	5.05
30	4.15	5.05	4.35	4.95	3.75	3.45	6.00	6.30	6.40	7.95	9.35	9.25
50	5.80	4.50	4.65	5.05	4.95	4.30	8.95	9.55	11.05	12.15	16.30	19.05
70	5.10	4.65	4.45	4.65	3.95	4.00	13.85	15.60	18.95	23.30	24.55	28.70
100	5.25	5.50	5.30	4.70	4.05	4.50	21.70	27.60	33.65	36.60	43.00	47.15
200	5.60	4.85	5.75	4.85	5.75	3.35	79.35	89.95	96.55	98.40	99.35	99.90
$CSB(\hat{p}, k = 1)$												
20	6.50	5.70	6.05	4.65	5.50	5.30	8.55	8.65	10.80	11.15	13.00	18.20
30	5.65	5.05	5.20	5.25	5.00	4.85	8.70	9.70	12.90	15.35	18.30	22.40
50	4.80	5.25	4.25	4.45	4.20	4.65	12.40	15.70	19.35	24.45	28.15	36.10
70	4.90	3.65	4.70	4.20	4.35	3.80	21.65	28.75	39.10	47.85	58.50	73.10
100	4.15	4.30	5.10	4.50	4.45	4.35	40.20	55.15	72.60	82.25	90.60	98.10
200	4.45	3.95	4.10	4.70	5.10	4.80	90.40	96.90	99.50	99.95	100.00	100.00
$P_{\hat{e}}(\hat{p})$												
20	20.60	24.95	30.60	35.50	41.70	54.60	21.60	24.95	32.00	35.55	42.00	55.35
30	15.35	16.20	18.35	22.85	23.65	32.60	17.95	20.15	24.00	30.10	32.50	48.65
50	12.90	13.70	14.80	14.65	17.50	23.95	23.00	27.40	38.85	45.45	54.80	78.95
70	9.70	10.75	11.85	13.10	15.40	16.95	33.20	41.60	56.75	69.80	81.65	96.60
100	7.95	8.65	8.75	8.05	8.85	8.50	48.00	62.65	82.15	89.95	95.85	99.85
200	8.45	8.30	7.95	8.65	9.15	8.60	96.25	99.55	100.00	100.00	100.00	100.00
$PMSB$												
20	0.75	1.45	4.05	5.90	9.30	15.05	0.70	1.10	3.00	4.80	6.35	11.75
30	0.70	1.50	2.35	4.05	3.50	6.70	1.25	1.75	2.75	4.85	5.65	9.55
50	1.55	1.50	1.75	1.60	2.10	3.60	6.30	9.75	16.70	22.50	32.30	58.90
70	1.75	1.65	2.15	1.75	2.55	2.45	17.25	26.45	46.40	61.05	75.75	95.05
100	2.15	2.95	2.60	2.80	2.30	2.55	43.50	63.25	85.20	94.45	96.85	99.85
200	3.05	2.65	2.70	3.25	3.60	3.10	93.05	98.40	99.90	99.95	100.00	100.00
P_b												
20	8.70	11.25	16.10	17.80	24.05	31.65	9.10	9.70	14.90	15.35	21.70	29.90
30	6.45	6.95	9.40	11.40	12.75	18.15	9.15	10.10	15.15	18.25	21.85	34.55
50	6.90	5.15	6.55	6.00	7.00	10.60	20.90	26.40	38.20	45.45	56.40	80.50
70	6.80	5.60	6.55	5.45	6.20	7.35	36.60	47.70	63.80	76.60	88.00	97.55
100	6.65	6.00	6.10	6.15	5.60	5.50	63.85	77.35	92.35	97.35	98.35	99.95
200	7.35	6.15	5.65	5.60	5.40	5.55	97.20	99.40	99.95	99.95	100.00	100.00
PP												
20	1.40	2.00	3.80	4.35	6.00	12.20	2.60	2.75	6.40	6.65	10.50	21.50
30	1.65	2.45	3.05	3.60	5.70	9.05	3.90	5.75	8.70	13.80	20.20	35.70
50	2.65	3.00	3.75	3.15	4.10	6.75	13.45	19.60	32.40	40.40	55.10	77.95
70	2.85	2.60	2.85	3.60	4.10	5.55	25.30	38.85	59.10	72.25	85.70	96.20
100	3.25	3.45	3.70	3.10	3.80	3.95	52.40	70.65	90.50	96.35	97.65	99.75
200	3.40	3.20	3.35	4.10	3.75	4.40	95.45	99.05	99.90	99.90	100.00	99.95
CPO												
20	0.05	0.00	0.00	0.00	0.00	0.00	0.05	0.00	0.00	0.00	0.00	0.00
30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
50	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
70	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
200	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\widetilde{CPO}$												
20	22.00	34.10	55.45	66.65	78.60	92.90	25.95	42.85	64.55	76.55	85.90	96.15
30	11.90	16.80	29.25	42.40	54.50	80.25	20.25	32.80	50.20	65.50	80.30	93.80
50	7.40	9.30	16.00	19.85	26.40	45.60	26.75	39.55	62.05	75.00	87.50	97.05
70	5.55	6.55	9.30	12.95	17.55	31.15	37.20	55.90	77.80	87.90	94.90	98.65
100	4.55	5.35	6.65	9.10	11.55	18.20	59.80	79.00	94.45	97.95	99.10	99.90
200	3.50	3.85	4.65	5.90	6.25	8.85	95.50	99.15	99.95	99.90	100.00	99.95

Notes: y_{it} is generated as described in Table 3, but $d_{iyt} = \mu_{iy} + (1-\rho_i)\delta_i t$ with $\mu_{iy} \sim iidU[0.0, 0.02]$ and $\delta_i \sim iidU[0.0, 0.02]$, and $d_{it} = \delta_{ix}$ with $\delta_{ix} \sim iidU[0.0, 0.02]$. See also the notes to Tables 1 and 3 for the specification of the rest of the parameters and the test statistics.

Table 5: Size and Power of Alternative Panel Unit Root Tests with Two Factors, Factors are Serially Uncorrelated but Idiosyncratic Errors are Negatively Serially Correlated, $m^0 = 2$ Known, With an Intercept Only

	Size: $\rho_i = \rho = 1$						**Power:** $\rho_i \sim iidU[0.90, 0.99]$					
(T, N)	20	30	50	70	100	200	20	30	50	70	100	200
$CIPS\ (\hat{p}, k=1)$												
20	6.65	7.20	6.35	6.60	7.40	6.85	8.45	11.55	11.60	13.95	13.20	15.60
30	6.60	7.10	6.45	6.70	7.40	7.45	12.10	14.90	17.05	18.00	19.15	22.95
50	5.10	6.20	6.40	6.90	5.70	6.05	17.60	21.90	26.95	28.30	31.90	38.50
70	5.95	5.50	5.35	6.00	6.30	5.65	28.80	33.45	41.55	47.65	50.95	56.85
100	6.30	7.40	7.00	5.40	6.15	5.95	46.40	56.25	68.20	70.10	79.35	83.35
200	6.20	5.15	6.15	5.65	5.45	5.20	97.75	99.50	99.90	99.90	100.00	100.00
$CSB(\hat{p}, k=1)$												
20	6.60	5.40	5.25	4.65	6.10	5.65	14.45	16.40	19.30	24.60	25.80	31.30
30	6.05	5.85	5.25	5.55	5.80	4.30	21.55	25.65	32.15	36.90	41.45	47.25
50	6.55	5.95	6.35	6.50	5.40	5.80	38.70	46.65	60.95	67.50	75.50	85.15
70	6.35	6.25	6.75	6.25	6.45	5.15	58.65	71.85	84.45	90.25	94.35	97.80
100	5.60	6.00	6.20	6.40	5.60	6.50	74.25	85.80	94.30	95.60	98.30	99.70
200	7.05	5.90	6.75	6.10	7.00	6.80	91.40	94.10	97.90	98.90	99.50	100.00
$P_{\hat{e}}(\hat{p})$												
20	9.15	10.35	11.15	12.45	12.40	15.35	23.20	27.05	35.15	41.70	50.70	71.05
30	9.65	9.40	8.80	8.35	8.65	10.35	30.20	37.05	49.35	60.10	71.10	87.35
50	9.25	8.45	8.45	9.90	10.45	11.45	55.75	66.15	82.40	90.15	94.45	97.60
70	6.90	8.05	8.20	8.05	7.95	8.85	71.50	84.30	93.15	96.20	98.60	99.60
100	7.25	6.90	7.40	6.55	6.95	6.05	86.65	95.45	99.15	99.35	99.55	99.95
200	7.35	6.95	6.65	6.15	6.00	6.50	99.45	99.95	99.95	100.00	100.00	100.00
$PMSB$												
20	0.75	0.90	1.10	1.05	1.00	1.65	5.00	7.90	15.05	20.75	27.60	51.10
30	1.30	1.40	1.55	1.50	1.20	0.95	12.25	22.30	36.30	49.45	62.45	81.35
50	1.70	1.70	2.05	2.10	1.95	1.35	38.10	56.65	77.50	86.40	91.05	94.55
70	2.20	2.45	2.90	2.20	2.20	1.60	62.95	78.40	90.90	94.30	97.25	98.55
100	2.20	3.10	3.50	2.60	3.05	2.35	81.30	92.60	96.65	98.00	98.70	99.55
200	3.30	3.15	3.25	3.70	3.70	2.55	98.05	99.45	99.95	99.90	100.00	100.00
P_b												
20	11.55	11.70	14.80	17.00	17.05	28.95	37.25	47.15	62.75	72.95	81.30	92.20
30	11.05	11.05	11.15	12.65	13.05	17.60	52.90	64.95	80.45	86.25	91.50	95.85
50	9.70	9.15	9.45	8.85	10.40	12.55	76.35	85.00	92.70	95.40	97.15	97.30
70	9.20	9.10	9.10	9.15	9.65	8.30	87.15	92.45	97.00	96.95	98.60	99.45
100	8.65	8.30	8.80	7.70	7.30	9.35	93.70	97.95	99.00	99.30	99.40	99.85
200	8.75	7.00	7.15	7.10	6.45	6.40	99.50	99.65	99.95	100.00	100.00	100.00
t_b^*												
20	11.75	12.25	15.40	18.15	21.20	26.35	79.90	88.15	94.00	95.05	96.55	97.70
30	12.10	12.55	13.65	15.20	17.05	22.30	90.55	93.15	96.55	97.65	98.60	98.95
50	9.10	9.20	10.70	10.80	14.05	17.35	95.15	98.15	98.40	99.35	99.50	99.95
70	9.10	9.60	9.00	10.30	10.95	15.00	98.45	99.25	99.50	99.50	99.85	100.00
100	9.40	8.65	9.20	8.75	10.45	10.95	99.15	99.90	99.90	99.95	100.00	100.00
200	8.00	6.75	7.65	7.90	7.80	9.35	100.00	100.00	100.00	100.00	100.00	100.00
CPO												
20	61.95	74.10	84.20	89.20	90.95	94.35	88.90	94.35	97.95	98.75	98.60	99.40
30	66.60	77.05	86.75	90.50	93.10	95.95	94.30	97.60	99.05	99.40	99.80	99.90
50	67.40	78.05	88.25	92.70	94.90	95.90	98.55	99.50	99.50	99.50	100.00	99.90
70	68.35	79.20	89.10	92.65	95.25	97.45	99.30	99.75	100.00	99.90	99.95	100.00
100	68.30	81.15	89.85	93.45	95.00	97.05	99.80	100.00	100.00	100.00	100.00	100.00
200	71.35	80.25	92.20	93.75	95.45	97.15	99.95	100.00	100.00	100.00	100.00	100.00
$\widetilde{CPO}$												
20	7.25	10.15	13.50	18.40	22.60	36.15	30.20	41.15	56.95	66.50	72.55	84.10
30	7.95	9.65	13.10	15.45	19.45	27.40	44.25	58.50	73.85	79.95	87.20	92.00
50	6.70	6.85	9.40	10.90	14.65	20.85	67.75	80.70	90.20	93.35	95.10	96.50
70	6.75	8.05	9.40	10.75	12.95	19.00	82.70	90.55	95.25	96.40	97.85	99.25
100	6.50	7.55	9.65	10.95	12.75	19.65	91.35	96.80	98.60	98.90	99.20	99.70
200	6.40	6.45	8.90	9.30	10.85	18.05	99.05	99.45	99.95	100.00	100.00	100.00

Notes: y_{it} is generated as described in the notes to Table 1, except that $\varepsilon_{iyt} = \rho_{iy\varepsilon}\varepsilon_{iyt-1} + (1-\rho_{iy\varepsilon}^2)^{1/2}\eta_{iyt}, \eta_{iyt} \sim iidN(0, \sigma_i^2)$, $\varepsilon_{iy,-50} = 0$, $\sigma_i^2 \sim iidU[0.5, 1.5]$, $\rho_{iy\varepsilon} \sim iidU[-0.4, -0.2]$. See also the notes to Table 1 for the specification of the rest of the parameters and the test statistics.

Table 6: Size and Power of Alternative Panel Unit Root Tests with Two Factors,
Factors are Serially Uncorrelated but Idiosyncratic Errors are Negatively Serially Correlated, $m^0 = 2$ Known,
With an Intercept and a Linear Trend

	Size: $\rho_i = \rho = 1$						**Power**: $\rho_i \sim iidU[0.90, 0.99]$					
(T, N)	20	30	50	70	100	200	20	30	50	70	100	200
$CIPS(\hat{p}, k = 1)$												
20	6.65	7.15	7.95	7.60	7.35	6.15	6.75	7.05	9.70	9.50	8.90	8.00
30	6.00	6.40	6.70	7.35	7.40	7.10	8.00	8.40	9.65	11.05	12.70	13.00
50	6.85	5.85	7.25	6.75	6.85	5.90	11.30	11.65	13.50	15.80	17.85	19.80
70	5.90	6.15	6.40	6.60	5.10	6.00	15.30	17.85	23.55	25.75	28.55	30.80
100	6.15	6.70	5.90	6.30	5.85	5.75	24.75	29.35	36.00	42.05	45.05	55.30
200	7.20	5.80	6.85	4.50	6.00	4.65	84.20	92.35	98.20	98.95	99.80	99.95
$CSB(\hat{p}, k = 1)$												
20	6.85	6.35	6.05	5.50	6.20	5.25	8.85	8.70	11.95	11.90	15.25	18.55
30	6.75	5.65	6.90	6.65	5.95	6.60	11.35	14.45	17.45	18.15	22.55	28.70
50	5.75	7.05	6.00	6.75	5.05	6.00	16.25	21.15	28.70	30.55	35.90	44.20
70	6.75	6.40	6.60	5.95	7.45	5.90	26.60	34.95	48.35	55.45	66.35	79.10
100	6.00	6.70	6.30	8.10	7.45	5.95	41.65	54.85	74.30	80.10	89.10	96.90
200	6.75	6.80	6.95	6.95	6.95	8.15	82.50	90.55	95.80	97.95	99.15	99.85
$P_{\hat{e}}(\hat{p})$												
20	12.10	12.95	13.75	16.05	16.75	18.90	13.65	13.75	15.90	17.70	19.45	24.70
30	9.10	8.85	8.90	9.20	10.90	11.05	12.10	13.25	13.85	16.10	17.90	22.45
50	8.35	9.45	9.85	10.15	9.70	11.35	18.40	20.05	26.50	30.55	39.15	56.30
70	8.70	8.15	8.30	8.60	9.00	9.35	25.20	32.25	40.05	50.65	58.75	80.50
100	7.15	6.95	5.25	5.60	6.10	5.55	39.15	48.90	64.35	72.50	83.35	92.30
200	6.95	7.90	6.15	6.65	6.65	6.75	89.25	95.20	99.15	99.40	99.90	100.00
$PMSB$												
20	0.20	0.35	0.25	0.40	0.55	0.30	0.65	0.95	0.60	0.95	0.70	0.70
30	1.25	1.15	1.15	0.75	0.60	0.20	2.05	2.30	2.45	2.90	3.60	5.90
50	1.90	1.50	1.70	1.55	1.65	0.80	6.65	11.00	17.45	22.45	30.60	51.55
70	2.20	2.20	1.90	2.15	2.60	1.75	16.75	24.10	40.75	49.45	59.90	79.40
100	3.00	3.20	3.00	3.25	2.80	2.50	37.95	51.80	70.65	78.85	85.25	91.60
200	3.10	3.85	3.60	3.35	4.10	3.30	84.00	91.10	96.00	96.85	97.70	98.90
P_b												
20	11.95	12.10	12.05	13.95	16.00	23.85	14.40	15.30	17.80	20.45	25.50	39.35
30	11.15	11.10	11.40	12.20	14.85	18.75	16.25	20.90	22.30	28.15	34.85	52.20
50	8.85	10.35	10.00	11.30	11.85	13.35	27.25	33.85	45.10	50.65	62.70	78.05
70	9.80	8.85	9.25	9.05	10.00	11.80	39.15	47.55	61.90	69.90	77.60	88.40
100	9.95	9.65	8.70	8.75	8.90	10.40	60.00	69.25	81.85	87.30	90.45	93.95
200	8.55	8.80	7.40	6.80	7.55	7.25	90.15	94.70	97.55	97.30	98.35	99.00
PP												
20	0.25	0.15	0.50	0.15	0.40	0.40	0.30	0.30	0.40	0.30	0.70	0.85
30	0.55	0.75	0.45	0.60	1.20	1.20	1.05	1.45	1.70	2.25	3.65	4.70
50	1.20	1.75	1.35	1.70	1.40	2.15	6.10	8.30	13.70	18.60	22.75	35.20
70	1.65	2.00	1.45	1.80	2.85	3.15	15.75	22.00	36.40	43.80	50.60	68.85
100	2.85	3.65	3.15	3.45	4.00	5.90	40.80	50.65	67.40	75.70	79.85	88.30
200	3.30	3.90	4.25	4.65	6.25	9.45	84.60	91.90	95.75	97.10	97.55	98.80
CPO												
20	99.45	99.80	100.00	100.00	100.00	100.00	99.60	99.85	100.00	100.00	100.00	100.00
30	99.90	100.00	100.00	100.00	100.00	100.00	99.95	100.00	100.00	100.00	100.00	100.00
50	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
70	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
100	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
200	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$\widetilde{CPO}$												
20	6.35	9.15	13.05	19.15	24.00	37.90	7.65	12.15	17.15	24.75	31.00	47.50
30	4.20	6.35	9.15	11.75	16.25	24.40	7.65	12.65	17.70	24.00	32.70	46.10
50	3.80	5.60	7.30	9.40	10.80	17.70	14.80	21.90	32.95	41.70	51.15	66.60
70	3.35	4.50	5.50	7.40	9.80	16.40	23.35	33.35	52.90	61.00	67.70	81.15
100	4.20	6.25	5.90	7.35	9.10	15.30	45.40	58.30	75.75	81.85	85.00	91.35
200	3.50	4.60	5.40	6.40	8.50	14.15	84.90	92.60	96.20	97.40	97.85	99.00

Notes: y_{it} is generated as described in Table 5, but $d_{iyt} = \mu_{iy} + (1-\rho_i)\delta_i t$ with $\mu_{iy} \sim iidU[0.0, 0.02]$ and $\delta_i \sim iidU[0.0, 0.02]$, and $d_{ix} = \delta_{ix}$ with $\delta_{ix} \sim iidU[0.0, 0.02]$. See also the notes to Tables 1 and 5 for the specification of the rest of the parameters and the test statistics.

Table 7: Results of the $CIPS$ and the CSB Panel Unit Root Tests Applied to Real Interest Rates for all $m^0 - 1$ Combinations of the 5 Additional Regressors ($1979Q2 - 2009Q4$)

Real Interest Rates ($N = 32, T = 118$) Intercept Only								
$m^0 = 1$			$m^0 = 3$			$m^0 = 4$		
	$CIPS(\hat{p})$	$CSB(\hat{p})$	$\bar{\mathbf{x}}_t$	$CIPS(\hat{p})$	$CSB(\hat{p})$	$\bar{\mathbf{x}}_t$	$CIPS(\hat{p})$	$CSB(\hat{p})$
	-2.983†	0.157†	$poil_t, \bar{r}^L_t$	-3.603†	0.319	$poil_t, \bar{r}^L_t, \overline{eq}_t$	-3.545†	0.282
			$poil_t, \overline{eq}_t$	-3.319†	0.166†	$poil_t, \bar{r}^L_t, \overline{ep}_t$	-3.566†	0.321
$m^0 = 2$			$poil_t, \overline{eq}_t$	-3.286†	0.158†	$poil_t, \bar{r}^L_t, \overline{gdp}_t$	-3.530†	0.284
$\bar{x}_t$	$CIPS(\hat{p})$	$CSB(\hat{p})$	$poil_t, \overline{gdp}_t$	-3.318†	0.161†	$poil_t, \overline{eq}_t, \overline{ep}_t$	-3.257†	0.164†
$poil_t$	-3.347†	0.163†	$\bar{r}^L_t, \overline{eq}_t$	-3.671†	0.234†	$poil_t, \overline{eq}_t, \overline{gdp}_t$	-3.325†	0.161†
$\bar{r}^L_t$	-3.604†	0.243†	$\bar{r}^L_t, \overline{ep}_t$	-3.457†	0.245*	$poil_t, \overline{ep}_t, \overline{gdp}_t$	-3.286†	0.160†
$\overline{eq}_t$	-2.878†	0.169†	$\bar{r}^L_t, \overline{gdp}_t$	-3.504†	0.243†	$\bar{r}^L_t, \overline{eq}_t, \overline{ep}_t$	-3.619†	0.233*
$\overline{ep}_t$	-2.866†	0.153†	$\overline{eq}_t, \overline{ep}_t$	-2.805†	0.165†	$\bar{r}^L_t, \overline{eq}_t, \overline{gdp}_t$	-3.684†	0.240*
$\overline{gdp}_t$	-3.117†	0.162†	$\overline{eq}_t, \overline{gdp}_t$	-3.354†	0.165†	$\bar{r}^L_t, \overline{ep}_t, \overline{gdp}_t$	-3.365†	0.234*
			$\overline{ep}_t, \overline{gdp}_t$	-3.129†	0.159†	$\overline{eq}_t, \overline{ep}_t, \overline{gdp}_t$	-3.407†	0.161†

Notes: † and $*$ denote rejections at 1% and 5% significance levels, respectively. For the selected lag order $\hat{p} = \left[4(T/100)^{1/4}\right]$, the critical values for the $CIPS$ test in the case where $m^0 = 1$ are -2.238, and -2.106 for the 1%, and 5% significance levels, respectively. For $m^0 = 2$ they are -2.486, and -2.335, for $m^0 = 3$ they are -2.669, and -2.504, and for $m^0 = 4$ they are -2.816, and -2.641. Similarly for the CSB test, the critical values for $m^0 = 1$ are 0.279, and 0.322, for $m^0 = 2$ they are 0.261, and 0.304, for $m^0 = 3$ they are 0.245, and 0.287, and for $m^0 = 4$ they are 0.231, and 0.270. The variables under the heading $\bar{\mathbf{x}}_t$ indicate the regressors used for cross section augmentation in addition to $\bar{y}_t$, where $y_{it} = r^S_{it} - \pi_{it}$. In the case where $m^0 = 1$ no additional regressors are used. The variables $poil_t, \overline{eq}_t, \overline{ep}_t$, and $\overline{gdp}_t$ are detrended.

Table 8: Results of $CIPS$ and CSB Panel Unit Root Tests Applied to Real Equity Prices for all $m^0 - 1$ Combinations of the Five Additional Regressors ($1979Q2 - 2009Q4$)

Real Equity Prices ($N = 26, T = 118$) Intercept and Trend								
$m^0 = 1$			$m^0 = 3$			$m^0 = 4$		
	$CIPS(\hat{p})$	$CSB(\hat{p})$	$\bar{\mathbf{x}}_t$	$CIPS(\hat{p})$	$CSB(\hat{p})$	$\bar{\mathbf{x}}_t$	$CIPS(\hat{p})$	$CSB(\hat{p})$
	-2.594	0.143	$poil_t, \bar{r}^L_t$	-3.022*	0.113	$poil_t, \bar{r}^L_t, \bar{\pi}_t$	-2.716	0.116
			$poil_t, \bar{\pi}_t$	-2.933*	0.125	$poil_t, \bar{r}^L_t, \overline{ep}_t$	-3.051*	0.094*
$m^0 = 2$			$poil_t, \overline{ep}_t$	-3.098†	0.122	$poil_t, \bar{r}^L_t, \overline{gdp}_t$	-2.423	0.106
$\bar{x}_t$	$CIPS(\hat{p})$	$CSB(\hat{p})$	$poil_t, \overline{gdp}_t$	-2.416	0.118	$poil_t, \bar{\pi}_t, \overline{ep}_t$	-3.044*	0.119
$poil_t$	-2.819*	0.125	$\bar{r}^L_t, \bar{\pi}_t$	-2.567	0.136	$poil_t, \bar{\pi}_t, \overline{gdp}_t$	-2.455	0.118
$\bar{r}^L_t$	-2.791*	0.132	$\bar{r}^L_t, \overline{ep}_t$	-2.711	0.124	$poil_t, \overline{ep}_t, \overline{gdp}_t$	-2.965	0.116
$\bar{\pi}_t$	-2.731	0.142	$\bar{r}^L_t, \overline{gdp}_t$	-2.433	0.116	$\bar{r}^L_t, \bar{\pi}_t, \overline{ep}_t$	-2.527	0.128
$\overline{ep}_t$	-2.759	0.135	$\bar{\pi}_t, \overline{ep}_t$	-2.664	0.135	$\bar{r}^L_t, \bar{\pi}_t, \overline{gdp}_t$	-2.249	0.116
$\overline{gdp}_t$	-2.297	0.132	$\bar{\pi}_t, \overline{gdp}_t$	-2.494	0.129	$\bar{r}^L_t, \overline{ep}_t, \overline{gdp}_t$	-2.624	0.101*
			$\overline{ep}_t, \overline{gdp}_t$	-2.665	0.125	$\bar{\pi}_t, \overline{ep}_t, \overline{gdp}_t$	-2.580	0.122

Notes: The critical values for the $CIPS$ test in the case where $m^0 = 1$ are -2.757, and -2.619 for the 1%, and 5% significance levels, respectively. For $m^0 = 2$ they are -2.926, and -2.773, for $m^0 = 3$ they are -3.075, and -2.911 and for $m^0 = 4$ they are -3.190, and -3.006. Similarly for the CSB test, the critical values for $m^0 = 1$ are 0.108, and 0.121, for $m^0 = 2$ they are 0.102, and 0.114, for $m^0 = 3$ they are 0.096, and 0.108, and for $m^0 = 4$ they are 0.090, and 0.101. Also see the notes to Table 7.

Table 9: Results for $P_{\hat{e}}(\hat{p})$, P_b, t^*_b, PP, $PMSB$ and $\widetilde{CPO}$ Panel Unit Root Tests for Real Interest Rates and Real Equity Prices ($1979Q2 - 2009Q4$)

	PANEL A					**PANEL B**				
	Real Interest Rates ($N = 32$) With an Intercept					Real Equity Prices ($N = 26$) With an Intercept and a Linear Trend				
m^0	$P_{\hat{e}}(\hat{p})$	P_b	t^*_b	$PMSB$	$\widetilde{CPO}$	$P_{\hat{e}}(\hat{p})$	P_b	PP	$PMSB$	$\widetilde{CPO}$
1	9.768	-9.140†	-19.187†	-1.804*	-4.114†	4.226	-1.776*	-2.084*	-1.349	-2.181*
2	8.826	-3.405†	-18.230†	-1.206	-4.124†	3.231	-2.232*	-1.916*	-1.689*	-2.019*
3	8.326	-3.263†	-22.527†	-1.547	-4.170†	3.430	-2.788†	-1.655*	-1.989*	-1.765*
4	9.197	-6.371†	-26.613†	-2.124*	-4.157†	3.082	-2.427†	-1.217	-1.778*	-1.347

For the $P_{\hat{e}}(\hat{p})$ test, the lag order is chosen according to $\hat{p} = \left[4(T/100)^{1/4}\right]$. The rest of the tests use the automatic lag-order selection for the estimation of the long-run variances following Andrews and Monahan (1992). Also see the notes to Table 7.