**A Note on Lemma 10**

# 1 Introduction

As it stands, **Lemma 10** on p 46 is only an approximation when $\boldsymbol{\theta}_2$ enters into the score. The exact result, as it applies to Corollary 9 on p 47 is set out below. This result necessitates modifications to some later results.

# 2 Information matrix for Corollary 9

The information matrix for $\boldsymbol{\psi}$, the parameters that appear only in the dynamic equation for a single time-varying parameter $\theta_{t|t-1}$, is given by $I_{\theta\theta}\mathbf{D}(\boldsymbol{\psi})$, as in Theorem 1 on p37. Lemma 10 extends Theorem 1 to include a second set of parameters, originally denoted $\boldsymbol{\theta}_2$ but now called $\boldsymbol{\xi}$, to avoid subscripts and because the result is different from what it was before. The score for $\boldsymbol{\xi}$ breaks down into two parts, the first is conditional on $\theta_{t|t-1}$ and the second is conditional on past observations, $Y_{t-1}$. The chain rule[1] gives

$$\frac{d\ln f(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\xi}} = \frac{\partial \ln f(y_t \mid \theta_{t|t-1};\boldsymbol{\xi})}{\partial\boldsymbol{\xi}} + \frac{\partial \ln f(y_t \mid Y_{t-1};\theta_{t|t-1})}{\partial\theta_{t|t-1}}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}}. \quad (1)$$

The first derivative of (1) is conditioned on $\theta_{t|t-1}$, as in the static model. Conditioning on $\theta_{t|t-1}$ automatically implies conditioning on $Y_{t-1}$ but the converse is not true because even with $Y_{t-1}$ fixed, $\theta_{t|t-1}$ may depend on $\boldsymbol{\xi}$ through $u_{t-1.}$ Hence the second term. (In the derivative of $\ln f(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})$ with respect to $\boldsymbol{\psi}$ only the second term appears.) The expectation of each part of (1) is zero.

*In what follows it is assumed that, for the static model, the score and its first derivative with respect to $\theta$ and $\xi$ have finite time invariant first and second moments that do not depend on $\theta$ and that the same is true for the cross-product of the score and these first derivatives. This condition is an extension of Condition 2 on p 35.*

[1] In our notation, the chain rule reads $\frac{d\,f(x,g(x))}{dx} = \frac{\partial f(x,g)}{\partial x} + \frac{\partial f(x,g)}{\partial g}\frac{dg(x)}{dx}$; i.e. the total derivative $d$ takes into account all dependencies, while the partial derivative $\partial$ treats all input parameters that are not differentiated as constants.

The information matrix, (2.55), for **Corollary 9** will be written

$$\mathbf{\Psi}\begin{pmatrix}\boldsymbol{\psi}\\ \boldsymbol{\xi}\end{pmatrix}=\begin{bmatrix}\mathbf{\Psi}_{\boldsymbol{\psi}\boldsymbol{\psi}} & \mathbf{\Psi}_{\boldsymbol{\psi}\boldsymbol{\xi}}\\ \mathbf{\Psi}_{\boldsymbol{\xi}\boldsymbol{\psi}} & \mathbf{\Psi}_{\boldsymbol{\xi}\boldsymbol{\xi}}\end{bmatrix}, \tag{2}$$

where

$$\mathbf{\Psi}_{\boldsymbol{\psi}\boldsymbol{\psi}}=E\left[\frac{d\ln f_t(y_t\mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\psi}}\frac{d\ln f_t(y_t\mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\psi}'}\right],$$

$$\begin{aligned}\mathbf{\Psi}_{\boldsymbol{\xi}\boldsymbol{\xi}} &= E\left[\frac{d\ln f_t(y_t\mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\xi}}\frac{d\ln f_t(y_t\mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\xi}'}\right]\\ &= \mathbf{I}_{\xi\xi}+I_{\theta\theta}E\left[\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\xi}}\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\xi}'}\right]\\ &\quad+\mathbf{I}_{\xi\theta}E\left[\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\xi}'}\right]+E\left[\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\xi}}\right]\mathbf{I}_{\theta\xi},\end{aligned} \tag{3}$$

where $\mathbf{I}_{\theta\xi}$ and $\mathbf{I}_{\xi\xi}$, like $I_{\theta\theta}$, are as in the static information matrix, and

$$\begin{aligned}\mathbf{\Psi}_{\boldsymbol{\xi}\boldsymbol{\psi}} &= E\left[\frac{d\ln f_t(y_t\mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\frac{d\ln f_t(y_t\mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{\partial\boldsymbol{\psi}'}\right]\\ &= I_{\theta\theta}E\left[\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\xi}}\frac{\partial\theta_{t\mid t-1}}{\partial\boldsymbol{\psi}'}\right]+\mathbf{I}_{\boldsymbol{\xi}\theta}E\left[\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\psi}'}\right]\end{aligned} \tag{4}$$

In (2.55) - which holds when $u_t$ does not depend on $\boldsymbol{\xi}-$ the entries are as in (2) but with $\mathbf{\Psi}_{\boldsymbol{\xi}\boldsymbol{\xi}}=\mathbf{I}_{\xi\xi}$ and $\mathbf{\Psi}_{\boldsymbol{\xi}\boldsymbol{\psi}}=\mathbf{I}_{\boldsymbol{\xi}\theta}E\left[d\theta_{t\mid t-1}/d\boldsymbol{\psi}'\right]$. Note that in the general case additional terms, involving $I_{\theta\theta}$, remain even when $\mathbf{I}_{\boldsymbol{\xi}\boldsymbol{\theta}}=\mathbf{0}$.

# 3 First-order model

In the first-order model $\mathbf{\Psi}_{\boldsymbol{\psi}\boldsymbol{\psi}}=I_{\theta\theta}\mathbf{D}(\boldsymbol{\psi})$ as in (2.56). The other terms require evaluation of the derivatives of $\theta_{t+1\mid t}$. If $x_t$ is defined as on p 35,

$$\begin{aligned}\frac{d\theta_{t+1\mid t}}{d\boldsymbol{\xi}} &= \phi\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\xi}}+\kappa\frac{\partial u_t}{\partial\boldsymbol{\xi}}+\kappa\frac{\partial u_t}{\partial\theta_{t\mid t-1}}\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\xi}}\\ &= x_t\frac{d\theta_{t\mid t-1}}{d\boldsymbol{\xi}}+\kappa\frac{\partial u_t}{\partial\boldsymbol{\xi}},\qquad t=1,...,T.\end{aligned} \tag{5}$$

The chain rule is used to take the total derivative of $u_t$; thus $\theta_{t|t-1}$ is fixed when $\partial u_t/\partial\boldsymbol{\xi}$ is evaluated. Taking expectations conditional on $Y_{t-1}$ gives

$$E_{t-1}\left(\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}}\right)=E_{t-1}\left(x_t\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}}+\kappa\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right)=a\frac{\partial\theta_{t|t-1}}{\partial\boldsymbol{\xi}}+\kappa E_{t-1}\left[\frac{\partial u_t(\theta_{t|t-1})}{\partial\boldsymbol{\xi}}\right].$$

We can take the unconditional expectation of $\partial u_t/\partial\boldsymbol{\xi}$ when, as assumed, it does not depend on $\theta_{t|t-1}$. Thus

$$E\left(\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}}\right)=\frac{\kappa}{1-a}E\left[\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right]=\frac{-\kappa}{1-a}\mathbf{I}_{\boldsymbol{\xi}\theta}. \tag{6}$$

(When $u_t$ is $k$ times the score, as on p32, rather than the score, the last term has to be multiplied by $k$.) Furthermore

$$\begin{aligned}E_{t-1}\left(\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}}\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}'}\right) &= E_{t-1}\left(x_t^2\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'}\right)+\kappa^2E_{t-1}\left(\frac{\partial u_t}{\partial\boldsymbol{\xi}}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right)\\ &\quad+\kappa E_{t-1}\left(x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'}\right)+\kappa E_{t-1}\left(x_t\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right)\\ &= b\left(\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'}\right)+\kappa^2E_{t-1}\frac{\partial u_t}{\partial\boldsymbol{\xi}}\frac{\partial u_t}{\partial\boldsymbol{\xi}'} \qquad (7)\\ &\quad+\kappa\mathbf{c}_\xi\frac{\partial\theta_{t|t-1}}{\partial\boldsymbol{\xi}'}+\kappa\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}}\mathbf{c}_\xi',\end{aligned}$$

where

$$\mathbf{c}_\xi=E_{t-1}\left(x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right)=E_{t-1}\left(\phi\frac{\partial u_t}{\partial\boldsymbol{\xi}}+\kappa\frac{\partial u_t}{\partial\theta_{t|t-1}}\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right)=-\phi\mathbf{I}_{\boldsymbol{\xi}\theta}+\kappa E_{t-1}\left(\frac{\partial u_t}{\partial\theta}\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right)$$

The conditional expectations can be replaced by the corresponding unconditional expectations so taking unconditional expectations in (7) gives

$$\begin{aligned}E\left(\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}}\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}'}\right) &= \frac{1}{1-b}\left[\kappa^2E\left(\frac{\partial u_t}{\partial\boldsymbol{\xi}}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right)+\kappa\mathbf{c}_\xi E\left(\frac{\partial\theta_{t|t-1}}{\partial\boldsymbol{\xi}'}\right)+\kappa E\left(\frac{\partial\theta_{t|t-1}}{\partial\boldsymbol{\xi}}\right)\mathbf{c}_\xi'\right]\\ &= \frac{\kappa^2}{1-b}\left[E\left(\frac{\partial u_t}{\partial\boldsymbol{\xi}}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right)-\frac{1}{1-a}\left(\mathbf{c}_\xi\mathbf{I}_{\boldsymbol{\xi}\theta}+\mathbf{I}_{\theta\xi}\mathbf{c}_\xi'\right)\right]. \qquad (8)\end{aligned}$$

Therefore

$$
\begin{aligned}
&E\left[\frac{d\ln f_t(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\xi}}\frac{d\ln f_t(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\xi}'}\right] \\
=\ & \mathbf{I}_{\xi\xi} + I_{\theta\theta}\frac{\kappa^2}{1-b}\left[E\left(\frac{\partial u_t}{\partial\boldsymbol{\xi}}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right)+\frac{1}{1-a}\left(2\phi\mathbf{I}_{\xi\theta}\mathbf{I}_{\theta\xi}-\kappa\mathbf{I}_{\xi\theta}E\frac{\partial u_t}{\partial\theta}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}-\kappa\mathbf{I}_{\xi\theta}\left[E\frac{\partial u_t}{\partial\theta}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right]'\right)\right] \\
&-\frac{2\kappa}{1-a}\mathbf{I}_{\xi\theta}\mathbf{I}_{\theta\xi} \qquad (9)
\end{aligned}
$$

The individual terms for $\kappa$, $\phi$ and $\omega$ in $\boldsymbol{\Psi}_{\boldsymbol{\xi\psi}}$ are shown in Appendix A to be

$$
\boldsymbol{\Psi}_{\boldsymbol{\xi\psi}} = \frac{I_{\theta\theta}}{1-b}\left[\mathbf{g} \quad \frac{\kappa}{1-a\phi}\mathbf{g} \quad \frac{1-\phi}{1-a}\mathbf{g}_\omega\right]+\left[\mathbf{0} \quad 0 \quad \frac{1-\phi}{1-a}\mathbf{I}_{\xi\theta}\right],
$$

with

$$
\mathbf{g} = \kappa E\left(u_t\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right)-\frac{\kappa c}{1-a}\mathbf{I}_{\boldsymbol{\xi\theta}} \quad \text{and} \quad \mathbf{g}_\omega = \kappa\left[\kappa E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right)-(1+\phi)\mathbf{I}_{\boldsymbol{\xi\theta}}\right]
$$

Thus evaluation of $\boldsymbol{\Psi}_{\boldsymbol{\xi\xi}}$ and $\boldsymbol{\Psi}_{\boldsymbol{\xi\psi}}$ requires that we determine

$$
E\left(\frac{\partial u_t}{\partial\boldsymbol{\xi}}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right), \quad E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right) \text{ and } E\left(u_t\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right),
$$

all of which are independent of $\theta$ by assumption.

When $u_t$ does not depend on $\boldsymbol{\xi}$, the information matrix in (2) has $\boldsymbol{\Psi}_{\boldsymbol{\xi\xi}} = \mathbf{I}_{\xi\xi}$ and

$$
\boldsymbol{\Psi}_{\boldsymbol{\xi\psi}} = \left[\mathbf{0} \quad 0 \quad \frac{1-\phi}{1-a}\mathbf{I}_{\xi\theta}\right], \qquad (10)
$$

which is as in (2.56).

# 4 Beta-t-EGARCH

In Beta-t-EGARCH $\theta_{t|t-1}$ becomes $\lambda_{t|t-1}$. The matrix in **Proposition 20** on p 116 needs to be modified by adding a term to $I_{\nu\nu} = h(\nu)/2$ so that

$$
\boldsymbol{\Psi}_{\nu\nu} = \frac{1}{2}h(\nu)+I_{\lambda\lambda}\frac{\kappa^2}{1-b}\left[E\left[\left(\frac{\partial u_t}{\partial\nu}\right)^2\right]-\frac{2\phi I_{\lambda\nu}^2}{1-a}-\frac{2\kappa I_{\lambda\nu}}{1-a}E\left(\frac{\partial u_t}{\partial\lambda}\frac{\partial u_t}{\partial\nu}\right)\right]-\frac{2\kappa}{1-a}I_{\lambda\nu}^2,
$$

where

$$I_{\lambda\nu} = \frac{-2}{(\nu+3)(\nu+1)} \qquad I_{\lambda\lambda} = \frac{2\nu}{\nu+3}$$

$$E\left[\left(\frac{\partial u_t}{\partial \nu}\right)^2\right] = Eb_t^4 - 2Eb_t^3(1-b_t)/\nu + Eb_t^2(1-b_t)^2/\nu^2$$

and

$$E\left(\frac{\partial u_t}{\partial \lambda}\frac{\partial u_t}{\partial \nu}\right) = -2(\nu+1)Eb_t^3(1-b_t) + 2(\nu+1)Eb_t^2(1-b_t)^2/\nu$$

The expectations of the terms involving the beta variables, $b_t$, depend only on $\nu$ and can be evaluated using (2.5) on p 23. The elements in the last column (and bottom row) of the information matrix, that is the elements of $\boldsymbol{\Psi}_{\lambda\nu}$, are

$$I_{\lambda\lambda}\frac{g}{1-b}, \qquad I_{\lambda\lambda}\frac{\kappa}{1-b}\frac{g}{1-a\phi}$$

and

$$I_{\lambda\nu}\frac{1-\phi}{1-a} + I_{\lambda\lambda}\frac{\kappa}{1-b}\frac{1-\phi}{1-a}\left[\kappa E\left(\frac{\partial u_t}{\partial \lambda}\frac{\partial u_t}{\partial \nu}\right) - (1+\phi)I_{\lambda\nu}\right],$$

where

$$g = \kappa E\left(u_t\frac{\partial u_t}{\partial \nu}\right) - \frac{\kappa c I_{\nu\lambda}}{1-a}$$

with

$$E\left(u_t\frac{\partial u_t}{\partial \nu}\right) = (\nu+1)b_t^3 - b_t^2 - (\nu+1)b_t^2(1-b_t)/\nu + b_t(1-b_t)/\nu.$$

Table 4.1 on p 116 needs to be amended. However, because the additional terms depend on $\kappa$, which is typically rather small, the differences are small. The original table also had $I_{\lambda\nu}$ set to the correct expression divided by minus 2. This correction was made when the approximate ASEs shown in brackets in the table below were calculated. As can be seen the effect of the omitted terms on the ASEs of $\kappa$ and $\phi$ is negligible. However, the exact ASE($\omega$) is very close to the RMSE. On the other hand ASE($\nu$) is smaller than the RMSE but this underestimation is not unusual with shape parameters and it also happens with location in the next section, where the exact results are much closer to the RMSEs. The extra terms in the information for $\nu$ must be positive and this can be expected to translate into a smaller ASE. Calculations for the other combinations of parameters show a similar pattern.

| Parameter | | ML estimates for T=10,000 | | | |
|---|---|---|---|---|---|
| $\phi$ | $\kappa$ | RMSE($\phi$) | ASE($\phi$) | RMSE($\kappa$) | ASE($\kappa$) |
| 0.95 | 0.10 | 0.0053 | 0.0051 | 0.0048 | 0.0050 |
| | | | (0.0053) | | (0.0048) |
| 0.99 | 0.05 | 0.0018 | 0.0018 | 0.0030 | 0.0031 |
| | | | (0.0018) | | (0.0031) |

| Parameter | | ML estimates for T=10,000 | | | |
|---|---|---|---|---|---|
| $\phi$ | $\kappa$ | RMSE($\omega$) | ASE($\omega$) | RMSE($\nu$) | ASE($\nu$) |
| 0.95 | 0.10 | 0.032 | 0.032 | 0.325 | 0.280 |
| | | | (0.043) | | (0.345) |
| 0.99 | 0.05 | 0.065 | 0.066 | 0.343 | 0.332 |
| | | | (0.091) | | (0.345) |

**Corollary 21** is affected in that the limiting distribution of $\widetilde{\nu}$ is not exact.

**Proposition 27** on p137-8 needs to be modified as follows. The matrix $\mathbf{I}(\boldsymbol{\psi},\nu)$ is as in the amended Proposition 20 and the term $\zeta(\nu+1)/(\nu+3)$ is multiplied by

$$1+\frac{\kappa^2}{1-b}8(\nu+1)E[b_t(1-b_t)^3]=1+\frac{\kappa^2}{1-b}\frac{8(\nu+4)(\nu+2)\nu}{(\nu+7)(\nu+5)(\nu+3)}$$

The block diagonality remains.

**Proof** There is some simplification because $I_{\lambda\mu}=0$ and

$$\frac{\partial u_t}{\partial\mu}=-2\varepsilon_t e^{-2\lambda}(1-b_t)^2(\nu+1)\nu^{-1}$$

is an odd function. Because

$$\left(\frac{\partial u_t}{\partial\mu}\right)^2=4(\nu+1)^2e^{-2\lambda}\nu^{-1}b_t(1-b_t)^3$$

$$I_{\mu\mu}+I_{\lambda\lambda}\frac{\kappa^2}{1-b}E\left[\left(\frac{\partial u_t}{\partial\mu}\right)^2\right]=\frac{\nu+1}{\nu+3}\zeta+\frac{\kappa^2}{1-b}\frac{8(\nu+1)^2\zeta}{(\nu+3)}E[b_t(1-b_t)^3]$$

The inflation in the information quantity coming from $\lambda_{t|t-1}$ is typically rather small, for example with $\kappa=0.05$, $\nu=6$, it is 4.03%. Since the information matrix is still block diagonal, this translates into a slightly smaller asymptotic variance for the ML estimator of $\boldsymbol{\beta}$.

Proposition 32 and 34 in Chapter 5 need to be modified in a similar way to Proposition 20.

**Remark 1** *In classic EGARCH $u_t$ is replaced by $|\varepsilon_t|$ and this does not depend on $\nu$. Thus $\mathbf{\Psi}_{\nu\nu} = h(\nu)/2$ and $\mathbf{\Psi}_{\boldsymbol{\nu\psi}} = [\mathbf{0} \quad 0 \quad I_{\lambda\nu}(1-\phi)/1-a]$.*

# 5 Dynamic Location

**Proposition 9** is modified by setting the elements in the information matrix for $\lambda$ and $\nu$ to

$$I_{\lambda\lambda} + I_{\mu\mu}\frac{\kappa^2}{1-b}E\left[\left(\frac{\partial u_t}{\partial \lambda}\right)^2\right] = \frac{2\nu}{\nu+3} + \frac{\nu+1}{\nu+3}\frac{\kappa^2}{1-b}4\nu E b_t^3(1-b_t)$$

$$I_{\lambda\nu} + I_{\mu\mu}\frac{\kappa^2}{1-b}E\left[\left(\frac{\partial u_t}{\partial \lambda}\right)\left(\frac{\partial u_t}{\partial \nu}\right)\right] = \frac{-2}{(\nu+1)(\nu+3)} + \frac{\nu+1}{\nu+3}\frac{\kappa^2}{1-b}2Eb_t^3(1-b_t)$$

and

$$I_{\nu\nu} + I_{\mu\mu}\frac{\kappa^2}{1-b}E\left[\left(\frac{\partial u_t}{\partial \nu}\right)^2\right] = \frac{h(\nu)}{2} + \frac{\nu+1}{\nu+3}\frac{\kappa^2}{1-b}\frac{1}{\nu}Eb_t^3(1-b_t)$$

As regards the last two columns, the entries are

$$\begin{bmatrix} \frac{\nu+1}{\nu+3}\frac{\kappa}{1-b}\nu 2Eb_t^2(1-b_t) & \frac{\nu+1}{\nu+3}\frac{\kappa}{1-b}Eb_t^2(1-b_t) \\ \frac{\nu+1}{\nu+3}\frac{\kappa}{1-b}\frac{\kappa}{1-a\phi}\nu 2Eb_t^2(1-b_t) & \frac{\nu+1}{\nu+3}\frac{\kappa}{1-b}\frac{\kappa}{1-a\phi}Eb_t^2(1-b_t) \\ 0 & 0 \end{bmatrix}$$

$$= \frac{\nu+1}{\nu+3}\frac{\kappa}{1-b}Eb_t^2(1-b_t)\begin{bmatrix} 2\nu & 1 \\ \frac{2\kappa\nu}{1-a\phi} & \frac{\kappa}{1-a\phi} \\ 0 & 0 \end{bmatrix}$$

The last two rows of the last two columns could be written as

$$\begin{bmatrix} I_{\lambda\lambda} & I_{\lambda\nu} \\ I_{\lambda\nu} & I_{\nu\nu} \end{bmatrix} + \frac{\nu+1}{\nu+3}\frac{\kappa^2}{1-b}Eb_t^3(1-b_t)\begin{bmatrix} 4\nu & 2 \\ 2 & 1/\nu \end{bmatrix}$$

**Proof** We have

$$\frac{\partial u_t}{\partial \nu} = (y-\mu)b_t(1-b_t)/\nu$$

so

$$E\left(\frac{\partial u_t}{\partial \nu}\right)^2 = e^{2\lambda}Eb_t^3(1-b_t)/\nu \quad \text{and} \quad E\left(u_t\frac{\partial u_t}{\partial \lambda}\right) = e^{2\lambda}E(1-b_t)b_t^2$$

Also

$$\frac{\partial u_t}{\partial \lambda} = 2(y-\mu)b_t(1-b_t) = 2\nu\frac{\partial u_t}{\partial \nu}$$

Functions of $\exp(2\lambda)$ cancel.

Note that the ML estimator of $\omega$ is distributed independently of the other parameters in large samples so Corollary 12 remains valid.

The Table shows some of the results from Harvey and Luati (2014) together with the correct ASEs. On the whole the new ASEs are closer to the RMSEs

| Parameter | | | ML estimates for $T = 1000$ | | | | |
|---|---|---|---|---|---|---|---|
| $\phi$ | $\kappa$ | | $\phi$ | $\kappa$ | $\lambda$ | $\omega$ | $\nu$ |
| 0.8 | 1.0 | RMSE | 0.025 | 0.067 | 0.031 | 0.144 | 0.920 |
| | | ASE jasa | 0.025 | 0.045 | 0.038 | 0.147 | 1.092 |
| | | ASE new | 0.022 | 0.058 | 0.032 | 0.147 | 0.855 |
| 0.95 | 1.3 | RMSE | 0.011 | 0.071 | 0.029 | 0.445 | 0.740 |
| | | ASE jasa | 0.010 | 0.043 | 0.038 | 0.596 | 1.092 |
| | | ASE new | 0.009 | 0.069 | 0.028 | 0.596 | 0.672 |

# A Derivation of the first-order information matrix

The terms involving $\boldsymbol{\psi}$ and $\boldsymbol{\xi}$. For $\kappa$

$$\begin{aligned} E_{t-1}\left[\frac{d\theta_{t+1|t}}{d\kappa}\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}'}\right] &= E_{t-1}\left[x_t^2\frac{d\theta_{t|t-1}}{d\kappa}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'}\right] + \kappa E_{t-1}\left[u_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right] \\ &\quad + E_{t-1}\left[x_t u_t\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'}\right] + \kappa E_{t-1}\left[\frac{d\theta_{t|t-1}}{d\kappa}x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right] \\ &= b\frac{d\theta_{t|t-1}}{d\kappa}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} + \kappa E_{t-1}\left[u_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right] \\ &\quad + E_{t-1}\left[x_t u_t\right]\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} + \kappa E_{t-1}\left[x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right]\frac{d\theta_{t|t-1}}{d\kappa} \end{aligned}$$

Taking unconditional expectations and noting that $E(d\theta_{t+1|t}/d\kappa) = 0$,

$$E\left[\frac{d\theta_{t+1|t}}{d\kappa}\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}'}\right] = \frac{\kappa}{1-b}\left[E\left(u_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right) - \frac{c}{1-a}\mathbf{I}_{\theta\xi}\right]$$

Hence

$$E\left[\frac{d\ln f_t(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\kappa}\frac{d\ln f_t(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\xi}'}\right] = I_{\theta\theta}\frac{\kappa}{1-b}\left[E\left(u_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right) - \frac{c}{1-a}\mathbf{I}_{\theta\boldsymbol{\xi}}\right] \tag{11}$$

For $\phi$

$$\begin{aligned}
E_{t-1}\left[\frac{d\theta_{t+1|t}}{d\phi}\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}'}\right] &= E_{t-1}\left[x_t^2\frac{d\theta_{t|t-1}}{d\phi}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'}\right] + \kappa E_{t-1}\left[(\theta_{t|t-1}-\omega)\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right] \\
&\quad + E_{t-1}\left[x_t(\theta_{t|t-1}-\omega)\frac{\partial\theta_{t|t-1}}{\partial\boldsymbol{\xi}'}\right] + \kappa E_{t-1}\left[\frac{d\theta_{t|t-1}}{d\phi}x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right] \\
&= b\frac{d\theta_{t|t-1}}{d\phi}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} + \kappa E\left[\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right](\theta_{t|t-1}-\omega) \\
&\quad + a(\theta_{t|t-1}-\omega)\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} + \kappa E_{t-1}\left[x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right]\frac{d\theta_{t|t-1}}{d\phi}
\end{aligned}$$

Now $Ed\theta_{t+1|t}/d\phi = 0$ and $E(\theta_{t|t-1} - \omega) = 0$. The third term is found as follows:

$$\begin{aligned}
E_{t-2}(\theta_{t|t-1}-\omega)\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} &= E_{t-2}\left[(x_{t-1}\frac{d\theta_{t-1|t-2}}{d\boldsymbol{\xi}} + \kappa\frac{\partial u_{t-1}}{\partial\boldsymbol{\xi}'})(\phi(\theta_{t-1|t-2}-\omega)+\kappa u_{t-1})\right] \\
&= a\phi\frac{d\theta_{t-1|t-2}}{d\boldsymbol{\xi}}(\theta_{t-1|t-2}-\omega) + \kappa c\frac{d\theta_{t-1|t-2}}{d\boldsymbol{\xi}} + \kappa^2 E_{t-2}u_{t-1}\frac{\partial u_{t-1}}{\partial\boldsymbol{\xi}'} \\
&\quad + \kappa\phi E_{t-2}(\frac{\partial u_{t-1}}{\partial\boldsymbol{\xi}'})(\theta_{t-1|t-2}-\omega)
\end{aligned}$$

On taking unconditional expectations, the last term disappears. As regards the first term

$$E(\theta_{t|t-1}-\omega)\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} = E_{t-2}\left[(x_{t-1}\frac{d\theta_{t-1|t-2}}{d\boldsymbol{\xi}} + \kappa\frac{\partial u_{t-1}}{\partial\boldsymbol{\xi}'})(\phi(\theta_{t-1|t-2}-\omega)+\kappa u_{t-1})\right]$$

Taking unconditional expectations

$$\begin{aligned}
E(\theta_{t|t-1}-\omega)\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} &= \frac{1}{1-a\phi}\left(\kappa cE\frac{d\theta_{t-1|t-2}}{d\boldsymbol{\xi}} + \kappa^2 Eu_{t-1}\frac{\partial u_{t-1}}{\partial\boldsymbol{\xi}'}\right) \\
&= \frac{1}{1-a\phi}\left(-\kappa c\frac{\kappa}{1-a}\mathbf{I}_{\theta\xi} + \kappa^2 Eu_{t-1}\frac{\partial u_{t-1}}{\partial\boldsymbol{\xi}'}\right)
\end{aligned}$$

Hence

$$E\left[\frac{d\ln f_t(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\phi}\frac{d\ln f_t(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\xi}'}\right] = I_{\theta\theta}\frac{1}{1-b}\frac{\kappa^2}{1-a\phi}\left(Eu_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'} - \frac{c}{1-a}\mathbf{I}_{\theta\xi}\right)$$

For $\omega$

$$\begin{aligned}
E_{t-1}\left[\frac{d\theta_{t+1|t}}{d\omega}\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}}\right] &= E_{t-1}\left[x_t^2\frac{d\theta_{t|t-1}}{d\omega}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'}\right] + \kappa E_{t-1}\left[(1-\phi)\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right] \\
&\quad + E_{t-1}\left[x_t(1-\phi)\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'}\right] + \kappa E_{t-1}\left[\frac{d\theta_{t|t-1}}{d\omega}x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right] \\
&= b\frac{d\theta_{t|t-1}}{d\kappa}\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} + \kappa(1-\phi)E_{t-1}\left[\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right] \\
&\quad + a(1-\phi)\frac{d\theta_{t|t-1}}{d\boldsymbol{\xi}'} + \kappa E_{t-1}\left[x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}}\right]\frac{d\theta_{t|t-1}}{d\omega}
\end{aligned}$$

so

$$E\left[\frac{d\theta_{t+1|t}}{d\omega}\frac{d\theta_{t+1|t}}{d\boldsymbol{\xi}'}\right] = \frac{\kappa}{1-b}\left[(1-\phi)E\left(\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right) + \frac{(1-\phi)a}{1-a}E\left(\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right) + E\left[x_t\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right]\frac{1-\phi}{1-a}\right]$$

Thus when $u_t$ is the score

$$\begin{aligned}
& E\left[\frac{d\ln f_t(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\omega}\frac{d\ln f_t(y_t \mid Y_{t-1};\boldsymbol{\psi},\boldsymbol{\xi})}{d\boldsymbol{\xi}'}\right] \\
= & \mathbf{I}_{\theta\xi}\frac{1-\phi}{1-a} + I_{\theta\theta}\frac{\kappa}{1-b}\frac{1-\phi}{1-a}\left[-(1+\phi)\mathbf{I}_{\theta\boldsymbol{\xi}} + \kappa E\left(\frac{\partial u_t}{\partial\theta}\frac{\partial u_t}{\partial\boldsymbol{\xi}'}\right)\right]
\end{aligned}$$