DYNAMIC OPTIMISATION EXERCISES

Exercise 1

An economy uses capital and labour to produce output which can be consumed or saved. Savings are equal to investment which enhances the capital stock. The labour force grows exponentially at a constant rate n. The social planner has to decide how much to save, or equivalently how much capital to acquire. Output is produced via a neo-classical production function

$$Y(t) = F\left(K(t), L(t)\right)$$

where $K(t)$ and $L(t)$ are stocks of capital and labour at time t. The production function is assumed twice differentiable and exhibits constant returns to scale. We denote $C(t)$ as consumption at time t and $I(t)$ investment. Let $s(t)$ denote the fraction of output at time t which is saved (and invested). So we have

$$\begin{aligned} Y &= C + I = (1-s)Y + sY \\ \dot{K} &= sF(K,L) - \mu K \\ \dot{L} &= nL \end{aligned}$$

where μ is the depreciation rate for capital and n the growth rate of the labour force. n and μ are assumed fixed. The planners objective is to maximise the discounted flow of utility from consumption per worker subject to the production function and the equations of motion for capital and labour. Letting lower case letters denote per-capita levels ($y = Y/L$, $k = K/L, c = C/L$) the problem is then to maximise

$$\int_0^\infty u(c)e^{-\delta t}dt$$

where the instantaneous utility function is $u(c)$ and δ is the discount rate the planner applies to future consumption.

Write down the Hamiltonian for this problem and show that the optimal paths satisfy

$$\frac{u''(c)c}{u'(c)}\frac{\dot{c}}{c} = \delta + \lambda - f'(k)$$

Draw the phase diagram for this problem.

Exercise 2

Consider an economy where the population is constant and the representative individual maximises

$$\max \int_0^\infty u(c_t)e^{-\delta t}dt$$

k is the capital stock of the economy and can be used for consumption or investment. Let x, $0 \leq x \leq 1$ be the proportion of capital used in the production of consumption goods. The two production functions for consumption and investment are given by

$$\begin{aligned} c &= f(xk) \\ f(0) &= 0 \quad f'(.) > 0 \quad f''(.) < 0 \\ \frac{dk}{dt} &= i = b(1-x)k \end{aligned}$$

Derive the first order conditions associated with the maximisation problem.

Assume that $f(xk) = A(xk)^\alpha$ where $0 < \alpha < 1$ and that $u(c) = \ln(c)$. Show that on the optimal path the growth of consumption is given by $\alpha(b-\delta)$. Notice that this economy can grow for ever if $b > \delta$