Supplementary Appendices to: Exponent of Cross-sectional Dependence: Estimation and Inference

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Supplementary Appendix I: Statement of technical Lemmas

Lemma 1 *Under Assumptions 2 and 3,* $\boldsymbol{f}_t, \{u_{it}\}_{t=1}^{\infty}$*, and* $\{u_{it}\}_{i=1}^{\infty}$ *are* L_r*-bounded,* L_2*-NED processes of size* $-\zeta$*, for some* $r > 2$*. This result holds uniformly over* i*, in the case of* $\{u_{it}\}_{t=1}^{\infty}$*, and over* t*, in the case of* $\{u_{it}\}_{i=1}^{\infty}$*.*

Lemma 2 *Under Assumptions 2 and 3,*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\boldsymbol{f}_t-\bar{\boldsymbol{f}}\right)\left(\sqrt{N}\bar{u}_t\right)\to_p N(\mathbf{0},\bar{\sigma}_N^2 I_m),$$

where

$$\bar{\sigma}_N^2=\lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N}\left(\sigma_i^2+\sum_{j=1,j\neq i}^{\infty}\sigma_{ij}\right)}{N}\right), \tag{B1}$$

$\sigma_i^2 = E(u_{it}^2)$*, and* $\sigma_{ij} = E(u_{it}u_{it-j})$*.*

Lemma 3 *Under Assumption 3,*

$$\frac{1}{\sqrt{2T}}\sum_{t=1}^{T}\left[\left(\sqrt{N}\bar{u}_t\right)^2-E\left(\sqrt{N}\bar{u}_t\right)^2\right]\to_d N(0,V),$$

where

$$V=\lim_{N\to\infty}\left(Var\left(\left(\sqrt{N}\bar{u}_t\right)^2\right)+\sum_{j=1}^{\infty}Cov\left(\left(\sqrt{N}\bar{u}_t\right)^2,\left(\sqrt{N}\bar{u}_{t-j}\right)^2\right)\right).$$

Lemma 4 *Under Assumptions 1-3, if* $m = 1$ *then* $\widehat{\bar{\sigma}_N^2}-\bar{\sigma}_N^2 = O_p\left(T^{-1}\right)+O_p\left((NT)^{-1/2}\right)$*. If* $m > 1$ *then* $\widehat{\bar{\sigma}_N^2}-\bar{\sigma}_N^2 = O_p\left(N^{\alpha-1}T^{-1/2}\right)$*.*

Lemma 5 *Under Assumptions 1-2 and* $m = 1$*,* $\sqrt{\min(N^{\alpha},T)}\left(\ln(s_{f_1}^2\bar{v}_{1N}^2)-\ln(\sigma_{f_1}^2\mu_{v_1}^2)\right)\to_d N\left(0,\omega\right).$

Lemma 6 *Under Assumption 5 and Assumptions 2-3 and* $m = 1$*,* $\sqrt{\min(N,T)}\left(\ln(s_{f_1}^2\bar{v}_{1N}^2)-\ln(\sigma_{f_1}^2\mu_{v_1}^2)\right)\to_d N\left(0,\omega\right).$

Lemma 7 *Under Assumptions 1-3, and as long as* $\alpha > 1/2$*,*

$$\sqrt{\min(N^{\alpha},T)}\left(\frac{\bar{\sigma}_N^2}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}-\frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}\right)=O_p\left(\sqrt{\min(N^{\alpha},T)}N^{2-4\alpha}\right).$$

Lemma 8 *Under Assumption 5 and Assumptions 2-3,*

$$\sqrt{\min(N,T)}\left(\frac{\bar{\sigma}_N^2}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}-\frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}\right)=o_p(1).$$

Lemma 9 *Under Assumptions 1-3, and as long as* $\alpha > 1/2$*,*

$$\sqrt{\min(N^{\alpha},T)}\ln(N)\left(\frac{\bar{\sigma}_N^2}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}-\frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}\left(1+\frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}\right)\right)=o_p\left(1\right).$$

Lemma 10 *Under Assumptions 1-3 and* $\boldsymbol{\Sigma}_{ff} = I$*, if* $\alpha = \alpha_2 = ... = \alpha_{q-1} > \alpha_q \geq ... \geq \alpha_m$*,*

$$\hat{\sigma}_{\bar{x}}^2-N^{2\alpha-2}\sum_{j=1}^{q}\mu_{v_j}^2\to_p 0.$$

In particular, if $\alpha > \alpha_2 \geq ... \geq \alpha_m$*,*

$$\hat{\sigma}_{\bar{x}}^2-N^{2\alpha-2}\mu_{v_1}^2\to_p 0.$$

Lemma 11 *Under Assumptions 1-2, and assuming $\alpha_j = \alpha$, for all $j = 1, ..., m$,*

$$\sqrt{\min(N^{\alpha}, T)} \left(\ln(\bar{\boldsymbol{v}}_N' \boldsymbol{S}_{ff} \bar{\boldsymbol{v}}_N) - \ln(\boldsymbol{\mu}_v' \boldsymbol{\Sigma}_{ff} \boldsymbol{\mu}_v) \right) \to_d N(0, \omega_m),$$

where $\boldsymbol{\mu}_v = E(\boldsymbol{v}_j)$, $\boldsymbol{\Sigma}_{ff} = E\left((\boldsymbol{f}_t - \boldsymbol{\mu}_f)' (\boldsymbol{f}_t - \boldsymbol{\mu}_f) \right)$,

$$\omega_m = \lim_{N,T\to\infty} \min(N^{\alpha}, T) E\left(\left\{ \left(\bar{\boldsymbol{v}}_N' \bar{\boldsymbol{f}}_T - \boldsymbol{\mu}_v' \boldsymbol{\mu}_f \right)^2 - E\left[\left(\bar{\boldsymbol{v}}_N' \bar{\boldsymbol{f}}_T - \boldsymbol{\mu}_v' \boldsymbol{\mu}_f \right)^2 \right] \right\}^2 \right),$$

$\boldsymbol{\mu}_f = E(\boldsymbol{f}_t)$ and $\bar{\boldsymbol{f}}_T = \frac{1}{T} \sum_{t=1}^T \boldsymbol{f}_t$.

Lemma 12 *Under Assumptions 1-2, and assuming $\alpha > \alpha_2 > ... > \alpha_m$,*

$$\sqrt{\min(N^{\alpha}, T)} \left(\ln(\bar{\boldsymbol{v}}_N' \boldsymbol{D}_N \boldsymbol{S}_{ff} \boldsymbol{D}_N \bar{\boldsymbol{v}}_N) - \ln(\boldsymbol{\mu}_v' \boldsymbol{D}_N \boldsymbol{\Sigma}_{ff} \boldsymbol{D}_N \boldsymbol{\mu}_v) \right) \to_d N(0, \omega).$$

Lemma 13 *Under Assumptions 1-3, and $\alpha > \alpha_2 \geq \alpha_3 \geq ... \geq \alpha_m$,*

$$\sqrt{\min(N^{\alpha}, T)} \ln(N) \ln(\boldsymbol{\mu}_v' \boldsymbol{D}_N \boldsymbol{\Sigma}_{ff} \boldsymbol{D}_N \boldsymbol{\mu}_v) - \ln\left(\mu_{v_1}^2 \sigma_{f_1}^2 \right) = o(1),$$

if either $\alpha_2 - \alpha < -0.25$ or, if $T^b = N$, $\alpha_2 < 3\alpha/4$ and

$$b > \frac{1}{4(\alpha - \alpha_2)}. \tag{B2}$$

Lemma 14 *Let $\beta_{i1} = N^{\alpha-1} v_{i1}$, $1/2 < \alpha \leq 1$, where $v_{i1} = v_{Ni} = \breve{v}_i + c_{Ni}$ and $\{\breve{v}_i\}_{i=1}^N$ is an i.i.d. sequence of random variables with mean $\mu_{\breve{v}} \neq 0$, and variance $\sigma_{\breve{v}}^2 < \infty$. Let $\bar{c}_N = \frac{1}{N} \sum_{i=1}^N c_{Ni}$. Under Assumptions 2-3, (a) $\hat{\alpha}$, $\tilde{\alpha}$ and $\breve{\alpha}$ are consistent estimators of α, if $\bar{c}_N = o_p(N^c)$ for all $c > 0$, (b) Corollary 1 holds, if $\sqrt{N} \bar{c}_N = o_p(1)$.*

Lemma 15 *Let $\hat{a}$ denote a generic estimator of α such that $\hat{a} - \alpha = O_p(h_N)$ where $h_N \to 0$. Then,*

$$N^{\hat{a}} - N^{\alpha} = O_p(N^{\alpha} h_N \ln N).$$

Lemma 16 *Denote the OLS estimator of the regression coefficient of x_{it} on $\tilde{x}_t = \bar{x}_t / \hat{\sigma}_{\bar{x}}$, by $\hat{v}_{i1}$, and let $\left\{ \hat{v}_{i1}^{(s)} \right\}_{i=1}^N$ be the reordering of $\{\hat{v}_{i1}\}_{i=1}^N$ where $\left| \hat{v}_{i1}^{(s)} \right| \geq \left| \hat{v}_{i1+1}^{(s)} \right|$, $\forall i$. Under Assumptions 1-3, $m = 1$ and assuming that $\lim_{T,N\to\infty} T^{-1} N^{\alpha} < \infty$ and that $\mu_{v_1} \neq 0$ is known, we have*

$$\frac{\sum_{i=1}^{N^{\hat{\alpha}}} \left(\hat{v}_{i1}^{(s)} - \frac{1}{N^{\hat{\alpha}}} \sum_{j=1}^{N^{\hat{\alpha}}} \hat{v}_{j1}^{(s)} \right)^2}{N^{\hat{\alpha}} - 1} - \frac{\sigma_{v_1}^2}{\mu_{v_1}^2} = o_p(1), \tag{B3}$$

if, further, $\hat{\alpha} - \alpha = o_p\left((\ln N)^{-1} \right)$.

Lemma 17 *Under Assumptions 1-3 and $m = 1$,*

$$\hat{\beta}_{i1} - \frac{\beta_{i1}}{\mu_{v_1}} = O_p\left(\frac{1}{N^{\alpha/2}} \right) + O_p\left(\frac{1}{T^{1/2}} \right).$$

Lemma 18 *Under Assumptions 1-2, we have $\hat{V}_{\overline{f_1^2}} - V_{\overline{f_1^2}} = o_p(1)$, as long as $l \to \infty$, $l = o(T)$ and $l = o\left(N^{\alpha-1/2} T^{1/2} \right)$.*

Supplementary Appendix II: Proof of Corollary 1

We reconsider (41) and $m = 1$. Under Assumption 5, $\bar{\beta}_{1N} = N^{(\alpha-1)/2} \bar{v}_{1N}$, where $\bar{v}_{1N} = N^{-1} \sum_{i=1}^N v_{1i}$, we have

$$\ln(\bar{\beta}_{1N}^2 s_{f_1}^2) = \ln\left(N^{(\alpha-1)/2} \bar{v}_{1N} s_{f_1} \right)^2 = (\alpha - 1) \ln(N) + \ln(s_{f_1}^2 \bar{v}_{1N}^2),$$

where $s_{f_i}^2 = T^{-1}\sum_{t=1}^T \left(f_{it} - T^{-1}\sum_{t=1}^T f_{it}\right)^2$ for $i = 1, ..., N$ and here $i = 1$. Hence, recalling from (11) that $\hat{\alpha} = 1 + \ln(\hat{\sigma}_{\bar{x}}^2)/2\ln(N)$,we have

$$\ln(N)\left(\hat{\alpha} - \alpha\right) - \ln(s_{f_1}^2 \bar{v}_{1N}^2) = \ln\left(1 + \frac{2\bar{\beta}_{1N}\left[\frac{1}{T}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\bar{u}_t\right] + \left[\frac{1}{T}\sum_{t=1}^T \bar{u}_t^2 - \bar{u}^2\right]}{\bar{\beta}_{1N}^2 s_{f_1}^2}\right). \tag{B4}$$

However,

$$\ln\left(1 + \frac{2\bar{\beta}_{1N}\left[\frac{1}{T}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\bar{u}_t\right] + \left[\frac{1}{T}\sum_{t=1}^T \bar{u}_t^2 - \bar{u}^2\right]}{\bar{\beta}_{1N}^2 s_{f_1}^2}\right) = \frac{2\bar{\beta}_{1N}\left[\frac{1}{T}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\bar{u}_t\right] + \left[\frac{1}{T}\sum_{t=1}^T \bar{u}_t^2 - \bar{u}^2\right]}{\bar{\beta}_{1N}^2 s_{f_1}^2} + o_p\left(B_{N,T}\right), \tag{B5}$$

where when $m = 1$,

$$B_{N,T} = \frac{2\bar{\beta}_{1N}\left[\frac{1}{T}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\bar{u}_t\right] + \left[\frac{1}{T}\sum_{t=1}^T \bar{u}_t^2 - \bar{u}^2\right]}{\bar{\beta}_{1N}^2 s_{f_1}^2}.$$

Consider the first term of the RHS of (42). We have,

$$\frac{2\bar{\beta}_{1N}\left[\frac{1}{T}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\bar{u}_t\right]}{\bar{\beta}_{1N}^2 s_{f_1}^2} = \frac{\frac{2}{\sqrt{TN}}\left[\frac{1}{\sigma_{f_1}\sqrt{T}}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\left(\sqrt{N}\bar{u}_t\right)\right]}{\left(s_{f_1}\bar{\beta}_{1N}\right)\left(s_{f_1}/\sigma_{f_1}\right)}.$$

We note that $s_{f_1}/\sigma_{f_1} = 1 + O_p(T^{-1/2})$. But, by Lemma 2 (as N and $T \to \infty$)

$$\frac{1}{\sigma_{f_1}\sqrt{T}}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\left(\sqrt{N}\bar{u}_t\right) \to_p N(0, \bar{\sigma}_N^2), \tag{B6}$$

where $\overline{\sigma}_N^2$ is as in (B1). Also, $1/\bar{\beta}_{1N} = N^{(1-\alpha)/2}\left(1/\bar{v}_{1N}\right)$. Hence,

$$\begin{aligned}\frac{2\bar{\beta}_{1N}\left[\frac{1}{T}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\bar{u}_t\right]}{\bar{\beta}_{1N}^2 s_{f_1}^2} &= \frac{\frac{2}{\sqrt{TN}}\left[\frac{1}{\sigma_{f_1}\sqrt{T}}\sum_{t=1}^T \left(f_{1t} - \bar{f}_1\right)\left(\sqrt{N}\bar{u}_t\right)\right]}{s_{f_1}\bar{\beta}_{1N}(s_{f_1}/\sigma_{f_1})} \\ &= O_p\left(T^{-1/2}N^{-\alpha/2}\right).\end{aligned} \tag{B7}$$

Consider now the second term on the RHS of (42). Note that since, by Lemma 1 and Theorems 17.5 and 19.11 of Davidson (1994), $\sqrt{NT}\bar{u} = O_p(1)$, and, since $s_{f_1}^2/\sigma_{f_1}^2 = 1 + O_p(T^{-1/2})$ where $0 < \sigma_{f_1}^2 < \infty$,

$$\frac{\bar{u}^2}{\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2} = \frac{\left(\sqrt{NT}\bar{u}\right)^2}{NT\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2} = O_p\left(T^{-1}N^{-\alpha}\right). \tag{B8}$$

Similarly,

$$\begin{aligned}\frac{\frac{1}{T}\sum_{t=1}^T \bar{u}_t^2}{\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2} &= \frac{\frac{1}{N\sqrt{T}}\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^T\left[(\sqrt{N}\bar{u}_t)^2 - \bar{\sigma}_N^2\right] + \sqrt{T}\bar{\sigma}_N^2\right\}}{\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2} = \frac{\frac{\bar{\sigma}_N^2}{N\sqrt{T}}\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^T\left[\left(\frac{\sqrt{N}\bar{u}_t}{\bar{\sigma}_N}\right)^2 - 1\right] + \sqrt{T}\right\}}{\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2} \\ &= \frac{\frac{\bar{\sigma}_N^2}{N\sqrt{T}}\frac{1}{\sqrt{T}}\sum_{t=1}^T\left[\left(\frac{\sqrt{N}\bar{u}_t}{\bar{\sigma}_N}\right)^2 - 1\right]}{\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2} + \frac{\bar{\sigma}_N^2}{N\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2}.\end{aligned} \tag{B9}$$

But, by Lemma 3,

$$\frac{1}{\sqrt{2T}}\sum_{t=1}^T\left[\left(\frac{\sqrt{N}\bar{u}_t}{\bar{\sigma}_N}\right)^2 - 1\right] \to_d N(0,1),$$

and

$$\frac{\frac{\bar{\sigma}_N^2}{N\sqrt{T}}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\left(\frac{\sqrt{N}\bar{u}_t}{\bar{\sigma}_N}\right)^2-1\right]\right)}{\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2}=O_p(T^{-1/2}N^{-\alpha}). \tag{B10}$$

Therefore, collecting all results derived above, and keeping the highest order terms of the RHS of (B7), (B8), and (B10), we have

$$2\ln(N)\left(\hat{\alpha}-\alpha\right)-\ln(s_{f_1}^2\bar{v}_{1N}^2)-\frac{\bar{\sigma}_N^2}{N^\alpha\bar{v}_{1N}^2 s_{f_1}^2}=O_p\left(T^{-1/2}N^{-\alpha/2}\right).$$

In the first instance, this implies that

$$\hat{\alpha}-\alpha=O_p\left(\frac{1}{\ln(N)}\right), \tag{B11}$$

which establishes the consistency of $\hat{\alpha}$ as an estimate of α as N and $T\to\infty$, in any order.

Consider now the derivation of the asymptotic distribution of $\hat{\alpha}$. We have

$$\ln(N)\left(\hat{\alpha}-\alpha\right)-\frac{\bar{\sigma}_N^2}{N^{2\alpha-1}\bar{v}_{1N}^2 s_{f_1}^2}\simeq\ln(s_{f_1}^2\bar{v}_{1N}^2)+\frac{\frac{2}{\sqrt{TN}}\left[\frac{1}{\sigma_f\sqrt{T}}\sum_{t=1}^{T}\left(f_{1t}-\bar{f}_1\right)\left(\sqrt{N}\bar{u}_t\right)\right]}{s_{f_1}N^{(\alpha-1)/2}\bar{v}_{1N}(s_{f_1}/\sigma_{f_1})}+$$
$$\frac{\left(\sqrt{NT}\bar{u}\right)^2}{NT\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2}+\frac{\frac{\bar{\sigma}_N^2}{N\sqrt{T}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\left(\frac{\sqrt{N}\bar{u}_t}{\bar{\sigma}_N}\right)^2-1\right]}{\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2}.$$

where $A\simeq B$ denotes that $A-B=o_p(B)$. We first examine $\ln(s_{f_1}^2\bar{v}_{1N}^2)$. By Lemma 6 we have

$$\sqrt{\min(N,T)}\left(\ln(s_{f_1}^2\bar{v}_{1N}^2)-\ln(\sigma_{f_1}^2\mu_{v_1}^2)\right)\to_d N\left(0,\omega\right).$$

Further,

$$\sqrt{\min(N,T)}\left(\frac{\frac{2}{\sqrt{TN}}\left[\frac{1}{\sigma_{f_1}\sqrt{T}}\sum_{t=1}^{T}\left(f_{1t}-\bar{f}_1\right)\left(\sqrt{N}\bar{u}_t\right)\right]}{s_{f_1}N^{(\alpha-1)/2}\bar{v}_{1N}(s_{f_1}/\sigma_{f_1})}\right)=O_p\left(\sqrt{\min(N,T)}T^{-1/2}N^{-\alpha/2}\right)=o_p(1).$$

Similarly,

$$\sqrt{\min(N,T)}\left(\frac{\left(\sqrt{NT}\bar{u}\right)^2}{NT\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2}\right)=O_p\left(\sqrt{\min(N,T)}T^{-1}N^{-\alpha}\right)=o_p(1),$$

and

$$\sqrt{\min(N,T)}\left(\frac{\frac{\bar{\sigma}_N^2}{N\sqrt{T}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\left(\frac{\sqrt{N}\bar{u}_t}{\bar{\sigma}_N}\right)^2-1\right]}{\left(N^{(\alpha-1)/2}\bar{v}_{1N}\right)^2 s_{f_1}^2}\right)=O_p\left(\sqrt{\min(N,T)}T^{-1/2}N^{-\alpha}\right)=o_p(1).$$

Thus,

$$\sqrt{\min(N,T)}\left(\ln(N)\left(\hat{\alpha}-\alpha_N^*\right)-\frac{\bar{\sigma}_N^2}{N^\alpha\bar{v}_{1N}^2 s_{f_1}^2}\right)\to_d N\left(0,\omega\right),$$

where $\alpha_N^*=\alpha+\ln(\mu_{v_1}^2)/2\ln(N)$, by setting $\sigma_f^2=1$ as normalisation. The second part of the Corollary follows by Lemma 8.

Supplementary Appendix III: Proofs of technical Lemmas

Proof of Lemma 1

The proof of this lemma considers the more general Assumption 4 for the error terms which incorporates Assumption 3. By the Marcinkiewicz–Zygmund inequality (see, e.g., (Stout, 1974, Theorem 3.3.6)),

$$\sup_i E(|u_{it}|^r) = \sup_i E\left(\left|\sum_{l=0}^{\infty}\left(\psi_{il}\sum_{s=-\infty}^{\infty}\xi_{is}v_{st-l}\right)\right|^r\right) \leq c\left(\sup_i\left(\sum_{l=0}^{\infty}|\psi_{il}|^2\right)\sup_i\left(\sum_{s=-\infty}^{\infty}|\xi_{is}|^2\right)\right)^{r/2}\left(\sup_{i,t}E(|v_{it}|^r)\right),$$

so u_{it} is L_r-bounded if $\sup_i \sup_t E(|v_t|^r) < \infty$ which holds by Assumption 4. Moreover, writing $\|\cdot\|_r$ for the L_r-norm, we have, by Minkowski's inequality,

$$\sup_i \left\|u_{it} - E(u_{it}|\mathcal{F}^{v_i}_{t,|m|})\right\|_2 = \sup_i\left\|\sum_{j=m+1}^{\infty}\psi_{ij}\left(\sum_{|s|\geq m}\xi_{is}\nu_{st}\right)\right\|_2 \leq \sup_{i,t}\|v_{it}\|_2\left(\sup_i\sum_{j=m+1}^{\infty}|\psi_{ij}|\right)\left(\sup_i\left(\sum_{|s|\geq m}|\xi_{is}|\right)\right), \tag{B12}$$

for any integer $m > 0$ where $\mathcal{F}^{\nu_i}_{t,|m|}$ is the σ-field generated by $\{v_{is}; i, s \leq t-m\} \cup \{v_{is}; i, s \geq t+m\}$. But, Assumption 4 implies that $\sup_i \lim_{m\to\infty} m^{\zeta}\sum_{j=m+1}^{\infty}|\psi_{ij}| = O(1)$ and $\sup_i \lim_{m\to\infty} m^{\zeta}\left(\sum_{|s|\geq m}|\xi_{is}|\right) = O(1)$. Consequently $\{u_{it}\}_{t=1}^{\infty}$ and $\{u_{it}\}_{i=1}^{\infty}$ are L_r-bounded, L_2-NED processes of size $-\zeta$, uniformly over i and t. Similarly, we can show that $\boldsymbol{f}_t$ are L_r-bounded ($r \geq 2$) L_2-NED processes of size $-\zeta$.

Proof of Lemma 2

We have $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\boldsymbol{f}_t - \bar{\boldsymbol{f}}\right)\left(\sqrt{N}\bar{u}_t\right) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\boldsymbol{z}_t$, where $\boldsymbol{z}_t = \left(\boldsymbol{f}_t - \bar{\boldsymbol{f}}\right)\left(\sqrt{N}\bar{u}_t\right)$. We have that $\boldsymbol{z}_t$ are stationary processes such that $E(\boldsymbol{z}_t) = \mathbf{0}$. We note that by Lemma 1 and Theorem 24.6 of Davidson (1994), we have that $E\left(\left(\sqrt{N}\bar{u}_t\right)^2\right) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2 < \infty$. Further, by Theorem 17.8 of Davidson (1994), we have that sums of L_2-bounded, L_2-NED triangular arrays of size $-\zeta$ are L_2-bounded, L_2-NED triangular arrays of size $-\zeta$ as well, implying, given Lemma 1, that $\sqrt{N}\bar{u}_t$ is an L_2-bounded, L_2-NED triangular arrays of size $-\zeta$. Further, by the Marcinkiewicz–Zygmund inequality,

$$E(\left|\sqrt{N}\bar{u}_t\right|^r) = E\left(\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{l=0}^{\infty}\left(\psi_{il}\sum_{s=-\infty}^{\infty}\xi_{is}v_{st-l}\right)\right|^r\right) \leq c\left(\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{l=0}^{\infty}|\psi_{il}|^2\right)\left(\sum_{s=-\infty}^{\infty}|\xi_{is}|^2\right)\right)^{r/2}\sup_{i,t}E(|v_{it}|^r) \leq \tag{B13}$$

$$c\left(\sup_i\left(\sum_{l=0}^{\infty}|\psi_{il}|^2\right)\sup_i\left(\sum_{s=-\infty}^{\infty}|\xi_{is}|^2\right)\right)^{r/2}\left(\sup_{i,t}E(|v_{it}|^r)\right) < \infty.$$

As a result, $\sqrt{N}\bar{u}_t$ is a L_r-bounded, L_2-NED triangular arrays of size $-\zeta$.

Finally, since $\{\sqrt{N}\bar{u}_t\}$ and $\{\boldsymbol{f}_t\}$ are L_r-bounded ($r \geq 2$) L_2-NED processes of size $-\zeta$ on a ϕ-mixing process of size $-\eta$ ($\eta > 1$), then, by Example 17.17 of Davidson (1994), $\{\boldsymbol{z}_t\}$ are L_2-NED of size $-\{\zeta(\varphi-2)\}/\{2(\varphi-1)\} \leq -1/2$ on a ϕ-mixing process of size $-\eta$. Since ν_{it} and $\nu_{\boldsymbol{f}t}$ are i.i.d. processes they are also ϕ-mixing processes of any size. In view of Theorem 17.5(ii) of Davidson (1994), this in turn implies that $\{\boldsymbol{z}_t\}$ are L_2-mixingale of size $-1/2$, if $2\eta > \zeta$, which automatically holds by the i.i.d. property of ν_{it} and $\nu_{\boldsymbol{f}t}$. This implies the result of the Lemma by Theorem 24.6 of Davidson (1994).

Proof of Lemma 3

By Lemma 2, $\sqrt{N}\bar{u}_t$ is a L_r-bounded, L_2-NED triangular arrays of size $-\zeta$. By Example 17.17 of Davidson (1994), and (B13), $\left(\sqrt{N}\bar{u}_t\right)^2$ is L_r-NED of size $-\{\zeta(\varphi-2)\}/\{2(\varphi-1)\} \leq -1/2$, $r > 4$. Then, by Theorem 24.6 of Davidson (1994), the result follows.

Proof of Lemma 4

We need to show that $\widehat{\bar{\sigma}_N^2} - \bar{\sigma}_N^2 = O_p\left(T^{-1}\right) + O_p\left((NT)^{-1/2}\right)$ if $m = 1$ and $\widehat{\bar{\sigma}_N^2} - \bar{\sigma}_N^2 = O_p\left(N^{\alpha-1}T^{-1/2}\right)$ otherwise. We have that $\widehat{\bar{\sigma}_N^2} = \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \hat{u}_{it}^2$, where $\hat{u}_{it}$ is the estimated residual. Then, $\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \hat{u}_{it}^2 = \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T u_{it}^2 + \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \left(\hat{u}_{it}^2 - u_{it}^2\right)$. Following similar lines to those of the proof of Lemma 3 we have that $\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T u_{it}^2 \to_p \bar{\sigma}_N^2$. Further, $\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \left(u_{it}^2 - \bar{\sigma}_N^2\right) = O\left((NT)^{-1/2}\right)$. Next, we examine $\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \left(\hat{u}_{it}^2 - u_{it}^2\right)$. It is sufficient to consider $\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T u_{it}\left(\hat{u}_{it} - u_{it}\right)$.

Single factor case: We note that the same residual is obtained irrespective of whether we regress x_{it} on $\bar{x}_t$, or $\tilde{x}_t$ or $N^{1-\alpha}\bar{x}_t$ or f_{1t}. We carry out the analysis by using $\tilde{x}_t$ as the regressor. We have that $\hat{u}_{it}=\frac{\tilde{x}_t\sum_{j=1}^T \tilde{x}_j u_{ij}}{\sum_{j=1}^T \tilde{x}_j^2}+u_{it}$. Then,

$$\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T u_{it}\left(\hat{u}_{it} - u_{it}\right) = \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T u_{it}\tilde{x}_t\left(\frac{\sum_{j=1}^T \tilde{x}_j u_{ij}}{\sum_{j=1}^T \tilde{x}_j^2}\right) = \left(\frac{1}{NT\sum_{j=1}^T \tilde{x}_j^2}\right)\sum_{i=1}^N\left(\sum_{j=1}^T \tilde{x}_j u_{ij}\right)\left(\sum_{t=1}^T \tilde{x}_t u_{it}\right) =$$

$$\frac{1}{NT}\left(\frac{1}{\frac{1}{T}\sum_{t=1}^T \tilde{x}_t^2}\right)\sum_{i=1}^N\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2 - E\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2\right)\right) +$$

$$+\frac{1}{NT}\left(\frac{1}{\frac{1}{T}\sum_{t=1}^T \tilde{x}_t^2}\right)\sum_{i=1}^N\left(E\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2\right)\right).$$

But $E\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2\right) < \infty$ uniformly over i and $\frac{1}{T}\sum_{t=1}^T \tilde{x}_t^2 = O_p(1)$, which implies that

$$\frac{1}{NT}\left(\frac{1}{\frac{1}{T}\sum_{t=1}^T \tilde{x}_t^2}\right)\sum_{i=1}^N\left(E\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2\right)\right) = O_p\left(\frac{1}{T}\right).$$

Further, $\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2 - E\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2\right)$ is a NED process over i, which implies that

$$\frac{1}{NT}\left(\frac{1}{\frac{1}{T}\sum_{t=1}^T \tilde{x}_t^2}\right)\sum_{i=1}^N\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2 - E\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T \tilde{x}_t u_{it}\right)^2\right)\right) = O_p\left(\frac{1}{T\sqrt{N}}\right),$$

proving the required result.

Multi factor case: We will focus on the case where $\alpha = \alpha_2 = ... = \alpha_m$ as the case $\alpha \geq \alpha_2 \geq ... \geq \alpha_m$ with at least one strict inequality can be treated similarly and has equal or lower rates for $\widehat{\bar{\sigma}_N^2} - \bar{\sigma}_N^2$. We have

$$\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T u_{it}\left(\hat{u}_{it} - u_{it}\right) = \frac{1}{NT}\sum_{i,\beta_{i1}\neq 0}^{N^\alpha}\sum_{t=1}^T u_{it}\left(\hat{u}_{it} - u_{it}\right) + \frac{1}{NT}\sum_{i,\beta_{i1}=0}^{N-N^\alpha}\sum_{t=1}^T u_{it}\left(\hat{u}_{it} - u_{it}\right) \tag{B14}$$

The second term of the RHS of (B14) can be treated as in the single factor case, giving

$$\frac{1}{NT}\sum_{i,\beta_{1i}=0}^{N-N^\alpha}\sum_{t=1}^T u_{it}\left(\hat{u}_{it} - u_{it}\right) = O_p\left(T^{-1}\right) + O_p\left(\frac{1}{T\sqrt{N}}\right)$$

For the first term of the RHS of (B14), we note that x_{it} can be written as $x_{it}= \varkappa_i\tilde{x}_t+\tilde{\beta}_i'\tilde{f}_t+u_{it}$, where $\tilde{f}_t$ is a zero mean process that is uncorrelated to $\tilde{x}_t$. Then, $\hat{u}_{it}=\frac{\tilde{x}_t\sum_{j=1}^T \tilde{x}_j\left(\tilde{\beta}_i'\tilde{f}_j+u_{ij}\right)}{\sum_{j=1}^T \tilde{x}_j^2}+\tilde{\beta}_i'\tilde{f}_t+u_{it}$ and

$$\frac{1}{NT}\sum_{i,\beta_{i1}\neq 0}^{N^\alpha}\sum_{t=1}^T u_{it}\left(\hat{u}_{it} - u_{it}\right) = \frac{1}{NT}\sum_{i,\beta_{i1}\neq 0}^{N^\alpha}\sum_{t=1}^T u_{it}\left(\tilde{\beta}_i'\tilde{f}_t + u_{it}\right) + R \tag{B15}$$

where R is of smaller order of probability than the first term of the RHS of (B15). Following similar arguments as above we obtain

$$\frac{1}{NT}\sum_{i,\beta_{i1}\neq 0}^{N^{\alpha}}\sum_{t=1}^{T}u_{it}\left(\tilde{\beta}_i'\tilde{f}_t+u_{it}\right)=O_p\left(N^{\alpha-1}T^{-1/2}\right),$$

which implies that

$$\widehat{\bar{\sigma}_N^2}-\bar{\sigma}_N^2=O_p\left(N^{\alpha-1}T^{-1/2}\right).$$

giving a lower rate of convergence than the single factor case.

Proof of Lemma 5

We have that

$$\ln(s_{f_1}^2\bar{v}_{1N}^2)-\ln(\sigma_{f_1}^2\mu_{v_1}^2)=\ln\left(\frac{s_{f_1}^2\bar{v}_{1N}^2}{\sigma_{f_1}^2\mu_{v_1}^2}\right)=\ln\left(\frac{s_{f_1}^2}{\sigma_{f_1}^2}\right)+\ln\left(\frac{\bar{v}_{1N}^2}{\mu_{v_1}^2}\right)=\left(\frac{s_{f_1}^2-\sigma_{f_1}^2}{\sigma_{f_1}^2}\right)+\left(\frac{\bar{v}_{1N}^2-\mu_{v_1}^2}{\mu_{v_1}^2}\right)+$$

$$O_p\left(\left(s_{f_1}^2-\sigma_{f_1}^2\right)^2\right)+O_p\left(\left(\bar{v}_{1N}^2-\mu_{v_1}^2\right)^2\right).$$

But, under Assumption 2, and setting $m=1$,

$$\sqrt{T}\left(\frac{s_{f_1}^2-\sigma_{f_1}^2}{\sigma_{f_1}^2}\right)=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{\left[(f_{1t}-\bar{f}_1)/\sigma_{f_1}\right]^2-1\right\}\to_d N\left(0,V_{f_1^2}\right),$$

where $\bar{f}_1=\frac{1}{T}\sum_{t=1}^{T}f_{1t}$, and

$$V_{f_1^2}=E\left(\left([(f_{1t}-\mu_{f_1})/\sigma_{f_1}]^2-1\right)^2\right)+\sum_{i=1}^{\infty}Cov\left(\left([(f_{1t}-\mu_{f_1})/\sigma_{f_1}]^2-1\right)\left([(f_{1,t-i}-\mu_{f_1})/\sigma_{f_1}]^2-1\right)\right).$$

Further, recalling that $\bar{v}_{1N}=\frac{1}{N^{\alpha}}\sum_{i=1}^{N^{\alpha}}v_{i1}$, $\sqrt{N^{\alpha}}\left(\frac{\bar{v}_{1N}^2-\mu_{v_1}^2}{\mu_{v_1}^2}\right)=\sqrt{N^{\alpha}}\left(\frac{\bar{v}_{1N}-\mu_{v_1}}{\mu_{v_1}}\right)\left(\frac{\bar{v}_{1N}+\mu_{v_1}}{\mu_{v_1}}\right)$. But $\frac{\bar{v}_{1N}+\mu_{v_1}}{\mu_{v_1}}\to_p 2$, and

$$\sqrt{N^{\alpha}}\left(\frac{\bar{v}_{1N}-\mu_{v_1}}{\mu_{v_1}}\right)\to_d N\left(0,\frac{\sigma_{v_1}^2}{\mu_{v_1}^2}\right). \tag{B16}$$

Further, $E\left[\left(\frac{s_{f_1}^2-\sigma_{f_1}^2}{\sigma_{f_1}^2}\right)\left(\frac{\bar{v}_{1N}^2-\mu_{v_1}^2}{\mu_{v_1}^2}\right)\right]=0$, implying that $\sqrt{\min(N^{\alpha},T)}\left(\ln(s_{f_1}^2\bar{v}_{1N}^2)-\ln(\sigma_{f_1}^2\mu_{v_1}^2)\right)\to_d N(0,\omega)$, where $\omega=\lim_{N,T\to\infty}\left[\frac{\min(N^{\alpha},T)}{T}V_{f_1^2}+\frac{\min(N^{\alpha},T)}{N^{\alpha}}\frac{4\sigma_{v_1}^2}{\mu_{v_1}^2}\right]$.

Proof of Lemma 6

The proof follows easily along the same lines as that of Lemma 5. In the present case under Assumption 5 we have $\bar{v}_{1N}=N^{-1}\sum_{i=1}^{N}v_{i1}$, and thus $\sqrt{N}\left(\frac{\bar{v}_{1N}^2-\mu_{v_1}^2}{\mu_{v_1}^2}\right)=\sqrt{N}\left(\frac{\bar{v}_{1N}-\mu_{v_1}}{\mu_{v_1}}\right)\left(\frac{\bar{v}_{1N}+\mu_{v_1}}{\mu_{v_1}}\right)$, and $\frac{\bar{v}_{1N}+\mu_{v_1}}{\mu_{v_1}}\to_p 2$. Therefore,$\sqrt{N}\left(\frac{\bar{v}_{1N}-\mu_{v_1}}{\mu_{v_1}}\right)\to_d N\left(0,\frac{\sigma_{v_1}^2}{\mu_{v_1}^2}\right)$.

Proof of Lemma 7

We need to show that

$$\sqrt{\min(N^{\alpha},T)}\left(\frac{\bar{\sigma}_N^2}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}-\frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}\right)=o_p(1). \tag{B17}$$

We have

$$\frac{\bar{\sigma}_N^2}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}-\frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}=\frac{\bar{\sigma}_N^2}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}-\frac{\widehat{\bar{\sigma}_N^2}}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}+\frac{\widehat{\bar{\sigma}_N^2}}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}-\frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}.$$

But, by lemma 4

$$\frac{\bar{\sigma}_N^2}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{\widehat{\bar{\sigma}_N^2}}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} = O_p\left(T^{-1/2}N^{-2\alpha}\right), \tag{B18}$$

which is negligible as a bias. Next,

$$\begin{aligned}\frac{\widehat{\bar{\sigma}_N^2}}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2} &= \bar{\sigma}_N^2\left(\frac{\widehat{\bar{\sigma}_N^2}}{\bar{\sigma}_N^2}\right)\left(\frac{1}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{1}{N\hat{\sigma}_{\bar{x}}^2}\right)\\ &= \bar{\sigma}_N^2\left(\frac{\widehat{\bar{\sigma}_N^2}}{\bar{\sigma}_N^2}\right)\left(\frac{1}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}\right)N^{2\alpha-1}\left(N^{2-2\alpha}\hat{\sigma}_{\bar{x}}^2 - \bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N\right)\left(\frac{1}{N\hat{\sigma}_{\bar{x}}^2}\right).\end{aligned}$$

But by the proof of Theorem 1, we have

$$\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N - N^{2-2\alpha}\hat{\sigma}_{\bar{x}}^2 = O_p\left(T^{-1/2}N^{-2\alpha}\right) + O_p(N^{1-3\alpha}) + O_p(N^{-\alpha}) + O_p\left(N^{1-2\alpha}\right).$$

So,

$$\begin{gathered}\bar{\sigma}_N^2\left(\frac{\widehat{\bar{\sigma}_N^2}}{\bar{\sigma}_N^2}\right)\left(\frac{1}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}\right)N^{2\alpha-1}\left(N^{2-2\alpha}\hat{\sigma}_{\bar{x}}^2 - \bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N\right)\left(\frac{1}{N\hat{\sigma}_{\bar{x}}^2}\right) =\\ O_p\left(T^{-1/2}N^{-2\alpha}N^{1-2\alpha}\right) + O_p(N^{1-3\alpha}N^{1-2\alpha}) + O_p(N^{-\alpha}N^{1-2\alpha}) + O_p\left(N^{1-2\alpha}N^{1-2\alpha}\right) =\\ O_p\left(T^{-1/2}N^{1-2\alpha}\right) + O_p(N^{2-5\alpha}) + O_p(N^{1-3\alpha}) + O_p\left(N^{2-4\alpha}\right).\end{gathered}$$

Therefore, for $\alpha > 1/2$, (B17) holds, which establishes the Lemma.

Proof of Lemma 8

We need to show that

$$\sqrt{\min(N,T)}\left(\frac{\bar{\sigma}_N^2}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}\right) = o_p(1). \tag{B19}$$

We have

$$\frac{\bar{\sigma}_N^2}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2} = \frac{\bar{\sigma}_N^2}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{\widehat{\bar{\sigma}_N^2}}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} + \frac{\widehat{\bar{\sigma}_N^2}}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2}.$$

But, by lemma 4

$$\frac{\bar{\sigma}_N^2}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{\widehat{\bar{\sigma}_N^2}}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} = O_p\left(T^{-1/2}N^{-\alpha}\right),$$

which is negligible as a bias. Next,

$$\begin{aligned}\frac{\widehat{\bar{\sigma}_N^2}}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{\widehat{\bar{\sigma}_N^2}}{N\hat{\sigma}_{\bar{x}}^2} &= \bar{\sigma}_N^2\left(\frac{\widehat{\bar{\sigma}_N^2}}{\bar{\sigma}_N^2}\right)\left(\frac{1}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N} - \frac{1}{N\hat{\sigma}_{\bar{x}}^2}\right)\\ &= \bar{\sigma}_N^2\left(\frac{\widehat{\bar{\sigma}_N^2}}{\bar{\sigma}_N^2}\right)\left(\frac{1}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}\right)N^{\alpha}\left(N^{1-\alpha}\hat{\sigma}_{\bar{x}}^2 - \bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N\right)\left(\frac{1}{N\hat{\sigma}_{\bar{x}}^2}\right).\end{aligned}$$

But by the proof of Theorem 1, we have

$$\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N - N^{1-\alpha}\hat{\sigma}_{\bar{x}}^2 = O_p\left(T^{-1/2}N^{-\alpha/2}\right).$$

So

$$\bar{\sigma}_N^2\left(\frac{\widehat{\bar{\sigma}_N^2}}{\bar{\sigma}_N^2}\right)\left(\frac{1}{N^{\alpha}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}\right)N^{\alpha}\left(N^{1-\alpha}\hat{\sigma}_{\bar{x}}^2 - \bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N\right)\left(\frac{1}{N\hat{\sigma}_{\bar{x}}^2}\right) = O_p\left(T^{-1/2}N^{-3\alpha/2}\right)$$

which establishes the Lemma.

Proof of Lemma 9

We have that

$$\bar{\sigma}_N^2\left(\frac{\widehat{\bar{\sigma}_N^2}}{\bar{\sigma}_N^2}\right)\left(\frac{1}{N^{2\alpha-1}\bar{\boldsymbol{v}}_N'\boldsymbol{D}_N\boldsymbol{S}_{ff}\boldsymbol{D}_N\bar{\boldsymbol{v}}_N}\right)N^{2\alpha-1}\left(N^{1-2\alpha}\left(\frac{1}{T}\sum_{t=1}^{T}(\sqrt{N}\bar{u}_t)^2-\widehat{\bar{\sigma}_N^2}\right)\right)\left(\frac{1}{N\hat{\sigma}_{\bar{x}}^2}\right)=O_p\left(N^{2-4\alpha}T^{-1/2}\right),$$

which is negligible as long as $\alpha>1/2$. To prove the above result we first note that

$$\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}u_{it}\right)^2\right)-\widehat{\bar{\sigma}_N^2}=$$

$$\left(\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{\sqrt{N}}\sum_{t=1}^{N}u_{it}\right)^2\right)-\bar{\sigma}_N^2\right)+\left(\bar{\sigma}_N^2-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}^2\right)-\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{u}_{it}^2-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}^2\right).$$

But, it is straightforward to show that $\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}u_{it}\right)^2\right)-\bar{\sigma}_N^2=O_p(T^{-1/2})$ and $\bar{\sigma}_N^2-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}^2=O_p(T^{-1/2})$. Finally, by Lemma 4, $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{u}_{it}^2-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}^2=o_p(T^{-1/2})$. So, $\frac{1}{T}\sum_{t=1}^{T}\left(\left(\frac{1}{\sqrt{N}}\sum_{t=1}^{N}u_{it}\right)^2\right)-\widehat{\bar{\sigma}_N^2}=O_p(T^{-1/2})$.

Proof of Lemma 10

We have that

$$\frac{1}{T}\sum_{t=1}^{T}\bar{x}_t^2=\frac{1}{T}\sum_{t=1}^{T}\left[\sum_{j=1}^{m}\left(f_{jt}\frac{N^{\alpha_j}}{N}\frac{1}{N^{\alpha_j}}\sum_{i=1}^{N^{\alpha_j}}v_{ij}\right)+\frac{1}{N}\sum_{i=1}^{N}u_{it}\right]^2=$$

$$\frac{1}{T}\sum_{t=1}^{T}\left\{\sum_{j=1}^{m}\left[f_{jt}^2\frac{N^{2\alpha_j}}{N^2}\left(\frac{1}{N^{\alpha_j}}\sum_{i=1}^{N^{\alpha_j}}v_{ij}\right)^2\right]\right\}+$$

$$\frac{1}{T}\sum_{t=1}^{T}\left\{\sum_{j=1,j\neq s}^{m}\left[f_{jt}\frac{N^{\alpha_j}}{N}\left(\frac{1}{N^{\alpha_j}}\sum_{i=1}^{N^{\alpha_j}}v_{ij}\right)\right]\sum_{s=1}^{m}\left[f_{st}\frac{N^{\alpha_s}}{N}\left(\frac{1}{N^{\alpha_s}}\sum_{i=1}^{N^{\alpha_s}}v_{is}\right)\right]\right\}+$$

$$\frac{1}{T}\sum_{t=1}^{T}\left\{\left[\sum_{j=1}^{m}\left(f_{jt}\frac{N^{\alpha_j}}{N}\frac{1}{N^{\alpha_j}}\sum_{i=1}^{N^{\alpha_j}}v_{ij}\right)\right]\left(\frac{1}{N}\sum_{i=1}^{N}u_{it}\right)\right\}+$$

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}u_{it}\right)^2.$$

But,

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}u_{it}\right)^2=O_p\left(N^{-1}\right)$$

$$\frac{1}{T}\sum_{t=1}^{T}\left[\left(f_{jt}\frac{N^{\alpha_j}}{N}\frac{1}{N^{\alpha_j}}\sum_{i=1}^{N^{\alpha_j}}v_{ij}\right)\left(\frac{1}{N}\sum_{i=1}^{N}u_{it}\right)\right]=O_p\left(N^{\alpha_j-3/2}T^{-1/2}\right),\ j=1,...,m$$

$$\frac{1}{T}\sum_{t=1}^{T}\left(\left(f_{jt}\frac{N^{\alpha_j}}{N}\frac{1}{N^{\alpha_j}}\sum_{i=1}^{N^{\alpha_j}}v_{ij}\right)\left(f_{st}\frac{N^{\alpha_s}}{N}\frac{1}{N^{\alpha_s}}\sum_{i=1}^{N^{\alpha_s}}v_{is}\right)\right)=O_p\left(N^{\alpha_j+\alpha_s-2}T^{-1/2}\right),\ j,s=1,...,m,\quad j\neq s,$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\left(f_{jt}^2\frac{N^{2\alpha_j}}{N^2}\left(\frac{1}{N^{\alpha_j}}\sum_{i=1}^{N^{\alpha_j}}v_{ij}\right)^2\right)-N^{2\alpha_j-2}\mu_{v_j}^2\to_p 0,\ j=1,...,m.$$

If $\alpha = \alpha_2 = ... = \alpha_{q-1} > \alpha_q \geq ... \geq \alpha_m$

$$\frac{1}{T}\sum_{t=1}^{T}\bar{x}_t^2 - N^{2\alpha-2}\sum_{j=1}^{q}\mu_{v_j}^2 \to_p 0.$$

In particular if $\alpha > \alpha_2 \geq ... \geq \alpha_m$

$$\frac{1}{T}\sum_{t=1}^{T}\bar{x}_t^2 - N^{2\alpha-2}\mu_{v_1}^2 \to_p 0.$$

Proof of Lemma 11

Without loss of generality, we consider the case of two factors. The result extends straightforwardly to m factors. We further assume, for simplicity, that factors are independent from each other. Then,

$$\ln\left(\bar{v}_{1N}^2 s_{f_1}^2 + 2\bar{v}_{1N}\bar{v}_{2N}s_{12,f} + \bar{v}_{2N}^2 s_{f_2}^2\right) - \ln\left(\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2\right) = \ln\left(\frac{\bar{v}_{1N}^2 s_{f_1}^2 + 2\bar{v}_{1N}\bar{v}_{2N}s_{12,f} + \bar{v}_{2N}^2 s_{f_2}^2}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2}\right).$$

Then,

$$\ln\left(\frac{\bar{v}_{1N}^2 s_{f_1}^2 + 2\bar{v}_{1N}\bar{v}_{2N}s_{12,f} + \bar{v}_{2N}^2 s_{f_2}^2}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2}\right) = \frac{\bar{v}_{1N}^2 s_{f_1}^2 + 2\bar{v}_{1N}\bar{v}_{2N}s_{12,f} + \bar{v}_{2N}^2 s_{f_2}^2}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} - 1 = \tag{B20}$$

$$\frac{\left(\bar{v}_{1N}^2 s_{f_1}^2 - \sigma_{f_1}^2\mu_{v_1}^2\right) + \left(\bar{v}_{2N}^2 s_{f_2}^2 - \sigma_{f_2}^2\mu_{v_2}^2\right) + 2\bar{v}_{1N}\bar{v}_{2N}s_{12,f}}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} =$$

$$\frac{\left(\bar{v}_{1N}^2 s_{f_1}^2 - \bar{v}_{1N}^2\sigma_{f_1}^2 + \bar{v}_{1N}^2\sigma_{f_1}^2 - \sigma_{f_1}^2\mu_{v_1}^2\right) + \left(\bar{v}_{2N}^2 s_{f_2}^2 - \bar{v}_{2N}^2\sigma_{f_2}^2 + \bar{v}_{2N}^2\sigma_{f_2}^2 - \sigma_{f_2}^2\mu_{v_2}^2\right) + 2\mu_{v_1}\mu_{v_2}s_{12,f}}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} =$$

$$\frac{\mu_{v_1}^2\left(s_{f_1}^2 - \sigma_{f_1}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} + \frac{\sigma_{f_1}^2\left(\bar{v}_{1N}^2 - \mu_{v_1}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} + \frac{\mu_{v_2}^2\left(s_{f_2}^2 - \sigma_{f_2}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} + \frac{\sigma_{f_2}^2\left(\bar{v}_{2N}^2 - \mu_{v_2}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} + \frac{2\mu_{v_1}\mu_{v_2}s_{12,f}}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2}.$$

Note that

$$\bar{v}_{1N}\bar{v}_{2N}s_{12,f} = \bar{v}_{1N}\bar{v}_{2N}s_{12,f} - \bar{v}_{1N}\mu_{v_2}s_{12,f} + \bar{v}_{1N}\mu_{v_2}s_{12,f} - \bar{v}_{1N}\mu_{v_2}\sigma_{12,f} + \bar{v}_{1N}\mu_{v_2}\sigma_{12,f} - 2\mu_{v_1}\mu_{v_2}\sigma_{12,f} =$$

$$s_{12,f}\bar{v}_{1N}\left(\bar{v}_{2N} - \mu_{v_2}\right) + \bar{v}_{1N}\mu_{v_2}\left(s_{12,f} - \sigma_{12,f}\right) + \sigma_{12,f}\mu_{v_2}\left(\bar{v}_{1N} - 2\mu_{v_1}\right) =$$

$$\left(s_{12f} - \sigma_{12,f}\right)\bar{v}_{1N}\left(\bar{v}_{2N} - \mu_{v_2}\right) + \sigma_{12,f}\bar{v}_{1N}\left(\bar{v}_{2N} - \mu_{v_2}\right) + \bar{v}_{1N}\mu_{v_2}\left(s_{12,f} - \sigma_{12,f}\right) + \sigma_{12,f}\mu_{v_2}\left(\bar{v}_{1N} - 2\mu_{v_1}\right).$$

But

$$\left(s_{12f} - \sigma_{12,f}\right)\bar{v}_{1N}\left(\bar{v}_{2N} - \mu_{v_2}\right) = o_p(T^{-1/2}),$$

and $\sigma_{12,} = 0$, and so

$$\begin{aligned}&\left(s_{12,f} - \sigma_{12,f}\right)\bar{v}_{1N}\left(\bar{v}_{2N} - \mu_{v_2}\right) + \sigma_{12,f}\bar{v}_{1N}\left(\bar{v}_{2N} - \mu_{v_2}\right) + \bar{v}_{1N}\mu_{v_2}\left(s_{12,f} - \sigma_{12,f}\right) + \sigma_{12,f}\mu_{v_2}\left(\bar{v}_{1N} - 2\mu_{v_1}\right)\\ &= \bar{v}_{1N}\mu_{v_2}s_{12,f} = \left(\frac{\bar{v}_{1N}}{\mu_{v_1}}\right)\mu_{v_1}\mu_{v_2}s_{12,f} + o_p\left(T^{-1/2}\right).\end{aligned}$$

Then,

$$\frac{\mu_{v_i}^2\left(s_{f_i}^2 - \sigma_{f_i}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} = \frac{\mu_{v_i}^2\sigma_{f_i}^2}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2}\frac{\left(s_{f_i}^2 - \sigma_{f_i}^2\right)}{\sigma_{f_i}^2}, \quad i = 1, 2,$$

$$\frac{\sigma_{f_i}^2\left(\bar{v}_{iN}^2 - \mu_{v_i}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2} = \frac{\mu_{v_i}^2\sigma_{f_i}^2}{\sigma_{f_1}^2\mu_{v_1}^2 + \sigma_{f_2}^2\mu_{v_2}^2}\frac{\left(\bar{v}_{iN}^2 - \mu_{v_i}^2\right)}{\mu_{v_i}^2}, \quad i = 1, 2.$$

Assuming loadings of factors and factors are independent of each other and across factors, gives

$$\mu_{v_i}^2\sigma_{f_i}^2\left(\sqrt{T}\frac{\left(s_{f_i}^2 - \sigma_{f_i}^2\right)}{\sigma_{f_i}^2}\right) = \mu_{v_i}^2\sigma_{f_i}^2\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{\left[(f_{it} - \bar{f}_i)/\sigma_{f_i}\right]^2 - 1\right\}\right) \to_d N\left(0, \left(\mu_{v_i}^2\sigma_{f_i}^2\right)^2\mu_i^{(4)}\right), \quad i = 1, 2,$$

$$\mu_{v_i}^2\sigma_{f_i}^2\left(\sqrt{N^\alpha}\left(\frac{\bar{v}_{iN}^2-\mu_{v_i}^2}{\mu_{v_i}^2}\right)\right)=\mu_{v_i}^2\sigma_{f_i}^2\left(\sqrt{N^\alpha}\left(\frac{\bar{v}_{iN}-\mu_{v_i}}{\mu_{v_i}}\right)\left(\frac{\bar{v}_{iN}+\mu_{v_i}}{\mu_{v_i}}\right)\right)\rightarrow_d N\left(0,4\sigma_{v_i}^2\mu_{v_i}^2\left(\sigma_{f_i}^2\right)^2\right),\quad i=1,2.$$

Further,

$$\mu_{v_1}\mu_{v_2}\sqrt{T}s_{12,f}=\mu_{v_1}\mu_{v_2}\frac{\sigma_{f_1}\sigma_{f_2}}{\sqrt{T}}\sum_{t=1}^T\left(\frac{f_{1t}-\bar{f}_1}{\sigma_{f_1}}\right)\left(\frac{f_{2t}-\bar{f}_2}{\sigma_{f_2}}\right)\rightarrow_d N\left(0,\mu_{v_1}^2\sigma_{f_1}^2\mu_{v_2}^2\sigma_{f_2}^2\right).$$

Further, by factor independence

$$E\left(\left[\frac{1}{\sqrt{T}}\sum_{t=1}^T\left\{\left[(f_{it}-\bar{f}_i)/\sigma_{f_i}\right]^2-1\right\}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^T\left(\frac{f_{1t}-\bar{f}_1}{\sigma_{f_1}}\right)\left(\frac{f_{2t}-\bar{f}_2}{\sigma_{f_2}}\right)\right]\right)=0,\quad i=1,2.$$

So

$$\sqrt{\frac{\min\left(N^\alpha,T\right)}{T}}\left(\mu_{v_1}^2\sigma_{f_1}^2\left(\sqrt{T}\frac{\left(s_{f_1}^2-\sigma_{f_1}^2\right)}{\sigma_{f_1}^2}\right)+\mu_{v_2}^2\sigma_{f_2}^2\left(\sqrt{T}\frac{\left(s_{f_2}^2-\sigma_{f_2}^2\right)}{\sigma_{f_2}^2}\right)+2\mu_{v_1}\mu_{v_2}\sqrt{T}s_{12,f}\right)+$$
$$\sqrt{\frac{\min\left(N^\alpha,T\right)}{N^\alpha}}\left(\mu_{v_1}^2\sigma_{f_1}^2\left(\sqrt{N^\alpha}\left(\frac{\bar{v}_{1N}^2-\mu_{v_1}^2}{\mu_{v_1}^2}\right)\right)+\mu_{v_2}^2\sigma_{f_2}^2\left(\sqrt{N^\alpha}\left(\frac{\bar{v}_{2N}^2-\mu_{v_2}^2}{\mu_{v_2}^2}\right)\right)\right)\rightarrow_d$$
$$N\left(\begin{array}{c}0,\frac{\min(N^\alpha,T)}{T}\left(\left(\mu_{v_1}^2\sigma_{f_1}^2\right)^2\mu_1^{(4)}+\left(\mu_{v_2}^2\sigma_{f_2}^2\right)^2\mu_2^{(4)}+4\mu_{v_1}^2\left(\sigma_{f_1}^2\right)^2\mu_{v_2}^2\left(\sigma_{f_2}^2\right)^2\right)\\+\frac{\min(N^\alpha,T)}{N^\alpha}\left(2\sigma_{v_1}^2\mu_{v_1}^2\sigma_{f_1}^2+2\sigma_{v_2}^2\mu_{v_2}^2\sigma_{f_2}^2\right)\end{array}\right).$$

Proof of Lemma 12

Again, without loss of generality, we look at the case of two factors. The result again extends straightforwardly. We further assume, for simplicity, that factors are independent from each other. Then,

$$\ln\left(\bar{v}_{1N}^2s_{f_1}^2+2N^{\alpha_2-\alpha}\bar{v}_{1N}\bar{v}_{2N}s_{12,f}+N^{2(\alpha_2-\alpha)}\bar{v}_{2N}^2s_{f_2}^2\right)-\ln\left(\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2\right)=$$
$$\ln\left(\frac{\bar{v}_{1N}^2s_{f_1}^2+2N^{\alpha_2-\alpha}\bar{v}_{1N}\bar{v}_{2N}s_{12,f}+N^{2(\alpha_2-\alpha)}\bar{v}_{2N}^2s_{f_2}^2}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}\right).$$

Then, similarly to the proof of Lemma 11

$$\ln\left(\frac{\bar{v}_{1N}^2s_{f_1}^2+2N^{\alpha_2-\alpha}\bar{v}_{1N}\bar{v}_{2N}s_{12,f}+N^{2(\alpha_2-\alpha)}\bar{v}_{2N}^2s_{f_2}^2}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}\right)=\frac{\mu_{v_1}^2\left(s_{f_1}^2-\sigma_{f_1}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}+\frac{\sigma_{f_1}^2\left(\bar{v}_{1N}^2-\mu_{v_1}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}+$$
$$\frac{N^{2(\alpha_2-\alpha)}\mu_{v_2}^2\left(s_{f_2}^2-\sigma_{f_2}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}+\frac{N^{2(\alpha_2-\alpha)}\sigma_{22,f}\left(\bar{v}_{2N}^2-\mu_{v_2}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}+\frac{2N^{\alpha_2-\alpha}\mu_{v_1}\mu_{v_2}s_{12,f}}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}.$$

Then,

$$\frac{\mu_{v_1}^2\left(s_{f_1}^2-\sigma_{f_1}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}=\frac{\mu_{v_1}^2\sigma_{f_1}^2}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}\frac{\left(s_{f_1}^2-\sigma_{f_1}^2\right)}{\sigma_{f_1}^2},$$
$$\frac{\sigma_{f_1}^2\left(\bar{v}_{1N}^2-\mu_{v_1}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}=\frac{\mu_{v_1}^2\sigma_{f_1}^2}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}\frac{\left(\bar{v}_{1N}^2-\mu_{v_1}^2\right)}{\mu_{v_1}^2},$$
$$\frac{N^{2(\alpha_2-\alpha)}\mu_{v_2}^2\left(s_{f_2}^2-\sigma_{f_2}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}=\frac{\mu_{v_2}^2\sigma_{f_2}^2}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}\frac{N^{2(\alpha_2-\alpha)}\left(s_{f_2}^2-\sigma_{f_2}^2\right)}{\sigma_{f_2}^2},\tag{B21}$$
$$\frac{N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\left(\bar{v}_{2N}^2-\mu_{v_2}^2\right)}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}=\frac{\mu_{v_2}^2\sigma_{f_2}^2}{\sigma_{f_1}^2\mu_{v_1}^2+N^{2(\alpha_2-\alpha)}\sigma_{f_2}^2\mu_{v_2}^2}\frac{N^{2(\alpha_2-\alpha)}\left(\bar{v}_{2N}^2-\mu_{v_2}^2\right)}{\mu_{v_2}^2}.\tag{B22}$$

But, it is obvious that the Lemma holds since (B21) and (B22) are $o_p(1)$, when multiplied by $\min\left(\sqrt{T},\sqrt{N^\alpha}\right)$ respectively, as well as $\min\left(\sqrt{T},\sqrt{N^\alpha}\right)N^{\alpha_2-\alpha}\mu_{v_1}\mu_{v_2}s_{12,f}$.

Proof of Lemma 13

We analyse the population counterpart of $\ln\left(\bar{\boldsymbol{v}}_N'\mathbf{D}_N\mathbf{S}_{ff}\mathbf{D}_N\bar{\boldsymbol{v}}_N\right)$ assuming for simplicity that $\boldsymbol{\Sigma}_{ff}$ is diagonal and $\alpha > \alpha_2 \geq \alpha_3 \geq ... \geq \alpha_m$. We have

$$\ln(\boldsymbol{\mu}_v'\mathbf{D}_N\boldsymbol{\Sigma}_{ff}\mathbf{D}_N\boldsymbol{\mu}_v) = \ln\left(\mu_{v_1}^2\sigma_{11,f} + N^{2(\alpha_2-\alpha)}\sum_{j=2}^{m} N^{2(\alpha_j-\alpha_2)}\mu_{v_j}^2\sigma_{f_j}^2\right).$$

Then,

$$\ln(\boldsymbol{\mu}_v'\mathbf{D}_N\boldsymbol{\Sigma}_{ff}\mathbf{D}_N\boldsymbol{\mu}_v) - \ln\left(\mu_{v_1}^2\sigma_{f_1}^2\right) = \ln\left(1 + \frac{N^{2(\alpha_2-\alpha)}\sum_{j=2}^{m}\left(N^{2(\alpha_j-\alpha_2)}\mu_{v_j}^2\sigma_{f_j}^2\right)}{\mu_{v_1}^2\sigma_{f_1}^2}\right) = \frac{\sum_{j=2}^{m}\left(N^{2(\alpha_j-\alpha_2)}\mu_{v_j}^2\sigma_{f_j}^2\right)}{\mu_{v_1}^2\sigma_{f_1}^2}N^{2(\alpha_2-\alpha)}.$$

So,

$$\sqrt{\min(N^\alpha, T)}\ln(N)\left(\ln(\bar{\boldsymbol{v}}_N'\mathbf{D}_N\mathbf{S}_{ff}\mathbf{D}_N\bar{\boldsymbol{v}}_N) - \ln(\boldsymbol{\mu}_v'\mathbf{D}_N\boldsymbol{\Sigma}_{ff}\mathbf{D}_N\boldsymbol{\mu}_v)\right) =$$

$$\sqrt{\min(N^\alpha, T)}\ln(N)\left[\ln(\bar{\boldsymbol{v}}_N'\mathbf{D}_N\mathbf{S}_{ff}\mathbf{D}_N\bar{\boldsymbol{v}}_N) - \ln(\mu_{v_1}^2\sigma_{f_1}^2)\right] - \sqrt{\min(N^\alpha, T)}\ln(N)\,N^{2(\alpha_2-\alpha)}\left(\frac{\sum_{j=2}^{m}\left(N^{2(\alpha_j-\alpha_2)}\mu_{v_j}^2\sigma_{f_j}^2\right)}{\mu_{v_1}^2\sigma_{f_1}^2}\right).$$

We need

$$N^{2(\alpha_2-\alpha)}\left(\frac{\sum_{j=2}^{m}\left(N^{2(\alpha_j-\alpha_2)}\mu_{v_j}^2\sigma_{f_j}^2\right)}{\mu_{v_1}^2\sigma_{f_1}^2}\right) = o\left(\min\left(N^\alpha, T\right)^{-1/2}\ln(N)^{-1}\right).$$

This holds if $\sqrt{\min(N^a, T)}N^{2(\alpha_2-\alpha)} = o(1)$. If $T < N^\alpha$ then a sufficient condition for the above to hold is $\alpha_2 - \alpha < -0.25$. Otherwise, the sufficient condition is $\alpha_2 < 3\alpha/4$. But, this condition is implied by $\alpha_2 - \alpha < -0.25$ as long as $\alpha \leq 1$. An alternative condition that relates to the relative rate of growth of N and T is that $\alpha_2 < 3\alpha/4$ and $T^b = N$ and $1/(4b) + \alpha_2 - \alpha < 0$ or $b > \frac{1}{4(\alpha-\alpha_2)}$

Proof of Lemma 14

We note that the first part of the Lemma holds if

$$\frac{\ln(s_{f_1}^2\bar{v}_{N1}^2)}{\ln(N)} = o_p(1). \tag{B23}$$

We have

$$\ln(s_{f_1}^2\bar{v}_{N1}^2) = \ln\left(s_{f_1}^2\right) + 2\ln\left(\bar{v}_{N1}\right) = \ln\left(s_{f_1}^2\right) + 2\ln\left(\frac{1}{N}\sum_{i=1}^{N}\breve{v}_i + \bar{c}_N\right).$$

So (B23) holds if $\frac{1}{N}\sum_{i=1}^{N}\breve{v}_i + \bar{c}_N = o_p\left(N^c\right)$ for all $c > 0$, which holds if $\bar{c}_N = o_p\left(N^c\right)$ for all $c > 0$, proving the first part of the Lemma. For the second part of the Lemma, we reconsider (B16). We have $\sqrt{N}\left(\bar{v}_{N1} - \mu_{\breve{v}}\right) = \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N}\breve{v}_i + \bar{c}_N - \mu_{\breve{v}}\right)$. But, $\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N}\breve{v}_i - \mu_{\breve{v}}\right) \to_d N\left(0, \sigma_{\breve{v}}^2\right)$. Therefore, $\sqrt{N}\bar{c}_N = o_p(1)$ is sufficient for the second part of the Lemma to hold.

Proof of Lemma 15

We have that

$$\frac{N^\alpha - N^{\hat{\alpha}}}{N^\alpha} = 1 - \frac{N^{\hat{\alpha}}}{N^\alpha} = \ln\left(\frac{N^{\hat{\alpha}}}{N^\alpha}\right) + o_p\left(\ln\left(\frac{N^{\hat{\alpha}}}{N^\alpha}\right)\right).$$

Then,

$$\ln\left(\frac{N^{\hat{\alpha}}}{N^\alpha}\right) = (\hat{\alpha} - \alpha)\ln N,$$

implying the result of the Lemma.

Proof of Lemma 16

The factor loadings of the cross-sectional units are partitioned into two groups by Assumption 1 and setting $m = 1$. The first group has non-zero loadings, denoted by v_{i1}, while the second group has loadings that are summable over the group. We do not observe the partition and need to estimate it. For this reason, we rank the estimated loadings as discussed in the statement of the Lemma. The first step in the proof is to show that the number of cross-sectional units that are misclassified, i.e., that are included in the variance calculation when their loading is not a function of any v_{i1}, is $o_p(N^\alpha)$. The first thing to note is that we abstract from the possibility that any $v_{i1} = 0$. By the fact that $\Pr(v_{i1} = 0) = 0$, it follows that the number of units with $v_{i1} = 0$ is $o_p(N^\alpha)$. Without loss of generality, we further assume that units whose loadings do not depend on any v_{i1} have zero loadings. There are two sources of errors in partitioning the loadings. The first arises because $N^{\hat{a}}$ is not equal to N^a. But by Lemma 15 this error is $o_p(N^\alpha)$ if $\hat{\alpha} - \alpha = o_p\left((\ln N)^{-1}\right)$ which is the case under the conditions of the Lemma. The fact that $\hat{\alpha} - \alpha = o_p\left((\ln N)^{-1}\right)$ justifies using the true α rather than the estimated one throughout the rest of the proof. The second source of error arises from the possibility that units are missclassified. We consider this source next assuming the true value of α is used. We know that the probability that any unit's coefficient is $\epsilon > 0$ away from its true value is of the order of N^{-a} (by Lemma (17) and the Markov inequality). We know that N^a units can be misclassified only if the estimated coefficients of any unordered and without replacement, sample of size N^a from the N units, jointly exceed their true value by ϵ. We know that since the v_{i1} are independent, that the event that an estimated coefficient will be away from its true value will be independent from the same event for another unit. So, the probability that a given set of N^a units can be jointly misclassified, is bounded from above by N^{-aN^a}. There are $\frac{N!}{N^a!(N-N^a)!}$ such sets. So, the probability that any set will behave thus, is bounded from above by $\frac{N^{-aN^a}N!}{N^a!(N-N^a)!}$. We need to aggregate across $i = N^a, ..., N$. So overall the probability is bounded from above by $\sum_{b>a} \frac{N^{-aN^b}N!}{N^b!\left(N-N^b\right)!}$. We replace this by $(N - N^a)\frac{N^{-aN^a}N!}{N^a!(N-N^a)!}$ and justify this step below. We have

$$(N - N^a)\frac{N^{-aN^a}N!}{N^a!\,(N - N^a)!} = \frac{N^{-aN^a}N!}{N^a!\,(N - N^a - 1)!}. \tag{B24}$$

We need the logarithm of the above quantity to have a limit of $-\infty$. We have using repeatedly Stirling's formula that ($\sim$ denotes equality up to an order of magnitude lower that any included terms)

$$\ln\left(\frac{N^{-aN^a}N!}{N^a!\,(N - N^a - 1)!}\right) = \ln\left(N^{-aN^a}N!\right) - \ln\left(N^a!\,(N - N^a - 1)!\right) =$$
$$-aN^a\ln(N) + \ln(N!) - \ln(N^a!) - \ln((N - N^a - 1)!) \sim$$
$$-aN^a\ln(N) + N\ln(N) - N - N^a\ln(N^a) + N^a - (N - N^a - 1)\ln(N - N^a - 1) + (N - N^a - 1) \sim$$
$$-aN^a\ln(N) + N\ln(N) - N - aN^a\ln(N) + N^a - (N - N^a - 1)\ln(N - N^a - 1) + (N - N^a - 1) =$$
$$-aN^a\ln(N) + N\ln(N) - N - aN^a\ln(N) + N^a - (N - N^a - 1)\ln\left(N(1 - N^{a-1} - N^{-1})\right) + (N - N^a - 1) =$$
$$-aN^a\ln(N) + N\ln(N) - N - aN^a\ln(N) + N^a - (N - N^a - 1)\ln(N) -$$
$$-(N - N^a - 1)\ln\left(1 - N^{a-1} - N^{-1}\right) + (N - N^a - 1) =$$
$$-aN^a\ln(N) - aN^a\ln(N) + N^a\ln(N) + N\ln(N) - N + N^a - N\ln(N) + \ln(N) -$$
$$+N^a + 1 - N^{2a-1} - N^{a-1} - N^{a-1} - N^{-1} + (N - N^a - 1) =$$
$$-(2a-1)N^a\ln(N) - N + N^a + \ln(N) -$$
$$-(N - N^a - 1)\left(-N^{a-1} - N^{-1}\right) + (N - N^a - 1).$$

The term $-(2a-1)N^a\ln(N)$ dominates other terms and tends to $-\infty$, as $N \to \infty$, for $a > 1/2$, proving the result. We now justify replacing $(N - N^a)\frac{N^{-aN^a}N!}{N^a!(N-N^a)!}$ for $\sum_{b>a}\frac{N^{-aN^b}N!}{N^b!\left(N-N^b\right)!}$ in (B24). We have

$$\ln\left(\frac{N^{-aN^b}N!}{N^b!\,(N - N^b)!}\right) = \ln\left(N^{-aN^b}N!\right) - \ln\left(N^b!\left(N - N^b\right)!\right) =$$

$$-aN^b \ln(N) + \ln(N!) - \ln\left(N^b!\right) - \ln\left(\left(N - N^b - 1\right)!\right) \sim$$

$$-aN^b \ln(N) + N\ln(N) - N - N^b \ln\left(N^b\right) + N^b - \left(N - N^b - 1\right)\ln\left(N - N^b - 1\right) + \left(N - N^b - 1\right) \sim$$

$$-aN^b \ln(N) + N\ln(N) - N - bN^b \ln(N) + N^b - \left(N - N^b - 1\right)\ln\left(N - N^b - 1\right) + \left(N - N^b - 1\right) \sim$$

$$-aN^b \ln(N) + N\ln(N) - N - bN^b \ln(N) + N^b - \left(N - N^b - 1\right)\ln\left(N(1 - N^{b-1} - N^{-1})\right) + \left(N - N^b - 1\right) =$$

$$-aN^b \ln(N) + N\ln(N) - N - bN^b \ln(N) + N^b - \left(N - N^b - 1\right)\ln(N) -$$

$$-\left(N - N^b - 1\right)\ln\left(1 - N^{b-1} - N^{-1}\right) + \left(N - N^b - 1\right).$$

The dominant term here is $-(a+b-1)N^a \ln(N)$ which for $b > a > 1/2$ is tending to $-\infty$ faster than $-(2a-1)N^a \ln(N)$ justifying the replacement.

Next, we prove the Lemma assuming that we observe which units have non-zero loadings. Recall that, assuming that units whose loadings do not depend on any v_{i1} have zero loadings, $x_{it} = \frac{v_{i1}}{\bar{v}_{1N}}\left(N^{1-\alpha}\bar{x}_t\right) + u_{it}$. We analyse $\hat{v}_{i1}$ by a slight abuse of notation whereby we define it to be the estimated regression coefficient of the regression of x_{it} on $N^{1-\alpha}\bar{x}_t$ rather than x_{it} on $\tilde{x}_t$. Since μ_{v_1} is assumed known, $\bar{v}_{1N} \to_p \mu_{v_1}$ and $\hat{\sigma}^2_{\bar{x}} \to_p N^{2\alpha-2}\mu^2_{v_1}$, by Lemma 10, this does not affect the analysis. Let $v^{(1)}_{i1} = \frac{v_{i1}}{\mu_{v_1}}$ and $v_{iN} = \frac{v_{i1}}{\bar{v}_{1N}}$. We need to show that $\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(\hat{v}_{i1} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}\hat{v}_{j1}\right)^2 - \frac{\sigma^2_{v_1}}{\mu^2_{v_1}} = o_p(1)$. We have

$$\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(\hat{v}_{i1} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}\hat{v}_{j1}\right)^2 - \frac{\sigma^2_{v_1}}{\mu^2_{v_1}} = \frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(\hat{v}_{i1} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}\hat{v}_{j1}\right)^2 - \frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(v_{iN} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}v_{jN}\right)^2 +$$

$$\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(v_{iN} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}v_{jN}\right)^2 - \frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(v^{(1)}_{i1} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}v^{(1)}_{j1}\right)^2 + \frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(v^{(1)}_{i1} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}v^{(1)}_{j1}\right)^2 - \frac{\sigma^2_{v_1}}{\mu^2_{v_1}}.$$

But by the law of large numbers for i.i.d. random variables with finite variance

$$\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(v^{(1)}_{i1} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}v^{(1)}_{j1}\right)^2 - \frac{\sigma^2_{v_1}}{\mu^2_{v_1}} = O_p\left(N^{-1/2}\right).$$

It is sufficient to show that

$$\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(\hat{v}_{i1} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}\hat{v}_{j1}\right)^2 - \frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(v_{iN} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}v_{jN}\right)^2 = o_p(1) \tag{B25}$$

and

$$\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(v_{iN} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}v_{jN}\right)^2 - \frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(v^{(1)}_{i1} - \frac{1}{N^\alpha}\sum_{j=1}^{N^\alpha}v^{(1)}_{j1}\right)^2 = o_p(1). \tag{B26}$$

For (B25), it is sufficient that $\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}(\hat{v}_{i1} - v_{iN}) = o_p(1)$. Recall that $x_{it} = \frac{v_{i1}}{\bar{v}_{1N}}\left(N^{1-\alpha}\bar{x}_t\right) + u_{it}$. So

$$\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}(\hat{v}_{i1} - v_{iN}) = \frac{\frac{1}{N^\alpha-1}\sum_{i=1}^{N^\alpha}\left(\sum_{t=1}^T \bar{x}_t u_{it}\right)}{\sum_{t=1}^T \bar{x}_t^2} = \frac{1}{(N^\alpha-1)\,\bar{v}_{1N}\left(\sum_{t=1}^T f_{1t}^2\right)}\sum_{i=1}^{N^\alpha}\sum_{t=1}^T f_{1t}u_{it}.$$

But $\sum_{i=1}^{N^\alpha}\sum_{t=1}^T f_{1t}u_{it} = O_p\left((N^\alpha T)^{1/2}\right)$ and $\sum_{t=1}^T f_{1t}^2 = O_p(T)$. So

$$\frac{\sum_{i=1}^{N^\alpha}\left(\sum_{t=1}^T \bar{x}_t u_{it}\right)}{N^\alpha - 1\left(\sum_{t=1}^T \bar{x}_t^2\right)} = O_p\left(T^{-1}N^{-\alpha}(N^\alpha T)^{1/2}\right) = O_p\left(T^{-1/2}N^{-\alpha/2}\right) = o_p(1).$$

For (B26), it is sufficient that $\frac{1}{N^{\alpha}-1}\sum_{i=1}^{N^{\alpha}}\left(v_{iN}-v_{i1}^{(1)}\right)=o_{p}\left(1\right)$. We have,

$$\frac{1}{N^{\alpha}-1}\sum_{i=1}^{N^{\alpha}}\left(v_{iN}-v_{i1}^{(1)}\right)=\frac{\left(\frac{1}{\bar{v}_{1N}}-\frac{1}{\mu_{v_1}}\right)\sum_{i=1}^{N^{\alpha}}v_{i1}}{N^{\alpha}-1}.$$

But $\frac{1}{N^{\alpha}-1}\sum_{i=1}^{N^{\alpha}}v_{i1}=O_{p}\left(1\right)$. Also $\frac{1}{\bar{v}_{1N}}-\frac{1}{\mu_{v_1}}=\frac{1}{\bar{v}_{1N}\mu_{v_1}}\left(\bar{v}_{1N}-\mu_{v_1}\right)=O_{p}\left(N^{-\alpha/2}\right)$. So, overall $\frac{1}{N-1}\sum_{i=1}^{N^{\alpha}}\left(\hat{v}_{i1}-\frac{1}{N}\sum_{j=1}^{N}\hat{v}_{j1}\right)^{2}-\frac{\sigma_{v_1}^{2}}{\mu_{v_1}^{2}}=o_{p}\left(1\right)$, proving the required result.

Proof of Lemma 17

Recall that $\beta_{i1}=v_{i1}$ for $i=1,2,...,N^{\alpha}$ and 0 for $i=N^{\alpha}+1,...,N$ (without loss of generality). Here we set $m=1$. Let

$$\tilde{x}_{t}=\frac{1}{N^{\alpha}}\sum_{i=1}^{N}x_{it},$$

where we have used the normalisation $N^{-\alpha}$ to ensure that $\bar{x}_{t}$ converges to f_{1t}. We have

$$\hat{\beta}_{i1}=\frac{\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{t}x_{it}}{\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{t}^{2}}. \tag{B27}$$

Then,

$$\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{t}x_{it}=\frac{1}{TN^{\alpha}}\sum_{t=1}^{T}\sum_{j=1}^{N}x_{jt}x_{it}=$$

$$\frac{1}{T}\sum_{t=1}^{T}\frac{1}{N^{\alpha}}\sum_{j=1}^{N}\left(\beta_{j1}f_{1t}+u_{jt}\right)\left(\beta_{i1}f_{1t}+u_{it}\right)=$$

$$\frac{1}{T}\sum_{t=1}^{T}\frac{1}{N^{\alpha}}\sum_{j=1}^{N}\left(\beta_{j1}\beta_{i1}f_{1t}^{2}+2\beta_{i1}\beta_{j1}f_{1t}u_{it}+u_{jt}u_{it}\right).$$

We have

$$\frac{1}{TN^{\alpha}}\sum_{t=1}^{T}\sum_{j=1}^{N}\beta_{j1}\beta_{i1}f_{1t}^{2}=\left(\frac{1}{T}\sum_{t=1}^{T}f_{1t}^{2}\right)\left(\frac{1}{N^{\alpha}}\sum_{j=1}^{N^{\alpha}}v_{j1}\beta_{i1}\right).$$

But,

$$\frac{1}{T}\sum_{t=1}^{T}f_{1t}^{2}\rightarrow_{p}1,\ \frac{\beta_{i1}}{N^{\alpha}}\sum_{j=1}^{N^{\alpha}}v_{j1}\rightarrow_{p}\beta_{i1}\mu_{v_1}.$$

Next,

$$\frac{1}{TN^{\alpha}}\sum_{t=1}^{T}u_{it}\sum_{j=1}^{N}u_{jt}=O_{p}\left(\frac{1}{T^{1/2}N^{\alpha-1/2}}\right),$$

and

$$\frac{2}{TN^{\alpha}}\sum_{t=1}^{T}\sum_{j=1}^{N}\beta_{i1}\beta_{j1}f_{1t}u_{it}=\frac{2}{T}\sum_{t=1}^{T}f_{1t}\beta_{i1}u_{it}\left(\left(\frac{1}{N^{\alpha}}\sum_{j=1}^{N^{\alpha}}v_{j1}\right)\right)=\left\{\begin{array}{c}O_{p}\left(\frac{1}{T^{1/2}}\right)\text{ if }i\leq N^{\alpha}\\ 0\ \text{ otherwise}\end{array}\right..$$

This concludes the analysis of the numerator of (B27). For the denominator we have,

$$\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{t}^{2}=\frac{1}{TN^{2\alpha}}\sum_{t=1}^{T}\sum_{j=1}^{N}\sum_{i=1}^{N}x_{jt}x_{it}=$$

$$\frac{1}{T}\sum_{t=1}^{T}\frac{1}{N^{2\alpha}}\sum_{j=1}^{N}\sum_{i=1}^{N}\left(\beta_{j1}f_{1t}+u_{jt}\right)\left(\beta_{i1}f_{1t}+u_{it}\right)=$$

$$\frac{1}{T}\sum_{t=1}^{T}\frac{1}{N^{2\alpha}}\sum_{j=1}^{N}\sum_{i=1}^{N}\left(\beta_{j1}\beta_{i1}f_{1t}^{2}+2\beta_{i1}\beta_{j1}f_{1t}u_{it}+u_{jt}u_{it}\right).$$

We have

$$\frac{1}{TN^{2\alpha}}\sum_{t=1}^{T}\sum_{j=1}^{N}\sum_{i=1}^{N}\beta_{j1}\beta_{i1}f_{1t}^{2}=\left(\frac{1}{T}\sum_{t=1}^{T}f_{1t}^{2}\right)\left(\frac{1}{N^{2\alpha}}\sum_{j=1}^{N^{\alpha}}\sum_{j=1}^{N^{\alpha}}v_{j1}v_{i1}\right).$$

But,

$$\frac{1}{T}\sum_{t=1}^{T}f_{1t}^{2}\to_{p}1,\ \frac{1}{N^{2\alpha}}\sum_{j=1}^{N^{\alpha}}\sum_{j=1}^{N^{\alpha}}v_{j1}v_{i1}\to_{p}\mu_{v_1}^{2},$$

$$\frac{1}{TN^{2\alpha}}\sum_{t=1}^{T}\sum_{j=1}^{N}\sum_{i=1}^{N}u_{it}u_{jt}=O_{p}\left(\frac{1}{T^{1/2}N^{2\alpha-1}}\right),$$

$$\frac{2}{TN^{2\alpha}}\sum_{t=1}^{T}\sum_{j=1}^{N}\sum_{i=1}^{N}\beta_{i1}\beta_{j1}f_{1t}u_{it}=$$

$$\frac{2}{T}\sum_{t=1}^{T}\left\{f_{1t}\left[\frac{1}{N^{\alpha}}\sum_{i=1}^{N^{\alpha}}v_{i1}u_{it}\left(\frac{1}{N^{\alpha}}\sum_{j=1}^{N^{\alpha}}v_{j1}\right)\right]\right\}=O_{p}\left(\frac{1}{T^{1/2}N^{\alpha/2}}\right).$$

Therefore,

$$\hat{\beta}_{i1}\to_{p}\frac{\beta_{i1}}{\mu_{v_1}}.$$

Next, we need to establish the rate at which $\frac{\frac{1}{T}\sum_{t=1}^{T}\bar{x}_{t}x_{it}}{\frac{1}{T}\sum_{t=1}^{T}\bar{x}_{t}^{2}}-\frac{\beta_{i1}\mu_{v_1}}{\mu_{v_1}^{2}}$ tends to zero. This is determined by the maximum of two rates:

$$\frac{1}{T}\sum_{t=1}^{T}\left(f_{1t}^{2}-1\right)=O_{p}\left(T^{-1/2}\right),$$

$$\frac{\beta_{i1}}{N^{\alpha}}\sum_{j=1}^{N^{\alpha}}v_{j1}-\beta_{i1}\mu_{v_1}=\left\{\begin{array}{c}O_{p}\left(\frac{1}{N^{\alpha/2}}\right)\text{ if }i\leq N^{\alpha}\\ 0\text{ otherwise}\end{array}\right.,$$

noting that

$$\frac{1}{N^{2\alpha}}\sum_{j=1}^{N^{\alpha}}\sum_{j=1}^{N^{\alpha}}\left(v_{j1}v_{i1}-\mu_{v_1}^{2}\right)=O_{p}\left(\frac{1}{N^{\alpha}}\right),$$

and

$$\frac{1}{TN^{2\alpha}}\sum_{t=1}^{T}\sum_{j=1}^{N}\sum_{i=1}^{N}u_{it}u_{jt}=O_{p}\left(\frac{1}{T^{1/2}N^{2\alpha-1}}\right).$$

Hence

$$\hat{\beta}_{i1}-\frac{\beta_{i1}}{\mu_{v_1}}=O_{p}\left(\frac{1}{N^{\alpha/2}}\right)+O_{p}\left(\frac{1}{T^{1/2}}\right).$$

Proof of Lemma 18

We need to show $\hat{V}_{\overline{f_1^2}}-V_{\overline{f_1^2}}=o_p(1)$ assuming a one factor setting (without loss of generality). The result extends straightforwardly to m factors. We have $\hat{V}_{\overline{f_1^2}}-V_{\overline{f_1^2}}=\hat{V}_{\overline{f_1^2}}-\bar{V}_{\overline{f_1^2}}+\bar{V}_{\overline{f_1^2}}-V_{\overline{f_1^2}}$ where

$$\bar{V}_{\overline{f_1^2}}=\frac{1}{T}\sum_{t=1}^{T}\left(q_{t}-\frac{1}{T}\sum_{t=1}^{T}q_{t}\right)^{2}+\sum_{j=1}^{l}\left(\frac{1}{T}\sum_{t=j+1}^{T}\left(q_{t-j}-\frac{1}{T}\sum_{t=1}^{T}q_{t}\right)\left(q_{t}-\frac{1}{T}\sum_{t=1}^{T}q_{t}\right)\right),$$

and $q_t = \left(\frac{f_{1t}-\bar{f}_1}{s_{f_1}}\right)^2$. But, by Theorem 25.3 of Davidson (1994) and Assumption 3, we have that $\bar{V}_{\overline{f_1^2}} - V_{\overline{f_1^2}} = o_p(1)$, as long as $l \to \infty$ and $l = o(T)$. Then, it is sufficient to examine

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{f_{1t}}{s_{f_1}} - \tilde{x}_t\right) = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{f_{1t}}{s_{f_1}} - \frac{\bar{x}_t}{\hat{\sigma}_{\bar{x}}}\right) = \frac{1}{s_{f_1}T}\sum_{t=1}^{T}\left(f_{1t} - \frac{\bar{x}_t}{N^{\alpha-1}\bar{v}_{1N}}\right) =$$

$$\frac{1}{s_{f_1}T}\sum_{t=1}^{T}\left(f_{1t} - \frac{N^{\alpha-1}\bar{v}_{1N}f_{1t} + \frac{1}{N}\sum_{i=1}^{N}u_{it}}{N^{\alpha-1}\bar{v}_{1N}}\right) = \frac{N^{-\alpha}}{s_{f_1}T}\sum_{t=1}^{T}\sum_{i=1}^{N}u_{it} = \frac{N^{-\alpha}}{\sigma_{f_1}T}\sum_{t=1}^{T}\sum_{i=1}^{N}u_{it} + o_p\left(\frac{N^{-\alpha}}{\sigma_{f_1}T}\sum_{t=1}^{T}\sum_{i=1}^{N}u_{it}\right).$$

But, $\sum_{t=1}^{T}\sum_{i=1}^{N}u_{it} = O_p\left((NT)^{1/2}\right)$. So, $\frac{1}{T}\sum_{t=1}^{T}\left(\frac{f_{1t}}{s_{f_1}} - \tilde{x}_t\right) = O_p\left(N^{1/2-\alpha}T^{-1/2}\right)$. Thus, $\hat{V}_{\overline{f_1^2}} - V_{\overline{f_1^2}} = O_p\left(lN^{1/2-\alpha}T^{-1/2}\right)$, proving the Lemma.

Supplementary Appendix IV: Justification of the use of the cumulative distribution function of the standard normal in the approach used to estimate μ_{v_1}

Consider the single factor model,

$$x_{it} = \beta_{i1}f_{1t} + u_{it}, \text{ for } i = 1, ..., N;\ t = 1, ..., T, \tag{B28}$$

and assume that $\bar{\beta}_{1N} = \frac{1}{N}\sum_{i=1}^{N}\beta_{i1} \neq 0$ for a finite N. Recall that $\beta_{i1} = \nu_{i1}$, for $i = 1, ..., N^{\alpha}$ and zero for $i = N^{\alpha}+1, ..., N$ (without loss of generality), so that

$$\bar{\beta}_{1N} = N^{\alpha-1}\bar{v}_{1N}, \text{ with } \bar{v}_{1N} = N^{-\alpha}\sum_{i=1}^{N^{\alpha}}v_{i1}.$$

Also letting $\bar{x}_t = \frac{1}{N}\sum_{i=1}^{N}x_{it}$, $\delta_i = \beta_{i1}/\bar{\beta}_{1N}$ and noting that $\bar{x}_t = \bar{\beta}_{1N}f_{1t} + \bar{u}_t$, we have

$$x_{it} = \delta_i\bar{x}_t + \xi_{it}, \text{ where } \xi_{it} = u_{it} - \delta_i\bar{u}_t. \tag{B29}$$

Consider now the t-ratio for testing $\delta_i = 0$ in the above regression and note that it is given by

$$z_i = z_{i,T,N} = \frac{\hat{\delta}_i}{\left(\sum_{t=1}^{T}\bar{x}_t^2\right)^{-1/2}\hat{\sigma}_{\xi i}} = \frac{\sum_{t=1}^{T}\bar{x}_t x_{it}}{\left(\sum_{t=1}^{T}\bar{x}_t^2\right)^{1/2}\hat{\sigma}_{\xi i}}, \tag{B30}$$

where

$$\hat{\sigma}_{\xi i}^2 = T^{-1}\sum_{t=1}^{T}(x_{it} - \hat{\delta}_i\bar{x}_t)^2,$$

$$\hat{\delta}_i = \frac{\sum_{t=1}^{T}\bar{x}_t x_{it}}{\sum_{t=1}^{T}\bar{x}_t^2} = \delta_i + \frac{\sum_{t=1}^{T}\bar{x}_t\xi_{it}}{\sum_{t=1}^{T}\bar{x}_t^2}.$$

But

$$\sum_{t=1}^{T}\bar{x}_t x_{it} = \sum_{t=1}^{T}\left(\bar{\beta}_{1N}f_t + \bar{u}_t\right)\left(\beta_{i1}f_{1t} + u_{it}\right)$$
$$= \beta_{i1}\bar{\beta}_{1N}\sum_{t=1}^{T}f_{1t}^2 + \beta_{i1}\sum_{t=1}^{T}\bar{u}_t f_{1t} + \bar{\beta}_{1N}\sum_{t=1}^{T}f_{1t}u_{it} + \sum_{t=1}^{T}\bar{u}_t u_{it}$$

and

$$\sum_{t=1}^{T}\bar{x}_t^2 = \bar{\beta}_{1N}^2\sum_{t=1}^{T}f_{1t}^2 + 2\bar{\beta}_{1N}\sum_{t=1}^{T}f_{1t}\bar{u}_t + \sum_{t=1}^{T}\bar{u}_t^2,$$

$$
\begin{aligned}
\hat{\sigma}_{\xi i}^2 &= T^{-1}\sum_{t=1}^{T}\left[x_{it}-\delta_i\bar{x}_t-(\hat{\delta}_i-\delta_i)\bar{x}_t\right]^2 \\
&= T^{-1}\sum_{t=1}^{T}\xi_{it}^2+(\hat{\delta}_i-\delta_i)^2T^{-1}\sum_{t=1}^{T}\bar{x}_t^2-2(\hat{\delta}_i-\delta_i)T^{-1}\sum_{t=1}^{T}\bar{x}_t\xi_{it} \\
&= T^{-1}\sum_{t=1}^{T}\xi_{it}^2+\left(\frac{T^{-1}\sum_{t=1}^{T}\bar{x}_t\xi_{it}}{T^{-1}\sum_{t=1}^{T}\bar{x}_t^2}\right)^2T^{-1}\sum_{t=1}^{T}\bar{x}_t^2-2\frac{T^{-1}\sum_{t=1}^{T}\bar{x}_t\xi_{it}}{T^{-1}\sum_{t=1}^{T}\bar{x}_t^2}T^{-1}\sum_{t=1}^{T}\bar{x}_t\xi_{it} \\
&= T^{-1}\sum_{t=1}^{T}\xi_{it}^2-\frac{\left(T^{-1}\sum_{t=1}^{T}\bar{x}_t\xi_{it}\right)^2}{T^{-1}\sum_{t=1}^{T}\bar{x}_t^2}.
\end{aligned}
$$

Also

$$
\begin{aligned}
z_i &= \frac{\beta_{i1}\bar{\beta}_{1N}\sum_{t=1}^{T}f_{1t}^2+\beta_{i1}\sum_{t=1}^{T}\bar{u}_tf_{1t}+\bar{\beta}_{1N}\sum_{t=1}^{T}f_{1t}u_{it}+\sum_{t=1}^{T}\bar{u}_tu_{it}}{\left(\bar{\beta}_{1N}^2\sum_{t=1}^{T}f_{1t}^2+2\bar{\beta}_{1N}\sum_{t=1}^{T}f_{1t}\bar{u}_t+\sum_{t=1}^{T}\bar{u}_t^2\right)^{1/2}\hat{\sigma}_{\xi i}} \\
&= \frac{\beta_{i1}\sum_{t=1}^{T}f_{1t}^2+\beta_{i1}\sum_{t=1}^{T}(\bar{u}_t/\bar{\beta}_{1N})f_{1t}+\sum_{t=1}^{T}f_{1t}u_{it}+\sum_{t=1}^{T}\left(\bar{u}_t/\bar{\beta}_{1N}\right)u_{it}}{\left[\sum_{t=1}^{T}f_{1t}^2+2\sum_{t=1}^{T}f_{1t}\left(\bar{u}_t/\bar{\beta}_{1N}\right)+\sum_{t=1}^{T}\left(\bar{u}_t/\bar{\beta}_{1N}\right)^2\right]^{1/2}\hat{\sigma}_{\xi i}}.
\end{aligned}
$$

Further, since $\bar{\beta}_{1N}= N^{\alpha-1}\bar{v}_{1N}$, we have $\bar{u}_t/\bar{\beta}_{1N}=N^{1-\alpha}(\bar{u}_t/\bar{v}_{1N})$ and

$$
T^{-1/2}z_i=\frac{\beta_{i1}T^{-1}\sum_{t=1}^{T}f_{1t}^2+T^{-1}\sum_{t=1}^{T}f_{1t}u_{it}+(\beta_{i1}/\bar{v}_{1N})\,N^{1-\alpha}T^{-1}\sum_{t=1}^{T}\bar{u}_tf_{1t}+(1/\bar{v}_{1N})\,N^{1-\alpha}T^{-1}\sum_{t=1}^{T}\bar{u}_tu_{it}}{\left[T^{-1}\sum_{t=1}^{T}f_{1t}^2+2\,(1/\bar{v}_{1N})\,N^{1-\alpha}T^{-1}\sum_{t=1}^{T}f_{1t}\bar{u}_t+(1/\bar{v}_{1N})^2\,N^{2(1-\alpha)}T^{-1}\sum_{t=1}^{T}\bar{u}_t^2\right]^{1/2}\hat{\sigma}_{\xi i}},
$$

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T}\xi_{it}^2 &= T^{-1}\sum_{t=1}^{T}u_{it}^2+\delta_i^2T^{-1}\sum_{t=1}^{T}\bar{u}_t^2-2\delta_iT^{-1}\sum_{t=1}^{T}\bar{u}_tu_{it} \\
&= T^{-1}\sum_{t=1}^{T}u_{it}^2+\beta_{i1}^2\,(1/\bar{v}_{1N})^2\,N^{2(1-\alpha)}T^{-1}\sum_{t=1}^{T}\bar{u}_t^2-2\beta_{i1}\,(1/\bar{v}_{1N})\,N^{1-\alpha}T^{-1}\sum_{t=1}^{T}\bar{u}_tu_{it}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\left(T^{-1}\sum_{t=1}^{T}\bar{x}_t\xi_{it}\right)^2}{T^{-1}\sum_{t=1}^{T}\bar{x}_t^2} &= \frac{\left(T^{-1}\sum_{t=1}^{T}\bar{x}_t(u_{it}-\beta_{i1}\left(\bar{u}_t/\bar{\beta}_{1N}\right)\right)^2}{\bar{\beta}_{1N}^2T^{-1}\sum_{t=1}^{T}f_{1t}^2+2\bar{\beta}_{1N}T^{-1}\sum_{t=1}^{T}f_{1t}\bar{u}_t+T^{-1}\sum_{t=1}^{T}\bar{u}_t^2} \\
&= \frac{\left(\left[T^{-1}\sum_{t=1}^{T}f_{1t}+\left(\bar{u}_t/\bar{\beta}_{1N}\right)\right]\left[u_{it}-\beta_{i1}\left(\bar{u}_t/\bar{\beta}_{1N}\right)\right]\right)^2}{T^{-1}\sum_{t=1}^{T}f_{1t}^2+2T^{-1}\sum_{t=1}^{T}f_{1t}\left(\bar{u}_t/\bar{\beta}_{1N}\right)+T^{-1}\sum_{t=1}^{T}\left(\bar{u}_t/\bar{\beta}_{1N}\right)^2} \\
&= \frac{\left(\left[T^{-1}\sum_{t=1}^{T}f_{1t}+\left(\bar{u}_t/\bar{\beta}_{1N}\right)\right]\left[u_{it}-\beta_{i1}\left(\bar{u}_t/\bar{\beta}_{1N}\right)\right]\right)^2}{T^{-1}\sum_{t=1}^{T}f_{1t}^2+2\,(1/\bar{v}_{1N})\,N^{1-\alpha}T^{-1}\sum_{t=1}^{T}f_{1t}\bar{u}_t+(1/\bar{v}_{1N})^2\,N^{2(1-\alpha)}T^{-1}\sum_{t=1}^{T}\bar{u}_t^2}.
\end{aligned}
$$

But we have

$$
T^{-1}\sum_{t=1}^{T}\left[f_{1t}+\left(\bar{u}_t/\bar{\beta}_{1N}\right)\right]\left[u_{it}-\beta_{i1}\left(\bar{u}_t/\bar{\beta}_{1N}\right)\right]=
$$

$$
T^{-1}\sum_{t=1}^{T}f_{1t}u_{it}-\beta_{i1}T^{-1}\sum_{t=1}^{T}f_{1t}\left(\bar{u}_t/\bar{\beta}_{1N}\right)-\beta_{i1}T^{-1}\sum_{t=1}^{T}\left(\bar{u}_t/\bar{\beta}_{1N}\right)^2+T^{-1}\sum_{t=1}^{T}\left(\bar{u}_t/\bar{\beta}_{1N}\right)u_{it}
$$

$$
=T^{-1}\sum_{t=1}^{T}f_{1t}u_{it}-(\beta_{i1}/\bar{v}_{1N})\,N^{1-\alpha}T^{-1}\sum_{t=1}^{T}f_{1t}\bar{u}_t-\beta_{i1}\,(1/\bar{v}_{1N})^2\,N^{2(1-\alpha)}T^{-1}\sum_{t=1}^{T}\bar{u}_t^2+(1/\bar{v}_{1N})\,N^{1-\alpha}T^{-1}\sum_{t=1}^{T}\bar{u}_tu_{it},
$$

$$N^{2(1-\alpha)}T^{-1}\sum_{t=1}^{T}\bar{u}_t^2 = O_p\left(N^{1-2\alpha}\right),\ N^{1-\alpha}T^{-1}\sum_{t=1}^{T}\bar{u}_t u_{it} = O_p\left(N^{1/2-\alpha}T^{-1/2}\right), \tag{B31}$$

(B32)

$$N^{1-\alpha}T^{-1}\sum_{t=1}^{T}f_{1t}\bar{u}_t = O_p\left(N^{1/2-\alpha}T^{-1/2}\right),\ T^{-1}\sum_{t=1}^{T}f_{1t}u_{it} = O_p\left(T^{-1/2}\right).$$

(B33)

Hence,

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\xi_{it}^2 &= T^{-1}\sum_{t=1}^{T}u_{it}^2 + \beta_{i1}^2\left(1/\bar{v}_{1N}\right)^2 N^{2(1-\alpha)}T^{-1}\sum_{t=1}^{T}\bar{u}_t^2 - 2\beta_{i1}\left(1/\bar{v}_{1N}\right)N^{1-\alpha}T^{-1}\sum_{t=1}^{T}\bar{u}_t u_{it} \\ &= \sigma_i^2 + O_p\left(T^{-1/2}\right) + O_p\left(N^{1/2-\alpha}T^{-1/2}\right) + O_p\left(N^{1-2\alpha}\right) \end{aligned}$$

$$T^{-1}\sum_{t=1}^{T}\left(f_{1t} + \left(\bar{u}_t/\bar{\beta}_{1N}\right)\right)\left[u_{it} - \beta_{i1}\left(\bar{u}_t/\bar{\beta}_{1N}\right)\right] = O_p\left(T^{-1/2}\right) + O_p\left(N^{1-2\alpha}\right) + O_p\left(N^{1/2-\alpha}T^{-1/2}\right).$$

$$\begin{aligned} \hat{\sigma}_{\xi i}^2 &= T^{-1}\sum_{t=1}^{T}\xi_{it}^2 - \frac{\left(T^{-1}\sum_{t=1}^{T}\bar{x}_t\xi_{it}\right)^2}{T^{-1}\sum_{t=1}^{T}\bar{x}_t^2} \\ &= \sigma_i^2 + O_p\left(T^{-1/2}\right) + O_p\left(N^{1/2-\alpha}T^{-1/2}\right) + O_p\left(N^{1-2\alpha}\right). \end{aligned}$$

Using the above results we now have

$$\begin{aligned} T^{-1/2}z_i &= \frac{T^{-1}\sum_{t=1}^{T}f_{1t}u_{it} + O_p\left(N^{1/2-\alpha}T^{-1/2}\right)}{\left[T^{-1}\sum_{t=1}^{T}f_{1t}^2 + O_p\left(N^{1/2-\alpha}T^{-1/2}\right) + O_p\left(N^{1-2\alpha}\right)\right]^{1/2}\hat{\sigma}_{\xi i}},\ \text{if } \beta_{i1} = 0 \\ &= \frac{T^{-1}\sum_{t=1}^{T}f_{1t}(u_{it}/\sigma_i)}{\left(T^{-1}\sum_{t=1}^{T}f_{1t}^2\right)^{1/2}} + O_p\left(N^{1/2-\alpha}T^{-1/2}\right) + O_p\left(N^{1-2\alpha}\right). \end{aligned} \tag{B34}$$

Therefore, under $\beta_{i1} = 0$, z_i is asymptotically distributed as $N(0,1)$ so long as N and T tend to infinity in any order and $\alpha > 1/2$. Also,

$$T^{-1/2}z_i = \frac{\beta_{i1}T^{-1}\sum_{t=1}^{T}f_{1t}^2 + T^{-1}\sum_{t=1}^{T}f_{1t}u_{it} + \left(\beta_{i1}/\bar{v}_{1N}\right)N^{1-\alpha}T^{-1}\sum_{t=1}^{T}\bar{u}_t f_{1t} + \left(1/\bar{v}_{1N}\right)N^{1-\alpha}T^{-1}\sum_{t=1}^{T}\bar{u}_t u_{it}}{\left[T^{-1}\sum_{t=1}^{T}f_{1t}^2 + O_p\left(N^{1/2-\alpha}T^{-1/2}\right) + O_p\left(N^{1-2\alpha}\right)\right]^{1/2}\hat{\sigma}_{\xi i}},$$

if $\beta_{i1} \neq 0$,

$$\begin{aligned} &= \left(\frac{\beta_{i1}}{\sigma_i}\right)\left(T^{-1}\sum_{t=1}^{T}f_{1t}^2\right)^{1/2} + \frac{T^{-1}\sum_{t=1}^{T}f_{1t}(u_{it}/\sigma_i)}{\left(T^{-1}\sum_{t=1}^{T}f_{1t}^2\right)^{1/2}} + \\ &O_p\left(N^{1/2-\alpha}T^{-1/2}\right) + O_p\left(N^{1-2\alpha}\right). \end{aligned} \tag{B35}$$

Thus, under $\beta_{i1} \neq 0$, and using the normalization $T^{-1}\sum_{t=1}^{T}f_{1t}^2 \to_p 1$, $(z_i - \frac{\sqrt{T}\beta_{i1}}{\sigma_i}) \to_d N(0,1)$ as N and $T \to \infty$, in any order, and if $\alpha > 1/2$. It is also easy to see that (B34) and (B35) also hold in mean square.

In the case of a multi-factor setting, (B29) can be re-written in the form shown in Lemma 4 so that the error term, ξ_{it},now is augmented by residuals from the regression of each of the m factors on $\bar{x}_t$. The rest of the analysis then follows through.

Supplementary Appendix V: Proof of consistency of $\hat{\mu}_{v_1}(c_{p,N})$ based on multiple testing

The proof is heuristic to the extent that a high level assumption is needed that may be difficult to establish using more primitive conditions. We make the following assumption.

Assumption 6 *1. β_{i1} is uniformly bounded over i .*

2. *$\bar{x}_t u_{it}$ is uniformly mixing over i, in the sense of the mixing assumption of Connor and Korajczyk (1993), with mixing coefficients $\phi_{t,m}$ that satisfy $\sup_t \lim_{m\to\infty}\phi_{t,m}=0$.*

3. *Let φ_i denote a standard normal variate. Then, if $z_{i,T,N}-\varphi_i=O_{m.s.}(N^{2-4\alpha})+O_{m.s.}(N^{1/2-\alpha}T^{-1/2})$, $\sup_i z_{i,T,N}-\varphi_i=O_p\left(N^{2-4\alpha}\right)+O_p\left(N^{1/2-\alpha}T^{-1/2}\right)$, and $\sup_i E\left(z_{i,T,N}-\varphi_i\right)^2=O(N^{2-4\alpha})+O(N^{1-2\alpha}T^{-1})$, where $O_{m.s.}()$ denotes order in mean square.*

4. *Let $\psi=(\psi_1,...,\psi_N)'$ denote an $N\times 1$ selector vector consisting of zeros and ones such that $\psi'\psi>N^{\alpha}$, for some $\alpha>1/2$. Define $u_t^{\psi}=(\psi'\psi)^{-1}\sum_{i=1}^{N}\psi_i u_{it}$. Then,*

$$\sup_t \sup_{\psi} E\left(u_t^{\psi}\right)^2=o(1).$$

Remark 2 *Condition 2 is a standard uniform mixing condition. Uniform mixing is a stronger form of mixing than strong mixing which is more widely used, but allows a CLT without any rates for the mixing coefficients and only the existence of $2+\delta$, $\delta>0$ moments. One could simplify further the assumption by imposing a uniform mixing condition on u_{it}, and thereby $f_{1t}u_{it}$ and proving that $\bar{x}_t u_{it}$ is uniform mixing with mixing coefficients that have mixing size -1/2, but we choose to make this slightly less primitive assumption for simplicity. Clearly, if u_{it} follow (16) then Condition 2 is satisfied. If u_{it}follow (28) then both (29) and assumptions on $\nu_{s,t}$ need to be strengthened. A discussion of these issues may be found in Section 14.3 of Davidson (1994) and, in particular, Theorem 14.14. Conditions 3 and 4 are uniform convergence technical conditions which again seem difficult to establish from more primitive conditions. A proof of the normality invoked in Condition 3 is provided in Supplementary Appendix IV and the assumption only strengthens the result to make it uniform. Condition 4 appears intuitive due to the weak cross-sectional dependence of the errors, although again uniformity is difficult to establish formally.*

Set

$$w_{it}=x_{it}I\left(|z_{i,T,N}|\geq c_{p_i,N}\right),\ \theta_i=\beta_{i1}I\left(|z_{i,T,N}|\geq c_{p_i,N}\right),\ \upsilon_{it}=u_{it}I\left(|z_{i,T,N}|\geq c_{p_i,N}\right),$$

where $c_{p_i,N}$ is the critical value of the i-th test. Then,

$$w_{it}=\theta_i f_{1t}+\upsilon_{it}, \text{for } i=1,...,N;\ t=1,...,T,$$

and

$$\bar{w}_t=\bar{\theta}f_{1t}+\bar{\upsilon}_t,$$

where

$$\bar{w}_t=\frac{\sum_{i=1}^{N}w_{it}}{\sum_{i=1}^{N}I\left(|z_{i,T,N}|\geq c_{p_i,N}\right)},\ \bar{\theta}=\frac{\sum_{i=1}^{N}\theta_i}{\sum_{i=1}^{N}I\left(|z_{i,T,N}|\geq c_{p_i,N}\right)},$$

and

$$\bar{\upsilon}_t=\frac{\sum_{i=1}^{N}\upsilon_{it}}{\sum_{i=1}^{N}I\left(|z_{i,T,N}|\geq c_{p_i,N}\right)}.$$

We take

$$\hat{\sigma}^2_{\bar{w}}=\frac{1}{T}\sum_{t=1}^{T}\left(\bar{w}_t-\bar{w}\right)^2,$$

and consider the limiting behaviour of $\sigma^2_{\bar{w}}/\mu_{v_1}$, where as before $\mu_{v_1}=E(v_{i1})$. Since

$$\bar{w}_t-\bar{w}=\bar{\theta}\left(f_{1t}-\bar{f}_1\right)+\left(\bar{\upsilon}_t-\bar{\upsilon}\right),$$

then

$$\frac{\hat{\sigma}_{\bar{w}}^2}{\mu_{v_1}^2} = \frac{\bar{\theta}^2}{\mu_{v_1}^2}\frac{1}{T}\sum_{t=1}^{T}\left(f_{1t}-\bar{f}_1\right)^2 + \frac{2\bar{\theta}}{\mu_{v_1}^2}\frac{1}{T}\sum_{t=1}^{T}\left(f_{1t}-\bar{f}_1\right)\left(\bar{v}_t-\bar{v}\right) + \frac{1}{\mu_{v_1}^2}\frac{1}{T}\sum_{t=1}^{T}\left(\bar{v}_t-\bar{v}\right)^2 \tag{B36}$$
$$= I + II + III.$$

We concentrate on I as we will prove that II and III tend to zero. For I, and since $\frac{1}{T}\sum_{t=1}^{T}\left(f_{1t}-\bar{f}_1\right)^2 \to_p 1$, we have

$$\bar{\theta} = \frac{\sum_{i=1}^{N^\alpha}\beta_{i1}I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}\neq 0\right) + \sum_{i=N^\alpha+1}^{N}\beta_{i1}I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}=0\right)}{\sum_{i=1}^{N^\alpha}I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}\neq 0\right) + \sum_{i=N^\alpha+1}^{N}I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}=0\right)}, \text{ or}$$
$$= \frac{\frac{1}{N^\alpha}\left(\sum_{i=1}^{N^\alpha}\beta_{i1}I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}\neq 0\right) + \sum_{i=N^\alpha+1}^{N}\beta_{i1}I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}=0\right)\right)}{\frac{1}{N^\alpha}\left(\sum_{i=1}^{N^\alpha}I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}\neq 0\right) + \sum_{i=N^\alpha+1}^{N}I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}=0\right)\right)}. \tag{B37}$$

We first consider the asymptotic behaviour of the following four terms:

$$A = \frac{1}{N^a}\sum_{i=1}^{N^\alpha}\beta_{i1}\left(I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}\neq 0\right) - \Pr\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}\neq 0\right)\right),$$

$$B = \frac{1}{N^a}\sum_{i=N^\alpha+1}^{N}\beta_{i1}\left(I\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}= 0\right) - \Pr\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}= 0\right)\right),$$

$$C = \frac{1}{N^a}\sum_{i=1}^{N^\alpha}\left(I\left(|z_{i,T,N}| \geq c_{p,N}|\beta_{i1}\neq 0\right) - \Pr\left(|z_{i,T,N}| \geq c_{p,N}|\beta_{i1}\neq 0\right)\right),$$

$$D = \frac{1}{N^a}\sum_{i=N^\alpha+1}^{N}\left(I\left(|z_{i,T,N}| \geq c_{p,N}|\beta_{i1}= 0\right) - \Pr\left(|z_{i,T,N}| \geq c_{p,N}|\beta_{i1}= 0\right)\right).$$

We need to show that the summands in $A-D$ follow a central limit theorem. It is sufficient to show that the summands are uniformly mixing. By Condition 2 of Assumption 6 it follows that $z_{i,T,N}$ is uniformly mixing over i. By the measurability of the indicator function (see, e.g., Theorem 3.27 of Davidson (1994)) and Theorem 14.1 of Davidson (1994), it follows that all summands in $A-D$, are uniformly mixing and, by Theorem 18.5.1 of Ibragimov and Linnik (1971), a central limit theorem holds. Then, it follows that

$$A = O\left(N^{-\alpha/2}\right), \quad B = O\left(N^{1/2-\alpha}\right), \quad C = O\left(N^{-\alpha/2}\right), \quad D = O\left(N^{1/2-\alpha}\right).$$

Next, we consider

$$\lim_{N,T\to\infty}\left(\frac{\sum_{i=1}^{N^\alpha}\Pr\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}\neq 0\right)}{N^\alpha}\right), \text{ and}$$
$$\lim_{N,T\to\infty}\left(\frac{\sum_{i=N^\alpha+1}^{N}\Pr\left(|z_{i,T,N}| \geq c_{p_i,N}|\beta_{i1}= 0\right)}{N^\alpha}\right).$$

We have that,

$$\Pr\left(|z_{i,T,N}| \geq c_{p,N}|\beta_{i1}\neq 0\right) = 1 - \left[\Phi\left(c_{p,N} - \frac{\beta_{i1}\sqrt{T}}{\sigma_i}\right) - \Phi\left(-c_{p,N} - \frac{\beta_{i1}\sqrt{T}}{\sigma_i}\right)\right] \tag{B38}$$
$$+ O\left(N^{1-2\alpha}T^{-1}\right) + O\left(N^{2-4\alpha}\right).$$
$$= 1 - \Phi\left(c_{p,N} - \frac{\beta_{i1}\sqrt{T}}{\sigma_i}\right) + \Phi\left(-c_{p,N} - \frac{\beta_{i1}\sqrt{T}}{\sigma_i}\right) + O\left(N^{2-4\alpha}\right) + O\left(N^{1-2\alpha}T^{-1}\right).$$

(B38) can be proven as follows. From Supplementary Appendix IV we have

$$z_{i,T,N} = z_i + O_p\left(N^{1-2\alpha}\right) + O_p\left(N^{1/2-\alpha}T^{-1/2}\right) = z_i + q_{i,N,T},$$

where z_i is distributed as $N(0,1)$ and $q_{i,N,T} = O_p\left(N^{1-2\alpha}\right) + O_p\left(N^{1/2-\alpha}T^{-1/2}\right)$. Then, we have

$$\Pr\left(|z_{i,T,N}| \leq c_{p_i,N}\right) - \Pr\left(|z_i| \leq c_{p_i,N}\right) = \Pr\left(|z_i + q_{i,N,T}| \leq c_{p_i,N}\right) - \Pr\left(|z_i| \leq c_{p_i,N}\right) \leq \Pr\left(|q_{i,N,T}| > 0\right)$$

$$\lim_{N,T\to\infty} \Pr\left(|q_{iN,T}| > 0\right) = \lim_{\epsilon\to 0} \lim_{N,T\to\infty} \Pr\left(|q_{i,N,T}| > \epsilon\right).$$

But

$$\Pr\left(|q_{i,N,T}| > \epsilon\right) \leq \frac{E\left(q_{i,N,T}^2\right)}{\epsilon^2}.$$

It is easy to see from the analysis of Supplementary Appendix IV that

$$E\left(q_{iN,T}^2\right) = O\left(N^{2-4\alpha}\right) + O\left(N^{1-2\alpha}T^{-1}\right),$$

then

$$\Pr\left(|z_{i,T,N}| \leq c_{p_i,N}\right) - \Phi(c_{p_i,N}) - \Phi(-c_{p_i,N}) = O\left(N^{2-4\alpha}\right) + O\left(N^{1-2\alpha}T^{-1}\right), \tag{B39}$$

proving (B38). Assumption 6 (3) strengthens this to

$$\sup_i \Pr\left(|z_{i,T,N}| \leq c_{p_i,N}\right) - \Phi(c_{p_i,N}) - \Phi(-c_{p_i,N}) = O\left(N^{2-4\alpha}\right) + O\left(N^{1-2\alpha}T^{-1}\right). \tag{B40}$$

Thus,

$$p \lim_{N,T\to\infty} \left(\frac{\sum_{i=1}^{[N^\alpha]} I\left(|z_{i,T,N}| \geq c_{p_i,N} | \beta_{i1} \neq 0\right)}{N^\alpha} \right) \to_p 1. \tag{B41}$$

as long as

$$c_{p_i,N} = o_p(T^{1/2}) \tag{B42}$$

uniformly over i. Also,

$$\begin{aligned}
\Pr\left(|z_{i,T,N}| \geq c_{p_i,N} | \beta_{i1} = 0\right) &= \left[1 - \Phi\left(c_{p_i,N}\right) + \Phi\left(-c_{p_i,N}\right)\right] + O\left(N^{1-2\alpha}T^{-1}\right) + O\left(N^{2-4\alpha}\right) \\
&= 2\left[1 - \Phi\left(c_{p_i,N}\right)\right] + O\left(N^{1-2\alpha}T^{-1}\right) + O\left(N^{2-4\alpha}\right).
\end{aligned}$$

Then,

$$\frac{\sum_{i=[N^\alpha]+1}^{N} \Pr\left(|z_{i,T,N}| \geq c_{p,N} | \beta_{i1} = 0\right)}{N^\alpha} = \frac{\sum_{i=[N^\alpha]+1}^{N} 2\left[1 - \Phi\left(c_{p_i,N}\right)\right]}{N^\alpha} + \frac{(N - N^\alpha)}{N^\alpha}\left[O\left(N^{1-2\alpha}T^{-1}\right) + O\left(N^{2-4\alpha}\right)\right],$$

and, as long as

$$\frac{\sum_{i=[N^\alpha]+1}^{N} 2\left[1 - \Phi\left(c_{p_i,N}\right)\right]}{N^\alpha} = o_p(1), \tag{B43}$$

then,[1]

$$\frac{\sum_{i=[N^\alpha]+1}^{N} I\left(|z_{i,T,N}| \geq c_{p_i,N} | \beta_{i1} = 0\right)}{N^\alpha} \to 0,$$

if either $\alpha > 2/3$ or $\alpha > 3/5$ and $N^{2-3\alpha}T^{-1} = o(1)$. The latter follows, if $\alpha > 3/5$ and $N = o(T^5)$. For simplicity, we will assume that $\alpha > 2/3$. Now, we check

$$\lim_{N,T\to\infty} \left(\frac{\sum_{i=1}^{N^\alpha} \beta_{i1} \Pr\left(|z_{i,T,N}| \geq c_{p_i,N} | \beta_{i1} \neq 0\right)}{N^\alpha} \right), \text{ and}$$

$$\lim_{N,T\to\infty} \left(\frac{\sum_{i=[N^\alpha]+1}^{N} \beta_{i1} \Pr\left(|z_{i,T,N}| \geq c_{p_i,N} | \beta_{i1} = 0\right)}{N^\alpha} \right).$$

[1] It is easy to see that both the Holm and Bonferroni multiple testing approaches discussed in Section 3.1 satisfy (B43). For Bonferroni, this is obvious. For Holm, we note that if $c_{p_i,N} = \Phi^{-1}(1 - p_i)$, $p_i = \frac{p}{2(N-i+1)}$, then $2\left[1 - \Phi\left(c_{p_i,N}\right)\right] = C_i \frac{p}{2(N-i+1)}$, for some uniformly bounded positive constants C_i. Since $\sum_{i=[N^\alpha]+1}^{N} 2\left[1 - \Phi\left(c_{p_i,N}\right)\right] \leq C \ln N$, (B43) holds.

We have,

$$\lim_{N,T\to\infty}\left(\frac{\sum_{i=[N^{\alpha}]+1}^{N}\beta_{i1}\Pr\left(|z_{i,T,N}|\geq c_{p_i,N}|\beta_{i1}=0\right)}{N^{\alpha}}\right)=0,$$

and

$$\lim_{N,T\to\infty}\left(\frac{\sum_{i=1}^{N^{\alpha}}\beta_{i1}\Pr\left(|z_{i,T,N}|\geq c_{p_i,N}|\beta_{i1}\neq 0\right)}{N^{\alpha}}\right)=\lim_{N,T\to\infty}\left(\frac{\sum_{i=1}^{N^{\alpha}}\beta_{i1}\left[1-\Phi\left(c_{p_i,N}-\frac{\beta_{i1}\sqrt{T}}{\sigma_i}\right)+\Phi\left(-c_{p_i,N}-\frac{\beta_{i1}\sqrt{T}}{\sigma_i}\right)\right]}{N^{\alpha}}\right) \tag{B44}$$

$$\to_p E(\bar{v})=\mu_{v_1},$$

using (B41), or

$$\bar{\theta}\to_p \mu_{v_1}. \tag{B45}$$

And therefore,

$$\frac{\hat{\sigma}^2_{\bar{w}}}{\mu^2_{v_1}}\to_p 1.$$

Finally, for II and III we first note that we have already established that $N^{-\alpha}\sum_{i=1}^{N}I\left(|z_{i,T,N}|\geq c_{p_i,N}\right)\to_p 1$, as N and $T\to\infty$, assuming that $\alpha>2/3$. But by the proofs of Lemma A.1 and A.2 of (Pesaran, 2006, Theorem 15.18) and using assumption 6 it immediately follows that $II=o_p(1)$ and $III=o_p(1)$ completing the proof. In summary, consistency is obtained under Assumption 6 and conditions (B42) and (B43) if $\alpha>2/3$.

Note that in the case of a multi-factor setting, (B44) can alter. If $\alpha=\alpha_1>\alpha_2>\ldots>\alpha_m$, then the denominator can potentially capture more elements than $N^{\alpha}$and so (B45) converges to $\sum_{j=1}^{m}c_j\mu_{v_j}$,where $0<c_j<1$.

Supplementary Appendix VI: Additional Monte Carlo simulation results

We provide some additional Monte Carlo simulation results in this appendix. First, we set $\mu_v=1$ and keep $\alpha=\alpha_1>\alpha_2$. In this case $\tilde{\alpha}$ consistently estimates α and has the asymptotic distribution as described in Theorem 1. Next, we present size and power of tests based on $\mathring{\alpha}$ as well. We use the same confidence bands as in the case of $\tilde{\alpha}$. From the results shown for experiments A-D, it is confirmed that $\mathring{\alpha}$ is super-consistent. Finally, we consider the two factor model of (36) for the case when $\alpha=\alpha_1=\alpha_2$ and depict bias and RMSE results for estimator $\mathring{\alpha}$.

A two factor model where $\mu_\nu=1$

In addition to the results analysed in Section 4, here we consider the instance when $\mu_\nu=1$ and show bias, RMSE, size and power results for estimator $\tilde{\alpha}$ which is asymptotically distributed in accordance to Theorem 1. We use the set up of experiment A of Section 4 and set $\mu_v=1$, $\mu_{\nu_2}=0.87$, $\mu_{\nu_1}=\sqrt{\mu^2_\nu-N^{2(\alpha_2-\alpha)}\mu^2_{\nu_2}}$ and $\varsigma^2=3/4$. Since the leading factor (f_{1t}) is serially uncorrelated, the statistic for making inference about α is given by

$$\left(\frac{1}{T}\hat{V}_{\overline{f_1^2}}+\frac{4}{N^{\tilde{\alpha}}}\widehat{\frac{\sigma^2_{v_1}}{\mu^2_{v_1}}}\right)^{-1/2}2\ln(N)\left(\tilde{\alpha}-\alpha^*\right)\to_d N(0,1). \tag{B46}$$

Note that when the leading factor is serially uncorrelated then $\hat{V}_{\overline{f_1^2}}=\widehat{E(f^4_{1t})/\sigma^4_{f_1}}-1$, where $\widehat{E(f^4_{1t})/\sigma^4_f}$ is consistently estimated by

$$\widehat{E(f^4_{1t})/\sigma^4_{f_1}}=\frac{\sum_{t=1}^{T}\left(\tilde{x}_t-\tilde{x}\right)^4}{T},$$

where $\tilde{x}_t=\left(N^{-1}\sum_{i=1}^{N}x_{it}\right)/\hat{\sigma}_{\bar{x}}$, and $\widehat{\sigma^2_{v_1}/\mu^2_{v_1}}$, the estimator of $\sigma^2_{v_1}/\mu^2_{v_1}$, is given by

$$\widehat{\frac{\sigma^2_{v_1}}{\mu^2_{v_1}}}=\frac{\sum_{i=1}^{N^{\tilde{\alpha}}}\left(\hat{v}^{(s)}_{i1}-\frac{1}{N^{\tilde{\alpha}}}\sum_{j=1}^{N^{\tilde{\alpha}}}\hat{v}^{(s)}_{j1}\right)^2}{N^{\tilde{\alpha}}-1},$$

where $\left\{\hat{v}_{i1}^{(s)}\right\}$ denotes the sequence of $\hat{v}_{i1}$ sorted according to their absolute values in a descending order, and $\hat{v}_{i1}$ is the OLS estimator of the regression coefficient of x_{it} on $\tilde{x}_t = \bar{x}_t/\hat{\sigma}_{\bar{x}}$ - see Lemma 16 for details. The above expressions apply irrespective of the number of factors included in model (36).

Further, though not depicted in these Monte Carlo simulation results (these are available upon request), we consider the case of serially correlated factors as it is being used in the empirical applications of Section 5. When $\rho_j \neq 0$, we use a corrected variance estimator of f_{1t}. The relevant formula for the test statistic is given by

$$\left[\frac{1}{T}\left[\hat{V}_{\overline{f_1^2}}(q)\right] + \frac{4}{N^{\tilde{\alpha}}}\frac{\widehat{\sigma_{v_1}^2}}{\mu_{v_1}^2}\right]^{-1/2} 2\ln(N)\left(\tilde{\alpha}-\alpha^*\right) \to_d N(0,1). \tag{B47}$$

$\hat{V}_{\overline{f_1^2}}(q)$ is computed by first estimating an AR(q) process for $\tilde{z}_t = z_t - \bar{z}$, where $z_t = \left(\tilde{x}_t - \tilde{x}\right)^2$, $\tilde{x}_t = \left(\frac{1}{N}\sum_{i=1}^N x_{it}\right)/\hat{\sigma}_{\bar{x}}$, $\tilde{x} = T^{-1}\sum_{t=1}^T \tilde{x}_t$ and $\bar{z} = T^{-1}\sum_{t=1}^T z_t$, and then $\hat{V}_{\overline{f_1^2}}(q) = \hat{\sigma}_z^2/(1-\widehat{\gamma}_1-\widehat{\gamma}_2-...-\widehat{\gamma}_q)^2$, where $\hat{\sigma}_z$ is the regression standard error and $\widehat{\gamma}_i$ is the i^{th} estimated AR coefficient fitted to $\tilde{z}_t$. The lag order is set to $q = T^{1/3}$, and $\widehat{\sigma_{v_1}^2/\mu_{v_1}^2}$ is computed as before. Note that this correction is not the standard Newey-West one but uses an estimated autoregressive filter. We found that this correction leads to better finite sample properties and hence we use this in both the Monte Carlo study and the empirical applications in Section 5.

Size of the tests is computed under $H_0 : \alpha = \alpha_0$, using a two-sided alternative where α_0 takes values in the range $[0.70, 1.00]$, as indicated previously. Power is computed under the alternatives $H_a : \alpha_a = \alpha_0 + 0.05$ (power+), and $H_a : \alpha_a = \alpha_0 - 0.05$ (power-). Again, all results are scaled up by 100.

Size and power of tests based on $\mathring{\alpha}$ estimator

Next, we conduct size and power tests based on estimator $\mathring{\alpha}$. We use the same variance estimates as in (B46) which constitute conservative bands for $\mathring{\alpha}$ and show results for the setting described in Section 4 for experiments A-D when $\mu_v^2 \neq 1$. The same specifications for the null and alternative hypotheses are imposed as in the section above.

A two factor model when $\alpha = \alpha_1 = \alpha_2$

Finally, we repeat the analysis of Section 4 for Experiment A using the less likely alternative of $\alpha = \alpha_1 = \alpha_2$. Here, we set $\mu_{v_1} = \mu_{v_2} = 0.5$ and $\varsigma^2 = 1/3$.

Additional results

Table A1 presents bias, RMSE, size and power statistics for experiment A in the case of the bias-corrected estimator, $\tilde{\alpha}$, and when $\mu_v = 1$. Results in Table A1 show more clearly the asymptotic distribution derived for $\tilde{\alpha}$ which is also used for $\mathring{\alpha}$. Again, we only report results for values of α over the range $[0.70, 1.0]$. Recall that α is identified only if $\alpha > 1/2$, and for asymptotically valid inference on α it is further required that $\alpha > 4/7$, unless $T^{1/2}/N^{(4\alpha-2)} \to 0$, as N and $T \to \infty$ in the case of $\tilde{\alpha}$ (see Theorem 1), or that $\alpha > 2/3$ in the case of $\mathring{\alpha}$ (see Supplementary Appendix V).

It appears that estimator $\tilde{\alpha}$ performs reasonably well in terms of bias and RMSE for values of α in the range $[0.70 - 0.85]$, when $\mu_v = 1$. To get a clearer picture of the asymptotics we turn to the right-hand-side of Table A1 that summarizes the size and power of the tests based on $\tilde{\alpha}$. There is evidence of some size distortion when α is below 0.75, but it tends towards the nominal 5% level as α is increased. The size distortion is also reduced as N and T are increased. The power of the test also rises in α, N and T, and approaches unity quite rapidly. However, the power function seems to be asymmetric with the power tending to be higher for alternatives above the null (denoted by Power+) as compared to the alternatives below the null (denoted by Power-). This asymmetry is particularly marked for low values of α and disappears as α is increased.

Turning to the size and power of the tests based on $\mathring{\alpha}$, its superior properties are verified by the results shown on the right-hand-side of Table A2. Indeed, in general size tends to zero as α increases towards 1 and as N and T increase. Similarly, power is uniformly close to unity irrespective of the value of α chosen or the N and T

combination considered (low power is only recorded for the smallest value of α considered and for small N and T combinations). Qualitatively similar conclusions are drawn for the remaining experiments B-D, as shown in Tables A3-A5.

Finally, we present results for Experiment A when $\alpha = \alpha_1 = \alpha_2$ in Table A6. Compared with Table A-B, both bias and RMSE results are more elevated for estimator $\mathring{\alpha}$ for all values of α when we impose the two factors to be of the same strength in the data generating process. This is expected given the discussion in Supplementary Appendix V. For this reason, size and power of $\mathring{\alpha}$ suffer and are therefore not shown in Table A6. Looking at the bias and RMSE of $\mathring{\alpha}$, they both fall gradually as N, T,and α are increased, consistent with the baseline case.

Calibration of $\bar{R}^2_N$

In order to select an appropriate $\bar{R}^2_N$ for the Monte Carlo simulation study of Section 4 and Supplementary Appendix VI, we computed $\bar{R}^2 s$ for the regressions, (36) summated based on data from a number of empirical applications. For each data set we first calculated $\mathring{\alpha}$ corresponding to $\breve{\alpha}$ and selected the strong $N^{\mathring{\alpha}}$ units. This resulted in a modified data set, $\boldsymbol{x}^{(s)} = \left[x^{(s)}_{it}\right]$ of dimension $T \times N^{\mathring{\alpha}}$ (elements of $\boldsymbol{x}^{(s)}$ were standardised to have unit variance). Then, we extracted the principal components (pc) from $\boldsymbol{x}^{(s)}$ and run the regression

$$x^{(s)}_{it} = a_i + \gamma_{ij} pc_j + \varepsilon_{it}, \tag{B48}$$

for $i = 1, 2, ..., N^{\mathring{\alpha}}$, and $t = 1, 2, ..., T$. We set the number of principal components to include in (B48) to $j = 1, 2, 3$, respectively. Finally, we computed the R^2 of each of the $N^{\mathring{\alpha}}$ regressions and took their average: $\bar{R}^2_N = \frac{1}{N^{\mathring{\alpha}}} \sum_{i=1}^{N^{\mathring{\alpha}}} R^2_i$. We conducted this analysis for a number of empirical applications, of which: (i) GVAR macro economic data sets (real GDP growth - $\bar{R}^2_N = 0.28, 0.37, 0.44$, inflation - $\bar{R}^2_N = 0.47, 0.57, 0.64$, real equity price change - $\bar{R}^2_N = 0.47, 0.59, 0.66$), and (ii) US - $\bar{R}^2_N = 0.30, 0.50, 0.60$ - and UK - $\bar{R}^2_N = 0.25, 0.43, 0.52$, all using $j = 1, 2, 3$ principal components, respectively. See also Section 5 for further details of the data sets.

Table A1: Bias, RMSE, size and power ($\times 100$) for the $\tilde{\alpha}$ estimate of the cross-sectional exponent - case of two serially independent factors and cross-sectionally independent idiosyncratic errors

$(\alpha_2 = 2\alpha/3,\ f_{jt}$ and $u_{it} \sim IIDN(0,1),\ v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2),\ j = 1,2,\ \mu_v = 1,\ \mu_{v_2} = 0.87,\ \mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_2 - \alpha)}\mu_{v_2}^2})$

N=50,100,200,500,1000 and T=100,200,500

N\T	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
					100			
50	Bias	-0.93	-1.21	-1.10	-0.91	-0.82	-0.33	-0.18
	RMSE	2.66	2.62	2.46	2.29	2.19	2.00	1.96
100	Bias	-0.47	-0.66	-0.73	-0.31	-0.28	-0.31	-0.19
	RMSE	2.07	1.97	1.90	1.71	1.66	1.63	1.59
200	Bias	-0.66	-0.23	-0.24	-0.17	-0.20	-0.14	-0.07
	RMSE	1.80	1.57	1.50	1.44	1.42	1.40	1.38
500	Bias	-0.17	-0.19	-0.10	-0.14	-0.10	-0.08	-0.06
	RMSE	1.32	1.25	1.20	1.19	1.17	1.16	1.16
1000	Bias	-0.22	-0.17	-0.11	-0.12	-0.09	-0.10	-0.08
	RMSE	1.16	1.10	1.06	1.04	1.03	1.03	1.02
					200			
50	Bias	-0.97	-1.25	-1.14	-0.92	-0.81	-0.31	-0.17
	RMSE	2.16	2.17	2.00	1.81	1.71	1.49	1.43
100	Bias	-0.49	-0.62	-0.65	-0.22	-0.18	-0.21	-0.09
	RMSE	1.57	1.50	1.43	1.23	1.19	1.17	1.13
200	Bias	-0.47	-0.14	-0.20	-0.14	-0.18	-0.13	-0.06
	RMSE	1.26	1.09	1.05	1.00	0.99	0.97	0.95
500	Bias	-0.17	-0.19	-0.10	-0.14	-0.10	-0.08	-0.06
	RMSE	0.96	0.92	0.87	0.86	0.84	0.83	0.83
1000	Bias	-0.09	-0.08	-0.02	-0.04	-0.01	-0.03	-0.01
	RMSE	0.79	0.76	0.74	0.73	0.72	0.72	0.72
					500			
50	Bias	-1.05	-1.26	-1.10	-0.85	-0.73	-0.22	-0.08
	RMSE	1.68	1.75	1.57	1.35	1.23	0.98	0.93
100	Bias	-0.17	-0.44	-0.56	-0.16	-0.14	-0.19	-0.06
	RMSE	1.00	1.02	1.03	0.83	0.80	0.79	0.75
200	Bias	-0.39	-0.08	-0.14	-0.10	-0.14	-0.08	-0.02
	RMSE	0.87	0.72	0.69	0.66	0.65	0.63	0.61
500	Bias	-0.13	-0.14	-0.05	-0.09	-0.05	-0.03	-0.01
	RMSE	0.63	0.59	0.55	0.54	0.52	0.51	0.51
1000	Bias	-0.19	-0.12	-0.04	-0.05	-0.02	-0.03	-0.01
	RMSE	0.56	0.51	0.49	0.48	0.47	0.46	0.46

N\T	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
					100			
50	Size	7.25	7.70	7.80	7.10	6.25	4.65	6.00
	Power+	60.95	70.00	74.50	76.45	76.80	72.80	70.55
	Power-	60.95	36.30	39.85	48.45	51.45	65.65	70.55
100	Size	2.65	3.55	5.15	3.80	4.45	4.65	6.10
	Power+	58.70	76.05	85.20	83.60	86.55	88.70	86.35
	Power-	58.70	55.65	62.90	78.20	81.55	83.25	86.35
200	Size	0.05	0.80	2.10	4.70	5.85	6.10	6.80
	Power+	55.75	77.85	90.45	93.65	95.10	95.30	94.65
	Power-	55.75	65.15	80.60	89.25	91.55	93.30	94.65
500	Size	7.35	7.25	6.90	7.05	6.85	6.85	7.20
	Power+	97.80	98.60	98.90	99.15	99.15	99.05	98.90
	Power-	97.80	97.20	98.10	98.25	98.70	98.95	98.90
1000	Size	3.95	5.80	6.05	6.65	6.40	6.65	6.75
	Power+	99.65	99.80	99.85	99.95	100.00	100.00	99.55
	Power-	99.65	99.00	99.45	99.40	99.45	99.50	99.55
					200			
50	Size	1.85	3.65	4.70	5.05	5.35	3.40	4.35
	Power+	55.15	80.35	91.40	92.60	93.65	91.40	90.60
	Power-	55.15	37.75	51.40	63.15	71.75	86.30	90.60
100	Size	0.00	0.05	0.70	1.20	2.20	3.25	4.50
	Power+	23.00	69.30	92.10	94.15	97.35	98.25	99.05
	Power-	23.00	39.75	63.20	87.95	95.75	98.00	99.05
200	Size	6.05	4.40	4.70	4.40	4.70	4.50	4.55
	Power+	99.25	99.45	99.85	99.90	99.95	99.95	100.00
	Power-	99.25	98.80	99.70	99.90	99.95	99.95	100.00
500	Size	5.35	6.10	6.00	6.15	5.80	5.80	6.50
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	99.95	99.95	100.00	100.00	100.00	100.00
1000	Size	6.60	5.50	5.10	5.10	5.20	5.25	5.30
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
					500			
50	Size	0.00	0.00	0.00	0.00	0.55	0.60	1.30
	Power+	0.60	8.00	41.25	80.60	95.45	97.25	99.85
	Power-	0.60	0.95	7.45	33.35	68.95	97.10	99.85
100	Size	3.40	6.45	9.10	4.30	4.85	4.80	5.45
	Power+	99.15	99.95	100.00	100.00	100.00	100.00	100.00
	Power-	99.15	99.20	99.85	100.00	100.00	100.00	100.00
200	Size	8.05	4.55	5.05	4.55	4.60	3.80	4.65
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
500	Size	0.25	1.50	2.85	4.20	4.05	4.45	5.00
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	0.00	0.20	1.55	4.05	5.05	5.70	6.15
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Notes: Size is computed under H_0: $\alpha = \alpha_0$, using a two-sided alternative where α_0 takes values in the range [0.70,1.00].
Power is computed under the alternatives H_a: $\alpha_a = \alpha_0 + 0.05$ (power+), and $\alpha_a = \alpha_0 + 0.05$ (power-).

Table A2: Bias, RMSE, size and power ($\times 100$) for the $\mathring{\alpha}$ estimate of the cross-sectional exponent - case of two serially independent factors and cross-sectionally independent idiosyncratic errors

$(\alpha_2 = 2\alpha/3,\ f_{jt}$ and $u_{it} \sim IIDN(0,1),\ v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2),\ j = 1,2,\ \mu_v = 0.87,\ \mu_{v_2} = 0.71,\ \mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_2 - \alpha)}\mu_{v_2}^2})$

N=50,100,200,500,1000 and T=100,200,500

	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
N\T					**100**			
50	Bias	2.20	1.04	0.42	0.09	-0.18	0.03	-0.23
	RMSE	3.54	2.47	1.80	1.28	0.90	0.54	0.24
100	Bias	0.97	0.45	0.02	0.27	0.15	-0.02	-0.04
	RMSE	2.02	1.44	0.93	0.71	0.48	0.29	0.06
200	Bias	0.52	0.43	0.19	0.15	0.04	0.03	0.03
	RMSE	1.48	0.99	0.63	0.46	0.29	0.17	0.03
500	Bias	0.13	0.05	0.09	0.03	0.04	0.03	0.05
	RMSE	0.68	0.46	0.35	0.23	0.16	0.10	0.06
1000	Bias	0.00	0.01	0.05	0.01	0.03	0.00	0.06
	RMSE	0.53	0.35	0.25	0.17	0.12	0.07	0.06
					200			
50	Bias	4.62	2.95	1.76	0.96	0.29	0.23	-0.29
	RMSE	5.41	3.77	2.58	1.75	1.10	0.71	0.30
100	Bias	2.51	1.55	0.62	0.60	0.31	0.05	-0.10
	RMSE	3.12	2.19	1.28	0.99	0.61	0.33	0.10
200	Bias	0.45	0.54	0.28	0.21	0.08	0.04	-0.02
	RMSE	1.14	0.96	0.61	0.45	0.28	0.15	0.02
500	Bias	0.24	0.12	0.14	0.06	0.06	0.04	0.02
	RMSE	0.61	0.41	0.31	0.19	0.14	0.08	0.02
1000	Bias	0.10	0.08	0.10	0.06	0.07	0.03	0.03
	RMSE	0.38	0.26	0.21	0.14	0.10	0.05	0.03
					500			
50	Bias	10.04	6.54	4.18	2.61	1.43	0.63	-0.35
	RMSE	10.36	6.91	4.54	2.96	1.80	0.97	0.35
100	Bias	4.02	3.06	1.75	1.38	0.79	0.31	-0.13
	RMSE	4.28	3.30	2.05	1.63	1.02	0.53	0.13
200	Bias	1.75	1.56	0.88	0.64	0.31	0.11	-0.05
	RMSE	2.08	1.83	1.14	0.86	0.51	0.23	0.05
500	Bias	0.90	0.47	0.31	0.13	0.09	0.04	-0.01
	RMSE	1.17	0.72	0.48	0.27	0.17	0.09	0.01
1000	Bias	0.66	0.23	0.14	0.07	0.06	0.02	0.00
	RMSE	0.90	0.42	0.25	0.15	0.10	0.04	0.00

	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
N\T					**100**			
50	Size	13.45	3.95	1.35	0.30	0.00	0.00	0.00
	Power+	19.90	32.60	49.40	67.95	81.40	89.75	98.30
	Power-	19.90	63.00	62.90	70.05	72.65	94.20	98.30
100	Size	3.00	1.25	0.05	0.00	0.00	0.00	0.00
	Power+	27.65	54.30	82.60	91.45	98.90	100.00	100.00
	Power-	27.65	72.35	83.95	97.95	99.90	100.00	100.00
200	Size	0.55	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	21.80	67.80	97.10	99.80	100.00	100.00	100.00
	Power-	21.80	87.25	99.15	100.00	100.00	100.00	100.00
500	Size	0.45	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	99.70	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.70	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	0.05	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	99.85	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.85	100.00	100.00	100.00	100.00	100.00	100.00
					200			
50	Size	30.55	22.95	15.20	4.10	0.40	0.05	0.00
	Power+	4.50	12.10	36.75	60.50	86.40	97.00	100.00
	Power-	4.50	86.80	91.60	93.35	97.75	99.95	100.00
100	Size	3.55	2.00	0.45	0.05	0.10	0.00	0.00
	Power+	0.55	13.10	59.00	85.15	99.10	100.00	100.00
	Power-	0.55	76.35	86.05	99.70	100.00	100.00	100.00
200	Size	4.00	3.60	0.40	0.10	0.00	0.00	0.00
	Power+	92.85	98.20	100.00	100.00	100.00	100.00	100.00
	Power-	92.85	100.00	100.00	100.00	100.00	100.00	100.00
500	Size	1.00	0.10	0.05	0.00	0.00	0.00	0.00
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	0.15	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
					500			
50	Size	78.00	40.30	17.55	6.20	1.15	0.00	0.00
	Power+	22.50	2.55	0.05	1.05	25.15	90.20	100.00
	Power-	22.50	94.55	92.30	92.85	96.20	99.90	100.00
100	Size	81.05	71.85	36.85	27.95	6.20	0.15	0.00
	Power+	13.20	33.75	83.00	96.95	99.95	100.00	100.00
	Power-	13.20	100.00	100.00	100.00	100.00	100.00	100.00
200	Size	47.45	47.55	20.05	10.10	1.20	0.00	0.00
	Power+	87.05	96.95	100.00	100.00	100.00	100.00	100.00
	Power-	87.05	100.00	100.00	100.00	100.00	100.00	100.00
500	Size	6.05	3.75	1.75	0.25	0.00	0.00	0.00
	Power+	99.10	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.10	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	0.65	0.10	0.00	0.00	0.00	0.00	0.00
	Power+	99.85	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.85	100.00	100.00	100.00	100.00	100.00	100.00

Notes: Size is computed under H_0: α=α_0, using a two-sided alternative where α_0 takes values in the range [0.70,1.00].
Power is computed under the alternatives H_a: $\alpha_a = \alpha_0 + 0.05$ (power+), and $\alpha_a = \alpha_0 + 0.05$ (power-).

Table A3: Bias, RMSE, size and power (×100) for the $\mathring{\alpha}$ estimate of the cross-sectional exponent - case of two serially correlated factors and cross-sectionally independent idiosyncratic errors

$(\alpha_2 = 2\alpha/3,\ \rho_j = 0.5,\ u_{it} \sim IIDN(0,1),\ v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2),\ j = 1,2,\ \mu_v = 0.87,\ \mu_{v_2} = 0.71,\ \mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_2-\alpha)}\mu_{v_2}^2})$

N=50,100,200,500,1000 and T=100,200,500

N\T	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
					100			
50	Bias	3.10	1.83	1.08	0.46	0.01	0.09	-0.26
	RMSE	4.62	3.35	2.39	1.68	1.12	0.68	0.29
100	Bias	2.52	1.64	0.89	0.82	0.50	0.16	-0.07
	RMSE	3.61	2.60	1.74	1.33	0.88	0.48	0.10
200	Bias	1.52	1.22	0.76	0.52	0.26	0.13	0.01
	RMSE	2.56	1.87	1.25	0.88	0.54	0.31	0.06
500	Bias	1.18	0.70	0.49	0.25	0.15	0.06	0.03
	RMSE	1.91	1.22	0.82	0.50	0.32	0.17	0.05
1000	Bias	0.84	0.48	0.32	0.18	0.10	0.02	0.03
	RMSE	1.42	0.87	0.57	0.36	0.23	0.12	0.05
					200			
50	Bias	5.54	3.78	2.35	1.39	0.64	0.44	-0.31
	RMSE	6.39	4.60	3.14	2.15	1.35	0.88	0.31
100	Bias	3.50	2.42	1.33	1.12	0.68	0.33	-0.10
	RMSE	4.25	3.08	1.94	1.51	0.98	0.58	0.10
200	Bias	1.88	1.44	0.88	0.62	0.32	0.16	-0.02
	RMSE	2.56	1.89	1.22	0.87	0.53	0.28	0.02
500	Bias	1.25	0.73	0.54	0.31	0.21	0.11	0.02
	RMSE	1.67	1.04	0.73	0.46	0.30	0.16	0.02
1000	Bias	0.86	0.53	0.39	0.24	0.17	0.08	0.03
	RMSE	1.20	0.74	0.52	0.33	0.23	0.11	0.03
					500			
50	Bias	9.57	6.63	4.66	3.02	1.57	0.74	-0.34
	RMSE	10.01	7.03	5.06	3.42	1.97	1.08	0.35
100	Bias	5.91	4.20	2.60	1.94	1.13	0.46	-0.13
	RMSE	6.28	4.50	2.89	2.19	1.35	0.65	0.13
200	Bias	3.25	2.54	1.55	1.06	0.56	0.26	-0.05
	RMSE	3.65	2.81	1.80	1.26	0.73	0.37	0.05
500	Bias	1.90	1.14	0.77	0.44	0.29	0.15	-0.01
	RMSE	2.19	1.36	0.93	0.56	0.37	0.19	0.01
1000	Bias	1.12	0.67	0.47	0.28	0.19	0.08	0.00
	RMSE	1.35	0.83	0.57	0.35	0.23	0.11	0.00

N\T	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
					100			
50	Size	22.40	12.65	6.70	2.50	1.30	0.25	0.00
	Power+	17.50	25.80	35.60	52.50	69.85	79.35	83.85
	Power-	17.50	66.05	66.80	68.65	70.80	83.80	83.85
100	Size	20.65	12.90	5.35	3.20	1.15	0.40	0.00
	Power+	24.15	38.35	58.55	69.80	84.15	91.80	94.25
	Power-	24.15	86.60	88.85	94.75	96.45	95.05	94.25
200	Size	16.15	11.20	4.65	2.35	0.85	0.20	0.00
	Power+	49.20	63.85	80.95	90.50	94.30	94.80	94.80
	Power-	49.20	96.65	97.35	97.45	96.70	95.70	94.80
500	Size	14.90	7.20	2.90	1.25	0.25	0.15	0.05
	Power+	73.75	89.70	94.95	96.60	96.30	95.90	95.10
	Power-	73.75	98.75	98.20	97.40	96.40	95.80	95.10
1000	Size	11.45	5.75	2.30	1.00	0.45	0.10	0.00
	Power+	89.40	96.10	96.80	96.45	95.70	95.45	95.35
	Power-	89.40	98.55	97.30	96.65	95.90	95.45	95.35
					200			
50	Size	50.95	34.55	18.55	7.75	1.30	0.45	0.00
	Power+	12.10	13.45	24.80	44.60	69.75	87.30	98.25
	Power-	12.10	89.90	89.55	90.30	93.35	98.10	98.25
100	Size	37.60	26.20	10.50	7.45	2.10	0.30	0.00
	Power+	20.60	33.45	61.25	79.20	94.40	98.90	99.90
	Power-	20.60	97.00	97.65	99.80	100.00	100.00	99.90
200	Size	20.45	16.40	6.50	2.55	0.45	0.05	0.00
	Power+	49.40	70.60	91.65	98.60	99.95	99.85	99.75
	Power-	49.40	99.85	100.00	100.00	100.00	99.85	99.75
500	Size	17.50	6.70	2.85	0.60	0.35	0.05	0.00
	Power+	83.55	98.10	99.80	99.90	99.90	99.80	99.70
	Power-	83.55	100.00	100.00	99.95	99.90	99.80	99.70
1000	Size	11.95	3.85	1.20	0.30	0.05	0.00	0.00
	Power+	96.50	99.80	99.85	99.80	99.50	99.35	99.25
	Power-	96.50	99.95	99.85	99.80	99.50	99.35	99.25
					500			
50	Size	93.35	80.80	63.85	40.30	12.55	1.75	0.00
	Power+	45.75	14.60	8.15	21.85	61.45	93.65	100.00
	Power-	45.75	99.60	99.40	99.45	99.55	99.95	100.00
100	Size	86.85	75.55	48.10	37.65	13.30	0.75	0.00
	Power+	12.20	12.95	43.10	72.85	97.70	100.00	100.00
	Power-	12.20	99.95	99.95	100.00	100.00	100.00	100.00
200	Size	60.50	58.95	35.20	20.65	4.70	0.10	0.00
	Power+	33.60	57.25	91.55	99.60	100.00	100.00	100.00
	Power-	33.60	100.00	100.00	100.00	100.00	100.00	100.00
500	Size	49.20	30.65	16.75	4.25	0.40	0.05	0.00
	Power+	81.45	98.80	100.00	100.00	100.00	100.00	99.95
	Power-	81.45	100.00	100.00	100.00	100.00	100.00	99.95
1000	Size	30.45	14.50	5.90	1.70	0.10	0.00	0.00
	Power+	98.10	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	98.10	100.00	100.00	100.00	100.00	100.00	100.00

Notes: Size is computed under H_0: $\alpha=\alpha_0$, using a two-sided alternative where α_0 takes values in the range [0.70,1.00].
Power is computed under the alternatives H_a: $\alpha_a = \alpha_0 + 0.05$ (power+), and $\alpha_a = \alpha_0 + 0.05$ (power-).

Table A4: Bias, RMSE, size and power ($\times 100$) for the $\mathring{\alpha}$ estimate of the cross-sectional exponent - case of two serially independent factors and cross-sectionally independent, non-normal idiosyncratic errors ($\varepsilon_{it} \sim IID\chi^2(2)$)

($\alpha_2 = 2\alpha/3$, f_{jt} and $u_{it} \sim IIDN(0,1)$, $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_v = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_2 - \alpha)}\mu_{v_2}^2}$)

N=50,100,200,500,1000 and T=100,200,500

N\T	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
					100			
50	Bias	2.26	1.09	0.43	0.07	-0.20	0.00	-0.24
	RMSE	3.64	2.59	1.83	1.29	0.91	0.53	0.25
100	Bias	1.24	0.60	0.12	0.30	0.17	-0.01	-0.05
	RMSE	2.29	1.56	0.98	0.73	0.49	0.29	0.06
200	Bias	0.27	0.36	0.17	0.13	0.02	0.02	0.03
	RMSE	1.26	0.89	0.57	0.41	0.27	0.16	0.04
500	Bias	0.14	0.04	0.08	0.02	0.03	0.02	0.05
	RMSE	0.72	0.49	0.35	0.24	0.17	0.10	0.06
1000	Bias	0.02	0.01	0.05	0.01	0.03	0.00	0.05
	RMSE	0.50	0.34	0.25	0.17	0.12	0.07	0.06
					200			
50	Bias	4.73	2.97	1.79	1.00	0.33	0.22	-0.29
	RMSE	5.61	3.82	2.60	1.77	1.10	0.68	0.30
100	Bias	2.10	1.29	0.49	0.51	0.30	0.04	-0.10
	RMSE	2.85	1.97	1.18	0.89	0.60	0.32	0.10
200	Bias	0.64	0.64	0.31	0.23	0.08	0.04	-0.02
	RMSE	1.40	1.09	0.66	0.48	0.29	0.16	0.02
500	Bias	0.28	0.13	0.14	0.06	0.06	0.04	0.02
	RMSE	0.67	0.42	0.31	0.20	0.14	0.08	0.02
1000	Bias	0.11	0.07	0.10	0.06	0.07	0.03	0.03
	RMSE	0.41	0.26	0.20	0.14	0.11	0.05	0.03
					500			
50	Bias	8.70	5.81	3.87	2.47	1.31	0.66	-0.34
	RMSE	9.16	6.23	4.26	2.86	1.68	1.00	0.34
100	Bias	4.64	3.33	1.92	1.45	0.84	0.30	-0.13
	RMSE	4.97	3.58	2.19	1.68	1.07	0.51	0.13
200	Bias	2.15	1.77	0.95	0.67	0.32	0.12	-0.05
	RMSE	2.52	2.03	1.21	0.88	0.50	0.25	0.05
500	Bias	0.88	0.45	0.30	0.13	0.09	0.04	-0.01
	RMSE	1.18	0.69	0.47	0.27	0.17	0.09	0.01
1000	Bias	0.30	0.15	0.13	0.07	0.06	0.02	0.00
	RMSE	0.55	0.32	0.22	0.14	0.09	0.04	0.00

N\T	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
					100			
50	Size	11.00	4.95	2.05	0.10	0.10	0.00	0.00
	Power+	17.95	30.05	44.75	60.90	81.25	90.20	96.80
	Power-	17.95	58.30	58.90	64.55	71.85	92.25	96.80
100	Size	5.30	1.95	0.25	0.05	0.00	0.00	0.00
	Power+	38.75	58.40	80.40	90.25	98.60	100.00	100.00
	Power-	38.75	81.15	87.05	98.15	99.85	100.00	100.00
200	Size	1.40	0.20	0.05	0.00	0.00	0.00	0.00
	Power+	72.25	88.80	98.65	99.95	100.00	100.00	100.00
	Power-	72.25	96.60	99.70	100.00	100.00	100.00	100.00
500	Size	0.35	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	97.75	99.80	100.00	100.00	100.00	100.00	100.00
	Power-	97.75	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	99.80	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.80	100.00	100.00	100.00	100.00	100.00	100.00
					200			
50	Size	41.15	22.70	11.50	3.85	0.25	0.00	0.00
	Power+	10.60	16.60	31.50	54.55	81.45	95.65	100.00
	Power-	10.60	85.40	85.20	89.50	93.75	99.65	100.00
100	Size	17.30	7.85	2.25	0.70	0.15	0.00	0.00
	Power+	34.00	54.50	84.50	95.25	99.55	100.00	100.00
	Power-	34.00	93.05	97.15	99.85	100.00	100.00	100.00
200	Size	4.65	3.30	0.55	0.15	0.00	0.00	0.00
	Power+	75.75	90.90	99.65	100.00	100.00	100.00	100.00
	Power-	75.75	99.75	100.00	100.00	100.00	100.00	100.00
500	Size	0.50	0.20	0.00	0.00	0.00	0.00	0.00
	Power+	98.65	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	98.65	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	0.20	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
					500			
50	Size	89.35	71.95	50.20	27.95	6.70	1.20	0.00
	Power+	34.70	8.40	10.15	32.10	72.65	96.45	100.00
	Power-	34.70	99.20	99.10	99.30	99.75	99.95	100.00
100	Size	71.20	57.85	29.15	21.20	6.95	0.15	0.00
	Power+	10.10	20.80	63.00	90.10	99.80	100.00	100.00
	Power-	10.10	99.90	99.95	100.00	100.00	100.00	100.00
200	Size	34.45	35.90	15.40	7.85	0.55	0.00	0.00
	Power+	54.15	77.85	98.90	99.95	100.00	100.00	100.00
	Power-	54.15	100.00	100.00	100.00	100.00	100.00	100.00
500	Size	14.20	5.60	2.30	0.45	0.00	0.00	0.00
	Power+	96.60	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	96.60	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	3.20	1.10	0.15	0.00	0.00	0.00	0.00
	Power+	99.95	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.95	100.00	100.00	100.00	100.00	100.00	100.00

Notes: Size is computed under H_0: α=α_0, using a two-sided alternative where α_0 takes values in the range [0.70,1.00].
Power is computed under the alternatives H_a: $\alpha_a = \alpha_0 + 0.05$ (power+), and $\alpha_a = \alpha_0 + 0.05$ (power-).

Table A5: Bias, RMSE, size and power ($\times 100$) for the $\mathring{\alpha}$ estimate of the cross-sectional exponent - case of two serially independent factors and cross-sectionally dependent idiosyncratic errors ($\theta = 0.2$)

($\alpha_2 = 2\alpha/3$, f_{jt} and $u_{it} \sim IIDN(0,1)$, $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_v = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_2 - \alpha)}\mu_{v_2}^2}$)

N=50,100,200,500,1000 and T=100,200,500

	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
N\T					**100**			
50	Bias	3.57	2.11	1.20	0.74	0.35	0.45	0.10
	RMSE	4.77	3.33	2.27	1.61	1.06	0.74	0.11
100	Bias	1.62	0.89	0.37	0.53	0.39	0.18	0.09
	RMSE	2.45	1.73	1.10	0.90	0.63	0.35	0.10
200	Bias	0.31	0.39	0.22	0.19	0.09	0.09	0.09
	RMSE	1.23	0.97	0.66	0.49	0.33	0.20	0.09
500	Bias	0.13	0.03	0.07	0.02	0.04	0.05	0.07
	RMSE	0.75	0.54	0.40	0.27	0.19	0.12	0.07
1000	Bias	-0.01	-0.02	0.03	0.00	0.03	0.01	0.06
	RMSE	0.56	0.39	0.28	0.20	0.14	0.08	0.07
					200			
50	Bias	6.83	4.52	2.95	1.89	0.98	0.70	0.04
	RMSE	7.52	5.17	3.57	2.46	1.50	0.99	0.04
100	Bias	2.78	1.79	0.85	0.81	0.53	0.23	0.05
	RMSE	3.42	2.39	1.43	1.14	0.76	0.40	0.05
200	Bias	0.87	0.77	0.42	0.33	0.17	0.12	0.04
	RMSE	1.52	1.19	0.74	0.55	0.35	0.21	0.04
500	Bias	0.28	0.12	0.14	0.07	0.08	0.07	0.04
	RMSE	0.69	0.45	0.33	0.22	0.16	0.10	0.04
1000	Bias	0.08	0.04	0.08	0.05	0.07	0.04	0.04
	RMSE	0.44	0.30	0.22	0.15	0.12	0.06	0.04
					500			
50	Bias	13.52	9.01	5.90	3.75	2.20	1.23	0.01
	RMSE	13.93	9.39	6.24	4.07	2.51	1.46	0.01
100	Bias	6.01	4.16	2.48	1.87	1.13	0.54	0.01
	RMSE	6.36	4.38	2.71	2.06	1.30	0.69	0.01
200	Bias	2.47	1.97	1.10	0.79	0.43	0.21	0.02
	RMSE	2.78	2.22	1.35	0.99	0.59	0.31	0.02
500	Bias	0.89	0.46	0.31	0.15	0.11	0.07	0.02
	RMSE	1.21	0.74	0.49	0.29	0.19	0.11	0.02
1000	Bias	0.27	0.13	0.11	0.06	0.06	0.03	0.01
	RMSE	0.53	0.31	0.21	0.14	0.10	0.05	0.01

	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
N\T					**100**			
50	Size	25.80	13.00	4.95	1.05	0.35	0.00	0.00
	Power+	13.85	24.15	34.40	46.70	67.50	79.10	99.50
	Power-	13.85	74.10	72.95	79.70	85.35	96.85	99.50
100	Size	8.10	3.40	0.35	0.15	0.00	0.00	0.00
	Power+	34.70	54.05	77.15	88.75	97.75	99.85	100.00
	Power-	34.70	88.00	92.10	98.85	99.95	100.00	100.00
200	Size	0.90	0.40	0.10	0.00	0.00	0.00	0.00
	Power+	75.75	89.30	98.60	99.85	100.00	100.00	100.00
	Power-	75.75	97.45	99.85	100.00	100.00	100.00	100.00
500	Size	0.00	0.05	0.00	0.00	0.00	0.00	0.00
	Power+	98.50	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	98.50	99.95	100.00	100.00	100.00	100.00	100.00
1000	Size	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	99.95	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.95	100.00	100.00	100.00	100.00	100.00	100.00
					200			
50	Size	71.10	50.45	28.40	12.55	2.10	0.25	0.00
	Power+	17.75	11.40	17.90	36.80	67.20	88.60	100.00
	Power-	17.75	96.45	95.40	96.75	97.65	99.95	100.00
100	Size	28.10	16.85	5.30	2.10	0.40	0.00	0.00
	Power+	29.10	48.00	79.20	91.95	99.45	100.00	100.00
	Power-	29.10	97.75	98.65	100.00	100.00	100.00	100.00
200	Size	5.35	4.30	0.80	0.00	0.00	0.00	0.00
	Power+	74.20	89.80	99.20	100.00	100.00	100.00	100.00
	Power-	74.20	99.90	100.00	100.00	100.00	100.00	100.00
500	Size	0.80	0.10	0.00	0.00	0.00	0.00	0.00
	Power+	99.05	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.05	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	0.05	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
					500			
50	Size	99.90	96.90	85.30	58.75	26.00	4.50	0.00
	Power+	90.15	46.80	9.80	13.35	48.10	88.45	100.00
	Power-	90.15	100.00	100.00	99.85	99.90	100.00	100.00
100	Size	92.55	81.90	49.80	37.30	11.90	0.85	0.00
	Power+	12.30	11.80	49.05	81.55	99.20	100.00	100.00
	Power-	12.30	100.00	100.00	100.00	100.00	100.00	100.00
200	Size	47.65	45.60	21.95	11.80	1.35	0.00	0.00
	Power+	51.70	75.45	98.45	100.00	100.00	100.00	100.00
	Power-	51.70	100.00	100.00	100.00	100.00	100.00	100.00
500	Size	15.95	7.05	3.05	0.45	0.00	0.00	0.00
	Power+	96.50	99.95	100.00	100.00	100.00	100.00	100.00
	Power-	96.50	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	2.25	0.65	0.15	0.00	0.00	0.00	0.00
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Notes: Size is computed under H_0: $\alpha=\alpha_0$, using a two-sided alternative where α_0 takes values in the range [0.70,1.00].
Power is computed under the alternatives H_a: $\alpha_a = \alpha_0 + 0.05$ (power+), and $\alpha_a = \alpha_0 + 0.05$ (power-).

Table A6: Bias, RMSE, size and power ($\times 100$) for the $\mathring{\alpha}$ estimate of the cross-sectional exponent - case of two serially independent factors and cross-sectionally dependent idiosyncratic errors ($\theta = 0.4$)

($\alpha_2 = 2\alpha/3$, f_{jt} and $u_{it} \sim IIDN(0,1)$, $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1,2$, $\mu_v = 0.87$, $\mu_{v_2} = 0.71$, $\mu_{v_1} = \sqrt{\mu_v^2 - N^{2(\alpha_2-\alpha)}\mu_{v_2}^2}$)

N=50,100,200,500,1000 and T=100,200,500

	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
N\T					**100**			
50	Bias	5.30	3.16	1.84	1.08	0.52	0.52	0.10
	RMSE	6.48	4.37	2.94	2.00	1.28	0.86	0.11
100	Bias	2.03	1.08	0.44	0.57	0.39	0.17	0.09
	RMSE	2.95	2.00	1.25	1.00	0.67	0.38	0.10
200	Bias	0.52	0.49	0.26	0.21	0.11	0.09	0.08
	RMSE	1.44	1.11	0.74	0.54	0.36	0.21	0.09
500	Bias	0.21	0.05	0.09	0.03	0.05	0.05	0.07
	RMSE	0.87	0.62	0.45	0.31	0.21	0.13	0.08
1000	Bias	0.04	-0.01	0.03	0.01	0.03	0.01	0.06
	RMSE	0.64	0.45	0.32	0.22	0.15	0.09	0.07
					200			
50	Bias	10.08	6.62	4.20	2.59	1.34	0.85	0.04
	RMSE	10.76	7.28	4.82	3.17	1.88	1.18	0.04
100	Bias	3.85	2.37	1.17	0.94	0.58	0.26	0.05
	RMSE	4.55	3.01	1.75	1.31	0.84	0.45	0.05
200	Bias	1.16	0.89	0.47	0.35	0.19	0.13	0.04
	RMSE	1.81	1.34	0.83	0.62	0.38	0.22	0.04
500	Bias	0.40	0.17	0.16	0.08	0.09	0.07	0.04
	RMSE	0.82	0.53	0.39	0.25	0.18	0.11	0.04
1000	Bias	0.11	0.05	0.08	0.05	0.07	0.04	0.04
	RMSE	0.48	0.33	0.24	0.17	0.12	0.07	0.04
					500			
50	Bias	19.84	13.93	9.02	5.40	3.03	1.55	0.01
	RMSE	20.11	14.29	9.40	5.78	3.39	1.84	0.01
100	Bias	8.91	5.64	3.25	2.25	1.30	0.61	0.01
	RMSE	9.28	5.91	3.51	2.46	1.49	0.78	0.01
200	Bias	3.35	2.32	1.28	0.88	0.46	0.23	0.02
	RMSE	3.72	2.57	1.53	1.10	0.64	0.33	0.02
500	Bias	1.03	0.50	0.32	0.14	0.11	0.07	0.02
	RMSE	1.33	0.77	0.51	0.29	0.19	0.10	0.02
1000	Bias	0.31	0.13	0.11	0.06	0.06	0.03	0.01
	RMSE	0.58	0.35	0.23	0.15	0.10	0.05	0.01

	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
N\T					**100**			
50	Size	48.40	26.20	12.25	3.50	0.95	0.15	0.00
	Power+	17.40	19.45	28.50	41.55	61.85	75.95	99.45
	Power-	17.40	84.15	80.40	82.85	86.10	96.65	99.45
100	Size	16.00	5.70	0.95	0.30	0.00	0.00	0.00
	Power+	32.55	51.25	76.10	85.80	97.10	99.70	100.00
	Power-	32.55	88.20	92.10	99.05	99.90	100.00	100.00
200	Size	2.20	0.90	0.20	0.00	0.00	0.00	0.00
	Power+	72.40	86.90	97.50	99.75	100.00	100.00	100.00
	Power-	72.40	97.60	99.80	100.00	100.00	100.00	100.00
500	Size	0.80	0.15	0.00	0.00	0.00	0.00	0.00
	Power+	97.55	99.90	100.00	100.00	100.00	100.00	100.00
	Power-	97.55	99.95	100.00	100.00	100.00	100.00	100.00
1000	Size	0.10	0.00	0.00	0.00	0.00	0.00	0.00
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
					200			
50	Size	91.85	77.25	51.95	27.45	6.70	0.80	0.00
	Power+	55.85	20.45	12.25	25.75	57.70	83.75	100.00
	Power-	55.85	98.65	97.95	98.25	98.55	99.95	100.00
100	Size	52.65	31.60	10.05	3.95	0.50	0.00	0.00
	Power+	23.45	40.55	72.10	89.25	98.90	100.00	100.00
	Power-	23.45	98.45	98.85	100.00	100.00	100.00	100.00
200	Size	11.20	6.80	1.75	0.45	0.00	0.00	0.00
	Power+	71.70	89.55	98.90	99.95	100.00	100.00	100.00
	Power-	71.70	99.85	100.00	100.00	100.00	100.00	100.00
500	Size	1.60	0.35	0.05	0.00	0.00	0.00	0.00
	Power+	98.45	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	98.45	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	0.25	0.05	0.00	0.00	0.00	0.00	0.00
	Power+	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	100.00	100.00	100.00	100.00	100.00	100.00	100.00
					500			
50	Size	100.00	99.95	98.15	83.10	50.30	12.85	0.00
	Power+	99.75	94.85	54.65	12.45	29.70	77.65	100.00
	Power-	99.75	100.00	100.00	100.00	100.00	100.00	100.00
100	Size	99.70	95.40	73.35	55.00	20.70	1.45	0.00
	Power+	61.05	12.05	32.10	69.90	98.25	100.00	100.00
	Power-	61.05	100.00	100.00	100.00	100.00	100.00	100.00
200	Size	73.25	61.80	30.65	18.00	2.35	0.05	0.00
	Power+	37.65	68.45	96.65	100.00	100.00	100.00	100.00
	Power-	37.65	100.00	100.00	100.00	100.00	100.00	100.00
500	Size	23.10	8.60	3.30	0.45	0.05	0.00	0.00
	Power+	96.30	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	96.30	100.00	100.00	100.00	100.00	100.00	100.00
1000	Size	4.00	1.50	0.15	0.00	0.00	0.00	0.00
	Power+	99.95	100.00	100.00	100.00	100.00	100.00	100.00
	Power-	99.95	100.00	100.00	100.00	100.00	100.00	100.00

Notes: Size is computed under H_0: α=α_0, using a two-sided alternative where α_0 takes values in the range [0.70,1.00].
Power is computed under the alternatives H_a: $\alpha_a = \alpha_0 + 0.05$ (power+), and $\alpha_a = \alpha_0 + 0.05$ (power-).

Table A7: Bias and RMSE ($\times 100$) for the $\mathring{\alpha}$ estimate of the cross-sectional exponent - case of two serially independent factors and cross-sectionally independent idiosyncratic errors ($\alpha_2 = \alpha$, f_{jt} and $u_{it} \sim IIDN(0,1)$, $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, $j = 1, 2$, $\mu_{v_1} = 0.5$, $\mu_{v_2} = 0.5$) N=50,100,200,500,1000 and T=100,200,500

	α	0.70	0.75	0.80	0.85	0.90	0.95	1.00
N\T					**100**			
50	Bias	10.13	8.92	8.12	7.06	5.48	3.46	-0.22
	RMSE	10.39	9.13	8.30	7.21	5.60	3.54	0.23
100	Bias	9.41	8.51	7.73	6.96	5.45	3.28	-0.04
	RMSE	9.55	8.63	7.82	7.03	5.51	3.33	0.06
200	Bias	8.92	8.41	7.54	6.49	5.04	3.12	0.03
	RMSE	9.00	8.47	7.59	6.53	5.08	3.14	0.04
500	Bias	7.39	7.01	6.52	5.66	4.50	2.81	0.05
	RMSE	7.45	7.06	6.56	5.69	4.53	2.83	0.06
1000	Bias	6.82	6.49	6.00	5.21	4.14	2.56	0.05
	RMSE	6.86	6.53	6.03	5.24	4.17	2.58	0.06
					200			
50	Bias	11.68	10.22	9.10	7.92	6.13	3.72	-0.27
	RMSE	11.86	10.38	9.22	8.02	6.21	3.77	0.28
100	Bias	10.95	9.84	8.91	8.00	6.29	3.72	-0.10
	RMSE	11.03	9.90	8.97	8.04	6.32	3.75	0.10
200	Bias	9.60	9.35	8.56	7.51	5.92	3.68	-0.02
	RMSE	9.64	9.38	8.58	7.53	5.93	3.69	0.02
500	Bias	8.76	8.36	7.81	6.88	5.56	3.50	0.02
	RMSE	8.78	8.37	7.82	6.89	5.56	3.51	0.02
1000	Bias	8.08	7.77	7.29	6.46	5.26	3.36	0.03
	RMSE	8.09	7.78	7.29	6.47	5.27	3.36	0.03
					500			
50	Bias	14.88	11.89	9.86	8.23	6.31	3.68	-0.35
	RMSE	15.03	12.03	9.98	8.32	6.38	3.72	0.35
100	Bias	11.54	10.39	9.39	8.35	6.52	3.84	-0.12
	RMSE	11.60	10.43	9.43	8.38	6.54	3.86	0.12
200	Bias	10.26	9.90	9.07	7.94	6.26	3.83	-0.04
	RMSE	10.28	9.92	9.09	7.96	6.27	3.84	0.04
500	Bias	9.40	8.97	8.42	7.44	6.00	3.75	-0.01
	RMSE	9.41	8.98	8.43	7.44	6.00	3.75	0.01
1000	Bias	8.89	8.43	7.93	7.08	5.77	3.67	0.00
	RMSE	8.89	8.43	7.94	7.08	5.77	3.67	0.00

Supplementary Appendix VII: Empirical exponents of cross-sectional dependence with confidence bands

The following two figures show the estimates of $\mathring{\alpha}_t$ and $\mathring{\alpha}_u$ displayed in Figures 1 and 4 including their respective 95% confidence bands.

Figure 5: $\mathring{\alpha}_t$ associated with S&P 500 securities' excess returns $-$ 5-yr rolling samples

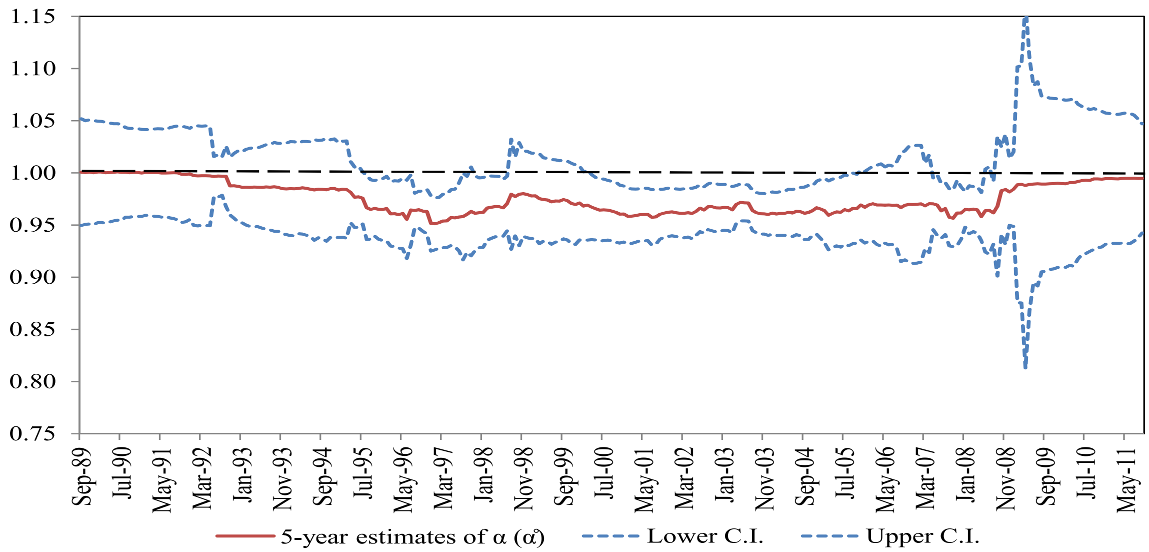
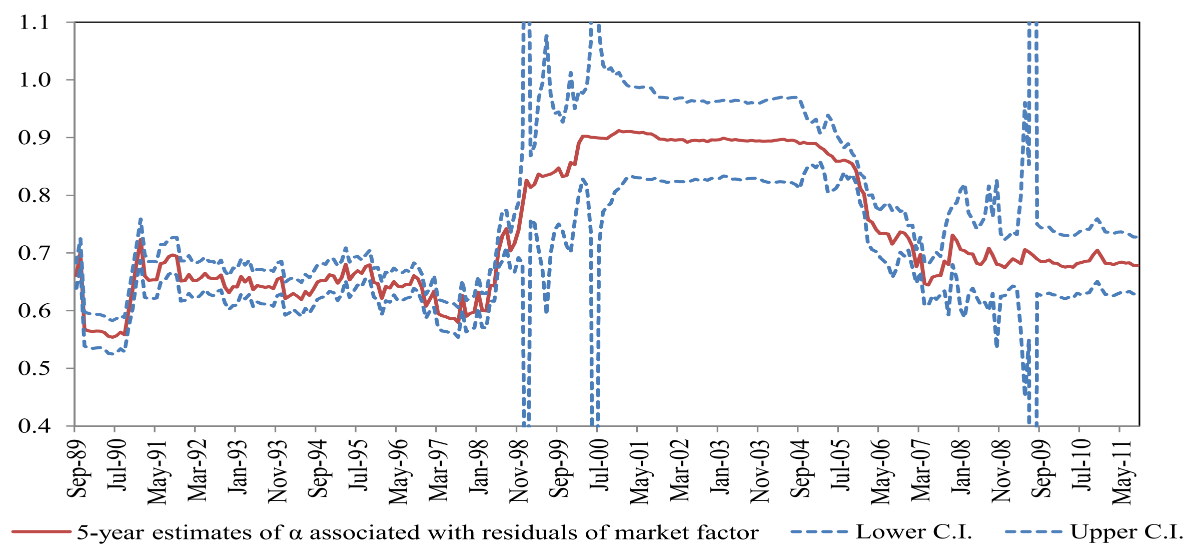


Figure 6: $\mathring{\alpha}_t$ associated with S&P 500 securities' excess returns $-$ 5-yr rolling samples



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