Estimation and Inference for Spatial Models with Heterogeneous Coefficients: An Application to U.S. House Prices*

Michele Aquaro
Joint Research Centre, European Commission

Natalia Bailey
Department of Econometrics and Business Statistics, Monash University

M. Hashem Pesaran
Department of Economics, University of Southern California,
and Trinity College, Cambridge

June 13, 2020

Abstract

This paper considers the estimation and inference of spatial panel data models with heterogeneous spatial lag coefficients, with and without weakly exogenous regressors, and subject to heteroskedastic errors. A quasi maximum likelihood (QML) estimation procedure is developed and the conditions for identification of the spatial coefficients are derived. The QML estimators of individual spatial coefficients, as well as their mean group estimators, are shown to be consistent and asymptotically normal. Small sample properties of the proposed estimators are investigated by Monte Carlo simulations and results are in line with the paper's key theoretical findings even for panels with moderate time dimensions and irrespective of the number of cross section units. A detailed empirical application to U.S. house price changes during the 1975-2014 period shows a significant degree of heterogeneity in spatio-temporal dynamics over the 338 Metropolitan Statistical Areas considered.

Keywords: Spatial panel data models, heterogeneous spatial lag coefficients, identification, quasi maximum likelihood (QML) estimators, house price changes, Metropolitan Statistical Areas.

JEL Codes: C21, C23

*The authors would like to thank Alex Chudik, Bernard Fingleton, Harry Kelejian, Matt Kahn, Ron Smith, Cynthia Fan Yang, Herman van Dijk (Co-editor), four anonymous referees and participants at the following conferences: 2015 (EC^2) conference: on Theory and Practice of Spatial Econometrics, Edinburgh, UK, 2015 conference on Endogenous Financial Networks and Equilibrium Dynamics: Addressing Challenges of Financial Stability and Monetary Policy, Bank of France, Paris, France, 2016 SEA conference, Rome, Italy, 2018 conference on Housing, Urban Development, and the Macroeconomy, USC, California, U.S., 2018 Billfest conference, Melbourne, Australia, 2018 ESAM conference, Auckland, New Zealand, 2019 IAAE conference, Nicosia, Cyprus, 2019 XIII World SEA-NARSC conference, Pittsburg, U.S. for valuable comments and suggestions. Financial support under ESRC Grant ES/I031626/1 is also gratefully acknowledged. All errors and omissions are our own. The scientific output expressed does not imply a policy position of the European Commission. Neither the European Commission nor any person acting on behalf of the Commission is responsible for the use which might be made of this paper.
Correspondence to: Natalia Bailey, Department of Econometrics and Business Statistics, Monash University, Monash University, Clayton, 3800, Melbourne, Australia. E-mail: natalia.bailey@monash.edu.

1 Introduction

This paper considers a heterogeneous version of the standard spatial autoregressive (SAR) panel data model whereby the spatial lag coefficients are allowed to differ over the cross section units, in addition to the fixed effects generally allowed for in the literature. We refer to this generalised specification as the heterogeneous SAR (or HSAR) model. The model also features weakly exogenous regressors, such as lagged values of the dependent variable, heteroskedastic error variances, and provides a reasonably general framework for the analysis of heterogeneous interactions, where it is important to distinguish between the average intensity of spill-over effects as characterised by standard spatial models, and the heterogeneity of such effects over different geographical units such as counties, regions or countries. Importantly, the framework studied in this paper allows for spatial dependence directly through contemporaneous dependence of individual units on their connections, and indirectly through possible cross-sectional dependence in the regressors. The econometric analysis of HSAR models presents new technical difficulties, both for identification and estimation of a large set of spatial lag coefficients that must be estimated simultaneously.

Our analysis builds on the existing literature on SAR models, pioneered by Whittle (1954) and Cliff and Ord (1973), and further advanced in a number of important directions. The maximum likelihood approach of Cliff and Ord, which was developed for a pure spatial model, has been extended to cover panel data models with fixed effects and dynamics. Other estimation and testing techniques, such as the generalised method of moments (GMM), also have been proposed. Some of the key references to this literature include Upton and Fingleton (1985), Anselin (1988), Cressie (1993), Kelejian and Robinson (1993), Ord and Getis (1995), Anselin and Bera (1998), and more recently, Haining (2003), Lee (2004), Kelejian and Prucha (1999), Kelejian and Prucha (2010), Lin and Lee (2010), Lee and Yu (2010), LeSage and Pace (2010), Arbia (2010), Cressie and Wikle (2011), and Elhorst (2014). Extensions to dynamic panels are provided by Anselin (2001), Baltagi et al. (2003), Kapoor et al. (2007), Baltagi et al. (2007), Yu et al. (2008) and Yu et al. (2012). Spatial techniques also have proved useful when analysing network effects as can be seen in the pioneering work of Case (1991) and Manski (1993).

Almost all these contributions (whether in the context of spatial or network models) assume that, apart from possibly fixed effects, spatial spill-over or network effects are homogeneous. However, even if all units in a network have the exact same number of connections, it can be the case that not all units are equally important or influential. Therefore, the assumption of a homogeneous spatial coefficient is likely to be restrictive, and should be relaxed when T is large. As shown in this paper, when T is large the heterogenous spatial model can be estimated for any N and it is not required that $N \to \infty$, which is needed for estimation of the traditional SAR model with a single spatial coefficient when T is small.

Examples of panel data sets with T large include panels that cover counties, regions, or countries in the analysis of economic variables such as house prices, real wages, employment and income. For instance, in the empirical applications by Baltagi and Levin (1986) on demand for tobacco consumption, and by Holly et al. (2010) on house price diffusion across states in the U.S., it is interesting to investigate whether the maintained assumption that spill-over effects from neighbouring states are the same across all the 48 mainland states in fact holds, particularly considering the large size of the U.S. and the uneven distribution of economic activity across it.

Whilst estimation of HSAR panel data models can be carried out using MLE and GMM approaches, in this paper we focus on the former and discuss identification, estimation and inference using the quasi maximum likelihood (QML) method. We derive conditions under which the QML estimators of the *individual* parameters are locally identified, and establish consistency and asymptotic normality of the estimators under certain regularity conditions. Asymptotic covariance matrices of the QML estimators are derived under both Gaussian and non-Gaussian errors, and consistent estimators of these covariances are proposed. Alternative estimation methods based on our HSAR model include the Bayesian Markov Chain Monte Carlo approach of LeSage and Chih (2018b) and the generalised Yule-Walker estimation method of Dou et al. (2016).

Although the estimation of individual coefficients of the HSAR model can be carried out for *any* N when T is large, the estimation of the mean of the coefficients across the units requires both N and T to be large. Accordingly, we propose an estimator of the cross section mean of the individual parameters (also known in the literature as the Mean Group, MG, estimator) assuming a random coefficient model, and show that the mean group estimators are consistent and asymptotically normal if both N and T tend to infinity jointly, such that $\sqrt{N}/T \to 0$, and the spatial dependence is sufficiently weak. Such estimators are helpful in two respects. They provide an overall average estimator of the spatial effects that could be compared to corresponding estimates obtained using standard homogeneous SAR models. They can also be used to obtain average estimators across sub-spatial groupings such as states or regions, or sub-groups within a production or financial network, such as industry types.

The small sample performance of the QML estimator is investigated by Monte Carlo simulations for different values of N and T and alternative choices of the spatial weight matrices. The simulation results are in line with the paper's key theoretical findings, and show that the proposed estimators have good small sample properties for panels with moderate time dimensions and irrespective of the number of cross section units in the panel, although under non-Gaussian errors, tests based on QML estimators of the spatial parameters can be slightly distorted when the time dimension is relatively small. We also investigate the small sample performance of the MG estimator and find its performance to be satisfactory with biases that are universally negligible, and RMSEs that decline with T and quite rapidly with N. Regarding size and power, tests based on the MG estimator exhibit some downward size distortions when T is small, but such distortions disappear as T rises for all values of N. The small sample bias of the MG estimator can be reduced using half-Jackknife procedure as discussed in Chudik and Pesaran (2019).

We provide an empirical application by modelling the spatial and temporal dimensions of quarterly U.S. house prices changes over the period 1975Q1-2014Q4 and across Metropolitan Statistical Areas (MSAs). Not surprisingly, we find a considerable degree of heterogeneity across the MSA specific estimates. As to be expected, the estimates of net spatial coefficients (contemporaneous and lagged) are mostly positive and statistically significant, suggesting a high degree of spill-over effects of house price changes to neighbouring MSAs. There were only 18 MSAs (out of 338 considered in our analysis) with statistically significant negative net spatial effects, and included Pittsfield (Massachusetts), Minneapolis (Minnesota) or Memphis (Arkansas). These MSAs tend to be relatively remote with outward migratory flows to the neighbouring regions.

We also consider mean estimates for six U.S. regions, and find the contemporaneous spatial lag coefficients to be all positive and statistically significant for all the regions. However, the net spatial effects, computed as the sum of the coefficients of contemporaneous and lagged spatial variables are positive in all regions but statistically significant only in the case half of these, namely Great Lakes, South East and Far West. This result clearly shows the importance of allowing for dynamics in the analysis of spatial effects, and is to be contrasted with the large (net) spatial effect of around 0.65 found by Yang (2020) who considers a homogeneous and static SAR specification. We also find positive and statistically significant effects of population and income growth on own-region house price changes, again with a high degree of heterogeneity across the regions. Using techniques analogous to the work of LeSage and Chih (2016), we also report direct and indirect partial effects of changes in population and income growth on house prices over time. The results continue to exhibit a high degree of heterogeneity across MSAs and regions with direct effects dominating, and the outcomes decaying quite rapidly.

The rest of the paper is organised as follows: Section 2 introduces the first order spatial autoregressive model with heterogeneous coefficients and some useful generalisations, and derives its log-likelihood function. Section 3 sets out the assumptions of the model, derives the identification conditions and proves consistency and asymptotic normality of the QML estimator when the time dimension is large. Section 4 outlines the Mean Group estimator derived from the heterogeneous spatial coefficients of the HSAR model. Section 5 presents the Monte Carlo design and reports small sample results (bias, root mean square errors, size and power) of the QML and MG estimators for different parameter values and sample size combinations. Section 6 reports the results of our empirical application to the U.S. house price changes across MSAs. Some concluding remarks are provided in Section 7. Mathematical proofs, data sources and additional empirical and Monte Carlo results are provided in an the online supplement.

Notations: We denote the largest and the smallest eigenvalues of the $N \times N$ matrix $A = (a_{ij})$ by $\lambda_{\max}(\boldsymbol{A})$ and $\lambda_{\min}(\boldsymbol{A})$, respectively, its trace by $\mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^{N} a_{ii}$, its spectral radius by $\rho(\boldsymbol{A}) = |\lambda_{\max}(\boldsymbol{A})|$, its spectral norm by $\|\boldsymbol{A}\| = \lambda_{\max}^{1/2}(\boldsymbol{A}'\boldsymbol{A})$, its maximum absolute column sum norm by $\|\boldsymbol{A}\|_1 = \max_{1\le j\le N}\left(\sum_{i=1}^{N}|a_{ij}|\right)$, and its maximum absolute row sum norm by $\|\boldsymbol{A}\|_\infty = \max_{1\le i\le N}\left(\sum_{j=1}^{N}|a_{ij}|\right)$. $\mathrm{Diag}(\boldsymbol{A}) = \mathrm{Diag}(a_{11}, a_{22}, \ldots, a_{NN})$ represents an $N \times N$ diagonal matrix formed by the diagonal elements of $\boldsymbol{A}$, while $\mathrm{diag}(\boldsymbol{A}) = (a_{11}, a_{22}, \ldots, a_{NN})'$ denotes an $N \times 1$ vector. We denote the ℓ_p-norm of the random variable x by $\|x\|_p = E(|x|^p)^{1/p}$ for $p \ge 1$, assuming that $E(|x|^p) < K$. $\odot$ stands for Hadamard product or element-wise matrix product operator, $\to_p$ denotes convergence in probability, $\overset{a.s.}{\to}$ almost sure convergence, $\to_d$ convergence in distribution, and $\overset{a}{\sim}$ asymptotic equivalence in distribution. K and c will be used to denote finite large and non-zero small positive numbers, respectively, that do not depend on N and T.

2 A heterogeneous spatial autoregressive model (HSAR)

2.1 Model specification

We consider the following SAR model with heterogeneous slopes:

$$y_{it} = \psi_{i0}\left(\sum_{j=1}^{N} w_{ij}y_{jt}\right) + \boldsymbol{\beta}_{i0}'\boldsymbol{x}_{it} + \varepsilon_{it}, \text{ for } i = 1, 2, \ldots, N;\ t = 1, 2, \ldots, T, \tag{1}$$

where y_{it} is the dependent variable for unit i observed at time t, $\boldsymbol{x}_{it} = (x_{i1,t}, x_{i2,t}, \ldots, x_{ik,t})'$ is a $k \times 1$ vector of (weakly) exogenous regressors, with the associated $k \times 1$ vector of slope parameters, and $\boldsymbol{\beta}_{i0} = (\beta_{i1,0}, \beta_{i2,0}, \ldots, \beta_{ik,0})'$. ε_{it} is the unexplained component of y_{it}, which we refer to as the error of the i^{th} cross section unit, or the 'error' for short, with variance $Var(\varepsilon_{it}) = \sigma_{i0}^2$. Finally, $y_{it}^* = \sum_{j=1}^{N} w_{ij}y_{jt} = \mathbf{w}_i'\mathbf{y}_t$ is the average effect of other units on unit i at time t, where $\mathbf{y}_t = (y_{1t}, y_{2t}, \ldots, y_{Nt})'$ and $\mathbf{w}_i'$ is the i^{th} row of the $N \times N$ spatial weight matrix, $\mathbf{W} = (w_{ij})$, with w_{ij}, for $i, j = 1, 2, \ldots, N$ being the spatial weights. Without loss of generality we set $w_{ii} = 0$, for all i, assume that $w_{ij} \ge 0$, and normalise the spatial weights so that $\sum_{j=1}^{N} w_{ij} = 1$.[1] When the weights are not normalised, (1) continues to hold with ψ_{i0}

[1] Strictly speaking, the weights, w_{ij}, are N-dependent and should be denoted as $w_{ij,N}$. The same also applies to y_{it}, β_{i0}, and ε_{it}. But we abstract from including the subscript N when denoting w_{ij}, y_{it} and ε_{it}, to keep the notations simple and manageable.

re-defined as ψ_{i0}/v_i, where $\sum_{j=1}^{N} w_{ij} = v_i$. Consequently, in the heterogeneous case the normalisation of the weights is innocuous, and can be viewed as an identifying restriction, so that ψ_{i0} can be distinguished from v_i, which is achieved by setting $v_i = 1$. The same is not true in the homogeneous case where $\psi_{i0} = \psi_0$ for all i, and the use of non-normalised weights is equivalent to setting $\psi_{i0} = \psi_0/v_i$, which is not an innocuous restriction. The HSAR model (1) can also be viewed as a generalisation of the random coefficient panel data model reviewed, for example, by Hsiao and Pesaran (2008). However, this is not a straightforward generalisation due to the endogeneity of $y_{it}^* = \mathbf{w}_i'\mathbf{y}_t$ in (1).

The assumption of non-negative weights ($w_{ij} \geq 0$) can be relaxed by replacing $\mathbf{W}$ with two weight matrices: one for positive weights, $\mathbf{W}^+ = (w_{ij}^+)$, where $w_{ij}^+ = w_{ij}$ if $w_{ij} > 0$ and zero otherwise, and one for negative weights, $\mathbf{W}^- = (w_{ij}^-)$, where $w_{ij}^- = -w_{ij}$ if $w_{ij} < 0$ and zero otherwise. Further, since regressors are allowed to be weakly exogenous, our analysis covers quite general spatio-temporal models, such as the following generalisation of (1):

$$y_{it} = \left(\sum_{q=1}^{h_\lambda} \lambda_{iq0} y_{i,t-q}\right) + \left[\sum_{q=0}^{h_\psi^+} \psi_{iq0}^+ \left(\sum_{j=1}^{N} w_{ij}^+ y_{j,t-q}\right)\right] + \left[\sum_{q=0}^{h_\psi^-} \psi_{iq0}^- \left(\sum_{j=1}^{N} w_{ij}^- y_{j,t-q}\right)\right] + \tilde{\boldsymbol{\beta}}_{i0}'\tilde{\boldsymbol{x}}_{it} + \varepsilon_{it}, \quad (2)$$

where h_λ, h_ψ^+ and h_ψ^- are fixed, and the slope coefficients, λ_{ik0}, ψ_{ik0}^+ and ψ_{ik0}^- measure the temporal effects and spatial impact effects for positively and negatively connected units. Such a model is analysed in Bailey et al. (2016).[2]

The HSAR model can be generalised further in two important directions. First, the assumption of zero diagonal elements for the weight matrix can be relaxed, which could be of interest in linking spatial models more closely to the GVAR approach, as discussed in Elhorst et al. (2018). Consider (1), with $w_{ii} \neq 0$ and $\sum_{j=1}^{N} w_{ij} = v_i$, where v_i are known constants. Then (1) can be repameterised as:

$$y_{it} = \dot{\psi}_{i0}\left(\sum_{j=1}^{N} \dot{w}_{ij} y_{jt}\right) + \dot{\boldsymbol{\beta}}_{i0}'\boldsymbol{x}_{it} + \dot{\varepsilon}_{it}, \text{ for } i = 1, 2, \ldots, N;\ t = 1, 2, \ldots, T, \quad (3)$$

where $\dot{\psi}_{i0} = (1 - \psi_{i0}w_{ii})^{-1}\psi_{i0}v_i$, $\dot{\boldsymbol{\beta}}_{i0} = (1 - \psi_{i0}w_{ii})^{-1}\boldsymbol{\beta}_{i0}$, and $Var(\dot{\varepsilon}_{it}) = \dot{\sigma}_{i0}^2 = (1 - \psi_{i0}w_{ii})^{-2}\sigma_{i0}^2$, respectively.[3] Second, the weights, w_{ij}, can be estimated so long as each unit has a finite number of known neighbours. In such a setting the HSAR model can be written as

$$y_{it} = \sum_{j=1}^{N} \psi_{ij0} I(w_{ij}) y_{jt} + \boldsymbol{\beta}_{i0}'\boldsymbol{x}_{it} + \varepsilon_{it}, \quad (4)$$

where $I(w_{ij}) = 1$ if $w_{ij} \neq 0$ and 0 otherwise, and $\sup_i \sum_{j=1}^{N} |\psi_{ij0}| I(w_{ij}) < K$. This specification only

[2] It is also possible to allow for spatial effects in the errors and the regressors. For example, ε_{it} can be replaced by $\varepsilon_{it} = \varphi_{i0}\left(\sum_{j=1}^{N} w_{\varepsilon,ij}\varepsilon_{jt}\right) + \nu_{it}$, and the regressors augmented with spatial effects such as $\tilde{x}_{i\ell,t}^* = \sum_{j=1}^{N} w_{\ell,ij}\tilde{x}_{j\ell,t}$, for $\ell = 1, 2, \ldots, k'$, where $w_{\varepsilon,ij}$ and $w_{\ell,ij}$ are the spatial weights. To simplify the exposition in this paper we abstract from spatial error and regressor processes and focus on the contemporaneous spatial effects in the dependent variable, y_{it}.

[3] The derivation for this modified HSAR specification can be found in the online supplement D.

exploits the qualitative information contained in $I\left(w_{ij}\right)$ which departs from the conventional homogeneous spatial model. In what follows we focus on the basic HSAR specification given by (1) and note that estimation and inference for models (2), (3) or (4) can be conducted along the lines set out in this paper.

Stacking the observations by the N individual units for each time period t, (1) can be written more compactly as

$$\left(\boldsymbol{I}_N - \boldsymbol{\Psi}_0 \boldsymbol{W}\right) \boldsymbol{y}_{\circ t} = \boldsymbol{B}_0 \boldsymbol{x}_{\circ t} + \boldsymbol{\varepsilon}_{\circ t},\ t = 1, 2, \ldots, T, \tag{5}$$

where $\boldsymbol{y}_{\circ t} = (y_{1t}, y_{2t}, \ldots, y_{Nt})'$, $\boldsymbol{I}_N$ is an $N \times N$ identity matrix, $\boldsymbol{\Psi}_0 = \text{Diag}\left(\boldsymbol{\psi}_0\right)$ with $\boldsymbol{\psi}_0 = (\psi_{10}, \psi_{20}, \ldots, \psi_{N0})'$, and $\boldsymbol{B}_0$ is the $N \times kN$ block diagonal matrix with elements $\boldsymbol{\beta}'_{i0}$, $i = 1, 2, \ldots, N$, on the main diagonal and zeros elsewhere, and $\boldsymbol{x}_{\circ t} = (\boldsymbol{x}'_{1t}, \boldsymbol{x}'_{2t}, \ldots, \boldsymbol{x}'_{Nt})'$ is the $kN \times 1$ vector of observations on the exogenous regressors. Finally, $Var\left(\boldsymbol{\varepsilon}_{\circ t}\right) = \boldsymbol{\Sigma}_0 = \text{Diag}\left(\boldsymbol{\sigma}_0^2\right)$, with $\boldsymbol{\sigma}_0^2 = (\sigma_{10}^2, \sigma_{20}^2, \ldots, \sigma_{N0}^2)'$. We set $\mathbf{S}(\boldsymbol{\psi}_0) = \mathbf{I}_N - \boldsymbol{\Psi}_0 \boldsymbol{W}$, and assuming that $\mathbf{S}(\boldsymbol{\psi}_0)$ is invertible, the reduced form of (5) can be expressed as

$$\boldsymbol{y}_{\circ t} = \mathbf{S}^{-1}(\boldsymbol{\psi}_0)\left[\boldsymbol{B}_0 \boldsymbol{x}_{\circ t} + \boldsymbol{\varepsilon}_{\circ t}\right],\ t = 1, 2, \ldots, T. \tag{6}$$

2.2 The log-likelihood function

To estimate the unit-specific coefficients we collect all the parameters of the N units in the $N(k+2) \times 1$ vector $\boldsymbol{\theta} = (\boldsymbol{\psi}', \boldsymbol{\beta}', \boldsymbol{\sigma}^{2\prime})'$ where $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_N)'$, $\boldsymbol{\beta} = (\boldsymbol{\beta}'_1, \boldsymbol{\beta}'_2, \ldots, \boldsymbol{\beta}'_N)'$ and $\boldsymbol{\sigma}^2 = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)'$, and denote the associated vector of true values by $\boldsymbol{\theta}_0 = (\boldsymbol{\psi}'_0, \boldsymbol{\beta}'_0, \boldsymbol{\sigma}_0^{2\prime})'$. The log-likelihood function of (6) can be written as

$$\ell_T(\boldsymbol{\theta}) = \ln L(\boldsymbol{\theta}) = -\frac{NT}{2}\ln(2\pi) - \frac{T}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + \frac{T}{2}\ln\left|\mathbf{S}'(\boldsymbol{\psi})\mathbf{S}(\boldsymbol{\psi})\right| - \frac{1}{2}\sum_{t=1}^{T}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right], \tag{7}$$

where $\boldsymbol{\Sigma} = \text{Diag}\left(\boldsymbol{\sigma}^2\right)$, $\boldsymbol{\Psi} = \text{Diag}\left(\boldsymbol{\psi}\right)$, and $\mathbf{S}(\boldsymbol{\psi}) = \boldsymbol{I}_N - \boldsymbol{\Psi}\boldsymbol{W}$.

The quasi maximum likelihood estimators (QMLE), $\hat{\boldsymbol{\theta}}$, are the extreme value estimators obtained by maximisation of (7). When the error terms, $\boldsymbol{\varepsilon}_{\circ t}\left(\boldsymbol{\theta}_0\right) = \mathbf{S}(\boldsymbol{\psi}_0)\boldsymbol{y}_{\circ t} - \boldsymbol{B}_0 \boldsymbol{x}_{\circ t}$, are normally distributed, then vector $\hat{\boldsymbol{\theta}}$ is the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$, while under non-Gaussian errors, $\hat{\boldsymbol{\theta}}$ is the QMLE of $\boldsymbol{\theta}$.

3 Asymptotic properties of QML estimators

3.1 Assumptions

In order to investigate the conditions under which $\boldsymbol{\theta}_0$ is identified, and to establish consistency and the asymptotic normality of $\hat{\boldsymbol{\theta}}$, we make the following assumptions, using the filtration $\mathcal{F}_t = (\boldsymbol{x}_{\circ t}, \boldsymbol{x}_{\circ t-1}, \boldsymbol{x}_{\circ t-2}, \ldots)$, where $\boldsymbol{x}_{\circ t} = (\boldsymbol{x}'_{1t}, \boldsymbol{x}'_{2t}, \ldots, \boldsymbol{x}'_{Nt})'$, which could also contain lagged values of $\boldsymbol{y}_{\circ t}$:

Assumption 1 *The error terms* $\{\varepsilon_{it},\ i=1,2,\ldots,N; t=1,2,\ldots,T\}$ *are independently distributed over* i *and* t; $E(\varepsilon_{it}|\mathcal{F}_t)=0$, $E(\varepsilon_{it}^2|\mathcal{F}_t)=\sigma_{i0}^2$, *for* $i=1,2,\ldots,N$, *where* $\inf_i\left(\sigma_{i0}^2\right)>c>0$, $\sup_i\left(\sigma_{i0}^2\right)<K<\infty$, *and* $E\left(|\varepsilon_{it}|^p\,|\mathcal{F}_t\right)=E\left(|\varepsilon_{it}|^p\right)=\varpi_{ip}<K$, *for all* i *and* t, *where* ϖ_{ip} *is a time-invariant constant,* $1\leq p\leq 4+\epsilon$, *for some* $\epsilon>0$.

Assumption 2 *(a)* $\boldsymbol{x}_{\circ t}$ *are stationary processes with mean zero, and satisfy the moment condition* $\sup_{i,\ell,t}E\left(|x_{i\ell,t}|^{2+c}\right)<K$, *for some* $c>0$, $i=1,2,\ldots,N$, $\ell=1,2,\ldots,k$, *and* $t=1,2,\ldots,T$. *(b)* $E\left(\boldsymbol{x}_{\circ t}\boldsymbol{x}_{\circ t}'\right)=\boldsymbol{\Sigma}_{\boldsymbol{xx}}=\left(\boldsymbol{\Sigma}_{ij}\right)$, *where* $\boldsymbol{\Sigma}_{ij}=E(\boldsymbol{x}_{it}\boldsymbol{x}_{jt}')$ *exists for all* i *and* j, *such that* $\sup_{i,j}\|\boldsymbol{\Sigma}_{ij}\|<K$, *and* $\boldsymbol{\Sigma}_{ii}$ *is a* $k\times k$ *non-singular matrix with* $\inf_i\left[\lambda_{\min}\left(\boldsymbol{\Sigma}_{ii}\right)\right]>c>0$, *and* $\sup_i\left[\lambda_{\max}\left(\boldsymbol{\Sigma}_{ii}\right)\right]<K$; *(c)* $T^{-1}\sum_{t=1}^T\boldsymbol{x}_{\circ t}\boldsymbol{x}_{\circ t}'\overset{a.s.}{\to}\boldsymbol{\Sigma}_{\boldsymbol{xx}}$, *as* $T\to\infty$.

Assumption 3 *The* $N(k+2)\times 1$ *parameter vector* $\boldsymbol{\theta}=(\boldsymbol{\psi}',\boldsymbol{\beta}',\boldsymbol{\sigma}^{2\prime})'$ *belongs to* $\boldsymbol{\Theta}=\boldsymbol{\Theta}_\psi\times\boldsymbol{\Theta}_\beta\times\boldsymbol{\Theta}_\sigma\subset\mathbb{R}^N\times\mathbb{R}^{Nk}\times\mathbb{R}^N$, *a sub-set of the* $N(k+2)$ *dimensional Euclidean space,* $\mathbb{R}^{N(k+2)}$. $\boldsymbol{\Theta}$ *is a closed and bounded (compact) set and includes the true value of* $\boldsymbol{\theta}$, *denoted by* $\boldsymbol{\theta}_0$, *which is an interior point of* $\boldsymbol{\Theta}$.

Assumption 4 *(a) The weight matrix* $\mathbf{W}=(w_{ij})$ *is known and time-invariant with* $w_{ii}=0$, *for* $i=1,2,\ldots,N$. *(b)* $\mathbf{W}$ *has bounded maximum row sum norm,* $\|\boldsymbol{W}\|_\infty<K<\infty$, *and*

$$\sup_{\psi_i\in\boldsymbol{\Theta}_\psi}|\psi_i|<\frac{1}{\|\boldsymbol{W}\|_\infty}. \tag{8}$$

Remark 1 *Assumption 1 implies that* $E(\varepsilon_{it})=0$, $E(\varepsilon_{it}^2)=\sigma_{i0}^2$, *for* $i=1,2,\ldots,N$, *and does not allow for conditional heteroskedasticity. But it is possible to allow for time variations in* $E\left(|\varepsilon_{it}|^{4+\epsilon}|\mathcal{F}_t\right)$ *by relaxing the moment conditions on* ε_{it} *and* $x_{i\ell,t}$.

Remark 2 *The HSAR model, (1), is quite general and encompasses many other models in the literature. Assumption 2 allows the regressors to be weakly exogenous and cross-sectionally correlated, namely the model can contain lagged dependent variables and observable common factors, such as time trends. It can also be modified to include an intercept (fixed effects) by setting one of the elements of* $\boldsymbol{x}_{it}$ *to unity, at the expense of complicating the algebra. It applies both when* N *is small or large, so long as* T *is sufficiently large. Small sample evidence on such settings is presented in Section 5.*

Remark 3 *Assumption 4 is sufficiently general and allows the spatial weights to take negative values. But, as noted above, in empirical applications one might wish to distinguish between positive and negative connections as they might have differential effects on the outcomes. This assumption does not require the weights to be normalised either, so long as condition (8) is met. In the case when a dense inverse distance matrix is adopted, for example when* w_{ij} *are set in terms of geodesic distance,* d_{ij}, *between the* i *and* j *units such that* $\mathbf{W}=\left(1/d_{ij}^\delta\right)$, *then for Assumption 4 to hold it is necessary that* $\delta>1$ *and is sufficiently large. See Elhorst (2014).*

Remark 4 *As shown in Lemma 1, under Assumption 4 we have* $\lambda_{\min}[\mathbf{S}(\boldsymbol{\psi})] > 0$ *and* $|\lambda_{\max}[\mathbf{S}(\boldsymbol{\psi})]| < 2$, *and boundedness of* $\|\boldsymbol{W}\|_1$ *is not needed. Condition (8) is required simply to ensure invertibility of matrix* $\mathbf{S}(\boldsymbol{\psi})$.

3.2 Identification

Here we focus on the problem of identification of the individual parameters in $N(k+2)\times 1$ vector $\boldsymbol{\theta}_0$ for a given N, and as $T \to \infty$. To highlight the main issues involved in the identification of spatial parameters under the heterogeneous setting, first we consider the HSAR model (5) without the exogenous regressors. Under Assumption 4(b), with $\mathbf{B}_0 = \mathbf{0}$, we have (see (6))

$$\boldsymbol{y}_{\circ t} = \mathbf{S}^{-1}(\boldsymbol{\psi})\boldsymbol{\varepsilon}_{\circ t},\ t = 1, 2, \ldots, T.$$

With a slight abuse of notation let $\boldsymbol{\theta} = (\boldsymbol{\psi}', \boldsymbol{\sigma}^{2\prime})'$, and note that in this case the log-likelihood function is given by

$$\ell_T(\boldsymbol{\theta}) = -\frac{NT}{2}\ln(2\pi) - \frac{T}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + \frac{T}{2}\ln|\mathbf{S}'(\boldsymbol{\psi})\mathbf{S}(\boldsymbol{\psi})| - \frac{1}{2}\sum_{t=1}^{T}\boldsymbol{y}_{\circ t}'\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t}.$$

It is also helpful to write the associated average log-likelihood function as

$$\bar{\ell}_T(\boldsymbol{\theta}) = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + \frac{1}{2}\ln|\mathbf{V}(\boldsymbol{\psi})| - \frac{1}{2}\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}_{\circ t}'\mathbf{P}(\boldsymbol{\theta})\boldsymbol{y}_{\circ t}\right),$$

where

$$\mathbf{V}(\boldsymbol{\psi}) = \mathbf{S}'(\boldsymbol{\psi})\mathbf{S}(\boldsymbol{\psi}),\ \mathbf{P}(\boldsymbol{\theta}) = \mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi}), \text{ and } \mathbf{S}(\boldsymbol{\psi}) = \boldsymbol{I}_N - \boldsymbol{\Psi}\boldsymbol{W}.$$

Let $Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta}) = \bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta})$, in which $\bar{\ell}_T(\boldsymbol{\theta}_0)$ is $\bar{\ell}_T(\boldsymbol{\theta})$ evaluated at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. Then, for a given N and as $T \to \infty$, we have (see Lemma 3 of the online supplement A when setting $\boldsymbol{B} = \mathbf{0}$ in (A.6)) $Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta}) - E_0[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})] \overset{a.s.}{\to} 0$, where

$$E_0[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})] = E_0\left[\bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta})\right] = -\frac{1}{2}\sum_{i=1}^{N}\ln\left(\sigma_{i0}^2/\sigma_i^2\right) - \frac{N}{2} + \frac{1}{2}\left[\ln\left(\frac{|\mathbf{V}(\boldsymbol{\psi}_0)|}{|\mathbf{V}(\boldsymbol{\psi})|}\right)\right] + \frac{1}{2}\operatorname{tr}\left[\mathbf{P}(\boldsymbol{\theta})\mathbf{P}^{-1}(\boldsymbol{\theta}_0)\right]. \tag{9}$$

Consider now the problem of identification of $\boldsymbol{\psi}_0$, which is the parameter vector of interest. Note that

$$|\mathbf{V}(\boldsymbol{\psi}_0)|/|\mathbf{V}(\boldsymbol{\psi})| = |\mathbf{S}(\boldsymbol{\psi}_0)|^2/|\mathbf{S}(\boldsymbol{\psi})|^2 = \left|\mathbf{S}(\boldsymbol{\psi}_0)\mathbf{S}^{-1}(\boldsymbol{\psi})\right|^2 = \left|\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\right|^{-2},$$

$$\operatorname{tr}\left[\mathbf{P}(\boldsymbol{\theta})\mathbf{P}^{-1}(\boldsymbol{\theta}_0)\right] = \operatorname{tr}\left[\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\Sigma}_0\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\right] = \operatorname{tr}\left[\boldsymbol{\Sigma}^{-1/2}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\Sigma}_0\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1/2}\right],$$

and rewrite (9) as

$$E_0[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})] = -\frac{N}{2} - \frac{1}{2}\sum_{i=1}^{N}\ln\left(\sigma_{i0}^2/\sigma_i^2\right) - \left[\ln\left(\left|\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\right|\right)\right] + \frac{1}{2}\operatorname{tr}\left[\boldsymbol{\Sigma}^{-1/2}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\Sigma}_0\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1/2}\right].$$

Further, we note that $\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0) = \boldsymbol{I}_N - \boldsymbol{D}\boldsymbol{G}_0$, where $\boldsymbol{G}_0 = \boldsymbol{W}(\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}$, and $\boldsymbol{D} = \boldsymbol{\Psi} - \boldsymbol{\Psi}_0$, is a diagonal matrix with elements $d_i = \psi_i - \psi_{i0}$. Using these results, the above expression for $E_0[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})]$

can be written equivalently as

$$E_0\left[Q_T\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\right]=-\frac{N}{2}-\frac{1}{2}\sum_{i=1}^{N}\ln\left(\sigma_{i0}^2/\sigma_i^2\right)-\ln\left|\boldsymbol{I}_N-\boldsymbol{D}\boldsymbol{G}_0\right|+\frac{1}{2}\operatorname{tr}\left[\boldsymbol{\Sigma}^{-1/2}\left(\boldsymbol{I}_N-\boldsymbol{D}\boldsymbol{G}_0\right)'\boldsymbol{\Sigma}_0\left(\boldsymbol{I}_N-\boldsymbol{D}\boldsymbol{G}_0\right)\boldsymbol{\Sigma}^{-1/2}\right]$$
$$=A_N+B_N,$$

where

$$A_N=\frac{1}{2}\sum_{i=1}^{N}\left[\left(\sigma_{i0}^2/\sigma_i^2\right)-\ln\left(\sigma_{i0}^2/\sigma_i^2\right)-1\right]-\ln\left|\boldsymbol{I}_N-\boldsymbol{D}\boldsymbol{G}_0\right|-\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_0\boldsymbol{D}\boldsymbol{G}_0\right), \tag{10}$$
$$B_N=\frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{G}_0'\boldsymbol{D}\boldsymbol{\Sigma}_0\boldsymbol{D}\boldsymbol{G}_0\boldsymbol{\Sigma}^{-1/2}\right). \tag{11}$$

We first note that $B_N\geq 0$, since we can write $B_N=(1/2)\operatorname{tr}\left(\mathbf{A}_0'\mathbf{A}_0\right)$, with $\mathbf{A}_0=\boldsymbol{\Sigma}_0^{1/2}\boldsymbol{D}\boldsymbol{G}_0\boldsymbol{\Sigma}^{-1/2}$. Consider now A_N, denote the i^{th} eigenvalue of $\boldsymbol{D}\boldsymbol{G}_0$ by μ_i, and note that since $\boldsymbol{I}_N-\boldsymbol{D}\boldsymbol{G}_0=\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)$, then the eigenvalues of $\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)$ are also given by $1-\mu_i$, for $i=1,2,\ldots,N$. Further, by Lemma 1 of the online supplement A, $\lambda_{\min}\left[\mathbf{S}(\boldsymbol{\psi})\right]>0$ for all ψ_i that satisfy condition (8). Hence, we must also have $1-\mu_i>0$, for all i. Using these results, A_N can now be written as

$$A_N=\frac{1}{2}\sum_{i=1}^{N}\left[\frac{\sigma_{i0}^2}{\sigma_i^2}-\ln\left(\sigma_{i0}^2/\sigma_i^2\right)-1\right]-\sum_{i=1}^{N}\ln(1-\mu_i)-\sum_{i=1}^{N}\left(\frac{\sigma_{i0}^2}{\sigma_i^2}\right)\mu_i.$$

Let $\delta_{\sigma i}=\sigma_{i0}^2/\sigma_i^2>0$ and $\delta_{\psi i}=(1-\mu_i)>0$, for all i. Then, write $E_0\left[Q_T\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\right]$ as

$$\begin{aligned}E_0\left[Q_T\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\right]=A_N+B_N&=\frac{1}{2}\sum_{i=1}^{N}\left[\delta_{\sigma i}-\ln\left(\delta_{\sigma i}\right)-1\right]-\sum_{i=1}^{N}\ln\delta_{\psi i}-\sum_{i=1}^{N}\delta_{\sigma i}(1-\delta_{\psi i})+B_N\\&=\frac{1}{2}\sum_{i=1}^{N}\left[\delta_{\sigma i}-\ln\left(\delta_{\sigma i}\right)-1\right]+\left[\sum_{i=1}^{N}\delta_{\sigma i}\left(\delta_{\psi i}-\ln\delta_{\psi i}-1\right)\right]\\&+\left[\sum_{i=1}^{N}\left(\delta_{\sigma i}-1\right)\ln\delta_{\psi i}\right]+B_N=A_{1,N}+A_{2,N}+\left(A_{3,N}+B_N\right).\end{aligned}$$

Since $\delta_{\sigma i}>0$, and $\delta_{\psi i}>0$ for all i, then $\delta_{\sigma i}-\ln\left(\delta_{\sigma i}\right)-1\geq 0$, and $\delta_{\psi i}-\ln\delta_{\psi i}-1\geq 0$ for all i, with equalities holding if and only if $\delta_{\sigma i}=1$ and $\delta_{\psi i}=1$ for all i. Hence, $A_{1,N}\geq 0$, and $A_{2,N}\geq 0$ for all values of N, and global identification of σ_{i0}^2 will be possible only if we are able to show that $A_{3,N}+B_N$ is non-negative. But it is easily seen that the non-negativity of $A_{3,N}+B_N$ can not be guaranteed without further restrictions. This follows since

$$A_{3,N}=\sum_{i=1}^{N}\left(\delta_{\sigma i}-1\right)\ln\delta_{\psi i},$$

and there are values of $\delta_{\sigma i}$ and $\delta_{\psi i}$ in $\boldsymbol{\Theta}=\boldsymbol{\Theta}_\psi\times\boldsymbol{\Theta}_\sigma$ for which $A_{3,N}<0$. Considering $(A_{3,N}+B_N)$ somewhat weakens the requirement since $B_N\geq 0$, but still does not guarantee that $(A_{3,N}+B_N)\geq 0$, for all values of $\delta_{\sigma i}>0$ and $\delta_{\psi i}>0$. Therefore, global identification of $\boldsymbol{\psi}_0$ can not be guaranteed. To investigate the possibility of local identification we introduce the following definition:

Definition 1 *Consider the set $\mathcal{N}_c(\boldsymbol{\sigma}_0^2)$ in the closed neighborhood of $\boldsymbol{\sigma}_0^2$ defined by*

$$\mathcal{N}_c(\boldsymbol{\sigma}_0^2) = \left\{\boldsymbol{\sigma}_0^2 \in \boldsymbol{\Theta}_\sigma,\ \left|\sigma_{i0}^2/\sigma_i^2 - 1\right| < c_i, \text{ for } i = 1, 2, \ldots, N\right\},$$

for some $c_i > 0$, $i = 1, 2, \ldots, N$, where $\boldsymbol{\Theta}_\sigma$ is a compact subset of $\mathbb{R}^N$.

We now show that $\boldsymbol{\theta}_0 = (\boldsymbol{\psi}_0', \boldsymbol{\sigma}_0^{2\prime})'$ is identified on $\boldsymbol{\Theta}_c = \boldsymbol{\Theta}_\psi \times \mathcal{N}_c(\boldsymbol{\sigma}_0^2)$. Consider values of $\delta_{\sigma i}$ within the local neighborhood of $\delta_{\sigma i} = 1$ for all i. Recall that $A_{1,N} + A_{2,N} \geq 0$, and the boundary values $A_{1,N} = 0$ or $A_{2,N} = 0$ can occur if and only if $\delta_{\sigma i} = 1$ and $\delta_{\psi i} = 1$ for all i, respectively. Therefore, $A_N \geq 0$ if $\delta_{\sigma i} = 1$, otherwise $A_{1,N} > 0$. Similarly, $A_N \geq 0$ if $\delta_{\psi i} = 1$, otherwise $A_{2,N} > 0$. Therefore, there must exist $\mathbf{c} = (c_1, c_2, \ldots, c_N) > \mathbf{0}$, such that $A_N = 0$ on $\boldsymbol{\Theta}_c$ if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, which in turn establishes that $\bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta}) \overset{a.s.}{\to} 0$, as $T \to \infty$, on the set $\boldsymbol{\Theta}_c$ *if and only if* $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.

Next, consider the HSAR model (5) with exogenous regressors. The average log-likelihood in this case is given by (see (7))

$$\bar{\ell}_T(\boldsymbol{\theta}) = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + \frac{1}{2}\ln|\mathbf{V}(\boldsymbol{\psi})| - \frac{1}{2}\left(\frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right]' \boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right]\right), \tag{12}$$

where $\boldsymbol{\theta}$ is now defined by $\boldsymbol{\theta} = (\boldsymbol{\psi}', \boldsymbol{\beta}', \boldsymbol{\sigma}^{2\prime})'$ and $\boldsymbol{B}$ has the same form as that used in (5). Following a similar line of reasoning as in the case without exogenous regressors (see Lemma 3 of the online supplement A), we have that $Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta}) = \bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta})$, where $\bar{\ell}_T(\boldsymbol{\theta})$ is now given by (12), and $Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta}) - E_0\left[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})\right] \overset{a.s.}{\to} 0$, (as $T \to \infty$) where

$$E_0\left[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})\right] = A_N + B_N + C_N. \tag{13}$$

A_N and B_N are defined as before by (10) and (11), and C_N is given by

$$\begin{aligned} C_N &= \frac{1}{2}\sum_{i=1}^{N}\left[(\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i0})' \boldsymbol{\Sigma}_{ii}(\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i0})\right]/\sigma_i^2 + \operatorname{tr}\left[\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{B} - \boldsymbol{B}_0)\boldsymbol{\Sigma}_{xx}\boldsymbol{\Xi}_0'\right] + \frac{1}{2}\operatorname{tr}\left(\boldsymbol{\Sigma}_{xx}\boldsymbol{\Xi}_0'\boldsymbol{\Xi}_0\right) \qquad (14) \\ &= C_{1,N} + C_{2,N} + C_{3,N}, \end{aligned}$$

where $\boldsymbol{\Xi}_0 = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{D}\boldsymbol{G}_0\boldsymbol{B}_0$, and as before $\mathbf{D} = \text{Diag}(\boldsymbol{\psi} - \boldsymbol{\psi}_0)$. Consider now $C_{3,N}$ and note that since $\boldsymbol{\Sigma}_{xx} = E(\boldsymbol{x}_{\circ t}\boldsymbol{x}_{\circ t}')$ and $\boldsymbol{\Xi}_0'\boldsymbol{\Xi}_0$ are both $kN \times kN$ positive semi-definite matrices, then by result (9) on p. 44 of Lütkepohl (1996),

$$\frac{1}{kN}\operatorname{tr}\left(\boldsymbol{\Sigma}_{xx}\boldsymbol{\Xi}_0'\boldsymbol{\Xi}_0\right) \geq \left[\det(\boldsymbol{\Sigma}_{xx})\right]^{1/kN}\left[\det\left(\boldsymbol{\Xi}_0'\boldsymbol{\Xi}_0\right)\right]^{1/kN} \geq 0,$$

and hence $C_{3,N} \geq 0$. Also, as shown above, on the subset $\boldsymbol{\Theta}_c = \boldsymbol{\Theta}_\psi \times \boldsymbol{\Theta}_\beta \times \mathcal{N}_c(\boldsymbol{\sigma}_0^2)$, $A_N + B_N = 0$ if and only if $\mathbf{D} = \text{Diag}(\boldsymbol{\psi} - \boldsymbol{\psi}_0) = \mathbf{0}$, and hence it must also follow that $C_{2,N} = 0$ on $\boldsymbol{\Theta}_c$. Thus, overall $\bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta}) \overset{a.s.}{\to} 0$ on $\boldsymbol{\Theta}_c$ if and only if

$$\sum_{i=1}^{N} (\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i0})' \boldsymbol{\Sigma}_{ii} (\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i0}) / \sigma_i^2 = 0. \tag{15}$$

This equality holds for all N if and only if $(\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i0})' \boldsymbol{\Sigma}_{ii} (\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i0}) = 0$, for all i, and since under Assumption 2(b) $\boldsymbol{\Sigma}_{ii}$ is a positive definite matrix this can occur if and only if $\boldsymbol{\beta}_i = \boldsymbol{\beta}_{i0}$ for all i.

Before we state the identification result for the general model (5), we require the following modification of Assumption 3:

Assumption 5 *The $N(k+2)\times 1$ parameter vector $\boldsymbol{\theta} = (\boldsymbol{\psi}', \boldsymbol{\beta}', \boldsymbol{\sigma}^{2\prime})'$ belongs to $\boldsymbol{\Theta}_c = \boldsymbol{\Theta}_\psi \times \boldsymbol{\Theta}_\beta \times \mathcal{N}_c(\boldsymbol{\sigma}_0^2)$, where $\boldsymbol{\Theta}_\psi$ and $\boldsymbol{\Theta}_\beta$ are compact subsets of $\mathbb{R}^N$ and $\mathbb{R}^{Nk}$, respectively, $\mathcal{N}_c(\boldsymbol{\sigma}_0^2)$ is given in Definition 1, and $\boldsymbol{\Theta}_c$ is a sub-set of the $N(k+2)$ dimensional Euclidean space, $\mathbb{R}^{N(k+2)}$.*

The main identification result of the paper is summarised in the following proposition:

Proposition 1 *Consider the heterogeneous spatial autoregressive (HSAR) model given by (5) with the associated log-likelihood function given by (7). Suppose that Assumptions 1, 2, 3, 4, and 5 hold. Then for a fixed N and k, the $N(k+2)$ dimensional true parameter vector $\boldsymbol{\theta}_0 = (\boldsymbol{\psi}_0', \boldsymbol{\beta}_0', \boldsymbol{\sigma}_0^{2\prime})'$ is almost surely locally identified on $\boldsymbol{\Theta}_c$.*

3.3 Consistency and asymptotic normality

We are now in a position to consider consistency and asymptotic normality of the QML estimator of $\boldsymbol{\theta}$, given by $\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \bar{\ell}_T(\boldsymbol{\theta})$, where $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\psi}}', \hat{\boldsymbol{\beta}}', \hat{\boldsymbol{\sigma}}^{2\prime})'$, which is estimated simultaneously. We establish the results for a given N, and as $T \to \infty$. First, we focus on the proof of consistency. Under Assumptions 1, 2, 4 and 5, we have: (i) $\boldsymbol{\Theta}_c$, being a subset of $\boldsymbol{\Theta}$, is compact, (ii) $\boldsymbol{\theta}_0$ is an interior point of $\boldsymbol{\Theta}_c$, (iii) $Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta}) \overset{a.s.}{\to} E_0[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})]$, with $Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta}) = \bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta})$ and $E_0[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})] = A_N + B_N + C_N$, where A_N, B_N and C_N are given by (10), (11) and (14), respectively, and (iv) $\boldsymbol{\theta}_0$ is a unique maximum of $E_0[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})]$ on $\boldsymbol{\Theta}_c$. The last result follows from the identification analysis of Section 3.2. It is clear that all conditions of Theorem 9.3.1 of Davidson (2000) are satisfied, therefore almost sure local consistency of $\hat{\boldsymbol{\theta}}$ is ensured, with $\hat{\boldsymbol{\theta}} \overset{a.s.}{\to} \boldsymbol{\theta}_0$ on $\boldsymbol{\Theta}_c$, as $T \to \infty$. To establish asymptotic normality of $\hat{\boldsymbol{\theta}}$, we apply the mean value theorem to $\bar{\ell}_T(\boldsymbol{\theta})$ such that

$$\bar{\ell}_T(\boldsymbol{\theta}) - \bar{\ell}_T(\boldsymbol{\theta}_0) = (\boldsymbol{\theta} - \boldsymbol{\theta}_0)' \bar{\boldsymbol{s}}_T(\boldsymbol{\theta}_0) - \frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)' \bar{\mathbf{H}}_T(\bar{\boldsymbol{\theta}}) (\boldsymbol{\theta} - \boldsymbol{\theta}_0), \tag{16}$$

where $\bar{\boldsymbol{s}}_T(\boldsymbol{\theta}) = \partial \bar{\ell}_T(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$, $\bar{\mathbf{H}}_T(\boldsymbol{\theta}) = -\partial^2 \bar{\ell}_T(\boldsymbol{\theta})/\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'$, and $\bar{\boldsymbol{\theta}}$ lies between $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_0$. By Lemma 5 of the online supplement A we have $\bar{\boldsymbol{s}}_T(\boldsymbol{\theta}_0) \overset{a.s.}{\to} \mathbf{0}$, and by the results of Section 3.2 we also have $\bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta}) \overset{a.s.}{\to} E_0\left[\bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta})\right] \geq 0$. Hence, in view of (16) it must also hold that (as $T \to \infty$)

$$(\boldsymbol{\theta} - \boldsymbol{\theta}_0)' \bar{\mathbf{H}}_T(\bar{\boldsymbol{\theta}}) (\boldsymbol{\theta} - \boldsymbol{\theta}_0) \overset{a.s.}{\to} E_0[Q_T(\boldsymbol{\theta}_0, \boldsymbol{\theta})],$$

where $E_0\left[Q_T\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}\right)\right]$ is given by (13). But we have already established that on $\boldsymbol{\Theta}_c$, the right hand side of the above expression can be equal to zero if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, and hence it must be that $\bar{\mathbf{H}}_T(\bar{\boldsymbol{\theta}}) \stackrel{a.s.}{\to} \bar{\mathbf{H}}(\boldsymbol{\theta}_0)$, where $\bar{\mathbf{H}}(\boldsymbol{\theta}_0)$ *must* be a positive definite matrix given by $\bar{\mathbf{H}}(\boldsymbol{\theta}_0) = \lim_{T\to\infty} E_0\left[-\partial^2 \bar{\ell}_T(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'\right]$. Next, for a given N we apply the mean value theorem to $\bar{\boldsymbol{s}}_T\left(\boldsymbol{\theta}\right) = 1/\sqrt{T}\boldsymbol{s}_T\left(\hat{\boldsymbol{\theta}}\right)$ so that

$$\mathbf{0} = \sqrt{T}\bar{\boldsymbol{s}}_T\left(\hat{\boldsymbol{\theta}}\right) = \sqrt{T}\bar{\boldsymbol{s}}_T\left(\boldsymbol{\theta}_0\right) - \bar{\mathbf{H}}_T(\breve{\boldsymbol{\theta}})\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right),$$

where $\boldsymbol{s}_T\left(\boldsymbol{\theta}\right) = \partial\ell_T(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$, $\mathbf{H}_T(\boldsymbol{\theta}) = -\frac{1}{T}\partial^2\ell_T(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'$, and $\breve{\boldsymbol{\theta}}$ lies between $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}_0$. Therefore, $\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) = \mathbf{H}_T^{-1}(\breve{\boldsymbol{\theta}})\left[\sqrt{T}\boldsymbol{s}_T\left(\boldsymbol{\theta}_0\right)\right]$, and since $\hat{\boldsymbol{\theta}}$ is consistent on $\boldsymbol{\Theta}_c$, then

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \stackrel{a}{\sim} \mathbf{H}^{-1}(\boldsymbol{\theta}_0)\left[\sqrt{T}\boldsymbol{s}_T\left(\boldsymbol{\theta}_0\right)\right],$$

where $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right) = \lim_{T\to\infty} E_0\left[-\frac{1}{T}\partial^2\ell_T(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'\right]$, with

$$E_0\left[-\frac{1}{T}\partial^2\ell_T(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'\right] = \begin{pmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} & \boldsymbol{H}_{13} \\ \boldsymbol{H}_{12}' & \boldsymbol{H}_{22} & \boldsymbol{H}_{23} \\ \boldsymbol{H}_{13}' & \boldsymbol{H}_{23}' & \boldsymbol{H}_{33} \end{pmatrix}_{N(k+2)\times N(k+2)}.$$

The expressions for $\mathbf{H}_{ij}$ can be obtained using the partial derivative $\partial^2\ell_T(\boldsymbol{\theta}_0)/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'$ given in the online supplement B. Specifically we have

$$\boldsymbol{H}\left(\boldsymbol{\theta}_0\right) = \begin{pmatrix} \left(\mathbf{G}_0 \odot \mathbf{G}_0'\right) + \boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\boldsymbol{G}_0\boldsymbol{\Sigma}_0\boldsymbol{G}_0'\right) + \boldsymbol{\Delta}_{\boldsymbol{\beta}_0} & \mathbb{E}_{\boldsymbol{\beta}_0} & \boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\mathbf{G}_0\right) \\ \mathbb{E}_{\boldsymbol{\beta}_0} & \boldsymbol{Z}_0 & \mathbf{0} \\ \boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\mathbf{G}_0\right) & \mathbf{0}' & \frac{1}{2}\boldsymbol{\Sigma}_0^{-2} \end{pmatrix}, \tag{17}$$

where $\boldsymbol{\Delta}_{\boldsymbol{\beta}_0}$, $\mathbb{E}_{\boldsymbol{\beta}_0}$, and $\boldsymbol{Z}_0$ are diagonal matrices given by

$$\boldsymbol{\Delta}_{\boldsymbol{\beta}_0} = \mathrm{Diag}\left[\sigma_{i0}^{-2}\sum_{r=1}^{N}\sum_{s=1}^{N} g_{0,is}g_{0,ir}\boldsymbol{\beta}_{r0}'\boldsymbol{\Sigma}_{rs}\boldsymbol{\beta}_{s0},\ i = 1, 2, \ldots, N\right], \tag{18}$$

$$\mathbb{E}_{\boldsymbol{\beta}_0} = \mathrm{Diag}\left[\sigma_{i0}^{-2}\sum_{s=1}^{N} g_{0,is}\boldsymbol{\beta}_{s0}'\boldsymbol{\Sigma}_{is},\ i = 1, 2, \ldots, N\right],$$

$$\boldsymbol{Z}_0 = \mathrm{Diag}\left[\sigma_{i0}^{-2}\boldsymbol{\Sigma}_{ii},\ i = 1, 2, \ldots, N\right].$$

Again by Lemma 5 of the online supplement A, we have that

$$\left[\frac{1}{\sqrt{T}}\boldsymbol{s}_T\left(\boldsymbol{\theta}_0\right)\right] \to_d N\left[\mathbf{0}, \mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\right]$$

where

$$\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right) = \lim_{T\to\infty}\begin{pmatrix} \begin{array}{c}\left(\mathbf{G}_0 \odot \mathbf{G}_0'\right) + \boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\boldsymbol{G}_0\boldsymbol{\Sigma}_0\boldsymbol{G}_0'\right) + \boldsymbol{\Delta}_{\boldsymbol{\beta}_0} \\ + (\gamma - 2)\,\mathrm{Diag}\left(\mathbf{G}_0 \odot \mathbf{G}_0'\right)\end{array} & \mathbb{E}_{\boldsymbol{\beta}_0} & \frac{\gamma}{2}\boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\mathbf{G}_0\right) \\ \mathbb{E}_{\boldsymbol{\beta}_0} & \boldsymbol{Z}_0 & \mathbf{0} \\ \frac{\gamma}{2}\boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\mathbf{G}_0\right) & \mathbf{0}' & \frac{\gamma}{4}\boldsymbol{\Sigma}_0^{-2} \end{pmatrix} \tag{19}$$

and

$$\gamma = \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} Var(\zeta_{it}^2) = \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} \left[E\left(\zeta_{it}^4\right) - 1\right], \tag{20}$$

with $\zeta_{it} = \varepsilon_{it}/\sigma_{i0}$, for $i = 1, 2, \ldots, N$. Hence, $\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \to_d N\left(\mathbf{0}, \mathbf{V}_{\boldsymbol{\theta}}\right)$, where $\mathbf{V}_{\boldsymbol{\theta}}$ has the usual sandwich formula

$$\mathbf{V}_{\boldsymbol{\theta}} = \mathbf{H}^{-1}(\boldsymbol{\theta}_0)\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\mathbf{H}^{-1}(\boldsymbol{\theta}_0). \tag{21}$$

In the case where the errors, ε_{it}, are Gaussian, $\gamma = 2$ and, as to be expected, $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right) = \mathbf{J}\left(\boldsymbol{\theta}_0, 2\right)$. This is easily verified by referring back to (17) which is equal to $\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)$ defined by (19) for $\gamma = 2$, as required.

Remark 5 *When no exogenous regressors are included in the HSAR specification (1), then the asymptotic variance,* $\mathbf{V}_{\boldsymbol{\theta}} = \boldsymbol{H}^{\mathbf{-1}}\left(\boldsymbol{\theta}_0\right)\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\boldsymbol{H}^{\mathbf{-1}}\left(\boldsymbol{\theta}_0\right),$ *simplifies so that:*

$$\boldsymbol{H}\left(\boldsymbol{\theta}_0\right) = \begin{pmatrix} \left(\boldsymbol{G}_0 \odot \boldsymbol{G}_0'\right) + \boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\boldsymbol{G}_0\boldsymbol{\Sigma}_0\boldsymbol{G}_0'\right) & \boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\mathbf{G}_0\right) \\ \boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\mathbf{G}_0\right) & \frac{1}{2}\boldsymbol{\Sigma}_0^{-2} \end{pmatrix}_{2N\times 2N},$$

and

$$\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right) = \begin{pmatrix} \begin{array}{c}\left(\boldsymbol{G}_0 \odot \boldsymbol{G}_0'\right) + \boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\boldsymbol{G}_0\boldsymbol{\Sigma}_0\boldsymbol{G}_0'\right) \\ +(\gamma - 2)\,\mathrm{Diag}\left(\boldsymbol{G}_0 \odot \boldsymbol{G}_0'\right)\end{array} & \frac{\gamma}{2}\boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\mathbf{G}_0\right) \\ \frac{\gamma}{2}\boldsymbol{\Sigma}_0^{-1}\,\mathrm{Diag}\left(\mathbf{G}_0\right) & \frac{\gamma}{4}\boldsymbol{\Sigma}_0^{-2} \end{pmatrix}.$$

Again, under Gaussian errors we have $\mathbf{J}\left(\boldsymbol{\theta}_0, 2\right) = \boldsymbol{H}\left(\boldsymbol{\theta}_0\right)$.

The main result of this section is summarised in the following proposition:

Proposition 2 *Consider the heterogeneous spatial autoregressive (HSAR) model given by (1). Suppose that Assumptions 1, 2, 3, 4, and 5 hold. Let N and k be fixed constants, and denote the $N(k+2)$ dimensional (quasi-) maximum likelihood estimator of $\boldsymbol{\theta}_0$ by $\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \bar{\ell}_T(\boldsymbol{\theta})$, where $\bar{\ell}_T(\boldsymbol{\theta})$ is given by (12). Then, $\hat{\boldsymbol{\theta}}$ is almost surely locally consistent for $\boldsymbol{\theta}_0$ on $\boldsymbol{\Theta}_c$, and has the following asymptotic distribution*

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \to_d N\left(\mathbf{0}, \mathbf{V}_\theta\right),$$

where $\mathbf{V}_{\boldsymbol{\theta}} = \mathbf{H}^{-1}(\boldsymbol{\theta}_0)\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\mathbf{H}^{-1}(\boldsymbol{\theta}_0)$, and $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)$ and $\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)$ are defined by (17) and (19), respectively.

Proof. See the online supplement B. ■

Focusing on the QML estimators of the spatial lag coefficients, $\hat{\boldsymbol{\psi}}$, we introduce the following partitioning of $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)$:

$$\boldsymbol{H}\left(\boldsymbol{\theta}_0\right) = \begin{pmatrix} \boldsymbol{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{12}' & \mathcal{H}_{22} \end{pmatrix},$$

where $\mathcal{H}_{12} = \left(\boldsymbol{H}_{12}, \boldsymbol{H}_{13}\right)$ is an $N \times (Nk + N)$ matrix, and since $\boldsymbol{H}_{23} = \boldsymbol{H}_{32} = \mathbf{0}$, then $\mathcal{H}_{22} = \mathrm{Diag}\left(\boldsymbol{H}_{22}, \boldsymbol{H}_{33}\right)$, which is an $(Nk + N) \times (Nk + N)$ matrix. Then, the inverse of $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)$ is given by

$$
\boldsymbol{H}^{-1}(\boldsymbol{\theta}_0) = \begin{pmatrix} \mathcal{H}_{11\cdot 2}^{-1} & -\mathcal{H}_{11\cdot 2}^{-1}\mathcal{H}_{12}\mathcal{H}_{22}^{-1} \\ -\mathcal{H}_{22}^{-1}\mathcal{H}_{21}\mathcal{H}_{11\cdot 2}^{-1} & \mathcal{H}_{22}^{-1} + \mathcal{H}_{22}^{-1}\mathcal{H}_{21}\mathcal{H}_{11\cdot 2}^{-1}\mathcal{H}_{12}\mathcal{H}_{22}^{-1} \end{pmatrix},
$$

and the inverse of the $N \times N$ information matrix $\mathcal{H}_{11\cdot 2}$ corresponds to the asymptotic covariance matrix of $\hat{\boldsymbol{\psi}}$. This result is summarised in the following proposition:

Corollary 1 *Consider the heterogeneous spatial autoregressive (HSAR) model given by (1).Suppose that Assumptions 1, 2, 3, 4, and 5 hold. Then for any fixed N and k, the $N \times N$ information matrix*

$$
\mathcal{H}_{11\cdot 2} = \left(\mathbf{G}_0 \odot \mathbf{G}_0'\right) + \mathrm{Diag}\left[-g_{0,ii}^2 + \sum_{s=1, s\neq i}^{N} \left(\sigma_{s0}^2/\sigma_{i0}^2\right) g_{0,is}^2,\ i = 1, 2, \ldots, N\right] \tag{22}
$$
$$
+ \mathrm{Diag}\left[\sigma_{i0}^{-2} \sum_{r=1}^{N}\sum_{s=1}^{N} g_{0,is} g_{0,ir} \boldsymbol{\beta}_{r0}' \left(\boldsymbol{\Sigma}_{rs} - \boldsymbol{\Sigma}_{ri}\boldsymbol{\Sigma}_{ii}^{-1}\boldsymbol{\Sigma}_{is}\right)\boldsymbol{\beta}_{s0},\ i = 1, 2, \ldots, N\right],
$$

is full rank, where $\mathbf{G}_0 = \boldsymbol{W}\left(\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W}\right)^{-1} = (g_{0,ij})$, $\boldsymbol{\Psi}_0 = \mathrm{Diag}(\boldsymbol{\psi}_0)$, $\boldsymbol{\psi}_0 = (\psi_{10}, \psi_{20}, \ldots, \psi_{N0})'$, and $\boldsymbol{W}$ is the spatial weight matrix, and $\varepsilon_{it} \sim IIDN(0, \sigma_{i0}^2)$. Then the maximum likelihood estimator of $\boldsymbol{\psi}_0$, denoted by $\hat{\boldsymbol{\psi}}$ and computed by maximizing (A.21), has the following asymptotic distribution,

$$
\sqrt{T}\left(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0\right) \to_d N\left(0, \mathbf{V}_{\boldsymbol{\psi}}\right), \tag{23}
$$

where $\mathbf{V}_{\boldsymbol{\psi}} = \left[\mathcal{H}_{11\cdot 2}\right]^{-1}$.

Proof. See the online supplement B. ■

Remark 6 *When $T \to \infty$, estimation of the HSAR model (1) can be conducted for any N, and $N \to \infty$ is not required. Proposition 2 describes the asymptotic distribution of each individual parameter in vector $\hat{\boldsymbol{\theta}}$. Yu et al. (2008) who study a similar panel data model to ours, with fixed effects but with homogeneous spatial and slope parameters, consider three cases for N: fixed, asymptotically proportional to T, and asymptotically large relative to T, as $T \to \infty$. The interest in distinguishing between these cases in their paper arises from the fact that different biases arise in the computation of their proposed QML estimators depending on the relative size of N and T and due to the homogeneity assumption imposed on their spatial and slope coefficients. On the other hand, the estimated fixed effects under their model specification converge to their respective true values at rate $\sqrt{T}$ irrespective of N - see Theorem 4 in Yu et al. (2008).*

Remark 7 *When the time dimension T is short and fixed, as N rises we are likely to encounter the well known 'incidental parameter' problem since the number of parameters rise with N and the standard asymptotic results do not hold; an issue originally highlighted by Neyman and Scott (1948). But the incidental parameter problem will not be present if T is large relatively to N, irrespective of whether N is* *fixed or rises with T. This is because the unit-specific parameters are estimated individually consistently, and the impact of initial conditions on the parameter estimates becomes negligible as $T \to \infty$. But, as discussed below in Section 4, if the object of interest is the mean of the individual coefficients, then we require both N and T to rise together such that $N/T \to \kappa$, for some strictly positive κ.*

Remark 8 *In the case where ε_{it} are non-Gaussian but $E(|\varepsilon_{it}|^{4+\epsilon}) < K$ holds for some $\epsilon > 0$, the quasi maximum likelihood estimator, $\hat{\boldsymbol{\psi}}$, continues to be normally distributed but its asymptotic covariance matrix is given by the upper $N \times N$ partition of $\boldsymbol{H}^{-1}(\boldsymbol{\theta}_0)\, \mathbf{J}(\boldsymbol{\theta}_0, \gamma)\, \boldsymbol{H}^{-1}(\boldsymbol{\theta}_0)$, where $\boldsymbol{H}(\boldsymbol{\theta}_0)$ and $\mathbf{J}(\boldsymbol{\theta}_0, \gamma)$ are defined by (17) and (19), respectively. Recall that γ is defined by (20), and under Gaussian errors it takes the value of $\gamma = 2$, so that we have $\mathbf{J}(\boldsymbol{\theta}_0, 2) = \boldsymbol{H}(\boldsymbol{\theta}_0)$.*

Remark 9 *The conditions that we have derived for local/global identification and consistency of the QML estimators have parallels in the GMM estimation of spatial models and correspond to the high level assumptions made under GMM requiring the moment conditions to have a unique solution (which might not be met and is often difficult to check). In practice, when computing QML and GMM estimators it is advisable that a number of different initial parameter vectors, $\boldsymbol{\theta}_{in}$, are considered in the optimisation procedure to make sure that the resultant estimates correspond to the global optimum, as far as possible.*

3.3.1 Consistent estimation of $\boldsymbol{V}_\theta$

The asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$ can be consistently estimated using the expressions given by (17) and (19), yielding the following standard and sandwich formulae

$$\boldsymbol{V}_\theta = \boldsymbol{H}^{-1}(\boldsymbol{\theta}_0) \text{ and } \boldsymbol{V}_\theta = \boldsymbol{H}^{-1}(\boldsymbol{\theta}_0)\, \mathbf{J}(\boldsymbol{\theta}_0, \gamma)\, \boldsymbol{H}^{-1}(\boldsymbol{\theta}_0),$$

with the information matrix equality holding in the case of $\varepsilon_{it} \sim IIDN(0, \sigma_{i0}^2)$ and $\gamma = 2$. Consistent estimators of $\mathbf{J}(\boldsymbol{\theta}_0, \gamma)$ and $\boldsymbol{H}(\boldsymbol{\theta}_0)$ can be obtained by replacing $\boldsymbol{\theta}_0$ with its QML estimator, $\hat{\boldsymbol{\theta}}$, and estimating γ by $\hat{\gamma} = (NT)^{-1} \sum_{t=1}^{T} \sum_{i=1}^{N} (\hat{\varepsilon}_{it}/\hat{\sigma}_i)^4 - 1$, where $\hat{\varepsilon}_{it} = y_{it} - \hat{\psi}_i \sum_{j=1}^{N} w_{ij} y_{jt} - \hat{\boldsymbol{\beta}}_i' \boldsymbol{x}_{it}$, with $\hat{\sigma}_i$, $\hat{\boldsymbol{\beta}}_i$ and $\hat{\psi}_i$ being the QML estimators of σ_{i0}, $\boldsymbol{\beta}_{i0}$ and ψ_{i0}, respectively.

Alternatively, one can use the sample counterparts of $\mathbf{J}(\boldsymbol{\theta}_0, \gamma)$ and $\boldsymbol{H}(\boldsymbol{\theta}_0)$ and estimate the covariance matrix of the QML estimators consistently by

$$\hat{\boldsymbol{V}}_{\hat{\theta}} = \hat{\boldsymbol{H}}_T^{-1}\left(\hat{\boldsymbol{\theta}}\right) \text{ and } \hat{\boldsymbol{V}}_{\hat{\theta}} = \hat{\boldsymbol{H}}_T^{-1}\left(\hat{\boldsymbol{\theta}}\right) \hat{\mathbf{J}}_T\left(\hat{\boldsymbol{\theta}}, \hat{\gamma}\right) \hat{\boldsymbol{H}}_T^{-1}\left(\hat{\boldsymbol{\theta}}\right), \tag{24}$$

where $\hat{\mathbf{J}}_T(\boldsymbol{\theta}) = T^{-1} \sum_{t=1}^{T} \left(\partial \ell_t(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}\right) \left(\partial \ell_t(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}\right)'$, $\ell_t(\boldsymbol{\theta})$ is defined by (A.20) and $\hat{\boldsymbol{H}}_T(\boldsymbol{\theta}) = -T^{-1}[\partial^2 \ell_T(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}']$. Consistency of $\hat{\mathbf{J}}(\hat{\boldsymbol{\theta}}, \hat{\gamma})$ for $\mathbf{J}(\boldsymbol{\theta}_0, \gamma)$ follows from consistency of $\hat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}_0$, of $\hat{\gamma}$ for γ and the independence of $\partial \ell_t(\boldsymbol{\theta}_0) / \partial \boldsymbol{\theta}$ over t, as shown in Lemma 5 of the online supplement A. The first and second derivatives are provided in the online supplement C.

4 Mean group estimators

So far we have focussed on estimation of the unit-specific parameters and have derived the asymptotic results for a given N and as $T \to \infty$. But in practice it is often of interest to obtain average estimates across all the units or a sub-group of the units in the panel, assuming that the individual coefficients follow a random coefficient model. In the context of the HSAR model (1), suppose that $\{\psi_{i0}, \boldsymbol{\beta}_{i0}, i = 1, 2, \ldots, N\}$ are randomly distributed around the common means, ψ_0 and $\boldsymbol{\beta}_0$, such that

$$\psi_{i0} = \psi_0 + \eta_{i\psi}, \text{ and } \boldsymbol{\beta}_{i0} = \boldsymbol{\beta}_0 + \boldsymbol{\eta}_{i\beta} \text{ for } i = 1, 2, \ldots, N, \tag{25}$$

where $\boldsymbol{\eta}_i = \left(\eta_{i\psi}, \boldsymbol{\eta}'_{i\beta}\right) \sim IID\left(\mathbf{0}, \boldsymbol{\Omega}_\eta\right)$, $\boldsymbol{\Omega}_\eta > 0$ is a positive definite matrix, and it is assumed that $E\left\|\boldsymbol{\eta}_i\right\|^{2+c} < K$, for some $c > 0$. The parameters of interest are ψ_0 and $\boldsymbol{\beta}_0$ which are the population means of spatial lags and slope parameters of the underlying HSAR model. For consistent estimation of ψ_0 and $\boldsymbol{\beta}_0$ we now need both N and T sufficiently large. Large T is required to consistently estimate the unit-specific coefficients, and large N is required for estimation of the common means, ψ_0 and $\boldsymbol{\beta}_0$. It is also possible to apply this procedure to subsets of the units, so long as the number of units in each set is reasonably large.

Consistent estimators of ψ_0 and $\boldsymbol{\beta}_0$ are given by the mean group (MG) estimators,

$$\hat{\psi}_{MG} = N^{-1}\sum_{i=1}^{N}\hat{\psi}_i, \text{ and } \hat{\boldsymbol{\beta}}_{MG} = N^{-1}\sum_{i=1}^{N}\hat{\boldsymbol{\beta}}_i, \tag{26}$$

where $\hat{\psi}_i$ and $\hat{\boldsymbol{\beta}}_i$ are the underlying unit-specific estimators. The MG estimator was originally developed by Pesaran and Smith (1995) who show that in the standard case where $\hat{\psi}_i$ and $\hat{\boldsymbol{\beta}}_i$ are independently distributed, then $\hat{\psi}_{MG}$ and $\hat{\boldsymbol{\beta}}_{MG}$ will be consistent and asymptotically normal. Recently, Chudik and Pesaran (2019) extend this analysis and consider MG estimators based on possibly cross correlated estimators and show that the standard MG estimation continues to apply so long as the underlying unit-specific estimators are weakly cross correlated.

The main result of this section is summarised in the following proposition:

Proposition 3 *Consider the heterogeneous spatial autoregressive (HSAR) model given by (1) where the coefficients $\{\psi_{i0}, \boldsymbol{\beta}_{i0}, i = 1, 2, \ldots, N\}$ are distributed randomly around the common means ψ_0 and $\boldsymbol{\beta}_0$ following (25). Suppose that Assumptions 1, 2, 3, 4, and 5 hold. Then for a fixed k, and as $N, T \to \infty$ jointly such that $\sqrt{N}/T \to 0$, the mean group estimators, $\hat{\psi}_{MG}$ and $\hat{\boldsymbol{\beta}}_{MG}$, defined by (26) have the following asymptotic distributions*

$$\sqrt{N}\left(\hat{\psi}_{MG} - \psi_0\right) \overset{a}{\sim} N\left(0, \omega_\psi^2\right), \text{ and } \sqrt{N}\left(\hat{\boldsymbol{\beta}}_{MG} - \boldsymbol{\beta}_0\right) \overset{a}{\sim} N\left(\mathbf{0}, \boldsymbol{\Omega}_\beta\right),$$

with consistent estimators of ω_{ψ}^{2} *and* $\mathbf{\Omega}_{\beta}$ *given by*

$$\hat{\omega}_{\psi}^{2}=\frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\psi}_{i}-\hat{\psi}_{MG}\right)^{2}\text{, and }\hat{\mathbf{\Omega}}_{\beta}=\frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\beta}}_{i}-\hat{\boldsymbol{\beta}}_{MG}\right)\left(\hat{\boldsymbol{\beta}}_{i}-\hat{\boldsymbol{\beta}}_{MG}\right)', \tag{27}$$

respectively, where $\hat{\psi}_{i}$ *and* $\hat{\boldsymbol{\beta}}_{i}$ *are the underlying unit-specific estimators.*

Proof. See the online supplement B. ■

Remark 10 *The asymptotic distributions of* $\hat{\psi}_{MG}$ *and* $\hat{\boldsymbol{\beta}}_{MG}$ *shown in Proposition 3 are carried out assuming* $\boldsymbol{\beta}_{i}$ *and* ψ_{i} *are heterogeneous, such that* $Var(\boldsymbol{\beta}_{i})$ *and* $Var\left(\psi_{i}\right)$ *are strictly non-zero, and do not apply to the case where the slopes are assumed to be homogeneous. For a formal exposition of the properties of the Mean Group estimators under heterogeneous (*$\sqrt{N}$*-consistent) and homogeneous slope coefficients (*$\sqrt{NT}$*-consistent) see Pesaran and Tosetti (2011).*

Remark 11 *In principle, it is possible to test the hypothesis of slope homogeneity using Hausman type tests. However, as shown in Pesaran and Yamagata (2008), such tests are likely to lack power. The development of more powerful tests of slope homogeneity in the context of HSAR models is beyond the scope of this paper and would be an interesting topic of further research.*

5 Small sample properties of the QMLE

We investigate the small sample properties of the proposed QML estimator and the associated MG estimator using Monte Carlo simulations. We consider the following data generating process (DGP)

$$y_{it}=a_{i}+\psi_{i}\sum_{j=1}^{N}w_{ij}y_{jt}+\beta_{i}x_{it}+\varepsilon_{it},\ i=1,2,\ldots,N;\quad t=1,2,\ldots,T. \tag{28}$$

We include one exogenous regressor, x_{it}, with coefficient β_{i} as well as fixed effects, a_{i}, in unit-specific regressions. Stacking these regressions we have

$$\boldsymbol{y}_{\circ t}=\boldsymbol{a}+\boldsymbol{\Psi}\boldsymbol{W}\boldsymbol{y}_{\circ t}+\boldsymbol{B}\boldsymbol{x}_{\circ t}+\boldsymbol{\varepsilon}_{\circ t},\ t=1,2,\ldots,T,$$

where $\boldsymbol{a}=(a_{1},a_{2},\ldots,a_{N})'$, $\boldsymbol{\Psi}=\mathrm{Diag}\left(\boldsymbol{\psi}\right)$ and $\boldsymbol{\psi}=(\psi_{1},\psi_{2},\ldots,\psi_{N})'$, $\boldsymbol{W}=(w_{ij})$, $i,j=1,2,\ldots,N$, $\boldsymbol{B}=\mathrm{Diag}\left(\boldsymbol{\beta}\right)$, with $\boldsymbol{\beta}=(\beta_{1},\beta_{2},\ldots,\beta_{N})'$, $\boldsymbol{x}_{\circ t}=(x_{1t},x_{2t},\ldots,x_{Nt})'$, and $\boldsymbol{\varepsilon}_{\circ t}=(\varepsilon_{1t},\varepsilon_{2t},\ldots,\varepsilon_{Nt})'$. Note that since we explicitly account for fixed effects which we separate out from the remaining regressors included in $\boldsymbol{x}_{\circ t}$, the unknown parameters are summarised in vector $\boldsymbol{\theta}$, as follows: $\boldsymbol{\theta}=\left(\boldsymbol{a}',\boldsymbol{\psi}',\boldsymbol{\beta}',\boldsymbol{\sigma}^{2\prime}\right)'$, $\boldsymbol{\sigma}^{2}=(\sigma_{1}^{2},\sigma_{2}^{2},\ldots,\sigma_{N}^{2})'$. In total there are $4N$ unknown parameters.

We allow for spatial dependence in the regressors, x_{it}, and generate them as

$$\boldsymbol{x}_{\circ t}=\left(\mathbf{I}_{N}-\boldsymbol{\Phi}\boldsymbol{W}_{x}\right)^{-1}\boldsymbol{v}_{\circ t}, \tag{29}$$

where $\mathbf{\Phi} = \text{Diag}(\phi_1, \phi_2, \ldots, \phi_N)$, and $\boldsymbol{v}_{\circ t} = (v_{1t}, v_{2t}, \ldots, v_{Nt})'$, with $v_{it} \sim IIDN(0, \sigma_v^2)$. We set $\phi_i = 0.5$ (representing a moderate degree of spatial dependence), and set $\sigma_v^2 = N/\operatorname{tr}\left[(\mathbf{I}_N - \mathbf{\Phi}\boldsymbol{W}_x)^{-1}(\mathbf{I}_N - \mathbf{\Phi}\boldsymbol{W}_x)'^{-1}\right]$, which ensures that $N^{-1}\sum_{i=1}^{N} Var(x_{it}) = 1$. We set $\boldsymbol{W}_x = \boldsymbol{W} = (w_{ij})$, $i, j = 1, 2, \ldots, N$, and use the 4-connection spatial matrix described below.

We consider Gaussian errors such that $\varepsilon_{it}/\sigma_{i0} \sim IIDN(0, 1)$, and non-Gaussian errors such that $\varepsilon_{it}/\sigma_{i0} \sim IID\left[\chi^2(2) - 2\right]/2$, for $i = 1, 2, \ldots, N$, and $t = 1, 2, \ldots, T$, where $\chi^2(2)$ is a chi-squared variate with 2 degrees of freedom. σ_{i0}^2 are generated as independent draws from $\chi^2(2)/4 + 0.50$, for $i = 1, 2, \ldots, N$, and kept fixed across the replications.

For the weight matrix, $\boldsymbol{W} = (w_{ij})$, first we use contiguity criteria to generate the non-normalised weights matrices, $\boldsymbol{W}^0 = (w_{ij}^o)$, and then row normalise these to obtain w_{ij}. More specifically, we consider $\boldsymbol{W}$ matrices with 2, 4 and 10 connections.[4] Since by construction $\|\boldsymbol{W}\|_\infty = 1$, then condition (8) is satisfied if $\sup_i |\psi_i| < 1$, and ensures that $\mathbf{I}_N - \mathbf{\Psi}\boldsymbol{W}$ is invertible. We generate the unit-specific coefficients of the HSAR model as $a_{i0} \sim IIDN(1, 1)$, $\beta_{i0} \sim IIDU(0, 1)$, and $\psi_{i0} \sim IIDU(0, 0.8)$, for $i = 1, 2, \ldots, N$.[5] Given the DGP in (28), values of y_{it} are now generated as $\boldsymbol{y}_{\circ t} = (\mathbf{I}_N - \mathbf{\Psi}\boldsymbol{W})^{-1}(\boldsymbol{a} + \boldsymbol{B}\boldsymbol{x}_{\circ t} + \boldsymbol{\varepsilon}_{\circ t})$.

Initially, to illustrate that our proposed estimator applies to both cases where N is small and large, we considered the two polar cases of $N = 5$ and $N = 100$, and set $T = 25, 50, 100, 200$, thus including (N, T) combinations such that $N < T$, $N = T$ and $N > T$, respectively. We then considered a more comprehensive set of N values, namely $N = 25, 50, 75, 100$. For each experiment we used $R = 2,000$ replications. Across the replications, $\boldsymbol{\theta}_0$, and the weight matrix, $\boldsymbol{W}$, are kept fixed, whilst the errors and the regressors, ε_{it} and x_{it} (and hence y_{it}), are re-generated randomly in each replication. Note that, as N increases, supplementary units are added to the original vector $\boldsymbol{\theta}_0$ generated initially for $N = 5$. Due to the problem of simultaneity, the degree of time variations in y_{it}^* for each unit i depends on the choice of $\boldsymbol{W}$ and the number of cross section units, N. Naturally, this is reflected in the performance of the estimators and the power properties of the tests based on them.

We report bias and RMSE of the QML estimators for individual cross section units, as well as their corresponding empirical sizes. In addition, we report power functions for three units with true spatial autoregressive parameters, ψ_{i0}, selected to be low, medium and large in magnitude. The experiments are carried out for spatial weight matrices, $\boldsymbol{W}$, with two, four and ten connections. The mean of simulated parameter estimates are computed as $\hat{\psi}_{i(R)} = R^{-1}\sum_{r=1}^{R}\hat{\psi}_{i,r}$, and $\hat{\beta}_{i(R)} = R^{-1}\sum_{r=1}^{R}\hat{\beta}_{i,r}$, where $\hat{\psi}_{i,r}$ and

[4] We generate $\boldsymbol{W}^0 = (w_{ij}^o)$ such that: (i) $w_{ij}^o = 1$ if $j = i-1, i+1$, and zero otherwise (2 connections), (ii) $w_{ij}^o = 1$ if $j = i-2, i-1, i+1, i+2$, and zero otherwise (4 connections), and (iii) $w_{ij}^o = 1$ if $j = i-5, i-4, \ldots, i-1, i+1, i+2, \ldots, i+5$, and zero otherwise (10 connections). By construction, the first and last units have fewer neighbours as compared to the other units.

[5] We also carried out experiments without exogenous regressors with $\beta_{i0} = 0$, for all i, corresponding to the simplified version of model (5) discussed in Section 3.2. The results of these experiments are available upon request.

$\hat{\beta}_{i,r}$ refer to the QML estimates of ψ_i and β_i in the r^{th} replication. The QML estimators are computed using the log-likelihood function (7). We also report small sample results for the MG estimators of ψ_0 and β_0, defined by (26), using the experiment described in Section 5.2 below.

5.1 Results for individual estimates

Since the results based on the Gaussian and non-Gaussian errors are very close, in what follows we only report the results for the non-Gaussian case where the errors are generated as iid $\chi^2(2)$ random variables, and use the sandwich formula (24) to compute standard errors. Also to save space, we focus on results based on the spatial weight matrix, $\boldsymbol{W}$, with four connections.[6] Initially, to highlight the applicability of the proposed estimators to small as well as large dimensional HSAR panels, we provide detailed results for the experiments with $N = 5$ and $N = 100$.

5.1.1 Two polar cases: $N = 5$ and $N = 100$

Table 1 reports the bias, RMSE, empirical size and power of the individual parameters, ψ_{i0} and β_{i0}, $i = 1, 2, \ldots, N$, for the experiments with $N = 5$. The bias of estimating ψ_{i0} tends to be small but negative when $T = 25$, whilst estimates of β_{i0} show an upward bias when T is small ($T = 25$). But the biases of both estimators fall quite rapidly with T, for all i. A similar pattern can be seen in the RMSEs, again declining with T reasonably fast. Turning to size and power of the tests based on the QML estimates, there is some evidence of over-rejection when T is small ($T = 25$). But the size distortion gets eliminated as T is increased, with the tests having the correct size for values of $T \geq 50$. This pattern is shared by both ψ_{i0} and β_{i0}. Similarly, power is low when $T = 25$ but improves markedly for all 5 units as T is increased.[7] Overall the small sample results are in line with our theoretical findings, and give satisfactory results for values of $T \geq 50$; a property which is repeated for other experiments considered in this paper.

For $N = 100$ we report the results only for a selected number of units, namely units with the three smallest and largest population values for ψ_{i0} and a few in between, and the associated β_{i0} values. The small sample results for these experiments are summarised in Table 2, and are qualitatively similar to those reported in Table 1 for $N = 5$, indicating that the theoretical framework of Section 3 can be applied equally to data sets with small and large numbers of cross section units.

[6] Results for Gaussian errors and other choices of spatial weight matrices are available upon request.

[7] Clearly, improvements in power can be achieved by reducing the error variances, σ^2_{i0}. Some supporting evidence is provided in Tables G1 and G2 in the online supplement G.

5.1.2 RMSE, size and power for all N and T combinations

We now turn to the rest of the results and consider all the combinations of $N \in \{25, 50, 75, 100\}$ and $T \in \{25, 50, 100, 200\}$. To save space we use boxplots to summarise the results for RMSE and size, and use empirical rejection frequency plots for power.[8] All results are shown in the online supplement G. The RMSE boxplots for ψ_{i0} and β_{i0} are given in Figures G1 and G2, respectively. Overall, the RMSE values are small for both parameters and fall with T but are not affected by changes in the cross section dimension, N, which is in line with the theory developed in Section 3.

The boxplots for the size of the tests based on the QML estimates of ψ_{i0} and β_{i0} are given in Figures G3 and G7, respectively. These results are based on the sandwich covariance matrix formula given by (24). As can be seen, in general the tests are correctly sized at 5 per cent for T relatively large, although for small values of T there are some size distortions. Once again the size estimates are not affected by N, and tend to 5 per cent as T increases, irrespective of the value of N.

To save space we only report the empirical power functions of the tests for three cross section units with low, medium and high parameter values. The power plots are computed for different values of ψ_i and β_i defined by $\psi_i = \psi_{i0} + \delta$, and $\beta_i = \beta_{i0} + \delta$, for $i = 1, 2, \ldots, N$, where $\delta = -0.800, -0.791, \ldots, 0.791, 0.800$. We only consider values of ψ_i that satisfy the condition $|\psi_i| < 1$.[9]

The power results for the spatial parameters, ψ_{i0}, are displayed in Figures G4-G6, that correspond to the low value ($\psi_{i0} = 0.3374$), the medium value ($\psi_{i0} = 0.5059$) and the high value ($\psi_{i0} = 0.7676$), respectively. As to be expected the power depends on the choice of ψ_{i0} and rises with T, but does not seem to be affected by N. Furthermore, perhaps not surprisingly, empirical power functions for ψ_{i0} become more and more asymmetrical as ψ_{i0}'s move closer and closer to the boundary value of 1. The power functions for the three associated values of β_{i0} are shown in Figures G8-G10 for the low value of β_{i0} ($\beta_{i0} = 0.0344$), the medium value ($\beta_{i0} = 0.4898$), and the high value ($\beta_{i0} = 0.9649$), respectively. Again the empirical power functions are similar across N and improve with T.

5.2 Small sample properties of the MG estimators

We employ the same data generating process, defined by (28), and set $a_{i0} = a_0 + \epsilon_{1i}$, with $a_0 = 1$ and $\epsilon_{1i} \sim IIDN(0, 1)$, $\psi_{i0} = \psi_0 + \epsilon_{2i}$, with $\psi_0 = 0.4$ and $\epsilon_{2i} \sim IIDU(-0.4, 0.4)$ and $\beta_{i0} = \beta_0 + \epsilon_{3i}$, with $\beta_0 = 0.5$ and $\epsilon_{3i} \sim IIDU\,(-0.5, 0.5)$. Parameters a_0, ψ_0 and β_0 are fixed while parameters a_{i0}, ψ_{i0} and

[8] The boxplots for bias of the estimators are similar to those of RMSE and are available upon request. The corresponding tables that show bias and RMSE results for the individuals estimates ($\hat{\psi}_{i(R)}$, and $\hat{\beta}_{i(R)}$, $i = 1, 2, \ldots, N$) are also available upon request.

[9] The empirical power functions are computed using the sandwich formula for the covariance matrix of the underlying estimators.

β_{i0} vary across replications, for $i = 1, 2, \ldots, N$, in accordance to the random coefficients model. The MG estimators and their standard errors are computed using (26) and (27), and the number of replications is set to $R = 2,000$. The small sample properties of the mean group estimators of ψ_0 and β_0 are summarised in Table G3 of the online supplement G. The top panel gives the results for Gaussian errors, and the bottom panel for non-Gaussian errors. As to be expected the bias and RMSE of the MG estimators decline steadily with both N and T, and it does not matter whether the errors are Gaussian or not. There are some small size distortions when $N = T = 25$, but the size rapidly converges to the nominal value of 5 percent as N and T are increased. For example for $T = 25$ the size is always within the simulation standard errors when $N \geq 50$.

6 Heterogeneous spatial spill-over effects in U.S. housing market

As an empirical application we estimate HSAR models for quarterly real house price changes in the United States at Metropolitan Statistical Areas (MSAs) over the period 1975Q1-2014Q4. Modelling and forecasting of cycles in housing markets are of paramount importance for prospective owners, investors, and real estate market participants such as insurers and mortgage lenders (Agnello et al., 2015). Some areas of interest include: (i) land use regulations which affect the elasticity of housing supply and thus the extent to which population growth translates into greater housing price growth (Saiz (2010)), (ii) REIT investment, by determining the optimal portfolios of real estate structures across space that have the highest returns whilst holding risk constant, (iii) equilibrium effects of public policies, such as the knock-on effects on real estate price growth due to local labour market distortions associated with increased import competition (Autor et al. (2013)).

Determinants of U.S. house price changes are numerous and well-documented in the literature; two prominent fundamentals being real per capita disposable income and population - see for example Malpezzi (1999) and Gallin (2006) among others. Factors such as differences in land use regulations, construction costs and real wages that vary rather slowly over time can be captured by fixed effects and slope heterogeneity. An important aspect of the modelling strategy is to account for the existence of co-movements in house prices within and across MSAs as well. Recently, Bailey et al. (2016) (hereafter BHP) highlight the importance of distinguishing between types of cross-sectional dependence in the analysis of U.S. house price changes, which if ignored can lead to biased parameter estimates. See, for example, the studies by Swoboda et al. (2015) and Munro (2018). BHP distinguish between spatial dependence that originates from economy-wide common shocks such as changes in interest rates, oil prices and technology, and the dependence across MSAs due to local spill-over effects arising from differences in house prices, incomes and demographics across MSAs.[10] Here, we use an extended version of the panel dataset employed by BHP and further augmented with population and per capita real income data by Yang (2020) to estimate HSAR models, after filtering out the effects of common factors on house price changes.[11] We provide MSA specific estimates of spill-over effects, temporal dynamics, as well as population and income elasticities of house prices and corresponding partial effects over time. Further, we report MG estimates of our individual parameter values both at the national and regional levels. As we shall see, we find considerable heterogeneity across MSAs and regions.

6.1 Data description and transformations

There exist 381 metropolitan statistical areas (MSAs) which fall under the February 2013 definition provided by the U.S. Office of Management and Budget (OMB). We consider $N = 377$ of these from the contiguous United States. Accordingly, we compile the following variables that are included in our model over the period 1975Q2-2014Q4 ($T = 160$ quarters): Π_{it} is the percent quarterly rate of change of real house prices of MSA i in quarter t (dependent variable), $GPOP_{it}$ is the percent quarterly rate of change of population (regressor), and $GINC_{it}$ is the percent quarterly rate of change in real per capita income (regressor). Details on data sources and transformations can be found in the online supplement F.

Our estimation strategy requires real house price changes, Π_{it}, to be cross-sectionally weakly dependent by Assumption 4. We first apply the CD test developed in Pesaran (2004, 2015) to Π_{it} in order to assess the strength of cross-sectional dependence (CSD) in Π_{it}. The CD statistic turns out to be 1621.22 which is substantially higher than the 1.96 critical value at 5 per cent level. With the null hypothesis of weak CSD soundly rejected, we then estimated the exponent of cross-sectional dependence, α, due to Bailey et al. (2016) which measures the degree of cross-sectional dependence of house price changes. Values of α close to unity are indicative of strong cross-sectional dependence. We obtained $\hat{\alpha} = 1.00$ (0.03), where the standard error of the estimate is given in brackets. It is clear that real house price changes, Π_{it}, are strongly correlated across MSAs, and before estimating local spill-over effects using the HSAR model, we must first purge Π_{it} of the common sources of their dependence, as suggested in BHP.

Accordingly, we de-seasonalise and de-factor the three variables that we use to estimate the HSAR specifications which we denote by π_{it}, $gpop_{it}$ and $ginc_{it}$, respectively.[12] The CD statistic for the filtered series, π_{it}, is -3.367, which is substantially lower than the value of 1621.22 obtained for the unfiltered

[10] For a theoretical analysis of the interactions between regional house prices, migration flows and income shocks, see Cun and Pesaran (2018).

[11] In order to decipher the relative importance of common factors and spill-overs in explaining the variation in house prices by MSA, one can extend Pesaran and Chudik (2014) to the case of heterogeneous spill-overs. This analysis is beyond the scope of this paper.

[12] Details of the de-seasonalising and de-factoring of Π_{it}, $GPOP_{it}$ and $GINC_{it}$ can be found in the online supplement F.

series, but is nevertheless too large, and could suggest that the filtering has not been effective in removing the strong sources of cross-sectional dependence. There are two reasons that this might not be the case. First, as Juodis and Reese (2019) point out, the CD test most likely over-rejects when it is applied to residuals from a regression on *unobserved* common factors. Second, as shown in Pesaran (2015), the implicit null of the CD test depends on the relative expansion rates of N and T, and for $T = O(N^{\epsilon})$, $0 \leq \epsilon \leq 1$, such that the exponent of cross-sectional dependence, α, which characterises the degree of cross-sectional dependence of π_{it}, falls in the range $0 \leq \alpha \leq (2-\epsilon)/4$. Hence, a mere rejection of the CD test does not necessarily mean that the de-factored series are not weakly correlated. This view point is supported by the estimates of spatial effects reported in the next sub-section which satisfy the stability conditions associated with weakly cross correlated processes.

6.2 Estimation of HSAR model for de-factored house price changes

We now consider the following HSAR specification for π_{it} which allows for spatio-temporal effects in house price inflation:

$$\pi_{it} = a_i + \psi_{0i}\sum_{j=1}^{N} w_{ij}\pi_{jt} + \psi_{1i}\sum_{j=1}^{N} w_{ij}\pi_{j,t-1} + \lambda_i \pi_{i,t-1} + \beta_i^{pop} gpop_{it} + \beta_i^{inc} ginc_{it} + \varepsilon_{it}, \tag{30}$$

for $i = 1, 2, \ldots, N$ and $t = 1, 2, \ldots, T$. The model incorporates fixed effects and full heterogeneity in the spatial and temporal autoregressive coefficients of real house price changes $(\psi_{0i}, \psi_{1i}, \lambda_i)$, as well as the slope coefficients for the two regressors $(\beta_i^{pop}, \beta_i^{inc})$. Innovations are assumed to be distributed as $\varepsilon_{it} \sim IID\left(0, \sigma_{i\varepsilon}^2\right)$.[13] (30) is in accordance with the theoretical model (1) analysed in Sections 2 and 3.[14]

6.2.1 Choice of weight matrix W

For the weight matrix $\boldsymbol{W} = (w_{ij})$, we consider the distance based weighting scheme implemented in Yang (2020), which is common in the spatial econometrics literature. More precisely, we compute the geodesic distance between each pair of latitude/longitude coordinates for the MSAs included in our sample using the Haversine formula. These coordinates correspond to the center of the polygon implied by each MSA. Then, we determine a specific radius threshold, d (miles), within which MSAs are considered to be neighbours. In this case, the relevant entries in the un-normalised weight matrix $\boldsymbol{W}^0$ are set to unity. The MSAs that fall outside this radius are labelled non-neighbours and their corresponding entries in

[13] In performing the data transformations of Section 6.1, we abstract from the sampling uncertainty related to using defactored series when estimating HSAR models. In principle, one could estimate the common and local effects simultaneously, instead of the two-stage procedure being followed. However, such an endeavour is beyond the scope of the present paper.

[14] We have considered alternative models to (30): one that allows time lags in the exogenous variables as well, and another assuming no spatially and temporaly lagged dependent variable. Overall, the results convey a similar message as that from running regression (30). For brevity of exposition, these results are not included in the paper, but are available upon request.

$\boldsymbol{W}^0$ are set to zero. Finally, we row-normalise $\boldsymbol{W}^0$ and obtain $\boldsymbol{W}$ which is used in (30).

We consider three versions of $\boldsymbol{W}$ constructed with the radius threshold values of $d = 75$, 100 and 125, miles. We name the adjacency matrices $\boldsymbol{W}_{75}$, $\boldsymbol{W}_{100}$ and $\boldsymbol{W}_{125}$, respectively. For brevity of exposition, in what follows we focus on the version of (30) that uses $\boldsymbol{W}_{75}$ which gives a reasonably sparse weight matrix with 0.88% non-zero elements. Other types of weighting schemes can also be entertained. For example, one can make use of the inverse of bilateral geodesic distances, d_{ij}, between MSAs to construct $\mathbf{W}$. Another alternative is considered by BHP who use two separate adjacency matrices determined by the statistically positive and negative pairwise correlations of de-factored real house price changes. A further scheme is proposed by Zhou et al. (2017) who use a sample-based adjacency matrix to approximate the true network structure by focusing on an estimation framework that incorporates just the degree (number of connections) of each unit in the network.

6.2.2 MSA specific estimates

First we present the estimates of individual contemporaneous and net spatial and temporal effects by MSAs. Note that when using adjacency matrix $\boldsymbol{W}_{75}$ in (30) there are 39 out of the total 377 MSAs that are completely isolated (have no neighbours) and are thus excluded from the analysis. This leaves us with a reduced sample of $N = 338$ MSAs. Out of these, 260 estimates (or 77%) of the contemporaneous spatial coefficients ($\hat{\psi}_{0i}$) were positive and statistically significant, with only 19 estimates being significantly negative. However, these positive ripple effects are negated somewhat after one quarter, with the coefficients of the lagged spatial effects ($\hat{\psi}_{1i}$) being largely negative, with 259 being negative and statistically significant, 14 significant and positive.[15] As a result the net spatial effects, computed as $\hat{\psi}_{0i} + \hat{\psi}_{1i}$, are smaller with fewer being statistically significant. Figure 1(a) displays the estimates of net effects. Each net spatial estimate, $\hat{\psi}_{0i} + \hat{\psi}_{1i}$, is matched to its corresponding MSA on the map of the U.S.. MSAs colored in blue depict positive net spatial lag coefficients, with different shades of blue corresponding to differing ranges within which each $(\hat{\psi}_{0i} + \hat{\psi}_{1i})$ falls: lighter shades refer to ranges closer to zero while darker shades relate to net spatial lag coefficient estimates closer to the boundary value of unity. Similarly, red areas are associated with negative net spatial lag coefficient estimates, with the lighter shade of red indicating $(\hat{\psi}_{0i} + \hat{\psi}_{1i})$ falling in ranges closer to zero, while darker red areas refer to more sizable net spatial coefficient estimates. Similarly, Figure 1(b) displays the estimates of the temporal coefficients, $\hat{\lambda}_i$, which are generally positive and highly significant.[16]

[15] A visual representation of the individual estimates, $\hat{\psi}_{0i}$ and $\hat{\psi}_{1i}$, is given in Figures F1(a) and F1(b) of the online supplement F.

[16] 42 MSAs have coefficient estimates that hit the upper or lower bound of 0.994/-0.994 set in our optimisation procedure. This occurs when $\left|\hat{\psi}_{0i}\right| > 0.994$ or $\left|\hat{\psi}_{1i}\right| > 0.994$ or $\left|\hat{\lambda}_i\right| > 0.994$. These MSAs are shown as a separate category in Figures

Though on average, spatial (contemporaneous and lagged) effects across the U.S. net out at a value of 0.08 (0.013), it is evident from Figure 1(a) that there are significant differences in these effects across individual MSAs. Indeed, 263 MSAs have positive net spatial lag coefficients of which 147 are statistically significant, while 75 MSAs have negative net spatial lag coefficients only 18 of which are significant. Overall, these net spatial lag coefficients point to the existence of important spill-over effects in the U.S. housing market even when the influence of national (common) factors are filtered out. It is easy to show spill-over effects in house price changes across MSAs without de-factoring, but such evidence suffers from the conjunctions of national and local influences, and can be misleading. The spatial display of the estimates in Figure 1(a) shows how the strength and sign of local spill-over effects changes as we move from less economically developed MSAs towards more vibrant neighbouring hubs. A distinction in relative spatial effects is evident between the sparsely populated areas in the middle of the U.S. (Plains, Rocky Mountains and South West), and the two coastal areas (South East, Mid East and Far West) which have a much higher population density - see Table F1 of the online supplement F. Next, Figure 1(b) shows that the temporal dynamics in house price changes are universally positive with 338 MSAs having positive temporal lag coefficients of which 328 are statistically significant. In general, these estimates are also reasonably large considering that de-factoring is likely to have removed most of the common dynamics in house price fluctuations. Still, heterogeneity across MSAs is evident, with parts of Mid East, South West and Far West showing stronger temporal feedback effects when compared to other U.S. regions.

Similar differences can also be seen in the estimates of the elasticities of house price changes to population and real per capita income changes, as shown in Figures 2(a) and 2(b). We observe that in 302 MSAs the contemporaneous population or income variables have a positive impact on house price changes, although the population effects tend to be more significant and sizable. Of these, more than 70% tend to coincide with areas also reporting positive estimates for the net spatial lag coefficients. Important examples of such MSAs include Los Angeles and San Francisco, Kansas City or New York-Newark-Jersey.[17] In contrast, the MSAs with negative estimates spread evenly across the United States and correspond to economically less active areas, such as Cheyenne (Wyoming), Pocatello (Idaho), Pittsfield (Massachusetts), Minneapolis (Minnesota) and Memphis (Arkansas). Interestingly, out of these 18 MSAs around half have in fact experienced relatively muted rise, stagnant or outright decline in population over our sample period, potentially contributing to their negative $(\psi_{0i} + \psi_{1i})$ estimates.

1(a) and 1(b).

[17] Adding lagged population and real per capita income variables in (30) produces estimates that are generally small and less statistically significant as compared to their contemporaneous effects. Hence, these estimates are omitted.

6.2.3 Regional estimates

The heterogeneity in the estimates we observe across the MSAs continue to be present at the regional level. Table 3 reports the mean group estimates of the parameters by six regions. We started with the standard eight regional classification, but combined New England and the Mid East, and South West and Rocky Mountains to ensure a reasonable number of MSAs (N_r in Table 3) per each region. As can be seen, the MG estimates of the contemporaneous and lagged spatial coefficients are quite close for Great Lakes, South East and Far West, but differ markedly for the other three regions, namely New England & Mid East, Plains, and South West & Rocky Mountains. These differences largely reflect the different degrees of population density across the U.S..

Temporal effects are positive and significant across all regions, and cluster within two broad groupings, namely (i) New England & Mid East, South West & Rocky Mountains and Far West, and (ii) Great Lakes, Plains and South East. We notice even larger differences in the MG estimates of population and real income variables across the regions, with much larger estimates for the effects of the population variable on house price changes as compared to the effects of the income variable. For the U.S. as a whole, the MG estimate of the net spatial and temporal effects amount to 0.088 (0.013) and 0.667 (0.010) respectively, where the net spatial effects are decomposed into the contemporaneous MG estimate 0.603 (0.027) and lagged spatial MG estimate -0.515 (0.020). These estimates point to the existence of non-negligible spatio-temporal effects in the U.S. even after conditioning on factors that generate strong correlation between disaggregate house price fluctuations over time and space. These results clearly show the importance of including dynamics in the analysis of spatial effects, which if omitted can lead to exaggeration of these effects. For example, Yang (2020) found (net) spatial effects of around 0.65 when considering a homogeneous and static SAR specification. Finally, the MG estimates of the contemporaneous effects of population and income variables for the U.S. as a whole are 0.250 (0.029) and 0.050 (0.006), respectively, both being statistically significant, with sizable long run effects.[18]

6.2.4 Direct and indirect spatial effects at different horizons

The estimated spatio-temporal model, (30), can also be used for impulse response analysis that focuses on the effects of MSA specific shocks (ε_{it}), and/or can be used to investigate the effects of changes in the exogenous variables, namely income and population. To save space we focus on the latter exercise. We closely follow Debarsy et al. (2012) and extend their analysis to our heterogeneous parameter specification.

[18]We repeated the above empirical analysis using as non-zero elements of the weight matrix $\boldsymbol{W}_0$ the inverse of the geodesic pairwise distances between MSAs instead. For brevity of exposition we report the regional MG parameter estimates of model (30) in Table F3 of the online supplement F.

Writing (30) in matrix notation and solving for $\boldsymbol{\pi}_{t+h} = (\pi_{1,t+h}, \pi_{2,t+h}, ..., \pi_{N,t+h})'$, we have

$$\boldsymbol{\pi}_{t+h} = \boldsymbol{\Phi}^{h+1}\boldsymbol{\pi}_{t-1} + \left(\sum_{s=0}^{h-1}\boldsymbol{\Phi}^{s}\right)\boldsymbol{c} + \sum_{s=0}^{h-1}\boldsymbol{\Phi}^{s}\boldsymbol{A}\boldsymbol{x}_{t+h-s} + \sum_{s=0}^{h-1}\boldsymbol{\Phi}^{s}\boldsymbol{u}_{t+h-s},$$

where $\boldsymbol{x}_t = (\boldsymbol{x}'_{1t}, \boldsymbol{x}'_{2t}, \ldots, \boldsymbol{x}'_{Nt})'$, $\boldsymbol{x}_{it} = (x_{i1,t}, x_{i2,t})' = (gpop_{it}, ginc_{it})'$,

$$\boldsymbol{c} = (\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{a},\ \boldsymbol{\Phi} = (\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}(\boldsymbol{\Psi}_1\boldsymbol{W} + \boldsymbol{\Lambda}),$$

$$\boldsymbol{A} = (\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{B}, \text{ and } \boldsymbol{u}_t = (\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{\varepsilon}_t.$$

The marginal effect of a unit change in $x_{j\ell,t}$ on $\pi_{i,t+h}$ is given by

$$\frac{\partial\boldsymbol{\pi}_{t+h}}{\partial x_{j\ell,t}} = \left[\boldsymbol{\Phi}^{h}(\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{e}_j\right]\beta_{j\ell},\ \text{for } i,j = 1,2,\ldots,N;\ \ell = 1,2;\ h = 0,1,\ldots.$$

The average marginal direct and indirect effects of a unit change in $x_{j\ell,t}$ are now given by

$$D_N(h,\ell) = \frac{1}{N}\sum_{i=1}^{N}\frac{\partial\pi_{i,t+h}}{\partial x_{i\ell,t}},\ \text{and } ID_N(h,\ell) = \frac{1}{N(N-1)}\sum_{i\neq j}^{N}\frac{\partial\pi_{i,t+h}}{\partial x_{j\ell,t}},$$

for $h = 0,1,\ldots$, and $\ell = 1,2$. Further, the average indirect effects can be decomposed into average spill-in and spill-out effects. For further details on the derivation and interpretation of these effects see the online supplement E, and related contributions by LeSage and Chih (2016), LeSage and Chih (2018a) and LeSage et al. (2019).

The direct and indirect effects (decomposed into spill-in and spill-out effects) at the MSA levels are displayed in Figures F2, F3 and F4 of the online supplement F. These figures give the estimates on impact, and at horizons 3 and 6 quarters following a one percent increase in population and real income growth. The relative importance of these effects across time and space is evident, also when we compute the equivalent average regional metrics, namely the within-region direct and indirect effects, and the between-region spill-in and spill-out effects as characterised by equations (E.63), (E.64), (E.65) and (E.66) of the online supplement E, respectively. Any significant effects on house prices from changes in either the population and income variables of own or neighbouring MSAs across regions and horizons are concentrated within-region while the between-region effects are by comparison negligible (universally less than 0.3% of direct effects). These are shown in Table F2 of the online supplement F. Focusing on the within-region effects, these are more sizable across the board following a change in population growth as compared to the real income growth, and decay quite rapidly over time. For both population and real income variables, direct effects dominate over indirect effects across the six regions but the ratio of indirect to direct effects remains remarkably similar for the population and income variables in each region. Nevertheless, there continue to be considerable heterogeneity across the regions. For example, indirect effects tend to be larger in more densely populated regions of New England & Mid East, Great

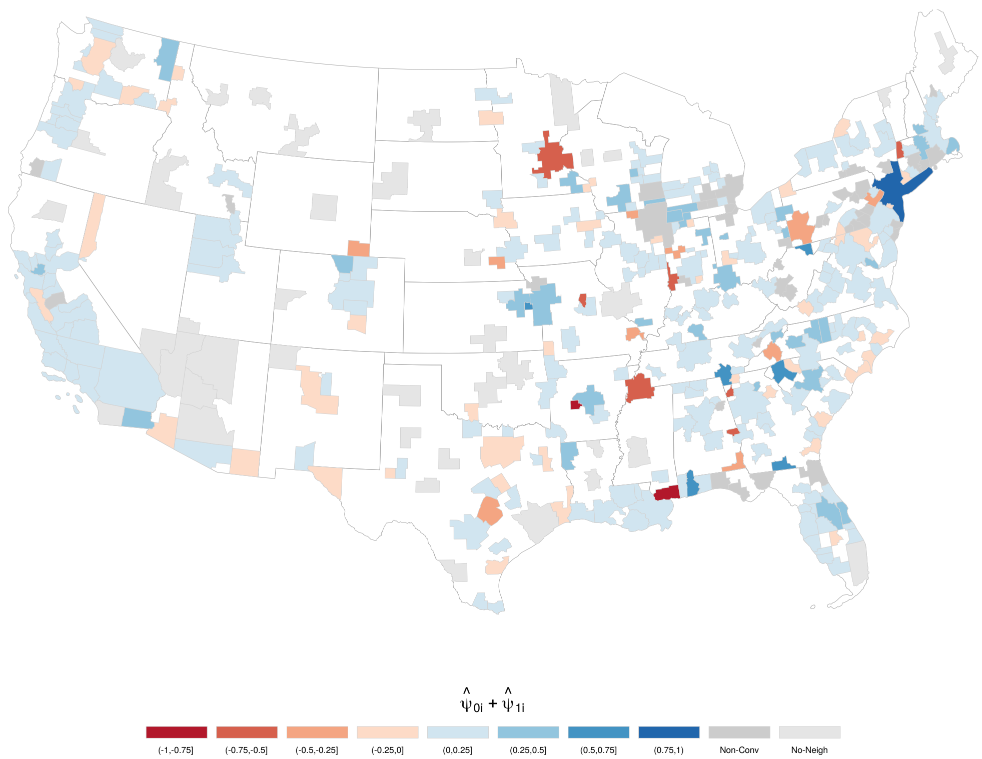
Lakes and Far West, especially on impact (see Table F1).

7 Conclusion

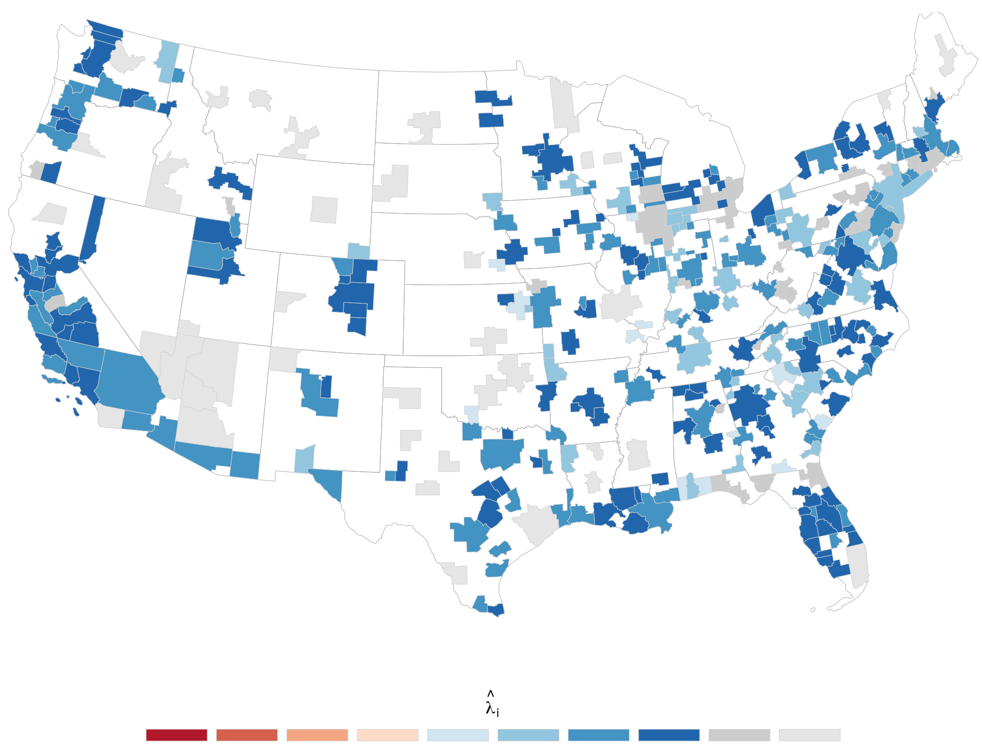
Standard spatial econometric models assume a single parameter to characterise the intensity or strength of spatial dependence across all units. In the case of pure cross section models or panel data models with a short time dimension, this assumption is inevitable. However, in a data rich environment where both the time (T) and cross section (N) dimensions are large, this can be relaxed. This paper investigates a spatial autoregressive panel data model with fully heterogeneous spatial parameters (HSAR) where the spatial dependence can arise directly through contemporaneous dependence of individual units on their neighbours, and indirectly through possible cross-sectional dependence in the regressors.

The asymptotic properties of the quasi maximum likelihood estimator are analysed assuming a sparse spatial structure with each individual unit having at least one connection. Conditions under which the QML estimator of spatial parameters are consistent and asymptotically normal are derived. It is also shown that under certain conditions on spatial coefficients and the spatial weights, the asymptotic properties of the individual estimates are not affected by the size of cross section dimension N. An estimator of the cross section mean of the individual parameters (MG estimators) is also analysed which can be used for comparisons with outcomes from standard homogeneous SAR models. It is shown that MG estimators are consistent and asymptotically normal as N and $T \to \infty$, jointly, so long as $\sqrt{N}/T \to 0$, and the spatial dependence is sufficiently weak. Monte Carlo simulation results provided are supportive of the theoretical findings. As an application of the HSAR model we investigate the potential heterogeneity in ripple effects, and the spatio-temporal direct and indirect effects of changes in population and income in the U.S. housing market at the MSA level.

Figure 1: Net spatial ($\hat{\psi}_{0i} + \hat{\psi}_{1i}$) and temporal ($\hat{\lambda}_i$) autoregressive parameter estimates for Metropolitan Statistical Areas in the Unites States



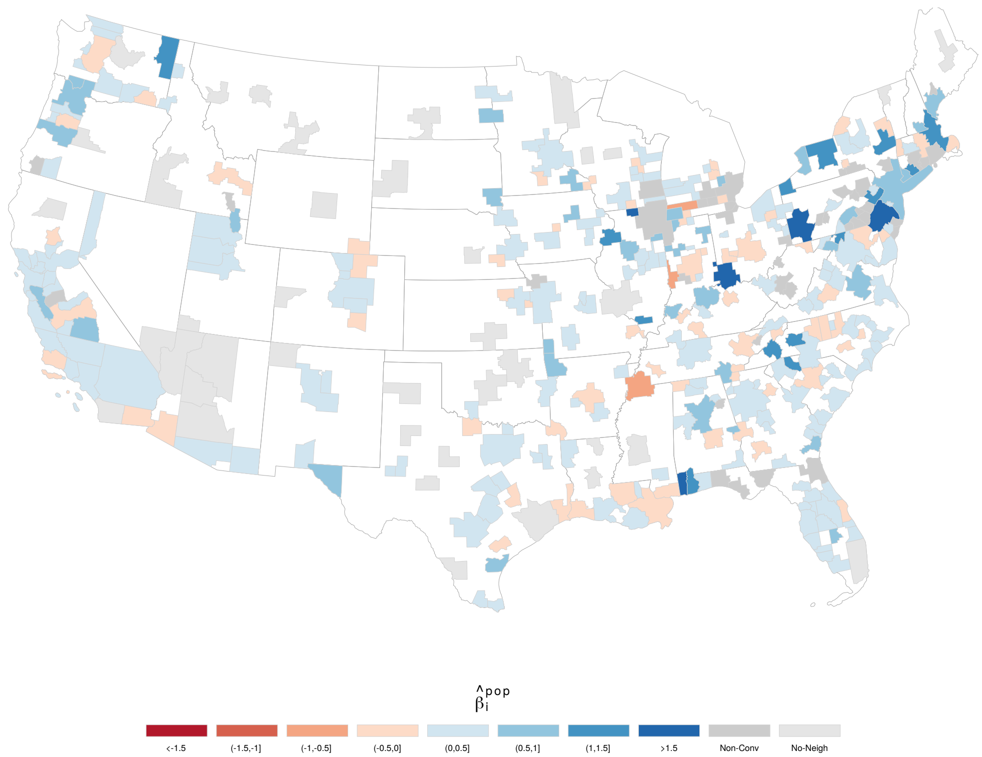
(a) Net spatial parameter estimates



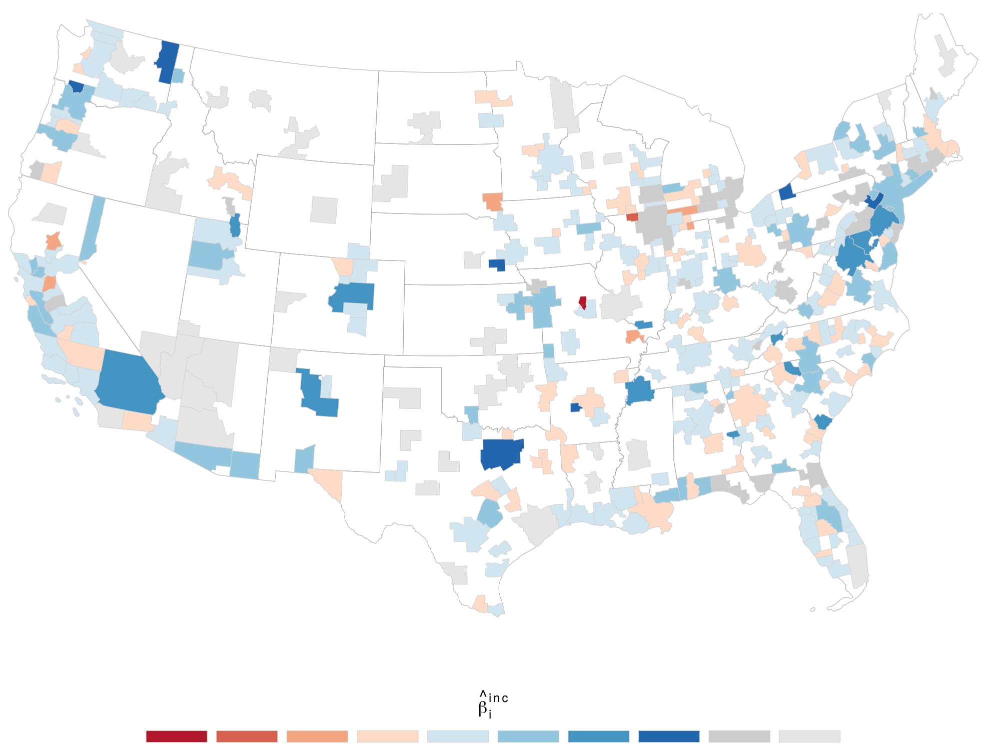
(b) Temporal parameter estimates

Notes: Each $\hat{\psi}_{0i}+\hat{\psi}_{1i}$ and $\hat{\lambda}_i$ is mapped to a Metropolitan Statistical Area (MSA) in the U.S.. A total of 338 MSAs are included in model (30). MSAs coloured in blue correspond to positive net spatial and temporal parameter estimates while MSAs coloured in red match to negative net spatial and temporal parameter estimates. Darker shades of blue or red indicate more sizable $\hat{\psi}_{0i} + \hat{\psi}_{1i}$ and $\hat{\lambda}_i$ while lighter shades related to $\hat{\psi}_{0i} + \hat{\psi}_{1i}$ and $\hat{\lambda}_i$ closer to zero in absolute terms. Category 'Non-conv' includes MSAs whose $\hat{\psi}_{0i}$, $\hat{\psi}_{1i}$, or $\hat{\lambda}_i$ estimates hit the upper/lower bound in the optimization procedure, while category 'No-Neigh' includes MSAs that have no neighbours when using $\boldsymbol{W}_{75}$.

Figure 2: Contemporaneous elasticities of house price changes to population growth ($\hat{\beta}_i^{pop}$) and real income growth ($\hat{\beta}_i^{inc}$) for Metropolitan Statistical Areas in the Unites States



(a) Population growth estimates



(b) Real income growth estimates

Notes: Each $\hat{\beta}_i^{pop}$ and $\hat{\beta}_i^{inc}$ is mapped to a Metropolitan Statistial Area (MSA) in the U.S.. A total of 338 MSAs are included in model (30). MSAs coloured in blue correspond to positive slope parameter estimates while MSAs coloured in red match to negative slope parameter estimates. Darker shades of blue or red indicate more sizable $\hat{\beta}_i^{pop}$ and $\hat{\beta}_i^{inc}$ while lighter shades related to $\hat{\beta}_i^{pop}$ and $\hat{\beta}_i^{inc}$ closer to zero in absolute terms.Category 'Non-conv' includes MSAs whose $\hat{\psi}_{0i}$, $\hat{\psi}_{1i}$ or $\hat{\lambda}_i$ estimates hit the upper/lower bound in the optimization procedure, while category 'No-Neigh' includes MSAs that have no neighbours when using $\boldsymbol{W}_{75}$.

Table 1: Bias, RMSE, size and power for parameters of individual units in the HSAR(1) model with one exogenous regressor and non-Gaussian errors for $N = 5$ and $T \in \{25, 50, 100, 200\}$.

T	25		50		100		200	
Parameter	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
ψ_{i0}								
$\psi_{1,0} = 0.1261$	-0.0056	0.1891	0.0005	0.1230	-0.0023	0.0851	0.0010	0.0592
$\psi_{2,0} = 0.3883$	-0.0051	0.2495	-0.0058	0.1687	-0.0006	0.1148	-0.0003	0.0803
$\psi_{3,0} = 0.4375$	-0.0115	0.2436	-0.0022	0.1499	0.0034	0.1041	-0.0003	0.0743
$\psi_{4,0} = 0.5059$	0.0050	0.1769	-0.0040	0.1221	-0.0028	0.0820	-0.0010	0.0571
$\psi_{5,0} = 0.7246$	-0.0109	0.2089	-0.0031	0.1502	-0.0009	0.1071	0.0006	0.0721
β_{i0}								
$\beta_{1,0} = 0.9649$	0.0125	0.2236	0.0069	0.1472	0.0024	0.1008	-0.0020	0.0717
$\beta_{2,0} = 0.9572$	0.0100	0.2674	0.0068	0.1833	-0.0022	0.1272	-0.0025	0.0892
$\beta_{3,0} = 0.2785$	0.0078	0.2908	-0.0012	0.1806	-0.0026	0.1252	0.0022	0.0907
$\beta_{4,0} = 0.9134$	-0.0020	0.2195	0.0072	0.1461	0.0012	0.1000	0.0000	0.0684
$\beta_{5,0} = 0.8147$	0.0104	0.2842	0.0108	0.1950	0.0081	0.1341	0.0003	0.0911

T	25	50	100	200	25	50	100	200
Parameter	Size				Power			
ψ_{i0}								
$\psi_{1,0} = 0.1261$	0.1040	0.0675	0.0535	0.0515	0.3410	0.4665	0.7060	0.9065
$\psi_{2,0} = 0.3883$	0.0950	0.0690	0.0560	0.0580	0.2515	0.3525	0.4900	0.7315
$\psi_{3,0} = 0.4375$	0.0935	0.0620	0.0560	0.0510	0.2245	0.3355	0.5115	0.7975
$\psi_{4,0} = 0.5059$	0.0835	0.0740	0.0660	0.0485	0.3430	0.5025	0.7355	0.9345
$\psi_{5,0} = 0.7246$	0.0660	0.0670	0.0645	0.0530	0.2450	0.3610	0.5410	0.7975
β_{i0}								
$\beta_{1,0} = 0.9649$	0.0900	0.0645	0.0525	0.0530	0.2845	0.3825	0.5360	0.8075
$\beta_{2,0} = 0.9572$	0.0930	0.0725	0.0625	0.0570	0.2165	0.2885	0.4380	0.6535
$\beta_{3,0} = 0.2785$	0.0960	0.0710	0.0515	0.0585	0.2585	0.3000	0.4565	0.6375
$\beta_{4,0} = 0.9134$	0.0865	0.0630	0.0565	0.0485	0.3055	0.3845	0.5715	0.8245
$\beta_{5,0} = 0.8147$	0.0890	0.0705	0.0555	0.0510	0.2005	0.2570	0.3700	0.6080

Notes: True parameter values are generated as $\psi_{i0} \sim IIDU(0, 0.8)$, $\alpha_{i0} \sim IIDN(1, 1)$, and $\beta_{i0} \sim IIDU(0, 1)$ for $i = 1, 2, \ldots, N$. Non-Gaussian errors are generated as $\varepsilon_{i0}/\sigma_{i0} \sim IID[\chi^2(2) - 2]/2$, with $\sigma^2_{i0} \sim IIDU[\chi^2(2)/4 + 0.5]$ for $i = 1, 2, \ldots, N$. The spatial weight matrix $\boldsymbol{W} = (w_{ij})$ has four connections so that $w_{ij} = 1$ if j is equal to: $i-2$, $i-1$, $i+1$, $i+2$, and zero otherwise, for $i = 1, 2, \ldots, N$. Biases and RMSEs are computed as $R^{-1}\sum_{r=1}^{R}(\hat{\psi}_{i,r} - \psi_{i0})$ and $\sqrt{R^{-1}\sum_{r=1}^{R}(\hat{\psi}_{i,r} - \psi_{i0})^2}$ for $i = 1, 2, \ldots, N$. Empirical size and empirical power are based on the sandwich formula given by (24). The nominal size is set to 5%. Size is computed under $H_{i0}\colon \psi_i = \psi_{i0}$, using a two-sided alternative, for $i = 1, 2, \ldots, N$. Power is computed under $\psi_i = \psi_{i0} + 0.2$, for $i = 1, 2, \ldots, N$. The number of replications is set to $R = 2,000$. Estimates are sorted in ascending order according to the true values of the spatial autoregressive parameters. Biases, RMSEs, sizes and powers for β_i, $i = 1, 2, \ldots, N$, are computed similarly, with power computed under $\beta_i = \beta_{i0} + 0.2$.

Table 2: Bias, RMSE, size and power for parameters of individual units in the HSAR(1) model with one exogenous regressor and non-Gaussian errors for $N = 100$ and $T \in \{25, 50, 100, 200\}$.

T	25		50		100		200	
Parameter	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
ψ_{i0}								
$\psi_{1,0} = 0.0244$	-0.0005	0.3152	-0.0049	0.2138	0.0021	0.1415	-0.0001	0.1010
$\psi_{2,0} = 0.0255$	-0.0330	0.5189	0.0034	0.3674	-0.0137	0.2641	-0.0033	0.1794
$\psi_{3,0} = 0.0397$	0.0129	0.3509	-0.0017	0.2448	-0.0014	0.1681	0.0013	0.1183
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{51,0} = 0.3927$	-0.0027	0.2912	0.0038	0.2056	0.0009	0.1395	0.0005	0.0960
$\psi_{52,0} = 0.3987$	0.0001	0.1994	-0.0031	0.1381	0.0029	0.0921	0.0005	0.0638
$\psi_{53,0} = 0.4004$	-0.0112	0.3063	0.0075	0.2049	0.0033	0.1392	-0.0015	0.0991
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{98,0} = 0.7695$	-0.0031	0.1621	0.0018	0.1149	0.0055	0.0824	-0.0003	0.0586
$\psi_{99,0} = 0.7705$	-0.0530	0.2903	-0.0126	0.1895	0.0002	0.1401	0.0003	0.1041
$\psi_{100,0} = 0.7904$	-0.0125	0.1716	-0.0094	0.1275	0.0011	0.0897	0.0008	0.0613
β_{i0}								
$\beta_{1,0} = 0.1978$	0.0089	0.2782	0.0017	0.1771	0.0007	0.1192	-0.0073	0.0824
$\beta_{2,0} = 0.7060$	0.0252	0.3699	0.0016	0.2359	-0.0005	0.1608	0.0049	0.1144
$\beta_{3,0} = 0.4173$	0.0107	0.2541	0.0034	0.1733	0.0000	0.1157	0.0028	0.0821
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{51,0} = 0.9448$	0.0060	0.1924	-0.0024	0.1294	0.0018	0.0896	0.0009	0.0634
$\beta_{52,0} = 0.1190$	0.0046	0.1824	0.0026	0.1259	-0.0005	0.0853	0.0021	0.0578
$\beta_{53,0} = 0.7127$	0.0026	0.2630	-0.0050	0.1654	0.0038	0.1201	0.0012	0.0831
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{98,0} = 0.1067$	0.0041	0.1688	-0.0031	0.1115	0.0010	0.0762	-0.0002	0.0550
$\beta_{99,0} = 0.4588$	0.0207	0.2643	0.0039	0.1788	0.0033	0.1232	0.0027	0.0888
$\beta_{100,0} = 0.3674$	0.0056	0.1691	0.0032	0.1179	0.0009	0.0830	0.0004	0.0560

T	25	50	100	200	25	50	100	200
Parameter	Size				Power			
ψ_{i0}								
$\psi_{1,0} = 0.0244$	0.0890	0.0810	0.0520	0.0590	0.1820	0.2200	0.3290	0.5480
$\psi_{2,0} = 0.0255$	0.0705	0.0595	0.0555	0.0490	0.0945	0.0895	0.1495	0.2140
$\psi_{3,0} = 0.0397$	0.0905	0.0745	0.0585	0.0575	0.1555	0.1895	0.2805	0.4450
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{51,0} = 0.3927$	0.0950	0.0645	0.0590	0.0535	0.1785	0.2625	0.3590	0.5810
$\psi_{52,0} = 0.3987$	0.0850	0.0620	0.0625	0.0505	0.3050	0.4390	0.6285	0.8660
$\psi_{53,0} = 0.4004$	0.0885	0.0785	0.0570	0.0585	0.1995	0.2490	0.3745	0.5800
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{98,0} = 0.7695$	0.0635	0.0630	0.0660	0.0610	0.3340	0.4755	0.6935	0.9145
$\psi_{99,0} = 0.7705$	0.0300	0.0285	0.0370	0.0495	0.1455	0.2045	0.3095	0.5205
$\psi_{100,0} = 0.7904$	0.0545	0.0570	0.0625	0.0505	0.3120	0.4665	0.6455	0.8845
β_{i0}								
$\beta_{1,0} = 0.1978$	0.1160	0.0700	0.0610	0.0505	0.2405	0.3040	0.4380	0.7115
$\beta_{2,0} = 0.7060$	0.1025	0.0580	0.0505	0.0545	0.1725	0.2095	0.2710	0.4510
$\beta_{3,0} = 0.4173$	0.0950	0.0780	0.0550	0.0595	0.2450	0.3160	0.4655	0.7080
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{51,0} = 0.9448$	0.0910	0.0685	0.0590	0.0570	0.3260	0.4665	0.6500	0.8880
$\beta_{52,0} = 0.1190$	0.0970	0.0800	0.0505	0.0440	0.3500	0.4840	0.7030	0.9150
$\beta_{53,0} = 0.7127$	0.1075	0.0660	0.0665	0.0515	0.2420	0.3150	0.4410	0.6810
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{98,0} = 0.1067$	0.0960	0.0660	0.0530	0.0545	0.3950	0.5500	0.7605	0.9475
$\beta_{99,0} = 0.4588$	0.0725	0.0615	0.0545	0.0595	0.2015	0.2775	0.4225	0.6415
$\beta_{100,0} = 0.3674$	0.0935	0.0660	0.0695	0.0540	0.3605	0.5025	0.7255	0.9370

Notes: See notes to Table 1.

Table 3: Mean group estimates (MGE) of spatial and temporal coefficients as well as elasticities of house price changes to population and real income growth by six major U.S. regions, and the U.S. as a whole

r	Name	N_r	$\hat{\psi}_{MG,r}$	$\hat{\psi}_{MG0,r}$	$\hat{\psi}_{MG1,r}$	$\hat{\lambda}_{MG,r}$	$\hat{\beta}^{pop}_{MG,r}$	$\hat{\beta}^{inc}_{MG,r}$
1 & 2	New England & Mideast	35	0.067	0.499‡	-0.432‡	0.645‡	0.629‡	0.085‡
			(0.044)	(0.087)	(0.069)	(0.026)	(0.172)	(0.020)
3	Great Lakes	48	0.115‡	0.714‡	-0.599‡	0.629‡	0.224‡	0.025‡
			(0.030)	(0.064)	(0.047)	(0.026)	(0.068)	(0.012)
4	Plains	26	0.040	0.525‡	-0.484‡	0.599‡	0.252‡	0.039
			(0.058)	(0.083)	(0.060)	(0.046)	(0.073)	(0.030)
5	Southeast	106	0.105‡	0.669‡	-0.564‡	0.655‡	0.194‡	0.038‡
			(0.024)	(0.044)	(0.032)	(0.019)	(0.034)	(0.008)
6 & 7	Southwest & Rocky Mountain	40	0.017	0.325‡	-0.308‡	0.713‡	0.162‡	0.072‡
			(0.021)	(0.078)	(0.064)	(0.020)	(0.038)	(0.016)
8	Far West	41	0.126‡	0.711‡	-0.585‡	0.759‡	0.183‡	0.067‡
			(0.017)	(0.040)	(0.035)	(0.012)	(0.047)	(0.018)
	US	296	0.088‡	0.603‡	-0.515‡	0.667‡	0.250‡	0.050‡
			(0.013)	(0.027)	(0.020)	(0.010)	(0.029)	(0.006)

Notes: * $p < 0.1$, † $p < 0.05$, ‡ $p < 0.01$. Non-parametric robust standard errors in parentheses (see below). For $r = 1, \ldots, 6$, $\hat{\psi}_{MG,r} = N_r^{-1} \sum_{i \in I_r} (\hat{\psi}_{0i} + \hat{\psi}_{1i})$, s.e.$(\hat{\psi}_{MG,r}) = \sqrt{[N_r(N_r-1)]^{-1} \sum_{i \in I_r} [(\hat{\psi}_{0i} + \hat{\psi}_{1i}) - \hat{\psi}_{MG,r}]^2}$, $\hat{\psi}_{MGj,r} = N_r^{-1} \sum_{i \in I_r} \hat{\psi}_{ji}$, and s.e.$(\hat{\psi}_{MGj,r}) = \sqrt{[N_r(N_r-1)]^{-1} \sum_{i \in I_r} (\hat{\psi}_{ji} - \hat{\psi}_{MGj,r})^2}$, for $j = 0, 1$. I_r is the set of units belonging to region r, $I_r = \{i\colon i \text{ is in region } r\}$, and N_r is the number of units per region, $N_r = \#(I_r)$. New England (9 MSAs) and Mid East (26 MSAs) as well as South West (26 MSAs) and Rocky Mountains (14 MSAs) have been merged in order to obtain a sufficiently large number of MSAs in the two broader regions. For the U.S. as a whole: $\hat{\psi}_{MG,US} = N^{-1} \sum_{i=1}^{N} (\hat{\psi}_{0i} + \hat{\psi}_{1i})$, and s.e.$(\hat{\psi}_{MG,US}) = \sqrt{[N(N-1)]^{-1} \sum_{i=1}^{N} [(\hat{\psi}_{0i} + \hat{\psi}_{1i}) - \hat{\psi}_{MG,US}]^2}$. The MGE of coefficient estimates of lagged house price changes ($\hat{\lambda}_{MG,r}$), as well as house price changes to population and real income changes ($\hat{\beta}^{pop}_{MG,r}$ and $\hat{\beta}^{inc}_{MG,r}$) are computed similarly to $\hat{\psi}_{MGj,r}$, $j = 0, 1$. The computations of all MG estimates exclude the MSAs whose spatial lag coefficients hit the upper/lower bound in the optimisation procedure.

References

Agnello, L., V. Castro, and R. M. Sousa (2015). Booms, busts, and normal times in the housing market. *Journal of Business and Economic Statistics 33*(1), 25–45.

Anselin, L. (1988). *Spatial Econometrics: Methods and Models*. Kluwer Academic: The Netherlands.

Anselin, L. (2001). Spatial econometrics. In B. H. Baltagi (Ed.), *A Companion to Theoretical Econometrics*. Blackwell: Massachusetts.

Anselin, L. and A. K. Bera (1998). Spatial dependence in linear regression models with an introduction to spatial econometrics. In A. Ullah and D. E. A. Giles (Eds.), *Handbook of Applied Economic Statistics*. Marcel Dekker: New York.

Arbia, G. (2010). *Spatial Econometrics: Statistical Foundations and Applications to Regional Convergence*. Springer: Berlin Heidelberg New York.

Autor, D., D. Dorn, and G. Hanson (2013). The China syndrome: Local labor market effects of import competition in the United States. *American Economic Review 103*(6), 2121–2168.

Bailey, N., S. Holly, and M. Pesaran (2016). A two-stage approach to spatio-temporal analysis with strong and weak cross-sectional dependence. *Journal of Applied Econometrics 31*(1), 249–280.

Baltagi, B. and D. Levin (1986). Estimating dynamic demand for cigarettes using panel data: The effects of bootlegging, taxation and advertising reconsidered. *The Review of Economics and Statistics 68*, 148–155.

Baltagi, B., S. H. Song, B. C. Jung, and W. Kon (2007). Testing for serial correlation, spatial autocorrelation and random effects using panel data. *Journal of Econometrics 140*, 5–51.

Baltagi, B., S. H. Song, and W. Kon (2003). Testing panel data regression models with spatial error correlation. *Journal of Econometrics 117*, 123–150.

Case, A. C. (1991). Spatial patterns in household demand. *Econometrica 59*, 953–965.

Chudik, A. and M. H. Pesaran (2019). Mean group estimator in presence of weakly cross-correlated estimators. *Economics Letters 175*, 101–105.

Cliff, A. D. and J. K. Ord (1973). *Spatial Autocorrelation*. Pion Ltd.: London.

Cressie, N. (1993). *Statistics for Spatial Data*. John Wiley & Sons: New York.

Cressie, N. and C. K. Wikle (2011). *Statistics for Spatio-Temporal Data*. Wiley-Blackwell: New York.

Cun, W. and M. H. Pesaran (2018). Land use regulations, migration and rising house price dispersion in the U.S. Available at $https://papers.ssrn.com/sol3/papers.cfm?abstract_id = 3162399$.

Davidson, J. (2000). *Econometric Theory*. Blackwell, Oxford, UK.

Debarsy, N., C. Ertur, and J. LeSage (2012). Interpreting dynamic space-time panel data models. *Statistical Methodology 9*, 158–171.

Dou, B., M. L. Parrella, and Q. Yao (2016). Generalized Yule-Walker estimation for spatio-temporal models with unknown diagonal coefficients. *Journal of Econometrics 194*(2), 369–382.

Elhorst, J., M. Gross, and E. Tereanu (2018). Spillovers in space and time: where spatial econometrics and Global VAR models meet. *ECB working paper no. 2134*.

Elhorst, J. P. (2014). *Spatial Econometrics: From Cross-Sectional Data to Spatial Panels*. Springer: Berlin Heidelberg.

Gallin, J. (2006). The long run relationship between housing prices and income: Evidence from local housing markets. *Real Estate Economics 34*(3), 417–438.

Haining, R. (2003). *Spatial Data Analysis: Theory and Practice*. Cambridge University Press.

Holly, S., M. H. Pesaran, and T. Yamagata (2010). A spatio-temporal model of house prices in the USA. *Journal of Econometrics 158*, 160–173.

Horn, R. A. and C. R. Johnson (1985). *Matrix analysis*. Cambridge University Press.

Hsiao, C. and M. H. Pesaran (2008). Random coefficient models. In L. Matyas and P. Sevestre (Eds.), *The Econometrics of Panel Data (Third Edition)*. Springer.

Juodis, A. and S. Reese (2019). The incidental parameters problem in testing for remaining cross-section correlation. *Available at: arXiv:1810.03715[econ.EM]*.

Kapoor, M., H. H. Kelejian, and I. R. Prucha (2007). Panel data models with spatially correlated error components. *Journal of Econometrics 140*, 97–130.

Kelejian, H. H. and I. R. Prucha (1999). A generalised moments estimator for the autoregressive parameter in a spatial model. *International Economic Review 40*, 509–533.

Kelejian, H. H. and I. R. Prucha (2001). On the asymptotic distribution of the Moran I test statistic with applications. *Journal of Econometrics 104*(2), 219–257.

Kelejian, H. H. and I. R. Prucha (2010). Specification and estimation of spatial autoregressive models with autoregressive and heteroskedastic disturbances. *Journal of Econometrics 157*, 53–67.

Kelejian, H. H. and D. Robinson (1993). A suggested method of estimation for spatial interdependent models with autocorrelated errors, and an application to a county expenditure model. *Papers in Regional Science 72*, 297–312.

Lee, L.-F. (2004). Asymptotic distributions of quasi-maximum likelihood estimators for spatial autoregressive models. *Econometrica 72*, 1899–1925.

Lee, L.-F. and J. Yu (2010). Estimation of spatial autoregressive panel data models with fixed effects. *Journal of Econometrics 154*, 165–185.

LeSage, J. and Y.-Y. Chih (2016). Interpreting heterogeneous coefficient spatial autoregressive panel models. *Economic Letters 142*, 1–5.

LeSage, J. and Y.-Y. Chih (2018a). A matrix exponential spatial panel data model with heterogeneous coefficients. *Geographical Analysis 50*, 422–453.

LeSage, J., Y.-Y. Chih, and C. Vance (2019). Markov chain monte carlo estimation of spatial dynamic panel models for large samples. *Computational Statistics and Data Analysis 138*, 107–125.

LeSage, J. and K. Pace (2010). Spatial econometrics. *Web Book of Regional Science Regional Research Institute 50*, 1014–1015.

LeSage, J. P. and Y.-Y. Chih (2018b). A bayesian spatial panel model with heterogeneous coefficients. *Regional Science and Urban Economics 72*, 58–73.

Lin, X. and L.-F. Lee (2010). GMM estimation of spatial autoregressive models with unknown heteroskedasticity. *Journal of Econometrics 157*, 34–52.

Lütkepohl, H. (1996). *Handbook of Matrices.* John Wiley and Sons.

Malpezzi, S. (1999). A simple error correction model of house prices. *Journal of Housing Economics 8*(1), 27–62.

Manski, C. F. (1993). Identification of endogenous social effects: the reflection problem. *The Review of Economic Studies 60*(3), 531–542.

Munro, A. (2018). Economic valuation of a rise in ambient radiation risks: hedonic pricing evidence from the accident in fukushima. *Journal of Regional Science 58*, 635–658.

Neyman, J. and E. Scott (1948). Consistent estimates based on partially consistent observations. *Econometrica 49*, 1417–1426.

Ord, J. K. and A. Getis (1995). Local spatial autocorrelation statistics: Distributional issues and an application. *Geographical Analysis 27*, 286–306.

Pesaran, H. and E. Tosetti (2011). Large panels with common factors and spatial correlation. *Journal of Econometrics 161*, 182–202.

Pesaran, M. and A. Chudik (2014). Aggregation in large dynamic panels. *Journal of Econometrics 178*, 273–285.

Pesaran, M. and T. Yamagata (2008). Testing slope homogeneity in large panels. *Journal of Econometrics 142*(1), 50–93.

Pesaran, M. H. (2004). General diagnostic tests for cross section dependence in panels. CESifo working paper series no. 1229. Available at $http://ssrn.com/abstract = 572504$.

Pesaran, M. H. (2015). Testing weak cross-sectional dependence in large panels. *Econometric Reviews 34*(6-10), 1089–1117.

Pesaran, M. H. and R. Smith (1995). Estimating long-run relationships from dynamic heterogeneous panels. *Journal of Econometrics 68*, 79–113.

Saiz, A. (2010). The geographic determinants of housing supply. *The Quarterly Journal of Economics 125*(3), 1253–1296.

Swoboda, A., T. Nega, and M. Timm (2015). Hedonic analysis over time and space: the case of house price and traffic noise. *Journal of Regional Science 55*, 644–670.

Upton, G. J. G. and B. Fingleton (1985). *Spatial Data Analysis by Example, Volume 1.* Wiley: Chichester.

White, H. (1984). *Asymptotic Theory for Econometricians.* Academic Press, Inc.

Whittle, P. (1954). On stationary processes on the plane. *Biometrika 41*, 434–449.

Yang, C. F. (2020). Common factors and spatial dependence: An application to US house prices. *Econometric Reviews, forthcoming. Available at: https://doi.org/10.1080/07474938.2020.1741785.*

Yu, J., R. de Jong, and L. F. Lee (2012). Estimation for spatial dynamic panel data with fixed effects: the case of spatial cointegration. *Journal of Econometrics 167*, 16–37.

Yu, J., R. M. de Jong, and L.-F. Lee (2008). Quasi-maximum likelihood estimators for spatial dynamic panel data with fixed effects when both n and T are large. *Journal of Econometrics 146*, 118–134.

Zhou, J., Y. Tu, Y. Chen, and H. Wang (2017). Estimating spatial autocorrelation with sampled network data. *Journal of Business and Economic Statistics 35*(1), 130–138.

Online Supplement
for

Estimation and Inference for Spatial Models with Heterogeneous Coefficients: An Application to U.S. House Prices

by
Michele Aquaro
Joint Research Centre, European Commission

Natalia Bailey
Monash University

M. Hashem Pesaran
Department of Economics,
University of Southern California and Trinity College, Cambridge

June 2020

Introduction

This online supplement is composed of Appendices A-G. Appendix A includes statements and proofs of lemmas used in the derivations of Sections 3.2, 3.3 and 4 of the paper. Appendix B provides proofs of Propositions 2 and 3, and Corollary 1 in Sections 3.3 and 4 of the paper, while Appendix C gives the first and second derivatives of the log likelihood function of the HSAR model with exogenous regressors. Appendix D derives the HSAR when relaxing Assumption 4(a) in terms of considering non-zero own weights, while Appendix E derives the expressions for direct and indirect effects implied by the HSAR model (1) of Section 2. Appendix F describes the data sources and transformations used in Section 6 and includes additional empirical results produced from running model (30) of Section 6. Finally, Appendix G displays additional Monte Carlo results based on the designs set out in Section 5 of the paper.

Appendix A Technical lemmas

Lemma 1 *Consider a given weight matrix* $\boldsymbol{W}$ *and suppose that Assumption 4 holds. Then (a) matrix* $\mathbf{S}(\boldsymbol{\psi}) = \mathbf{I}_N - \boldsymbol{\Psi}\boldsymbol{W}$ *is non-singular with positive eigenvalues, namely* $\lambda_{\min}[\mathbf{S}(\boldsymbol{\psi})] > 0$*, and (b)* $|\lambda_{\max}[\mathbf{S}(\boldsymbol{\psi})]| < 2$.

Proof. Let $\varrho(\boldsymbol{\Psi}\boldsymbol{W})$ be the spectral radius of matrix $\boldsymbol{\Psi}\boldsymbol{W}$. Non-singularity of $\mathbf{S}(\boldsymbol{\psi}) = \mathbf{I}_N - \boldsymbol{\Psi}\boldsymbol{W}$ is ensured if

$$\varrho(\boldsymbol{\Psi}\boldsymbol{W}) < 1. \tag{A.1}$$

However, since for any matrix norm $\|\mathbf{A}\|$, $\varrho(\boldsymbol{A}) \leq \|\mathbf{A}\|$, then using the maximum row sum matrix norm we have

$$\varrho(\boldsymbol{\Psi}\boldsymbol{W}) \leq \|\boldsymbol{\Psi}\boldsymbol{W}\|_\infty \leq \|\boldsymbol{\Psi}\|_\infty \|\boldsymbol{W}\|_\infty = \sup_{\psi_i \in \boldsymbol{\Theta}_\psi} |\psi_i| \, \|\boldsymbol{W}\|_\infty < K, \tag{A.2}$$

and from (A.1) we have

$$\sup_{\psi_i \in \boldsymbol{\Theta}_\psi} |\psi_i| \, \|\boldsymbol{W}\|_\infty < 1, \tag{A.3}$$

where we have used the result $\|\boldsymbol{\Psi}\|_\infty = \sup_{\psi_i \in \boldsymbol{\Theta}_\psi} |\psi_i|$. Therefore, matrix $\mathbf{S}(\boldsymbol{\psi}) = \boldsymbol{I}_N - \boldsymbol{\Psi}\boldsymbol{W}$ is invertible under condition (8) of Assumption 4(b). Also all eigenvalues of $\mathbf{S}(\boldsymbol{\psi})$ are necessarily positive, since $\lambda_{\min}[\mathbf{S}(\boldsymbol{\psi})] = 1 - \lambda_{\max}(\boldsymbol{\Psi}\boldsymbol{W}) \geq 1 - |\lambda_{\max}(\boldsymbol{\Psi}\boldsymbol{W})| = 1 - \varrho(\boldsymbol{\Psi}\boldsymbol{W}) > 0$. To establish part (b) we note that $|\lambda_{\max}[\mathbf{S}(\boldsymbol{\psi})]| = \varrho[\mathbf{S}(\boldsymbol{\psi})] \leq \|\mathbf{S}(\boldsymbol{\psi})\|_\infty = \|\mathbf{I}_N - \boldsymbol{\Psi}\boldsymbol{W}\|_\infty \leq 1 + \sup_{\psi_i \in \boldsymbol{\Theta}_\psi} |\psi_i| \, \|\boldsymbol{W}\|_\infty$, and in view of (A.3), we have $|\lambda_{\max}[\mathbf{S}(\boldsymbol{\psi})]| < 2$, as required. ∎

Lemma 2 *Let* $\mathbf{G}(\boldsymbol{\psi}) = \boldsymbol{W}(\boldsymbol{I}_N - \boldsymbol{\Psi}\boldsymbol{W})^{-1}$*. (a) Suppose that Assumption 4 holds. Then*

$$\|\mathbf{G}(\boldsymbol{\psi})\|_\infty < K, \tag{A.4}$$

and

$$\left\|\mathbf{G}(\boldsymbol{\psi}) \odot \mathbf{G}'(\boldsymbol{\psi})\right\|_\infty < K \ , \tag{A.5}$$

for all values of $\boldsymbol{\psi} \in \boldsymbol{\Theta}_\psi$ *that satisfy condition (8).*

Proof. Under condition (8), we have

$$\mathbf{G}(\boldsymbol{\psi}) = \boldsymbol{W} + \boldsymbol{W}\boldsymbol{\Psi}\boldsymbol{W} + \boldsymbol{W}\boldsymbol{\Psi}\boldsymbol{W}\boldsymbol{\Psi}\boldsymbol{W} + \ldots,$$

and

$$\|\mathbf{G}(\boldsymbol{\psi})\|_\infty \leq \|\boldsymbol{W}\|_\infty + \|\boldsymbol{W}\|_\infty^2 \|\boldsymbol{\Psi}\|_1 + \|\boldsymbol{W}\|_\infty^3 \|\boldsymbol{\Psi}\|_1^2 + \ldots.$$

But $\|\boldsymbol{\Psi}\|_1^s \leq \left[\sup_{\psi_i \in \boldsymbol{\Theta}_\psi} |\psi_i|\right]^s$, and under condition (8) we have $\sup_{\psi_i \in \boldsymbol{\Theta}_\psi} |\psi_i| \, \|\boldsymbol{W}\|_\infty < 1$. Hence,

$$\|\mathbf{G}(\boldsymbol{\psi})\|_\infty \leq \|\boldsymbol{W}\|_\infty \left(\frac{1}{1 - \sup_{\psi_i \in \boldsymbol{\Theta}_\psi} |\psi_i| \, \|\boldsymbol{W}\|_\infty} \right),$$

and (A.4) follows since under Assumption 4(b) $\|\boldsymbol{W}\|_\infty$ is bounded, and $\sup_{\psi_i\in\boldsymbol{\Theta}_\psi}|\psi_i|\,\|\boldsymbol{W}\|_\infty<1$. Finally,

$$\left\|\mathbf{G}\left(\boldsymbol{\psi}\right)\odot\mathbf{G}'\left(\boldsymbol{\psi}\right)\right\|_\infty=\max_{1\leq i\leq N}\left(\sum_{j=1}^{N}|g_{ij}g_{ji}|\right)\leq\max_{1\leq i\leq N}\left(\sup_i|g_{ij}|\sum_{j=1}^{N}|g_{ji}|\right),$$

where by (A.4), $\sup_i\sum_{j=1}^N|g_{ji}|=\|\mathbf{G}\left(\boldsymbol{\psi}\right)\|_\infty<K$, and $\sup_{i,j}|g_{ij}|<K$. Hence (A.5) follows as desired. ■

Lemma 3 *Consider the average log-likelihood function of (5):*

$$\bar{\ell}_T(\boldsymbol{\theta})=T^{-1}\ell_T(\boldsymbol{\theta})=-\frac{N}{2}\ln(2\pi)-\frac{1}{2}\sum_{i=1}^{N}\ln\sigma_i^2+\frac{1}{2}\ln|\mathbf{V}(\boldsymbol{\psi})| \tag{A.6}$$
$$-\frac{1}{2T}\sum_{t=1}^{T}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t}-\boldsymbol{B}\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t}-\boldsymbol{B}\boldsymbol{x}_{\circ t}\right],$$

where $\boldsymbol{\theta}=(\boldsymbol{\psi}',\boldsymbol{\beta}',\boldsymbol{\sigma}^{2\prime})'$, $\boldsymbol{B}$ is the $N\times kN$ block diagonal matrix with elements $\boldsymbol{\beta}_i'$, $i=1,2,\ldots,N$, on the main diagonal and zeros elsewhere, $\bar{\ell}_T(\boldsymbol{\theta})=T^{-1}\ell_T(\boldsymbol{\theta})$ and $\ell_T(\boldsymbol{\theta})$ is defined by (7). Also, $\mathbf{V}\left(\boldsymbol{\psi}\right)=\mathbf{S}'(\boldsymbol{\psi})\mathbf{S}(\boldsymbol{\psi})$, where $\mathbf{S}(\boldsymbol{\psi})=\mathbf{I}_N-\boldsymbol{\Psi}\boldsymbol{W}$. Then, under Assumptions 1 to 4 for a given fixed N and as $T\to\infty$ we have

$$\bar{\ell}_T(\boldsymbol{\theta}_0)-\bar{\ell}_T(\boldsymbol{\theta})\overset{a.s.}{\to}\lim_{T\to\infty}E_0\left[\bar{\ell}_T(\boldsymbol{\theta}_0)-\bar{\ell}_T(\boldsymbol{\theta})\right], \tag{A.7}$$

where E_0 represents expectations taken under $\boldsymbol{\theta}=\boldsymbol{\theta}_0$.

Proof. Let $Q_T\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)=\bar{\ell}_T(\boldsymbol{\theta}_0)-\bar{\ell}_T(\boldsymbol{\theta})$, and evaluating (A.6) at $\boldsymbol{\theta}=\boldsymbol{\theta}_0$, note that

$$Q_T\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)=-\frac{1}{2}\sum_{i=1}^{N}\ln\left(\sigma_{i0}^2/\sigma_i^2\right)+\frac{1}{2}\left[\ln\left(\frac{|\mathbf{V}\left(\boldsymbol{\psi}_0\right)|}{|\mathbf{V}\left(\boldsymbol{\psi}\right)|}\right)\right] \tag{A.8}$$
$$-\frac{1}{2}\left\{\frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{S}(\boldsymbol{\psi}_0)\boldsymbol{y}_{\circ t}-\boldsymbol{B}_0\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}_0^{-1}\left[\mathbf{S}(\boldsymbol{\psi}_0)\boldsymbol{y}_{\circ t}-\boldsymbol{B}_0\boldsymbol{x}_{\circ t}\right]\right\}$$
$$+\frac{1}{2}\left\{\frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t}-\boldsymbol{B}\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t}-\boldsymbol{B}\boldsymbol{x}_{\circ t}\right]\right\},$$

where $|\mathbf{V}\left(\boldsymbol{\psi}\right)|=|\mathbf{S}'(\boldsymbol{\psi})\mathbf{S}(\boldsymbol{\psi})|=|\mathbf{S}(\boldsymbol{\psi})|^2$. Also, by first taking conditional expectations with respect to $\mathcal{F}_t$, and then taking expectations with respect to $\boldsymbol{x}_{\circ t}$, we have

$$\sum_{t=1}^{T}E_0\left\{\operatorname{tr}\left[\boldsymbol{y}_{\circ t}'\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t}\right]\right\}=T\operatorname{tr}\left[\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\boldsymbol{\Sigma}_{y0}\right],$$
$$\sum_{t=1}^{T}E_0\left\{\operatorname{tr}\left[\boldsymbol{y}_{\circ t}'\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\boldsymbol{x}_{\circ t}\right]\right\}=T\operatorname{tr}\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\boldsymbol{\Sigma}_{\boldsymbol{xx}}\boldsymbol{B}_0'\boldsymbol{S}_0'^{-1}\mathbf{S}'(\boldsymbol{\psi})\right],$$
$$\sum_{t=1}^{T}E_0\left\{\operatorname{tr}\left[\boldsymbol{x}_{\circ t}'\boldsymbol{B}'\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\boldsymbol{x}_{\circ t}\right]\right\}=T\operatorname{tr}\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\boldsymbol{\Sigma}_{\boldsymbol{xx}}\boldsymbol{B}'\right],$$

and hence

$$\frac{1}{T}\sum_{t=1}^{T}E_0\left\{\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t}-\boldsymbol{B}\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t}-\boldsymbol{B}\boldsymbol{x}_{\circ t}\right]\right\}=\left\{\begin{array}{c}\operatorname{tr}\left[\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\boldsymbol{\Sigma}_{y0}\right]\\-2\operatorname{tr}\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\boldsymbol{\Sigma}_{\boldsymbol{xx}}\boldsymbol{B}_0'\mathbf{S}_0'^{-1}\mathbf{S}'(\boldsymbol{\psi})\right]\\+\operatorname{tr}\left[\boldsymbol{\Sigma}^{-1/2}\boldsymbol{B}\boldsymbol{\Sigma}_{xx}\boldsymbol{B}'\boldsymbol{\Sigma}^{-1/2}\right]\end{array}\right\}.$$

Using the above results in (A.8) we now obtain

$$
\begin{aligned}
E_0\left[Q_T\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}\right)\right] &= E_0\left[\bar{\ell}_T(\boldsymbol{\theta}_0) - \bar{\ell}_T(\boldsymbol{\theta})\right] \\
&= -\frac{1}{2}\sum_{i=1}^{N} \ln\left(\sigma_{i0}^2/\sigma_i^2\right) - \frac{N}{2} + \frac{1}{2}\left[\ln\left(\frac{|\mathbf{V}\left(\boldsymbol{\psi}_0\right)|}{|\mathbf{V}\left(\boldsymbol{\psi}\right)|}\right)\right] \\
&+ \frac{1}{2}\operatorname{tr}\left[\mathbf{P}(\boldsymbol{\theta})\mathbf{P}^{-1}(\boldsymbol{\theta}_0)\right] \\
&+ \frac{1}{2}\operatorname{tr}\left\{\boldsymbol{\Sigma}^{-1/2}\left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]\boldsymbol{\Sigma}_{\boldsymbol{xx}}\left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]'\boldsymbol{\Sigma}^{-1/2}\right\},
\end{aligned}
$$

where $\mathbf{P}\left(\boldsymbol{\theta}\right) = \mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})$,

$$
\frac{|\mathbf{V}\left(\boldsymbol{\psi}_0\right)|}{|\mathbf{V}\left(\boldsymbol{\psi}\right)|} = \frac{|\mathbf{S}(\boldsymbol{\psi}_0)|^2}{|\mathbf{S}(\boldsymbol{\psi})|^2} = \left|\mathbf{S}(\boldsymbol{\psi}_0)\mathbf{S}^{-1}(\boldsymbol{\psi})\right|^2 = \left|\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\right|^{-2},
$$

and

$$
\begin{aligned}
\operatorname{tr}\left[\mathbf{P}\left(\boldsymbol{\theta}\right)\mathbf{P}^{-1}\left(\boldsymbol{\theta}_0\right)\right] &= \operatorname{tr}\left[\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\Sigma}_0\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\right] \\
&= \operatorname{tr}\left[\boldsymbol{\Sigma}^{-1/2}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\Sigma}_0\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1/2}\right].
\end{aligned}
$$

To establish (A.7) we show that

$$
Q_T\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}\right) - E_0\left[Q_T\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}\right)\right] \xrightarrow{a.s.} 0. \tag{A.9}
$$

To this end we note that under (5),

$$
\frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{S}(\boldsymbol{\psi}_0)\boldsymbol{y}_{\circ t} - \boldsymbol{B}_0\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}_0^{-1}\left[\mathbf{S}(\boldsymbol{\psi}_0)\boldsymbol{y}_{\circ t} - \boldsymbol{B}_0\boldsymbol{x}_{\circ t}\right] = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\zeta}_{\circ t}'\boldsymbol{\zeta}_{\circ t},
$$

where $\boldsymbol{\zeta}_{\circ t} = (\zeta_{1t}, \zeta_{2t}, \ldots, \zeta_{Nt}) \sim IID(\mathbf{0}, \mathbf{I}_N)$, $\zeta_{it} = \varepsilon_{it}/\sigma_{i0}$, for $i = 1, 2, \ldots, N$. Also,

$$
\begin{aligned}
&\frac{1}{2}\left\{\frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right]\right\} \\
&= \frac{1}{2}\left[\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\varepsilon}_{\circ t}'\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\mathbf{P}\left(\boldsymbol{\theta}\right)\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\varepsilon}_{\circ t}\right] \\
&+ \frac{1}{2}\left[\frac{1}{T}\sum_{t=1}^{T}\left\{\boldsymbol{\Sigma}^{-1/2}\left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]\boldsymbol{x}_{\circ t}\right\}'\left\{\boldsymbol{\Sigma}^{-1/2}\left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]\boldsymbol{x}_{\circ t}\right\}\right] \\
&+ \left[\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{\circ t}'\left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]'\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{\varepsilon}_{\circ t}\right].
\end{aligned}
$$

Using the above results in (A.8) and after some simplifications we have

$$
\begin{aligned}
Q_T\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}\right) - E_0\left[Q_T\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}\right)\right] &= -\frac{1}{2}\left[\frac{1}{T}\sum_{t=1}^{T} z_{1t,N}\left(\boldsymbol{\theta}_0\right)\right] + \frac{1}{2}\left[\frac{1}{T}\sum_{t=1}^{T} z_{2t,N}\left(\boldsymbol{\theta}\right)\right] \\
&+ \frac{1}{2}\left[\frac{1}{T}\sum_{t=1}^{T} z_{3t,N}\left(\boldsymbol{\theta}_0, \boldsymbol{\theta}\right)\right] + \frac{1}{2}\left[\frac{1}{T}\sum_{t=1}^{T} z_{4t,N}\left(\boldsymbol{\theta}\right)\right],
\end{aligned} \tag{A.10}
$$

where

$$z_{1t,N}\left(\boldsymbol{\theta}_0\right) = \boldsymbol{\zeta}'_{\circ t}\boldsymbol{\zeta}_{\circ t} - N = \sum_{i=1}^{N}\left(\zeta_{it}^2 - 1\right), \tag{A.11}$$

$$\begin{aligned} z_{2t,N}\left(\boldsymbol{\theta},\boldsymbol{\psi}_0\right) &= \boldsymbol{\varepsilon}'_{\circ t}\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\mathbf{P}\left(\boldsymbol{\theta}\right)\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\varepsilon}_{\circ t} - E_0\left[\boldsymbol{\varepsilon}'_{\circ t}\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\mathbf{P}\left(\boldsymbol{\theta}\right)\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\varepsilon}_{\circ t}\right] \\ &= \boldsymbol{\zeta}'_{\circ t}\mathcal{A}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\boldsymbol{\zeta}_{\circ t} - \operatorname{tr}\left[\mathcal{A}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\right], \end{aligned} \tag{A.12}$$

in which $\mathcal{A}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right) = \boldsymbol{\Sigma}_0^{1/2}\mathbf{S}'^{-1}(\boldsymbol{\psi}_0)\mathbf{P}\left(\boldsymbol{\theta}\right)\mathbf{S}^{-1}(\boldsymbol{\psi}_0)\boldsymbol{\Sigma}_0^{1/2}$,

$$z_{3t,N}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right) = \boldsymbol{x}'_{\circ t}\mathcal{B}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\boldsymbol{x}_{\circ t} - \operatorname{tr}\left[\mathcal{B}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\boldsymbol{\Sigma}_{\boldsymbol{xx}}\right], \tag{A.13}$$

in which $\mathcal{B}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right) = \left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]'\boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]$, and

$$z_{4t,N}\left(\boldsymbol{\theta}\right) = \boldsymbol{x}'_{\circ t}\left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]'\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{\varepsilon}_{\circ t}. \tag{A.14}$$

We establish that each $z_{jt,N}$, $j = 1,2,3,4$ is a martingale difference process with finite second order moments. But first we note that

$$\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1} = \mathbf{S}(\boldsymbol{\psi})\mathbf{S}^{-1}(\boldsymbol{\psi}_0) = \boldsymbol{I}_N - \boldsymbol{D}\boldsymbol{G}_0,$$

where $\boldsymbol{G}_0 = \boldsymbol{W}\left(\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W}\right)^{-1}$, and $\boldsymbol{D} = \boldsymbol{\Psi} - \boldsymbol{\Psi}_0$, is a diagonal matrix with elements $d_i = \psi_i - \psi_{i0}$. Hence, substituting the $\mathbf{P}\left(\boldsymbol{\theta}\right) = \mathbf{S}'(\boldsymbol{\psi})\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})$ in the expression above given for$\mathcal{A}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)$ we have (after some simplifications)

$$\mathcal{A}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right) = \boldsymbol{\Sigma}_0^{1/2}\left(\boldsymbol{I}_N - \boldsymbol{D}\boldsymbol{G}_0\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{I}_N - \boldsymbol{D}\boldsymbol{G}_0\right)\boldsymbol{\Sigma}_0^{1/2}.$$

Consider the first term of (A.10) and note that under Assumption 1, $\sup_{it} E\left|\zeta_{it}\right|^{4+\epsilon} < K$, for some $\epsilon > 0$. Then the elements in (A.11) are L_2 bounded, in the sense that $\sup_{it} E\left|\zeta_{it}^2 - 1\right|^2 < K$. Hence,

$$\frac{1}{T}\sum_{t=1}^{T}\left(\zeta_{it}^2 - 1\right) \overset{a.s.}{\to} 0,$$

and for a fixed N, we have (as $T \to \infty$)

$$\frac{1}{T}\sum_{t=1}^{T} z_{1t,N}\left(\boldsymbol{\theta}_0\right) = \sum_{i=1}^{N}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\zeta_{it}^2 - 1\right)\right] \overset{a.s.}{\to} 0. \tag{A.15}$$

Consider now (A.12), and note that $z_{2t,N}\left(\boldsymbol{\theta}\right)$ is serially independent (over t), and has mean zero. Further, note that $z_{2t,N}$ is a de-meaned quadratic form in ζ_{it} and Theorem 1 of Kelejian and Prucha (2001) applies to $z_{2t,N}$. Denote the (i,j) element of $\mathcal{A}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)$, by a_{ij}, and note that $a_{ij} = a_{ji}$. Then using (3.2) in Kelejian and Prucha (2001), we have (recall that $E\left(\zeta_{it}^2\right) = 1$)

$$\begin{aligned} Var\left[z_{2t,N}\left(\boldsymbol{\theta}\right)\right] &= Var\left[\boldsymbol{\zeta}'_{\circ t}\mathcal{A}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\boldsymbol{\zeta}_{\circ t}\right] \\ &= 4\sum_{i=1}^{N}\sum_{j=1}^{i-1} a_{ij}^2 + \sum_{i=1}^{N} a_{ii}^2\left[E\left(\zeta_{it}^4\right) - 1\right]. \end{aligned}$$

But for a fixed N, $\sum_{i=1}^{N} a_{ii}^2 < K$ and $\sum_{i=1}^{N}\sum_{j=1}^{i-1} a_{ij}^2 < K$. Furthermore,

$$\left|\sum_{i=1}^{N} a_{ii}^2\left[E\left(\zeta_{it}^4\right) - 1\right]\right| < \sup_{it}\left|E\left(\zeta_{it}^4\right) - 1\right|\sum_{i=1}^{N} a_{ii}^2,$$

and under Assumption 1, $\sup_{it} E\left(\zeta_{it}^4\right) < K$, and hence $Var\left[z_{2t,N}\left(\boldsymbol{\theta}\right)\right] < K$. Further, since $z_{2t,N}\left(\boldsymbol{\theta}\right)$ are independently distributed over t, then we have (see, for example, White (1984))

$$\frac{1}{T}\sum_{t=1}^{T} z_{2t,N}\left(\boldsymbol{\theta}\right) \overset{a.s.}{\to} 0. \tag{A.16}$$

Next, using (A.13),

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} z_{3t,N}\left(\boldsymbol{\theta}\right) &= \text{tr}\left[\mathcal{B}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\left(T^{-1}\sum_{t=1}^{T}\boldsymbol{x}_{\circ t}\boldsymbol{x}_{\circ t}'\right)\right] - \text{tr}\left[\mathcal{B}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\boldsymbol{\Sigma}_{\boldsymbol{xx}}\right] \\ &= \text{tr}\left\{\mathcal{B}\left(\boldsymbol{\theta}_0,\boldsymbol{\theta}\right)\left[T^{-1}\sum_{t=1}^{T}\left(\boldsymbol{x}_{\circ t}\boldsymbol{x}_{\circ t}' - \boldsymbol{\Sigma}_{\boldsymbol{xx}}\right)\right]\right\}.\end{aligned}$$

But by Assumption 2(b) and (c) we have that $E\left(\boldsymbol{x}_{\circ t}\boldsymbol{x}_{\circ t}' - \boldsymbol{\Sigma}_{\boldsymbol{xx}}|\mathcal{F}_t\right) = \mathbf{0}$, and $T^{-1}\sum_{t=1}^{T}\boldsymbol{x}_{\circ t}\boldsymbol{x}_{\circ t}' \overset{a.s.}{\to} \boldsymbol{\Sigma}_{\boldsymbol{xx}}$, as $T\to\infty$, which establishes that $T^{-1}\sum_{t=1}^{T} z_{3t,N}\left(\boldsymbol{\theta}\right) \overset{a.s.}{\to} 0$, as required. Finally, using (A.14),

$$\frac{1}{T}\sum_{t=1}^{T} z_{4t,N}\left(\boldsymbol{\theta}\right) = \text{tr}\left\{\left[\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}\boldsymbol{B}_0 - \boldsymbol{B}\right]'\boldsymbol{\Sigma}^{-1}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}_0^{-1}T^{-1}\sum_{t=1}^{T}\boldsymbol{\varepsilon}_{\circ t}\boldsymbol{x}_{\circ t}'\right\}.$$

But by Assumption 2(a), we have that $E\left(\boldsymbol{\varepsilon}_{\circ t}\boldsymbol{x}_{\circ t}'|\mathcal{F}_t\right) = 0$ and $E\left(\left|\boldsymbol{\varepsilon}_{\circ t}\boldsymbol{x}_{\circ t}'\right|^p \mid \mathcal{F}_t\right) \leq E\left(\left|\boldsymbol{\varepsilon}_{\circ t}\right|^p\left|\boldsymbol{x}_{\circ t}'\right|^p \mid \mathcal{F}_t\right)$, for $p=2$, which is bounded. Hence, $T^{-1}\sum_{t=1}^{T}\boldsymbol{\varepsilon}_{\circ t}\boldsymbol{x}_{\circ t}' \overset{a.s.}{\to} \mathbf{0}$, as $T\to\infty$ and $\frac{1}{T}\sum_{t=1}^{T} z_{4t,N}\left(\boldsymbol{\theta}\right) \overset{a.s.}{\to} 0$. (see, for example, White (1984)). Finally, (A.9) and (A.7) follow similarly. ■

Lemma 4 *Let*

$$\eta_{it} = \sigma_{i0}^{-1}\mathbf{e}_{i,N}'\boldsymbol{G}_0\mathbf{B}_0\boldsymbol{x}_{\circ t}\zeta_{it} + \sigma_{i0}^{-1}\mathbf{e}_{i,N}'\boldsymbol{G}_0\boldsymbol{\Sigma}_0^{1/2}\boldsymbol{\zeta}_{\circ t}\zeta_{it} - g_{0,ii}, \tag{A.17}$$

where $\boldsymbol{G}_0 = \boldsymbol{W}\left(\mathbf{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W}\right)^{-1} = \left(g_{0,ij}\right)$ *and* $\mathbf{e}_{i,N}$ *is an* N *dimensional vector with its* i^{th} *element unity and zeros elsewhere, and* $\boldsymbol{\zeta}_{\circ t} = \left(\zeta_{1t},\zeta_{2t},\ldots,\zeta_{Nt}\right)' = \left(\varepsilon_{1t}/\sigma_{10},\varepsilon_{2t}/\sigma_{20},\ldots,\varepsilon_{Nt}/\sigma_{N0}\right)'$*. Then under Assumptions 1, 2, 3, 4, and 5,* η_{it} *is a martingale difference process with respect to the filtration,* $\mathcal{F}_t = \left(\boldsymbol{x}_{\circ t},\boldsymbol{x}_{\circ t-1},\boldsymbol{x}_{\circ t-2},\ldots\right)$*, namely* $E\left(\eta_{it}|\mathcal{F}_t\right) = 0$*, and*

$$\sup_{i,t} E\left|\eta_{it}\right|^p < K, \text{ for } 1 \leq p \leq 2+c, \text{ and some } c > 0. \tag{A.18}$$

Proof. We first recall that $E\left(\boldsymbol{\zeta}_t|\mathcal{F}_t\right) = \mathbf{0}$, and hence $E\left(\boldsymbol{\zeta}_t\right) = \mathbf{0}$. Also $E\left(\boldsymbol{\zeta}_{\circ t}\zeta_{it}\right) = \mathbf{e}_{i,N}$ and $Var(\boldsymbol{\zeta}_{\circ t}) = \mathbf{I}_N$. Now under Assumption 1 it follows that

$$\begin{aligned}E\left(\eta_{it}|\mathcal{F}_t\right) &= E\left(\sigma_{i0}^{-1}\mathbf{e}_{i,N}'\boldsymbol{G}_0\mathbf{B}_0\boldsymbol{x}_{\circ t}\zeta_{it}|\mathcal{F}_t\right) + E\left(\sigma_{i0}^{-1}\mathbf{e}_{i,N}'\boldsymbol{G}_0\boldsymbol{\Sigma}_0^{1/2}\boldsymbol{\zeta}_{\circ t}\zeta_{it}|\mathcal{F}_t\right) - g_{0,ii} \\ &= 0 + g_{0,ii} - g_{0,ii} = 0,\end{aligned}$$

and establishes that η_{it} is a martingale difference process with respect to $\mathcal{F}_t$, as required. To establish (A.18), since $\zeta_{it} = \varepsilon_{it}/\sigma_{i0}$ then by Minkowski's inequality for $p \geq 1$ we have:

$$\left\|\eta_{it}\right\|_p \leq \sigma_{i0}^{-2}\left\|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\varepsilon_{it}\right\|_p + \sigma_{i0}^{-1}\left\|\boldsymbol{\vartheta}_i'\boldsymbol{\zeta}_{\circ t}\zeta_{it}\right\|_p + \left|g_{0,ii}\right|, \tag{A.19}$$

where $\boldsymbol{\varphi}_i' = \mathbf{e}_{i,N}'\boldsymbol{G}_0\mathbf{B}_0$, $\boldsymbol{\vartheta}_i' = \mathbf{e}_{i,N}'\boldsymbol{G}_0\boldsymbol{\Sigma}_0^{1/2} = \left(g_{i1}\sigma_{10}, g_{i2}\sigma_{20},\ldots,g_{iN}\sigma_{N0}\right)$, and $\left|g_{0,ii}\right| < K$. Consider now the first term of (A.19), and note that since conditional on $\mathcal{F}_t$, $\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}$ is given, and noting that by Assumption 1 $E\left(\left|\varepsilon_{it}\right|^p|\mathcal{F}_t\right) = \varpi_{ip} < K$, then

$$\left\|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\varepsilon_{it}\right\|_p^p = E\left[E\left(\left|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\varepsilon_{it}\right|^p \mid \mathcal{F}_t\right)\right] \leq E\left[\left|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\right|^p\left|E\left(\left|\varepsilon_{it}\right|^p \mid \mathcal{F}_t\right)\right.\right] = E\left(\left|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\right|^p\right)\varpi_{ip},$$

and hence $\left\|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\varepsilon_{it}\right\|_p \leq \varpi_{ip}^{1/p}\left\|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\right\|_p$. Also

$$\begin{aligned}\left\|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\right\|_p &= \left\|\sum_{j=1}^{N} g_{ij,0}\boldsymbol{\beta}_{j0}'\boldsymbol{x}_{jt}\right\|_p \leq \sum_{j=1}^{N}\left|g_{ij,0}\right|\left\|\boldsymbol{\beta}_{j0}'\boldsymbol{x}_{jt}\right\|_p \\ &\leq \left(\sup_{j,t} E\left\|\boldsymbol{\beta}_{j0}'\boldsymbol{x}_{jt}\right\|_p\right)\sum_{j=1}^{N}\left|g_{ij,0}\right|.\end{aligned}$$

The first term on the right hand side is bounded by Assumption 2(a), for $p \leq 2+c$, and $\sup_i \sum_{j=1}^{N}\left|g_{ij,0}\right|$ is bounded by Lemma 2. Hence, $\sup_{i,t}\left\|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\right\|_p < K$, and overall we have $\sup_{i,t}\left\|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\varepsilon_{it}\right\|_p < K$. Consider now the second term of (A.19) and note that

$$\left\|\boldsymbol{\vartheta}_i'\boldsymbol{\zeta}_{\circ t}\zeta_{it}\right\|_p \leq \sum_{j=1}^{N}\left\|g_{ij,0}\sigma_{j0}\zeta_{jt}\zeta_{it}\right\|_p \leq \frac{1}{\sigma_{i0}}\sum_{j=1}^{N}\left|g_{ij,0}\right|\left|\varepsilon_{jt}\varepsilon_{it}\right|_p = \frac{1}{\sigma_{i0}}\sum_{j=1}^{N}\left|g_{ij,0}\right|\left[E\left|\varepsilon_{jt}\varepsilon_{it}\right|^p\right]^{1/p}.$$

But $\sup_i\left(1/\sigma_{i0}\right) < K$ by Assumption 1, and using Cauchy–Schwarz inequality we obtain[19]

$$\begin{aligned}\left\|\boldsymbol{\vartheta}_i'\boldsymbol{\zeta}_{\circ t}\zeta_{it}\right\|_p &\leq K\sum_{j=1}^{N}\left|g_{ij,0}\right|\left[E\left(\varepsilon_{jt}^{2p}\right)\right]^{1/2p}\left[E\left(\varepsilon_{it}^{2p}\right)\right]^{1/2p} \\ &\leq K\left\{\sup_{i,t}\left[E\left(\varepsilon_{it}^{2p}\right)\right]^{1/p}\right\}\sum_{j=1}^{N}\left|g_{ij,0}\right|.\end{aligned}$$

Again $\sup_i \sum_{j=1}^{N}\left|g_{ij,0}\right| < K$ under Lemma 2 , and $E\left(\varepsilon_{it}^{2p}\right) < K$ for $2p = 4+\epsilon$ under Assumption 1, and hence $\left\|\boldsymbol{\vartheta}_i'\boldsymbol{\zeta}_{\circ t}\zeta_{it}\right\|_p < K$. Using this result together with $\sup_{i,t}\left\|\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\varepsilon_{it}\right\|_p < K$ (established above) in (A.19) now yields (A.18) by setting $c = 2\epsilon$. ■

Lemma 5 *Let*

$$\begin{aligned}\ell_t(\boldsymbol{\theta}) = &-\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + \frac{1}{2}\ln\left|\mathbf{V}(\boldsymbol{\psi})\right| \\ &-\frac{1}{2}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right],\end{aligned} \tag{A.20}$$

where $\mathbf{V}(\boldsymbol{\psi}) = \mathbf{S}'(\boldsymbol{\psi})\mathbf{S}(\boldsymbol{\psi})$, *and note that the log-likelihood function is given by*

$$\begin{aligned}\ell_T(\boldsymbol{\theta}) = \sum_{t=1}^{T}\ell_t(\boldsymbol{\theta}) = &-\frac{NT}{2}\ln(2\pi) - \frac{T}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + \frac{T}{2}\ln\left|\mathbf{V}(\boldsymbol{\psi})\right| \\ &-\frac{1}{2}\sum_{t=1}^{T}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right]'\boldsymbol{\Sigma}^{-1}\left[\mathbf{S}(\boldsymbol{\psi})\boldsymbol{y}_{\circ t} - \boldsymbol{B}\boldsymbol{x}_{\circ t}\right],\end{aligned}$$

[19]Note that since by Assumption 1 $E\left(\left|\varepsilon_{it}\right|^p\left|\mathcal{F}_t\right.\right) = \varpi_{ip} < K$, then for a given i we also have $E\left(\left|\varepsilon_{it}\right|^p\right) = \varpi_{ip}$, unconditionally.

(see also (7)) which can be written equivalently as

$$\ell_T(\boldsymbol{\theta}) = -\frac{NT}{2}\ln(2\pi) - \frac{T}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + \frac{T}{2}\ln|\mathbf{V}(\boldsymbol{\psi})| \tag{A.21}$$
$$-\frac{1}{2}\left\{\sum_{i=1}^{N}\frac{(\boldsymbol{y}_{i\circ}-\psi_i\boldsymbol{y}_{i\circ}^* - \mathbf{X}_{i\circ}\boldsymbol{\beta}_i)'(\boldsymbol{y}_{i\circ}-\psi_i\boldsymbol{y}_{i\circ}^* - \mathbf{X}_{i\circ}\boldsymbol{\beta}_i)}{\sigma_i^2}\right\},$$

where $\boldsymbol{y}_{i\circ} = (y_{i1}, y_{i2}, \ldots, y_{iT})'$ and $\boldsymbol{y}_{i\circ}^* = (y_{i1}^*, y_{i2}^*, \ldots, y_{iT}^*)'$ are $T\times 1$ vectors, and $\mathbf{X}_{i\circ} = (\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \ldots, \boldsymbol{x}_{iT})'$ is the $T\times k$ matrix of observations on regressors specific to the i^{th} cross section unit. Suppose that Assumptions 1, 2, 3, 4,and 5 hold. Denote the score function by $\boldsymbol{s}_T(\boldsymbol{\theta}) = \partial\ell_T(\boldsymbol{\theta})/\partial\boldsymbol{\theta} = \sum_{t=1}^{T}\partial\ell_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta}$. Then

$$T^{-1}\boldsymbol{s}_T(\boldsymbol{\theta}_0) \overset{a.s.}{\to} \mathbf{0}, \tag{A.22}$$

and

$$T^{-1/2}\boldsymbol{s}_T(\boldsymbol{\theta}_0) \to_d N\left[\mathbf{0}, \mathbf{J}(\boldsymbol{\theta}_0,\gamma)\right], \tag{A.23}$$

where

$$\mathbf{J}(\boldsymbol{\theta}_0,\gamma) = (J_{0,ij}) = \lim_{T\to\infty}\sum_{t=1}^{T}E_0\left[\frac{1}{T}\left(\frac{\partial\ell_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)\left(\frac{\partial\ell_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)'\right] \tag{A.24}$$

and

$$\gamma = \left[\lim_{T\to\infty}T^{-1}\sum_{t=1}^{T}E\left(\zeta_{it}^4\right) - 1\right] = \lim_{T\to\infty}T^{-1}\sum_{t=1}^{T}Var(\zeta_{it}^2),$$

with $\zeta_{it}\sim IID(0,1)$, $\zeta_{it} = \varepsilon_{it}/\sigma_{i0}$, for $i = 1, 2, \ldots, N$. A consistent estimator of $\mathbf{J}(\boldsymbol{\theta}_0,\gamma)$ is given by

$$\hat{\mathbf{J}}(\hat{\boldsymbol{\theta}},\hat{\gamma}) = \frac{1}{T}\left\{\left[\sum_{t=1}^{T}\frac{\partial\ell_t\left(\hat{\boldsymbol{\theta}}\right)}{\partial\boldsymbol{\theta}}\right]\left[\sum_{t=1}^{T}\frac{\partial\ell_t\left(\hat{\boldsymbol{\theta}}\right)}{\partial\boldsymbol{\theta}}\right]'\right\},$$

where $\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}}\bar{\ell}_T(\boldsymbol{\theta})$ and

$$\hat{\gamma} = (NT)^{-1}\sum_{t=1}^{T}\sum_{i=1}^{N}\left(\frac{\hat{\varepsilon}_{it}}{\hat{\sigma}_i}\right)^4 - 1,$$

with $\hat{\varepsilon}_{it} = y_{it} - \hat{\psi}_i\sum_{j=1}^{N}w_{ij}y_{jt} - \hat{\boldsymbol{\beta}}_i'\boldsymbol{x}_{it}$. $\hat{\sigma}_i$, $\hat{\boldsymbol{\beta}}_i$ and $\hat{\psi}_i$ are the QML estimators of σ_{i0}, $\boldsymbol{\beta}_{i0}$ and ψ_{i0}, respectively.

Proof. For a given N, the $N(k+2)\times 1$ score vector $\boldsymbol{s}_T(\boldsymbol{\theta}_0) = \left(\frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\psi}'}, \frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\beta}'}, \frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\sigma}^{2\prime}}\right)'$, where

$$\begin{pmatrix}\frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\psi}}\\ \frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\beta}}\\ \frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\sigma}^2}\end{pmatrix}_{N(k+2)\times 1} = \begin{pmatrix}\left[-T\,\mathrm{Diag}\left(\boldsymbol{G}_0\right) + \mathrm{Diag}\left(\frac{\boldsymbol{y}_{i\circ}^{*\prime}\boldsymbol{\varepsilon}_{i\circ}}{\sigma_{i0}^2}, i=1,2,\ldots,N\right)\right]\boldsymbol{\tau}_N\\ \mathrm{Diag}\left(\frac{\mathbf{X}_{i\circ}'\boldsymbol{\varepsilon}_{i\circ}}{\sigma_{i0}^2}, i=1,2,\ldots,N\right)\boldsymbol{\tau}_{Nk}\\ \mathrm{Diag}\left[-\frac{T}{2\sigma_{i0}^2} + \frac{1}{2\sigma_{i0}^4}\left(\boldsymbol{\varepsilon}_{i\circ}'\boldsymbol{\varepsilon}_{i\circ}\right), i=1,2,\ldots,N\right]\boldsymbol{\tau}_N\end{pmatrix}, \tag{A.25}$$

$\boldsymbol{y}_{i\circ}^* = (y_{i1}^*, y_{i2}^*, \ldots, y_{iT}^*)$, $\boldsymbol{\varepsilon}_{i\circ} = (\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{iT})$, $\boldsymbol{\varepsilon}_{i\circ} = \boldsymbol{y}_{i\circ} - \psi_{i0}\boldsymbol{y}_{i\circ}^* - \mathbf{X}_{i\circ}\boldsymbol{\beta}_{i0}$, and $\boldsymbol{\tau}_\kappa$ is a $\kappa\times 1$ vector of ones. Consider first the i^{th} component of $\partial\ell_T(\boldsymbol{\theta}_0)/\partial\boldsymbol{\psi}$, and note that it can be written as

$$\frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\psi_i} = -Tg_{0,ii} + \frac{1}{\sigma_{i0}^2}\sum_{t=1}^{T}y_{it}^*\varepsilon_{it}.$$

Also $y_{it}^* = \mathbf{e}_{i,N}'\boldsymbol{G}_0(\mathbf{B}_0\boldsymbol{x}_{\circ t} + \boldsymbol{\varepsilon}_{\circ t})$, where $\boldsymbol{G}_0 = \boldsymbol{W}(\mathbf{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}$ and $\boldsymbol{e}_{i,N}$ is an N dimensional vector with its i^{th} element unity and zeros elsewhere. Then

$$\frac{1}{T}\frac{\partial \ell_T(\boldsymbol{\theta}_0)}{\partial \psi_i}=\frac{1}{T}\sum_{t=1}^{T}\eta_{it}, \tag{A.26}$$

where η_{it} is already defined by (A.19) which we write as

$$\eta_{it}=\sigma_{i0}^{-1}\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\zeta_{it}+\sigma_{i0}^{-1}\sigma_{i0}^{-1}\boldsymbol{\vartheta}_i'\boldsymbol{\zeta}_{\circ t}\zeta_{it}-g_{0,ii}, \tag{A.27}$$

and as in proof of Lemma 4, $\boldsymbol{\varphi}_i'=\mathbf{e}_{i,N}'\boldsymbol{G}_0\mathbf{B}_0$, $\boldsymbol{\vartheta}_i'=\mathbf{e}_{i,N}'\boldsymbol{G}_0\boldsymbol{\Sigma}_0^{1/2}=(g_{i1}\sigma_{10},g_{i2}\sigma_{20},\ldots,g_{iN}\sigma_{N0})$, $\boldsymbol{\zeta}_{\circ t}=(\zeta_{1t},\zeta_{2t},\ldots,\zeta_{Nt})'$, and $\zeta_{it}=\varepsilon_{it}/\sigma_{i0}$. Also recall that by Lemma 4, $E(\eta_{it}|\mathcal{F}_t)=0$, and $\sup_{i,t}E|\eta_{it}|^{2+c}<K$, for some $c>0$. Therefore, using (A.26) by the strong law of large numbers for martingales we have (see, for example, White (1984))

$$\frac{1}{T}\frac{\partial \ell_T(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\psi}}\overset{a.s.}{\to}0. \tag{A.28}$$

Further, since $E(\eta_{it}|\mathcal{F}_t)=0$, then using (A.27)

$$\begin{aligned}Var(\eta_{it})=E[Var(\eta_{it}\mid\mathcal{F}_t)]&=\sigma_{i0}^{-2}\mathbf{g}_{0,i}'\mathbf{B}_0\boldsymbol{\Sigma}_{xx}\mathbf{B}_0'\mathbf{g}_{0,i}+\sigma_{i0}^{-2}\sum_{j=1}^{N}\sigma_{j0}^2g_{0,ij}^2+g_{0,ii}^2\left[E\left(\zeta_{it}^4\right)-2\right]\\&=\sigma_{i0}^{-2}\sum_{r=1}^{N}\sum_{s=1}^{N}g_{0,is}g_{0,ir}\boldsymbol{\beta}_{r0}'\boldsymbol{\Sigma}_{rs}\boldsymbol{\beta}_{s0}+\sigma_{i0}^{-2}\sum_{j=1}^{N}\sigma_{j0}^2g_{0,ij}^2+g_{0,ii}^2\left[E\left(\zeta_{it}^4\right)-2\right].\end{aligned} \tag{A.29}$$

Consider now the limiting distribution of $\frac{1}{\sqrt{T}}\frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\psi_i}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\eta_{it}$, and note that by Lemma 4, $\sup_{i,t}E|\eta_{it}|^{2+c}<K$ for some $c>0$, and by Corollary 5.25 in White (1984) it follows that

$$\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta}_0)}{\partial \psi_i}\to_d N(0,\omega_{ii})\text{, as }T\to\infty, \tag{A.30}$$

where (using (A.29))

$$\begin{aligned}\omega_{ii}&=\lim_{T\to\infty}T^{-1}\sum_{t=1}^{T}Var(\eta_{it})\\&=g_{0,ii}^2\left[\lim_{T\to\infty}T^{-1}\sum_{t=1}^{T}E\left(\zeta_{it}^4\right)-2\right]+\sigma_{i0}^{-2}\mathbf{g}_{0,i}'\mathbf{B}_0\boldsymbol{\Sigma}_{xx}\mathbf{B}_0'\mathbf{g}_{0,i}+\sigma_{i0}^{-2}\sum_{j=1}^{N}\sigma_{j0}^2g_{0,ij}^2,\end{aligned}$$

which exists and is finite under Assumptions 1 and 2(b).
Similarly, consider the i^{th} component of $\frac{\partial\ell_T(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\beta}}$. Then, write

$$\frac{1}{T}\frac{\partial \ell_T(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\beta}_i}=\frac{1}{\sigma_{i0}^2}\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{it}\varepsilon_{it}=\frac{1}{\sigma_{i0}}\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{it}\zeta_{it}. \tag{A.31}$$

But by Assumption 2(a), $E(\boldsymbol{x}_{it}\zeta_{it}\mid\mathcal{F}_t)=(1/\sigma_{i0})\boldsymbol{x}_{it}E(\varepsilon_{it}|\mathcal{F}_t)=\mathbf{0}$, and $Var(\boldsymbol{x}_{it}\zeta_{it}\mid\mathcal{F}_t)=\boldsymbol{x}_{it}\boldsymbol{x}_{it}'E\left(\zeta_{it}^2\mid\mathcal{F}_t\right)=\boldsymbol{x}_{it}\boldsymbol{x}_{it}'$. Hence $E(\boldsymbol{x}_{it}\zeta_{it})=\mathbf{0}$, and $Var(\boldsymbol{x}_{it}\zeta_{it})=\boldsymbol{\Sigma}_{ii}<K$. Therefore, noting that $\boldsymbol{x}_{it}\zeta_{it}$ is a martingale difference process with finite second-order moments, it follows that

$$\frac{1}{T}\frac{\partial \ell_T(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\beta}}\overset{a.s.}{\to}\mathbf{0}\text{, as }T\to\infty. \tag{A.32}$$

Denote the ℓ^{th} element of $\boldsymbol{x}_{it}\zeta_{it}$ by $z_{i\ell,t}=x_{i\ell,t}\zeta_{it}$ for $\ell=1,2,\ldots,k$, and note that the ℓ^{th} element of

$\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\beta}_i}$ is given $\frac{1}{\sigma_{i0}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} z_{i\ell,t}$, where $z_{i\ell,t}$ is a martingale difference process with respect to $\mathcal{F}_t$. Also, by Assumptions 1 and 2(a),

$$\begin{aligned}\sup_{i,\ell,t} E\left|z_{i\ell,t}\right|^p &= \sup_{i,\ell,t} E\left|x_{i\ell,t}\zeta_{it}\right|^p = \sup_{i,\ell,t} E\left[E\left(\left|x_{i\ell,t}\zeta_{it}\right|^p \left|\mathcal{F}_t\right.\right)\right] \leq \sup_{i,\ell,t} E\left[\left|x_{i\ell,t}\right|^p E\left(\left|\zeta_{it}\right|^p \left|\mathcal{F}_t\right.\right)\right] \\ &= \sup_{i,\ell,t}\left(E\left|x_{i\ell,t}\right|^p\right)\sigma_{i0}^{-p}\varpi_{ip} < K,\end{aligned}$$

for $p = 2 + c$, $c > 0$. Hence, by Corollary 5.25 in White (1984) it follows that for each i and ℓ and as $T \to \infty$, $\frac{1}{\sigma_{i0}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} z_{i\ell,t}$ tends to a normal distribution and as whole we have

$$\frac{1}{\sqrt{T}}\frac{\partial \ell_T\left(\boldsymbol{\theta}_0\right)}{\partial \boldsymbol{\beta}_i} \to_d N\left(\mathbf{0}, \boldsymbol{\Omega}_i\right), \tag{A.33}$$

where

$$\boldsymbol{\Omega}_i = \frac{1}{\sigma_{i0}^2}\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E\left(\boldsymbol{x}_{it}\boldsymbol{x}_{it}'\right). \tag{A.34}$$

Finally, consider the i^{th} component of $\frac{\partial \ell_T(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\sigma}^2}$, and note that

$$\frac{1}{T}\frac{\partial \ell_T\left(\boldsymbol{\theta}_0\right)}{\partial \sigma_i^2} = \frac{1}{2\sigma_{i0}^2}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\varepsilon_{it}^2}{\sigma_{i0}^2} - 1\right) = \frac{1}{2\sigma_{i0}^2}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\zeta_{it}^2 - 1\right)\right].$$

Let $\xi_{it} = \zeta_{it}^2 - 1$,where $\zeta_{it} = \varepsilon_{it}/\sigma_{i0}$. Then

$$\frac{1}{T}\frac{\partial \ell_T\left(\boldsymbol{\theta}_0\right)}{\partial \sigma_i^2} = \frac{1}{2\sigma_{i0}^2}\left[\frac{1}{T}\sum_{t=1}^{T}\xi_{it}\right]. \tag{A.35}$$

We have $E\left(\xi_{it} \mid \mathcal{F}_t\right) = E\left(\zeta_{it}^2 \mid \mathcal{F}_t\right) - 1 = 0$, and $E\left(\xi_{it}^2 \mid \mathcal{F}_t\right) = E\left(\zeta_{it}^4 \mid \mathcal{F}_t\right) - 1$, so that, since under Assumption 1 ξ_{it}'s are martingale difference processes and $E(\left|\varepsilon_{it}\right|^{4+\epsilon}\left|\mathcal{F}_t\right.) < K$, for some small positive ϵ, then $\sup_i E\left|\xi_{it}\right|^2 < K$ and by the strong law of large numbers for martingale processes we have

$$\frac{1}{T}\frac{\partial \ell_T\left(\boldsymbol{\theta}_0\right)}{\partial \boldsymbol{\sigma}^2} \overset{a.s.}{\to} 0, \text{ as } T \to \infty. \tag{A.36}$$

Similarly, since $\sup_i E\left|\xi_{it}\right|^{2+c} < K$ for some $c > 0$, then as before

$$T^{-1/2}\frac{\partial \ell_T\left(\boldsymbol{\theta}_0\right)}{\partial \sigma_i^2} \to_d N(0, v_{ii}), \tag{A.37}$$

where

$$v_{ii} = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\left[\frac{1}{4\sigma_{i0}^4}Var(\xi_{it})\right] = \left(\frac{1}{4\sigma_{i0}^4}\right)\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} Var(\zeta_{it}^2). \tag{A.38}$$

Now results (A.28), (A.32) and (A.36) establish (A.22), and results (A.30), (A.33) and (A.37) establish (A.23), as required, with $\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right) = \lim_{T\to\infty}\sum_{t=1}^{T} E_0\left[\frac{1}{T}\left(\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)\left(\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)'\right]$. Consistency of $\hat{\mathbf{J}}(\hat{\boldsymbol{\theta}}, \hat{\gamma})$ for $\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)$ follows from consistency of $\hat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}_0$, and $\hat{\gamma}$ for γ, and independence of $\frac{\partial \ell_t(\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}}$ over t. Further, since $\hat{\boldsymbol{\theta}} \overset{a.s.}{\to} \boldsymbol{\theta}_0$ on $\boldsymbol{\Theta}_c$, as $T \to \infty$, as shown in Section 3.3, we have

$$\hat{\varepsilon}_{it} = \varepsilon_{it} + O_p\left(\frac{1}{\sqrt{T}}\right), \text{ and } \hat{\sigma}_i^2 = \sigma_i^2 + O_p\left(\frac{1}{\sqrt{T}}\right),$$

which establishes that

$$\hat{\gamma} = (NT)^{-1} \sum_{t=1}^{T} \sum_{i=1}^{N} \left(\frac{\hat{\varepsilon}_{it}}{\hat{\sigma}_i} \right)^4 - 1 \to_p \gamma, \text{ as } T \to \infty, \text{ for any } N.$$

■

Appendix B Proofs of Propositions 2 and 3, and Corollary 1

Proof of Proposition 2. First, we consider the information matrix $\boldsymbol{H}(\boldsymbol{\theta}_0)$ given by

$$\boldsymbol{H}(\boldsymbol{\theta}_0) = \lim_{T\to\infty} E_0 \left[-\frac{1}{T} \frac{\partial^2 \ell_T(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right], \tag{B.39}$$

where

$$E_0 \left[-\frac{1}{T} \frac{\partial^2 \ell_T(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right] = \begin{pmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} & \boldsymbol{H}_{13} \\ \boldsymbol{H}'_{12} & \boldsymbol{H}_{22} & \boldsymbol{H}_{23} \\ \boldsymbol{H}'_{13} & \boldsymbol{H}'_{23} & \boldsymbol{H}_{33} \end{pmatrix}_{N(k+2)\times N(k+2)}.$$

We evaluate each partial derivative in (B.39):
$\boldsymbol{H}_{11} = E_0 \left[-\frac{1}{T} \frac{\partial^2 \ell_T(\boldsymbol{\theta})}{\partial \boldsymbol{\psi} \partial \boldsymbol{\psi}'} \right]$ is given by the $N \times N$ matrix

$$\boldsymbol{H}_{11} = \left(\mathbf{G}_0 \odot \mathbf{G}'_0 \right) + \text{Diag} \left[\frac{1}{\sigma_{i0}^2} \frac{1}{T} \sum_{t=1}^{T} E_0 \left(y_{it}^{*2} \right), \, i = 1, 2, \ldots, N \right],$$

where $\mathbf{G}_0 = \boldsymbol{W} (\boldsymbol{I}_N - \boldsymbol{\Psi}_0 \boldsymbol{W})^{-1}$ with its i^{th} row denoted by $\boldsymbol{g}'_{0i}$, and

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^{T} E_0 \left(y_{it}^{*2} \right) &= \boldsymbol{w}'_i (\mathbf{I}_N - \boldsymbol{\Psi}_0 \boldsymbol{W})^{-1} \left[\boldsymbol{B}_0 E \left(\boldsymbol{x}_{\circ t} \boldsymbol{x}'_{\circ t} \right) \boldsymbol{B}'_0 + \boldsymbol{\Sigma}_0 \right] (\mathbf{I}_N - \boldsymbol{W}' \boldsymbol{\Psi}_0)^{-1} \boldsymbol{w}_i \\ &= \boldsymbol{g}'_{0i} \left(\boldsymbol{B}_0 \boldsymbol{\Sigma}_{xx} \boldsymbol{B}'_0 + \boldsymbol{\Sigma}_0 \right) \boldsymbol{g}_{0i} \\ &= \sum_{r=1}^{N} \sum_{s=1}^{N} g_{0,is} g_{0,ir} \boldsymbol{\beta}'_{r0} \boldsymbol{\Sigma}_{rs} \boldsymbol{\beta}_{s0} + \sum_{s=1}^{N} g_{0,is}^2 \sigma_{s0}^2. \end{aligned}$$

Note that as shown in Lemmas 2 and 5,

$$\|\mathbf{G}_0\|_\infty < K, \ \ \left\| \mathbf{G}_0 \odot \mathbf{G}'_0 \right\|_\infty < K \text{ and } \left\| \sum_{r=1}^{N} \sum_{s=1}^{N} g_{0,is} g_{0,ir} \boldsymbol{\beta}'_{r0} \boldsymbol{\Sigma}_{rs} \boldsymbol{\beta}_{s0} \right\|_\infty < K.$$

$\boldsymbol{H}_{12} = E_0 \left[-\frac{1}{T} \frac{\partial^2 \ell_T(\boldsymbol{\theta})}{\partial \boldsymbol{\psi} \partial \boldsymbol{\beta}'} \right]$ is an $N \times kN$ matrix with its i^{th} row given by a $1 \times kN$ vector of zeros except for its i^{th} block which is given by the $1 \times k$ vector $\sigma_{i0}^{-2} E_0(T^{-1} \boldsymbol{y}_i^{*\prime} \mathbf{X}_i)$, namely

$$\boldsymbol{H}_{12} = \begin{pmatrix} \sigma_{10}^{-2} E_0(T^{-1} \boldsymbol{y}_1^{*\prime} \mathbf{X}_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sigma_{20}^{-2} E_0(T^{-1} \boldsymbol{y}_2^{*\prime} \mathbf{X}_2) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \sigma_{N0}^{-2} E_0(T^{-1} \boldsymbol{y}_N^{*\prime} \mathbf{X}_N) \end{pmatrix},$$

where

$$
\begin{aligned}
E_0(T^{-1}\boldsymbol{y}_i^{*\prime}\mathbf{X}_i) &= E_0\left(T^{-1}\sum_{t=1}^{T} y_{it}^{*}\boldsymbol{x}_{it}'\right) = E_0\left(T^{-1}\sum_{t=1}^{T}\boldsymbol{w}_i'\boldsymbol{y}_{\circ t}\boldsymbol{x}_{it}'\right) \\
&= \boldsymbol{w}_i'(\mathbf{I}_N-\boldsymbol{\Psi}_0\mathbf{W})^{-1}\mathbf{B}E\left(\boldsymbol{x}_{\circ t}\boldsymbol{x}_{it}'\right) \\
&= \boldsymbol{g}_{0i}'\left(\boldsymbol{\Sigma}_{i1}\boldsymbol{\beta}_{10},\boldsymbol{\Sigma}_{i2}\boldsymbol{\beta}_{20},\ldots,\boldsymbol{\Sigma}_{iN}\boldsymbol{\beta}_{N0}\right)' = \sum_{s=1}^{N} g_{0,is}\boldsymbol{\beta}_{s0}'\boldsymbol{\Sigma}_{is}.
\end{aligned}
$$

Again by Assumptions 2(b), 3 and 5, $\sup_s \|\boldsymbol{\beta}_{s0}\|_1$ and $\boldsymbol{\Sigma}_{is}$ exist and are finite. Also, $\max_i \sum_{s=1}^{N}|g_{0,is}| = \|\boldsymbol{G}_0\|_\infty$ which is bounded under our assumptions. $\mathbf{H}_{13} = E_0\left[-\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\sigma}^{2\prime}}\right]$ is an $N\times N$ diagonal matrix with its i^{th} element given by $\sigma_{i0}^{-2}\boldsymbol{w}_i'(\mathbf{I}_N-\boldsymbol{\Psi}_0\mathbf{W})^{-1}\mathbf{e}_{i,N} = \sigma_{i0}^{-2}g_{0,ii}$, where $\mathbf{e}_{i,N}$ is an N dimensional vector with its i^{th} element unity and zeros elsewhere. $\boldsymbol{H}_{22} = E_0\left[-\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right]$ is an $Nk\times Nk$ block diagonal matrix with its i^{th} block given by $\sigma_{i0}^{-2}\boldsymbol{\Sigma}_{ii}$. $\boldsymbol{H}_{23} = \mathbf{0}$, and finally $\boldsymbol{H}_{33} = E_0\left[-\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial(\boldsymbol{\sigma}^2)\partial(\boldsymbol{\sigma}^{2\prime})}\right] = \mathrm{Diag}(1/2\sigma_{10}^4, 1/2\sigma_{20}^4,\ldots,1/2\sigma_{N0}^4)$. Collecting all terms, we obtain (B.39).
Next, recalling from Lemma 4 that

$$
\eta_{it} = \sigma_{i0}^{-1}\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\zeta_{it} + \sigma_{i0}^{-1}\boldsymbol{\vartheta}_i'\boldsymbol{\zeta}_{\circ t}\zeta_{it} - g_{0,ii},
$$

where $\boldsymbol{\varphi}_i' = \mathbf{e}_{i,N}'\boldsymbol{G}_0\mathbf{B}_0$ and $\boldsymbol{\vartheta}_i' = \mathbf{e}_{i,N}'\boldsymbol{G}_0\boldsymbol{\Sigma}_0^{1/2}$, and using (A.26) and (A.29) of Lemma 5, we have the following cross-products (for $i \neq j$)

$$
\begin{aligned}
E_0\left[\frac{1}{\sqrt{T}}\frac{\partial\ell_T(\boldsymbol{\theta})}{\partial\psi_i}\frac{1}{\sqrt{T}}\frac{\partial\ell_T(\boldsymbol{\theta})}{\partial\psi_j}\right] &= \frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E_0\left(\eta_{it}\eta_{jt'}\right) \\
&= \frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E\left\{\begin{array}{c}\left[\sigma_{i0}^{-1}\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\zeta_{it}+\sigma_{i0}^{-1}\boldsymbol{\vartheta}_i'\boldsymbol{\zeta}_{\circ t}\zeta_{it}-g_{0,ii}\right] \\ \left[\sigma_{j0}^{-1}\boldsymbol{\varphi}_j'\boldsymbol{x}_{\circ t'}\zeta_{jt'}+\sigma_{j0}^{-1}\boldsymbol{\vartheta}_j'\boldsymbol{\zeta}_{\circ t'}\zeta_{jt'}-g_{0,jj}\right]\end{array}\right\} \\
&= \frac{1}{T}\sum_{t=1}^{T}E\left\{\begin{array}{c}\left[\sigma_{i0}^{-1}\boldsymbol{\varphi}_i'\boldsymbol{x}_{\circ t}\zeta_{it}+\sigma_{i0}^{-1}\boldsymbol{\vartheta}_i'\boldsymbol{\zeta}_{\circ t}\zeta_{it}-g_{0,ii}\right] \\ \left[\sigma_{j0}^{-1}\boldsymbol{\varphi}_j'\boldsymbol{x}_{\circ t}\zeta_{jt}+\sigma_{j0}^{-1}\boldsymbol{\vartheta}_j'\boldsymbol{\zeta}_{\circ t}\zeta_{jt}-g_{0,jj}\right]\end{array}\right\} \\
&= \frac{1}{T}\sum_{t=1}^{T}\left\{\begin{array}{c}\sigma_{i0}^{-1}\sigma_{j0}^{-1}\boldsymbol{\varphi}_i'E\left(\boldsymbol{x}_{\circ t}\boldsymbol{x}_{\circ t}'\zeta_{it}\zeta_{jt}\right)\boldsymbol{\varphi}_j+\sigma_{i0}^{-1}\sigma_{j0}^{-1}\boldsymbol{\vartheta}_i'E\left(\zeta_{it}\zeta_{jt}\boldsymbol{\zeta}_{\circ t}\boldsymbol{\zeta}_{\circ t}'\right)\boldsymbol{\vartheta}_j \\ -g_{0,ii}g_{0,jj}+\sigma_{i0}^{-1}\sigma_{j0}^{-1}\boldsymbol{\varphi}_i'E_0\left(\boldsymbol{x}_{\circ t}\boldsymbol{\zeta}_{\circ t}'\boldsymbol{\vartheta}_j\zeta_{it}\zeta_{jt}\right) \\ +\sigma_{j0}^{-1}\sigma_{i0}^{-1}\boldsymbol{\varphi}_j'E_0\left(\boldsymbol{x}_{\circ t}\boldsymbol{\zeta}_{\circ t}'\boldsymbol{\vartheta}_j\zeta_{jt}\zeta_{it}\right)\end{array}\right. \\
&= \left\{\begin{array}{ll}\left\{\begin{array}{c}g_{0,ii}^2\left[\frac{1}{T}\sum_{t=1}^{T}E\left(\zeta_{it}^4\right)-2\right]+\sigma_{i0}^{-2}\mathbf{g}_{0,i}'\mathbf{B}_0\boldsymbol{\Sigma}_{xx}\mathbf{B}_0'\mathbf{g}_{0,i} \\ +\sigma_{i0}^{-2}\sum_{j=1}^{N}\sigma_{j0}^2 g_{0,ij}^2\end{array}\right. & \text{, for } i=j \\ g_{0,ij}g_{0,ji}, & \text{, for } i\neq j\end{array}\right. .
\end{aligned}
$$

Further, using (A.31) and (A.34) of Lemma 5 we have

$$
\begin{aligned}
E_0\left[\frac{1}{\sqrt{T}}\frac{\partial\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}_i}\frac{1}{\sqrt{T}}\frac{\partial\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}_j'}\right] &= \frac{1}{\sigma_{i0}\sigma_{j0}}\frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E\left(\boldsymbol{x}_{it}\zeta_{it}\zeta_{jt'}\boldsymbol{x}_{jt'}'\right) \\
&= \left\{\begin{array}{c}\frac{1}{\sigma_{i0}^2 T}\sum_{t=1}^{T}E\left(\boldsymbol{x}_{it}\boldsymbol{x}_{it}'\right),\ \text{for } i=j \\ \mathbf{0},\ \text{for } i\neq j\end{array}\right. ,
\end{aligned}
$$

and using (A.35) and (A.38) of Lemma 5 we have

$$\begin{aligned}
E_0\left[\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta})}{\partial \sigma_i^2}\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta})}{\partial \sigma_j^2}\right] &= \frac{1}{4\sigma_{i0}^2\sigma_{j0}^2}\frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E\left(\xi_{it}\xi_{jt'}\right) \\
&= \frac{1}{4\sigma_{i0}^2\sigma_{j0}^2}\frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E\left[\left(\zeta_{it}^2-1\right)\left(\zeta_{jt'}^2-1\right)\right] \\
&= \frac{1}{4\sigma_{i0}^2\sigma_{j0}^2}\frac{1}{T}\sum_{t=1}^{T}E\left(\zeta_{it}^2\zeta_{jt}^2-\zeta_{it}^2-\zeta_{jt}^2+1\right) \\
&= \begin{cases} \frac{1}{4\sigma_{i0}^4 T}\left[\sum_{t=1}^{T}E\left(\zeta_{it}^4\right)-1\right], \text{ for } i=j \\ 0, \text{ for } i\neq j \end{cases}.
\end{aligned}$$

In addition,

$$\begin{aligned}
E_0\left[\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}_i}\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta})}{\partial \psi_j}\right] &= \frac{1}{\sigma_{i0}T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E_0\left[\left(\boldsymbol{x}_{it}\zeta_{it}\right)\eta_{jt'}\right] \\
&= \frac{1}{\sigma_{i0}T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E_0\left[\boldsymbol{x}_{it}\zeta_{it}\left(\sigma_{j0}^{-1}\boldsymbol{\varphi}_j'\boldsymbol{x}_{\circ t'}\zeta_{jt'}+\sigma_{j0}^{-1}\boldsymbol{\vartheta}_j'\boldsymbol{\zeta}_{\circ t'}\zeta_{jt'}-g_{0,jj}\right)\right] \\
&= \frac{1}{\sigma_{i0}T}\sum_{t=1}^{T}\left[\begin{array}{c}\sigma_{j0}^{-1}E_0\left(\boldsymbol{x}_{it}\boldsymbol{\varphi}_j'\boldsymbol{x}_{\circ t}\zeta_{it}\zeta_{jt}\right)+\sigma_{j0}^{-1}E_0\left(\boldsymbol{x}_{it}\zeta_{it}\boldsymbol{\vartheta}_j'\boldsymbol{\zeta}_{\circ t}\zeta_{jt}\right) \\ -g_{0,jj}E\left(\boldsymbol{x}_{it}\zeta_{it}\right)\end{array}\right] \\
&= \begin{cases} \sigma_{i0}^{-2}\boldsymbol{g}_{0i}'\left(\boldsymbol{\Sigma}_{i1}\boldsymbol{\beta}_{10},\boldsymbol{\Sigma}_{i2}\boldsymbol{\beta}_{20},\ldots,\boldsymbol{\Sigma}_{iN}\boldsymbol{\beta}_{N0}\right)', \text{ for } i=j \\ \mathbf{0}, \text{ for } i\neq j \end{cases}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
E_0\left[\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta})}{\partial \sigma_i^2}\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta})}{\partial \psi_j}\right] &= \frac{1}{2\sigma_{i0}^2}\frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E\left(\xi_{it}\eta_{jt'}\right) \\
&= \frac{1}{2\sigma_{i0}^2}\frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E\left[\left(\zeta_{it}^2-1\right)\left(\sigma_{j0}^{-1}\boldsymbol{\varphi}_j'\boldsymbol{x}_{\circ t'}\zeta_{jt'}+\sigma_{j0}^{-1}\boldsymbol{\vartheta}_j'\boldsymbol{\zeta}_{\circ t'}\zeta_{jt'}-g_{0,jj}\right)\right] \\
&= \frac{1}{2\sigma_{i0}^2}\frac{1}{T}\sum_{t=1}^{T}E\left[\zeta_{it}^2\left(\sigma_{j0}^{-1}\boldsymbol{\varphi}_j'\boldsymbol{x}_{\circ t}\zeta_{jt}+\sigma_{j0}^{-1}\boldsymbol{\vartheta}_j'\boldsymbol{\zeta}_{\circ t}\zeta_{jt}-g_{0,jj}\right)\right] \\
&= \begin{cases} \frac{g_{0,ii}\left[\frac{1}{T}\sum_{t=1}^{T}E\left(\zeta_{it}^4\right)-1\right]}{2\sigma_{i0}^2}, \text{ for } i=j \\ 0, \text{ for } i\neq j \end{cases},
\end{aligned}$$

and finally,

$$\begin{aligned}
E_0\left[\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}_i}\frac{1}{\sqrt{T}}\frac{\partial \ell_T(\boldsymbol{\theta})}{\partial \sigma_j^2}\right] &= \frac{1}{2\sigma_{i0}\sigma_{j0}^2}\frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E\left[\left(\boldsymbol{x}_{it}\zeta_{it}\right)\xi_{jt'}\right] \\
&= \frac{1}{2\sigma_{i0}\sigma_{j0}^2}\frac{1}{T}\sum_{t=1}^{T}\sum_{t'=1}^{T}E\left[\left(\boldsymbol{x}_{it}\zeta_{it}\right)\left(\zeta_{jt'}^2-1\right)\right] \\
&= 0, \text{ for all } i,j=1,2,\ldots,N.
\end{aligned}$$

Overall, let

$$\gamma = \left[\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E\left(\zeta_{it}^{4}\right) - 1\right] = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} Var(\zeta_{it}^{2}). \tag{B.40}$$

We can collect the various terms and construct matrix

$$\mathbf{J}\left(\boldsymbol{\theta}_0,\gamma\right) = \left(J_{0,ij}\right) = \lim_{T\to\infty} E_0\left[\frac{1}{T}\left(\sum_{t=1}^{T}\frac{\partial \ell_t\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\theta}}\right)\left(\sum_{t=1}^{T}\frac{\partial \ell_t\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\theta}}\right)'\right] = \begin{pmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} & \boldsymbol{J}_{13} \\ . & \boldsymbol{J}_{22} & \boldsymbol{J}_{23} \\ . & . & \boldsymbol{J}_{33}\end{pmatrix}_{N(k+2)\times N(k+2)}, \tag{B.41}$$

where $\ell_t\left(\boldsymbol{\theta}\right)$ is defined in (A.20) and

$$\boldsymbol{J}_{11} = \left\{\begin{array}{c} \left(\boldsymbol{G}_0\odot\boldsymbol{G}_0'\right) + (\gamma-2)\,\mathrm{Diag}\left(\boldsymbol{G}_0\odot\boldsymbol{G}_0'\right) \\ +\,\mathrm{Diag}\left[\sigma_{i0}^{-2}\boldsymbol{g}_{0i}'\left(\boldsymbol{B}_0\boldsymbol{\Sigma}_{xx}\boldsymbol{B}_0' + \boldsymbol{\Sigma}_0\right)\boldsymbol{g}_{0i},\ i=1,2,\ldots,N\right]\end{array}\right.,$$

$$\boldsymbol{J}_{12} = \mathrm{Diag}\left[\sigma_{i0}^{-2}\sum_{s=1}^{N} g_{0,is}\boldsymbol{\beta}_{s0}'\boldsymbol{\Sigma}_{is},\ i=1,2,\ldots,N\right],$$

$$\boldsymbol{J}_{13} = \frac{\gamma}{2}\,\mathrm{Diag}\left(\sigma_{i0}^{-2}g_{0,ii},\ i=1,2,\ldots,N\right),\ \ \boldsymbol{J}_{22} = \mathrm{Diag}\left(\sigma_{i0}^{-2}\boldsymbol{\Sigma}_{ii},\ i=1,2,\ldots,N\right),$$

$$\boldsymbol{J}_{23} = \mathbf{0},\ \ \boldsymbol{J}_{33} = \frac{\gamma}{4}\,\mathrm{Diag}\left(1/\sigma_{i0}^{4},\ i=1,2,\ldots,N\right).$$

Having established that the score vector is asymptotically normally distributed, it is now easily seen that as $T\to\infty$,

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \to_d N\left(\mathbf{0},\boldsymbol{V}_{\boldsymbol{\theta}}\right), \tag{B.42}$$

where $\boldsymbol{V}_{\boldsymbol{\theta}} = \boldsymbol{H}^{-1}\left(\boldsymbol{\theta}_0\right)\mathbf{J}\left(\boldsymbol{\theta}_0,\gamma\right)\boldsymbol{H}^{-1}\left(\boldsymbol{\theta}_0\right)$. ■

Proof of Proposition 3. Suppose that $\{\psi_{i0},\ \boldsymbol{\beta}_{i0},\ i=1,2,\ldots,N\}$ are randomly distributed around the common means, ψ_0 and $\boldsymbol{\beta}_0$, following (25). First, the asymptotic validity of the MG estimator, $\hat{\psi}_{MG}$, defined by (26), can be established by first noting that

$$\sqrt{N}\left(\hat{\psi}_{MG} - \psi_0\right) = N^{-1/2}\sum_{i=1}^{N}\left(\hat{\psi}_i - \psi_{i0}\right) + N^{-1/2}\sum_{i=1}^{N}\left(\psi_{i0} - \psi_0\right), \tag{B.43}$$

where $\hat{\psi}_i$ are the underlying unit-specific estimators. Upon using (25), (B.43) can also be written as

$$\begin{aligned}\sqrt{N}\left(\hat{\psi}_{MG} - \psi_0\right) &= \frac{\sqrt{N}}{T}\left[N^{-1}\sum_{i=1}^{N} T\left(\hat{\psi}_i - \psi_{i0}\right)\right] + N^{-1/2}\sum_{i=1}^{N}\eta_{i\psi} \\ &= q_{NT} + \xi_{NT}.\end{aligned} \tag{B.44}$$

Consider now q_{NT}, the

first term of the above expression, and recall that under the regularity conditions set out in Section 3, $\sqrt{T}\left(\hat{\psi}_i - \psi_{i0}\right) \overset{a}{\sim} N(0,\omega_{\psi_i}^2)$, where $\sup_i \omega_{\psi_i}^2 < K$, and $E\left(\hat{\psi}_i - \psi_{i0}\right) = O\left(T^{-1}\right)$, for all i. Hence

$$E\left(q_{NT}\right) = O\left(\frac{\sqrt{N}}{T}\right). \tag{B.45}$$

Furthermore we note that q_{NT} can also be written as

$$q_{NT} = N^{-1/2}\sum_{i=1}^{N}\left(\hat{\psi}_i - \psi_{i0}\right) = T^{-1/2}N^{-1/2}\boldsymbol{\tau}_N'\left[\sqrt{T}\left(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0\right)\right],$$

where $\hat{\boldsymbol{\psi}} = (\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_N)'$, and $\boldsymbol{\tau}_N$ is an $N \times 1$ vector of ones. Denote the $N \times N$ asymptotic covariance matrix of $\hat{\boldsymbol{\psi}}$ by $\mathbf{V}_{\boldsymbol{\psi}} = AsyVar\left[\sqrt{T}\left(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0\right)\right]$, then

$$\begin{aligned}
\lim_{N,T\to\infty} Var\left(q_{NT}\right) &= \lim_{N,T\to\infty} Var\left[T^{-1/2}N^{-1/2}\boldsymbol{\tau}_N'\sqrt{T}\left(\hat{\boldsymbol{\psi}} - \boldsymbol{\psi}_0\right)\right] \\
&= \lim_{N,T\to\infty}\left(\frac{\boldsymbol{\tau}_N'\mathbf{V}_{\boldsymbol{\psi}}\boldsymbol{\tau}_N}{NT}\right) \leq \lim_{N,T\to\infty}\left[\frac{\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\psi}}\right)}{T}\right]. \qquad \text{(B.46)}
\end{aligned}$$

Suppose now that $\sqrt{N}/T \to 0$, and $\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\psi}}\right) < K$ as N and $T \to \infty$. Then using (B.45) and (B.46) it readily follows that

$$\lim_{N,T\to\infty} E\left(q_{NT}\right) = 0, \text{ and } \lim_{N,T\to\infty} Var\left(q_{NT}\right) = 0,$$

and hence as $\sqrt{N}/T \to 0$, $q_{NT} = o_p(1)$, and in view of (B.44) $\sqrt{N}\left(\hat{\psi}_{MG} - \psi_0\right) \overset{a}{\sim} \xi_{NT} = N^{-1/2}\sum_{i=1}^{N}\eta_{i\psi}$. Finally, under the random coefficient model where $\{\eta_{i\psi}$, for $i = 1, 2, \ldots, N\}$ are assumed to be independently distributed with zero means and finite variances, $\xi_{NT} \overset{a}{\sim} N\left[0, Var\left(\eta_{i\psi}\right)\right]$, and therefore under the additional conditions, $\sqrt{N}/T \to 0$ and $\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\psi}}\right) < K$, we have (as N,$T \to \infty$, jointly):

$$\sqrt{N}\left(\hat{\psi}_{MG} - \psi_0\right) \overset{a}{\sim} N\left(0, \omega_\psi^2\right), \qquad \text{(B.47)}$$

where $\omega_\psi^2 = Var\left(\eta_{i\psi}\right)$. It is also easily seen that ω_ψ^2 can be consistently estimated by

$$\hat{\omega}_\psi^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\psi}_i - \hat{\psi}_{MG}\right)^2.$$

Similarly, for the MG estimator, $\hat{\boldsymbol{\beta}}_{MG}$, again defined by (26), we have

$$\sqrt{N}\left(\hat{\boldsymbol{\beta}}_{MG} - \boldsymbol{\beta}_0\right) \overset{a}{\sim} N\left(\mathbf{0}, \boldsymbol{\Omega}_\beta\right), \qquad \text{(B.48)}$$

so long as $\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\beta}}\right) < K$, where $\mathbf{V}_{\boldsymbol{\beta}}$ is the $kN \times kN$ asymptotic covariance matrix of $\sqrt{T}\ \hat{\boldsymbol{\beta}} = (\sqrt{T}\hat{\boldsymbol{\beta}}_1', \sqrt{T}\hat{\boldsymbol{\beta}}_2', \ldots, \sqrt{T}\hat{\boldsymbol{\beta}}_N')'$ and $\hat{\boldsymbol{\beta}}_i$ are the underlying unit-specific estimators. A consistent estimator of $\boldsymbol{\Omega}_\beta$ is given by

$$\hat{\boldsymbol{\Omega}}_\beta = \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\beta}}_i - \hat{\boldsymbol{\beta}}_{MG}\right)\left(\hat{\boldsymbol{\beta}}_i - \hat{\boldsymbol{\beta}}_{MG}\right)',$$

It now remains to establish conditions under which $\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\psi}}\right) < K$ and $\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\beta}}\right) < K$ hold. We first note that $\mathbf{V}_{\boldsymbol{\psi}}$ and $\mathbf{V}_{\boldsymbol{\beta}}$ are sub-matrices of $\mathbf{V}_\theta$ defined by (21) which we reproduce here for convenience:

$$\mathbf{V}_{\boldsymbol{\theta}} = \mathbf{H}^{-1}(\boldsymbol{\theta}_0)\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\mathbf{H}^{-1}(\boldsymbol{\theta}_0),$$

where $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)$ and $\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)$ are given by (17) and (19), respectively. Hence, it is sufficient to show that $\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\theta}}\right)$ is bounded in N. To this end we first note that

$$\left\|\mathbf{V}_{\boldsymbol{\theta}}\right\| \leq \left\|\mathbf{H}^{-1}(\boldsymbol{\theta}_0)\right\|^2 \left\|\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\right\|, \qquad \text{(B.49)}$$

where $\|\mathbf{A}\| = \lambda_{\max}^{1/2}\left(\mathbf{A}'\mathbf{A}\right)$ is the spectral norm of $\mathbf{A}$. However, since $\mathbf{V}_{\boldsymbol{\theta}}$,$\mathbf{H}^{-1}(\boldsymbol{\theta}_0)$ and $\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)$ are symmetric matrices, then $\left\|\mathbf{V}_{\boldsymbol{\theta}}\right\| = \lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\theta}}\right)$, $\left\|\mathbf{H}^{-1}(\boldsymbol{\theta}_0)\right\|^2 = \lambda_{\max}^2\left[\boldsymbol{H}^{-1}\left(\boldsymbol{\theta}_0\right)\right]$, and $\left\|\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\right\| = \lambda_{\max}\left[\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\right]$, and (B.49) can also be written as

$$\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\theta}}\right) \leq \lambda_{\max}^2\left[\boldsymbol{H}^{-1}\left(\boldsymbol{\theta}_0\right)\right]\lambda_{\max}\left[\mathbf{J}\left(\boldsymbol{\theta}_0, \gamma\right)\right]. \qquad \text{(B.50)}$$

But $\lambda_{\max}\left[\boldsymbol{H}^{-1}\left(\boldsymbol{\theta}_0\right)\right] = 1/\lambda_{\min}\left[\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)\right]$, and under the identification conditions established in Section

3.2, we have $\lambda_{\min}\left[\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)\right] > 0$, which ensures that $\lambda_{\max}\left[\boldsymbol{H}^{-1}\left(\boldsymbol{\theta}_0\right)\right] < K$ is bounded in N. Finally, we note that by Theorem 5.6.9 of Horn and Johnson (1985),

$$\lambda_{\max}\left[\mathbf{J}\left(\boldsymbol{\theta}_0,\gamma\right)\right] \leq \left\|\mathbf{J}\left(\boldsymbol{\theta}_0,\gamma\right)\right\|_{\infty}, \tag{B.51}$$

and using (19) it is easily seen that the column (row) norm of $\mathbf{J}\left(\boldsymbol{\theta}_0,\gamma\right)$ is dominated by matrices $\left(\boldsymbol{G}_0 \odot \boldsymbol{G}_0'\right)$, $\boldsymbol{\Delta}_{\boldsymbol{\beta}_0}$ and $\mathbb{E}_{\boldsymbol{\beta}_0}$, where the latter two matrices are diagonal. The other matrices in $\mathbf{J}\left(\boldsymbol{\theta}_0,\gamma\right)$, namely $\boldsymbol{\Sigma}_0$ and $\mathbf{Z}_0$, are also diagonal matrices whose elements do not vary with N. Consider $\boldsymbol{\Delta}_{\boldsymbol{\beta}_0}$ defined by (18) and note that

$$\sup_i\left(\sum_{s=1}^{N} g_{0,is}^2\sigma_{s0}^2\right) \leq \sup_s\left(\sigma_{s0}^2\right)\sup_i\left(\sum_{s=1}^{N}\left|g_{0,is}\right|\right) = \sup_s\left(\sigma_{s0}^2\right)\left\|\boldsymbol{G}_0\right\|_{\infty}.$$

Similarly,

$$\begin{aligned}
\sup_i\left|\sum_{r=1}^{N}\sum_{s=1}^{N} g_{0,is}g_{0,ir}\boldsymbol{\beta}_{r0}'\boldsymbol{\Sigma}_{rs}\boldsymbol{\beta}_{s0}\right| &\leq \sup_i\sum_{r=1}^{N}\sum_{s=1}^{N}\left|g_{0,is}\right|\left|g_{0,ir}\right|\left\|\boldsymbol{\beta}_{r0}'\boldsymbol{\Sigma}_{rs}\boldsymbol{\beta}_{s0}\right\| \\
&\leq \sup_{r,s}\left\|\boldsymbol{\beta}_{r0}'\boldsymbol{\Sigma}_{rs}\boldsymbol{\beta}_{s0}\right\|\sup_i\sum_{r=1}^{N}\sum_{s=1}^{N}\left|g_{0,is}\right|\left|g_{0,ir}\right| \\
&= \sup_s\left\|\boldsymbol{\beta}_{s0}\right\|\sup_r\left\|\boldsymbol{\beta}_{r0}\right\|\sup_{r,s}\left\|\boldsymbol{\Sigma}_{rs}\right\|\left\|\boldsymbol{G}_0\right\|_{\infty}^2.
\end{aligned}$$

However, under Assumptions 2(b) and 3 we have $\sup_s\left\|\boldsymbol{\beta}_{s0}\right\| < K$ and $\sup_{r,s}\left\|\boldsymbol{\Sigma}_{rs}\right\| < K$, and under Assumption 4(b) it follows that $\left\|\boldsymbol{G}_0\right\|_{\infty} < K$ (see Lemma 2). Hence, $\left\|\boldsymbol{\Delta}_{\boldsymbol{\beta}_0}\right\| < K$. Similarly, it is also easily established that all elements of $\mathbb{E}_{\boldsymbol{\beta}_0}$ are bounded in N. Finally, again under Assumption 4 and as shown in Lemma 2, $\left\|\boldsymbol{G}_0 \odot \boldsymbol{G}_0'\right\|_{\infty} < K$. Consequently, $\left\|\mathbf{J}\left(\boldsymbol{\theta}_0,\gamma\right)\right\|_{\infty} < K$, and in view of (B.51) it follows that $\lambda_{\max}\left[\mathbf{J}\left(\boldsymbol{\theta}_0,\gamma\right)\right] < K$. Using this result in (B.50) and recalling that $\lambda_{\max}\left[\boldsymbol{H}^{-1}\left(\boldsymbol{\theta}_0\right)\right] < K$, then overall we have $\lambda_{\max}\left(\mathbf{V}_{\boldsymbol{\theta}}\right) < K$, as required. Note that this result does not need the exogenous regressors to be weakly cross-correlated; it is sufficient that $\sup_{r,s}\left\|\boldsymbol{\Sigma}_{rs}\right\| < K$. ■

Proof of Corollary 1. Consider the information matrix $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)$ given by (17). We partition $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)$ as follows:

$$\boldsymbol{H}\left(\boldsymbol{\theta}_0\right) = \begin{pmatrix} \boldsymbol{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{12}' & \mathcal{H}_{22} \end{pmatrix},$$

where $\mathcal{H}_{12} = \left(\boldsymbol{H}_{12},\boldsymbol{H}_{13}\right)$ is an $N \times (Nk+N)$ matrix, and since $\boldsymbol{H}_{23} = \boldsymbol{H}_{32} = \mathbf{0}$, then $\mathcal{H}_{22} = \text{Diag}\left(\boldsymbol{H}_{22},\boldsymbol{H}_{33}\right)$, which is an $(Nk+N)\times(Nk+N)$ matrix. Then, the inverse of $\boldsymbol{H}\left(\boldsymbol{\theta}_0\right)$ is given by

$$\boldsymbol{H}^{-1}\left(\boldsymbol{\theta}_0\right) = \begin{pmatrix} \mathcal{H}_{11\cdot 2}^{-1} & -\mathcal{H}_{11\cdot 2}^{-1}\mathcal{H}_{12}\mathcal{H}_{22}^{-1} \\ -\mathcal{H}_{22}^{-1}\mathcal{H}_{21}\mathcal{H}_{11\cdot 2}^{-1} & \mathcal{H}_{22}^{-1} + \mathcal{H}_{22}^{-1}\mathcal{H}_{21}\mathcal{H}_{11\cdot 2}^{-1}\mathcal{H}_{12}\mathcal{H}_{22}^{-1} \end{pmatrix}.$$

We focus on the $N \times N$ information matrix $\mathcal{H}_{11\cdot 2}$ which is written as

$$\begin{aligned}
\mathcal{H}_{11\cdot 2} = \boldsymbol{H}_{11} - \mathcal{H}_{12}\mathcal{H}_{22}^{-1}\mathcal{H}_{21} &= \boldsymbol{H}_{11} - \boldsymbol{H}_{12}\boldsymbol{H}_{22}^{-1}\boldsymbol{H}_{21} - \boldsymbol{H}_{13}\boldsymbol{H}_{33}^{-1}\boldsymbol{H}_{31} \\
&= \left(\mathbf{G}_0 \odot \mathbf{G}_0'\right) + \text{Diag}\left[-g_{0,ii}^2 + \sum_{s=1,s\neq i}^{N}\left(\sigma_{s0}^2/\sigma_{i0}^2\right)g_{0,is}^2,\ i = 1,2,\ldots,N\right] \\
&+ \text{Diag}\left[\sigma_{i0}^{-2}\sum_{r=1}^{N}\sum_{s=1}^{N} g_{0,is}g_{0,ir}\boldsymbol{\beta}_{r0}'\left(\boldsymbol{\Sigma}_{rs} - \boldsymbol{\Sigma}_{ri}\boldsymbol{\Sigma}_{ii}^{-1}\boldsymbol{\Sigma}_{is}\right)\boldsymbol{\beta}_{s0},\ i = 1,2,\ldots,N\right].
\end{aligned}$$

Then, by Assumptions 1, 2, 4, and 5, the matrix $\mathcal{H}_{11\cdot 2}$ is full rank, where $\mathbf{G}_0 = \boldsymbol{W}\left(\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W}\right)^{-1} = \left(g_{0,ij}\right)$, $\boldsymbol{\Psi}_0 = \text{Diag}(\boldsymbol{\psi}_0)$, $\boldsymbol{\psi}_0 = (\psi_{10},\psi_{20},\ldots,\psi_{N0})'$, and $\boldsymbol{W}$ is the spatial weight matrix, and $\varepsilon_{it} \sim IIDN(0,\sigma_{i0}^2)$. Hence, the maximum likelihood estimator of $\boldsymbol{\psi}_0$, denoted by $\hat{\boldsymbol{\psi}}$ and computed by maximising (A.21), is asymptotically normally distributed with asymptotic covariance matrix $\mathbf{V}_{\psi} = [\mathcal{H}_{11 \cdot 2}]^{-1}$. ■

Appendix C Consistent estimation of $V\left(\hat{\boldsymbol{\theta}}\right)$

Derivatives of the log-likelihood function

The vector of maximum likelihood estimates, $\hat{\boldsymbol{\theta}}_T$, in Section 2 is obtained by maximising the log-likelihood function (A.21) which we reproduce here for convenience[20]

$$\ell_T(\boldsymbol{\theta}) = -\frac{NT}{2}\ln(2\pi) - \frac{T}{2}\sum_{i=1}^{N}\ln\sigma_i^2 + T\ln|\boldsymbol{I}_N - \boldsymbol{\Psi}\boldsymbol{W}| - \frac{1}{2}\sum_{i=1}^{N}\frac{(\boldsymbol{y}_{i\circ} - \psi_i\boldsymbol{y}^*_{i\circ} - \mathbf{X}_{i\circ}\boldsymbol{\beta}_i)'(\boldsymbol{y}_{i\circ} - \psi_i\boldsymbol{y}^*_{i\circ} - \mathbf{X}_{i\circ}\boldsymbol{\beta}_i)}{\sigma_i^2}, \tag{C.52}$$

where $\boldsymbol{\theta} = (\boldsymbol{\psi}', \boldsymbol{\beta}', \boldsymbol{\sigma}^{2\prime})'$.

First derivatives

We have

$$\frac{\partial\ell_T(\boldsymbol{\theta})}{\partial\psi_i} = -T\,\mathrm{tr}[(\boldsymbol{I}_N - \boldsymbol{\Psi}\boldsymbol{W})^{-1}\boldsymbol{E}_{ii}\boldsymbol{W}] + \frac{\boldsymbol{y}^{*\prime}_{i\circ}(\boldsymbol{y}_{i\circ} - \psi_i\boldsymbol{y}^*_{i\circ} - \mathbf{X}_{i\circ}\boldsymbol{\beta}_i)}{\sigma_i^2}, \text{ for } i = 1, 2, \ldots, N,$$

$$\frac{\partial\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}_i} = \frac{\mathbf{X}'_{i\circ}(\boldsymbol{y}_{i\circ} - \psi_i\boldsymbol{y}^*_{i\circ} - \mathbf{X}_{i\circ}\boldsymbol{\beta}_i)}{\sigma_i^2}, \text{ for } i = 1, 2, \ldots, N,$$

$$\frac{\partial\ell_T(\boldsymbol{\theta})}{\partial\sigma_i^2} = -\frac{T}{2\sigma_i^2} + \frac{1}{2\sigma_i^4}(\boldsymbol{y}_{i\circ} - \psi_i\boldsymbol{y}^*_{i\circ} - \mathbf{X}_{i\circ}\boldsymbol{\beta}_i)'(\boldsymbol{y}_{i\circ} - \psi_i\boldsymbol{y}^*_{i\circ} - \mathbf{X}_{i\circ}\boldsymbol{\beta}_i), \text{ for } i = 1, 2, \ldots, N,$$

where $\boldsymbol{E}_{ii}$ is the $N \times N$ matrix whose (i, i) element is 1 and zero elsewhere.

Second derivatives

We have

$$\hat{\boldsymbol{H}}(\boldsymbol{\theta}) = -\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'} = \begin{pmatrix} \hat{\boldsymbol{H}}_{11} & \hat{\boldsymbol{H}}_{12} & \hat{\boldsymbol{H}}_{13} \\ . & \hat{\boldsymbol{H}}_{22} & \hat{\boldsymbol{H}}_{23} \\ . & . & \hat{\boldsymbol{H}}_{33} \end{pmatrix},$$

$$= \begin{pmatrix} -\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'} & -\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\beta}'} & -\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\sigma}^{2\prime}} \\ . & -\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} & -\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\sigma}^{2\prime}} \\ . & . & -\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial(\boldsymbol{\sigma}^2)\partial(\boldsymbol{\sigma}^{2\prime})} \end{pmatrix}.$$

With the (i, j) or i^{th} element of associated matrix or vector given in $\{\}$, we have

$$\hat{\boldsymbol{H}}_{11} = -\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\psi}\partial\boldsymbol{\psi}'} = \left\{-\frac{1}{T}\frac{\partial^2\ell_{NT}(\boldsymbol{\theta})}{\partial\psi_i\partial\psi_j}\right\},$$

$$-\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\psi_i\partial\psi_j} = \begin{cases} \mathrm{tr}\left[(\boldsymbol{I} - \boldsymbol{\Psi}\boldsymbol{W})^{-1}\boldsymbol{E}_{ii}\boldsymbol{W}(\boldsymbol{I} - \boldsymbol{\Psi}\boldsymbol{W})^{-1}\boldsymbol{E}_{ii}\boldsymbol{W}\right] + \frac{1}{\sigma_i^2}\frac{\boldsymbol{y}^{*\prime}_{i\circ}\boldsymbol{y}^*_{i\circ}}{T} & \text{if } i = j \\ \mathrm{tr}\left[(\boldsymbol{I} - \boldsymbol{\Psi}\boldsymbol{W})^{-1}\boldsymbol{E}_{jj}\boldsymbol{W}(\boldsymbol{I} - \boldsymbol{\Psi}\boldsymbol{W})^{-1}\boldsymbol{E}_{ii}\boldsymbol{W}\right] & \text{if } i \neq j \end{cases},$$

$$-\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\psi_i\partial\psi_j} = \begin{cases} g_{ii}^2 + \frac{1}{\sigma_i^2}\frac{\boldsymbol{y}^{*\prime}_{i\circ}\boldsymbol{y}^*_{i\circ}}{T} & \text{if } i = j \\ g_{ij}g_{ji} & \text{if } i \neq j \end{cases},$$

[20] Note that $\ln|\boldsymbol{S}'(\psi)\mathbf{S}(\psi)| = 2\ln|\boldsymbol{I}_N - \boldsymbol{\Psi}\boldsymbol{W}|$.

where $\boldsymbol{E}_{ij}$ is the $N\times N$ matrix whose (i,j) element is 1 and zero elsewhere and $\mathbf{G}=(g_{ij})=\boldsymbol{W}\left(\boldsymbol{I}_N-\boldsymbol{\Psi}\boldsymbol{W}\right)^{-1}$. Further,

$$\hat{\boldsymbol{H}}_{12}=-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\psi}\partial\boldsymbol{\beta}'}=\left\{-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\psi_i\partial\boldsymbol{\beta}_j'}\right\}=\left\{\frac{1}{\sigma_i^2}\frac{\boldsymbol{y}_{i\circ}^{*\prime}\boldsymbol{X}_{i\circ}}{T}\text{, if } i=j\text{, and }\mathbf{0}\text{, if } i\neq j\right\},$$

$$\hat{\boldsymbol{H}}_{13}=-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\psi}\partial\boldsymbol{\sigma}^{2\prime}}=\left\{-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\psi_i\partial\sigma_j^2}\right\}=\left\{\frac{1}{\sigma_i^4}\frac{\boldsymbol{y}_{i\circ}^{*\prime}\left(\boldsymbol{y}_{i\circ}-\psi_i\boldsymbol{y}_{i\circ}^*-\mathbf{X}_{i\circ}\boldsymbol{\beta}_i\right)}{T}\text{, if } i=j\text{, and }0\text{, if } i\neq j\right\},$$

$$\hat{\boldsymbol{H}}_{22}=-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}=\left\{-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\beta}_i\partial\boldsymbol{\beta}_j'}\right\}=\left\{\frac{1}{\sigma_i^2}\frac{\boldsymbol{X}_{i\circ}'\boldsymbol{X}_{i\circ}}{T}\text{, if } i=j\text{, and }\mathbf{0}\text{, if } i\neq j\right\},$$

$$\hat{\boldsymbol{H}}_{23}=-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\beta}\partial\boldsymbol{\sigma}^{2\prime}}=\left\{-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\beta}_i\partial\sigma_j^2}\right\}=\left\{\frac{1}{\sigma_i^4}\frac{\mathbf{X}_{i\circ}'(\boldsymbol{y}_{i\circ}-\psi_i\boldsymbol{y}_{i\circ}^*-\boldsymbol{X}_{i\circ}\boldsymbol{\beta}_i)}{T}\text{, if } i=j\text{, and }\mathbf{0}\text{, if } i\neq j\right\},$$

$$\begin{aligned}\hat{\boldsymbol{H}}_{33}&=-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\left(\boldsymbol{\sigma}^2\right)\partial\left(\boldsymbol{\sigma}^{2\prime}\right)}=\left\{-\frac{1}{T}\frac{\partial^2\ell_T\left(\boldsymbol{\theta}\right)}{\partial\sigma_i^2\partial\sigma_j^2}\right\}\\&=\left\{-\frac{1}{2\sigma_i^4}+\frac{1}{\sigma_i^6}\frac{1}{T}(\boldsymbol{y}_{i\circ}-\psi_i\boldsymbol{y}_{i\circ}^*-\mathbf{X}_{i\circ}\boldsymbol{\beta}_i)'(\boldsymbol{y}_{i\circ}-\psi_i\boldsymbol{y}_{i\circ}^*-\mathbf{X}_{i\circ}\boldsymbol{\beta}_i),\ \text{if } i=j\text{, and }0\text{, if } i\neq j\right\}.\end{aligned}$$

Finally, from the above results we obtain:

$$\hat{\boldsymbol{J}}\left(\boldsymbol{\theta}\right)=\frac{1}{T}\left\{\left[\sum_{t=1}^{T}\frac{\partial\ell_t\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\theta}}\right]\left[\sum_{t=1}^{T}\frac{\partial\ell_t\left(\boldsymbol{\theta}\right)}{\partial\boldsymbol{\theta}}\right]'\right\}\text{ and }\hat{\boldsymbol{H}}\left(\boldsymbol{\theta}\right)=-\frac{1}{T}\frac{\partial^2\ell_T(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'},$$

from which the standard and sandwich covariance matrix estimators in (24) are given by

$$\hat{\boldsymbol{V}}_{\hat{\theta}}=\hat{\boldsymbol{H}}^{-1}\left(\hat{\boldsymbol{\theta}}\right)\text{ and }\hat{\boldsymbol{V}}_{\hat{\theta}}=\hat{\boldsymbol{H}}^{-1}\left(\hat{\boldsymbol{\theta}}\right)\hat{\boldsymbol{J}}\left(\hat{\boldsymbol{\theta}}\right)\hat{\boldsymbol{H}}^{-1}\left(\hat{\boldsymbol{\theta}}\right).$$

Appendix D HSAR model with non-zero diagonal weights

Consider the following HSAR model:

$$y_{it}=\psi_{i0}\left(\sum_{j=1}^{N}w_{ij}y_{jt}\right)+\boldsymbol{\beta}_{i0}'\boldsymbol{x}_{it}+\varepsilon_{it},\ \text{for } i=1,2,\ldots,N;\ t=1,2,\ldots,T,\tag{D.53}$$

where $w_{ii}\neq 0$, $\sum_{j=1}^{N}w_{ij}=v_i$, v_i are known constants, and $Var\left(\varepsilon_{it}\right)=\sigma_{i0}^2$. This implies that $\boldsymbol{W}=(w_{ij})$ are non-standardised and need not have zero diagonal elements. Assume that $|v_i|<K$ and $\sum_{j=1}^{N}|w_{ij}|<K$. Then, (D.53) can be reparameterised such that:

$$y_{it}=\left(\frac{1}{1-\psi_{i0}w_{ii}}\right)\left[\psi_{i0}\left(\sum_{j\neq i}^{N}w_{ij}y_{jt}\right)+\boldsymbol{\beta}_{i0}'\boldsymbol{x}_{it}+\varepsilon_{it}\right].\tag{D.54}$$

Let

$$\dot{w}_{ij}=\begin{cases}v_i^{-1}w_{ij}\text{, if } i\neq j\\ 0\text{, if } i=j\end{cases},$$

and note that $\sum_{j\neq i}^{N} w_{ij}y_{jt} = v_i \sum_{j\neq=i}^{N} \dot{w}_{ij}y_{jt}$, such that $\sum_{j\neq=i}^{N} \dot{w}_{ij} = 1$ and $\dot{w}_{ii} = 0$. Then, (D.54) can be written equivalently as:

$$y_{it} = \dot{\psi}_{i0}\left(\sum_{j=1}^{N} \dot{w}_{ij}y_{jt}\right) + \dot{\boldsymbol{\beta}}_{i0}'\boldsymbol{x}_{it} + \dot{\varepsilon}_{it},\ \text{for } i = 1, 2, \ldots, N;\ t = 1, 2, \ldots, T, \tag{D.55}$$

where $\dot{\psi}_{i0} = \frac{\psi_{i0}v_i}{1-\psi_{i0}w_{ii}}$, $\dot{\boldsymbol{\beta}}_{i0} = \frac{\boldsymbol{\beta}_{i0}}{1-\psi_{i0}w_{ii}}$ and $Var(\dot{\varepsilon}_{it}) = \dot{\sigma}_{i0}^2 = \frac{\sigma_{i0}^2}{(1-\psi_{i0}w_{ii})^2}$, respectively. The parameters in (D.55) can be estimated using the QML approach developed in Section 3 of the main paper, and parameters in (D.53) can be recovered such that $\psi_{i0} = \frac{\dot{\psi}_{i0}}{v_i+\dot{\psi}_{i0}w_{ii}}$, $\boldsymbol{\beta}_{i0} = \dot{\boldsymbol{\beta}}_{i0}(1-\psi_{i0}w_{ii})$ and $\sigma_{i0}^2 = (1-\psi_{i0}w_{ii})^2\dot{\sigma}_{i0}^2$, respectively.

Appendix E Direct and indirect effects in HSAR model

Consider the theoretical HSAR model (2) of the main paper where the time dynamics are made explicit. This specification is used in our empirical application of Section 6:

$$y_{it} = a_i + \psi_{i0}\sum_{j=1}^{N} w_{ij}y_{jt} + \psi_{i1}\sum_{j=1}^{N} w_{ij}y_{j,t-1} + \lambda_i y_{i,t-1} + \boldsymbol{\beta}_i'\boldsymbol{x}_{it} + \varepsilon_{it},\ \text{for } i = 1, 2, \ldots, N;\ t = 1, 2, \ldots, T, \tag{E.56}$$

where $\boldsymbol{x}_{it} = (x_{i1,t}, x_{i2,t}, \ldots, x_{ik,t})'$ is a $k \times 1$ vector of exogenous regressors, with the associated $k \times 1$ vector of slope parameters, $\boldsymbol{\beta}_i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{ik})'$. We can re-write (E.56) as

$$\boldsymbol{y}_t = \boldsymbol{a} + \boldsymbol{\Psi}_0\boldsymbol{W}\boldsymbol{y}_t + \boldsymbol{\Psi}_1\boldsymbol{W}\boldsymbol{y}_{t-1} + \boldsymbol{\Lambda}\boldsymbol{y}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{\varepsilon}_t,\ \text{for } t = 1, 2, \ldots, T,$$

where $\boldsymbol{a} = (a_1, a_2, \ldots, a_N)'$, $\boldsymbol{\Psi}_0 = \text{Diag}(\boldsymbol{\psi}_0)$ with $\boldsymbol{\psi}_0 = (\psi_{10}, \psi_{20}, \ldots, \psi_{N0})'$, $\boldsymbol{\Psi}_1 = \text{Diag}(\boldsymbol{\psi}_1)$ with $\boldsymbol{\psi}_0 = (\psi_{11}, \psi_{21}, \ldots, \psi_{N1})'$, $\boldsymbol{\Lambda} = \text{Diag}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)'$ and $\boldsymbol{B}$ is the $N \times kN$ block diagonal matrix with elements $\boldsymbol{\beta}_i'$, $i = 1, 2, \ldots, N$, on the main diagonal and zeros elsewhere. Also, $\boldsymbol{y}_t = (y_{1t}, y_{2t}, \ldots, y_{Nt})'$ and $\boldsymbol{x}_t = (\boldsymbol{x}_{1t}', \boldsymbol{x}_{2t}', \ldots, \boldsymbol{x}_{Nt}')'$. Then, we have

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Phi}\boldsymbol{y}_{t-1} + \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{u}_t,\ \text{for } t = 1, 2, \ldots, T, \tag{E.57}$$

where $\boldsymbol{c} = (\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{a}$, $\boldsymbol{\Phi} = (\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}(\boldsymbol{\Psi}_1\boldsymbol{W} + \boldsymbol{\Lambda})$, $\boldsymbol{A} = (\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{B}$ and $\boldsymbol{u}_t = (\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{\varepsilon}_t$. Following LeSage and Chih (2016), we focus on

$$\frac{\partial y_{i,t+h}}{\partial x_{j\ell,t}},\ \text{for } i, j = 1, 2, \ldots, N;\ \ell = 1, 2, \ldots, k;\ h = 0, 1, 2, \ldots$$

where $x_{j\ell,t}$ is the ℓ^{th} regressor of the j^{th} unit, and $h = 0, 1, 2, \ldots$ represents the horizon relative to the partial change of $x_{j\ell,t}$ at $h = 0$. Using (E.57) and solving forward from time t, we have

$$\boldsymbol{y}_{t+h} = \boldsymbol{\Phi}^{h+1}\boldsymbol{y}_{t-1} + \left(\sum_{s=0}^{h-1}\boldsymbol{\Phi}^s\right)\boldsymbol{c} + \sum_{s=0}^{h-1}\boldsymbol{\Phi}^s\boldsymbol{A}\boldsymbol{x}_{t+h-s} + \sum_{s=0}^{h-1}\boldsymbol{\Phi}^s\boldsymbol{u}_{t+h-s}, \tag{E.58}$$

Therefore,

$$\frac{\partial \boldsymbol{y}_{t+h}}{\partial x_{j\ell,t}} = \left[\boldsymbol{\Phi}^h(\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{e}_j\right]\beta_{j\ell}, \tag{E.59}$$

where $\boldsymbol{e}_j$ is an $N \times 1$ vector of zeros except for its j^{th} element which is unity. The i^{th} element of (E.59) is given by

$$\frac{\partial y_{i,t+h}}{\partial x_{j\ell,t}} = \left[\boldsymbol{e}_i'\boldsymbol{\Phi}^h(\boldsymbol{I}_N - \boldsymbol{\Psi}_0\boldsymbol{W})^{-1}\boldsymbol{e}_j\right]\beta_{j\ell}, \tag{E.60}$$

where $\boldsymbol{e}_i$ is an $N \times 1$ vector of zeros except for its i^{th} element which is unity. The direct effects are given by setting $i = j$, while the indirect effects focus on the off-diagonal elements of the partial derivatives matrix for each $h = 0, 1, \ldots$, such that $i \neq j$. The latter can be differentiated into spill-in effects (cumulative row sums) and spill-out effects (cumulative column sums).

As summary measures, we define the average direct effects to be

$$D_N(h, \ell) = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial y_{i,t+h}}{\partial x_{i\ell,t}}, \text{ for } \ell = 1, 2, \ldots, k; \; h = 0, 1, \ldots. \tag{E.61}$$

In terms of indirect effects, averaging over spill-in and spill-out effects produces the same result and can be used interchangeably:

$$ID_N(h, \ell) = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} \frac{\partial y_{i,t+h}}{\partial x_{j\ell,t}}, \text{ for } \ell = 1, 2, \ldots, k; \; h = 0, 1, \ldots. \tag{E.62}$$

The same computations can be used to evaluate average direct and indirect effects at regional level. In this case, the indirect effects can be further decomposed into within-region and between-region indirect effects. Though the within-region average spill-in and spill-out effects again produce the same result, the between-region average spill-in and spill-out effects may differ. More specifically, suppose that there are N_r MSAs in region r $(= 1, 2, \ldots, R)$, then we have

$$D_{N_r}(h, \ell) = \frac{1}{N_r} \sum_{i=1}^{N_r} \frac{\partial y^r_{i,t+h}}{\partial x^r_{i\ell,t}}, \tag{E.63}$$

$$ID_{W,N_r}(h, \ell) = \frac{1}{N_r(N_r-1)} \sum_{\substack{i=1 \\ i \neq j}}^{N_r} \sum_{j=1}^{N_r} \frac{\partial y^r_{i,t+h}}{\partial x^r_{j\ell,t}}, \tag{E.64}$$

$$ID_{B_{in},N_r}(h, \ell) = \frac{1}{N_r(N-N_r)} \sum_{i=1}^{N_r} \sum_{\substack{j=1 \\ i \neq j}}^{N-N_r} \frac{\partial y^r_{i,t+h}}{\partial x^{\notin r}_{j\ell,t}}, \tag{E.65}$$

and

$$ID_{B_{out},N_r}(h, \ell) = \frac{1}{N_r(N-N_r)} \sum_{\substack{i=1 \\ i \neq j}}^{N-N_r} \sum_{j=1}^{N_r} \frac{\partial y^{\notin r}_{i,t+h}}{\partial x^r_{j\ell,t}}, \tag{E.66}$$

for $\ell = 1, 2, \ldots, k$; $r = 1, 2, \ldots R$; $h = 0, 1, \ldots$, where ID_{W,N_r}, ID_{W,N_r} and ID_{B_{out},N_r} stand for within-region indirect, between-region spill-in and between-region spill-out effects, respectively. A further refinement entails the units that do not belong to region r, $x^{\notin r}_{ik,t}$, to be further categorised into the remainder regions.

Appendix F Data sources, transformations and additional empirical results

Data sources

Monthly data for U.S. house prices over the period January 1975 to December 2014 are obtained from the Freddie Mac House Price Index (FMHPI). These data are available at:
http://www.freddiemac.com/research/.

Annual data on nominal income per capita and population at MSA level are acquired from the Bureau of Economic Analysis website for the same period. These data are available at: https://www.bea.gov/data/.

Annual State level Consumer Price Index data are obtained from the Bureau of Labour Statistics: https://www.bls.gov/cpi/. These are matched to the corresponding MSAs. In some cases where area data are missing then the U.S. average CPI is used instead.

Data transformations

There exist 381 metropolitan statistical areas (MSAs) which fall under the February 2013 definition provided by the U.S. Office of Management and Budget (OMB), of which 337 are located in the contiguous United States.[21,22] Accordingly, we compile quarterly nominal house prices (HP) for these 377 MSAs over the period 1975Q1-2014Q4. In addition, we obtain nominal income per capita (INC) and population (POP) at the MSA level over the same period. Both real house prices and real per capita income for all MSAs are then computed by deflating their nominal values by State level Consumer Price Index data (CPI) which are matched to the corresponding MSAs.[23]

The corresponding percent quarterly rates of change in real house prices (Π_{it}), population ($GPOP_{it}$) and real per capita income ($GINC_{it}$) of MSA i in quarter t are computed as follows:

$$\begin{aligned}
\Pi_{it} &= 100 \times \left[\ln\left(\frac{HP_{it}}{CPI_{it}}\right) - \ln\left(\frac{HP_{i,t-1}}{CPI_{i,t-1}}\right)\right], \\
GPOP_{it} &= 100 \times \left[\ln\left(POP_{it}\right) - \ln\left(POP_{it-1}\right)\right], \\
GINC_{it} &= 100 \times \left[\ln\left(\frac{INC_{it}}{CPI_{it}}\right) - \ln\left(\frac{INC_{i,t-1}}{CPI_{i,t-1}}\right)\right],
\end{aligned}$$

for $i = 1, 2, \ldots, N$, and $t = 1975Q1 - 2014Q4$. In total $N = 377$ MSAs and $T = 160$ quarters.

Further, we de-seasonalise and de-factor the three variables that we use to estimate the HSAR specifications, and use residuals from OLS regressions of Π_{it}, $GPOP_{it}$ and $GINC_{it}$ on: (i) an intercept, (ii) 3 quarterly dummies and (iii) national, regional and local cross-sectional averages of Π_{it}, $GPOP_{it}$ and $GINC_{it}$ respectively, in line with Yang (2020).[24]

Additional empirical results

All individual spatial and slope coefficient estimates with their standard errors from model (30) are available upon request.

Figures F1(a) and F1(b) show the individual contemporaneous ($\hat{\psi}_{0i}$) and lagged ($\hat{\psi}_{1i}$) spatial coefficient estimates for the $N = 338$ MSAs in our sample, which form the net spatial coefficient estimates ($\hat{\psi}_{0i}+\hat{\psi}_{1i}$) depicted in Figure 1(a) of Section 6.

Figures F2, F3 and F4 display the direct, spill-in and spill-out effects on individual MSA house price changes from a one percent change in population and real income growth at different time horizons, $h = 0, 3, 6$. These are produced from empirical model (30) of Section 6. The equivalent average regional metrics, namely the within-region direct and indirect effects, are shown in Table F2. The between-region averages have been omitted as they are negligible by comparison and account for less that 0.3% of direct effects in any one instance.

[21]The February 2013 delineation states that 'metropolitan statistical areas have at least one urbanised area of 50,000 or more population, plus adjacent territory that has a high degree of social and economic integration with the core as measured by commuting ties'. For further details see:
https://www.whitehouse.gov/sites/whitehouse.gov/files/omb/bulletins/2013/b13-01.pdf

[22]This excludes the non-contiguous states of Alaska (2 MSAs) and Hawaii (2 MSAs) and all other off-shore insular areas.

[23]The quarterly figures for nominal house prices (HP) are arithmetic averages of monthly observations of HP. Further, per capita income (INC), population (POP) and consumer price index (CPI) are annual data which are converted into quarterly observations by following the interpolation method provided in the GVAR Toolbox User Guide which can be found at: https://sites.google.com/site/gvarmodelling/gvar-toolbox.

[24]We partition the MSAs into $R = 8$ regions, following the Bureau of Economic Analysis classification, each region $r = 1, 2, \ldots, R$, containing a total of N_r MSAs. The eight regions are: New England (15 MSAs), Mid East (41 MSAs), South East (120 MSAs), Great Lakes (59 MSAs), Plains (33 MSAs), South West (39 MSAs), Rocky Mountains (22 MSAs) and Far West (48 MSAs).

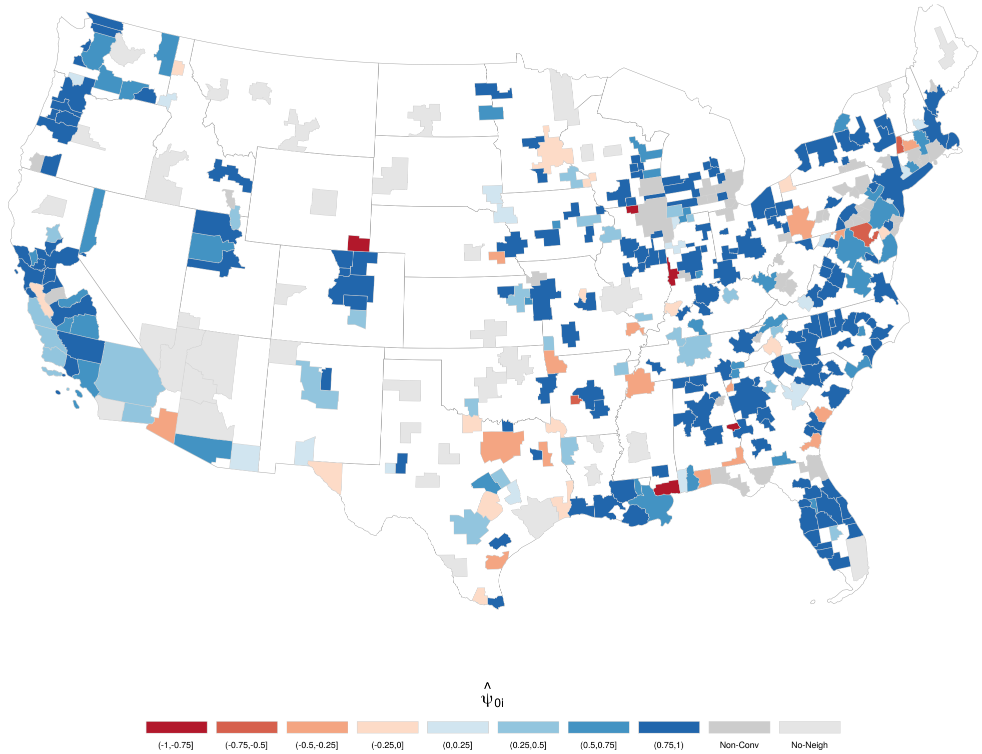
Table F1 displays the population densities across six major regions in the U.S.. These are computed by aggregating MSA-level population and area estimates for the $N = 338$ MSAs in our sample.

Finally, Table F3 replicates the MGE regional and national results of Table 3 in the main paper when using as un-normalised sparse weights matrix, $\boldsymbol{P} = (p_{ij})$, where

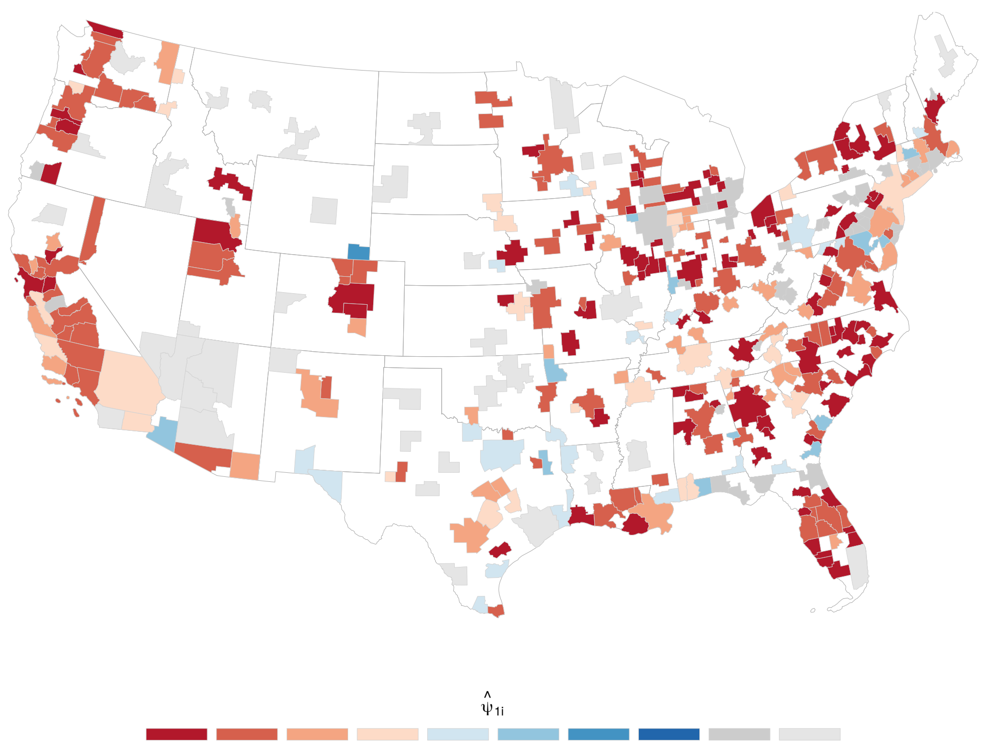
$$p_{ij}\left(d_{ij}\right) = \begin{cases} \frac{1}{d_{ij}}, \text{ if } d_{ij} \leq 75 \text{ miles} \\ 0, \qquad\qquad \text{otherwise} \end{cases}, \tag{F.67}$$

and d_{ij} stands for geodesic distance between MSAs $i, j = 1, 2, \ldots, N$. Geodesic distances are computed by applying the Haversine formula. $\boldsymbol{W} = (w_{ij})$ then becomes the row-normalised version of $\boldsymbol{P}$, namely $w_{ij} = p_{ij}/\sum_{r=1}^{N} p_{ir}$. We use the same 75 mile radius for distinguishing between neighbours and non-neighbours as in the main paper. Results are qualitatively very similar to those in Table 3. Similar results are also obtained when assuming a dense inverse distance matrix such that $\mathbf{P} = (p_{ij}) = \left(1/d_{ij}^{\delta}\right)$, for $i, j = 1, 2, \ldots, N$ and $\delta = 6$ in (F.67). These results are available upon request.

Figure F1: Contemporaneous ($\hat{\psi}_{0i}$) and lagged ($\hat{\psi}_{1i}$) spatial autoregressive parameter estimates for Metropolitan Statistical Areas in the Unites States



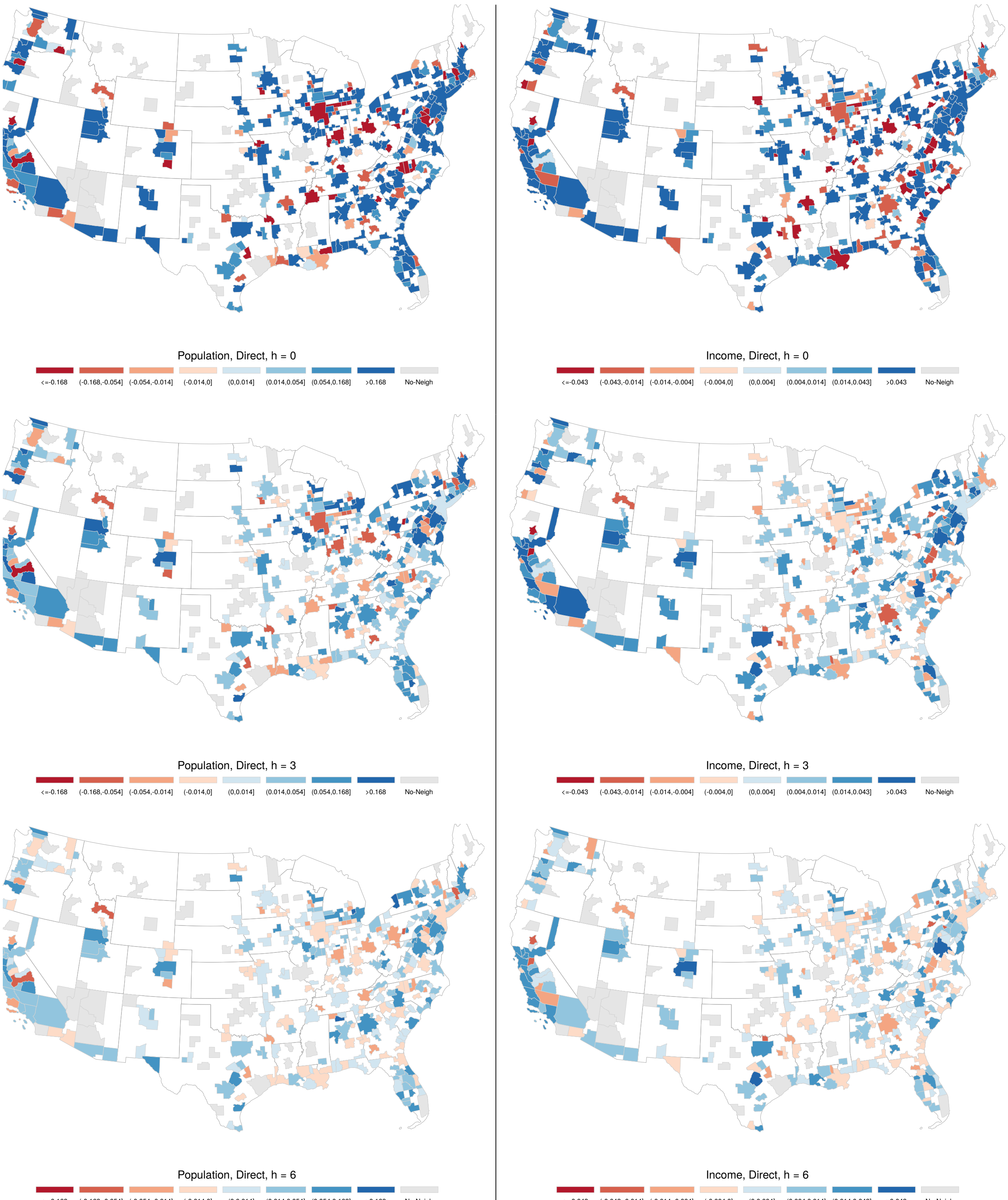
(a) Contemporaneous spatial parameter estimates



(b) Lagged spatial parameter estimates

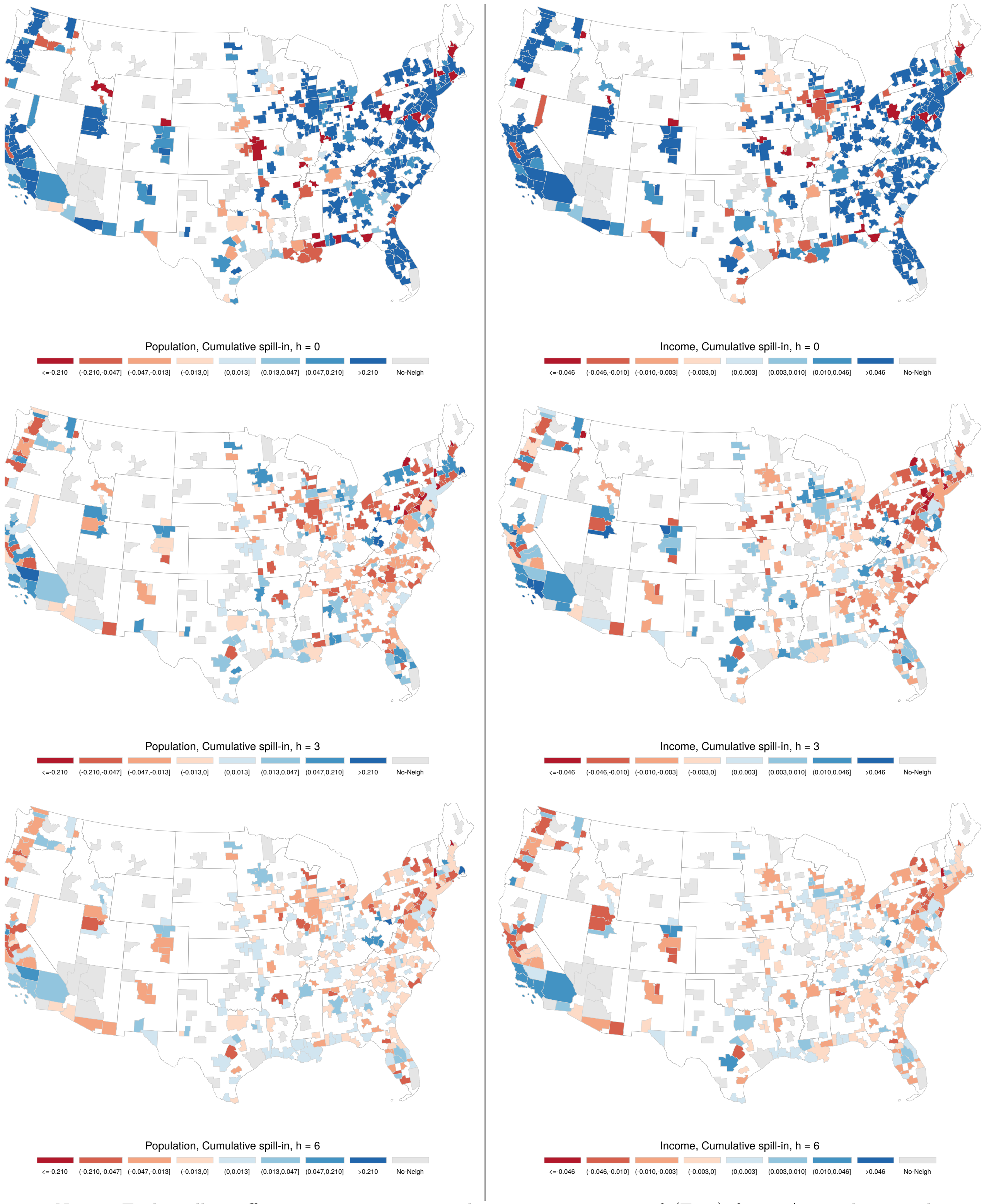
Notes: Each $\hat{\psi}_{0i}$ and $\hat{\psi}_{1i}$ is mapped to a Metropolitan Statistial Area (MSA) in the U.S.. A total of 338 MSAs are included in model (30). MSAs coloured in blue correspond to positive contemporaneous and lagged spatial parameter estimates while MSAs coloured in red match to negative contemporaneous and lagged spatial parameter estimates. Darker shades of blue or red indicate more sizable $\hat{\psi}_{0i}$ and $\hat{\psi}_{1i}$ while lighter shades related to $\hat{\psi}_{0i}$ and $\hat{\psi}_{1i}$ closer to zero in absolute terms. Category 'Non-conv' includes MSAs whose $\hat{\psi}_{0i}$, $\hat{\psi}_{1i}$ or $\hat{\lambda}_i$ estimates hit the upper/lower bound in the optimization procedure, while category 'No-Neigh' includes MSAs that have no neighbours when using $\boldsymbol{W}_{75}$.

Figure F2: Direct effects for Metropolitan Statistical Areas in the Unites States at time horizons $h = 0, 3, 6$ quarters after a one percent change to own MSA population or real income growth at time t.



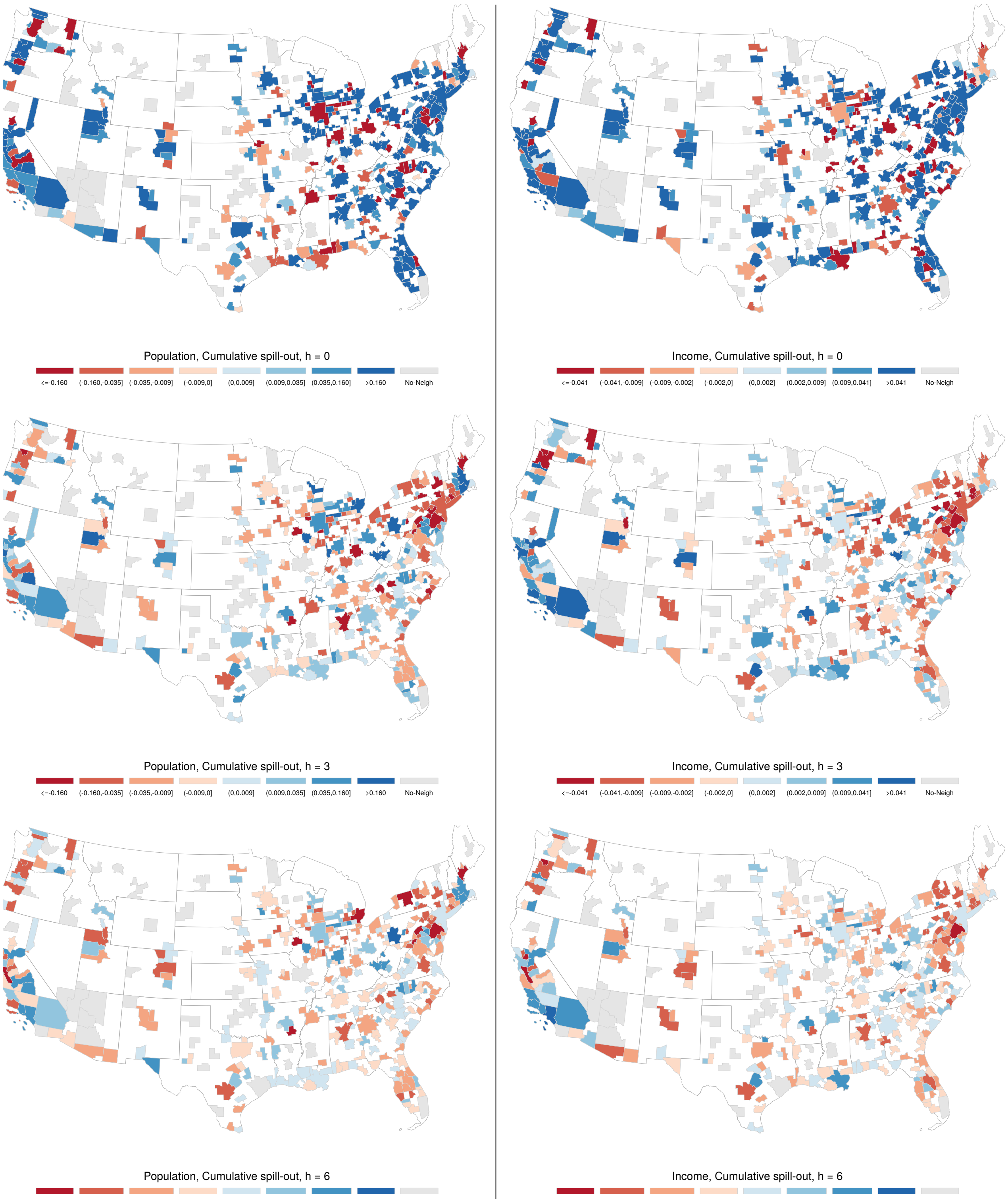
Notes: Each direct effect estimate is computed using (E.60) for $i = j$ and mapped to a Metropolitan Statistical Area (MSA) in the U.S.. A total of 338 MSAs are included in model (30). MSAs coloured in blue correspond to positive direct effect estimates while MSAs coloured in red match to negative direct effect estimates. Darker shades of blue or red indicate more sizable effects while lighter shades relate to effects closer to zero in absolute terms. Category 'No-Neigh' includes MSAs that have no neighbours when using $\boldsymbol{W}_{75}$.

Figure F3: Spill-in effects for Metropolitan Statistical Areas in the Unites States at time horizons $h = 0, 3, 6$ quarters after a one percent change to neighbouring population or real income growth at time t.



Notes: Each spill-in effect estimate is computed using sums over j of (E.60) for $i \neq j$ and mapped to a Metropolitan Statistical Area (MSA) in the U.S.. A total of 338 MSAs are included in model (30). MSAs coloured in blue correspond to positive spill-in effect estimates while MSAs coloured in red match to negative spill-in effect estimates. Darker shades of blue or red indicate more sizable effects while lighter shades relate to effects closer to zero in absolute terms. Category 'No-Neigh' includes MSAs that have no neighbours when using $\boldsymbol{W}_{75}$.

Figure F4: Spill-out effects for Metropolitan Statistical Areas in the Unites States at time horizons $h = 0, 3, 6$ quarters after a one percent change to own MSA population or real income growth at time t.



Notes: Each spill-out effect estimate is computed using sums over i of (E.60) for $i \neq j$ and mapped to a Metropolitan Statistical Area (MSA) in the U.S.. A total of 338 MSAs are included in model (30). MSAs coloured in blue correspond to positive spill-out effect estimates while MSAs coloured in red match to negative spill-out effect estimates. Darker shades of blue or red indicate more sizable effects while lighter shades relate to effects closer to zero in absolute terms. Category 'No-Neigh' includes MSAs that have no neighbours when using $\boldsymbol{W}_{75}$.

Table F1: Population, area and population density estimates by six major U.S. regions

r	Name	N_r	Population	Area (sqr miles)	Density (per sqr mile)
1 & 2	New England & Mideast	54	55275714	83111	665.1
3	Great Lakes	57	34050860	77400	439.9
4	Plains	27	9320365	64266	145.0
5	Southeast	116	42770914	215211	198.7
6 & 7	Southwest & Rocky Mountain	41	18339459	145292	126.2
8	Far West	43	35590563	145666	244.3

Notes: I_r is the set of units belonging to region r, $I_r = \{i\colon i \text{ is in region } r\}$, and N_r is the number of units per region, $N_r = \#(I_r)$. New England (13 MSAs) and Mid East (41 MSAs) as well as South West (26 MSAs) and Rocky Mountains (15 MSAs) have been merged in order to obtain a sufficiently large number of MSAs in the two broader regions. MSA-level population estimates are averages over the sample period 1975Q1-2014Q4. MSA-level area estimates are obtained from the US Census Bureau. Region-level population and area estimates are sums over the number of MSAs per region r (N_r). The summary statistics are based on all MSAs included in our analysis ($N = 338$) which excludes 39 MSAs that are completely isolated (have no neighbours).

Table F2: Average within-region partial effects on house price changes at horizons $h = 0, 3, 6$ quarters following a one percent change in population or real income growth by six major U.S. regions

			Within-region effects					
			Direct (in levels)			Indirect (as % of Direct)		
r	Name	$N_r \backslash h$	0	3	6	0	3	6
			Population					
1 & 2	New England & Mideast	35	0.558	0.112	0.035	2.17	-1.42	-2.33
3	Great Lakes	48	0.232	0.042	0.017	3.66	-0.22	-1.13
4	Plains	26	0.267	0.034	0.012	2.57	-1.71	-2.17
5	Southeast	106	0.224	0.041	0.016	1.51	-0.04	-0.32
6 & 7	Southwest & Rocky Mountain	40	0.176	0.044	0.015	1.76	0.63	-0.67
8	Far West	41	0.223	0.061	0.023	4.70	0.98	-1.24
			Income					
1 & 2	New England & Mideast	35	0.096	0.021	0.007	2.39	-1.67	-2.21
3	Great Lakes	48	0.035	0.007	0.003	3.28	-0.02	-0.81
4	Plains	26	0.046	0.007	0.003	4.44	-1.71	-2.59
5	Southeast	106	0.045	0.008	0.003	1.64	-0.06	-0.33
6 & 7	Southwest & Rocky Mountain	40	0.082	0.020	0.008	1.83	0.51	-0.75
8	Far West	41	0.083	0.022	0.008	4.12	1.27	-0.08

Notes: I_r is the set of units belonging to region r, $I_r = \{i\colon i \text{ is in region } r\}$, and N_r is the number of units per region, $N_r = \#(I_r)$. New England (9 MSAs) and Mid East (26 MSAs) as well as South West (26 MSAs) and Rocky Mountains (14 MSAs) have been merged in order to obtain a sufficiently large number of MSAs in the two broader regions. The average within-region direct and indirect effects are computed using equations (E.63) and (E.64), respectively. $h = 0, 3, 6$ quarters stand for the horizons relative to the partial change in population or income at time $h = 0$. The computations of all effects exclude the MSAs whose spatial lag coefficients hit the upper/lower bound in the optimisation procedure.

Table F3: Mean group estimates (MGE) of spatial and temporal coefficients, and elasticities of house price changes to population and real income growth by six major regions and the U.S. as a whole, using a sparse inverse of distance adjacency matrix

r	Name	N_r	$\hat{\psi}_{MG,r}$	$\hat{\psi}_{MG0,r}$	$\hat{\psi}_{MG1,r}$	$\hat{\lambda}_{MG,r}$	$\hat{\beta}^{pop}_{MG,r}$	$\hat{\beta}^{inc}_{MG,r}$
1 & 2	New England & Mideast	36	0.080	$0.537^{\ddagger}$	$-0.456^{\ddagger}$	$0.646^{\ddagger}$	$0.628^{\ddagger}$	$0.082^{\ddagger}$
			(0.042)	(0.084)	(0.068)	(0.027)	(0.166)	(0.019)
3	Great Lakes	48	$0.151^{\ddagger}$	$0.759^{\ddagger}$	$-0.609^{\ddagger}$	$0.635^{\ddagger}$	$0.198^{\ddagger}$	$0.029^{\ddagger}$
			(0.027)	(0.050)	(0.040)	(0.024)	(0.060)	(0.010)
4	Plains	27	0.009	$0.472^{\ddagger}$	$-0.463^{\ddagger}$	$0.608^{\ddagger}$	$0.217^{\ddagger}$	0.043
			(0.059)	(0.097)	(0.063)	(0.041)	(0.084)	(0.028)
5	Southeast	106	$0.125^{\ddagger}$	$0.742^{\ddagger}$	$-0.617^{\ddagger}$	$0.674^{\ddagger}$	$0.170^{\ddagger}$	$0.032^{\ddagger}$
			(0.020)	(0.033)	(0.026)	(0.017)	(0.031)	(0.007)
6 & 7	Southwest & Rocky Mountain	39	0.009	$0.313^{\ddagger}$	$-0.304^{\ddagger}$	$0.718^{\ddagger}$	$0.158^{\ddagger}$	$0.074^{\ddagger}$
			(0.020)	(0.079)	(0.066)	(0.020)	(0.038)	(0.015)
8	Far West	42	$0.104^{\ddagger}$	$0.682^{\ddagger}$	$-0.578^{\ddagger}$	$0.759^{\ddagger}$	$0.177^{\ddagger}$	$0.066^{\ddagger}$
			(0.025)	(0.048)	(0.033)	(0.012)	(0.047)	(0.019)
	US	298	$0.095^{\ddagger}$	$0.631^{\ddagger}$	$-0.536^{\ddagger}$	$0.676^{\ddagger}$	$0.234^{\ddagger}$	$0.049^{\ddagger}$
			(0.012)	(0.025)	(0.019)	(0.010)	(0.028)	(0.006)

Notes: See notes of Table 3 in the main paper. We consider adjacency matrix $\mathbf{W} = (w_{ij}), i.j = 1, 2, \ldots, N$ which is the row-normalised version of matrix $\mathbf{P} = (p_{ij}) = 1/d_{ij}$, if $d_{ij} \leq 75$ miles and zero otherwise. Pairwise geodesic distances, d_{ij}, between MSAs are computed using the Haversine formula, as described in Appendix C of Bailey et al. (2016). The computations of all MG estimates exclude the MSAs whose spatial lag coefficients hit the upper/lower bound in the optimisation procedure.

Appendix G Additional Monte Carlo results

The Monte Carlo results provided in the tables and plots below are based on the designs set out in Section 5 of the paper.

Table G1: Bias, RMSE, size and power for parameters of individual units in the HSAR(1) model with one exogenous regressor and non-Gaussian errors for $N = 5$ and $T \in \{25, 50, 100, 200\}$.

T	25		50		100		200	
Parameter	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
ψ_{i0}								
$\psi_{1,0} = 0.1261$	-0.0063	0.1537	0.0001	0.1012	-0.0023	0.0707	0.0007	0.0490
$\psi_{2,0} = 0.3883$	-0.0035	0.2078	-0.0049	0.1392	-0.0006	0.0955	-0.0002	0.0666
$\psi_{3,0} = 0.4375$	-0.0096	0.1852	-0.0016	0.1155	0.0020	0.0807	0.0000	0.0578
$\psi_{4,0} = 0.5059$	0.0022	0.1478	-0.0039	0.1018	-0.0020	0.0686	-0.0008	0.0481
$\psi_{5,0} = 0.7246$	-0.0040	0.1747	-0.0016	0.1248	-0.0008	0.0880	0.0001	0.0597
β_{i0}								
$\beta_{1,0} = 0.9649$	0.0097	0.1663	0.0047	0.1102	0.0020	0.0758	-0.0015	0.0540
$\beta_{2,0} = 0.9572$	0.0065	0.1968	0.0048	0.1349	-0.0017	0.0938	-0.0018	0.0657
$\beta_{3,0} = 0.2785$	0.0054	0.2088	-0.0012	0.1316	-0.0017	0.0918	0.0013	0.0666
$\beta_{4,0} = 0.9134$	-0.0011	0.1643	0.0054	0.1098	0.0007	0.0749	0.0000	0.0515
$\beta_{5,0} = 0.8147$	0.0039	0.2079	0.0069	0.1440	0.0056	0.0989	0.0003	0.0674
T	25	50	100	200	25	50	100	200
Parameter	Size				Power			
ψ_{i0}								
$\psi_{1,0} = 0.1261$	0.1025	0.0665	0.0545	0.0505	0.4350	0.6010	0.8260	0.9725
$\psi_{2,0} = 0.3883$	0.1000	0.0690	0.0575	0.0555	0.3015	0.4520	0.6205	0.8535
$\psi_{3,0} = 0.4375$	0.0990	0.0630	0.0570	0.0520	0.3395	0.4910	0.7365	0.9400
$\psi_{4,0} = 0.5059$	0.0795	0.0670	0.0615	0.0460	0.4265	0.6165	0.8475	0.9780
$\psi_{5,0} = 0.7246$	0.0770	0.0720	0.0700	0.0560	0.3270	0.4680	0.6765	0.9125
β_{i0}								
$\beta_{1,0} = 0.9649$	0.0920	0.0645	0.0535	0.0610	0.3985	0.5485	0.7610	0.9500
$\beta_{2,0} = 0.9572$	0.0965	0.0725	0.0595	0.0545	0.3100	0.4265	0.6445	0.8690
$\beta_{3,0} = 0.2785$	0.0965	0.0695	0.0500	0.0625	0.3370	0.4380	0.6435	0.8525
$\beta_{4,0} = 0.9134$	0.0890	0.0685	0.0550	0.0470	0.4230	0.5305	0.7830	0.9545
$\beta_{5,0} = 0.8147$	0.0940	0.0730	0.0590	0.0525	0.2885	0.3880	0.5590	0.8350

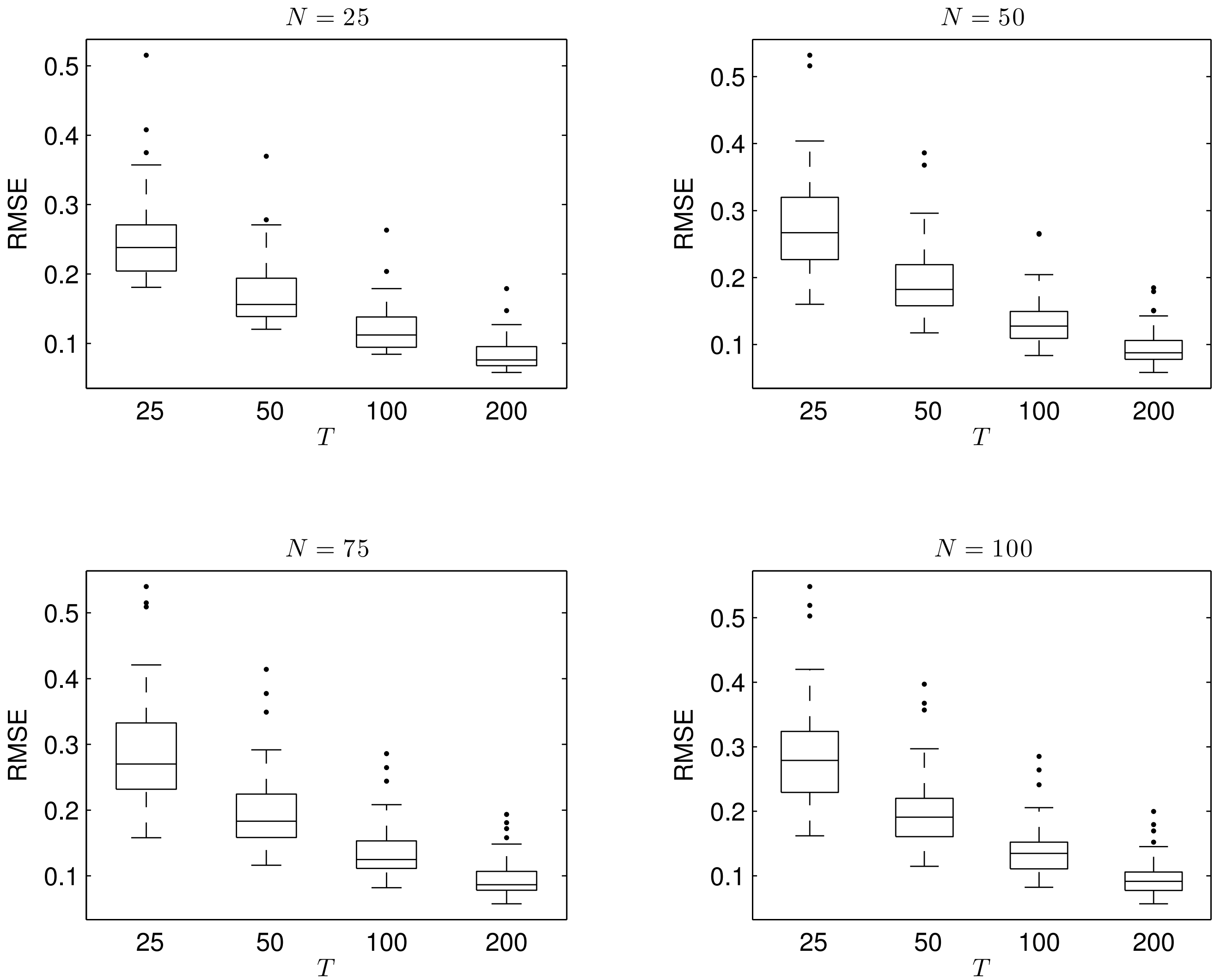
Notes: True parameter values are generated as $\psi_{i0} \sim IIDU(0, 0.8)$, $\alpha_{i0} \sim IIDN(1, 1)$, and $\beta_{i0} \sim IIDU(0, 1)$ for $i = 1, 2, \ldots, N$. Non-Gaussian errors are generated as $\varepsilon_{i0}/\sigma_{i0} \sim IID[\chi^2(2) - 2]/2$, with $\sigma^2_{i0} \sim IIDU[\chi^2(2)/8 + 0.25]$ for $i = 1, 2, \ldots, N$. The spatial weight matrix $\boldsymbol{W} = (w_{ij})$ has four connections so that $w_{ij} = 1$ if j is equal to: $i-2$, $i-1$, $i+1$, $i+2$, and zero otherwise, for $i = 1, 2, \ldots, N$. Biases and RMSEs are computed as $R^{-1}\sum_{r=1}^{R}(\hat{\psi}_{i,r} - \psi_{i0})$ and $\sqrt{R^{-1}\sum_{r=1}^{R}(\hat{\psi}_{i,r} - \psi_{i0})^2}$ for $i = 1, 2, \ldots, N$. Empirical size and empirical power are based on the sandwich formula given by (24). The nominal size is set to 5%. Size is computed under $H_{i0}\colon \psi_i = \psi_{i0}$, using a two-sided alternative, for $i = 1, 2, \ldots, N$. Power is computed under $\psi_i = \psi_{i0} + 0.2$, for $i = 1, 2, \ldots, N$. The number of replications is set to $R = 2{,}000$. Estimates are sorted in ascending order according to the true values of the spatial autoregressive parameters. Biases, RMSEs, sizes and powers for β_i, $i = 1, 2, \ldots, N$, are computed similarly, with power computed under $\beta_i = \beta_{i0} + 0.2$.

Table G2: Bias, RMSE, size and power for parameters of individual units in the HSAR(1) model with one exogenous regressor and non-Gaussian errors for $N = 100$ and $T \in \{25, 50, 100, 200\}$.

T	25		50		100		200	
Parameter	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
ψ_{i0}								
$\psi_{1,0} = 0.0244$	-0.0021	0.2572	-0.0036	0.1741	0.0009	0.1160	0.0000	0.0828
$\psi_{2,0} = 0.0255$	-0.0356	0.4532	0.0022	0.3105	-0.0114	0.2191	-0.0022	0.1499
$\psi_{3,0} = 0.0397$	0.0078	0.3029	-0.0032	0.2099	-0.0006	0.1438	0.0011	0.1025
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{51,0} = 0.3927$	-0.0022	0.2698	0.0020	0.1914	-0.0003	0.1305	0.0001	0.0896
$\psi_{52,0} = 0.3987$	0.0002	0.1694	-0.0039	0.1166	0.0021	0.0777	0.0004	0.0545
$\psi_{53,0} = 0.4004$	-0.0097	0.2691	0.0054	0.1770	0.0025	0.1205	-0.0017	0.0855
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{98,0} = 0.7695$	-0.0012	0.1433	0.0019	0.1013	0.0046	0.0724	-0.0001	0.0510
$\psi_{99,0} = 0.7705$	-0.0397	0.2546	-0.0088	0.1692	0.0020	0.1248	0.0007	0.0920
$\psi_{100,0} = 0.7904$	-0.0084	0.1514	-0.0070	0.1113	0.0008	0.0771	0.0006	0.0521
β_{i0}								
$\beta_{1,0} = 0.1978$	0.0066	0.2012	0.0010	0.1280	0.0006	0.0868	-0.0052	0.0598
$\beta_{2,0} = 0.7060$	0.0202	0.2711	0.0005	0.1720	-0.0001	0.1176	0.0033	0.0837
$\beta_{3,0} = 0.4173$	0.0077	0.1852	0.0027	0.1254	-0.0002	0.0842	0.0019	0.0597
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{51,0} = 0.9448$	0.0043	0.1415	-0.0015	0.0962	0.0015	0.0665	0.0007	0.0471
$\beta_{52,0} = 0.1190$	0.0030	0.1324	0.0023	0.0913	-0.0004	0.0619	0.0014	0.0421
$\beta_{53,0} = 0.7127$	0.0019	0.1941	-0.0036	0.1226	0.0025	0.0893	0.0010	0.0615
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{98,0} = 0.1067$	0.0024	0.1221	-0.0024	0.0807	0.0005	0.0553	-0.0002	0.0399
$\beta_{99,0} = 0.4588$	0.0147	0.1909	0.0026	0.1300	0.0017	0.0899	0.0017	0.0650
$\beta_{100,0} = 0.3674$	0.0035	0.1239	0.0022	0.0865	0.0006	0.0607	0.0002	0.0408
T	25	50	100	200	25	50	100	200
Parameter	Size				Power			
ψ_{i0}								
$\psi_{1,0} = 0.0244$	0.0915	0.0805	0.0560	0.0600	0.2255	0.3025	0.4560	0.7070
$\psi_{2,0} = 0.0255$	0.0830	0.0645	0.0580	0.0525	0.1225	0.1170	0.1940	0.2890
$\psi_{3,0} = 0.0397$	0.0905	0.0785	0.0605	0.0630	0.1815	0.2390	0.3450	0.5405
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{51,0} = 0.3927$	0.0995	0.0640	0.0595	0.0530	0.1975	0.2915	0.4020	0.6325
$\psi_{52,0} = 0.3987$	0.0865	0.0660	0.0620	0.0525	0.3785	0.5380	0.7440	0.9395
$\psi_{53,0} = 0.4004$	0.0960	0.0810	0.0520	0.0590	0.2400	0.3020	0.4540	0.6760
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{98,0} = 0.7695$	0.0710	0.0665	0.0675	0.0660	0.4015	0.5760	0.7930	0.9650
$\psi_{99,0} = 0.7705$	0.0390	0.0320	0.0405	0.0535	0.1750	0.2380	0.3820	0.6095
$\psi_{100,0} = 0.7904$	0.0690	0.0655	0.0575	0.0510	0.3845	0.5705	0.7605	0.9500
β_{i0}								
$\beta_{1,0} = 0.1978$	0.1085	0.0710	0.0570	0.0485	0.3320	0.4715	0.6715	0.9195
$\beta_{2,0} = 0.7060$	0.1055	0.0580	0.0495	0.0570	0.2315	0.3110	0.4530	0.6680
$\beta_{3,0} = 0.4173$	0.0935	0.0805	0.0520	0.0590	0.3585	0.4815	0.6985	0.9120
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{51,0} = 0.9448$	0.0940	0.0705	0.0535	0.0545	0.4710	0.6580	0.8445	0.9745
$\beta_{52,0} = 0.1190$	0.0950	0.0845	0.0525	0.0470	0.5250	0.6900	0.8940	0.9920
$\beta_{53,0} = 0.7127$	0.1055	0.0690	0.0685	0.0545	0.3615	0.4800	0.6485	0.8910
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{98,0} = 0.1067$	0.0945	0.0685	0.0520	0.0570	0.5775	0.7650	0.9320	0.9960
$\beta_{99,0} = 0.4588$	0.0745	0.0625	0.0505	0.0580	0.2845	0.4355	0.6460	0.8600
$\beta_{100,0} = 0.3674$	0.1000	0.0710	0.0680	0.0540	0.5480	0.7205	0.9070	0.9935

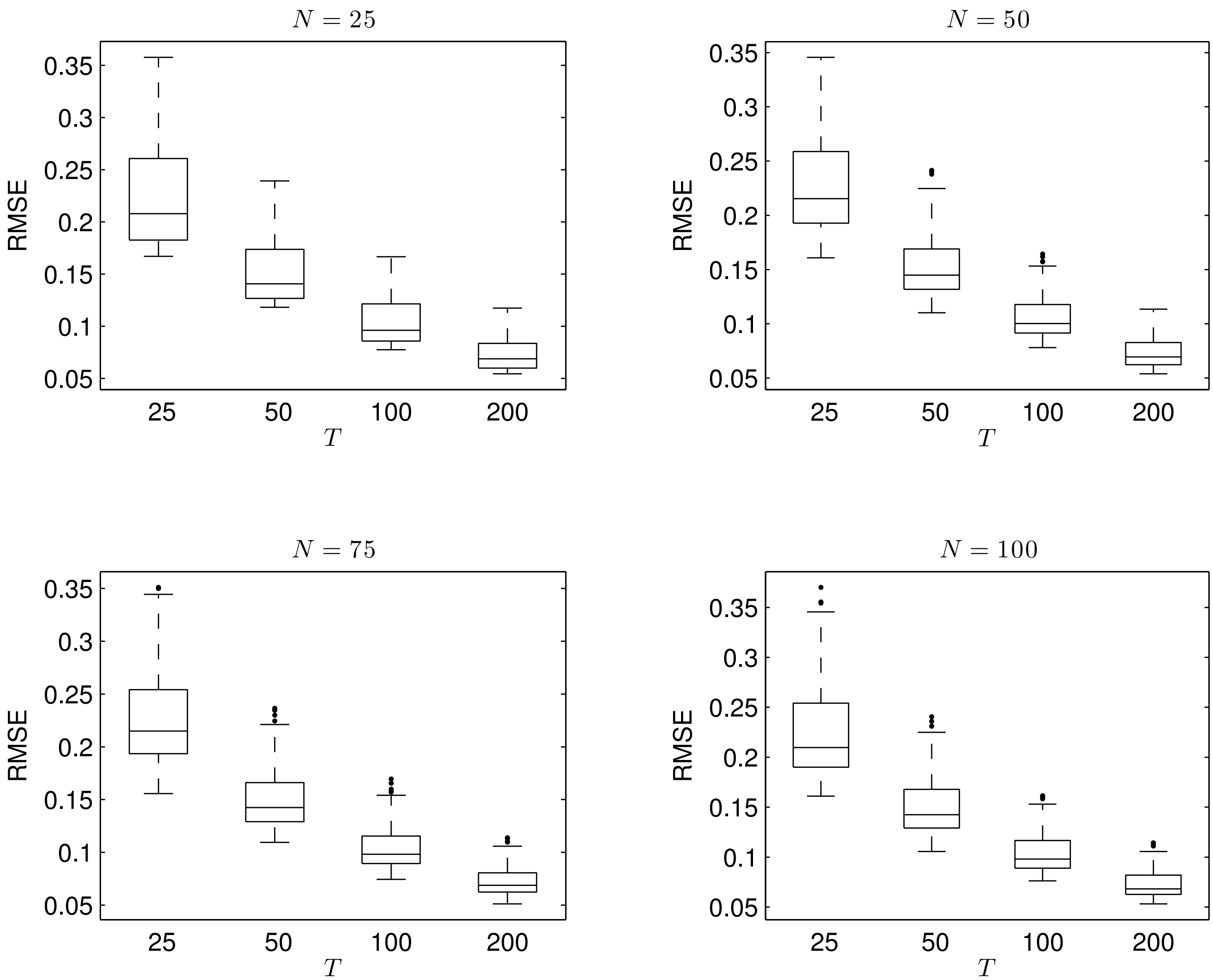
Notes: See notes to Table G1.

Figure G1: Boxplots of RMSEs for the individual autoregressive spatial parameter estimates from the HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections for different N and T combinations



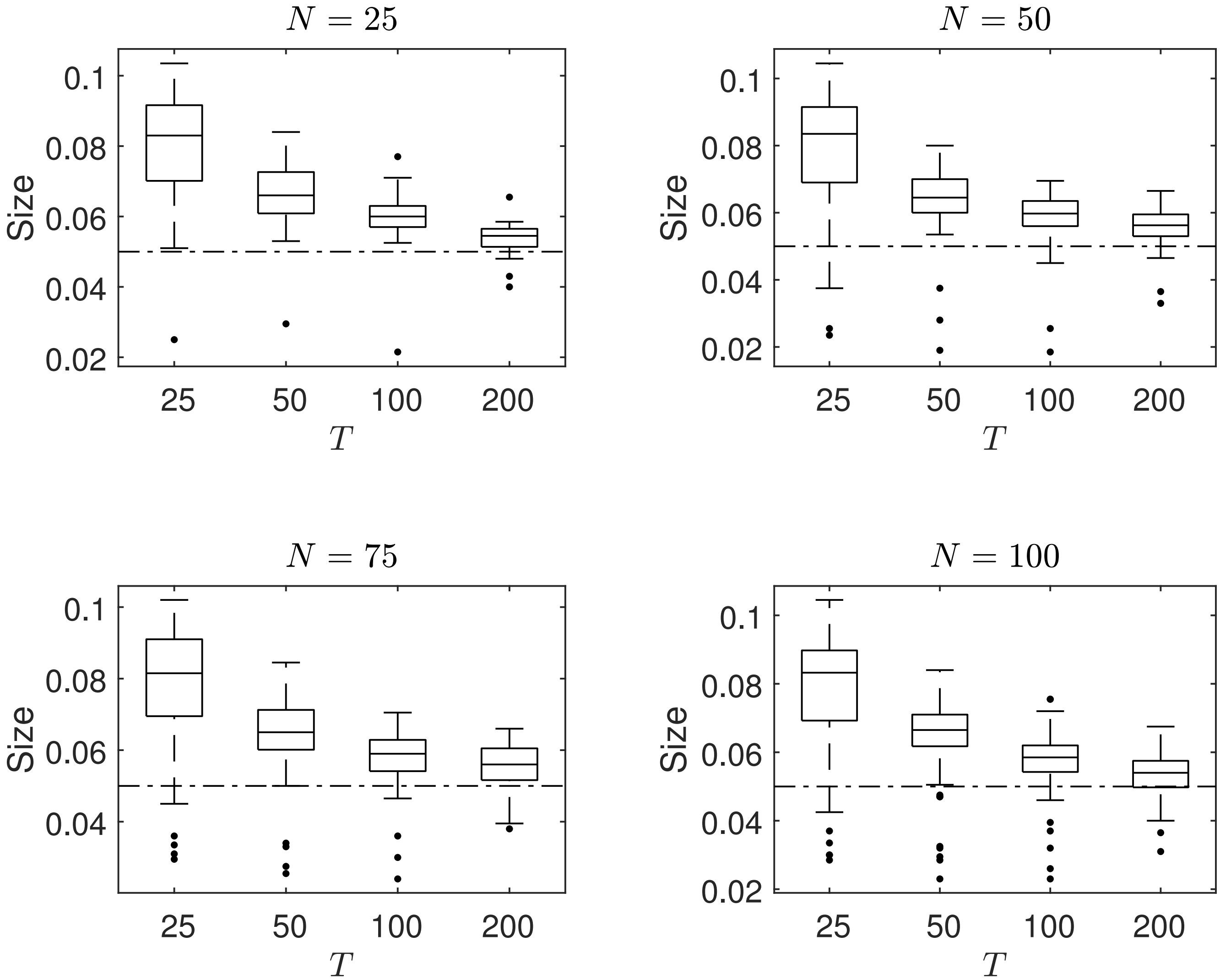
Notes: True parameter values are generated as $\psi_{i0} \sim IIDU(0, 0.8)$, $a_{i0} \sim IIDN(1, 1)$ and $\beta_{i0} \sim IIDU(0, 1)$, for $i = 1, 2, \dots, N$. Non-Gaussian errors are generated as $\varepsilon_{it}/\sigma_{i0} \sim IID[\chi^2(2) - 2]/2$, with $\sigma^2_{i0} \sim IID[\chi^2(2)/4 + 0.5]$, for $i = 1, 2, \dots, N$. Exogenous regressors are spatially correlated across i and generated by (29), with $\phi_i = 0.5$. The spatial weight matrix $\boldsymbol{W} = (w_{ij})$ has four connections so that $w_{ij} = 1$ if j is equal to $i - 2$, $i - 1$, $i + 1$, $i + 2$, and zero otherwise, for $i = 1, 2, \dots, N$. RMSEs are computed as $\sqrt{R^{-1} \sum_{r=1}^{R} (\hat{\psi}_{i,r} - \psi_{i0})^2}$ for $i = 1, 2, \cdots, N$. The number of replications is set to $R = 2,000$.

Figure G2: Boxplots of RMSEs for the individual slope parameter estimates from the HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections for different N and T combinations



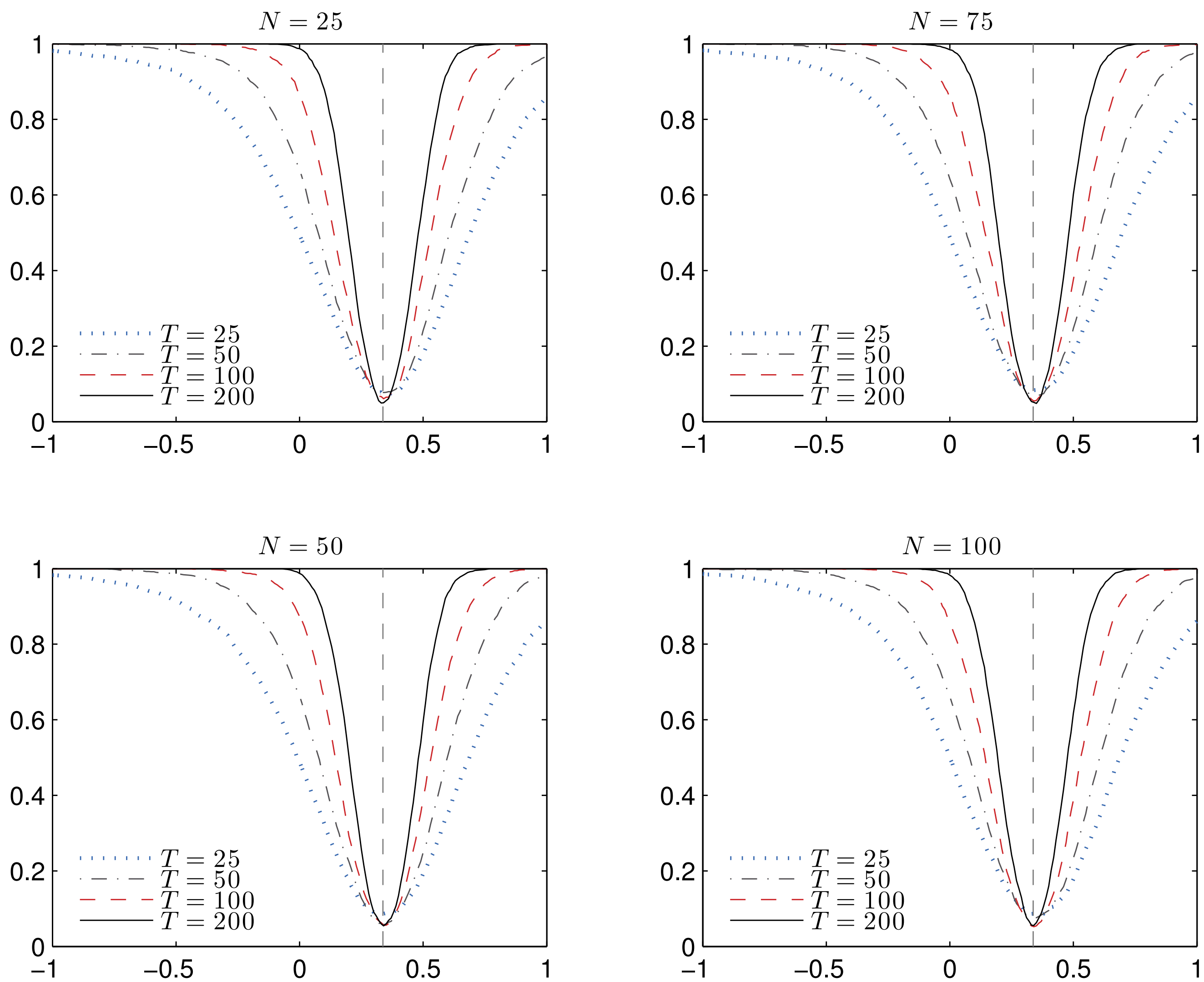
Notes: RMSEs are computed as $\sqrt{R^{-1}\sum_{r=1}^{R}(\hat{\beta}_{i,r}-\beta_{i0})^2}$ for $i = 1, 2, \cdots, N$. See the notes to Figure G1 for details of the data generating process. The number of replications is set to $R = 2,000$.

Figure G3: Boxplots of empirical sizes of tests for individual spatial parameters from HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections for different N and T combinations, using the sandwich formula for the variance



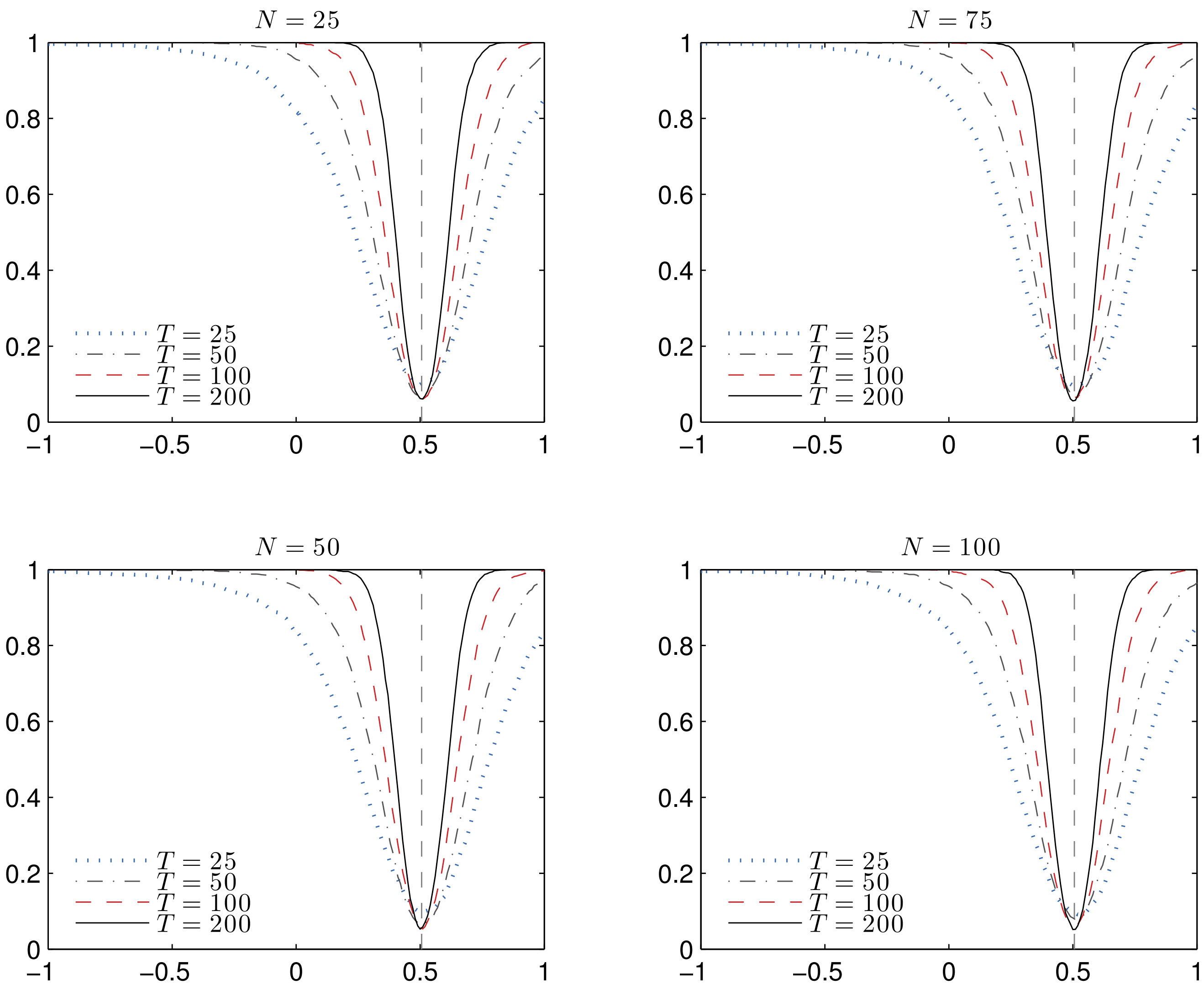
Notes: Nominal size is set to 5%. The sandwich formula is given by (24). See the notes to Figure G1 for details of the data generating process. Size is computed under H_0: $\psi_i = \psi_{i0}$, using a two-sided alternative where ψ_{i0} takes values in the range [0.0, 0.8] for $i = 1, 2, \ldots, N$. The number of replications is set to $R = 2,000$.

Figure G4: Empirical power functions for different N and T combinations, associated with testing the spatial parameter value $\psi_{i0} = 0.3374$ from HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections, using the sandwich formula for the variance



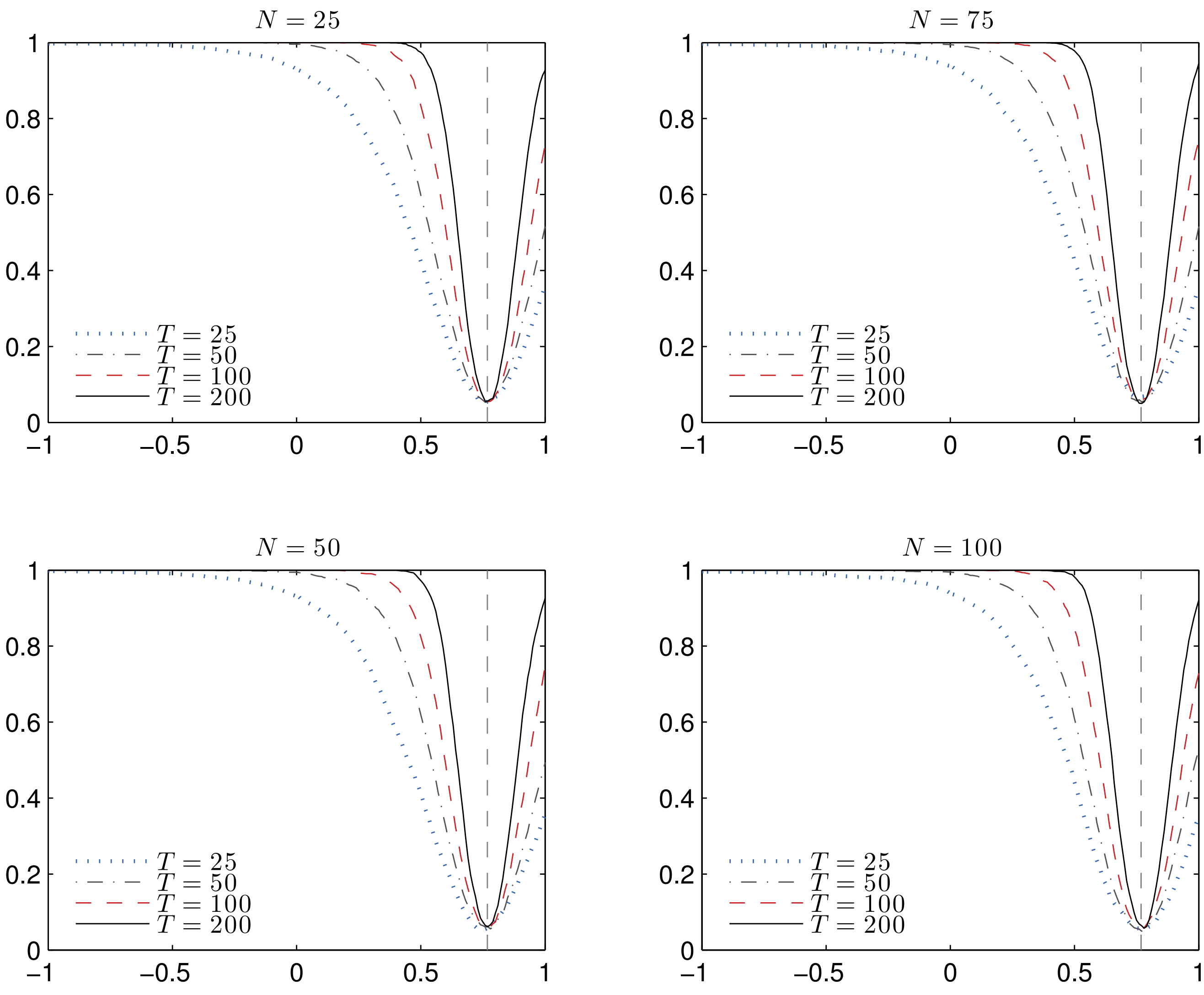
Notes: The power functions are based on the sandwich formula given by (24). See the notes to Figure G1 for details of the data generating process. Power is computed under $\psi_i = \psi_{i0} + \delta$, where $\delta = -0.8, -0.791, \ldots, 0.791, 0.8$ or until the parameter space boundaries of -1 and 1 are reached. The number of replications is set to $R = 2,000$.

Figure G5: Empirical power functions for different N and T combinations, associated with testing the spatial parameter value $\psi_{i0} = 0.5059$ from HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections, using the sandwich formula for the variance



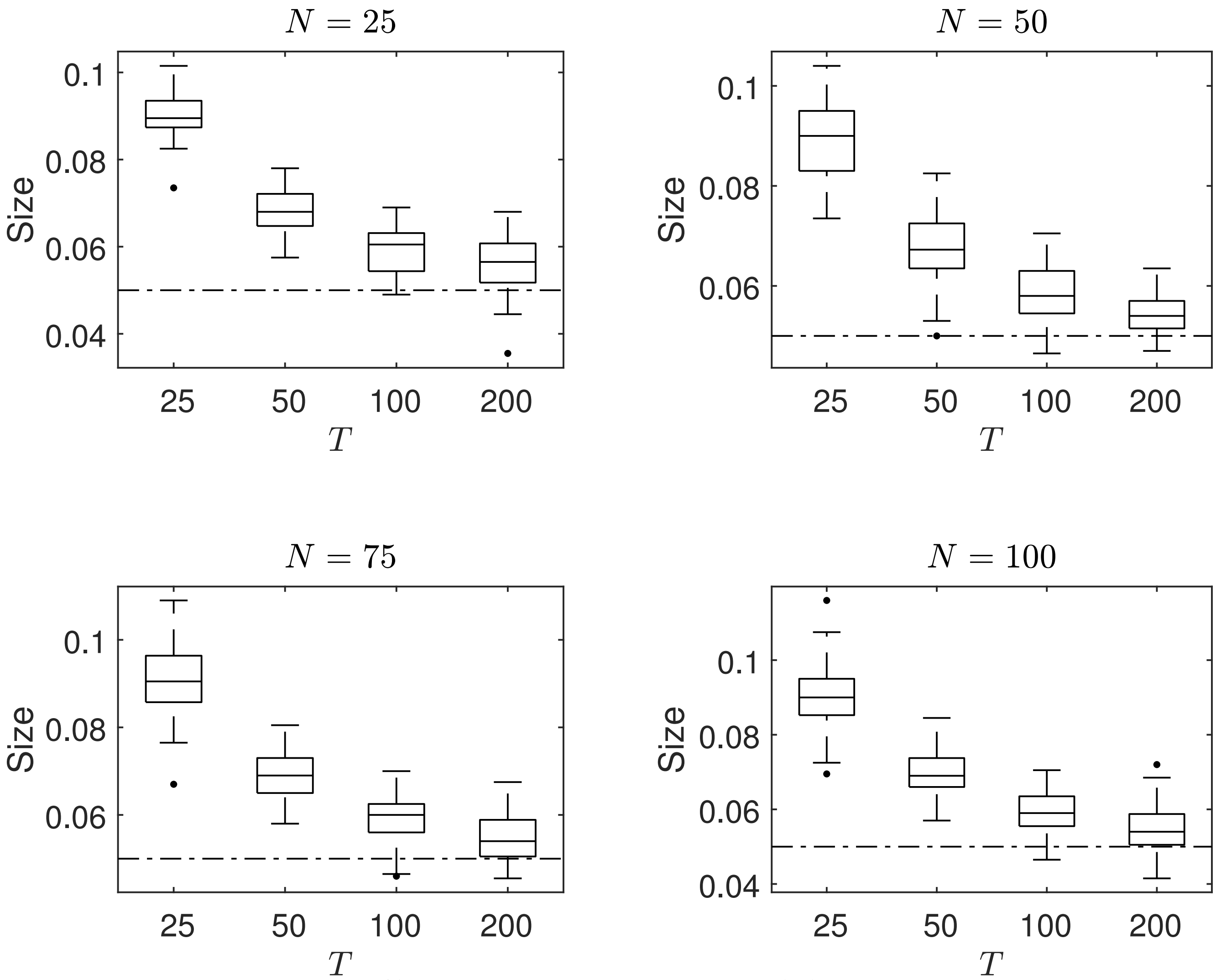
Notes: See the notes to Figure G4.

Figure G6: Empirical power functions for different N and T combinations, associated with testing the spatial parameter value $\psi_{i0} = 0.7676$ from HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections, using the sandwich formula for the variance



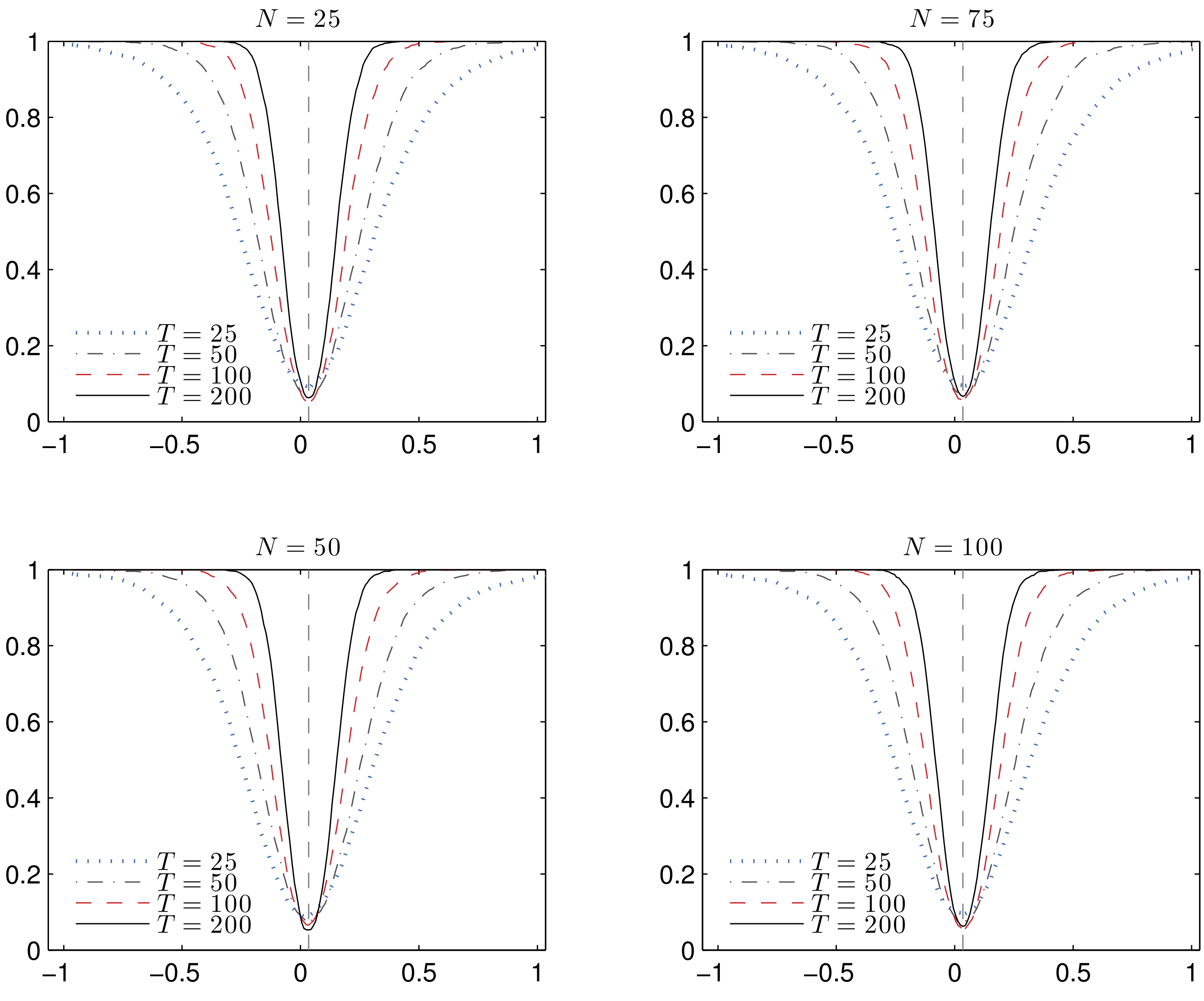
Notes: See the notes to Figure G4.

Figure G7: Boxplots of empirical sizes of tests for individual slope parameters from HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections for different N and T combinations, using the sandwich formula for the variance



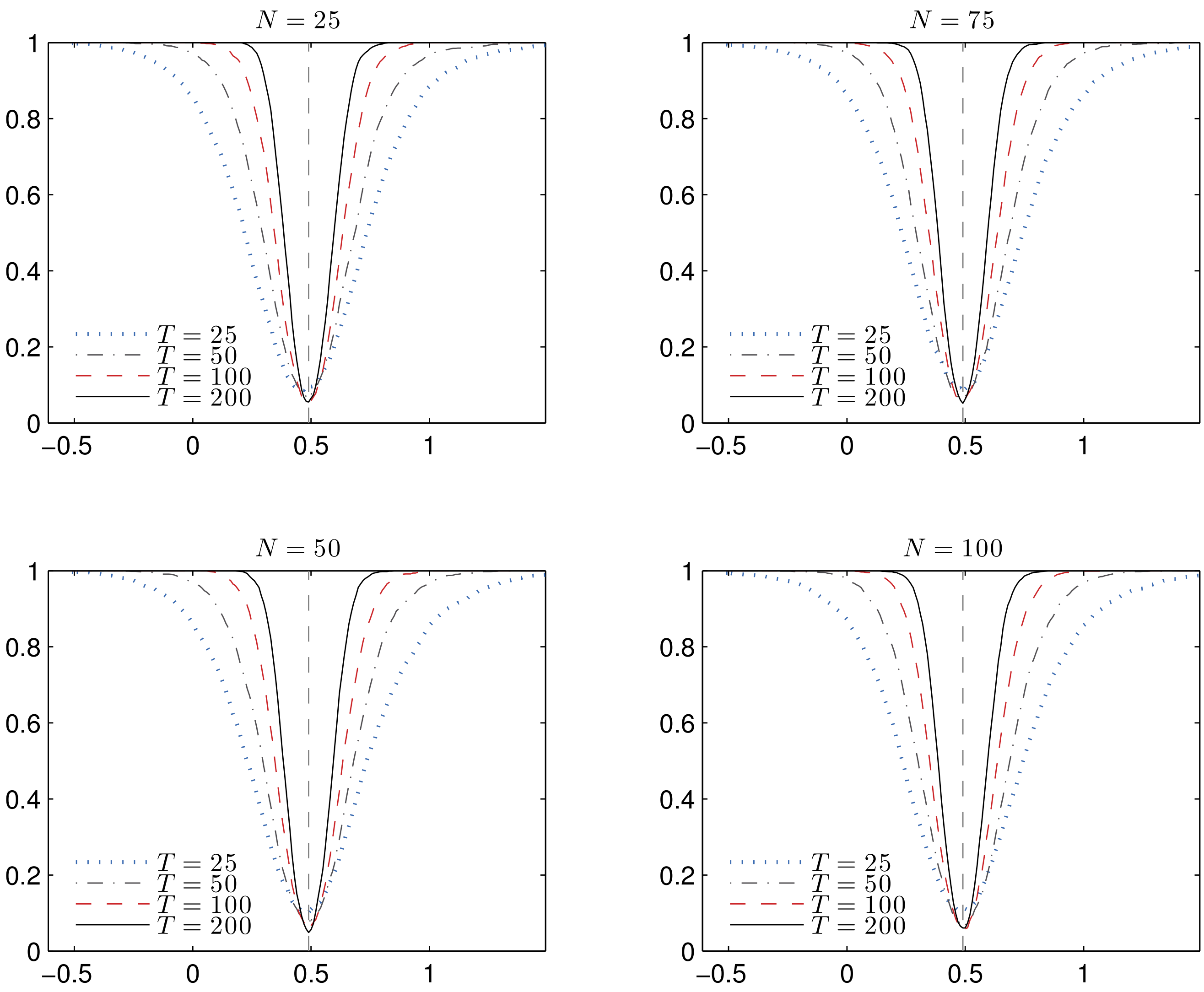
Notes: Nominal size is set to 5%. The sandwich formula is given by (24). See the notes to Figure G1 for details of the data generating process. Size is computed under H_0: $\beta_i = \beta_{i0}$, using a two-sided alternative where β_{i0} takes values in the range [0.0, 1.0] for $i = 1, 2, \ldots, N$. The number of replications is set to $R = 2,000$.

Figure G8: Empirical power functions for different N and T combinations, associated with testing the slope parameter value $\beta_{i0} = 0.0344$ from HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections, using the sandwich formula for the variance



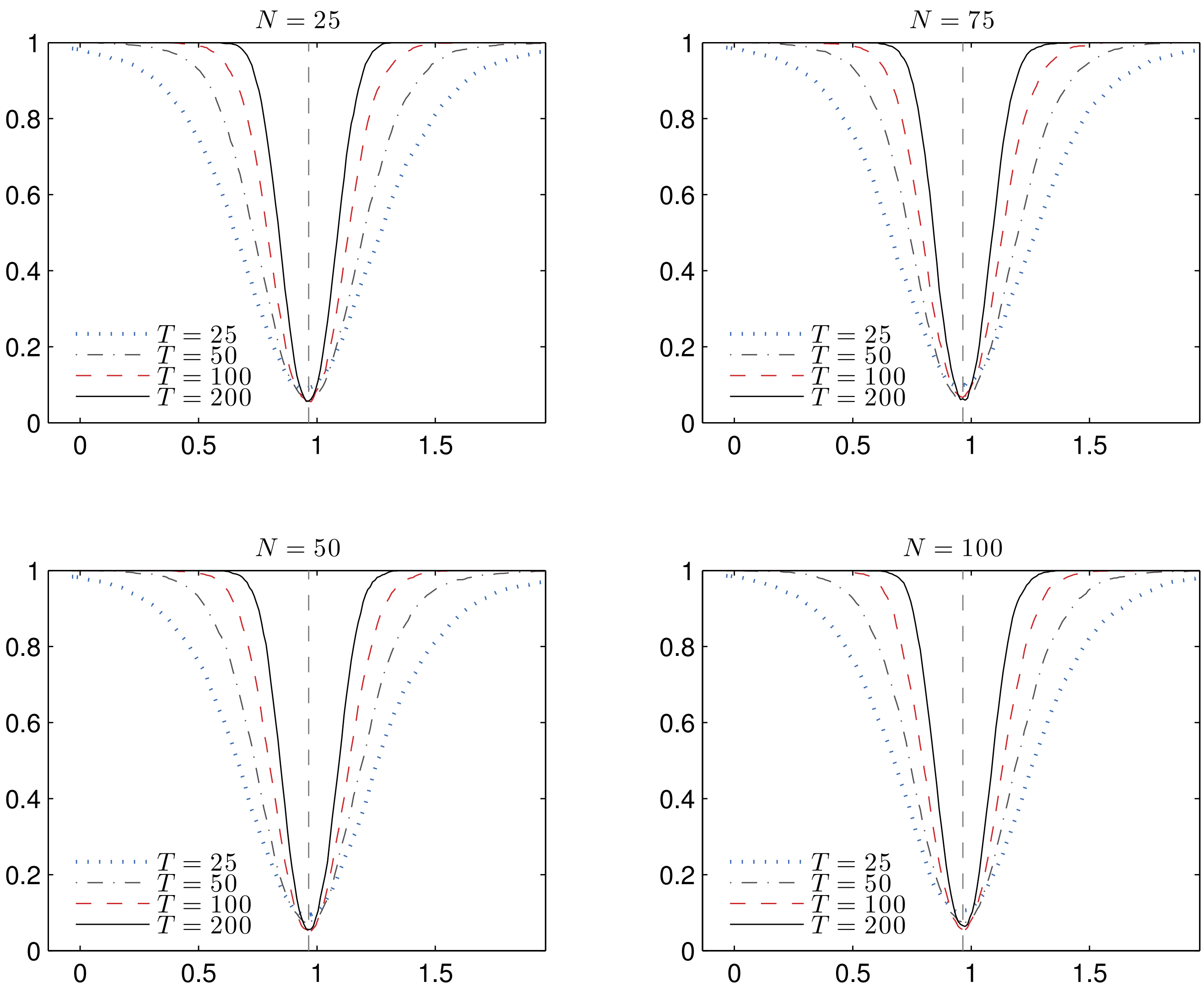
Notes: The power functions are based on the sandwich formula given by (24). See the notes to Figure G1 for details of the data generating process. Power is computed under $\beta_i = \beta_{i0} + \delta$, where $\delta = -1.0, -0.991, \ldots, 0.991, 1.0$. The number of replications is set to $R = 2,000$.

Figure G9: Empirical power functions for different N and T combinations, associated with testing the slope parameter value $\beta_{i0} = 0.4898$ from HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections, using the sandwich formula for the variance



Notes: See the notes to Figure G8.

Figure G10: Empirical power functions for different N and T combinations, associated with testing the slope parameter value $\beta_{i0} = 0.9649$ from HSAR(1) model with non-Gaussian errors, one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections, using the sandwich formula for the variance



Notes: See the notes to Figure G8.

Table G3: Bias, RMSE and size for the Mean Group (MG) estimator from the HSAR(1) model with one exogenous regressor and spatial weight matrix $\boldsymbol{W}$ having 4 connections for different N and T combinations

N	25				50				75				100			
T	25	50	100	200	25	50	100	200	25	50	100	200	25	50	100	200
Gaussian errors																
Bias																
$\hat{\hat{\psi}}_{MG}$	-0.0060	-0.0005	-0.0013	-0.0007	-0.0053	-0.0013	-0.0008	0.0009	-0.0075	-0.0014	0.0006	-0.0005	-0.0070	-0.0012	-0.0007	-0.0001
$\hat{\hat{\beta}}_{MG}$	0.0061	0.0011	0.0023	-0.0008	0.0052	0.0005	0.0028	0.0012	0.0065	0.0024	0.0010	0.0014	0.0055	0.0033	0.0020	-0.0001
RMSE																
$\hat{\hat{\psi}}_{MG}$	0.0599	0.0542	0.0508	0.0482	0.0446	0.0381	0.0357	0.0339	0.0358	0.0311	0.0286	0.0276	0.0310	0.0270	0.0251	0.0237
$\hat{\hat{\beta}}_{MG}$	0.0752	0.0662	0.0602	0.0595	0.0522	0.0477	0.0433	0.0422	0.0435	0.0378	0.0360	0.0340	0.0371	0.0324	0.0311	0.0298
Size																
$\hat{\hat{\psi}}_{MG}$	0.0255	0.0415	0.0435	0.0585	0.0270	0.0335	0.0430	0.0475	0.0205	0.0285	0.0370	0.0445	0.0200	0.0325	0.0335	0.0350
$\hat{\hat{\beta}}_{MG}$	0.0595	0.0635	0.0530	0.0630	0.0510	0.0625	0.0625	0.0545	0.0560	0.0515	0.0525	0.0485	0.0485	0.0475	0.0580	0.0520
non-Gaussian errors																
Bias																
$\hat{\hat{\psi}}_{MG}$	-0.0102	-0.0018	-0.0010	-0.0003	-0.0088	-0.0018	-0.0007	0.0008	-0.0097	-0.0017	0.0001	-0.0007	-0.0079	-0.0010	-0.0010	-0.0005
$\hat{\hat{\beta}}_{MG}$	0.0087	0.0016	0.0025	-0.0011	0.0048	0.0013	0.0026	0.0010	0.0067	0.0023	0.0016	0.0011	0.0047	0.0028	0.0014	0.0001
RMSE																
$\hat{\hat{\psi}}_{MG}$	0.0638	0.0558	0.0508	0.0480	0.0451	0.0389	0.0361	0.0336	0.0369	0.0313	0.0285	0.0271	0.0319	0.0275	0.0250	0.0237
$\hat{\hat{\beta}}_{MG}$	0.0755	0.0648	0.0610	0.0593	0.0528	0.0461	0.0440	0.0420	0.0436	0.0372	0.0362	0.0338	0.0380	0.0325	0.0307	0.0300
Size																
$\hat{\hat{\psi}}_{MG}$	0.0330	0.0410	0.0520	0.0535	0.0280	0.0330	0.0425	0.0480	0.0260	0.0350	0.0360	0.0390	0.0225	0.0320	0.0350	0.0375
$\hat{\hat{\beta}}_{MG}$	0.0625	0.0620	0.0600	0.0600	0.0540	0.0505	0.0625	0.0560	0.0555	0.0515	0.0580	0.0510	0.0515	0.0495	0.0525	0.0540

Notes: True parameter values are generated as $a_{i0} = a_0 + \epsilon_{1i}$, with $a_0 = 1$ and $\epsilon_{1i} \sim IIDN(0,1)$, $\psi_{i0} = \psi_0 + \epsilon_{2i}$, with $\psi_0 = 0.4$ and $\epsilon_{2i} \sim IIDU(-0.4, 0.4)$ and $\beta_{i0} = \beta_0 + \epsilon_{3i}$, with $\beta_0 = 0.5$ and $\epsilon_{3i} \sim IIDU(-0.5, 0.5)$, where ϵ_{1i}, ϵ_{2i}, and ϵ_{3i} are drawn for each replication. Gaussian errors are generated as $\varepsilon_{it}/\sigma_{i0} \sim N(0,1)$, while non-Gaussian errors are generated as $\varepsilon_{it}/\sigma_{i0} \sim IID[\chi^2(2)-2]/2$, where $\sigma^2_{i0} \sim IID[\chi^2(2)/4+0.5]$, for $i = 1, 2, \ldots, N$ in both instances. Exogenous regressors are spatially correlated across i and generated by (29), with $\phi_i = 0.5$. Let $\hat{\psi}_r$ denote the MG estimator of ψ_0 for replication r, $\hat{\psi}_r = N^{-1}\sum_{i=1}^{N}\hat{\psi}_{i,r}$. $r = 1, \ldots, R$, Bias and RMSE are computed as $R^{-1}\sum_{r=1}^{R}(\hat{\psi}_r - \psi_0)$ and $\sqrt{R^{-1}\sum_{r=1}^{R}(\hat{\psi}_r - \psi_0)^2}$; bias and RMSE for $\hat{\beta}_r$ are computed accordingly. The spatial weight matrix $\boldsymbol{W} = (w_{ij})$ has four connections so that $w_{ij} = 1$ if j is equal to $i-2$, $i-1$, $i+1$, $i+2$, and zero otherwise, for $i = 1, 2, \ldots, N$. The number of replications is set to $R = 2,000$.