

On Identification of Bayesian DSGE Models*

Gary Koop, University of Strathclyde
M. Hashem Pesaran, Cambridge University, and USC
Ron P. Smith, Birkbeck College

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Abstract

There has been increasing concern about parameter identification in dynamic stochastic general equilibrium (DSGE) models. Given their structure it may be difficult to determine whether a parameter is identified. When using Bayesian methods, a lack of identification may not be evident since the posterior may differ from its prior even if the parameter is unidentified. We suggest two Bayesian identification indicators that do not suffer from this difficulty and are relatively easy to compute. The first applies where the parameters can be partitioned into those that are known to be identified and the rest where it is not known whether they are identified. In such cases the marginal posterior of an unidentified parameter will equal the posterior expectation of the prior for that parameter conditional on the identified parameters. The second indicator is more generally applicable and considers the rate at which the posterior precision gets updated as the sample size (T) is increased. For identified parameters the posterior precision rises with T , whilst for an unidentified parameter its posterior precision may be updated but its rate of update will be slower than T . These results are illustrated by means of simple DSGE models.

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1 Introduction

Soon after rational expectations (RE) models were widely adopted in economics there was concern about the issue of observational equivalence (Sargent, 1976, McCallum, 1979) and the identification of the parameters of the RE models (Wallis, 1980, Pesaran, 1981, 1987, Pudney, 1982). Observational equivalence concerns whether one can distinguish different models, such as RE and non-RE models; the closely related issue of identification concerns the conditions under which it is possible to estimate the parameters of a particular model from available data. During the 1990s interest in identification waned, partly because of the shift in focus to calibration, where it is assumed that the parameters are known *a priori*, perhaps from microeconomic evidence.¹ Kydland and Prescott (1996) argue that the task of computational experiments of the sort they conduct is to derive the quantitative implications of the theory rather than to measure economic parameters, one of the primary objects of econometric analysis.

Over the past ten years it has become more common to estimate, rather than calibrate, dynamic stochastic general equilibrium (DSGE) models, often using Bayesian techniques (see, among many others, DeJong, Ingram and Whiteman, 2000, Smets and Wouters, 2003, 2007 and An and Schorfheide, 2007). In this context the issue of identification has attracted renewed attention. Questions have been raised about the identification of particular equations of the standard new Keynesian DSGE model, such as the Phillips curve (Mavroeidis, 2005, Nason and Smith, 2008, Kleibergen and Mavroeidis, 2009, Dees et al., 2009, and others), or the Taylor rule, Cochrane (2011). There have also been questions about the identification of DSGE systems as a whole. Canova and Sala (2009) conclude: “it appears that a large class of popular DSGE structures are only very weakly identified”. Iskrev (2010b) concludes “the results indicate that the parameters of the Smets and Wouters (2007) model are quite poorly identified in most of the parameter space”. Other recent papers which consider determining the identification of DSGE systems are Andrieu (2010), Iskrev (2010a), Iskrev and Ratto (2010), Komunjer and Ng (2011), who

¹The calibrators’ practice of basing the estimates of the structural parameters of macro models on macroeconomic evidence has been criticized by microeconomists, such as Hansen and Heckman (1996).

provide rank and order conditions for local identification based on the spectral density matrix, and Muller (2010), who suggests measures of prior sensitivity and prior informativeness based on the derivative of the posterior mean with respect to a particular parameterization of the prior mean. Andrews and Cheng (2011) provide a review of more general issues relating to weak identification. Whereas papers like Iskrev (2010a,b) and Komunjer and Ng (2011) provide classical procedures for determining identification based on the rank of particular matrices, our objective is to provide Bayesian indicators, since these models are usually estimated by Bayesian methods.

The 1980s literature on the identification of RE models tended to assume that the system included observed exogenous variables, whereas most current DSGE systems do not contain such variables. While most of the DSGE literature has focussed on the regular or determinate case where there is a unique solution to the linear RE system, there has been some interest in the indeterminate case, where there are multiple solutions (e.g. Clarida, Gali, and Gertler, 2000, Beyer and Farmer, 2004 and Lubik and Schorfheide, 2004). The indeterminate case also raises interesting identification issues.

Unlike the simpler simultaneous equations model (SEM) the non-linear nature of the cross equation restrictions in DSGE models makes it more difficult to analytically check identification. The RE structure means that they require more identifying restrictions than comparable SEMs. Although the approximate solution of DSGE models is taken to be linear, the structural parameters are complicated non-linear functions of the parameters of the linearized (reduced form) model and as a result the likelihood function for the structural parameters may be very badly behaved. This has led many to look at other features than the likelihood, such as impulse response functions or impact effects. The form of the likelihood may also cause problems for understanding certain features of the posterior, e.g. Herbst (2010). When the model involves unobserved variables the solution is of a VARMA form rather than a VAR. Thus some of the associated reduced form parameters may not be identified. The requirement for a determinate solution also puts restrictions on the joint parameter space, which may create dependence between identified and unidentified

parameters.

Faced with these difficulties, it is common practice in Bayesian DSGE modelling to compare posteriors to priors as informal indicators of identification. We discuss how this can be misleading, since, as we show, priors can differ from posteriors even for unidentified parameters. We then propose two different Bayesian indicators of identification that do not suffer from this drawback. The first draws on results from Poirier (1998) and concerns the case where the parameters can be partitioned into those known to be identified and those where it is uncertain whether they are identified. Then the marginal posterior of an unidentified parameter will equal the posterior expectation of the prior for that parameter conditional on the identified parameters. The marginal posterior and posterior expectation of the prior can be computed as a by-product of estimating a DSGE model (e.g. using MCMC methods) and compared. However, this indicator relies on the assumption that parameters can be separated into those which the researcher knows are identified and those for which identification is uncertain. Unfortunately, as we show in this paper, for the researcher working with the structural parameters of DSGE models, this assumption may not hold. Hence, we propose a second Bayesian indicator of local identification. This uses the fact that whilst for identified parameters the posterior precision rises with T , for an unidentified parameter its posterior precision may be updated but its rate of update will be slower than T . This suggests a strategy where the researcher simulates larger and larger data sets and observes the behavior of the posterior as sample size increases. Empirical illustrations show the usefulness of both these approaches for checking for the presence and strength of identification. This procedure is easy to implement in standard software packages such as Dynare (<http://www.dynare.org/>). In a recent paper Caglar et al. (2011) use this strategy to examine the identification of the parameters of the Bayesian DSGE model due to Smets and Wouters (2007), and find that most parameters of this often used model do not appear to be well identified.

The paper is organized as follows. Section 2 discusses the theory of rational expectations DSGE models and provides some simple theoretical examples. Section 3 discusses the econometrics. It

is broken into sub-sections on i) general identification issues; ii) existing Bayesian approaches to identification in DSGE models; iii) how (and when) these existing approaches can be used to check for presence and/or strength of identification; and iv) asymptotic results which show the behavior of the posterior for non-identified parameters in large samples and how these can be used to check for identification. Section 4 provides several empirical illustrations of the methods developed or discussed in Section 3. Section 5 concludes.

2 The Theory of Rational Expectations DSGE Models

2.1 A General Framework

Most macroeconomic DSGE models are constructed by linearizing an underlying non-linear model around its steady state, where θ is a vector of deeper parameters of this underlying model.² Consider a linearized rational expectations model for an $n \times 1$ vector of stationary variables of interest, y_t , $t = 1, 2, \dots, T$. These would usually be measured as deviations from their steady states. Denote expectations as $E_t(y_{t+1}) = E(y_{t+1} | \mathcal{I}_t)$ where \mathcal{I}_t is the information set available at time t .³ There is also a vector of observed exogenous variables⁴ x_t and an $n \times 1$ vector of unobserved variables u_t . We assume that both the exogenous and unobserved variables follow VAR(1) processes. Then the system can be written

$$A_0(\theta)y_t = A_1(\theta)E_t(y_{t+1}) + A_2(\theta)y_{t-1} + A_3(\theta)x_t + u_t, \quad (1)$$

$$x_t = \Phi_x x_{t-1} + v_t, \quad u_t = \Phi_u u_{t-1} + \varepsilon_t.$$

We treat the VAR(1) parameters for x_t or u_t , Φ_i , $i = x, u$, as not being specified by macroeconomic theory. This structure assumes that there is no feedback from y_t to x_t or u_t . ε_t is a vector of mean zero, serially uncorrelated, structural shocks, with $E(\varepsilon_t \varepsilon_t') = \Omega(\theta)$. It is common in the literature to assume that $\Omega(\theta) = I_n$. For Bayesian or maximum likelihood estimation, ε_t is typically assumed to be normally distributed. Notice that if $\Phi_u = 0$, the structural shocks enter the equations directly.

²We focus on the linearized case where the estimated parameters are non-linear functions of the structural parameters. Our analysis should continue to be more generally applicable were an exact solution to be available, since again there would be a non-linear relation between the estimated and structural parameters.

³In some papers in the literature, expectations are taken using information at time $t - 1$.

⁴Many DSGE models are closed, without exogenous variables, but we include them for completeness.

If $A_0(\theta)$ is nonsingular, then (1) can be written

$$y_t = A_0(\theta)^{-1}A_1(\theta)E_t(y_{t+1}) + A_0(\theta)^{-1}A_2(\theta)y_{t-1} + A_0(\theta)^{-1}A_3(\theta)x_t + A_0(\theta)^{-1}u_t. \quad (2)$$

The solution of such systems is discussed in Binder and Pesaran (1995, 1997) and Sims (2002). Details of how this solution proceeds are given in the Technical Appendix. Suffice it to note here that the unique stationary solution, if it exists, takes the form:

$$y_t = C(\theta)y_{t-1} + G_1(\theta, \phi_x)x_t + G_2(\theta, \phi_u)u_t, \quad (3)$$

where $\phi_i = \text{vec}(\Phi_i)$, $i = x, u$. The matrices $G_i(\theta, \phi_i)$ $i = x, u$, can be obtained using the method of undetermined coefficients (see Blinder and Pesaran, 1997, for details). Notice that the coefficient matrix for the lagged dependent variable vector is just a function of θ , and not ϕ_x or ϕ_u .

Likelihood-based estimation of this model is straightforward. If $\Phi_u = 0$, this is just a VAR with exogenous variables and the likelihood function is easily obtained. In general where the unobserved components of the model are serially correlated, the rational expectations solution will involve moving average components and it is more convenient to write the model as a state space model where Kalman filtering techniques can be used to evaluate the likelihood function. In such cases a simple analytical relationship between the structural and reduced form parameters might not be available, which further complicates the analysis of identification of the structural parameters. In the next sub-sections, we use some simple special cases of DSGE models where the RE solution is available analytically to clearly show how identification issues arise. For notational simplicity, we do not make the dependence on θ explicit.

2.2 DSGE models without lags

Abstracting from lagged values and exogenous regressors (1) simplifies to

$$A_0y_t = A_1E_t(y_{t+1}) + \varepsilon_t, \quad (4)$$

$$E(\varepsilon_t) = 0, \quad E(\varepsilon_t\varepsilon_t') = \Omega,$$

where ε_t are serially uncorrelated. If A_0 is non-singular using (4) we have

$$y_t = A_0^{-1}A_1E_t(y_{t+1}) + A_0^{-1}\varepsilon_t = QE_t(y_{t+1}) + A_0^{-1}\varepsilon_t. \quad (5)$$

The regular case, where there is a unique stationary solution, arises if the non-zero eigenvalues of Q lie within the unit circle. In this case, the unique solution of the model is given by

$$y_t = \sum_{j=0}^{\infty} Q^j A_0^{-1} E_t(\varepsilon_{t+j}). \quad (6)$$

Hence, $E_t(y_{t+1}) = 0$ and the unique RE solution is given by

$$A_0 y_t = \varepsilon_t. \quad (7)$$

and

$$y_t = A_0^{-1}\varepsilon_t = u_t$$

$$E(u_t u_t') = \Sigma = A_0^{-1}\Omega A_0^{-1'}. \quad (8)$$

Notice that (8) provides us with a likelihood function which does not depend on A_1 and, therefore, the parameters that are unique to A_1 (i.e. the coefficients on the forward variables) are not identified. Furthermore, the RE model is observationally equivalent to a model without forward variables which takes the form of (7). Since what can be estimated from the data, namely Σ , is not a function of A_1 , all possible choices of A_1 are observationally equivalent in the sense that they lead to the same observed data covariance matrix. Although the coefficients in the forward solution (6) are functions of A_1 , this does not identify them because $E_t(\varepsilon_{t+j}) = 0$. Elements of A_1 could be identified by certain sorts of *a priori* restrictions, but these are likely to be rather special, rather limited in number and cannot be tested.

If the parameters of the DSGE model were thought to be known *a priori* from calibration, there would be no identification problem and the structural errors ε_{it} could be recovered and used, for instance, in calculating impulse response functions, IRFs. However, suppose someone else believed that the true model was just a set of random errors $y_t = u_t$, with different IRFs.

There is no information in the data that a proponent of the DSGE could use to persuade the other person that the DSGE model was correct relative to the random error model. This is exactly the same point that Sargent (1976) made with respect to "natural and unnatural rate theories".

The above result generalizes to higher order RE models. Consider for example the model

$$A_0 y_t = \sum_{i=1}^q A_i E_t(y_{t+i}) + \varepsilon_t.$$

Once again the unique stable solution of this model is also given by $A_0 y_t = \varepsilon_t$, and none of the elements of A_1, A_2, \dots, A_q that are variation free with respect to the elements of A_0 are identified.

2.2.1 Example 1. A New Keynesian (NK) system without lags

As an illustration consider a standard three equation NK-DSGE model used in Benati (2010) that involves only current and future variables:

$$R_t = \psi \pi_t + \varepsilon_{1t}, \tag{9}$$

$$x_t = E_t(x_{t+1}) - \sigma(R_t - E_t(\pi_{t+1})) + \varepsilon_{2t}, \tag{10}$$

$$\pi_t = \beta E_t(\pi_{t+1}) + \gamma x_t + \varepsilon_{3t}. \tag{11}$$

where $E_t(x_{t+1}) = E(x_{t+1} | \mathfrak{J}_t)$. The model contains a monetary policy rule determining the interest rate, R_t , an IS curve determining the output gap, x_t , and a Phillips Curve determining inflation, π_t , all measured as deviations from their steady states. The errors, which are assumed to be white noise, are a monetary policy shock, ε_{1t} , a demand shock, ε_{2t} , and a supply or cost shock, ε_{3t} . These are assumed to be orthogonal. The discount factor is β and σ is the inter-temporal elasticity of substitution. This is a highly restricted system with many parameters set to zero *a priori*. For instance, output does not appear in the monetary policy rule and the coefficient of future output is exactly equal to unity in the IS equation. In terms of (4), $y_t = (R_t, x_t, \pi_t)'$ and

$$A_0 = \begin{pmatrix} 1 & 0 & -\psi \\ \sigma & 1 & 0 \\ 0 & -\gamma & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \sigma \\ 0 & 0 & \beta \end{pmatrix}.$$

Hence

$$A_0^{-1} = \frac{1}{\gamma\sigma\psi + 1} \begin{pmatrix} 1 & \gamma\psi & \psi \\ -\sigma & 1 & -\sigma\psi \\ -\gamma\sigma & \gamma & 1 \end{pmatrix}$$

$$Q = A_0^{-1}A_1$$

$$= \frac{1}{\gamma\sigma\psi + 1} \begin{pmatrix} 0 & \gamma\psi & \psi(\beta + \gamma\sigma) \\ 0 & 1 & \sigma(1 - \beta\psi) \\ 0 & \gamma & \beta + \gamma\sigma \end{pmatrix}$$

and the two non-zero eigenvalues of \mathbf{Q} are

$$\kappa_1 = \frac{1}{2(\gamma\sigma\psi + 1)} (1 + \beta + \gamma\sigma + \Psi), \quad (12)$$

$$\kappa_2 = \frac{1}{2(\gamma\sigma\psi + 1)} (1 + \beta + \gamma\sigma - \Psi),$$

$$\Psi = \sqrt{\beta^2 - 2\beta + \gamma^2\sigma^2 + 2\gamma\sigma + 2\gamma\sigma\beta - 4\gamma\sigma\beta\psi + 1}.$$

Assuming that $|\kappa_i| < 1$ for $i = 1, 2$ then the solution is given by (7), which in this case is:

$$R_t = \psi\pi_t + \varepsilon_{1t}, \quad (13)$$

$$x_t = -\sigma R_t + \varepsilon_{2t},$$

$$\pi_t = \gamma x_t + \varepsilon_{3t}.$$

This illustrates some of the features of DSGE models. First, the RE model parameter matrices, A_0 and A_1 , are written in terms of deeper parameters $\theta = (\gamma, \sigma, \psi, \beta)'$. Second, the parameters which appear only in A_1 do not enter the RE solution and, thus, do not enter the likelihood function. In this example, β does not appear in the likelihood function.⁵ Third, the restrictions necessary to ensure regularity (i.e. $|\kappa_i| < 1$ for $i = 1, 2$), imply bounds involving the structural parameters, including the unidentified β . Thus, the parameter space is not variation free. Fourth, if β is fixed at some pre-selected value for the discount rate (as would be done by a calibrator), then the model is identified. Canova and Sala (2009) make similar points with a similar model.

2.3 DSGE models with lags

In order to reproduce the dynamics that are typically observed with macroeconomic data, most empirical DSGE models include lagged values of endogenous or exogenous (observed or unob-

⁵Notice, though, that σ which appears in A_1 does appear in the likelihood function because it also appears in A_0 .

served) variables. For instance Clarida, Gali and Gertler (1999) assume that the errors in the IS and Phillips curve equations follow AR(1) processes and derive an optimal feedback policy for the interest rate based on the forecasts from these autoregressions. In this case, there is a predictable component in expected inflation because of the serial correlation in the equation errors.

Consider the special case of (1), where $A_3 = \Phi_u = 0$ so that the model only contains lagged endogenous variables

$$A_0 y_t = A_1 E_t(y_{t+1}) + A_2 y_{t-1} + \varepsilon_t. \quad (14)$$

In this case the unique solution is given by

$$y_t = C y_{t-1} + A_0^{-1} \varepsilon_t, \quad (15)$$

where C solves the quadratic matrix equation $A_1 C^2 - A_0 C + A_2 = 0$. The solution is unique and stationary if all the eigenvalues of C and $(I - A_1 C)^{-1} A_1$ lie strictly inside the unit circle.

Therefore, the RE solution is observationally equivalent to the non-RE structural model :

$$A_0 y_t = A_2 y_{t-1} + \varepsilon_t,$$

where, in the case of the SEM, $C = A_0^{-1} A_2$.

Again whereas the order condition for identification of the SEM requires n^2 restrictions, the RE model requires $2n^2$ restrictions. Not only is the RE model observationally equivalent to a purely backward looking SEM, it is observationally equivalent (in the sense of having the same reduced form), to any other model of expectations where in (14) $E_t(y_{t+1})$ is replaced by $D y_{t-1}$. More specifically, knowing the form of the solution, (15), does not, on its own, provide information on the cross equation parametric restrictions. In either case, the identifying cross-equation restrictions are lost.

Thus, in models with lags, the same problem of observational equivalence between RE and other models recurs. One may be able to distinguish the reduced forms of particular RE models from other observationally equivalent models, because the RE models impose particular types of

cross-equation restriction on the reduced form, which arise from the nature of the rational expectations. But such restrictions are subject to the objection made by Sims (1980), who criticized identification by ‘incredible’ dynamic restrictions on the coefficients and lag lengths. RE models, which depend on restrictions on the form of the dynamics, such as AR(1) errors, are equally vulnerable to such objections.

2.3.1 Example 2: A Hybrid New Keynesian Phillips Curve (NKPC)

A specific example where it is well known how identification depends on assumptions about the dynamics is the hybrid NKPC with an exogenous driving process:

$$\pi_t = \beta_b \pi_{t-1} + \beta_f E_{t-1} \pi_{t+1} + \gamma x_t + \varepsilon_t, \quad (16)$$

where it is assumed that there is no feedback from π_t to x_t and that x_t can be written as a finite order autoregression.

The parameters of (16) are nonlinear functions of underlying structural parameters. For instance, following Gali, Gertler and Lopez-Salido (2005) suppose that there is staggered price setting, with a proportion of firms, $(1 - \alpha)$, resetting prices in any period, and a proportion, α , keeping prices unchanged. Of those firms able to adjust prices only a fraction $(1 - \omega)$ set prices optimally on the basis of expected marginal costs. A fraction ω use a rule of thumb based on lagged inflation. Then for a subjective discount factor, β , we have

$$\beta_f = \beta \alpha \delta^{-1}, \quad \beta_b = \omega \delta^{-1}, \quad \gamma = (1 - \omega)(1 - \alpha)(1 - \lambda \alpha) \delta^{-1},$$

where $\delta = \alpha + \omega[1 - \alpha(1 - \beta)]$. If $\omega = 0$, all those who adjust prices do so optimally, then $\beta_f = \beta$, and $\beta_b = 0$. If the discount factor, $\beta = 1$, then $\beta_f + \beta_b = 1$ in either case. We will consider the identification of the three parameters, β_f , β_b , and γ ; but one could also consider identification of the four deeper parameters, β , α , ω , and δ . If the three parameters are not identified, then the four deeper parameters will not be.

If we assume ε_t is a martingale difference process; x_t follows a stationary time series process; there are no feedbacks from inflation to the output gap, $\beta_b, \beta_f \geq 0$, $\beta_f \beta_b \leq 1/4$ and $\beta_b + \beta_f \leq 1$,

then the NKPC (16) has the unique solution,

$$\begin{aligned} \pi_t &= \kappa_b \pi_{t-1} + \frac{\gamma}{1 - \kappa_b \beta_f} \sum_{j=0}^{\infty} \kappa_f^{-j} E_{t-1}(x_{t+j}) \\ &+ \gamma [x_t - E_{t-1}(x_t)] + \varepsilon_t, \end{aligned} \quad (17)$$

where κ_b and κ_f are roots of $\beta_f \kappa^2 - \kappa + \beta_b = 0$. The RE solution is unique if $|\kappa_b| \leq 1$ and $|\kappa_f| > 1$, which are satisfied if $\beta_b + \beta_f < 1$. In the case where $\beta_b + \beta_f = 1$, then $\kappa_b = 1$ and $\kappa_f = \beta_f^{-1}(1 - \beta_f) > 1$ if $\beta_f < 1/2$. Inflation will be $I(1)$ in this case. Finally, if $\beta_b + \beta_f > 1$, the RE solution will be indeterminate and there exists a multiplicity of solutions. Analysis of identification in this latter case is beyond the scope of the present paper and will not be considered.

As noted originally in Pesaran (1981, 1987, Ch. 7) and emphasized recently by Mavroeidis (2005), Beyer et al (2007) and Nason and Smith (2008) among others, identification of the structural parameters critically depends on the process generating x_t . For example, suppose that x_t follows the $AR(1)$ process

$$x_t = \rho x_{t-1} + v_t. \quad (18)$$

Then the RE solution is given by

$$\pi_t = \alpha_1(\theta) \pi_{t-1} + \alpha_2(\theta) x_{t-1} + u_t, \quad (19)$$

where $\theta = (\beta_b, \beta_f, \gamma, \rho)'$, $u_t = (\varepsilon_t + \gamma v_t)$, and

$$\begin{aligned} \alpha_1 &= \alpha_1(\beta_b, \beta_f) = \kappa_b = \frac{1 - \sqrt{1 - 4\beta_f \beta_b}}{2\beta_f}, \\ \alpha_2 &= \alpha_2(\beta_b, \beta_f, \gamma, \rho) = \frac{1}{1 - \kappa_b \beta_f} \left(\frac{\gamma \rho}{1 - \rho \kappa_f^{-1}} \right) = \frac{\gamma \rho}{1 - \beta_f(\kappa_b + \rho)}, \end{aligned}$$

The reduced form for (π_t, x_t) is a restricted $VAR(1)$ that allows consistent estimation of the three parameters, α_1, α_2 , and ρ , whilst we have four unknown coefficients, β_f, β_b, γ , and ρ . In this case the structural parameters β_f, β_b and γ are not identified.

For identification we need the order of the $AR(p)$ process for the output gap to be at least equal to two. In general if the output gap, x_t , is $AR(p)$, the form for the RE solution is $ARDL(1, p)$ in

π_t . In the case where x_t follows the $AR(2)$ process

$$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + v_t,$$

then the extra instrument x_{t-2} exactly identifies the model. But the identification can be weak if ρ_2 is not sufficiently large. Weak instruments make GMM and the usual tests for over-identification unreliable, e.g., Stock, Wright and Yogo (2002). We return to this example below.

3 Identification Issues in Rational Expectations DSGE Models: The Econometrics

3.1 Identification: General Issues

We begin with a brief overview of identification in a general context, before focussing on DSGE models. We use notation where $\theta = (\theta_1, \theta_2)'$ lies in a region Φ , $p(\theta)$ is the prior, $p(\theta|y)$ is the posterior and $L(\theta; y)$ is the likelihood function. We define identification as follows: θ is identified if $L(\theta^{(1)}; y) = L(\theta^{(2)}; y)$ implies that $\theta^{(1)} = \theta^{(2)}$. It can be seen that non-identification can occur in several ways as was illustrated in Examples 1 and 2. As we shall stress below, some of the traditional Bayesian results relate only to particular types of non-identification.

It is worth emphasizing that θ is our notation for the structural parameters and, in many cases one may have other parameters, α (e.g. reduced form parameters) which are functions of the structural parameters. It may be that α is identified, but elements of θ are not, e.g. if $\alpha = \theta_1 \theta_2$. Typically, economists are interested in structural parameters, ones that are invariant to a class of policy interventions. As Marschak (1953) noted, for many economic purposes it may be easier, and equally useful for policy purposes, to identify policy invariant combinations of structural parameters, such as α , rather than the individual parameters themselves, a point Heckman (2010) also makes.

Poirier (1998), building on earlier contributions (e.g. Kadane, 1974) sets out a framework for discussing identification in Bayesian models and we describe here a few of his key results which we will use later in our discussion of identification in DSGE models. Consider the case where the parameters in θ_2 are identified but the scalar, θ_1 , is not. In DSGE models, the parameter space

is often not variation free, so care must be taken with the bounds of the parameter space. Hence, we introduce notation where $\Lambda_1(\theta_2)$ and $\Lambda_2(\theta_1)$ defines the permissible range of values of θ_1 for given θ_2 , and θ_2 for given θ_1 , respectively. If the parameter space is variation free then we define $\Lambda_1 \equiv \Lambda_1(\theta_2)$ and $\Lambda_2 \equiv \Lambda_2(\theta_1)$.

Result 1: Suppose θ_1 is not identified in the sense that $L(\theta; y)$ is flat over $\theta_1 \in \Lambda_1(\theta_2)$. In this case, the likelihood function can be written as depending only on θ_2 (although θ_1 can still enter the bounds of the parameter space as $\Lambda_2(\theta_1)$).

It is straightforward to use Result 1 and Bayes' theorem to show:

Result 2: If there is prior independence between θ_1 and θ_2 such that $p(\theta_1, \theta_2) = p(\theta_1)p(\theta_2)$ and the parameter space is a product space (i.e. $\Phi = \Lambda_1 \times \Lambda_2$) then $p(\theta_1|y) = p(\theta_1)$.

This is the commonly cited result that "posterior equals prior for unidentified parameters". Note, however, that this result only holds for the particular type of non-identification defined in Result 1 and assumes prior independence and a variation free parameter space. If any of these conditions is not satisfied then $p(\theta_1|y) \neq p(\theta_1)$. Informally speaking, data based learning about θ_2 can "spill over" onto the unidentified θ_1 (see Koop and Poirier, 1997, for an example).

As we shall discuss below, a better metric for investigating identification can be constructed based on Proposition 2 of Poirier (1998) which we state here.

Result 3: Assume the conditions of Result 1 apply and let $p(\theta_1, \theta_2) = p(\theta_1|\theta_2)p(\theta_2)$ be the prior (which may exhibit correlation between θ_1 and θ_2), then

$$\begin{aligned} p(\theta_1|y) &= \int_{\Phi(\theta_2)} p(\theta_1|\theta_2, y) p(\theta_2|y) d\theta_2 \\ &= \int_{\Phi(\theta_2)} p(\theta_1|\theta_2) p(\theta_2|y) d\theta_2 \\ &= E_{\theta_2|y} [p(\theta_1|\theta_2)]. \end{aligned} \tag{20}$$

In words, the marginal posterior for the non-identified θ_1 will always be the posterior expectation of the conditional prior, $p(\theta_1|\theta_2)$.

The concepts discussed so far can be used with any econometric model, but we will use them below with DSGE models.

3.2 Bayesian Identification in DSGE Models

From the material in Section 2, it can be seen that some types of DSGE models are either simultaneous equations models, or closely related to them. For such models, of course, identification issues are well-understood. In the Bayesian literature on identification in the simultaneous equations model influential papers include Drèze (1976), Drèze and Richard (1983) and Kleibergen and van Dijk (1998). And Bayesian instrumental variable methods are well established (see, among many others, Kleibergen and Zivot, 2003, Hoogerheide, Kleibergen and van Dijk, 2007). Insofar as the DSGE model can be written as a conventional SEM, conventional methods can be used for Bayesian estimation and checking for identification. The NK-DSGE and NKPC models above fall in this category. There is also a literature relating to specific models such as the NKPC (e.g. Mavroeidis, 2005 and Kleibergen and Mavroeidis, 2009, 2010). For DSGEs which can be written in structural VAR form Rubio-Ramirez, Waggoner and Zha (2010) provides an exhaustive treatment.

Iskrev (2010a,b) are also important contributions. He uses the Jacobian of the transformation from the first two moments of the data to the structural parameters (in Iskrev, 2010a) and the information matrix (in Iskrev, 2010b) to investigate local identification. More informally, directly looking at the likelihood function and whether it is flat (or nearly so) can reveal a lack of identification or weak identification and this is sometimes done. For instance, An and Schorfheide (2007, Figures 14 and 15) also present plots of the log-likelihood function.⁶

It is also worth noting that the methods in Canova and Sala (2009), Iskrev (2010a,b) and Komunjer & Ng (2011) are not hypothesis testing procedures, but are better thought of as diagnostic procedures or indicators. Canova and Sala (2009) use the term diagnostics in reference to their methods. However, since diagnostic tests for misspecification are common in econometrics, we use the term indicator, to reinforce the point that these are not tests.

⁶Canova and Sala (2009) consider matching impulse response functions, so the diagnostics for detecting the existence and source of identification problems are not expressed in likelihood terms. But the basic ideas and concepts they use transfer to likelihood function terms discussed below.

Bayesians typically use posterior simulation algorithms to estimate DSGE models. Our first proposed indicator can be calculated as part of such a posterior simulation algorithm without the need for additional steps such as the coding of analytical derivatives. Our second indicator involves using artificial data but it, as well, will involve standard posterior simulation algorithms. The Bayesian who uses proper priors will (under weak conditions) obtain a proper posterior, allowing for valid statistical inference. Since the parameters in DSGE models have a structural interpretation, sensible proper priors are usually available. These priors may be purely subjective or could reflect data from other sources (e.g. the priors could reflect estimates of structural parameters produced in microeconomic studies or could be based on a training sample of macroeconomic data). Given such prior information, there is a sense in which identification is not a worry for the Bayesian DSGE modeler.⁷ However, if a parameter is not identified, then there is the possibility that there is no data-based learning about it and its posterior can solely reflect prior information. In complicated models such as DSGEs, where it can be hard to analytically disentangle identification issues, this can lead to the case where the researcher believes she is presenting posterior estimates but is really simply reproducing her prior.

Even if parameters are identified, weak identification can lead to relatively flat regions of the likelihood function where the prior is extremely influential. Such concerns have led to a recent interest in identification issues in Bayesian DSGE modelling. Consider, for instance, Canova (2007, page 190) which states “while it is hard to ‘cheat’ in a classical framework, it is not very difficult to give the impression that identification problems are absent in a Bayesian framework by choosing tight enough priors, presenting well-behaved posterior distributions and entirely side-stepping the comparison between priors and posteriors”. In response to this, an increasingly common practice is to compare priors and posteriors for structural parameters, a practice which Canova (2007, page 191) refers to as “necessary [but] by no means sufficient” to reveal identification problems in DSGE models. We will draw on our earlier discussion of Bayesian identification (see Section

⁷In one of the classic Bayesian texts, identification is mentioned only in an oft-quoted footnote: “In passing it might be noted that unidentifiability causes no real difficulty in the Bayesian approach” (Lindley, 1971, page 46 footnote 34).

3.1) to discuss why this is so and introduce an alternative method for investigating identification in Bayesian DSGE models.

Result 2 of Section 3.1 underlies some informal discussion of identification in the Bayesian DSGE literature. For instance, An and Schorfheide (2007, page 127) say that: “A direct comparison of priors and posteriors can often provide valuable insights about the extent to which data provide information about parameters of interest.” This is true, but can be an imperfect way of formally investigating identification issues, since the posterior for an unidentified parameter can differ substantially from its prior if the non-identified parameter is, a priori, correlated with identified ones or if the parameter space is not a product space. Both of these are likely to hold with DSGE models. In fact, papers such as Del Negro and Schorfheide (2008) make a strong case that priors for structural parameters in DSGE models should not exhibit prior independence. Thus, informally comparing priors to posteriors could be useful to see *if* learning about parameters occurs, but may not be able tell the researcher *why* it is occurring. That is, the researcher may be unable to distinguish between learning via the likelihood function and learning solely due to the fact that the prior does not exhibit independence or the parameter space is not variation free.

Nevertheless, it is common in the Bayesian DSGE literature to use such informal comparisons of priors and posteriors, as the quote from An and Schorfheide above indicates. Among many others, Smets and Wouters (2007, page 594) compare prior and posteriors and note that the mean of the posterior distribution is typically quite close to the mean of the prior assumptions and later note that "It appears that the data are quite informative on the behavioral parameters, as indicated by the lower variance of the posterior distribution relative to the prior distribution." As a recent example, Guerron-Quintana (2010, page 782) says “Initial estimation attempts showed that the posteriors of [certain structural parameters] sat on top of their priors. Hence those parameters are fixed to the values [taken from another paper]”. Statements similar to this implicitly suggest a comparison of prior to posterior is useful for checking identification in complicated DSGE models where it is not easy to analytically check identification.

Result 3 of Section 3.1 offers a promising way of formally investigating identification issues. In cases where a subset of structural parameters, say θ_2 , is known to be identified, but there is doubt regarding the identification of another sub-set, θ_1 , then $p(\theta_1|y)$ should be equal to $E_{\theta_2|y}[p(\theta_1|\theta_2)]$ where we use the notation of Section 3.1. For DSGE modelers interested in indicators that may shed light on identification issues, we would recommend comparing the properties of $p(\theta_1|y)$ and $E_{\theta_2|y}[p(\theta_1|\theta_2)]$ in addition to (or instead of) comparing $p(\theta_1|y)$ to $p(\theta_1)$.

In terms of computation, note that our proposed indicator is typically easy to calculate. That is, the Bayesian DSGE modeler will typically be using an MCMC algorithm and, thus, posterior draws of θ_2 will be available. Calculation of $E_{\theta_2|y}[p(\theta_1|\theta_2)]$ simply evolves evaluating $p(\theta_1|\theta_2)$ at each draw of θ_2 at a grid of values for θ_1 and averaging across the posterior draws of θ_2 . In many cases an analytical form for $p(\theta_1|\theta_2)$ will be available. For instance, if $p(\theta_1, \theta_2)$ is normal then $p(\theta_1|\theta_2)$ is also normal with textbook formula for its mean and variance. For priors which do not admit of analytical results, adding a prior simulation step at each posterior draw would only slightly add to the computational burden.

Formally, if a parameter is unidentified then $p(\theta_1|y)$ and $E_{\theta_2|y}[p(\theta_1|\theta_2)]$ should be identical, apart from MCMC approximation error. Hence, the two densities cannot be used as a test for identification. That is, any difference between $p(\theta_1|y)$ and $E_{\theta_2|y}[p(\theta_1|\theta_2)]$ beyond MCMC approximation error means identification is present. However, we can use the magnitude of the difference between these densities as an indicator revealing the strength of identification.

We will illustrate the usefulness of this indicator below. However, for DSGE models, it has one substantive drawback. For the theory underlying Result 3 to hold, the parameters in θ_2 must all be identified and θ_1 must not enter the likelihood function (this point is stressed on page 489 of Poirier, 1998). When working with a DSGE model, we would like to simply set θ to be the structural parameters. But typically we will not be able to do so (in the sense that the indicator defined in this way will not necessarily be zero for non-identified parameters). This is because the parameters in θ_2 defined in this way may not all be identified.

To see how this can happen in practice, we return to our example involving the NKPC (see Example 2 in Section 2) with an AR(1) process for the output gap. The two reduced form parameters in the inflation equations, a_1 and a_2 , depend on three structural parameters β_f, β_b and γ and, thus, there is an identification problem involving these three structural parameters.⁸ In contrast to Example 1, the identification problem does not manifest itself simply in terms of a single structural parameter which does not enter the likelihood function. Using the notation of Result 3, we might be tempted to set $\theta_1 = \beta_f$ and $\theta_2 = (\beta_b, \gamma, \rho)$ in order to investigate the identification of β_f . However, it can be easily seen that the derivations in (20) used to prove Result 3 are no longer valid. In such cases, we will not have $p(\theta_1|y) = E_{\theta_2|y}[p(\theta_1|\theta_2)]$, even though θ_1 is only partially identified.

The advice given by Poirier (1998) in such cases is to re-parameterize the model so that θ_2 contains only identified parameters. In some DSGE cases, a simple way of choosing θ_2 suggests itself: let θ_2 be some or all of the reduced form parameters and θ_1 be one of the structural parameters. We know the reduced form parameters are identified and, thus, the conditions under which Result 3 holds are satisfied. In such cases, we can recommend a comparison of $p(\theta_1|y)$ and $E_{\theta_2|y}[p(\theta_1|\theta_2)]$ as shedding light on the identification of θ_1 .

3.3 A Bayesian Identification Indicator Based on Large Sample Results

The advantage of the identification indicator of the preceding sub-section is that it can be calculated as a by-product of estimating a DSGE model (on the actual data) using MCMC methods. The disadvantage is that we typically cannot simply work with the structural parameters of a model. A second Bayesian identification indicator can be obtained based on the asymptotic theory of non-identified models written in terms of the structural parameters. Thus, we can focus on identification of a single structural parameter without worrying about whether the other structural parameters are identified or not. Empirically, this indicator involves simulating artificial data sets of increasing size and then estimating the DSGE model using these data sets. Since the

⁸ a_2 also depends on ρ but, since this parameter identified through the AR(1) process for the output gap, this dependence is irrelevant for the argument we are making here.

generation of simulated data is fairly standard in the DSGE literature, this strategy fits in with existing empirical methodologies.

To explain the theory underlying this second indicator of identification, note that standard Bayesian results for stationary models (see, e.g., Berger, 1985, page 224), imply that, under certain regularity conditions, the Bayesian asymptotic theory relating to the posterior is numerically identical to the asymptotic distribution theory for the maximum likelihood estimator. Thus, for instance, the posterior for θ will asymptotically converge to its true value and the role of the prior will vanish. One of the regularity conditions is that θ is identified. In this sub-section, we relax this assumption and show that this asymptotic convergence will not occur. This result holds even for cases of partial identification such as the NKPC of our Example 2.

To derive this result, let $\theta = (\theta_1, \theta_2, \dots, \theta_p)'$ be a $p \times 1$ vector of structural parameters of interest in a DSGE model of the type set out in Section 2. Let $\theta \in \Theta \subset \mathbb{R}^p$, and suppose that the likelihood function for a sample of T observations can be written as $L_T(\alpha(\theta); y)$, where $\alpha(\theta) = (\alpha_1(\theta), \alpha_2(\theta), \dots, \alpha_k(\theta))'$ is a $k \times 1$ vector-valued function of θ . Further suppose that $\alpha_s(\theta)$, for $s = 1, 2, \dots, k$ are differentiable at each point of the open convex set Θ^0 containing Θ , and denote the first derivatives of $\alpha_s(\theta)$ by $\partial\alpha_s(\theta)/\partial\theta_i$, $i = 1, 2, \dots, p$. Then by the mean value theorem for all pairs θ^a and $\theta^b \in \Theta^0$ we have (see, for example, Jennrich (1969, lemma 3), and Davidson (1994, p. 340))⁹

$$\alpha_s(\theta^a) - \alpha_s(\theta^b) = \sum_{i=1}^p \frac{\partial\alpha_s(\hat{\theta})}{\partial\theta_i} (\theta_i^a - \theta_i^b), \quad (21)$$

where $\hat{\theta} \in \Theta^0$ is a point on the line segment joining θ^a and θ^b .

Let $\ell_T(\alpha) = \ln [L_T(\alpha; y)]$, and assume that

$$Q_T(\alpha) = \frac{-1}{T} \frac{\partial^2 \ell_T(\alpha)}{\partial\alpha \partial\alpha'} \quad (22)$$

is a positive definite matrix for all values of $\alpha \in \mathbf{A} \subset \mathbb{R}^k$. Denote the maximum likelihood

⁹Note that Jennrich formulation of the mean value theorem allows θ^a or θ^b to depend on y , as will be required in our application below.

estimator of α by $\hat{\alpha}_T$ and assume that

$$\sqrt{T}(\hat{\alpha}_T - \alpha_0) \rightarrow_d N(0, Q^{-1}), \quad (23)$$

where α_0 is the true value of $\alpha \in \mathbf{A}$, and $Q = \text{plim}_{T \rightarrow \infty} Q_T(\alpha_0) = Q > 0$, is a symmetric positive definite matrix. Similarly, corresponding to α_0 select $\theta_0 \in \Theta$ such that $\alpha(\theta_0) = \alpha_0$. The choice of θ_0 need not be unique so long as $\alpha_0 = \alpha(\theta_0)$. Here we assume that α is identified. Since the quadratic approximation underlying (23) might work poorly if α is weakly identified, we rule out this case here. However, our analysis can be appropriately modified if one or more elements of the reduced form parameters, α , are weakly identified. Since our focus is on identification of the structural parameters, θ , it seems reasonable to begin with the standard case where the reduced form parameters are identified and their identification is not weak. In such a setting, using standard results from \sqrt{T} -consistent ML estimators, we have (noting that $\partial \ell_T(\hat{\alpha}_T)/\partial \alpha_s = \mathbf{0}$)

$$\ell_T(\alpha_0) = \ell_T(\hat{\alpha}_T) + \frac{1}{2} \sum_{s=1}^k \sum_{r=1}^k \frac{\partial^2 \ell_T(\hat{\alpha}_T)}{\partial \alpha_r \partial \alpha_s} (\alpha_{0r} - \hat{\alpha}_{rT})(\alpha_{0s} - \hat{\alpha}_{sT}) + O_p(T^{-1/2}). \quad (24)$$

The object of the exercise is to derive the posterior distribution of θ_1 (one of the elements of θ) assuming that the prior density of θ is given by

$$p(\theta) = (2\pi)^{-p/2} |\mathbf{H}|^{1/2} \exp -\frac{1}{2}(\theta - \underline{\theta})' \mathbf{H}(\theta - \underline{\theta}), \quad (25)$$

where $\underline{\theta}$ and \mathbf{H} are prior mean vector and prior precision matrix of θ defined on Θ . The posterior density of θ , evaluated at θ_0 , can be written as

$$p(\theta_0 | y) = \exp\{\ln [p(\theta_0)] + \ell_T(\alpha(\theta_0)) - \ln(p(y))\}. \quad (26)$$

When T is finite, assuming a proper prior, the posterior of θ is well defined even if θ is not identified. Denote the $k \times p$ matrix of derivatives $\frac{\partial \alpha(\theta)}{\partial \theta'} = (\partial \alpha_s(\theta)/\partial \theta_i)$ by $R(\theta)$. Then in the classical sense θ is globally identified if $\text{Rank}(R(\theta_0)) = k \geq p$ for all $\theta_0 \in \Theta$, and θ is identified locally in the neighborhood of $\underline{\theta}$ if $\text{Rank}(R(\underline{\theta})) = k \geq p$. A necessary condition for identification is given by $p \leq k$.

Suppose that $p = k + 1$, and θ_1 is not identified. But for simplicity assume that the remaining $(p - 1) \times 1$ structural parameters, $\theta_* = (\theta_2, \dots, \theta_p)'$, are identified. The generalization to the case where two or more structural parameters are unidentified is discussed below.

Note that, if θ_1 does not enter the likelihood function and θ_1 is, *a priori*, uncorrelated with θ_* , then Result 2 of Section 3.1 says that $p(\theta_1) = p(\theta_1|y)$ for all T . In this case, the posterior precision of θ_1 does not get updated at all as sample size increases (i.e. it remains equal to prior precision). Formally, if we define average posterior precision as posterior precision divided by T , then average posterior precision will tend to zero with T in this case. The derivations below show that the statement “average posterior precision will tend to zero with T ” will hold whenever θ_1 is unidentified (even if the nature of the identification problem is more complicated than that assumed in Result 2). However, when θ_1 is identified, then the average posterior precision will tend to a strictly positive constant which is independent of the prior precision.

To see why the result arises, note that although θ is not identified, there exists $\hat{\theta}_T$ such that $\hat{\alpha}_T = \alpha(\hat{\theta}_T)$. The choice of $\hat{\theta}_T$ need not be unique but as we shall see this is of no consequence for the derivation of posterior precision of θ_1 . Using the mean value theorem in (21) for $\hat{\theta}_T$ and $\theta_0 \in \Theta^0$, we have

$$\alpha_s(\theta_0) - \alpha_s(\hat{\theta}_T) = \sum_{i=1}^p \frac{\partial \alpha_s(\hat{\theta})}{\partial \theta_i} (\theta_{0i} - \hat{\theta}_{iT}),$$

where $\hat{\theta}$ is on the line segment joining $\hat{\theta}_T$ and θ_0 . Using this result in (24) we have (recalling that $\alpha_0 = \alpha(\theta_0)$)

$$\ell_T(\theta_0) = \ell_T(\hat{\theta}_T) + \frac{1}{2} \sum_{s=1}^k \sum_{r=1}^k \sum_{i=1}^p \sum_{j=1}^p \frac{\partial^2 \ell_T(\hat{\alpha}_T)}{\partial \alpha_r \partial \alpha_s} \frac{\partial \alpha_s(\hat{\theta})}{\partial \theta_i} \frac{\partial \alpha_r(\hat{\theta})}{\partial \theta_j} (\theta_{0i} - \hat{\theta}_{iT})(\theta_{0j} - \hat{\theta}_{jT}) + O_p(T^{-1/2}).$$

Written more compactly we have

$$\ell_T(\theta_0) = \ell_T(\hat{\theta}_T) - \frac{T}{2} (\theta_0 - \hat{\theta}_T)' \hat{S}_T (\theta_0 - \hat{\theta}_T) + O_p(T^{-1/2}),$$

where $\hat{S}_T = \hat{R}' Q_T \hat{R}$ and $\hat{R} = R(\hat{\theta})$, for all θ_0 and $\hat{\theta} \in \Theta^0$. Using this result and (25) in (26) we

have (up to terms of $O_p(T^{-1/2})$)

$$p(\theta_0 | y) \propto \exp\left\{-\frac{1}{2}(\theta_0 - \underline{\theta})' \mathbf{H}(\theta_0 - \underline{\theta}) - \frac{T}{2}(\theta_0 - \hat{\theta}_T)' \hat{S}_T(\theta_0 - \hat{\theta}_T)\right\}.$$

Using textbook results for combining a normal prior with normal likelihood, the posterior distribution of θ is approximately normal with mean $\bar{\theta}_T$ and the precision matrix \bar{H}_T , where

$$\bar{\theta}_T = \left(T \hat{S}_T + \mathbf{H}\right)^{-1} (T \hat{S}_T \hat{\theta}_T + \mathbf{H} \underline{\theta}), \text{ and } \bar{H}_T = T \hat{S}_T + \mathbf{H}.$$

It is clear that $\bar{\theta}_T$ is defined even if \hat{R} , or equivalently \hat{S}_T , fails the rank condition. Since the marginals of multivariate normal are also normally distributed, the posterior of θ_1 is (approximately) normally distributed with mean $\bar{\theta}_{1T}$, where $\bar{\theta}_{1T}$ is the first element of $\bar{\theta}_T$, and the posterior precision of θ_1 is (approximately) given by (suppressing the T subscript to simplify the exposition)

$$\bar{h}_{11} = \bar{H}_{11} - \bar{H}_{12} \bar{H}_{22}^{-1} \bar{H}_{21},$$

where

$$\bar{H} = \begin{pmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & \bar{H}_{22} \end{pmatrix} = T \begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{pmatrix} + \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{pmatrix}.$$

Hence

$$\bar{h}_{11} = (T \hat{S}_{11} + \mathbf{H}_{11}) - \left(T \hat{S}_{12} + \mathbf{H}_{12}\right) \left(T \hat{S}_{22} + \mathbf{H}_{22}\right)^{-1} \left(T \hat{S}_{21} + \mathbf{H}_{21}\right).$$

It is clear that when T is finite \bar{h}_{11} is well defined irrespective of whether \hat{R} is a full rank matrix or not. Note that, even if $\mathbf{H}_{21} = 0$, the posterior of θ_1 may not be independent of the posterior of θ_* . This is because in general $\hat{S}_{12} \neq 0$. In the case where $p = k$ posterior independence follows if $\hat{R}_{12} = Q_{12} = 0$.

Consider now the case where $T \rightarrow \infty$, and note that since

$$T^{-1} \bar{h}_{11} = \left(\hat{S}_{11} + T^{-1} \mathbf{H}_{11}\right) - \left(\hat{S}_{12} + T^{-1} \mathbf{H}_{12}\right) \left(\hat{S}_{22} + T^{-1} \mathbf{H}_{22}\right)^{-1} \left(\hat{S}_{21} + T^{-1} \mathbf{H}_{21}\right),$$

then as $T \rightarrow \infty$ we have

$$\lim_{T \rightarrow \infty} (T^{-1} \bar{h}_{11}) = S_{11} - S_{12} S_{22}^{-1} S_{21},$$

where

$$S = p \lim_{T \rightarrow \infty} (\hat{R}' Q_T \hat{R}) = R' Q R = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

In general $R = R(\theta)$, where θ can take any value in Θ^0 . Note that the probability limit of $\hat{\theta}$ is not unique but must lie in Θ^0 . Therefore, without loss of generality and to simplify the notations in what follows we use the generic symbol θ to represent the limiting value of $\hat{\theta}$.

In the case where R is full rank, S is a positive definite matrix. Hence, $S_{11} - S_{12} S_{22}^{-1} S_{21} > 0$, and $\lim_{T \rightarrow \infty} (T^{-1} \bar{h}_{11})$ is strictly positive and does not depend on the prior precision. But when the rank condition is not satisfied the above result does not follow. For example, suppose that $p = k + 1$ and consider the following partition of R

$$R = \begin{pmatrix} r & R_* \end{pmatrix},$$

where r is a $k \times 1$ vector and R_* is $k \times k$. Recall that $p = 1 + k$, and by assumption R_* is a non-singular matrix. Then

$$S = R' Q R = \begin{pmatrix} r' \\ R_*' \end{pmatrix} Q \begin{pmatrix} r & R_* \end{pmatrix} = \begin{pmatrix} r' Q r & r' Q R_* \\ R_*' Q r & R_*' Q R_* \end{pmatrix}.$$

Hence

$$S_{11} - S_{12} S_{22}^{-1} S_{21} = r' Q r - r' Q R_* (R_*' Q R_*)^{-1} R_*' Q r.$$

But since R_* and Q are both non-singular then it readily follows that $S_{11} - S_{12} S_{22}^{-1} S_{21} = 0$, and

$$\lim_{T \rightarrow \infty} (T^{-1} \bar{h}_{11}) = 0.$$

Namely the posterior precision of θ_1 must change at a rate slower than T when θ_1 is non-identified. In terms of the posterior variance this result confirms that the posterior variance of a non-identified parameter need not tend to zero, and in cases that it does its rate of decline must be slower than T .

The above result readily generalizes when two or more of the structural parameters are unidentified. Consider the case where $\theta = (\theta'_1, \theta'_2)'$ with θ_1 the $s \times 1$ vector of unidentified parameters and

θ_2 the $(p-s) \times 1$ vector of the identified parameters where $p-s = k$. Partition R as $R = (R_1, R_2)$, where R_1 and R_2 are $k \times s$ and $k \times k$ matrices, where R_2 is non-singular and write S as

$$S = R'QR = \begin{pmatrix} R_1'QR_1 & R_1'QR_2 \\ R_2'QR_1 & R_2'QR_2 \end{pmatrix}.$$

Then noting that by assumption Q and R_2 are non-singular matrices, it readily follows that $S_{11} - S_{12}S_{22}^{-1}S_{21} = 0$, as required.

We propose to use these results as a second Bayesian indicator of identification which can be used when our first Bayesian indicator of identification is not applicable. That is, we recommend the following strategy: Suppose that it is of interest to investigate if one or more elements of θ are identified, for example in the locality of the prior mean vector, $\underline{\theta}$. First, the researcher should generate an artificial data set of size T from the DSGE model at $\theta = \underline{\theta}$. T should be chosen to be a large value where asymptotic results are expected to be very good approximations (e.g. the empirical illustrations below set $T = 10,000$). Second, the researcher should estimate the DSGE model using sample sizes $\tau = cT$ for a grid of values for c (e.g. $c = 0.0001, 0.001, 0.01, 0.1, 1.0$) and calculate a measure which relates to the posterior precision for every parameter. By comparing the behavior of the measure over different sample sizes, the researcher can see which parameters are identified and which are not. For instance, the posterior variance should be going to zero with sample size for identified parameters, but will not be doing so for unidentified parameters, while the posterior precision, divided by the sample size will go to a constant for identified parameters, and go to zero for unidentified parameters. Notice that for identified parameters, the variance need not go to zero monotonically, the variance may increase before falling, depending on the prior.

This approach provides an indicator of local identification (i.e. it will indicate identification at the parameter values used to generate the artificial data). Hence, the researcher may wish to carry out the procedure for various artificial data sets generated with different parameter values. This strategy is comparable to the one used by Iskrev (2010a), who draws parameter values from the prior and checks identification at each of the draws.

3.3.1 Example 3: Regressions with exactly collinear regressors

As an example of the preceding derivations, consider the following simple regression model

$$y_t = \theta_1 x_{1t} + \theta_2 x_{2t} + u_t,$$

where θ_1 and θ_2 are the parameters of interest. Suppose that $x_{2t} = \rho x_{1t}$ where ρ is a known non-zero constant. Then

$$y_t = \alpha(\theta) x_{1t} + u_t,$$

where $\alpha = \alpha(\theta) = \theta_1 + \rho\theta_2$. Assuming a normal prior for the parameters of interest, as in (25), causes α and θ_1 to be dependent on one another (unless θ_1 and θ_2 are a priori dependent and $\rho = -v_{11}/v_{12}$, where v_{ij} for $i, j = 1, 2$ denote the prior variance-covariances of θ_1 and θ_2). Therefore, in cases where θ_1 and θ_2 are a priori independent (i.e. $v_{12} = 0$) or when $v_{12} \neq 0$ but $\rho \neq -v_{11}/v_{12}$, then α and θ_1 are a priori dependent and Result 2 of Section 3.1. does not apply.

Using our previous results, the posterior precision of θ_1 for a finite T is given by

$$T^{-1}\bar{h}_{11} = (\hat{s}_{11} + T^{-1}\mathbf{h}_{11}) - (\hat{s}_{12} + T^{-1}\mathbf{h}_{12}) (\hat{s}_{22} + T^{-1}\mathbf{h}_{22})^{-1} (\hat{s}_{21} + T^{-1}\mathbf{h}_{21}),$$

where

$$\hat{S} = R'Q_T R, R = \begin{pmatrix} 1 & \rho \end{pmatrix}, Q_T = T^{-1} \sum_{t=1}^T x_{1t}^2 = s_{T1}^2 = s_1^2 > 0,$$

and hence

$$\hat{s}_{11} = s_1^2, \hat{s}_{12} = \hat{s}_{21} = \rho s_1^2 \text{ and } \hat{s}_{22} = \rho^2 s_1^2.$$

Therefore

$$\begin{aligned} T^{-1}\bar{h}_{11} &= s_1^2 \left\{ 1 + T^{-1}(\mathbf{h}_{11}/s_1^2) - \frac{(\rho + T^{-1}\mathbf{h}_{21}/s_1^2)^2}{\rho^2 + T^{-1}(\mathbf{h}_{22}/s_1^2)} \right\} \\ &= T^{-1}s_1^2 \left\{ \frac{(\mathbf{h}_{22}/s_1^2) + (\mathbf{h}_{11}/s_1^2)\rho^2 + (\mathbf{h}_{11}/s_1^2)T^{-1}(\mathbf{h}_{22}/s_1^2) - 2\rho\mathbf{h}_{21}/s_1^2 - T^{-1}(\mathbf{h}_{21}/s_1^2)^2}{\rho^2 + T^{-1}(\mathbf{h}_{22}/s_1^2)} \right\}. \end{aligned}$$

When the priors of θ_1 and θ_2 are independent the above expression simplifies to

$$\bar{h}_{11} = \frac{\mathbf{h}_{22} + \rho^2\mathbf{h}_{11} + T^{-1}\mathbf{h}_{11}\mathbf{h}_{22}/s_1^2}{\rho^2 + T^{-1}(\mathbf{h}_{22}/s_1^2)}.$$

Hence, posterior precision (\bar{h}_{11}) of the unidentified parameter, θ_1 , differs from its prior precision (\underline{h}_{11}) for all T , and as $T \rightarrow \infty$, even though θ_1 and θ_2 are assumed to be a priori independent. For T sufficiently large we have

$$\lim_{T \rightarrow \infty} \bar{h}_{11} = \underline{h}_{11} + \rho^{-2} \underline{h}_{22}.$$

Hence, the posterior precision is bounded in T , in contrast to the posterior precision of an identified parameter.

The extent to which the posterior precision deviates from the prior precision is determined by $\underline{h}_{22}/\rho^2$. It is also worth noting, however, that as T increases the posterior precision declines. This could be viewed as an indication that θ_1 is not identified. In the case where a parameter is identified we would expect the posterior precision to rise with T and eventually dominate the prior precision.

3.3.2 Example 4: The NKPC with no Backward Looking Behavior

Consider the NKPC (see Example 2 in Section 2.3.1) and, for simplicity, assume that there is no backward looking behavior ($\beta_b = 0$) and simplify notation by defining $\beta \equiv \beta_f$. The solution of this model can be written as:

$$y_t = \alpha(\gamma, \beta)x_t + u_t,$$

where

$$\alpha \equiv \alpha(\gamma, \beta) = \frac{\gamma\rho}{1 - \rho\beta},$$

and assume that ρ is known (it is identified, so asymptotically its estimator will converge to the true value). For this example we have

$$R = \left(\begin{array}{cc} \frac{\rho}{1 - \rho\beta} & \frac{\gamma\rho^2}{(1 - \rho\beta)^2} \end{array} \right).$$

It is clear that the rank condition is not satisfied and neither of the structural parameters is identified.

This example is complicated by the fact that R depends on the unknown parameters. With this in mind we note that R can also be written as

$$R = \frac{\rho}{1 - \rho\beta} \begin{pmatrix} 1 & \alpha \end{pmatrix}.$$

Furthermore

$$\hat{R} = \frac{\rho}{1 - \rho\hat{\beta}} \begin{pmatrix} 1 & \hat{\alpha} \end{pmatrix}, \quad Q_T = T^{-1} \sum_{t=1}^T x_t^2 = s_x^2 > 0,$$

$$\hat{S} = \hat{R}' Q_T \hat{R} = \frac{\rho^2 s_x^2}{(1 - \rho\hat{\beta})^2} \begin{pmatrix} 1 & \hat{\alpha} \\ \hat{\alpha} & \hat{\alpha}^2 \end{pmatrix}.$$

$\hat{\alpha}$ could be selected to be the OLS estimator of α in the regression of y_t on x_t . But the choice of $\hat{\beta}$ is arbitrary so long as it lies in the range of $[0, 1)$ - but as we shall see below in the limit $\hat{\beta}$ gets eliminated from the posterior precisions.

It can now be seen that the following approximate results hold:

$$T^{-1} \bar{h}_{\gamma\gamma} \approx \left(\frac{\rho^2 s_x^2}{(1 - \rho\hat{\beta})^2} + T^{-1} \mathbf{h}_{\gamma\gamma} \right) - \frac{\left(\frac{\rho^2 s_x^2 \hat{\alpha}}{(1 - \rho\hat{\beta})^2} + T^{-1} \mathbf{h}_{\gamma\beta} \right)^2}{\frac{\rho^2 s_x^2 \hat{\alpha}^2}{(1 - \rho\hat{\beta})^2} + T^{-1} \mathbf{h}_{\beta\beta}},$$

and similarly

$$T^{-1} \bar{h}_{\beta\beta} \approx \frac{\rho^2 \hat{\alpha}^2 s_x^2}{(1 - \rho\hat{\beta})^2} + T^{-1} \mathbf{h}_{\beta\beta} - \frac{\left(\frac{\rho^2 s_x^2 \hat{\alpha}}{(1 - \rho\hat{\beta})^2} + T^{-1} \mathbf{h}_{\gamma\beta} \right)^2}{\frac{\rho^2 s_x^2}{(1 - \rho\hat{\beta})^2} + T^{-1} \mathbf{h}_{\gamma\gamma}}.$$

In the case where priors of γ and β are independent of one another, $\mathbf{h}_{\gamma\beta} = 0$, we obtain

$$\bar{h}_{\gamma\gamma} \approx \mathbf{h}_{\gamma\gamma} + \frac{\frac{\rho^2 s_x^2}{(1 - \rho\hat{\beta})^2} \mathbf{h}_{\beta\beta}}{\frac{\rho^2 s_x^2 \hat{\alpha}^2}{(1 - \rho\hat{\beta})^2} + T^{-1} \mathbf{h}_{\beta\beta}}$$

and as $T \rightarrow \infty$

$$\lim_{T \rightarrow \infty} (\bar{h}_{\gamma\gamma}) = \mathbf{h}_{\gamma\gamma} + \frac{\mathbf{h}_{\beta\beta}}{\alpha^2},$$

which does not depend on the nuisance parameter β . Similarly

$$\bar{h}_{\beta\beta} \approx \mathbf{h}_{\beta\beta} + \frac{\frac{\rho^2 \hat{\alpha}^2 s_x^2}{(1 - \rho\hat{\beta})^2} \mathbf{h}_{\gamma\gamma}}{\frac{\rho^2 s_x^2}{(1 - \rho\hat{\beta})^2} + T^{-1} \mathbf{h}_{\gamma\gamma}},$$

and

$$\lim_{T \rightarrow \infty} (\bar{h}_{\beta\beta}) = \mathbf{h}_{\beta\beta} + \alpha^2 \mathbf{h}_{\gamma\gamma}.$$

In the case of both structural parameters the posterior precisions are dominated by the prior precisions even if T is sufficiently large. In neither case do the prior and the posterior precisions coincide despite the prior independence of the structural parameters.

This example illustrates the drawback of our first Bayesian indicator of identification discussed above. That is, Results 2 and 3 of Section 3.1 (or the propositions of Poirier, 1998, Section 2), obtain when the vector of parameters is written in terms of those which are identified and those which are not. In cases of partial identification such as this one, the vector of structural parameters cannot be written in this way (i.e. β and γ are both unidentified and thus, neither can be included in what we called θ_2 in Results 2 and 3). However, it also shows how our large sample derivations can be used as a second Bayesian indicator of identification which is valid even in cases such as this.

4 Applications

In this section, we illustrate both of our Bayesian identification indicators in the context of the two examples of DSGE models introduced in Section 2. These are the NK-DSGE (see Section 2.2.1) and the NKPC (see Section 2.3.1). For the NKPC, we use the simplified version of the model with no backward looking behavior (used in Section 3.3.2).

4.1 Example 1 (cont.): Bayesian identification of the simple NK-DSGE model

Previously, we introduced a simple NK-DSGE in (9), (10) and (11). We will illustrate some issues relating to Bayesian inference and identification in this simple and easily understood model where the identification of the model can be immediately seen. This example involves four structural parameters, σ , γ , ψ and β . The rational expectations solution given in (13) does not involve β so this parameter is unidentified. However, the bounds given in (12) which ensure regularity such that there is a unique stationary solution do involve β .

We generated one artificial data set of $T = 10,000$ observations from (13) with $\sigma = 0.4$, $\gamma = 0.75$ and $\psi = 2.0$. These values were chosen so as to be not too far from the boundaries

given in (12), but also not too near.¹⁰ The errors, ε_{jt} for $j = 1, 2, 3$ are all standard normal and independent of one another.

We estimate the model using different sample sizes and two different priors. Both priors are normal with prior means: $E(\sigma) = 0.4$, $E(\gamma) = 0.75$, $E(\psi) = 2.0$ and $E(\beta) = 0.9$. The two priors differ in their prior variances. Let $\theta = (\sigma, \gamma, \psi, \beta)'$. The first prior (which we call the Independent Prior) has $\text{var}(\theta) = I$. The second prior (the Dependent Prior) has the same prior covariance matrix except for a single element: this is the covariance between β and σ which is set to 0.9. These priors are combined with the likelihood function based on the three equation system in (13). We use a random walk Metropolis-Hastings algorithm to do posterior simulation in this model.¹¹

We begin by illustrating the properties of our first Bayesian identification indicator with $T = 100$ (i.e. we use only the first 100 of the artificially generated observations). Figures 1 and 2 graph various priors and posteriors for β and σ , respectively, for the Independent Prior. Figures 3 and 4 do the same for the Dependent Prior. For the sake of brevity, we only present graphs for one identified and one non-identified parameter and, thus, do not present graphs for γ and ψ .

Consider first the priors and posteriors for β . Since β is unidentified, a naive researcher may expect its posterior to equal its prior. For the reasons discussed in Section 3, this may not be the case. The top panels of Figures 1 and 3 illustrate this empirically. Even with the Independent Prior (where β is, a priori, uncorrelated with the other parameters in the model), the fact that β enters the bounds for the regularity region given in (12) has an appreciable impact on the posterior in Figure 1. In Figure 3 (which uses a prior where the unidentified β is strongly correlated with the identified σ), this effect is even more noticeable. The posterior for β has a much smaller variance than its prior, indicating how information about σ is spilling over onto β .

The priors and posterior for σ show (as expected) that learning is occurring about this identified parameter. The posteriors in the top panels of Figures 2 and 4 are concentrated near the

¹⁰The posterior simulation algorithm rejected 3.1% of the draws for violating the bounds.

¹¹In this small model, with only four parameters, this algorithm works well. In larger models, depending on the form of the prior, more efficient posterior simulation algorithms could be used.

true value used to generate the data.

This example illustrates an important point we have made previously: An informal comparison of priors and posteriors of structural parameters in DSGE models can be a useful way of investigating if learning is occurring about a parameter. However, such a comparison will not tell the researcher why the learning is occurring. Our figures show posteriors can differ from priors, even for parameters which do not enter the likelihood function, either when the parameter space is not variation free or through prior correlations with identified parameters. Since DSGE models will often exhibit such features, this illustration shows how caution should be taken when interpreting comparisons of priors with posteriors.

In Section 3.2, we recommended using an alternative indicator based on (20). If interest centers on identification issues relating to θ_1 then this indicator involved comparing $p(\theta_1|y)$ to $E_{\theta_2|y}[p(\theta_1|\theta_2)]$. The bottom panels of Figures 1 through 4 present such a comparison for β and σ for our two priors. Clearly our indicator is working well. For the non-identified parameter, $p(\beta|y)$ and $E_{\sigma,\gamma,\psi|y}[p(\beta|\sigma,\gamma,\psi)]$ are the same density. For the identified parameter, $p(\sigma|y)$ and $E_{\beta,\gamma,\psi|y}[p(\sigma|\beta,\gamma,\psi)]$ are massively different, indicating the parameter is strongly identified.

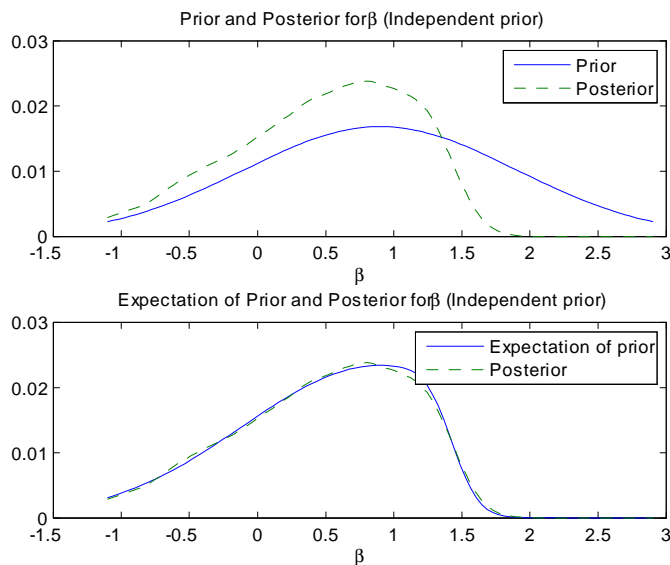


Figure 1

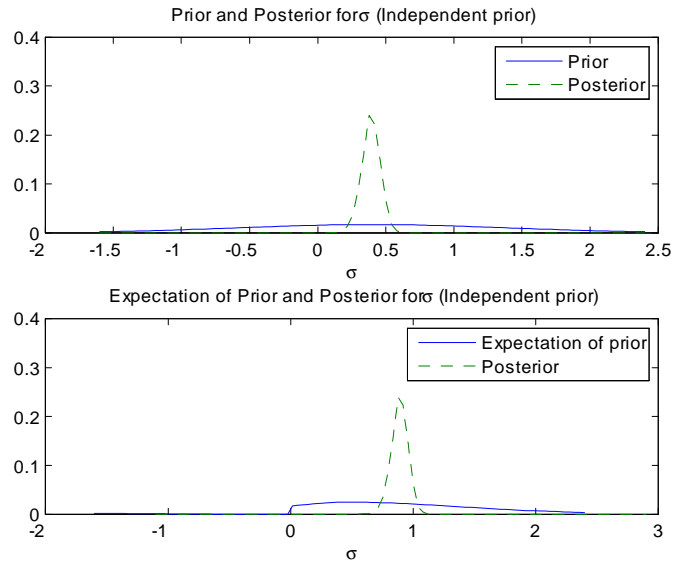


Figure 2

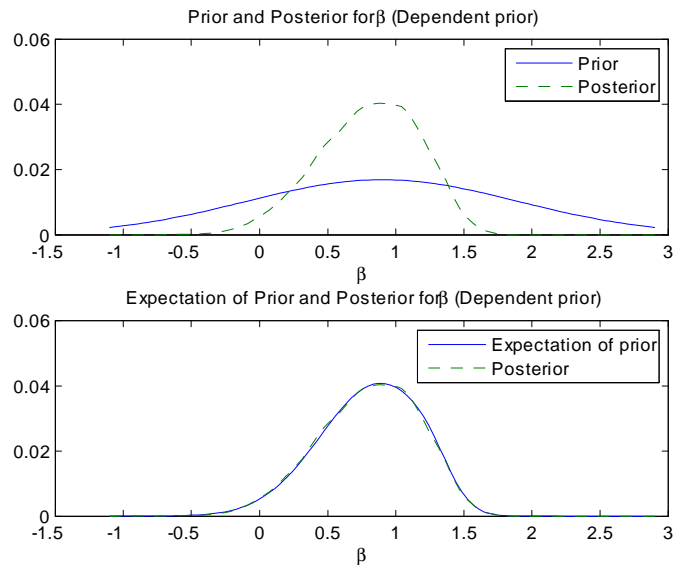


Figure 3

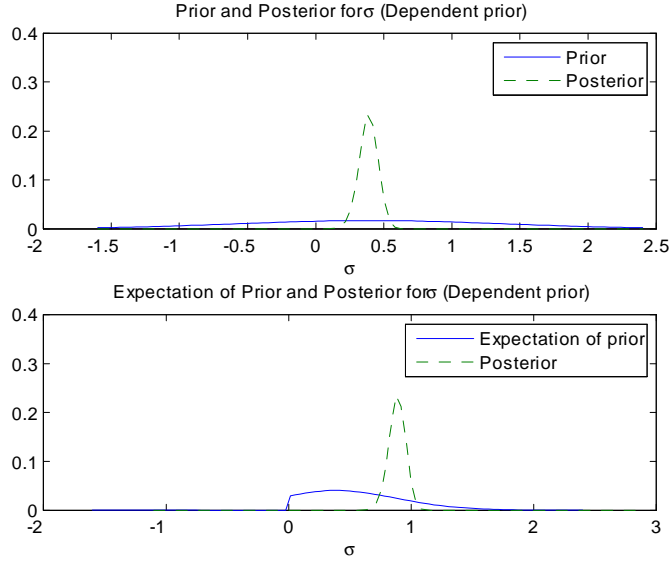


Figure 4

To illustrate our second Bayesian identification indicator, based on large sample derivations, Table 1 presents the posterior precisions of the parameters divided by T in the model using larger and larger data sets. Remember that the theoretical derivations underlying our second Bayesian identification indicator imply that the posterior precision of the identified parameters, σ , γ and ψ , should be increasing at a rate of T . But the posterior precision of the unidentified parameter, β , will be increasing (if at all) at a slower rate. These properties can be clearly seen in Table 1. In contrast to the identified parameters, the posterior precision of β divided by sample size is heading towards zero. This result holds irrespective of whether the prior distribution of β depends on the other parameters or not. For example, in the case of independent priors, when $T = 10,000$ the average posterior precision of β is 2×10^{-4} as compared to 2.941, 0.909 and 0.689 for σ , γ and ψ , respectively.

Table 1: Posterior Precisions Divided by Sample Size				
Number of observations	σ	γ	ψ	β
	Independent Prior			
T=10	0.991	0.209	0.641	0.194
T=20	1.191	0.233	0.625	0.107
T=50	2.105	0.741	0.571	0.044
T=100	2.222	0.769	0.588	0.023
T=1,000	2.439	0.833	0.556	0.002
T=10,000	2.941	0.909	0.689	2×10^{-4}
	Dependent Prior			
T=10	1.409	0.312	0.617	0.336
T=20	1.163	0.225	0.625	0.263
T=50	1.539	1.333	0.556	0.131
T=100	1.961	0.741	0.588	0.072
T=1,000	2.941	1.052	0.526	0.008
T=10,000	2.778	1.102	0.476	8×10^{-4}

4.2 Example 2 (cont.): The NKPC model

For the reasons discussed in Section 3.2 and in the theoretical derivations of Section 3.3.2, our first Bayesian indicator of identification will not work reliably when we work with the NKPC and parameterize the model in terms of its structural parameters. However, our second Bayesian identification indicator of Section 3.3, based on large sample theory, should still work. Accordingly, we use the NKPC (as in Section 3.3.2, we assume there is no backward looking behavior) to investigate the performance of this second identification indicator. We consider both an unidentified and identified version of the NKPC. The unidentified version assumes an AR(1) process for the output gap. The identified version assumes an AR(2) process for the output gap. Including the identified version allows us to investigate issues relating to the strength of identification.

4.2.1 The NKPC with AR(1) Process for the Output Gap

When no backward looking behavior exists ($\beta_b = 0$), then the hybrid NKPC, with AR(1) process (18), is parameterized in terms of three structural parameters β, γ and ρ (where $\beta \equiv \beta_f$ is the forward-looking coefficient in the NKPC). The RE solution given in (19) simplifies and depends on two reduced form parameters, α and ρ . The lack of identification reveals itself through the mapping from structural to reduced form parameters: $\alpha = \rho\gamma/(1 - \rho\beta)$.

We generated one artificial data set of 10,000 observations from the NKPC with $\beta = 0.6$, $\gamma = 0.9$, $\rho = 0.3$. In addition, u_t is i.i.d. $N(0, 0.25)$ and v_t is i.i.d. $N(0, 1)$. For γ and ρ , the prior is normal: $N(0.5, 0.1I_2)$. This prior is chosen so that the prior means are a bit different (but not too different) from the true values and prior variance is fairly informative. For β , we use a Uniform prior over $(0, 1)$ which is the region which ensures a unique RE solution. This prior is combined with the likelihood function based on the two equations for π_t and x_t . We use a random walk Metropolis-Hastings algorithm to do posterior simulation using the first T of the artificially generated observations for $T = 10, 20, 50, 100, 1,000$ and $10,000$.

Table 2 presents posterior variances for various sample sizes. The results in Table 2 not only confirm the large sample theory derived in this paper, but also show that it is empirically useful. That is, in this RE model, it can clearly be seen that the posterior precision of ρ is going to zero at the expected rate of T . In contrast, the posterior precisions divided by sample size of both non-identified parameters are declining. However, this decline is slower for γ than β . To see why this is occurring, remember that we have $\alpha = \gamma\rho / (1 - \rho\beta)$ and impose $0 < \beta < 1$. Since we have a precise (positive) estimate of ρ , then the sign of γ is the same as the sign of α , providing some information. In addition, even though γ is not identified we can test $\gamma = 0$ by testing $\alpha = 0$. Thus in some sense there is more information about γ in the estimate of α than about β and this is reflected in the posterior precisions.

Number of observations	ρ	γ	β
T=10	3.704	1.695	1.266
T=20	3.571	0.926	0.633
T=50	2.857	0.400	0.253
T=100	2.632	0.233	0.127
T=1,000	1.429	0.053	0.013
T=10,000	1.149	0.011	0.001

The posterior precision is not the only possible feature that the researcher could use as an indicator of identification. An alternative is simply to plot the posteriors for different choices of T . This is done in Figures 5, 6 and 7. In Figure 5, the posterior for ρ , clearly is converging in the manner implied by the asymptotic theory for identified models. However, in Figures 6 and 7,

the posteriors for γ and β are converging much more slowly. The posterior for γ changes with T more rapidly than the posterior for β . However, there is clearly some updating of beliefs about β occurring (remember that the prior for this parameter is uniform).

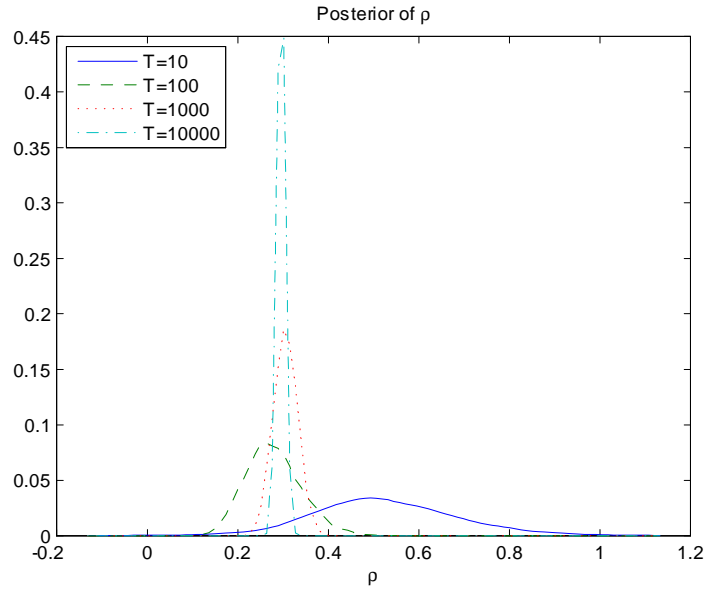


Figure 5

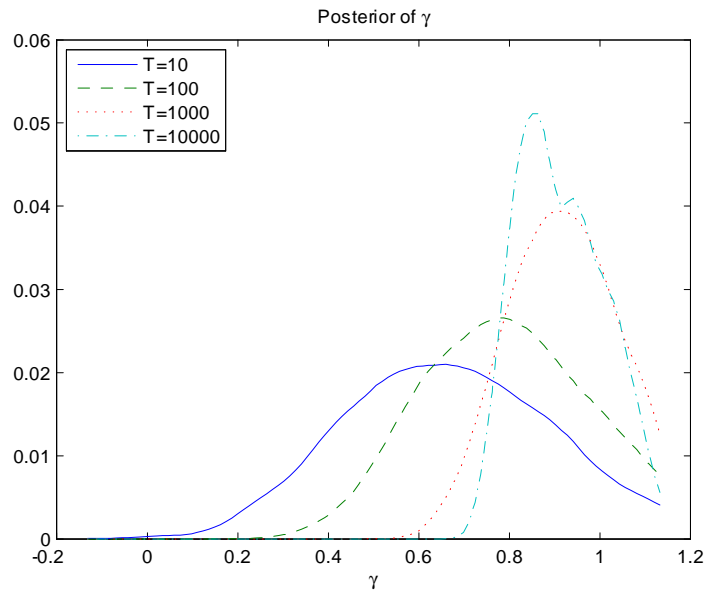


Figure 6

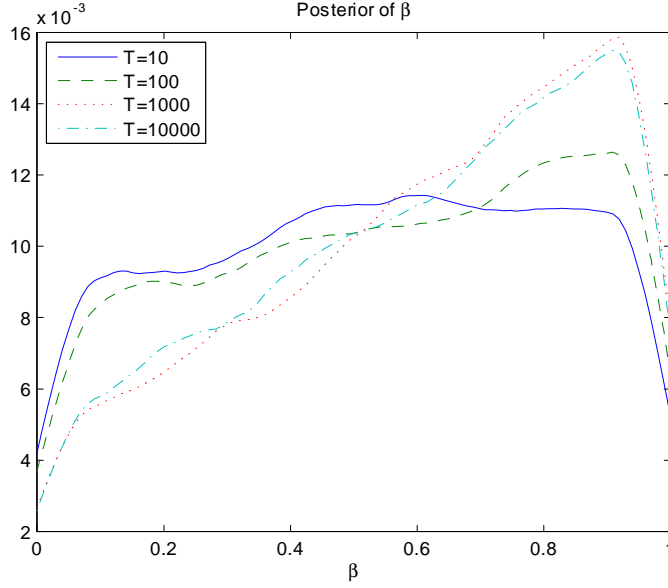


Figure 7

4.2.2 The NKPC with AR(2) Process for Output Gap

We now turn to the identified version of the NKPC, where we have an AR(2) process for the output gap. The structural parameters of this model (with no backward looking behavior) are $(\beta, \gamma, \rho_1, \rho_2)$ and the reduced form parameters are $(\alpha_1, \alpha_2, \rho_1, \rho_2)$. There is a one-to-one mapping between reduced form and structural form:

$$\alpha_1 = \frac{\gamma(\rho_1 + \beta_f \rho_2)}{1 - \beta_f(\rho_1 + \beta_f \rho_2)}$$

and

$$\alpha_2 = \frac{\gamma \rho_2}{1 - \beta_f(\rho_1 + \beta_f \rho_2)}.$$

Thus, all of the structural parameters are identified. However, if ρ_2 is near zero then identification will be weak. Accordingly, we use this example to investigate issues relating to the strength of identification.

The data generating process (DGP) is the same as in our previous example, except for its

treatment of ρ_2 . We generate four different data sets with $\rho_2 = 0, 0.01, 0.1, 0.6$, respectively.¹² The prior for (β, γ, ρ_1) is the same as that used in the AR(1) example. To this we add an independent $N(0.5, 0.1)$ prior for ρ_2 . Just as with the AR(1) example, this prior is combined with the likelihood function based on the two equations for π_t and x_t . We use a random walk Metropolis-Hastings algorithm to do posterior simulation using the first T of the artificially generated observations for $T = 10, 20, 50, 100, 1,000$ and $10,000$.

Table 3 reports posterior precisions divided by T for the case where $\rho_2 = 0$. This artificial data set is the same as that used to produce Table 2. However, the model being estimated differs in this case. Since we are estimating an additional parameter, ρ_2 , it is not surprising that the posterior precisions (especially for small sample sizes) are slightly smaller in Table 3 than Table 2. However, the general pattern revealed by Table 3 is the same as Table 2. Table 4 (where $\rho_2 = 0.01$) also exhibits a similar pattern. These findings suggest that the Bayesian identification indicator based on large sample results will present useful information even if our estimating model is identified. Thus, it can be a useful indicator of weak identification.

In this model, the weakness of identification seems to impact mostly on β . That is, its posterior precision is increasing very little over time in the cases where $\rho_2 = 0$ or 0.01 . With $\rho_2 = 0.1$ (see Table 5), we can begin to see clear signs that the posterior precision of β is increasing with T . However, even in this case, the increase of the posterior precision is quite slow. However, when $\rho_2 = 0.6$ (see Table 6), the posterior of all the parameters can be seen to be converging at roughly the same rate. This is reassuring since, in this case, all the parameters are strongly identified.

Number of observations	ρ_1	ρ_2	γ	β
T=10	2.941	3.571	1.429	1.266
T=20	2.778	3.333	0.820	0.633
T=50	2.564	3.356	0.364	0.256
T=100	2.326	3.360	0.213	0.128
T=1,000	1.017	4.167	0.050	0.013
T=10,000	1.091	4.348	0.009	0.001

¹²In order to make these data sets as comparable as possible, we use the same seed for the random number generator for all DGPs.

Number of observations	ρ_1	ρ_2	γ	β
T=10	2.941	3.704	1.515	1.282
T=20	2.778	3.329	0.794	0.641
T=50	2.857	2.912	0.357	0.256
T=100	2.500	3.23	0.217	0.128
T=1,000	1.062	4.562	0.053	0.014
T=10,000	1.178	4.712	0.009	0.001

Number of observations	ρ_1	ρ_2	γ	β
T=10	3.125	4.167	1.563	1.389
T=20	2.632	3.571	0.769	0.735
T=50	2.201	3.298	0.341	0.323
T=100	2.013	2.597	0.189	0.169
T=1,000	1.103	3.232	0.033	0.021
T=10,000	1.190	2.560	0.008	0.003

Number of observations	ρ_1	ρ_2	γ	β
T=10	2.702	3.571	1.429	2.128
T=20	3.321	4.167	0.877	2.273
T=50	3.309	3.287	0.466	1.429
T=100	2.459	2.478	0.303	0.987
T=1,000	1.144	1.098	0.083	0.376
T=10,000	1.760	1.603	0.098	0.251

5 Concluding Remarks

This paper has examined the identification of the parameters of DSGE models, in the light of the widespread concern in the literature that the parameters may be either not identified or only weakly identified. In purely forward looking models, with no lags, the coefficients of the expectational variables are generically not identified since they do not enter the likelihood function. In forward looking models with lags, identification is dependent on the assumed structure of the dynamics, making it vulnerable to the Sims (1980) critique of ‘incredible’ identifying restrictions. In more complicated models with unobserved variables and no analytical solution, it is difficult to determine whether the models are identified. When the DSGE models are estimated by Bayesian methods, this lack of identification may not be evident since the posterior may differ from the prior even if the parameter is not identified and the posterior for unidentified parameters may also be updated as the sample size increases. These properties have been demonstrated both

analytically and numerically, using familiar examples of unidentified or weakly identified rational expectations DSGE models.

We propose two Bayesian identification indicators. The first involves comparing the marginal posterior of a parameter with the posterior expectation of the prior for that parameter conditional on the other parameters. This can be computed as part of the MCMC estimation of a DSGE model using whatever real data set the researcher is working with. However, this indicator can be applied only in situations where parameters can be partitioned into a set that are known to be identified and another set for which identification is uncertain. This may not be possible when the researcher is working with the structural parameters of a DSGE model. Our second Bayesian indicator is more generally applicable and considers the rate at which the posterior precision gets updated as the sample size (T) is increased. For identified parameters the posterior precision rises with T , whilst for an unidentified parameter its posterior precision may be updated but its rate of update will be slower than T . This result assumes that the identified parameters are \sqrt{T} -consistent, but similar differential rates of updates for identified and unidentified parameters can be established in the case of weak (or super) consistent estimators. This suggests a strategy where the researcher simulates larger and larger data sets and observes the behavior of the posterior as sample size increases.

We present an empirical illustration which shows the effectiveness of the first Bayesian identification indicator, in cases where it is applicable. Further empirical illustrations show the usefulness of our second Bayesian identification indicator, both for checking for the presence and the strength of identification.

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Technical Appendix

Solving DSGE Models

The solution method proposed by Binder and Pesaran involves finding an $n \times n$ matrix $C(\theta)$ such that in terms of the quasi-difference transformation $Y_t = y_t - C(\theta)y_{t-1}$, the model only involves future expectations, where $C(\theta)$ is a solution of the following quadratic matrix equation

$$A_1(\theta)C(\theta)^2 - A_0(\theta)C(\theta) + A_2(\theta) = 0. \quad (27)$$

Then assuming $[I_n - A_0(\theta)^{-1}A_1(\theta)C(\theta)]$ is non-singular, we obtain

$$Y_t = F(\theta)E_t(Y_{t+1}) + W_t,$$

where

$$F(\theta) = [A_0(\theta) - A_1(\theta)C(\theta)]^{-1} A_1(\theta),$$

$$W_t = [A_0(\theta) - A_1(\theta)C(\theta)]^{-1} [A_3(\theta)x_t + u_t].$$

There will be a unique solution if there exists a real matrix solution to (27) such that all the eigenvalues of $C(\theta)$ lie inside or on the unit circle, and all the eigenvalues of $F(\theta)$ lie strictly inside the unit circle. In such cases the unique solution is given by

$$y_t = C(\theta)y_{t-1} + \sum_{h=0}^{\infty} F(\theta)^h E_t(W_{t+h}).$$

When, as assumed above, x_t and u_t follow a VAR(1) process, and the roots of the Φ_i , $i = x, u$ lie on or inside the unit circle, then

$$E_t(x_{t+h}) = \Phi_x^h x_t$$

$$E_t(u_{t+h}) = \Phi_u^h u_t$$

and since x_t and u_t are independent

$$E_t(W_{t+h}) = [A_0(\theta) - A_1(\theta)C(\theta)]^{-1} [A_3(\theta)\Phi_x^h x_t + \Phi_u^h u_t]$$

therefore the solution has the form

$$y_t = C(\theta)y_{t-1} + \sum_{h=0}^{\infty} F^h(\theta)[A_0(\theta) - A_1(\theta)C(\theta)]^{-1} [A_3(\theta)\Phi_x^h x_t + \Phi_u^h u_t], \quad (28)$$

which we can write as in (3).