Counterfactual Analysis in Macroeconometrics: An Empirical Investigation into the Effects of Quantitative Easing*

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Abstract

The policy innovations that followed the recent Great Recession, such as unconventional monetary policies, prompted renewed interest in the question of how to measure the effectiveness of such policy interventions. To test policy effectiveness requires a model to construct a counterfactual for the outcome variable in the absence of the policy intervention and a way to determine whether the differences between the realised outcome and the model-based counterfactual outcomes are larger than what could have occurred by chance in the absence of policy intervention. Pesaran & Smith (2014b) propose tests of policy ineffectiveness in the context of macroeconometric rational expectations dynamic stochastic general equilibrium models. When we are certain of the specification, estimation of the complete system imposing all the cross-equation restrictions implied by the full structural model is more efficient. But if the full model is misspecified, one may obtain more reliable estimates of the counterfactul outcomes from a parsimonious reduced form policy response equation, which conditions on lagged values, and on the policy measures and variables known to be invariant to the policy intervention. We propose policy ineffectiveness tests based on such reduced forms and illustrate the tests with an application to the unconventional monetary policy known as quantitative easing (QE) adopted in the UK.

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1 Introduction

The Great Recession that followed the financial crisis starting in 2007 prompted a range of policy innovations. These included unconventional monetary policies and increased fiscal activism, through either stimulus or austerity measures depending on the country concerned. These innovations have prompted renewed interest in the question of how one measures the effectiveness of such policy interventions. To test policy effectiveness requires a model to construct a counterfactual for the outcome variable in the absence of the policy intervention and a way to determine whether the difference between the realised outcome and the counterfactual outcome is larger than one would expect by chance. In a companion paper, Pesaran & Smith (2014b), PS, we propose asymptotic tests for the null hypothesis of policy ineffectiveness in the context of complete systems of macroeconometric dynamic stochastic general equilibrium (DSGE), rational expectations, RE, models. In this paper we consider testing for the effectiveness of policy using "reduced form policy response equations" rather than full structural models. We pay particular attention to the specification of the counterfactual and the role of endogenous and exogenous conditioning variables. As an illustration, we test for the effectiveness of the unconventional monetary policy known as quantitative easing (QE) adopted in March 2009 in the UK.

When we are reasonably confident in the specification, estimation of the complete system imposing all the cross-equation restrictions implied by the structural DSGE model yields more reliable estimates of the counterfactual outcomes as compared to using the reduced form specifications. However, we are rarely certain of the correct specification for the complete system and if the full model is misspecified, more robust estimates of the counterfactuals may be obtained from a reduced form policy response equation for the variable of interest. Since a counterfactual is a type of forecast, and parsimonious models tend to forecast better, using a small policy model to construct the counterfactual outcomes might even be preferable to using a large structural models, where one may be more liable to misspecification of some equations with adverse consequences for the quality of counterfactual outcomes reliability of the other equations. Accordingly, we propose tests for policy ineffectiveness based on such reduced forms.

As in PS, we consider a policy intervention which takes the form of a change in one or more of the parameters of a policy rule. The tests are then based on the differences, over a given policy evaluation horizon, between the post-intervention realizations of the policy target(s) and associated counterfactual outcomes based on the parameters estimated using data before the policy intervention. The Lucas Critique is not an issue since the counterfactual, given by the predictions from the model estimated on pre-intervention data, will embody pre-intervention parameters while the actual post-intervention outcomes will embody any effect of the change in policy, the change in parameters and the consequent change in expectations. The development of the test does not require knowing the post-intervention parameters.

We are concerned with *ex post* evaluation of a policy intervention on a single unit (country), where data are available before and after the intervention. Different issues are involved in *ex ante* policy formulation where post-intervention data are not available and the Lucas Critique could be an issue since the possible effects of the policy change on parameters and expectations must be taken into account. We develop tests of policy ineffectiveness and derive their asymptotic distributions both when the post-intervention sample is fixed as the pre-intervention sample expands, and when both samples rise jointly but at different rates and investigate the power of the tests. In the case of static reduced forms we also derive an exact test.

The concept of counterfactual used in the development of our proposed test refers to hypothetical outcomes obtained under the null of policy ineffectiveness and is, therefore, more narrowly defined than in the literature. The term "counterfactual" has a variety of distinct, though connected, uses in philosophy, history, economics and statistics. In philosophy counterfactual scenarios are often used in the analysis of causality, e.g. Lewis (1973). Pearl (2009) provides an overview of the concepts and develops an analysis of causality based on structural models. In history counterfactuals are posed by "what if" questions, such as "what would the U.S. economy have been like in 1890 had there been no railroads?", Fogel (1964). In economics alternative counterfactuals (hypothetical states of the world) are considered in decision making under uncertainty. In statistics and econometrics counterfactuals are used in medical trials and microeconometric program evaluations. These uses, whilst connected, are quite distinct and the appropriate definition of a counterfactual crucially depends on the context.

Counterfactuals have been used to examine a range of macroeconometric questions. Abadie and Gardeazabal (2003) examine the effect of terrorism on the Basque country using a "synthetic control region" to provide a counterfactual. Pesaran, Smith and Smith (2007) examine what would have happened to the economies of the UK and the eurozone had the UK joined the euro in 1999, using "euro" restrictions on a GVAR model to construct counterfactual outcomes. Hsiao, Ching and Wan (2011) examine the effect on output growth in Hong Kong of political and economic integration with mainland China, constructing counterfactuals based on predictions from similar economies. Fagan, Lothian and McNelis (2013) examine whether the Gold Standard was really destablising, constructing the counterfactual by replacing the pre-1914 US money supply process with a Taylor rule in a DSGE model.

Whereas there is a large literature on microeconometric policy evaluation that focuses on the measurement and testing of treatment effects, surveyed, for example, by Imbens and Wooldridge (2009), there has been less systematic methodological discussion of macroeconometric policy evaluation. The micro and macro issues are rather different. For instance, the endogeneity and sample selection bias that arise due to correlated heterogeneity across the units in the micro-treatment case is not a problem in the macro case when the focus of the policy evaluation is on a single unit, and the "policy on/policy off" comparisons are done over time rather than across units. In micro terminology, the parameter of interest in the macro cases is the effect of treatment on the treated: it makes no sense to consider either the effect of Hong Kong joining the euro or of the UK being integrated with China.

Another recent approach to macro policy evaluation borrows techniques from the micro literature to obtain an estimate of an average treatment effect. Angrist, Jorda and Kuersteiner (2013, AJK), drawing on Angrist and Kuersteiner (2011), estimate the effect of monetary policy, while Jorda and Taylor (2013) use similar procedures to estimate the effect of fiscal policy. AJK use local linear projection type estimators to measure the average effect of policy changes on future values of the outcome variables (inflation, industrial production, and unemployment), weighted inversely by policy propensity scores in a way similar to that used to adjust non-random samples. Their approach differs from the one proposed in this paper in two important respects. First, they rely on outcomes averaged across different (possibly heterogenous) policy episodes whilst we consider a single policy intervention and average the counterfactual outcomes for the same post intervention sample. Second, AJK do not use a structural model and their analysis is subject to the Lucas Critique. Their approach requires that the underlying parameters are invariant to policy changes, since it is only policy changes within the same regime that are identified in their framework (see AJK, p.5). In addition, matching estimators of this sort require a lot of data whereas macroeconometric samples tend to be data-poor relative to microeconometric samples. This is reflected in the large confidence bands AJK report around the measures of their estimated effects of target rate changes on macro variables.

We use the policy ineffectiveness tests to investigate the effects of the QE introduced in the UK after March 2009 To this end we employ an autoregressive distributed lag (ARDL) equation in the target variable, output growth (y_t) , the policy variable, the spread between long and short rates (x_t), and US and euro area output variables, $\mathbf{w}_t$, that we assume to be invariant to the policy change. We exclude endogenous variables, $\mathbf{z}_t$, that could influence y_t both directly or through the changes in x_t. For instance, it would be wrong to include the exchange rate in the equation, because if QE was effective in reducing the spread then the exchange rate would almost certainly have been changed by it and we would have needed to allow for that effect by considering a separate equation that links the exchange rate to x_t. By excluding the exchange rate from the policy response equation we are in effect replacing the exchange rate by its determinants The same argument also applies to any other endogenous variable which is affected by the policy change. This is the reverse of the usual misspecification argument, since we wish to attribute to x_t the effects that are transmitted through $\mathbf{z}_t$. It is not a question of *ceteris paribus*, other things being equal, held constant, but *mutatis mutandis*, changing what needs to be changed.[1] We find that a 100 basis points reduction in the spread (due to the QE) has an impact effect on output growth of about one percentage point, but the policy impact is very quickly reversed.

The rest of the paper is organized as follows: Section 2 sets up a DSGE model with exogenous variables and derives its solution which is the basis for the reduced form policy response equations we estimate. Section 3 develops the policy ineffectiveness tests. Section 4 considers the empirical application to investigate the effectiveness of the QE in the UK. Section 5 ends with some concluding remarks. The more technical derivations are given in the Appendix.

2 Derivation of the reduced form policy response equation

Following PS, we consider a standard rational expectations (RE) model, with exogenous variables. We suppose that the target variable, y_t, is affected directly by a vector of variables, $\mathbf{z}_t$, and assume that the $(k_z+1)\times 1$ vector $\mathbf{q}_t=(y_t,\mathbf{z}_t')'$ are the endogenous variables, which may include policy variables. Endogenous policy rules, such as the Taylor rule, follow closed loop control with feedback, but there may be open loop control without feedbacks, such as fixed money supply rules, where the policy variable x_t is exogenous. There may also be non-policy exogenous variables, $\mathbf{w}_t$, such as global variables that affect $\mathbf{z}_t$ and/or y_t but are invariant to changes in the policy variables, x_t.

The exogenous policy and non-policy variables are included in $\mathbf{s}_t=(x_t,\mathbf{w}_t')'$, a $(1+k_w)\times 1$ vector. The RE model is

$$\mathbf{A}_0\mathbf{q}_t=\mathbf{A}_1E_t(\mathbf{q}_{t+1})+\mathbf{A}_2\mathbf{q}_{t-1}+\mathbf{A}_3\mathbf{s}_t+\mathbf{u}_t, \tag{1}$$

and suppose, for illustration, that the forcing variables, $\mathbf{s}_t$, follow the VAR(1) specification

$$\mathbf{s}_t=\mathbf{R}\mathbf{s}_{t-1}+\eta_t, \tag{2}$$

where

$$\mathbf{R}=\begin{pmatrix}\rho & 0\\ 0 & \mathbf{R}_w\end{pmatrix},\ \eta_t=\begin{pmatrix}\eta_{xt}\\ \eta_{wt}\end{pmatrix},$$

so that the $\mathbf{w}_t$ process is invariant to changes in x_t. The errors, $\mathbf{u}_t$ and η_t are assumed to be serially and cross sectionally uncorrelated, with zero means and constant variances, $\mathbf{\Sigma}_u$, and $\mathbf{\Sigma}_\eta$, respectively.

Initially we abstract from parameter estimation uncertainty and denote the vector of parameters by $\boldsymbol{\theta}$, which includes $\mathbf{a}=vec(\mathbf{A}_0,\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)$, and and the parameters of the processes

[1] A similar argument is developed in a continuous time regression context in Pesaran and Smith (2014a).

generating the exogenous variables, $\phi = (\rho, vec(\mathbf{R}_w)')'$. We assume that $\mathbf{\Sigma}_u$ and $\mathbf{\Sigma}_\eta$ remain fixed. The parameter vector, $\boldsymbol{\theta}$**,** is composed of a set of policy parameters, $\boldsymbol{\theta}_p$, and a set of structural parameters, $\boldsymbol{\theta}_s$, that are invariant to changes in $\boldsymbol{\theta}_p$. A policy intervention is defined in terms of a change in one or more elements of θ_p. The null hypothesis of our test will be that the intervention was ineffective, there was no change in policy parameters. We assume that the model is known by economic agents, the announcement and implementation of the intervention are credible, and no further change is expected. We suppose that the policy intervention occurs at the end of time $t = T_0$, and we have pre-intervention sample that covers the period $t = M, M+1, ..., T_0$, and the post-intervention sample for $t = T_0+1, T_0+2, .., T_0+H$. Therefore, the post-intervention horizon is H and the sample size for estimation of the pre-intervention parameters is $T = T_0 - M + 1$. This notation allows us to increase the sample size T (by letting $M \to -\infty$), while keeping the time of intervention, T_0, fixed.

Initially, consider the case where there are no dynamics, namely $\mathbf{A}_2 = \mathbf{0}$, and all eigenvalues of $\mathbf{A}_0^{-1}\mathbf{A}_1$ lie within the unit circle. Then the unique solution of (1) is given by

$$\mathbf{A}_0\mathbf{q}_t = \mathbf{G}(\theta)\mathbf{s}_t + \mathbf{u}_t, \tag{3}$$

where, suppressing the dependence on $\boldsymbol{\theta}$,

$$vec(\mathbf{G}) = \left[(\mathbf{I}_{k_w+1} \otimes \mathbf{I}_{k_z+1}) - \left(\mathbf{R}' \otimes \mathbf{A}_1\mathbf{A}_0^{-1}\right)\right]^{-1} vec\left(\mathbf{A}_3\right).$$

Equation (3) is the structural form of a standard simultaneous equations model. The reduced form is

$$\begin{aligned} \mathbf{q}_t &= \mathbf{A}_0^{-1}\mathbf{G}(\theta)\mathbf{s}_t + \mathbf{A}_0^{-1}\mathbf{u}_t \\ &= \mathbf{\Pi}\left(\theta\right)\mathbf{s}_t + \mathbf{\Gamma}(\theta)\mathbf{u}_t. \end{aligned} \tag{4}$$

If the intervention at T_0 is fully understood and expectations adjust immediately, then the process switches from

$$\mathbf{q}_t = \mathbf{\Pi}(\theta^0)\mathbf{s}_t + \mathbf{\Gamma}(\theta^0)\mathbf{u}_t, \ t = M, M+1, M+2, ..., T_0,$$

to

$$\mathbf{q}_t = \mathbf{\Pi}(\theta^1)\mathbf{s}_t + \mathbf{\Gamma}(\theta^1)\mathbf{u}_t, \ t = T_0+1, T_0+2, ..., T_0+H.$$

In the general case, $\mathbf{A}_2 \neq \mathbf{0}$, the RE solution is

$$\mathbf{q}_t = \mathbf{\Phi}\left(\theta\right)\mathbf{q}_{t-1} + \mathbf{\Psi}_x\left(\theta\right)\mathbf{x}_t + \mathbf{\Psi}_w\left(\theta\right)\mathbf{w}_t + \mathbf{\Gamma}(\theta)\mathbf{u}_t, \tag{5}$$

where θ contains $\mathbf{a} = vec(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$ and $\phi = (\rho, vec(\mathbf{R}_w)')'$.

3 Tests of policy ineffectiveness using reduced forms

Section 2 assumed a fully specified RE structural models. When we are certain of the specification, estimation of the complete system imposing all the cross-equation restrictions implied by the structural RE model is more efficient. In practice, we are rarely certain about the specification of the complete model, and given the consequences of misspecification transmitting between equations, one may obtain more robust estimates of the counterfactuals from reduced form, policy response, single equation. estimates. We first consider the static case. While dynamics are likely to be important in practice, the static case illuminates certain aspects of the procedure and enables us to derive an exact test that allows for parameter estimation uncertainty. For simplicity, we shall assume that the policy change is formulated in terms of changes in ρ, the parameter of the policy equation x_t. Also to simplify the exposition we abstract from the policy implementation errors, η_{xt}.

3.1 The static case

In the static case, using (3), the reduced form equation is given by

$$\mathbf{q}_t = \mathbf{\Pi}\left(\theta\right)\mathbf{s}_t + \varepsilon_t, \tag{6}$$

where $\varepsilon_t = \mathbf{\Gamma}(\theta)\mathbf{u}_t$. Recalling that y_t is the first element of $\mathbf{q}_t$, we obtain the following policy response equation

$$y_t = \pi_y'\left(\theta\right)\mathbf{s}_t + \varepsilon_{yt} = \pi_{yx}\left(\theta\right)x_t + \pi_{yw}'\left(\theta\right)\mathbf{w}_t + \varepsilon_{yt}, \tag{7}$$

which does not depend on $\mathbf{z}_t$. But it is clear that the parameters of (7) depend on the structural and policy coefficients.

The policy intervention, changing ρ^0 to ρ^1, will cause the policy response equation to exhibit a break at time $T_0 + 1$:

$$y_t = \pi_{yx}(\rho^0)x_t + \pi_{yw}'(\rho^0)\mathbf{w}_t + \varepsilon_{yt},\ t = M, M+1, M+2, ..., T_0; \tag{8}$$

$$y_t = \pi_{yx}(\rho^1)x_t + \pi_{yw}'(\rho^1)\mathbf{w}_t + \varepsilon_{yt},\ t = T_0+1, T_0+2, ..., T_0+H. \tag{9}$$

In the static case the policy response equation is given by (8). The counterfactual outcomes of y_t under the joint null hypothesis of (i) no policy intervention and (ii) no change in the other parameters is just the H period forecast made at time T_0 conditional on $x^0_{T_0+h}$, what we would expect policy to be if the policy parameter was ρ^0, and the realised $\mathbf{w}_{T+h}$

$$y^0_{T_0+h} = \pi_{yx}(\rho^0)x^0_{T_0+h} + \pi_{yw}'(\rho^0)\mathbf{w}_{T_0+h},\ h = 1, 2, ..., H. \tag{10}$$

The policy effects are $d_{T_0+h} = y_{T_0+h} - y^0_{T_0+h}$, and while we need to know $\pi_i(\rho^0)$, $i = yx, yw$, to construct $y^0_{T_0+h}$ and d_{T_0+h} we do not need to know $\pi_i(\rho^1)$.

It is instructive to decompose the policy effects into the part due to the change in the policy variable and the part that arises due to the policy-induced parameter changes. Using (9) and (10) we have

$$\begin{aligned} d_{T_0+h} &= \left[\pi_{yx}(\rho^1)x_{T_0+h} - \pi_{yx}(\rho^0)x^0_{T_0+h}\right] + \left[\pi_{yw}(\rho^1) - \pi_{yw}(\rho^0)\right]' \mathbf{w}_{T_0+h} + v_{y,T_0+h} \\ &= \pi^0_{yx}\left(x_{T_0+h} - x^0_{T_0+h}\right) + \left(\pi^1_{ys} - \pi^0_{ys}\right)' \mathbf{s}_{T_0+h} + v_{y,T_0+h}, \end{aligned}$$

for $h = 1, 2, ..., H$, where (as before) $\mathbf{s}_t = (x_t, \mathbf{w}'_t)'$, $\pi^0_{yx} = \pi_{yx}(\rho^0)$, and $\pi^i_{ys} = \left[\pi_{yx}(\rho^i), \pi'_{yw}(\rho^i)\right]'$, for $i = 0, 1$. The first term captures the effects of the change in the policy variable, x_t, whilst the second term captures the effects of the policy-induced parameter changes. Only the first term would be present in the case of *ad hoc* policy changes that do not induce parameter change in the policy response equation. In either case, the pure policy effect is diluted due to the post-intervention random errors, v_{y,T_0+h}. But we can reduce the importance of such random influences by using the mean policy effect, $\bar{d}_H = H^{-1}\sum_{h=1}^{H} d_{T_0+h}$. The relative importance of the random errors, v_{y,T_0+h}, can also be reduced by using additional policy invariant variables, $\mathbf{w}_t$, when available.

In the static case it is possible to develop an exact test of policy ineffectiveness. Suppose that we have k_x policy variables, $\mathbf{x}_t$, and let $\mathbf{X}_{(0)}$ be the $T \times k_x$ matrix of observations on the policy variables before the intervention, and let $\mathbf{X}_{(1)}$ be the $H \times k_x$ matrix of observations (realized values) on $\mathbf{x}_t$ after the intervention. Similarly, let $\mathbf{W}_{(0)}$ to be the $T \times k_w$ matrix of observations on the policy invariant variables, $\mathbf{w}_t$, pre-intervention and let $\mathbf{W}_{(1)}$ be the $H \times k_w$ matrix of observations on $\mathbf{w}_t$ post-intervention. Initially, we assume that H is fixed.

The vector of policy effects, $\mathbf{d}_{(1)} = (d_{T_0+1}, d_{T_0+2}, ..., d_{T_0+H})'$, can be written as

$$\mathbf{d}_{(1)} = \mu_{(1)} + \mathbf{v}_{(1)}, \tag{11}$$

where, $\mathbf{v}_{(1)} = (v_{y,T_0+1}, v_{y,T_0+2}, ..., v_{y,T_0+H})'$,

$$\begin{aligned} \mu_{(1)} &= \left(\mathbf{X}_{(1)} - \mathbf{X}^0_{(1)}\right)\pi^0_{yx} + \mathbf{S}_{(1)}\left(\pi^1_{ys} - \pi^0_{ys}\right) \\ &= \mathbf{S}_{(1)}\pi^1_{ys} - \mathbf{S}^0_{(1)}\pi^0_{ys} \end{aligned} \tag{12}$$

$\mathbf{S}_{(1)} = \left(\mathbf{X}_{(1)}, \mathbf{W}_{(1)}\right)$, $\mathbf{S}^0_{(1)} = \left(\mathbf{X}^0_{(1)}, \mathbf{W}_{(1)}\right)$,and $\mathbf{X}^0_{(1)}$ is the $H \times k_x$ matrix of observations on the counterfactual values of $\mathbf{x}_t$ over the post-intervention sample, namely the values of $\mathbf{x}_t$ that would have materialized in the absence of the policy change. For example, in the case of the an AR(1) policy rule, $x_t = \rho x_{t-1} + \eta_{xt}$, we have $\mathbf{X}^0_{(1)} = \left[\rho^0 x_{T_0}, \left(\rho^0\right)^2 x_{T_0}, ..., \left(\rho^0\right)^H x_{T_0}\right]'$, where ρ^0 can be estimated using the pre-intervention sample. In what follows we assume that $\mathbf{X}^0_{(1)}$ is given.

We now consider two different specifications of y_t for $t = T_0+1, T_0+2, ..., T_0+H$: (i) realized values of y_t post-intervention, defined by $\mathbf{y}_{(1)} = (y_{T_0+1}, y_{T_0+2}, ..., y_{T_0+H})'$, and (ii) the associated estimated counterfactual, $\hat{\mathbf{y}}^0_{(1)} = (\hat{y}^0_{T_0+1}, \hat{y}^0_{T_0+2}, ..., \hat{y}^0_{T_0+H})'$. Then an estimate of $\mathbf{d}_{(1)}$ can now be written as

$$\hat{\mathbf{d}}_{(1)} = \mathbf{y}_{(1)} - \hat{\mathbf{y}}^0_{(1)}, \tag{13}$$

where

$$\hat{\mathbf{y}}^0_{(1)} = \mathbf{X}^0_{(1)}\hat{\pi}^0_{yx} + \mathbf{W}_{(1)}\hat{\pi}^0_{yw}, \tag{14}$$

and $\hat{\pi}^0_{ys} = \left(\hat{\pi}^{0\prime}_{yx}, \hat{\pi}^{0\prime}_{yw}\right)'$ are the least squares estimates of the coefficients in the regression of y_t on $\mathbf{s}_t = (\mathbf{x}'_t, \mathbf{w}'_t)'$, using the pre-intervention sample. More specifically,

$$\hat{\pi}^0_{ys} = \left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1}\mathbf{S}'_{(0)}\mathbf{y}_{(0)}, \tag{15}$$

where $\mathbf{y}_{(0)} = (y_M, y_{M+1}, ..., y_{T_0})'$, and $\mathbf{S}_{(0)} = \left(\mathbf{X}_{(0)}, \mathbf{W}_{(0)}\right)$ is the $T \times (k_x + k_w)$ matrix of pre-intervention observations on $\mathbf{s}_t$. It is useful also to note that $\hat{\mathbf{y}}^0_{(1)}$ can be equivalently computed as

$$\begin{aligned}\hat{\mathbf{y}}^0_{(1)} &= \left(\mathbf{X}^0_{(1)} - \mathbf{X}_{(1)}\right)\hat{\pi}^0_{yx} + \mathbf{S}_{(1)}\hat{\pi}^0_{ys}, \\ &= \mathbf{S}^0_{(1)}\hat{\pi}^0_{ys},\end{aligned} \tag{16}$$

which decomposes the counterfactual outcomes to a part due to the change in the policy variables, and the *ex ante* forecasts based on pre-intervention parameter estimates.

Different tests of policy ineffectiveness can now be derived by testing the statistical significance of the individual elements of $\hat{\mathbf{d}}_{(1)}$, or a linear combination of its elements. To this end we suppose that all the classical assumptions apply to the policy response equation during pre-intervention sample ($t = M, M+1, ..., T_0$), namely $\mathbf{s}_t$ and $v_{yt'}$ are uncorrelated for all t and t', and v_{yt} are serially uncorrelated with a constant variance, σ^2_v. Post-intervention, we assume the same policy response equation holds, albeit with different parameter values, namely

$$\mathbf{y}_{(1)} = \mathbf{S}_{(1)}\pi^1_{ys} + \mathbf{v}_{(1)}, \tag{17}$$

$\mathbf{W}_{(1)}$ and $\mathbf{v} = (\mathbf{v}'_{(0)}, \mathbf{v}'_{(1)})'$ are uncorrelated, and $E\left(\mathbf{v}_{(1)}\mathbf{v}'_{(0)}\right) = \mathbf{0}$, $E\left(\mathbf{v}_{(0)}\mathbf{v}'_{(0)}\right) = \sigma^2_{0v}\mathbf{I}_T$, and $E\left(\mathbf{v}_{(1)}\mathbf{v}'_{(1)}\right) = \sigma^2_{1v}\mathbf{I}_H$. Note that we do not need to make any assumptions concerning the realized values of $\mathbf{x}_t$, over the post-intervention sample.

Using (16) and (17) in (13) and after some simplifications we have

$$\hat{\mathbf{d}}_{(1)} = \mu_{(1)} + \mathbf{v}_{(1)} - \xi_{(1)}, \tag{18}$$

where

$$\xi_{(1)} = \mathbf{S}^0_{(1)}\left(\hat{\pi}^0_{ys} - \pi^0_{ys}\right) \tag{19}$$

and $\mathbf{S}^0_{(1)} = \left(\mathbf{X}^0_{(1)}, \mathbf{W}_{(1)}\right)$, which differs from $\mathbf{S}_{(1)}$ in that the post-intervention realizations of $\mathbf{x}_t$, namely $\mathbf{X}_{(1)}$, are replaced by their counterfactual values, $\mathbf{X}^0_{(1)}$.

The implicit null of the policy ineffectiveness test is given by

$$H_{s,0} : \mu_{(1)} = \mathbf{0},\ \sigma^2_{0v} - \sigma^2_{1v} = 0. \tag{20}$$

The latter condition, $\sigma^2_{0v} = \sigma^2_{1v}$, is required in the implementation of the test. Under H_0, and assuming that the above classical assumptions hold we have

$$\hat{\mathbf{d}}_{(1)} = \mathbf{v}_{(1)} - \mathbf{S}^0_{(1)} \left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1} \mathbf{S}'_{(0)}\mathbf{v}_{(0)}, \tag{21}$$

and it readily follows that $\hat{\mathbf{d}}_{(1)} \sim (\mathbf{0}, \mathbf{\Omega}_d)$, where

$$\mathbf{\Omega}_d = \sigma^2_{0v} \left[\mathbf{I}_H + \mathbf{S}^0_{(1)} \left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1} \mathbf{S}^{0\prime}_{(1)}\right].$$

A test can now be based on all the individual H elements of $\hat{\mathbf{d}}_{(1)}$ which yields the joint test statistic

$$\chi^2_{d,H} = \frac{\hat{\mathbf{d}}'_{(1)} \left[\mathbf{I}_H + \mathbf{S}^0_{(1)} \left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1} \mathbf{S}^{0\prime}_{(1)}\right]^{-1} \hat{\mathbf{d}}_{(1)}}{\sigma^2_{0v}}. \tag{22}$$

Under the policy ineffectiveness hypothesis, H_0, and assuming that $\mathbf{v} = (\mathbf{v}'_{(0)}, \mathbf{v}'_{(1)})'$ is normally distributed then $\chi^2_{d,H}$ is distributed as a chi-square variate with H degrees of freedom. If σ^2_{0v} is replaced by its unbiased estimator based on the pre-intervention sample:

$$\hat{\sigma}^2_{0v} = \frac{\left(\mathbf{y}_{(0)} - \mathbf{S}_{(0)}\hat{\pi}^0_{ys}\right)' \left(\mathbf{y}_{(0)} - \mathbf{S}_{(0)}\hat{\pi}^0_{ys}\right)}{T - k_x - k_w}, \tag{23}$$

we obtain the feasible test statistic

$$\mathcal{F}_{d,H} = \frac{\hat{\mathbf{d}}'_{(1)} \left[\mathbf{I}_H + \mathbf{S}^0_{(1)} \left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1} \mathbf{S}^{0\prime}_{(1)}\right]^{-1} \hat{\mathbf{d}}_{(1)}}{H\hat{\sigma}^2_{0v}}, \tag{24}$$

which under the null hypothesis, $H_{s,0}$ defined by (20), is distributed as F with H and $T - k_x - k_w$ degrees of freedom. A proof is provided in Appendix A1.

Alternatively, one can base a test on linear combinations of the elements of $\hat{\mathbf{d}}_{(1)}$, such as the mean $\bar{\hat{d}}_H = H^{-1}\tau'_H\hat{\mathbf{d}}_{(1)}$, where τ_H is a vector of ones of length H. The policy ineffectiveness test statistic for this case is given by

$$t_{d,H} = \frac{\sqrt{H}\bar{\hat{d}}_H}{\hat{\sigma}_{0v}\sqrt{1 + H^{-1}T^{-1}\tau'_H\mathbf{S}^0_{(1)} \left[T^{-1}\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right]^{-1} \mathbf{S}^{0\prime}_{(1)}\tau_H}}. \tag{25}$$

For this test, the assumption that $\mathbf{v}_{(1)}$ is normally distributed can be relaxed, so long as H is sufficiently large. This result holds irrespective of the relative rate at which H and $T \to \infty$.

It is now easily seen that under the null of policy ineffectiveness, $t_{d,H}$ is $N(0,1)$ for H and T sufficiently large. In the case where T is large relative to H, the estimation uncertainty will be relatively negligible and the test statistic simplifies to

$$t_{d,H}^{a} = \frac{\sqrt{H}\bar{\hat{d}}_H}{\hat{\sigma}_{0v}} \overset{a}{\sim} N(0,1), \tag{26}$$

where $\hat{\sigma}_{0v}$ is defined by (23).

3.2 The dynamic case

In the general case, with dynamics, the RE solution is given by (5) above

$$\mathbf{q}_t = \mathbf{\Phi}\left(\theta\right)\mathbf{q}_{t-1} + \mathbf{\Psi}_x\left(\theta\right)\mathbf{x}_t + \mathbf{\Psi}_w\left(\theta\right)\mathbf{w}_t + \varepsilon_t.$$

where θ contains $\mathbf{a}$ as well as the parameters of the processes generating the exogenous variables, $\mathbf{s}_t = (\mathbf{x}_t', \mathbf{w}_t')'$.

The effect of policy on the target variable is the difference between the realised values, y_{T_0+h}, and the counterfactual values, $y_{T_0+h}^0$,

$$d_{T_0+h} = y_{T_0+h} - y_{T_0+h}^0,\ h = 1, 2, ..., H. \tag{27}$$

These measured policy effects will be subject to the post intervention random errors, $\varepsilon_{y,T_0+h.}$.

Introducing $\mathfrak{s}$, a the $(k_z + 1) \times 1$ selection vector with all its elements zero except for its first element which is set to unity, the counterfactual values of y_{T_0+h}, are given by

$$y_{T_0+h}^0 = \mathfrak{s}'\left[\mathbf{\Phi}\left(\theta^0\right)\right]^h \mathbf{q}_{T_0} + \mathfrak{s}'\sum_{j=0}^{h-1}\left[\mathbf{\Phi}\left(\theta^0\right)\right]^j\left[\mathbf{\Psi}_x\left(\theta^0\right)\mathbf{x}_{T_0+h-j}^0 + \mathbf{\Psi}_w\left(\theta^0\right)\mathbf{w}_{T_0+h-j}\right],$$

where $x_{T_0+h}^0$ for $h = 1, 2, ..., H$ denote the counterfactual values of the policy variable, and $\mathbf{w}_{T+h}$, for $h = 1, 2, ..., H$, are the realized values of the policy invariant variables.

In general where the correct specification of the RE model is not known, a more robust specification for the policy response equation can be derived by eliminating the lagged values of $\mathbf{z}_t$, as set out in Zellner and Palm (1974), and obtain the following $ARDL(p_y, p_x, p_w)$ specification for pre and post-intervention samples:[2]

$$y_t = \sum_{i=1}^{p_y}\lambda_i(\theta^0)y_{t-i} + \sum_{i=0}^{p_x}\pi_{yx,i}(\theta^0)x_{t-i} + \sum_{i=0}^{p_w}\pi_{yw,i}'(\theta^0)\mathbf{w}_{t-i} + v_{yt},\ t = M, M+1, M+2, ..., T_0; \tag{28}$$

$$y_t = \sum_{i=1}^{p_y}\lambda_i(\theta^1)y_{t-i} + \sum_{i=0}^{p_x}\pi_{yx,i}(\theta^1)x_{t-i} + \sum_{i=0}^{p_w}\pi_{yw,i}'(\theta^1)\mathbf{w}_{t-i} + v_{yt},\ t = T_0+1, T_0+2, ..., T_0+H, \tag{29}$$

[2] It is well known that univariate representations of variables in a VAR are ARMA (autoregressive moving average) processes. For example, in the case where $\mathbf{z}_t$ is a scalar variable and the RE model does not contain any exogenous variables, the univariate representation of y_t will be an ARMA(2,1) process. However, in practice such ARMA processes are approximated by high order AR processes.

where the lag orders, p_y, p_x, p_w, are selected to be sufficiently long to ensure that the reduced form residuals, v_{yt}, are serially uncorrelated.

The derivation of the tests above readily extend to the dynamic specification, (28) and (29). First, set $p_y = p_x = 1$, $p_w = 0$, and consider the ARDL(1,1) specification that we shall be using in the empirical application

$$y_t = \lambda^0 y_{t-1} + \pi^0_{yx0} x_t + \pi^0_{yx1} x_{t-1} + \pi^{0\prime}_{yw} \mathbf{w}_t + v_{yt}, \text{ for } t = M, M+1, M+2, ..., T_0, \tag{30}$$

$$y_t = \lambda^1 y_{t-1} + \pi^1_{yx0} x_t + \pi^1_{yx1} x_{t-1} + \pi^{1\prime}_{yw} \mathbf{w}_t + v_{yt}, \text{ for } t = T_0+1, ..., T_0+H, \tag{31}$$

where $\left|\lambda^j\right| < 1$ for $j = 0, 1$. We will also assume that the estimate of λ^0, denoted by $\hat{\lambda}^0$, satisfies the stationary condition, $\left|\hat{\lambda}^0\right| < 1$.

To allow for the endogeniety of policy, suppose also that the policy variable x_t is generated as

$$x_t = b_1(L) x_{t-1} + b_2(L) y_{t-1} + v_{xt}, \quad b_j(L) = b_{j0} + b_{j1} L + ... + b_{js_j} L^{s_j}.$$

with v_{yt} and v_{xt} being correlated. To correct for the endogeneity, following Pesaran and Shin (1999), we can model the contemporaneous correlation between v_{yt} and v_{xt}, by $v_{yt} = \delta v_{xt} + \eta_t$, where by construction v_{xt} and η_t are uncorrelated. The parametric correction for the endogeniety of x_t is equivalent to augmenting the ARDL specification with an adequate number of lagged values of x_t before estimation of the policy reduced form equation is carried out.

In what follows we continue to use the lag orders $p_y = p_x = 1$ and $p_w = 0$, keep the notations simple and rewrite the ARDL specifications for the pre-intervention sample as

$$\mathbf{y}_{(0)} = \lambda^0 \mathbf{y}_{-1,(0)} + \mathbf{S}_{(0)} \pi^0_{ys} + \mathbf{v}_{(0)},$$

where $\mathbf{y}_{(0)}$ and $\mathbf{v}_{(0)}$ are defined as before, $\mathbf{y}_{-1,(0)} = (y_{M-1}, y_M, ..., y_{T_0-1})'$, $\mathbf{S}_{(0)} = \left(\mathbf{X}_{(0)}, \mathbf{W}_{(0)}\right)$, $\mathbf{X}_{(0)} = \left(\mathbf{x}_{(0)}, \mathbf{x}_{-1,(0)}\right)$, $\mathbf{x}_{(0)} = (x_M, x_{M+1}, ..., x_{T_0})'$, $\mathbf{x}_{-1,(0)} = (x_{M-1}, x_M, ..., x_{T_0-1})'$, and $\pi^0_{ys} = \left(\pi^0_{yx0}, \pi^0_{yx1}, \pi^{0\prime}_{yw}\right)'$. Further lagged values of x_t as well as deterministic components such as intercept and linear trends can also be included in $\mathbf{S}_{(0)}$.

Based on this specification and given counterfactual values of the policy variables and their lagged values over the post-intervention sample, which as before we denote by $\mathbf{X}^0_{(1)}$, by forward iterations of the dynamic equations from $t = T_0$ we obtain the following counterfactual outcomes

$$\begin{aligned} \hat{\mathbf{y}}^0_{(1)} &= \hat{\mathbf{\Lambda}}^0_H \left[\mathbf{e}_1 \hat{\lambda}^0 y_{T_0} + \mathbf{X}^0_{(1)} \hat{\pi}^0_{yx} + \mathbf{W}_{(1)} \hat{\pi}^0_{yw},\right] \\ &= \hat{\mathbf{\Lambda}}^0_H \left[\mathbf{e}_1 \hat{\lambda}^0 y_{T_0} + \mathbf{S}^0_{(1)} \hat{\pi}^0_{ys}\right], \end{aligned} \tag{32}$$

where $\hat{\Lambda}^0_H$ is the $H \times H$ lower triangular matrix

$$\hat{\mathbf{\Lambda}}^0_H = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \hat{\lambda}^0 & 1 & 0 & \cdots & 0 & 0 \\ \left(\hat{\lambda}^0\right)^2 & \hat{\lambda}^0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(\hat{\lambda}^0\right)^{H-2} & \left(\hat{\lambda}^0\right)^{H-3} & & \cdots & 1 & 0 \\ \left(\hat{\lambda}^0\right)^{H-1} & \left(\hat{\lambda}^0\right)^{H-2} & & \cdots & \hat{\lambda}^0 & 1 \end{pmatrix}, \tag{33}$$

$\mathbf{e}_1 = (1, 0, ..., 0)'$, and $\hat{\lambda}^0$, $\hat{\pi}^0_{ys} = (\hat{\pi}^{0\prime}_{yx}, \hat{\pi}^{0\prime}_{yw})'$ are least square estimates of λ^0,$\pi^0_{ys} = (\pi^{0\prime}_{yx}, \pi^{0\prime}_{yw})'$ in the dynamic policy impulse equation, (30), based on the pre-intervention sample.[3] More specifically, setting $\varphi^0 = (\lambda^0, \pi^{0\prime}_{ys})'$, and $\mathbf{Q}_{(0)} = \left(\mathbf{y}_{-1,(0)}, \mathbf{S}_{(0)}\right)$, we have

$$\hat{\varphi}^0 = \left(\mathbf{Q}_{(0)}\mathbf{Q}'_{(0)}\right)^{-1} \mathbf{Q}'_{(0)}\mathbf{y}_{(0)}. \tag{34}$$

For future reference we also note that under fairly general conditions on the error terms, $\mathbf{v}_{(0)}$ and assuming that $\left|\lambda^0\right| < 1$, and $\left|\hat{\lambda}^0\right| < 1$, then as $T \to \infty$ we have

$$\sqrt{T}\left(\hat{\varphi}^0 - \varphi^0\right) \to_d N(\mathbf{0}, \sigma^2_{0v}\mathbf{\Sigma}^{-1}_0), \tag{35}$$

where as before, $E(\mathbf{v}\mathbf{v}') = \sigma^2_{0v}\mathbf{I}_{T+H}$, H is finite and $\mathbf{\Sigma}_0 = p\lim_{T\to\infty}\left(T^{-1}\mathbf{Q}_{(0)}\mathbf{Q}'_{(0)}\right)$ is a positive definite matrix.

The estimates of the policy effects are now given by

$$\hat{\mathbf{d}}_{(1)} = \mathbf{y}_{(1)} - \hat{\mathbf{\Lambda}}^0_H\left[y_{T_0}\hat{\lambda}^0\mathbf{e}_1 + \mathbf{S}^0_{(1)}\hat{\pi}^0_{ys}\right].$$

As before, this can be decomposed into a systematic effect of the policy, the random components due to $\mathbf{v}_{(1)}$ and the sampling uncertainty in estimation of $\hat{\lambda}^0$, $\hat{\pi}^0_{yx}$, and $\hat{\pi}^0_{yw}$.

Using the forward recursive approach, we first note that

$$\begin{aligned} \mathbf{y}_{(1)} &= \mathbf{\Lambda}^1_H\left[y_{T_0}\lambda^1\mathbf{e}_1 + \mathbf{X}_{(1)}\pi^1_{yx} + \mathbf{W}_{(1)}\pi^1_{yw} + \mathbf{v}_{(1)}\right] \\ &= \mathbf{\Lambda}^1_H\left[y_{T_0}\lambda^1\mathbf{e}_1 + \mathbf{S}_{(1)}\pi^1_{ys} + \mathbf{v}_{(1)}\right]. \end{aligned}$$

Using the above results we have

$$\hat{\mathbf{d}}_{(1)} = \mathbf{\Lambda}^1_H\left[y_{T_0}\lambda^1\mathbf{e}_1 + \mathbf{S}_{(1)}\pi^1_{ys}\right] - \hat{\mathbf{\Lambda}}^0_H\left[y_{T_0}\hat{\lambda}^0\mathbf{e}_1 + \mathbf{S}^0_{(1)}\hat{\pi}^0_{ys}\right] + \mathbf{\Lambda}^1_H\mathbf{v}_{(1)},$$

which can be written as

$$\hat{\mathbf{d}}_{(1)} = \mu_{(1)} - \xi_{(1)} + \mathbf{\Lambda}^1_H\mathbf{v}_{(1)}, \tag{36}$$

[3] The above counterfactual outcomes reduce to the ones obtained for the static case, (16), if $\hat{\lambda}^0 = 0$.

where

$$\mu_{(1)} = y_{T_0}\left(\mathbf{\Lambda}_H^1\lambda^1 - \mathbf{\Lambda}_H^0\lambda^0\right)\mathbf{e}_1 + \left[\mathbf{\Lambda}_H^1\mathbf{S}_{(1)}\pi_{ys}^1 - \mathbf{\Lambda}_H^0\mathbf{S}_{(1)}^0\pi_{ys}^0\right], \tag{37}$$

and

$$\xi_{(1)} = \hat{\mathbf{\Lambda}}_H^0\left[y_{T_0}\hat{\lambda}^0\mathbf{e}_1 + \mathbf{S}_{(1)}\hat{\pi}_{ys}^0\right] - \mathbf{\Lambda}_H^0\left[y_{T_0}\lambda^0\mathbf{e}_1 + \mathbf{S}_{(1)}\pi_{ys}^0\right]. \tag{38}$$

In the dynamic case, the implicit null of the policy ineffectiveness hypothesis is given by

$$\mu_{(1)} = y_{T_0}\left(\mathbf{\Lambda}_H^1\lambda^1 - \mathbf{\Lambda}_H^0\lambda^0\right)\mathbf{e}_1 + \left[\mathbf{\Lambda}_H^1\mathbf{S}_{(1)}\pi_{ys}^1 - \mathbf{\Lambda}_H^0\mathbf{S}_{(1)}^0\pi_{ys}^0\right] = \mathbf{0}.$$

The third term of (36), $\mathbf{\Lambda}_H^1\mathbf{v}_{(1)}$, is the vector of the random shocks during post-intervention period, and the implementation of the test of policy ineffectiveness hypothesis in the dynamic case requires making the additional assumption that under H_0, we also have $\lambda^1 = \lambda^0$, as well as $E(\mathbf{v}_{(1)}\mathbf{v}_{(1)}') = \sigma_{0v}^2\mathbf{I}_H$, the assumption already made in the static case. Finally, $\xi_{(1)}$ captures the effects of sampling uncertainty associated with the estimation λ^0 and π_{ys}. In the dynamic case the null hypothesis of policy ineffectiveness is given by

$$H_0 : \mu_{(1)} = \mathbf{0},\ \sigma_{0v}^2 = \sigma_{1v}^2,\ \lambda^0 = \lambda^1. \tag{39}$$

We now derive the asymptotic distribution of $\hat{\mathbf{d}}_{(1)}$ under H_0, initially assuming that H is fixed. Under H_0

$$\hat{\mathbf{d}}_{(1)} = \mathbf{\Lambda}_H^1\mathbf{v}_{(1)} - \xi_{(1)}, \tag{40}$$

and using (38) we have

$$\begin{aligned}\xi_{(1)} &= y_{T_0}\left(\hat{\lambda}^0 - \lambda^0\right)\left(\hat{\mathbf{\Lambda}}_H^0 - \mathbf{\Lambda}_H^0\right)\mathbf{e}_1 + \left(\hat{\mathbf{\Lambda}}_H^0 - \mathbf{\Lambda}_H^0\right)\mathbf{S}_{(1)}^0\left(\hat{\pi}_{ys}^0 - \pi_{ys}^0\right)\\ &+ y_{T_0}\lambda^0\left(\hat{\mathbf{\Lambda}}_H^0 - \mathbf{\Lambda}_H^0\right)\mathbf{e}_1 + \left(\hat{\mathbf{\Lambda}}_H^0 - \mathbf{\Lambda}_H^0\right)\mathbf{S}_{(1)}^0\pi_{ys}^0\\ &+ y_{T_0}\left(\hat{\lambda}^0 - \lambda^0\right)\mathbf{\Lambda}_H^0\mathbf{e}_1 + \mathbf{\Lambda}_H^0\mathbf{S}_{(1)}^0\left(\hat{\pi}_{ys}^0 - \pi_{ys}^0\right).\end{aligned}$$

Also estimating the dynamic regression model, (30), by least squares under standard assumption we have

$$\hat{\lambda}^0 = \lambda^0 + a_T T^{-1/2},\ \text{and } \hat{\pi}_{ys}^0 = \pi_{ys}^0 + \mathbf{b}_T T^{-1/2}, \tag{41}$$

where a_T and $\mathbf{b}_T$ are random variables bounded in T. Hence, under H_0

$$\begin{aligned}\hat{\mathbf{d}}_{(1)} &= \mathbf{\Lambda}_H^1\mathbf{v}_{(1)} - y_{T_0}\lambda^0\left(\hat{\mathbf{\Lambda}}_H^0 - \mathbf{\Lambda}_H^0\right)\mathbf{e}_1 - y_{T_0}\left(\hat{\lambda}^0 - \lambda^0\right)\mathbf{\Lambda}_H^0\mathbf{e}_1\\ &- \left(\hat{\mathbf{\Lambda}}_H^0 - \mathbf{\Lambda}_H^0\right)\mathbf{S}_{(1)}\pi_{ys}^0 - \mathbf{\Lambda}_H^0\mathbf{S}_{(1)}\left(\hat{\pi}_{ys}^0 - \pi_{ys}^0\right) + O_p(T^{-1}).\end{aligned} \tag{42}$$

Using a Taylor series expansion we have (for H fixed)

$$\hat{\mathbf{\Lambda}}_H^0 - \mathbf{\Lambda}_H^0 = \frac{\partial\mathbf{\Lambda}_H^0}{\partial\lambda^0}\left(\hat{\lambda}^0 - \lambda^0\right) + O_p\left(\frac{1}{T}\right).$$

where

$$\frac{\partial \mathbf{\Lambda}_H^0}{\partial \lambda^0} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 2\lambda_0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (H-2)\left(\lambda^0\right)^{H-3} & (H-3)\left(\lambda^0\right)^{H-4} & \cdots & 1 & 0 & 0 \\ (H-1)\left(\lambda^0\right)^{H-2} & (H-2)\left(\lambda^0\right)^{H-3} & \cdots & 2\lambda_0 & 1 & 0 \end{pmatrix},$$

Using this result in (42) now yields

$$\hat{\mathbf{d}}_{(1)} = \mathbf{\Lambda}_H^1 \mathbf{v}_{(1)} - \mathbf{D}_{(1)}^0 \left(\hat{\lambda}^0 - \lambda^0\right) - \mathbf{\Lambda}_H^0 \mathbf{S}_{(1)}^0 \left(\hat{\pi}_{ys}^0 - \pi_{ys}^0\right) + O_p\left(\frac{1}{T}\right),$$

where

$$\mathbf{D}_{(1)}^0 = y_{T_0}\left(\lambda^0 \frac{\partial \mathbf{\Lambda}_H^0}{\partial \lambda^0} + \mathbf{\Lambda}_H^0\right)\mathbf{e}_1 + \frac{\partial \mathbf{\Lambda}_H^0}{\partial \lambda^0}\mathbf{S}_{(1)}^0 \pi_{ys}^0.$$

Writing the above result more compactly, we have

$$\hat{\mathbf{d}}_{(1)} = \mathbf{\Lambda}_H^1 \mathbf{v}_{(1)} - \mathbf{\Psi}_{(1)}^0 \left(\hat{\varphi}^0 - \varphi^0\right) + O_p\left(\frac{1}{T}\right).$$

where $\mathbf{\Psi}_{(1)}^0 = \left(\mathbf{D}_{(1)}^0, \mathbf{\Lambda}_H^0 \mathbf{S}_{(1)}^0\right)$.

When H is fixed and T sufficiently large a test can be based on all the elements of $\hat{\mathbf{d}}_{(1)}$ if $\mathbf{v}_{(1)}$ has a known distribution. But in general, as in the static case, we need to base the test of policy ineffectiveness on some average of $\hat{\mathbf{d}}_{(1)}$. Again using the policy mean effect statistic, $\bar{\hat{d}}_H = H^{-1}\tau_H' \hat{\mathbf{d}}_{(1)}$, under H_0 defined by (39) we have

$$\bar{\hat{d}}_H = \frac{1}{H}\tau_H' \mathbf{\Lambda}_H^0 \mathbf{v}_{(1)} - \frac{1}{H}\tau_H' \mathbf{\Psi}_{(1)}^0 \left(\hat{\varphi}^0 - \varphi^0\right) + O_p\left(\frac{1}{T}\right).$$

For a finite H and a sufficiently large T the distribution of $\bar{\hat{d}}_H$ depends on the distribution of $\mathbf{v}_{(1)}$. In the case where $\mathbf{v}_{(1)}$ is normally distributed we can use the following test statistic

$$\mathcal{T}_{d,H}^a = \frac{\sqrt{H}\bar{\hat{d}}_H}{\hat{\sigma}_{0v}\left[\left(\frac{\tau_H' \hat{\mathbf{\Lambda}}_H^0 \hat{\mathbf{\Lambda}}_H^{0\prime} \tau_H}{H}\right) + \frac{\tau_H' \hat{\mathbf{\Psi}}_{(1)}^0 \left(T^{-1}\mathbf{Q}_{(0)}\mathbf{Q}_{(0)}'\right)^{-1} \hat{\mathbf{\Psi}}_{(1)}^{0\prime} \tau_H'}{TH}\right]^{1/2}} \to_d N(0,1), \tag{43}$$

where $\hat{\lambda}^0$ and $\hat{\sigma}_{0v}$ are the least squares estimates of λ and σ_v based on the pre-intervention sample,

$$\hat{\mathbf{\Psi}}_{(1)}^0 = \left[y_{T_0}\left(\hat{\lambda}^0 \frac{\partial \hat{\mathbf{\Lambda}}_H^0}{\partial \lambda^0} + \hat{\mathbf{\Lambda}}_H^0\right)\mathbf{e}_1 + \frac{\partial \hat{\mathbf{\Lambda}}_H^0}{\partial \lambda^0}\mathbf{S}_{(1)}^0 \hat{\pi}_{ys}^0,\ \hat{\mathbf{\Lambda}}_H^0 \mathbf{S}_{(1)}^0\right],$$

and $\mathbf{Q}_{(0)} = \left(\mathbf{y}_{-1,(0)}, \mathbf{S}_{(0)}\right)$. In the case where T is reasonably large relative to H, the second term in the denominator of (43) will be negligible and the test statistic simplifies to

$$\mathcal{T}_{d,H}^a = \frac{\sqrt{H}\bar{\hat{d}}_H}{\hat{\sigma}_{0v}\left(\frac{\tau_H' \hat{\mathbf{\Lambda}}_H^0 \hat{\mathbf{\Lambda}}_H^{0\prime} \tau_H}{H}\right)^{1/2}} \to_d N(0,1), \tag{44}$$

where $\hat{\sigma}_{0v}$ is the estimate of σ_{0v} computed using the pre-intervention sample, and

$$\frac{\tau_H' \hat{\mathbf{\Lambda}}_H^0 \hat{\mathbf{\Lambda}}_H^{0\prime} \tau_H}{H} = \frac{1}{\left(1-\hat{\lambda}^0\right)^2}\left[1 - \frac{2}{H}\left(\frac{\left(\hat{\lambda}^0\right)^{H+1} - \hat{\lambda}^0}{1-\hat{\lambda}^0}\right) + \frac{1}{H}\left(\frac{\left(\hat{\lambda}^0\right)^{2H+2} - \left(\hat{\lambda}^0\right)^2}{\left[1-\left(\hat{\lambda}^0\right)^2\right]}\right)\right]. \tag{45}$$

In the case where H rises with T, the derivations are best carried out in terms of the individual elements of $\hat{\mathbf{d}}_{(1)}$ defined by (36), which we write as

$$\hat{d}_{T_0+h} = -\left[\left(\hat{\lambda}^0\right)^h - \left(\lambda^1\right)^h\right] y_{T_0} - \sum_{j=0}^{h-1}\left[\left(\hat{\lambda}^0\right)^h \hat{\pi}_{ys}^0 - \left(\lambda^1\right)^h \pi_{ys}^1\right]' \mathbf{s}_{T_0+h-j} + \sum_{j=0}^{h-1}\left(\lambda^1\right)^h v_{T_0+h-j},$$

where $\mathbf{s}_t = (x_t, x_{t-1}, w_t, w_{t-1})'$. The mean policy effect test statistic can now be written as

$$\begin{aligned}\sqrt{H}\bar{\hat{d}}_H = & -H^{-1/2}\sum_{h=1}^{H}\left[\left(\hat{\lambda}^0\right)^h - \left(\lambda^1\right)^h\right] y_{T_0} \\ & - H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\left[\left(\hat{\lambda}^0\right)^h \hat{\pi}_{ys}^0 - \left(\lambda^1\right)^h \pi_{ys}^1\right]' \mathbf{s}_{T_0+h-j} \\ & + H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\left(\lambda^1\right)^h v_{T_0+h-j}.\end{aligned} \tag{46}$$

Under the null hypothesis $\lambda^1 = \lambda^0$, $\pi_{ys}^1 = \pi_{ys}^0$, and $\sigma_{1v}^2 = \sigma_{0v}^2$, we first note that

$$\sum_{h=1}^{H}\left[\left(\hat{\lambda}^0\right)^h - \left(\lambda^0\right)^h\right] = \frac{\hat{\lambda}^0 - \lambda^0 + \hat{\lambda}^0\lambda^0\left[\left(\hat{\lambda}^0\right)^H - \left(\lambda^0\right)^H\right] - \left[\left(\hat{\lambda}^0\right)^{H+1} - \left(\lambda^0\right)^{H+1}\right]}{\left(1-\lambda^0\right)\left(1-\hat{\lambda}^0\right)}.$$

Also using results in Lemma 3 in PS, and since $\hat{\lambda}^0 - \lambda^0 = a_T^0/\sqrt{T}$, $(\lambda^0 \neq 0)$

$$H^{-1/2}\left|\left(\hat{\lambda}^0\right)^H - \left(\lambda^0\right)^H\right| \leq H^{1/2}\left|\hat{\lambda}^0\right|^{H-1}\left|\hat{\lambda}^0 - \lambda^0\right| \leq \left(\frac{H}{T}\right)^{1/2}\left|a_T^0\right|\left|\lambda^0\right|^H\left|1 + \frac{a_T^0}{\lambda^0\sqrt{T}}\right|^H,$$

where $\left|a_T^0\right|$ is bounded in T. But $\left|1 + a_T^0/\lambda^0\sqrt{T}\right|^H$ tends to a bounded random variable if $H/\sqrt{T}$ tends to a fixed constant , or equivalently if $H = \kappa T^\epsilon$, with $\epsilon \leq 1/2$ as T and $H \to \infty$, jointly. Under this condition we have $H^{-1/2}\left|\left(\hat{\lambda}^0\right)^H - \left(\lambda^0\right)^H\right| \to_p 0$ since $\left|\lambda^0\right| < 1$. Similarly, under the null hypothesis

$$\begin{aligned} & H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\left[\left(\hat{\lambda}^0\right)^h \hat{\pi}_{ys}^0 - \left(\lambda^1\right)^h \pi_{ys}^1\right]' \mathbf{s}_{T_0+h-j} \\ & = H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\left[\begin{array}{c}\left[\left(\hat{\lambda}^0\right)^h - \left(\lambda^0\right)^h\right]\left(\hat{\pi}_{ys}^0 - \pi_{ys}^0\right) + \left[\left(\hat{\lambda}^0\right)^h - \left(\lambda^0\right)^h\right]\pi_{ys}^0 \\ + \left(\lambda^0\right)^h\left(\hat{\pi}_{ys}^0 - \pi_{ys}^0\right)\end{array}\right]' \mathbf{s}_{T_0+h-j},\end{aligned}$$

we note that (using Lemma 1 in PS)

$$
\begin{aligned}
&H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\left[\left(\hat{\lambda}^{0}\right)^{h}-\left(\lambda^{0}\right)^{h}\right]\pi_{ys}^{0\prime}\mathbf{s}_{T_0+h-j} \\
&=\left[\frac{H^{1/2}\left(\hat{\lambda}^{0}-\lambda^{0}\right)}{\left(1-\lambda^{0}\right)\left(1-\hat{\lambda}^{0}\right)}\right]\left(H^{-1}\sum_{j=1}^{H}\pi_{ys}^{0\prime}\mathbf{s}_{T_0+j}\right) \\
&-\left(H^{-1/2}\sum_{j=1}^{H}\left[\left(1-\hat{\lambda}^{0}\right)^{-1}\left(\hat{\lambda}^{0}\right)^{H-j+1}-\left(1-\lambda^{0}\right)^{-1}\left(\lambda^{0}\right)^{H-j+1}\right]\pi_{ys}^{0\prime}\mathbf{s}_{T_0+j}\right).
\end{aligned}
$$

Since by assumption $\left|H^{-1}\sum_{j=1}^{H}\pi_{ys}^{0\prime}\mathbf{s}_{T_0+j}\right| < K$, and $\hat{\lambda}^{0}-\lambda^{0} = O_p(T^{-1/2})$, then the first term of the above expression tend to zero in probability of $H/T \to 0$, which is satisfied if $H=\kappa T^{\epsilon}$, with $\epsilon \le 1/2$. Consider the second term of the above expression and note that

$$
\begin{aligned}
&\left|H^{-1/2}\sum_{j=1}^{H}\left[\left(1-\hat{\lambda}^{0}\right)^{-1}\left(\hat{\lambda}^{0}\right)^{H-j+1}-\left(1-\lambda^{0}\right)^{-1}\left(\lambda^{0}\right)^{H-j+1}\right]\pi_{ys}^{0\prime}\mathbf{s}_{T_0+j}\right| \\
&\le \sup_j\left|\pi_{ys}^{0\prime}\mathbf{s}_{T_0+j}\right| H^{-1/2}\sum_{j=1}^{H}\left|\left(1-\hat{\lambda}^{0}\right)^{-1}\left(\hat{\lambda}^{0}\right)^{H-j+1}-\left(1-\lambda^{0}\right)^{-1}\left(\lambda^{0}\right)^{H-j+1}\right| \\
&\le \frac{\sup_j\left|\pi_{ys}^{0\prime}\mathbf{s}_{T_0+j}\right|}{\left|\left(1-\lambda^{0}\right)\left(1-\hat{\lambda}^{0}\right)\right|}\left[H^{1/2}\sum_{j=1}^{H}\left|\left(\hat{\lambda}^{0}\right)^{j}-\left(\lambda^{0}\right)^{j}\right|+\left|\lambda^{0}\right|\left|\hat{\lambda}^{0}\right|H^{1/2}\sum_{j=1}^{H}\left|\left(\hat{\lambda}^{0}\right)^{j-1}-\left(\lambda^{0}\right)^{j-1}\right|\right].
\end{aligned}
$$

But using results in Lemma 3 and 4 in PS we have

$$
\begin{aligned}
H^{1/2}\sum_{j=1}^{H}\left|\left(\hat{\lambda}^{0}\right)^{j}-\left(\lambda^{0}\right)^{j}\right| &\le H^{1/2}\left|\hat{\lambda}^{0}-\lambda^{0}\right|\sum_{j=1}^{H}j\left|\hat{\lambda}^{0}\right|^{j-1} \\
&= H^{1/2}\left|\hat{\lambda}^{0}-\lambda^{0}\right|\left[\frac{1-\left|\hat{\lambda}^{0}\right|^{H}}{\left(1-\left|\hat{\lambda}^{0}\right|\right)^{2}}-\frac{H\left|\hat{\lambda}^{0}\right|^{H+1}}{\left(1-\left|\hat{\lambda}^{0}\right|\right)}\right],
\end{aligned}
$$

and using similar arguments as before it follows that

$$
H^{1/2}\sum_{j=1}^{H}\left|\left(\hat{\lambda}^{0}\right)^{j}-\left(\lambda^{0}\right)^{j}\right| \to_p 0,
$$

if $\sup_j\left|\pi_{ys}^{0\prime}\mathbf{s}_{T_0+j}\right|$ is bounded in T, and $H=\kappa T^{\epsilon}$, with $\epsilon \le 1/2$, as T and $H\to\infty$, jointly. Notice that since, by assumption, both $\hat{\lambda}^{0}$ and λ^{0} are less than one in absolute value, $\left|\left(\hat{\lambda}^{0}\right)^{j}-\left(\lambda^{0}\right)^{j}\right|$ declines exponentially in j.

Under these conditions and employing the above results in (46), and using (45) with $H\to\infty$, we finally obtain the large T and H test statistic

$$
\mathcal{T}_{d,H}^{b}=\frac{\left|1-\hat{\lambda}^{0}\right|\sqrt{H}\bar{\hat{d}}_{H}}{\hat{\sigma}_{0v}} \to_d N(0,1), \tag{47}
$$

which is the large H version of (44).

The above test statistics can also be readily generalized to the higher order ARDL specification given by (28) and (29). See Appendix A2 for the details.

4 An empirical application: testing the effects of quantitative easing

We will illustrate the policy-ineffectiveness test with an investigation into the effect of an unconventional monetary policy (UMP), known as quantitative easing (QE), in the UK introduced in March 2009.[4] We will use a reduced form approach and an ARDL(1,1) model, as in Section 3.2.

UMPs have tended to be adopted when central banks have hit the zero lower bound for the policy interest rate, though in principle they could be adopted even if interest rates are not at the lower bound. The term quantitative easing was used by the Bank of Japan to describe its policies from 2001. See, for example, Bowman et al. (2011). During the financial crisis, starting in 2007, and particularly after the failure of Lehman Brothers in 2008, many central banks adopted UMP. Examples include the large scale asset purchase programme by Federal Reserve in US and the long term repo operations and emergency liquidity assistance by the European Central Bank. The central banks differed in the specific measures used and had different theoretical perceptions of what the policy interventions were designed to achieve and the transmission mechanisms involved.[5] Borio and Disyatat (2010) classify such policies as balance sheet policies, as distinct from interest rate policies, and describe the variety of different types of measures adopted by seven central banks during the financial crisis. There has been considerable controversy over two questions: (a) what was the effects of UMP on various interest rates? usually answered using "event studies" and (b) what was the effect of those interest rate changes on output and inflation? We shall consider question (b) to illustrate our reduced form test taking the answer to (a) as given.

In the UK QE involved exchanging one liability of the state - government bonds (gilts) - for another - claims on the central bank. That change in the quantities of the two assets could cause a rise in the price of gilts, decline in their yields, but also cause a rise in the prices of substitute assets such as corporate bonds and equities. The Bank of England believed that QE boosted demand by increasing wealth and by reducing the cost of finance to companies.[6] It also increases banks liquidity and may have prompted more lending. Event studies documented in Joyce et al. (2011) suggest that QE reduced the spread of long over short term government interest rates (the

[4] See also the November 2012 Special Issue of the Economic Journal on *Unconventional Monetary Policy after the Financial Crisis*.

[5] For instance Giannone et al. (2011), who discuss the euro area, distinguish the Eurosystem's actions from the QE adopted by other Central Banks.

[6] For instance see the Financial Times article 4 May 2012 by Charlie Bean, then the Bank's Deputy Governor.

"spread") by 100 basis points from its introduction in March 2009. Thus the counterfactual we consider is the effect on log real output, Y_t, of there not having been a 100 basis points reduction in the spread. The estimate that QE reduced the spread by 100 basis points is not uncontroversial, Meaning and Zhu (2011) estimate a smaller impact of about 25 basis points, but our estimates could be easily scaled downwards to match this alternative estimate. This assumes a deterministic change in policy parameters, our counterfactual value for policy is $x^0_{T_0+h} = x_{T_0+h} - \delta$, where δ is a constant, so that the variance of the policy implementation errors, discussed in PS, is zero.

In examining QE we model the growth rate of output, $y_t = Y_t - Y_{t-1}$, because log output appears to have a unit root (and there is no long-run relationship between log output and the spread). The test will then be based on a mean policy effect computed over the post-intervention horizon $T_0 + h$, for $h = 1, 2, ..., H$, namely $\bar{\hat{d}}_H$ given by

$$\bar{\hat{d}}_H = \frac{1}{H}\sum_{h=1}^{H}\hat{d}_{T_0+h},$$

where

$$d_{T_0+h} = y_{T_0+h} - \hat{y}^0_{T_0+h}, \ \ h = 1, 2, ..., H.$$

Since log output seems to have a unit root and there is no long-run relationship between log output, Y_t and the spread, we make our dependent variable $y_t = Y_t - Y_{t-1}$, the growth rate. However, our analysis still applies to log output. Consider

$$\begin{aligned}
\bar{\hat{d}}_H &= [\sum_{h=1}^{H}(y_{T_0+h} - \hat{y}^0_{T_0+h})]/H \\
&= [(Y_{T_0+H} - Y_{T_0+H-1}) + (Y_{T_0+H-1} - Y_{T_0+H-2}) + ... + (Y_{T_0+1} - Y_{T_0}) \\
&- \left(\widehat{Y}^0_{T_0+H} - \widehat{Y}^0_{T_0+H-1}\right) - \left(\widehat{Y}^0_{T_0+H-1} - \widehat{Y}^0_{T_0+H-2}\right) + ... + (\widehat{Y}^0_{T_0+1} - Y_{T_0})]/H \\
&= (Y_{T_0+H} - \hat{Y}^0_{T_0+H})/H.
\end{aligned}$$

Thus $\bar{\hat{d}}_H$ measures the total level effect of the policy over the horizon period. Thus $\bar{\hat{d}}_H$ measures the total level effect of the policy over the horizon period. The policy ineffectiveness test statistics are given by (47) or (44), depending whether H is sufficiently large.

Kapetanios et al. (2012), who examine the effects of QE on UK output growth and inflation, also use a reduction in spread of 100 basis points. They use three time-varying vector autoregressions, VARs, that include other $\mathbf{z}_t$ type variables and allow for parameter change in different ways. Baumeister and Benati (2013) also use time varying VARs to assess the macroeconomic effects of QE in the US and UK, assuming the effect of QE in the UK was to reduce the spread by 50 basis points. But as our theoretical analysis highlights, the effects of structural breaks due to factors other than the policy change must be distinguished from the structural breaks that could result from the policy intervention. Goodhart and Ashworth (2012) challenge the view that the official long rate is the proper measure of the effect of QE on the economy, and argue that the transmission was through other variables such as credit risk spreads. We do not rule out that QE might have had an impact on other such variables, the $\mathbf{z}_t$ in our notation, with effects on output growth, but such effects are indirectly accommodated in our approach. Goodhart and Ashworth (2012) also argue that external effects are important and these may be accommodated through the inclusion of foreign variables among the $\mathbf{w}_t$.

Here we re-examine the effects of QE on UK output growth, and for reasons explained above we shall be using the policy impulse equation, like (30), rather than a full structural model. The data are taken from the Global VAR data set, starting in 1979Q2 and recently extended to 2011Q2.[7] Growth, y_t, is measured by the quarterly change in the logarithm of real GDP. In calculating, x_t, the spread between the short and long government interest rates, the rates are expressed as $0.25 \log(1 + R_t/100)$, where R_t is the annual percent rate. For the conditioning variables, $\mathbf{w}_t = (y_t^{US}, y_t^{Euro})'$, we use US and euro area output growth as they are unlikely to have been significantly affected by UK QE, but their inclusion allows for the possible indirect effects of UMPs implemented in US and euro area on UK output growth. Over the full sample the correlation between UK growth and US growth is 0.47, in the post 1999 sample it is 0.76. For euro growth, the correlations are 0.36 and 0.73. Like Kapetanios et al. (2012) we assume that the reduction in the spread is permanent. But other time profiles for the policy effects of QE on spreads could also be considered.

We use an ARDL in output growth (y_t) and the spread between long and short government interest rates (x_t) augmented by current euro and US growth rates. Pesaran and Shin (1999) show that ARDL estimates are robust to endogeneity and robust to the fact that y_t (stationary) and x_t (near unit root) have different degrees of persistence. The ARDL may be more robust to structural change, than models like VARs with a large number of variables. Since more parsimonious models tend to forecast better, the ARDL may reduce forecast uncertainty due to parameter estimation error. The ARDL is also preferable to VAR models for counterfactual analysis since it allows efficiency gains by conditioning on contemporaneous policy and exogenous variables.

We choose the specification on the full sample 1980Q3-2011Q2, since the change in policy, while it may change the parameters, is unlikely to influence the lag length. With potential structural instability there is an issue of whether the variance or the mean shifts. When error variances are falling, as occurred during the period before the financial crisis (the so-called great moderation), it is optimal to place more weights on the most recent observations, Pesaran, Pick and Pranovich (2013). Both AIC and SBC indicated one lag. Thus the ARDL for the pre-intervention period is

[7] Described in Dees et al. (2007), with updates available at https://sites.google.com/site/gvarmodelling/

given by (30),

$$y_t = \lambda^0 y_{t-1} + \pi^0_{yx0} x_t + \pi^0_{yx1} x_{t-1} + \pi^{0\prime}_{yw} \mathbf{w}_t + v_{yt}, \text{ for } t = 1, 2, ..., T_0,$$

The equation passes diagnostic tests for serial correlation and heteroskedasticity, but fails (at 5% level) tests for normality and functional form. The restriction that it is the spread, rather than short and long government interest rates separately, that matters is not rejected (pval=0.23). It is clearly the change in spread that is important and the long-run effect of the spread is positive, which is implausible, but insignificant - the restriction $\pi_{yx0} + \pi_{yx1} = 0$ cannot be rejected on the full sample, t=1.61. This restriction is imposed on the model used in obtaining the pre-policy estimates. The estimates for the full sample and the pre-policy sample, using the change in spread and lagged spread, are shown in Table 1.

Table 1: ARDL in UK growth (y) and spread (x) augmented with US and Euro area growth rates, t ratios in parentheses.

	1980Q3-2008Q4	1980Q3-2011Q2
y_{t-1}	0.34688	0.3822
	(3.94)	(5.12)
Δx_t	-0.94559	-0.8197
	(-3.05)	(-2.63)
x_{t-1}	-	0.18960
	-	(1.61)
y_t^{US}	0.15509	0.1465
	(2.07)	(2.00)
y_t^{Euro}	0.11040	0.1636
	(1.89)	(3.047)
$\overline{R}^2$	0.333	0.446
LM test Res. Serial Corr.	0.414	0.332
$\widehat{\sigma}_v$	0.0051	0.0050

The model indicates that a permanent 100 basis points reduction in the spread increases predicted growth by almost 1% on impact, although this effect is quickly reversed and disappears altogether within two years. Although they do not emphasise this feature, the estimates of Kapetanios et al. (2012), tell very much the same story: the beneficial effects of QE on growth are of a similar size and rather short-lived. However this predicted positive effect on growth of reducing the spread is small relative to the large negative equation errors, the estimated equation over-predicts growth during the recession. So the actual is below the counterfactual outcome without QE in all but one post-intervention quarter. For $H = 10$, $(2009Q1 - 2011Q2)$, $\bar{\hat{d}}_H = -0.00315$; and from Table 1, $\hat{\lambda}^0 = 0.347$ and $\hat{\sigma}_{0v} = 0.0051$. So the test statistic (44)

$$\mathcal{T}^a_{d,H} = \frac{\sqrt{H}\bar{\hat{d}}_H}{\hat{\sigma}_{0v}\left(\frac{\tau'_H \hat{\mathbf{\Lambda}}^0_H \hat{\mathbf{\Lambda}}^{0\prime}_H \tau_H}{H}\right)^{1/2}},$$

is -1.17.

The small H adjustment used in (44) does not make very much difference and the test statistic without it (47)

$$\mathcal{T}^b_{d,H} = \frac{\left|1 - \hat{\lambda}^0\right| \sqrt{H}\bar{\hat{d}}_H}{\hat{\sigma}_{0v}},$$

is -1.28. Also taking account of the sampling uncertainty associated with parameter estimation, captured by the second term in the denominator of (43), namely

$$(TH)^{-1}\tau_H'\hat{\mathbf{\Psi}}^0_{(1)}\left(T^{-1}\mathbf{Q}_{(0)}\mathbf{Q}_{(0)}'\right)^{-1}\hat{\mathbf{\Psi}}^{0\prime}_{(1)}\tau_H' > 0,$$

will not alter the test outcome. Firstly, with $T = 114$ this second term is likely to be small, and secondly given that it is positive its inclusion can only reduce the statistical significance of the test.

As a result we can safely conclude that given our model specification, the null that the QE policy intervention was ineffective cannot be rejected. But at the same time we need to bear in mind that, as with all statistical tests, the null hypothesis being tested is a joint null, assuming that under the null hypothesis either no other major policy changes were put into effect, or such additional policy changes were also ineffective. Separating the effects of QE from other policy developments, such as the austerity measures that were put into effect by the Coalition Government in the UK would be difficult. However, since the austerity measures were not introduced till after 2010Q2, the joint null problem might not be that serious in the present application.

5 Conclusion

In this paper we have derived tests for the null hypothesis of the ineffectiveness of a policy intervention, defined as a change in the parameters of a policy rule for reduced form policy response equations, which are simpler to implement and could be more robust than the tests based on possibly misspecified complete structural models. In such cases we propose estimating an unrestricted reduced form policy equation which makes the target variable a function of lagged values of the target variable, as well as current and lagged values of the policy and policy-invariant exogenous variables (if any).

The tests are based on the differences, over a given policy evaluation horizon, between the post-intervention realizations of the target variable and the associated counterfactual outcomes based on the parameters estimated using data before the policy intervention. The Lucas Critique is not an issue since the counterfactual, given by the predictions from the model estimated on pre-intervention data, will embody pre-intervention parameters, while the actual post-intervention outcomes will embody any effect of the change in policy, the change in parameters and the consequent change in expectations. The tests do not require knowing the post-intervention parameters.

We derive the asymptotic distribution of the policy ineffectiveness tests under alternative assumptions concerning the type of model, the future error processes and the pre and post-intervention sample sizes. We also develop a policy ineffectiveness test based on the mean policy effect which is robust to the distribution of future errors, but requires the post-intervention, policy evaluation horizon to be reasonably large. In the case of a static model we also derive an exact test allowing for the estimation uncertainty.

We illustrate some of the issues that arise in counterfactual policy evaluation with an empirical application to Quantitative Easing which was introduced in the UK in March 2009. The UK QE involved exchanging one liability of the state - government bonds (gilts) - for another - claims on the central bank. That change in the quantities of the two assets would cause a rise in the price of guilts, decline in their yields, but also cause a rise in the prices of substitute assets such as corporate bonds and equities. We estimate models explaining UK output growth over two sample periods, one ending in 2008Q4 (before QE), and the other ending in 2011Q2. We use an ARDL(1,1) between growth and the change in the spread of long government interest rates over short rates, augmented by current values of US and euro area output growth. We follow the Bank of England in assuming that QE caused a permanent 100 basis points reduction in the spread of long interest rates over short interest rates after March 2009. The model indicates that QE had an immediate positive effect on growth, but this effect tends to disappear quite quickly, certainly within a year. The estimates of Kapetanios et al. (2012) for the time profiles of the effects of the QE tell very much the same story, namely the beneficial effects of QE are rather short-lived. We then apply the tests we have suggested and the null hypothesis of policy ineffectiveness is not rejected. QE did not have a significant effect on UK growth.

References

Abadie, A., and J. Gardeazabal (2003): The Economic cost of conflict a case study of the Basque country, *American Economic Review,* 93, 113-132.

Angrist, J. D., and G. Kuersteiner (2011): Causal effects of monetary policy shocks: semiparametric conditional independence tests with a multinomial propensity score, *Review of Economics and Statistics*, 93, 725-747.

Angrist, J. D., O. Jorda and G. Kuersteiner (2013): Semiparametric estimates of monetary policy effects: string theory revisited, NBER Working Paper No. 19355.

Baumeister, C., and L. Benati (2013): Unconventional monetary policy and the great recession:

estimating the macroeconomic effects of spread compression at the zero lower bound, *International Journal of Central Banking,* 9, 165-212.

Borio, C., and P. Disayatat (2010): Unconventional monetary policies: an appraisal, *Manchester School*, Supplement, 53-89.

Bowman, D., F. Cai, S. Davies and S. Kamin (2011): Quantitative Easing and Bank Lending: evidence from Japan, Board of Governors of the Federal Reserve System, International Finance Discussion Papers No. 1018.

Dees, S., F. di Mauro, M. H. Pesaran, and L. V. Smith (2007): Exploring the international linkages of the euro area: a global VAR analysis, *Journal of Applied Econometrics*, 22, 1-38.

Fagan, G., J. R. Lothian and P. D. McNelis (2013): Was the Gold Standard really Destabilising *Journal of Applied Econometrics* 28, 231-249.

Fogel, R. W. (1964): *Railroads and American Economic Growth*, Johns Hopkins Press, Baltimore.

Giannone, D., M. Lenza, H. Pill and L. Reichlin (2011): Non-standard monetary policy measures and monetary developments, ECB Working Paper No. 1290.

Goodhart, C. A. E., and J. P. Ashworth (2012): QE: A successful start may be running into diminishing returns, *Oxford Review of Economic Policy* 28, 640-670.

Hsiao, C., H. S. Ching and S. K. Wan (2012): A panel data approach for program evaluation: measuring the benefits of political and economic integration of Hong kong with mainland China. *Journal of Applied Econometrics*, 27, 705-740.

Imbens, G, and J. M. Wooldridge (2009): Recent developments in the econometrics of program evaluation, *Journal of Economic Literature*, 47, 5-86.

Jorda, O., and A. M. Taylor (2013): The time for austerity: estimating the average treatment effect of fiscal policy, NBER Working Paper No. 19414.

Joyce, M. A. S., A. Lasaosa, I. Stevens and M. Tong (2011): The financial market impact of Quantitative Easing in the UK, *International Journal of Central Banking*, 7, 113-161.

Joyce, M. A. S., D. Miles, A. Scott and D. Vayanos (2012): Quantitative Easing and Unconventional Monetary Policy - An Introduction, *Economic Journal*, 122, F271-F288.

Kapetanios, G., H. Mumtaz, I. Stevens and K. Theodoridis (2012): Assessing the economy wide effects of quantitative easing, *Economic Journal*, 122, F316-347.

Lewis, D. (1973): *Counterfactuals*, Blackwells, Oxford.

Meaning, J., and F. Zhu (2011): The impact of recent central bank asset purchase programs, *BIS Quarterly Review*, December, 73-83.

Pearl, J. (2009): *Causality, models, reasoning and inference*, 2nd edition, *Cambridge University Press*, Cambridge.

Pesaran, M. H., A. Pick, and M. Pranovich (2013): Optimal forecasts in the presence of structural breaks, *Journal of Econometrics*, 177, 134–152.

Pesaran, M.H., and Y. Shin (1999): An autoregressive distributed-lag modelling approach to cointegration analysis, in (ed) S Strom, *Econometrics and Economic Theory in the 20th Century: The Ragnar Frisch Centennial Symposium*, 1999, Chapter 11, pp. 371-413. Cambridge University Press, Cambridge.

Pesaran, M. H., L. V. Smith and R. P. Smith (2007): What if the UK or Sweden had joined the euro in 1999? An empirical evaluation using a Global VAR. *International Journal of Finance and Economics*, 12, 55-87.

Pesaran, M.H. and R. P Smith, (2014a), Signs of Impact Effects in Time Series Regression Models, *Economics Letters*, 122, 150-153.

Pesaran, M.H. and R.P. Smith (2014b) Tests of Policy Ineffectiveness in Macroeconometrics, USC Center for Applied Financial Economics, CAFE Research Paper No. 14.07.

Zellner, A., and F. Palm (1974), Time series analysis and simultaneous equation econometric models, in *The Structural Econometric Time Series Analysis Approach*, edited by A. Zellner and F.C. Palm, *Cambridge University Press*, Cambridge.

Appendices

Appendix A1: **Derivation of the distribution of $\mathcal{F}_{d,H}$ defined by (24)**

Under the null hypothesis H_{s0} defined by (**20**)

$$\hat{\mathbf{d}}_{(1)} = \mathbf{v}_{(1)} - \mathbf{S}^0_{(1)}\left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1}\mathbf{S}'_{(0)}\mathbf{v}_{(0)} = \mathbf{G}'\mathbf{v},$$

$$\mathbf{y}_{(0)} - \mathbf{S}_{(0)}\hat{\pi}^0_{ys} = \left[\mathbf{I}_T - \mathbf{S}_{(0)}\left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1}\mathbf{S}_{(0)}\right]\mathbf{v}_{(0)} = \mathbf{Q}\mathbf{v},$$

where $\mathbf{v} = (\mathbf{v}'_{(0)}, \mathbf{v}'_{(1)})'$, and

$$\mathbf{G}' = \left[-\mathbf{S}^0_{(1)}\left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1}\mathbf{S}'_{(0)},\ \mathbf{I}_H\right],$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{I}_T - \mathbf{S}_{(0)}\left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1}\mathbf{S}_{(0)} & \mathbf{0}_{T\times H} \\ \mathbf{0}_{H\times T} & \mathbf{0}_{H\times H} \end{pmatrix}.$$

Using these results in $\mathcal{F}_{d,H}$ defined by (24), we have

$$\mathcal{F}_{d,H} = \frac{T-k_x-k_w}{H}\frac{\hat{\mathbf{d}}'_{(1)}\left[\mathbf{I}_H + \mathbf{S}^0_{(1)}\left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1}\mathbf{S}^{0\prime}_{(1)}\right]^{-1}\hat{\mathbf{d}}_{(1)}}{\mathbf{v}'_{(0)}\left[\mathbf{I}_H - \mathbf{S}_{(0)}\left(\mathbf{S}'_{(0)}\mathbf{S}_{(0)}\right)^{-1}\mathbf{S}_{(0)}\right]\mathbf{v}_{(0)}}$$

$$= \frac{T-k_x-k_w}{H}\frac{\xi'\mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\xi}{\xi'\mathbf{Q}\xi},$$

where $\xi = \mathbf{v}/\sigma_{0v} \sim N(\mathbf{0}, \mathbf{I}_{T+H})$. It is also easily seen that $\mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'$ and $\mathbf{Q}$ are idempotent matrices that are orthogonal (namely, $\mathbf{G}'\mathbf{Q} = \mathbf{0}$) with ranks H and $T - k_x - k_w$, which establish that $\mathcal{F}_{d,H}$ has a F distribution with H and $T - k_x - k_w$ degrees of freedom.

Appendix A2: Derivation of the distribution of $\mathcal{T}_{d,H}$ in the general dynamic case

Consider the general ARDL specification given by (28) and (29). In this case, setting $p_y = p$ to simplify the notation, the post-intervention model can be written as

$$\mathbf{\Gamma}_H^1 \mathbf{y}_{(1)} = \mathbf{\Upsilon}_H^1 \mathbf{y}_p^* + \mathbf{S}_{(1)} \pi_{ys}^1 + \mathbf{v}_{(1)},$$

where $\mathbf{y}_p^*$ is the $H \times 1$ vector containing the p initial observations, $\mathbf{y}_p^* = (y_{T_0}, y_{T_0-1}, ..., y_{T_0-p+1}, 0, ..., 0)'$, $\mathbf{\Gamma}_H^1 = \mathbf{\Gamma}(\lambda^1)$, $\mathbf{\Upsilon}_H^1 = \mathbf{\Upsilon}_H(\lambda^1)$, $\lambda = (\lambda_1.\lambda_2, ..., \lambda_p)'$,

$$\mathbf{\Gamma}_H(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_2 & -\lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 \\ -\lambda_p & -\lambda_{p-1} & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda_p & -\lambda_{p-1} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -\lambda_p & \cdots & -\lambda_1 & 1 \end{pmatrix}$$

$$\mathbf{\Upsilon}_H(\lambda) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & 0 & \lambda_p \\ \lambda_2 & \lambda_3 & \cdots & \lambda_p & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and as before, $\mathbf{S}_{(1)}$ represents the $H \times k$ matrix of post-intervention observations on the exogenous variables, $\mathbf{x}_t$ and $\mathbf{w}_t$, and their lagged values, $k = (1 + p_x) k_x + (1 + p_w) k_w$. Also

$$\mathbf{\Gamma}_H^{-1}(\lambda) = \mathbf{\Lambda}_H(\lambda) = \begin{pmatrix} a_0 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{H-2} & a_{H-3} & \cdots & a_0 & 0 \\ a_{H-1} & a_{H-2} & \cdots & a_1 & a_0 \end{pmatrix}, \tag{48}$$

where α_h is obtained recursively using the difference equation

$$a_h = \sum_{i=1}^{p} \lambda_i a_{h-i}, \;\; h = 1, 2, ..., H,$$

with $a_0 = 1$ and $a_i = 0$, for all $i < 0$. It is easily verified that for $p = 1$, $\mathbf{\Lambda}_H(\lambda)$ defined by (48) reduces to the matrix defined by (33). Using the above set up, the estimated counterfactual outcomes are given by

$$\begin{aligned}\hat{\mathbf{y}}^0_{(1)} &= \mathbf{\Lambda}_H(\hat{\lambda}^0)\left[\mathbf{\Upsilon}_H(\hat{\lambda}^0)\mathbf{y}^*_p + \mathbf{X}^0_{(1)}\hat{\pi}^0_{yx} + \mathbf{W}_{(1)}\hat{\pi}^0_{yw},\right] \\ &= \hat{\mathbf{\Lambda}}^0_H\left[\hat{\mathbf{\Upsilon}}^0_H\mathbf{y}^*_p + \mathbf{S}^0_{(1)}\hat{\pi}^0_{ys}\right],\end{aligned}$$

where $\hat{\varphi}^0 = \left(\hat{\lambda}^{0\prime}, \hat{\pi}^{0\prime}_{ys}\right)'$ is obtained estimating the ARDL regression using the pre-intervention sample. The mean policy ineffectiveness test can be defined as before, $\mathcal{T}_{d,H} = \sqrt{H}\bar{\hat{d}}_H/\hat{\omega}_0$, where $\bar{\hat{d}}_H = H^{-1}\tau'_H\hat{\mathbf{d}}_{(1)}$, $\hat{\mathbf{d}}_{(1)} = \mathbf{y}_{(1)} - \hat{\mathbf{y}}^0_{(1)}$,

$$\hat{\omega}^2_0 = \hat{\sigma}^2_{0v}\left[H^{-1}\tau'_H\mathbf{\Lambda}_H(\hat{\lambda}^0)\mathbf{\Lambda}'_H(\hat{\lambda}^0)\tau_H\right],$$

and $\hat{\sigma}^2_{0v}$ is the estimate of σ^2_{0v} based on the pre policy sample. Following the same line of reasoning as in sub-section 3.2, it follows that $\mathcal{T}_{d,H} \to_d N(0,1)$, under the null hypothesis of policy ineffectiveness (including the restrictions $\lambda^0 = \lambda^1$ and $\sigma^2_{0v} = \sigma^2_{1v}$), and as T and $H \to \infty$, jointly such that $H = \kappa T^{\epsilon}$, with $\epsilon \geq 1/2$.