Supplementary appendix to: A multiple testing approach to the regularisation of large sample correlation matrices

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Supplementary Appendix A

Technical Lemmas

A.1 Statement of technical lemmas

We begin by stating a few technical lemmas that are needed for the proof of the main results.

Lemma 1 *Consider the sample correlation coefficient, $\hat{\rho}_{ij,T}$, defined by (4) and suppose that Assumptions 2 and 3 hold. Then*

$$\lim_{a_{ij,T}\to\pm\infty}\left\{e^{\frac{1-\epsilon}{2}a_{ij,T}^2}\left[F_{ij,T}(a_{ij,T}\,|\mathcal{P}_{ij})-\Phi(a_{ij,T})\right]\right\}=0, \tag{A.1}$$

for some small positive ϵ.

Lemma 2 *Suppose that $z\sim N(0,1)$, then*

$$E\left[zI(L\leq z\leq U)\right]=\phi\left(L\right)-\phi(U), \tag{A.2}$$

and

$$E\left[z^2I(L\leq z\leq U)\right]=\left[\Phi\left(U\right)-\Phi\left(L\right)\right]+L\phi(L)-U\phi(U). \tag{A.3}$$

Also for $L<0$, and $U>0$, we have

$$E\left[\left|z\right|I(L\leq z\leq U)\right]=2\phi(0)-\phi\left(L\right)-\phi(U), \tag{A.4}$$

and hence $E\left(|z|\right)=2\phi(0)=\sqrt{2/\pi}$.

Lemma 3 *Consider the critical value function*[1]

$$c_p(N)=\Phi^{-1}\left(1-\frac{p}{2f(N)}\right),$$

where $\Phi^{-1}\left(.\right)$ is the inverse function of the cumulative standard normal distribution, $0<p<1$, $f(N)=c_\delta N^\delta$, where c_δ and δ are finite positive constants, and suppose there exists finite N_0 such that for all $N>N_0$

$$1-\frac{p}{2f(N)}>0, \tag{A.5}$$

Then:

(a) $c_p(N)=O\left(\left[\ln\left(N\right)\right]^{1/2}\right)$,

(b) $\exp\left[-\varkappa c_p^2\left(N\right)/2\right]=\ominus\left(N^{-\delta\varkappa}\right)$, *and*

(c) if $\delta>1/\varkappa$, then $N\exp\left[-\varkappa c_p^2\left(N\right)/2\right]\to 0$ as $N\to\infty$, where $0<\varkappa\leq 1$.

[1] We would like to thank George Kapetanios for his help with the proof of (*b*) and (*c*) of this Lemma.

Lemma 4 *Consider the standardised sample correlation coefficient* $z_{ij,T} = \frac{\left[\hat{\rho}_{ij,T} - E\left(\hat{\rho}_{ij,T}\right)\right]}{\sqrt{Var\left(\hat{\rho}_{ij,T}\right)}}$, *where* $\hat{\rho}_{ij,T}$ *is defined by (4) and* $E\left(\hat{\rho}_{ij,T}\right)$ *and* $Var\left(\hat{\rho}_{ij,T}\right) > 0$ *are given by (8) and (9), respectively. Suppose that* $c_p(N) = \Phi^{-1}\left(1 - \frac{p}{2f(N)}\right)$, *and condition (A.5) holds. Then for all* i *and* j, *there exist* N_0 *and* T_0 *such that for* $N > N_0$ *and* $T > T_0$

$$\begin{aligned} \lim_{T\to\infty} E\left\{ z_{ij,T}^s \left[I\left(\left|\hat{\rho}_{ij,T}\right| \leq \frac{c_p(N)}{\sqrt{T}} \right) \right] \right\} &= \lim_{T\to\infty} E\left[z_{ij,T}^s I\left(L_{ij,T} \leq z_{ij,T} \leq U_{ij,T}\right)\right] \\ &= \lim_{T\to\infty} E\left[z^s I\left(L_{ij,T} \leq z \leq U_{ij,T}\right)\right], \end{aligned} \tag{A.6}$$

and

$$\lim_{T\to\infty} E\left\{ \left|z_{ij,T}\right|^s \left[I\left(\left|\hat{\rho}_{ij,T}\right| \leq \frac{c_p(N)}{\sqrt{T}} \right) \right] \right\} = \lim_{T\to\infty} E\left[|z|^s I\left(L_{ij,T} \leq z \leq U_{ij,T}\right)\right], \tag{A.7}$$

for $s = 0, 1, 2, \ldots,$ *where*

$$U_{ij,T} = \frac{c_p(N) - \sqrt{T}E\left(\hat{\rho}_{ij,T}\right)}{\sqrt{Var\left(\sqrt{T}\hat{\rho}_{ij,T}\right)}},\ L_{ij,T} = \frac{-c_p(N) - \sqrt{T}E\left(\hat{\rho}_{ij,T}\right)}{\sqrt{Var\left(\sqrt{T}\hat{\rho}_{ij,T}\right)}} \tag{A.8}$$

and $z \sim N(0,1)$.

Lemma 5 *Consider the cumulative distribution function of a standard normal variate, defined by*

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

Then for $x > 0$

$$\Phi(-x) = 1 - \Phi(x) \leq \frac{1}{2}\exp(-\frac{x^2}{2}). \tag{A.9}$$

Lemma 6 *Consider the sample correlation coefficient,* $\hat{\rho}_{ij,T}$, *defined by (4) and suppose that Assumptions 2 and 3 hold, then there exists* N_0 *and* T_0 *such that for all* $N > N_0$ *and* $T > T_0$

$$\Pr\left(\left| \sqrt{T}\hat{\rho}_{ij,T} \right| > c_p(N) \left| \rho_{ij} = 0 \right.\right) \leq K e^{-\frac{1-\epsilon}{2} \frac{c_p^2(N)}{\varphi_{ij}}} \left[1 + o(1)\right] \tag{A.10}$$

where $\varphi_{ij} = E\left(y_{it}^2 y_{jt}^2 \left| \rho_{ij} = 0\right.\right)$, *and* ϵ *is a small positive constant.*[2] *Further, if* $\left|\rho_{ij}\right| > c_p(N)/\sqrt{T}$ *we have*

$$\Pr\left(\left| \sqrt{T}\hat{\rho}_{ij,T} \right| < c_p(N) | \rho_{ij} \neq 0 \right) \leq K e^{\frac{-1}{2} \frac{T\left[\left|\rho_{ij}\right| - \frac{c_p(N)}{\sqrt{T}}\right]^2}{K_v(\boldsymbol{\theta}_{ij})}} \left[1 + o(1)\right], \tag{A.11}$$

where $K_v(\boldsymbol{\theta}_{ij})$ *is given by (11),*

$$c_p(N) = \Phi^{-1}\left(1 - \frac{p}{2f(N)}\right) > 0, \tag{A.12}$$

$0 < p < 1$, $f(N) = c_\delta N^\delta$, *where* c_δ *and* δ *are finite positive constants, and*

$$\ln f(N)/T \to 0, \text{ as } N \text{ and } T \to \infty. \tag{A.13}$$

[2] To simplify the notation we have dropped explicit reference to $\mathcal{P}_{ij}$, the underlying bivariate distribution of the observations.

Lemma 7 *Consider the data generating process*

$$\mathbf{y}_t = \mathbf{P}\mathbf{u}_t,$$

where $\mathbf{y}_t$ *and* $\mathbf{u}_t$ *are* $N \times 1$ *vectors of random variables, and* $\mathbf{P}$ *is an* $N \times N$ *matrix of fixed constants, such that* $\mathbf{P}\mathbf{P}' = \mathbf{R}$*, where* $\mathbf{R}$ *is a correlation matrix. Suppose that* $\mathbf{u}_t$ *follows a multivariate t-distribution with v degrees of freedom generated as*

$$\mathbf{u}_t = \left(\frac{\mathrm{v}-2}{\chi^2_{v,t}}\right)^{1/2} \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{Nt})' \sim IIDN(\mathbf{0}, \mathbf{I}_N)$*, and* $\chi^2_{\mathrm{v},t}$ *is a chi-squared random variate with* $\mathrm{v} > 4$ *degrees of freedom distributed independently of* $\boldsymbol{\varepsilon}_t$*. Then we have that*

$$\mu_{ij}(2,2) = E(y_{it}^2 y_{jt}^2) = \frac{(\mathrm{v}-2)\left[(\mathbf{p}_i'\mathbf{p}_i)^2 + (\mathbf{p}_i'\mathbf{p}_j)^2\right]}{(\mathrm{v}-4)},$$

where $\mathbf{p}_i'$ *is the* i^{th} *row of* $\mathbf{P}$*. In the case where* $\mathbf{P} = \mathbf{I}_N$*,* $E(y_{it}^2 y_{jt}^2) = (\mathrm{v}-2)/(\mathrm{v}-4)$ *and*

$$E(y_{it}^2 y_{jt}) = E(y_{jt}^2 y_{it}) = 0.$$

Lemma 8 *Fat-tailed shocks do not necessarily generate* $E(y_{it}^2 y_{jt}^2) > 1$*.*

A.2 Proofs of lemmas for the MT estimator

Proof of Lemma 1. Under (14), and noting that

$$e^{\frac{1-\epsilon}{2}a_{ij,T}^2}\phi\left(a_{ij,T}\right) = e^{\frac{1-\epsilon}{2}a_{ij,T}^2}(2\pi)^{-1/2}\exp\left(-\frac{1}{2}a_{ij,T}^2\right) = (2\pi)^{-1/2}\exp\left(-\frac{\epsilon}{2}a_{ij,T}^2\right),$$

we have

$$\begin{aligned} e^{\frac{1-\epsilon}{2}a_{ij,T}^2}\left[F_{ij,T}(a_{ij,T}\,|\mathcal{P}_{ij}) - \Phi(a_{ij,T})\right] &= (2\pi)^{-1/2}\exp\left(-\frac{\epsilon}{2}a_{ij,T}^2\right) \\ &\times \left[\sum_{r=1}^{s-2} T^{-r/2} G_r\left(a_{ij,T}\,|\mathcal{P}_{ij}\right) + O\left(T^{-(s-1)/2}\right)\right], \end{aligned}$$

and the desired result follows since $G_r\left(a_{ij,T}\,|\mathcal{P}_{ij}\right)$, $r = 1, 2, \ldots, s-2$, are polynomials in $a_{ij,T}$, and noting that $a_{ij,T}^r \exp\left(-\frac{\epsilon}{2}a_{ij,T}^2\right) \to 0$ as $a_{ij,T} \to \pm\infty$, for all $r \geq 1$. This result holds for a fixed T, and as $T \to \infty$. ■

Proof of Lemma 2. Denote the density of the standard normal distribution by $\phi(z) = (2\pi)^{-1/2}e^{-(1/2)z^2}$, then

$$E\left[zI(L \leq z \leq U)\right] = \int_L^U z(2\pi)^{-1/2}e^{-(1/2)z^2}dz = \left[-\phi(z)\right]_L^U = \phi\left(L\right) - \phi(U).$$

Similarly, to prove (A.3) note that $E\left[z^2 I(L \leq z \leq U)\right] = \int_L^U z^2\phi(z)dz$. Hence, integrating by parts, we have

$$\int_L^U z^2\phi(z)dz = \left[-z\phi(z)\right]_L^U + \int_L^U \phi(z)dz = \left[\Phi\left(U\right) - \Phi\left(L\right)\right] + L\phi(L) - U\phi(U),$$

as required. Finally, to prove (A.4) note that since $L < 0$ and $U > 0$ and given the symmetry of the density function, $\phi(z) = \phi(-z)$, then

$$\begin{aligned} E\left[|z|\, I(L \le z \le U)\right] &= \int_L^U |z|\,\phi(z)dz \\ &= \int_L^0 |z|\,\phi(z)dz + \int_0^U |z|\,\phi(z)dz = \int_0^{-L} z\phi(z)dz + \int_0^U z\phi(z)dz \\ &= [\phi(0) - \phi(-L)] + [\phi(0) - \phi(U)] \\ &= 2\phi(0) - \phi(L) - \phi(U), \end{aligned}$$

as required. ■

Proof of Lemma 3. First note that

$$\Phi^{-1}(z) = \sqrt{2}\,\mathrm{erf}^{-1}(2z-1),\ \ z \in (0,1),$$

where $\Phi(x)$ is cumulative distribution function of a standard normal variate, and $\mathrm{erf}(x)$ is the error function defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\textstyle\int_0^x e^{-u^2}du. \tag{A.14}$$

Consider now the inverse complementary error function $\mathrm{erfc}^{-1}(x)$ given by

$$\mathrm{erf\,c}^{-1}(1-x) = \mathrm{erf}^{-1}(x).$$

Using results in Chiani et al. (2003) on p.842, we have

$$\mathrm{erf\,c}^{-1}(x) \le \sqrt{-\ln(x)}.$$

Applying the above results to $c_p(N)$ we have

$$\begin{aligned} c_p(N) &= \Phi^{-1}\left(1 - \frac{p}{2f(N)}\right) \\ &= \sqrt{2}\,\mathrm{erf}^{-1}\left\{2\left[1 - \frac{p}{2f(N)}\right] - 1\right\} \\ &= \sqrt{2}\,\mathrm{erf}^{-1}\left[1 - \frac{p}{f(N)}\right] = \sqrt{2}\,\mathrm{erf\,c}^{-1}\left[\frac{p}{f(N)}\right] \\ &\le \sqrt{2}\sqrt{-\ln\left[\frac{p}{f(N)}\right]} = \sqrt{2\left[\ln f(N) - \ln(p)\right]}. \end{aligned}$$

Therefore, for $f(N) = c_\delta N^\delta$ we have

$$c_p^2(N) \le 2\left[\delta\ln(N) - \ln(p)\right] = O\left[\ln(N)\right],$$

which establishes result (a). It follows straightforwardly that $\frac{\ln f(N)}{T} \to 0$, as N and $T \to \infty$, and given that p is fixed, then $c_p(N)/\sqrt{T}$ is bounded in N and T, and $c_p(N)/\sqrt{T} \to 0$, as N and $T \to \infty$, since $c_p(N)/\sqrt{T} \le \sqrt{2\left[\ln f(N) - \ln(p)\right]/T} \to 0$, as N and $T \to \infty$.

Further, by Proposition 24 of Dominici (2003) we have that

$$\lim_{N\to\infty} c_p(N)/LW\left\{\frac{1}{2\pi\left[\left(1-\frac{p}{2f(N)}\right)-1\right]^2}\right\}^{1/2}=1,$$

where LW denotes the Lambert W function which satisfies $\lim_{N\to\infty} LW(N)/\left\{\ln(N)-\ln\left[\ln(N)\right]\right\}=1$ as $N\to\infty$. We note that $\lim_{N\to\infty}\ln(N)/\left\{\ln(N)-\ln\left[\ln(N)\right]\right\}=1$ as $N\to\infty$. So

$$\lim_{N\to\infty}\frac{LW\left\{\frac{1}{2\pi\left[\left(1-\frac{p}{2f(N)}\right)-1\right]^2}\right\}^{1/2}}{\left\{2\ln\left(\frac{\sqrt{2}f(N)}{\sqrt{\pi}p}\right)\right\}^{1/2}}=1.$$

Hence, for any $0<\varkappa\leq 1$,

$$\lim_{N\to\infty}\frac{\exp\left[-\varkappa c_p^2(N)/2\right]}{\exp\left[-\frac{\varkappa\left\{\left[2\ln\left(\frac{\sqrt{2}f(N)}{\sqrt{\pi}p}\right)\right]^{1/2}\right\}^2}{2}\right]}=\lim_{N\to\infty}\frac{\exp\left[-\varkappa c_p^2(N)/2\right]}{\left[f\left(N\right)\right]^{-\varkappa}\pi^{\varkappa}p^{2\varkappa}2^{-\varkappa}}=1 \text{ as } N\to\infty,$$

and substituting N^{δ} for $f\left(N\right)$ yields,

$$\lim_{N\to\infty}\frac{\exp\left[-\varkappa c_p^2\left(N\right)/2\right]}{N^{-\delta\varkappa}}\to\frac{2^{\varkappa}}{\pi^{\varkappa}p^{2\varkappa}}. \tag{A.15}$$

It follows from (A.15) that $\exp\left[-\varkappa c_p^2\left(N\right)/2\right]=\ominus\left(N^{-\delta\varkappa}\right)$, as required. This completes the proof of result (b). Finally, it readily follows from (b) that $N\exp\left[-\varkappa c_p^2\left(N\right)/2\right]=\ominus\left(N^{1-\delta\varkappa}\right)$, and therefore $N\exp\left[-\varkappa c_p^2\left(N\right)/2\right]\to 0$ when $\delta>1/\varkappa$, as desired. This completes the proof of the last result (c). ■

Proof of Lemma 4. We first note that since $Var\left(\hat{\rho}_{ij,T}\right)>0$

$$\begin{aligned}
I\left(\left|\hat{\rho}_{ij,T}\right|\leq\frac{c_p(N)}{\sqrt{T}}\right) &= I\left(\frac{-c_p(N)}{\sqrt{T}}\leq\hat{\rho}_{ij,T}\leq\frac{c_p(N)}{\sqrt{T}}\right)\\
&= I\left(\frac{\frac{-c_p(N)}{\sqrt{T}}-E\left(\hat{\rho}_{ij,T}\right)}{\sqrt{Var\left(\hat{\rho}_{ij,T}\right)}}\leq\frac{\hat{\rho}_{ij,T}-E\left(\hat{\rho}_{ij,T}\right)}{\sqrt{Var\left(\hat{\rho}_{ij,T}\right)}}\leq\frac{\frac{c_p(N)}{\sqrt{T}}-E\left(\hat{\rho}_{ij,T}\right)}{\sqrt{Var\left(\hat{\rho}_{ij,T}\right)}}\right)\\
&= I\left(L_{ij,T}\leq z_{ij,T}\leq U_{ij,T}\right).
\end{aligned} \tag{A.16}$$

Also, since $\hat{\rho}_{ij,T}$ is a correlation coefficient, $\left|\hat{\rho}_{ij,T}\right|<1$, and for a finite $T>T_0$, $Var\left(\hat{\rho}_{ij,T}\right)>0$, then

$$\left|z_{ij,T}\right|<\frac{\left|\hat{\rho}_{ij,T}\right|+\left|E\left(\hat{\rho}_{ij,T}\right)\right|}{\sqrt{Var\left(\hat{\rho}_{ij,T}\right)}}<2\sup_{i,j}\left(\frac{1}{\sqrt{Var\left(\hat{\rho}_{ij,T}\right)}}\right)<K.$$

Hence all moments of $z_{ij,T}$ exist for T finite. Furthermore, it is well known that $z_{ij,T} \rightarrow_d N(0,1)$ as $T \rightarrow \infty$. Therefore, all moments of $z_{ij,T}$ exist for all values of $T > T_0$, and by the *second limit-theorem* (see, for example, Rao and Kendall (1950) on p. 228)

$$E\left(z_{ij,T}^s\right) \rightarrow E\left(z^s\right), \text{ as } T \rightarrow \infty, \text{ for all } s = 1, 2, \ldots.$$

Furthermore, since $I\left(L_{ij,T} \leq z_{ij,T} \leq U_{ij,T}\right) = I\left(\left|\hat{\rho}_{ij,T}\right| \leq \frac{c_p(N)}{\sqrt{T}}\right) \leq c_p(N)/\sqrt{T}$, and under condition (A.5), $c_p(N)/\sqrt{T}$ is bounded (see Lemma 3). Then for all $N > N_0$ we must also have

$$\lim_{T\rightarrow\infty} E\left[z_{ij,T}^s I\left(\left|\hat{\rho}_{ij,T}\right| \leq \frac{c_p(N)}{\sqrt{T}}\right)\right] = \lim_{T\rightarrow\infty} E\left[z^s I\left(L_{ij,T} \leq z \leq U_{ij,T}\right)\right],$$

as required. Results in (A.7) follow similarly. ■

Proof of Lemma 5. Using results in Chiani et al. (2003) - eq. (5), we have

$$\text{erf c}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \leq \exp(-x^2), \tag{A.17}$$

where erf $c(x)$ is the complement of the erf(x) function defined by (A.14). But

$$1 - \Phi(x) = (2\pi)^{-1/2} \int_x^\infty e^{-\frac{u^2}{2}} du = \frac{1}{2} \text{erf c}\left(\frac{x}{\sqrt{2}}\right),$$

and using (A.17) we have

$$1 - \Phi(x) = \frac{1}{2} \text{erf c}\left(\frac{x}{\sqrt{2}}\right) \leq \frac{1}{2} \exp\left[-\left(\frac{x}{\sqrt{2}}\right)^2\right] = \frac{1}{2} \exp\left(-\frac{x^2}{2}\right).$$

■

Proof of Lemma 6. We first note that

$$\begin{aligned} \Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| \leq c_p(N)\right) &= \Pr\left(-c_p(N) \leq \sqrt{T}\hat{\rho}_{ij,T} \leq c_p(N)\right) \\ &= \Pr\left(L_{ij} \leq \frac{\sqrt{T}\left[\hat{\rho}_{ij,T} - E\left(\hat{\rho}_{ij,T}\right)\right]}{\sqrt{Var\left(\sqrt{T}\hat{\rho}_{ij,T}\right)}} \leq U_{ij}\right), \end{aligned}$$

where

$$U_{ij} = \frac{c_p(N) - \sqrt{T}E\left(\hat{\rho}_{ij,T}\right)}{\sqrt{Var\left(\sqrt{T}\hat{\rho}_{ij,T}\right)}}, \; L_{ij} = \frac{-c_p(N) - \sqrt{T}E\left(\hat{\rho}_{ij,T}\right)}{\sqrt{Var\left(\sqrt{T}\hat{\rho}_{ij,T}\right)}}. \tag{A.18}$$

Using (8) and (9), we also note that under $\rho_{ij} = 0$,

$$\begin{aligned} E\left(\hat{\rho}_{ij,T} \left| \rho_{ij} = 0\right.\right) &= \frac{\psi_{ij}}{T} + O\left(T^{-2}\right), \\ Var\left(\hat{\rho}_{ij,T} \left| \rho_{ij} = 0\right.\right) &= \frac{\varphi_{ij}}{T} + O\left(T^{-2}\right), \end{aligned}$$

where ψ_{ij} and φ_{ij} are given by (13) and (12) respectively, and

$$\Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| \leq c_p(N) \left|\rho_{ij} = 0\right.\right) = F_{ij,T}\left[U_{ij,T}(0)\right] - F_{ij,T}\left[L_{ij,T}(0)\right]$$

where

$$U_{ij,T}(0) = \frac{c_p(N) - \frac{\psi_{ij}}{\sqrt{T}} + O\left(T^{-3/2}\right)}{\sqrt{\varphi_{ij} + O\left(T^{-1}\right)}},\ L_{ij,T}(0) = \frac{-c_p(N) - \frac{\psi_{ij}}{\sqrt{T}} + O\left(T^{-3/2}\right)}{\sqrt{\varphi_{ij} + O\left(T^{-1}\right)}}. \tag{A.19}$$

Hence,

$$\Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| > c_p(N) \left|\rho_{ij} = 0\right.\right) = 1 - F_{ij,T}\left[U_{ij,T}(0)\right] + F_{ij,T}\left[L_{ij,T}(0)\right]. \tag{A.20}$$

Setting $a_{ij,T} = U_{ij,T}(0)$ we have that (recall by assumption $\sup_{ij}\left|\psi_{ij}\right| < K$)

$$a_{ij,T}^2 = \frac{c_p^2(N)}{\varphi_{ij}} + O\left[\frac{c_p(N)}{\sqrt{T}}\right] + O\left(T^{-1}\right).$$

By Lemma 3, $c_p(N)/\sqrt{T} = o(1)$, as N and $T \to \infty$, and hence

$$a_{ij,T}^2 = \frac{c_p^2(N)}{\varphi_{ij}} + o(1). \tag{A.21}$$

Therefore, in view of (A.1) established in Lemma 1 and (A.21), we have (for some small positive ϵ)

$$\begin{aligned} F_{ij,T}\left[U_{ij,T}(0)\right] &= \Phi\left[U_{ij,T}(0)\right] + Ke^{-\frac{1-\epsilon}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\left[1 + o(1)\right], \\ F_{ij,T}\left[L_{ij,T}(0)\right] &= \Phi\left[L_{ij,T}(0)\right] + Ke^{-\frac{1-\epsilon}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\left[1 + o(1)\right]. \end{aligned}$$

Substituting the above results in (A.20) yields

$$\begin{aligned} \Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| > c_p(N) \left|\rho_{ij} = 0\right.\right) &= 1 - \Phi\left[U_{ij,T}(0)\right] + \Phi\left[L_{ij,T}(0)\right] \\ &\quad + Ke^{-\frac{1-\epsilon}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\left[1 + o(1)\right], \end{aligned}$$

or

$$\begin{aligned} \Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| > c_p(N) \left|\rho_{ij} = 0\right.\right) &= \Phi\left[-U_{ij,T}(0)\right] + \Phi\left[L_{ij,T}(0)\right] \\ &\quad + Ke^{-\frac{1-\epsilon}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\left[1 + o(1)\right]. \end{aligned} \tag{A.22}$$

Since by assumption $\left|\psi_{ij}\right| < K$, and $c_p(N)$ is an increasing function of N then there must exist N_0 and T_0 such that for values of $N > N_0$ and $T > T_0$

$$-U_{ij,T}(0) = \frac{-c_p(N) + \frac{\psi_{ij}}{\sqrt{T}} + O\left(T^{-3/2}\right)}{\sqrt{\varphi_{ij} + O\left(T^{-1}\right)}} < 0, \tag{A.23}$$

and

$$L_{ij,T}(0) = \frac{-c_p(N) - \frac{\psi_{ij}}{\sqrt{T}} + O\left(T^{-3/2}\right)}{\sqrt{\varphi_{ij} + O\left(T^{-1}\right)}} < 0, \tag{A.24}$$

and by Lemma 5 we have

$$\begin{aligned}
\Phi\left[-U_{ij,T}(0)\right] &\leq \frac{1}{2}\exp\left\{-\frac{\left[c_p(N) - \frac{\psi_{ij}}{\sqrt{T}} + O\left(T^{-3/2}\right)\right]^2}{2\left[\varphi_{ij} + O\left(T^{-1}\right)\right]}\right\} \\
&= \frac{1}{2}e^{-\frac{1}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\left[1 + O\left(\frac{c_p(N)}{\sqrt{T}}\right) + O\left(T^{-1}\right)\right] \qquad (A.25)\\
&= \frac{1}{2}e^{-\frac{1}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\left[1 + o(1)\right].
\end{aligned}$$

Similarly,

$$\Phi\left[L_{ij,T}(0)\right] \leq \frac{1}{2}e^{-\frac{1}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\left[1 + o(1)\right]. \tag{A.26}$$

Substituting the above results in (A.22) now yields

$$\Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| > c_p(N)\left|\rho_{ij} = 0\right.\right) \leq \left[e^{-\frac{1}{2}\frac{c_p^2(N)}{\varphi_{ij}}} + Ke^{-\frac{1-\epsilon}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\right]\left[1 + o(1)\right],$$

or[3]

$$\Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| > c_p(N)\left|\rho_{ij} = 0\right.\right) \leq Ke^{-\frac{1-\epsilon}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\left[1 + o(1)\right],$$

as required.

Consider now the case where $\rho_{ij} \neq 0$ and note that

$$\Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| < c_p(N)|\rho_{ij} \neq 0\right) = F_{ij,T}\left[U_{ij,T}(\rho_{ij})\right] - F_{ij,T}\left[L_{ij,T}(\rho_{ij})\right], \tag{A.27}$$

where

$$U_{ij,T}(\rho_{ij}) = \frac{c_p(N) - \sqrt{T}\rho_{ij} - \frac{K_{\mathrm{m}}(\boldsymbol{\theta}_{ij})}{\sqrt{T}} + O\left(T^{-3/2}\right)}{\sqrt{K_v(\boldsymbol{\theta}_{ij}) + O\left(T^{-1}\right)}}, \tag{A.28}$$

$$L_{ij,T}(\rho_{ij}) = \frac{-c_p(N) - \sqrt{T}\rho_{ij} - \frac{K_{\mathrm{m}}(\boldsymbol{\theta}_{ij})}{\sqrt{T}} + O\left(T^{-3/2}\right)}{\sqrt{K_v(\boldsymbol{\theta}_{ij}) + O\left(T^{-1}\right)}}, \tag{A.29}$$

$|K_{\mathrm{m}}(\boldsymbol{\theta}_{ij})| < K$, and $0 < K_v(\boldsymbol{\theta}_{ij}) < K$. Suppose that $\rho_{ij} > 0$. Then $\sqrt{T}\rho_{ij} + c_p(N) \to \infty$ and $\sqrt{T}\rho_{ij} - c_p(N) \to \infty$, as N and $T \to \infty$ (recall that $c_p(N)/\sqrt{T} \to 0$ with N and $T \to \infty$).

[3] Note that

$$\left[e^{-\frac{1}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\right] / \left[e^{-\frac{1-\epsilon}{2}\frac{c_p^2(N)}{\varphi_{ij}}}\right] = e^{-\frac{\epsilon}{2}\frac{c_p^2(N)}{\varphi_{ij}}} \to 0, \text{ as } c_p^2(N) \to \infty.$$

Again using (A.28) and (A.29) for $a_{ij,T}$ in (A.1) we have

$$\begin{aligned} F_{ij,T}\left[U_{ij,T}(\rho_{ij})\right] &= \Phi\left[U_{ij,T}(\rho_{ij})\right] + Ke^{\frac{-1}{2}\frac{\left[c_p(N)-\sqrt{T}\rho_{ij}\right]^2}{K_{\rm v}(\boldsymbol{\theta}_{ij})}}\left[1+o(1)\right], \\ F_{ij,T}\left[L_{ij,T}(\rho_{ij})\right] &= \Phi\left[L_{ij,T}(\rho_{ij})\right] + Ke^{\frac{-1}{2}\frac{\left[c_p(N)+\sqrt{T}\rho_{ij}\right]^2}{K_{\rm v}(\boldsymbol{\theta}_{ij})}}\left[1+o(1)\right]. \end{aligned}$$

Hence

$$\begin{aligned} \Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| < c_p(N)|\rho_{ij}\neq 0\right) &= \Phi\left[U_{ij,T}(\rho_{ij})\right] - \Phi\left[L_{ij,T}(\rho_{ij})\right] \\ &\quad + Ke^{\frac{-1}{2}\frac{\left[c_p(N)-\sqrt{T}\rho_{ij}\right]^2}{K_v(\boldsymbol{\theta}_{ij})}}\left[1+o(1)\right] \\ &\quad + Ke^{\frac{-1}{2}\frac{\left[c_p(N)+\sqrt{T}\rho_{ij}\right]^2}{K_v(\boldsymbol{\theta}_{ij})}}\left[1+o(1)\right]. \end{aligned}$$

Further, since $\Phi\left[L_{ij,T}(\rho_{ij})\right] \geq 0$, then

$$\Phi\left(\left[U_{ij,T}(\rho_{ij})\right]\right) - \Phi\left(\left[L_{ij,T}(\rho_{ij})\right]\right) \leq \Phi\left(\frac{c_p(N)-\sqrt{T}\rho_{ij}-\frac{K_{\rm m}(\boldsymbol{\theta}_{ij})}{\sqrt{T}}+O\left(T^{-3/2}\right)}{\sqrt{K_v(\boldsymbol{\theta}_{ij})+O\left(T^{-1}\right)}}\right).$$

Also, there exists N_0 and T_0 such that for $\rho_{ij} > 0$, and all $N > N_0$ and $T > T_0$, we have (using Lemma 5)

$$\Phi\left(\frac{c_p(N)-\sqrt{T}\rho_{ij}-\frac{K_{\rm m}(\boldsymbol{\theta}_{ij})}{\sqrt{T}}+O\left(T^{-3/2}\right)}{\sqrt{K_v(\boldsymbol{\theta}_{ij})+O\left(T^{-1}\right)}}\right) \leq \frac{1}{2}e^{\frac{-1}{2}\frac{\left[c_p(N)-\sqrt{T}\rho_{ij}\right]^2}{K_v(\boldsymbol{\theta}_{ij})}}\left[1+o(1)\right],$$

and hence

$$\Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| < c_p(N)|\rho_{ij} > 0\right) \leq Ke^{\frac{-1}{2}\frac{\left[c_p(N)-\sqrt{T}\rho_{ij}\right]^2}{K_v(\boldsymbol{\theta}_{ij})}}\left[1+o(1)\right].$$

A similar result can also be obtained for $\rho_{ij} < 0$, yielding the overall result

$$\Pr\left(\left|\sqrt{T}\hat{\rho}_{ij,T}\right| < c_p(N)|\rho_{ij}\neq 0\right) \leq Ke^{\frac{-1}{2}\frac{T\left[\left|\rho_{ij}\right|-\frac{c_p(N)}{\sqrt{T}}\right]^2}{K_v(\boldsymbol{\theta}_{ij})}}\left[1+o(1)\right].$$

■

Proof of Lemma 7. We first note that

$$\begin{aligned} E\left(\frac{1}{\chi^2_{{\rm v},t}}\right) &= \frac{1}{{\rm v}-2},\ Var\left(\frac{1}{\chi^2_{{\rm v},t}}\right) = \frac{2}{({\rm v}-2)^2({\rm v}-4)} \\ E\left(\frac{1}{\chi^2_{{\rm v},t}}\right)^2 &= \frac{2}{({\rm v}-2)^2({\rm v}-4)} + \left(\frac{1}{{\rm v}-2}\right)^2 = \frac{{\rm v}-2}{({\rm v}-2)^2({\rm v}-4)}. \end{aligned} \quad (A.30)$$

Then

$$E\left(\mathbf{u}_t\mathbf{u}_t'\right) = E\left[\left(\frac{{\rm v}-2}{\chi^2_{\rm v}}\right)\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\right] = E\left(\frac{{\rm v}-2}{\chi^2_{{\rm v},t}}\right)E\left(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\right) = \mathbf{I}_N,$$

and

$$E(\mathbf{y}_t) = \mathbf{0},\ E\left(\mathbf{y}_t\mathbf{y}_t'\right) = \mathbf{PP}' = \mathbf{R}.$$

It is clear that y_{it} has mean zero and a unit variance. Denote the i^{th} row of $\mathbf{P}$ by $\mathbf{p}_i'$ and note that $y_{it} = \mathbf{p}_i'\mathbf{u}_t = \left(\frac{\text{v}-2}{\chi^2_{\text{v},t}}\right)^{1/2}\mathbf{p}_i'\boldsymbol{\varepsilon}_t$, and hence

$$\mu_{ij}(2,2) = E(y_{it}^2y_{jt}^2) = E\left[\left(\frac{\text{v}-2}{\chi^2_{\text{v},t}}\right)^2\left(\mathbf{p}_i'\boldsymbol{\varepsilon}_t\right)^2\left(\mathbf{p}_j'\boldsymbol{\varepsilon}_t\right)^2\right],$$

and since $\boldsymbol{\varepsilon}_t$ and $\chi^2_{\text{v},t}$ are distributed independently using (A.30) we have

$$E(y_{it}^2y_{jt}^2) = \frac{(\text{v}-2)^3}{(\text{v}-2)^2(\text{v}-4)}E\left[\left(\boldsymbol{\varepsilon}_t'\mathbf{A}_i\boldsymbol{\varepsilon}_t\right)\left(\boldsymbol{\varepsilon}_t'\mathbf{A}_j\boldsymbol{\varepsilon}_t\right)\right],$$

where $\mathbf{A}_i = \mathbf{p}_i\mathbf{p}_i'$. But since $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{I}_N)$, using results in Magnus (1978) we have

$$\begin{aligned} E\left[\left(\boldsymbol{\varepsilon}_t'\mathbf{A}_i\boldsymbol{\varepsilon}_t\right)\left(\boldsymbol{\varepsilon}_t'\mathbf{A}_j\boldsymbol{\varepsilon}_t\right)\right] &= tr\left(\mathbf{p}_i\mathbf{p}_i'\right)tr\left(\mathbf{p}_j\mathbf{p}_j'\right) + tr\left(\mathbf{p}_i\mathbf{p}_i'\mathbf{p}_j\mathbf{p}_j'\right) \\ &= \left(\mathbf{p}_i'\mathbf{p}_i\right)^2 + \left(\mathbf{p}_i'\mathbf{p}_j\right)^2. \end{aligned}$$

Hence

$$E(y_{it}^2y_{jt}^2) = \frac{(\text{v}-2)\left[\left(\mathbf{p}_i'\mathbf{p}_i\right)^2 + \left(\mathbf{p}_i'\mathbf{p}_j\right)^2\right]}{(\text{v}-4)}.$$

When $\mathbf{P}$ is an identity matrix then $\mathbf{p}_i'\mathbf{p}_i = 1$ and $\mathbf{p}_i'\mathbf{p}_j = 0$, and hence $E(y_{it}^2y_{jt}^2) = (\text{v}-2)/(\text{v}-4)$. Also

$$E(y_{it}^2y_{jt}) = E\left[\left(\frac{\text{v}-2}{\chi^2_{\text{v},t}}\right)^{3/2}\right]E\left[\left(\boldsymbol{\varepsilon}_t'\mathbf{A}_i\boldsymbol{\varepsilon}_t\right)\mathbf{p}_j'\boldsymbol{\varepsilon}_t\right] = 0.$$

■

Proof of Lemma 8. Consider the data generating process $\mathbf{y}_t = \mathbf{Pu}_t$ where the elements of $\mathbf{u}_t = (u_{1t}, u_{2t}, \ldots, u_{Nt})'$, u_{it}, are generated as a standardized independent chi-squared distribution with v_i degrees of freedom, namely

$$u_{it} = \frac{\chi^2_{it}(\text{v}_i) - \text{v}_i}{\sqrt{2\text{v}_i}}, \text{ for all } i \text{ and } t.$$

Then it is clear that $E(u_{it}) = 0$, $E(u_{it}^2) = 1$, and also $E(u_{it}^2u_{jt}^2) = E(u_{it}^2)E(u_{jt}^2) = 1$, and $E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{I}_N$. Let $\mathbf{p}_i'$ be the i^{th} row of $\mathbf{P}$ and note that

$$\begin{aligned} E\left(y_{it}y_{jt}\right) &= \mathbf{p}_i'E\left(\mathbf{u}_t\mathbf{u}_t'\right)\mathbf{p}_j = \mathbf{p}_i'\mathbf{p}_j = \rho_{ij} \\ \mathbf{p}_i'\mathbf{p}_i &= \sum_{r=1}^{N}p_{ir}^2 = 1. \end{aligned}$$

Also

$$\begin{aligned} E\left(y_{it}^2y_{jt}^2\right) &= E\left[\left(\mathbf{p}_i'\mathbf{u}_t\mathbf{u}_t'\mathbf{p}_i\right)\left(\mathbf{p}_j'\mathbf{u}_t\mathbf{u}_t'\mathbf{p}_j\right)\right] \\ &= \sum_{r}\sum_{r'}\sum_{s}\sum_{s'}p_{ir}p_{ir'}p_{js}p_{js'}E(u_{rt}u_{r't}u_{st}u_{s't}). \end{aligned}$$

But

$$\begin{aligned} E(u_{rt}u_{r't}u_{st}u_{s't}) &= 0 \text{ if } r \neq r' \text{ or } s \neq s' \\ &= E(u_{rt}^2 u_{st}^2) = 1 \text{ if } r = r' \text{ and } s = s', \end{aligned}$$

and hence

$$E\left(y_{it}^2 y_{jt}^2\right) = \sum_r \sum_s p_{ir}^2 p_{js}^2 = \left(\sum_{r=1}^{N} p_{ir}^2\right)^2 = 1.$$

Therefore, fat-tailed shocks do not necessarily generate $\mu_{ij}(2,2) = E\left(y_{it}^2 y_{jt}^2\right) > 1$. ■

Supplementary Appendix B

An overview of key regularisation techniques

Here we provide an overview of three main covariance estimators proposed in the literature which we use in our Monte Carlo experiments for comparative analysis, namely the thresholding methods of Bickel and Levina (2008), and Cai and Liu (2011), and the shrinkage approach of Ledoit and Wolf (2004).

B.1 Bickel-Levina (BL) thresholding

The method developed by Bickel and Levina (2008) - BL - employs 'universal' thresholding of the sample covariance matrix $\hat{\mathbf{\Sigma}} = (\hat{\sigma}_{ij})$, $i, j = 1, 2, \ldots, N$. Under this approach $\mathbf{\Sigma}$ is required to be sparse as they define on p. 2580. The BL thresholding estimator is given by

$$\widetilde{\mathbf{\Sigma}}_{BL,C} = \left(\hat{\sigma}_{ij} I\left[|\hat{\sigma}_{ij}| \geq C\sqrt{\frac{\log(N)}{T}}\right]\right), \ i = 1, 2, \ldots, N-1, \ j = i+1, i+2, \ldots, N \quad \text{(B.31)}$$

where $I(.)$ is an indicator function and C is a positive constant which is unknown. The choice of thresholding function - $I(.)$ - implies that (B.31) implements 'hard' thresholding. The consistency rate of the BL estimator is $m_N\sqrt{\frac{\log(N)}{T}}$ under the spectral norm of the error matrix $\left(\widetilde{\mathbf{\Sigma}}_{BL,C} - \mathbf{\Sigma}\right)$. The potential computational burden in the implementation of this approach is the estimation of the thresholding parameter, C. This is usually calibrated by a separate cross-validation (CV) procedure. The quality of the performance of the BL estimator is rooted in the specification chosen for the implementation of CV.[4] Details of the BL cross-validation procedure are given in Section B.3.

As argued by BL, thresholding maintains the symmetry of $\hat{\mathbf{\Sigma}}$ but does not ensure positive definiteness of $\widetilde{\mathbf{\Sigma}}_{BL,\hat{C}}$ in finite samples. BL show that their threshold estimator is positive definite if

$$\left\|\widetilde{\mathbf{\Sigma}}_{BL,C} - \widetilde{\mathbf{\Sigma}}_{BL,0}\right\|_{spec} \leq \epsilon \text{ and } \lambda_{\min}(\mathbf{\Sigma}) > \epsilon, \quad \text{(B.32)}$$

[4]Fang et al. (2013) provide useful guidelines regarding the specification of various parameters used in cross-validation through an extensive simulation study.

where $\|.\|_{spec}$ is the spectral or operator norm and ϵ is a small positive constant. This condition is not met unless T is sufficiently large relative to N. 'Universal' thresholding on $\hat{\mathbf{\Sigma}}$ performs best when the units x_{it}, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$ are assumed homoskedastic (i.e. $\sigma_{11} = \sigma_{22} = \ldots = \sigma_{NN}$).

B.2 Cai and Liu (CL) thresholding

Cai and Liu (2011) - CL - proposed an improved version of the BL approach by incorporating the unit specific variances in their 'adaptive' thresholding procedure. In this way, unlike 'universal' thresholding on $\hat{\mathbf{\Sigma}}$, their estimator is robust to heteroscedasticity. Specifically, the thresholding estimator $\widetilde{\mathbf{\Sigma}}_{CL,C}$ is defined as

$$\widetilde{\mathbf{\Sigma}}_{CL,C} = \left(\hat{\sigma}_{ij} s_{\tau_{ij}} \left[|\hat{\sigma}_{ij}| \geq \tau_{ij}\right]\right),\ i = 1, 2, \ldots, N-1,\ j = i+1, i+2, \ldots, N \tag{B.33}$$

where $\tau_{ij} > 0$ is an entry-dependent adaptive threshold such that $\tau_{ij} = \sqrt{\hat{\theta}_{ij}}\omega_T$,with $\hat{\theta}_{ij} = T^{-1}\sum_{i=1}^{T}(x_{it}x_{jt} - \hat{\sigma}_{ij})^2$ and $\omega_T = C\sqrt{\log(N)/T}$, for some constant $C > 0$. CL implement their approach using the general thresholding function $s_\tau(.)$ rather than $I(.)$, but point out that all their theoretical results continue to hold for the hard thresholding estimator. The consistency rate of the CL estimator is $C_0 m_N \sqrt{\log(N)/T}$ under the spectral norm of the error matrix $\left(\widetilde{\mathbf{\Sigma}}_{CL,C} - \mathbf{\Sigma}\right)$. The parameter C can be fixed to a constant implied by theory ($C = 2$ in CL) or chosen via cross-validation. Details of the CL cross-validation procedure are provided in Section B.3.

As with the BL estimator, thresholding in itself does not ensure positive definiteness of $\widetilde{\mathbf{\Sigma}}_{CL,\hat{C}}$. In light of condition (B.32), Fan et al. (2013) - FLM - extend the CL approach and propose setting a lower bound on the cross-validation grid when searching for C such that the minimum eigenvalue of their threshold estimator is positive, $\lambda_{\min}\left(\widetilde{\mathbf{\Sigma}}_{FLM,\hat{C}}\right) > 0$. This idea originated from Fryzlewicz (2013). Further details of this procedure can be found in Section B.3. We apply this extension to both BL and CL procedures (see Section B.3 for the relevant expressions).

B.3 Cross-validation

We perform a grid search for the choice of C over a specified range: $C = \{c : C_{\min} \leq c \leq C_{\max}\}$. In the BL procedure, we set $C_{\min} = \left|\min_{ij} \hat{\sigma}_{ij}\right| \sqrt{\frac{T}{\log N}}$ and $C_{\max} = \left|\max_{ij} \hat{\sigma}_{ij}\right| \sqrt{\frac{T}{\log N}}$ and impose increments of $\frac{(C_{\max} - C_{\min})}{N}$. In CL cross-validation, we set $C_{\min} = 0$ and $C_{\max} = 4$, and impose increments of c/N. In MT cross-validation set $\delta_{\min} = 1$ and $\delta_{\max} = 2.5$ and impose either fixed increments of 0.1 or N-dependent increments of $1/N$. In each point of this range, c, we use x_{it}, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$ and select the $N \times 1$ column vectors $\boldsymbol{x}_t = (x_{1t}, x_{2t}, \ldots, x_{Nt})'$, $t = 1, 2, \ldots, T$ which we randomly reshuffle over the t-dimension. This gives rise to a new set of $N \times 1$ column vectors $\boldsymbol{x}_t^{(s)} = \left(x_{1t}^{(s)}, x_{2t}^{(s)}, \ldots, x_{Nt}^{(s)}\right)'$ for the first shuffle $s = 1$. We repeat this reshuffling S times in total where we set $S = 50$. We consider this to be sufficiently large (FLM suggested $S = 20$ while BL recommended $S = 100$ - see also Fang et al. (2013)). In each shuffle $s = 1, 2, \ldots, S$, we divide $\boldsymbol{x}^{(s)} = \left(\boldsymbol{x}_1^{(s)}, \boldsymbol{x}_2^{(s)}, \ldots, \boldsymbol{x}_T^{(s)}\right)$

into two subsamples of size $N \times T_1$ and $N \times T_2$, where $T_2 = T - T_1$. A theoretically 'justified' split suggested in BL is given by $T_1 = T\left(1 - \frac{1}{\log(T)}\right)$ and $T_2 = \frac{T}{\log(T)}$. In our simulation study we set $T_1 = \frac{2T}{3}$ and $T_2 = \frac{T}{3}$. Let $\hat{\mathbf{\Sigma}}_1^{(s)} = \left(\hat{\sigma}_{1,ij}^{(s)}\right)$, with elements $\hat{\sigma}_{1,ij}^{(s)} = T_1^{-1}\sum_{t=1}^{T_1} x_{it}^{(s)} x_{jt}^{(s)}$, and $\hat{\mathbf{\Sigma}}_2^{(s)} = \left(\hat{\sigma}_{2,ij}^{(s)}\right)$ with elements $\hat{\sigma}_{2,ij}^{(s)} = T_2^{-1}\sum_{t=T_1+1}^{T} x_{it}^{(s)} x_{jt}^{(s)}$, $i, j = 1, 2, \ldots, N$, denote the sample covariance matrices generated using T_1 and T_2 respectively, for each split s. We threshold $\hat{\mathbf{\Sigma}}_1^{(s)}$ as in (B.31), (B.33) or (5) using $I(.)$ as the thresholding function, where for CL both $\hat{\theta}_{ij}$ and ω_T are adjusted to

$$\hat{\theta}_{1,ij}^{(s)} = \frac{1}{T_1}\sum_{t=1}^{T_1}(x_{it}^{(s)} x_{jt}^{(s)} - \hat{\sigma}_{1,ij}^{(s)})^2,$$

and

$$\omega_{T_1}(c) = c\sqrt{\frac{\log(N)}{T_1}}.$$

Then (B.33) becomes

$$\widetilde{\mathbf{\Sigma}}_1^{(s)}(c) = \left(\hat{\sigma}_{1,ij}^{(s)} I\left[\left|\hat{\sigma}_{1,ij}^{(s)}\right| \geq \tau_{1,ij}^{(s)}(c)\right]\right),$$

for each c, where

$$\tau_{1,ij}^{(s)}(c) = \sqrt{\hat{\theta}_{1,ij}^{(s)}}\omega_{T_1}(c) > 0,$$

and $\hat{\theta}_{1,ij}^{(s)}$ and $\omega_{T_1}(c)$ are defined above.

The following expression is computed for BL, CL or MT,

$$\hat{G}(c) = \frac{1}{S}\sum_{s=1}^{S}\left\|\widetilde{\mathbf{\Sigma}}_1^{(s)}(c) - \widetilde{\mathbf{\Sigma}}_2^{(s)}\right\|_F^2, \tag{B.34}$$

for each c and

$$\hat{C} = \arg\min_{C_{\min} \leq c \leq C_{\max}} \hat{G}(c). \tag{B.35}$$

If several values of c attain the minimum of (B.35), then $\hat{C}$ is chosen to be the smallest one. The final estimator of the covariance matrix is then given by $\widetilde{\mathbf{\Sigma}}_{\hat{C}}$. The thresholding approach does not necessarily ensure that the resultant estimate, $\widetilde{\mathbf{\Sigma}}_{\hat{C}}$, is positive definite. To ensure that the threshold estimator is positive definite Fan et al. (2013) propose setting a lower bound on the cross-validation grid for the search of C such that $\lambda_{\min}\left(\widetilde{\mathbf{\Sigma}}_{\hat{C}}\right) > 0$ - see Fryzlewicz (2013). Therefore, for BL and CL we modify (B.35) so that

$$\hat{C}^* = \arg\min_{C_{pd}+\epsilon \leq c \leq C_{\max}} \hat{G}(c), \tag{B.36}$$

where C_{pd} is the lowest c such that $\lambda_{\min}\left(\widetilde{\mathbf{\Sigma}}_{C_{pd}}\right) > 0$ and ϵ is a small positive constant. We do not conduct thresholding on the diagonal elements of the covariance matrices which remain in tact.

B.4 Ledoit and Wolf (LW) shrinkage

Ledoit and Wolf (2004) - LW - considered a shrinkage estimator for regularisation which is based on a linear combination of the sample covariance matrix, $\hat{\boldsymbol{\Sigma}}$, and an identity matrix $\boldsymbol{I}_N$, and provide formulae for the appropriate weights. The LW shrinkage is expressed as

$$\hat{\boldsymbol{\Sigma}}_{LW} = \hat{\rho}_1 \boldsymbol{I}_N + \hat{\rho}_2 \hat{\boldsymbol{\Sigma}}, \tag{B.37}$$

with the estimated weights given by

$$\hat{\rho}_1 = m_T b_T^2 / d_T^2, \;\; \hat{\rho}_2 = a_T^2 / d_T^2$$

where

$$\begin{aligned} m_T &= N^{-1} \operatorname{tr}\left(\hat{\boldsymbol{\Sigma}}\right), \; d_T^2 = N^{-1} \operatorname{tr}\left(\hat{\boldsymbol{\Sigma}}^2\right) - m_T^2, \\ a_T^2 &= d_T^2 - b_T^2, \; b_T^2 = \min(\bar{b}_T^2, d_T^2), \end{aligned}$$

and

$$\bar{b}_T^2 = \frac{1}{NT^2} \sum_{t=1}^{T} \left\| \dot{\boldsymbol{x}}_t \dot{\boldsymbol{x}}_t' - \hat{\boldsymbol{\Sigma}} \right\|_F^2 = \frac{1}{NT^2} \sum_{t=1}^{T} \operatorname{tr}\left[\left(\dot{\boldsymbol{x}}_t \dot{\boldsymbol{x}}_t'\right) \left(\dot{\boldsymbol{x}}_t \dot{\boldsymbol{x}}_t'\right) \right] - \frac{2}{NT^2} \sum_{t=1}^{T} \operatorname{tr}\left(\dot{\boldsymbol{x}}_t' \hat{\boldsymbol{\Sigma}} \dot{\boldsymbol{x}}_t \right) + \frac{1}{NT} \operatorname{tr}\left(\hat{\boldsymbol{\Sigma}}^2\right),$$

and noting that $\sum_{t=1}^{T} \operatorname{tr}\left(\dot{\boldsymbol{x}}_t' \hat{\boldsymbol{\Sigma}} \dot{\boldsymbol{x}}_t \right) = \sum_{t=1}^{T} \operatorname{tr}\left(\hat{\boldsymbol{\Sigma}} \sum_{t=1}^{T} \dot{\boldsymbol{x}}_t \dot{\boldsymbol{x}}_t' \right) = T \sum_{t=1}^{T} \operatorname{tr}\left(\hat{\boldsymbol{\Sigma}}^2\right)$, we have

$$\bar{b}_T^2 = \frac{1}{NT^2} \sum_{t=1}^{T} \left(\sum_{i=1}^{N} \dot{x}_{it}^2 \right)^2 - \frac{1}{NT} \operatorname{tr}\left(\hat{\boldsymbol{\Sigma}}^2\right),$$

with $\dot{\boldsymbol{x}}_t = (\dot{x}_{1t}, \dot{x}_{2t}, \ldots, \dot{x}_{Nt})'$ and $\dot{x}_{it} = (x_{it} - \bar{x}_i)$.[5]

$\hat{\boldsymbol{\Sigma}}_{LW}$ is positive definite by construction. Thus, the inverse $\hat{\boldsymbol{\Sigma}}_{LW}^{-1}$ exists and is well conditioned.

Supplementary Appendix C

Shrinkage on MT estimator (S-MT)

Recall the shrinkage on the multiple testing estimator (S-MT) expression displayed in Section 3.1,

$$\widetilde{\boldsymbol{R}}_{S\text{-}MT}(\xi) = \xi \boldsymbol{I}_N + (1 - \xi) \widetilde{\boldsymbol{R}}_{MT},$$

where the $N \times N$ identity matrix $\boldsymbol{I}_N$ is set as benchmark target, the shrinkage parameter is denoted by $\xi \in (\xi_0, 1]$, and ξ_0 is the minimum value of ξ that produces a non-singular $\widetilde{\boldsymbol{R}}_{S\text{-}MT}(\xi_0)$ matrix. Note that shrinkage is deliberately implemented on the correlation matrix $\widetilde{\boldsymbol{R}}_{MT}$ rather than on $\widetilde{\boldsymbol{\Sigma}}_{MT}$. In this way we ensure that no shrinkage is applied to the variances.

[5] Note that LW scale the Frobenius norm by $1/N$, and use $\|\boldsymbol{A}\|_F^2 = \operatorname{tr}(\boldsymbol{A}'\boldsymbol{A})/N$. See Definition 1 of Ledoit and Wolf (2004) (p. 376). Here we use the standard notation for this norm.

Further, shrinkage is applied to the non-zero elements of $\widetilde{\boldsymbol{R}}_{MT}$, and as a result the shrinkage estimator, $\widetilde{\boldsymbol{R}}_{S\text{-}MT}$, also consistently recovers the support of $\mathbf{R}$, since it has the same support recovery property as $\widetilde{\boldsymbol{R}}_{MT}$. With regard to the calibration of the shrinkage parameter, ξ, we solve the following optimisation problem

$$\xi^* = \arg \min_{\xi_0+\epsilon\leq\xi\leq 1} \left\| \boldsymbol{R}_0^{-1} - \widetilde{\boldsymbol{R}}_{S\text{-}MT}^{-1}(\xi) \right\|_F^2,$$

where ϵ is a small positive constant, and $\mathbf{R}_0$ is a reference invertible correlation matrix. Let $\boldsymbol{A} = \boldsymbol{R}_0^{-1}$ and $\boldsymbol{B}(\xi) = \widetilde{\boldsymbol{R}}_{S\text{-}MT}^{-1}(\xi)$. Note that since $\boldsymbol{R}_0$ and $\widetilde{\boldsymbol{R}}_{S\text{-}MT}$ are symmetric

$$\left\| \boldsymbol{R}_0^{-1} - \widetilde{\boldsymbol{R}}_{S\text{-}MT}^{-1}(\xi) \right\|_F^2 = \operatorname{tr}\left(\boldsymbol{A}^2\right) - 2\operatorname{tr}[\boldsymbol{A}\boldsymbol{B}(\xi)] + \operatorname{tr}[\boldsymbol{B}^2(\xi)].$$

The first order condition for the above optimisation problem is given by

$$\frac{\partial \left\| \boldsymbol{R}_0^{-1} - \widetilde{\boldsymbol{R}}_{S\text{-}MT}^{-1}(\xi) \right\|_F^2}{\partial \xi} = -2\operatorname{tr}\left(\boldsymbol{A}\frac{\partial \boldsymbol{B}(\xi)}{\partial \xi}\right) + 2\operatorname{tr}\left(\boldsymbol{B}(\xi)\frac{\partial \boldsymbol{B}(\xi)}{\partial \xi}\right),$$

where

$$\begin{aligned}\frac{\partial \boldsymbol{B}(\xi)}{\partial \xi} &= -\widetilde{\boldsymbol{R}}_{S\text{-}MT}^{-1}(\xi)\left(\boldsymbol{I}_N - \tilde{\boldsymbol{R}}_{MT}\right)\widetilde{\boldsymbol{R}}_{S\text{-}MT}^{-1}(\xi) \\ &= -\boldsymbol{B}(\xi)\left(\boldsymbol{I}_N - \tilde{\boldsymbol{R}}_{MT}\right)\boldsymbol{B}(\xi).\end{aligned}$$

Hence, ξ^* is obtained as the solution of

$$f(\xi) = -\operatorname{tr}\left[(\mathbf{A} - \boldsymbol{B}(\xi))\boldsymbol{B}(\xi)\left(\mathbf{I}_N - \widetilde{\boldsymbol{R}}_{MT}\right)\boldsymbol{B}(\xi)\right] = 0,$$

where $f(\xi)$ is an analytic differentiable function of ξ for values of ξ close to unity, such that $\boldsymbol{B}(\xi)$ exists.

The resulting $\widetilde{\boldsymbol{R}}_{S\text{-}MT}(\xi^*)$ is guaranteed to be positive definite since

$$\lambda_{\min}\left[\widetilde{\boldsymbol{R}}_{S\text{-}MT}(\xi)\right] = \xi\lambda_{\min}(\boldsymbol{I}_N) + (1-\xi)\lambda_{\min}\left(\widetilde{\boldsymbol{R}}_{MT}\right) > 0,$$

for any $\xi \in [\xi_0, 1]$, where $\xi_0 = \max\left(\frac{\epsilon - \lambda_{\min}\left(\widetilde{\boldsymbol{R}}_{MT}\right)}{1-\lambda_{\min}\left(\widetilde{\boldsymbol{R}}_{MT}\right)}, 0\right)$.

C.1 Derivation of S-MT shrinkage parameter

We need to solve $f(\xi) = 0$ for ξ^* such that $f(\xi^*) = 0$ for a given choice of $\mathbf{R}_0$.[6]

Abstracting from the subscripts, note that

$$f(1) = -\operatorname{tr}\left[\left(\boldsymbol{R}^{-1} - \boldsymbol{I}_N\right)\left(\boldsymbol{I}_N - \widetilde{\boldsymbol{R}}\right)\right],$$

[6] The code for computing $\mathbf{R}_0$ of our choice is available upon request (see Section C.2).

or

$$
\begin{aligned}
f(1) &= -\operatorname{tr}\left[-\boldsymbol{R}^{-1}\widetilde{\boldsymbol{R}}+\boldsymbol{R}^{-\mathbf{1}}-\boldsymbol{I}_N+\widetilde{\boldsymbol{R}}\right] \\
&= \operatorname{tr}\left(\boldsymbol{R}^{-1}\widetilde{\boldsymbol{R}}\right)-tr\left(\boldsymbol{R}^{-\mathbf{1}}\right),
\end{aligned}
$$

which is generally non-zero. Also, $\xi = 0$ is ruled out, since $\widetilde{\boldsymbol{R}}_{S\text{-}MT}(0) = \widetilde{\boldsymbol{R}}$ need not be non-singular.

Thus we need to assess whether $f(\xi) = 0$ has a solution in the range $\xi_0 < \xi < 1$, where ξ_0 is the minimum value of ξ such that $\widetilde{\boldsymbol{R}}_{S\text{-}MT}(\xi_0)$ is non-singular. First, we can compute ξ_0 by implementing naive shrinkage as an initial estimate:

$$
\widetilde{\boldsymbol{R}}_{S\text{-}MT}(\xi_0) = \xi_0\boldsymbol{I}_N + (1-\xi_0)\widetilde{\boldsymbol{R}}.
$$

The shrinkage parameter $\xi_0 \in [0,1]$ is given by

$$
\xi_0 = \max\left(\frac{\epsilon-\lambda_{\min}\left(\widetilde{\boldsymbol{R}}\right)}{1-\lambda_{\min}\left(\widetilde{\boldsymbol{R}}\right)}, 0\right),
$$

where in our simulation study we set $\epsilon = 0.01$. Here, $\lambda_{\min}(\boldsymbol{A})$ stands for the minimum eigenvalue of matrix $\boldsymbol{A}$. If $\widetilde{\boldsymbol{R}}$ is already positive definite and $\lambda_{\min}\left(\widetilde{\boldsymbol{R}}\right) > 0$, then ξ_0 is automatically set to zero. Conversely, if $\lambda_{\min}\left(\widetilde{\boldsymbol{R}}\right) \leq 0$, then ξ_0 is set to the smallest possible value that ensures positivity of $\lambda_{\min}\left(\widetilde{\boldsymbol{R}}_{S\text{-}MT}(\xi_0)\right)$.

Second, we implement the optimisation procedure. In our simulation study we employ a grid search for $\xi^* = \{\xi : \xi_0 + \epsilon \leq \xi \leq 1\}$ with increments of 0.005. The final ξ^* is given by

$$
\xi^* = \arg\min_{\xi}\left[f(\xi)\right]^2.
$$

C.2 Specification of reference matrix $\mathbf{R}_0$

Implementation of the above procedure requires the use of a suitable reference matrix $\boldsymbol{R}_0$. Our experimentations suggested that the shrinkage estimator of Ledoit and Wolf (2004) - LW - applied to the correlation matrix is likely to work well in practice, and is to be recommended. Schäfer and Strimmer (2005) consider LW shrinkage on the correlation matrix. In our application we also take account of the small sample bias of the correlation coefficients in what follows. We set as reference matrix $\mathbf{R}_0$ the shrinkage estimator of LW applied to the sample correlation matrix:

$$
\hat{\mathbf{R}}_0 = \theta\boldsymbol{I}_N + (1-\theta)\hat{\mathbf{R}},
$$

with shrinkage parameter $\theta \in [0,1]$, and $\hat{\mathbf{R}} = (\hat{\rho}_{ij})$. The optimal value of the shrinkage parameter that minimizes the expectation of the squared Frobenius norm of the error of estimating $\mathbf{R}$ by $\hat{\mathbf{R}}_0$:

$$
E\left\|\hat{\mathbf{R}}_0-\mathbf{R}\right\|_F^2 = \sum_{i\neq j}\sum E\left(\hat{\rho}_{ij}-\rho_{ij}\right)^2 + \theta^2\sum_{i\neq j}\sum E\left(\hat{\rho}_{ij}^2\right) - 2\theta\sum_{i\neq j}\sum E\left[\hat{\rho}_{ij}\left(\hat{\rho}_{ij}-\rho_{ij}\right)\right], \quad \text{(C.38)}
$$

is given by

$$\theta^* = \frac{\sum\sum_{i\neq j} E\left[\hat{\rho}_{ij}\left(\hat{\rho}_{ij}-\rho_{ij}\right)\right]}{\sum\sum_{i\neq j} E\left(\hat{\rho}_{ij}^2\right)} = 1 - \frac{\sum\sum_{i\neq j} E\left(\hat{\rho}_{ij}\rho_{ij}\right)}{\sum\sum_{i\neq j} E\left(\hat{\rho}_{ij}^2\right)}, \tag{C.39}$$

with

$$\hat{\theta}^* = 1 - \frac{\sum\sum_{i\neq j} \hat{\rho}_{ij}\left[\hat{\rho}_{ij} - \frac{\hat{\rho}_{ij}(1-\hat{\rho}_{ij}^2)}{2T}\right]}{\frac{1}{T}\sum\sum_{i\neq j}(1-\hat{\rho}_{ij}^2)^2 + \sum\sum_{i\neq j}\left[\hat{\rho}_{ij} - \frac{\hat{\rho}_{ij}(1-\hat{\rho}_{ij}^2)}{2T}\right]^2}.$$

Note that $\lim_{T\to\infty}(\hat{\theta}^*) = 0$ for any N. However, in small samples values of $\hat{\theta}^*$ can be obtained that fall outside the range $[0,1]$. To avoid such cases, if $\hat{\theta}^* < 0$ then $\hat{\theta}^*$ is set to 0, and if $\hat{\theta}^* > 1$ it is set to 1, or $\hat{\theta}^{**} = \max(0, \min(1, \hat{\theta}^*))$.

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