Supplement to "Testing for Alpha in Linear Factor Pricing Models with a Large Number of Securities"
by

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This supplement consists of two parts. The first part establishes a number of lemmas used in the proofs of theorems in Section 4 of the paper. The second part provides additional documentation of the Monte Carlo experiments, specifically regarding the simulation of multivariate non-Gaussian random variables, details of the alternative test statistics considered in Section 5, and additional Monte Carlo results.

Notations

We use K and c to denote finite and small positive constants. If $\{f_t\}_{t=1}^{\infty}$ is any real sequence and $\{g_t\}_{t=1}^{\infty}$ is a sequences of positive real numbers, then $f_t = O(g_t)$, if there exists a positive finite constant K such that $|f_t|/g_t \leq K$ for all t. $f_t = o(g_t)$ if $f_t/g_t \rightarrow 0$ as $t \rightarrow \infty$. For two $N \times N$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, the Hadamard product $\mathbf{A} \odot \mathbf{B} = \mathbf{B} \odot \mathbf{A}$ is an $N \times N$ matrix with elements given by $a_{ij}b_{ij}$. The minimum and maximum eigenvalues of matrix $\mathbf{A}$ is denoted by $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$, respectively, its trace by $Tr(\mathbf{A})$, its maximum absolute column and row sum matrix norms by $\|\mathbf{A}\|_{\infty} = \max_{1\leq i\leq N}\left\{\sum_{j=1}^{N}|a_{ij}|\right\}$, and,$\|\mathbf{A}\|_1 = \max_{1\leq j\leq N}\left\{\sum_{i=1}^{N}|a_{ij}|\right\}$, respectively, its Frobenius and spectral norms by $\|\mathbf{A}\|_F = \sqrt{Tr(\mathbf{A}'\mathbf{A})}$, and $\|\mathbf{A}\| = \lambda_{\max}^{1/2}(\mathbf{A}'\mathbf{A})$, respectively. For an $N \times 1$ dimensional vector, $\boldsymbol{\alpha}$, $\|\boldsymbol{\alpha}\| = (\boldsymbol{\alpha}'\boldsymbol{\alpha})^{1/2}$. We set

$$\mathbf{M}_G = (m_{tt'}) = \mathbf{I}_T - \mathbf{P}_G,\ \mathbf{P}_G = \mathbf{G}\left(\mathbf{G}'\mathbf{G}\right)^{-1}\mathbf{G}',\ \mathbf{G} = (\mathbf{F}, \boldsymbol{\tau}_T),\ v = Tr(\mathbf{M}_G) = T - m - 1, \qquad \text{(S.1)}$$

$$\begin{aligned} \mathbf{M}_F &= (m_{F,tt'}) = \mathbf{I}_T - \mathbf{F}\left(\mathbf{F}'\mathbf{F}\right)^{-1}\mathbf{F}',\ \mathbf{H}_F = \mathbf{h}\mathbf{h}' = (h_t h_{t'}) \\ \text{with } \mathbf{h} &= (h_t) = \mathbf{M}_F\boldsymbol{\tau}_T,\ w_T = Tr(\mathbf{H}_F) = \mathbf{h}'\mathbf{h} = \boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T, \end{aligned} \qquad \text{(S.2)}$$

where $\mathbf{F}$ is a $T \times m$ matrix, and $\boldsymbol{\tau}_T = (1, 1, ..., 1)'$ is a $T \times 1$ vector of ones. To simplify the algebra all derivations are made conditional on $\mathbf{F}$.

S1 Statement of lemmas and their proofs

Lemma 2 *(Moments of linear functions) Consider $w = \sum_{i=1}^{N} a_i\epsilon_i$, which is a linear combination of independently distributed random variables, ϵ_i, for $i = 1, 2, ..., N$, with mean zero and a unit variance, and the weights, a_i, that satisfy $\sum_{i=1}^{N} a_i^2 = 1$. Then, the r^{th} moment of w exists if ϵ_i has the r^{th} moment.*

Proof. We first note that since $\sum_{i=1}^{N} a_i^2 = 1$, then it must be that $|a_i| \leq 1$, and hence $|a_i|^r \leq |a_i|$, for $r \geq 1$. Therefore,

$$\sum_{i=1}^{N} a_i^3 \leq \sum_{i=1}^{N} |a_i|^3 \leq \sum_{i=1}^{N} a_i^2 = 1,\ \sum_{i=1}^{N} a_i^4 \leq \sum_{i=1}^{N} a_i^2 = 1,$$

or more generally, $\sum_{i=1}^{N} |a_i|^r \leq 1$, for $r = 2, 3, ...$. Consider now moments of w, and note that $E(w) = 0$, $E(w^2) = \sum_{i=1}^{N} a_i^2 = 1$,

$$E(w^3) = E\left(\sum_{i=1}^{N} a_i\epsilon_i\right)^3 = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N} a_i a_j a_\ell E\left(\epsilon_i\epsilon_j\epsilon_\ell\right) = \left(\sum_{i=1}^{N} a_i^3\right)E\left(\epsilon_i^3\right) \leq \sup_i E\left(\epsilon_i^3\right),$$

$$\begin{aligned} E(w^4) = E\left(\sum_{i=1}^{N} a_i\epsilon_i\right)^4 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}\sum_{n=1}^{N} a_i a_j a_\ell a_n E\left(\epsilon_i\epsilon_j\epsilon_\ell\epsilon_n\right) = 3\sum_{i\neq j} a_i^2 a_j^2 E\left(\epsilon_i^2\right)E\left(\epsilon_j^2\right) + \sum_i a_i^4 E\left(\epsilon_i^4\right) \\ &= 3\left[\left(\sum_{i=1}^{N} a_i^2 E(\epsilon_i^2)\right)^2 - \left(\sum_{i=1}^{N} a_i^4\left[E(\epsilon_i^2)\right]^2\right)\right] + \left(\sum_{i=1}^{N} a_i^4 E\left(\epsilon_i^4\right)\right) \\ &= 3\left(\sum_{i=1}^{N} a_i^2 E(\epsilon_i^2)\right)^2 + \sum_{i=1}^{N} a_i^4\left\{E\left(\epsilon_i^4\right) - 3\left[E(\epsilon_i^2)\right]^2\right\} \end{aligned}$$

$$= 3+\sum_{i=1}^{N} a_i^4 \left[E\left(\epsilon_i^4\right)-3\right] \leq 3+\sup_i \left[E\left(\epsilon_i^4\right)-3\right]\left(\sum_{i=1}^{N} a_i^4\right) \leq 3+\sup_i \left[E\left(\epsilon_i^4\right)-3\right].$$

Note that $E\left(\epsilon_i^r\right)$ need not be the same across i, it is only required that $E\left(\epsilon_i^r\right) < K < \infty$.

$$\begin{aligned}
E(w^5) &= E\left(\sum_{i=1}^{N} a_i\epsilon_i\right)^5 = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}\sum_{n=1}^{N}\sum_{p=1}^{N} a_i a_j a_\ell a_n a_p E\left(\epsilon_i\epsilon_j\epsilon_\ell\epsilon_n\epsilon_p\right) \\
&= 10\sum_{i\neq j} a_i^2 a_j^3 E\left(\epsilon_i^2\right) E\left(\epsilon_j^3\right) + \sum_i a_i^5 E\left(\epsilon_i^5\right)
\end{aligned}$$

$$\begin{aligned}
&= 10\left[\left(\sum_{i=1}^{N} a_i^2 E\left(\epsilon_i^2\right)\right)\left(\sum_{i=1}^{N} a_i^3 E\left(\epsilon_i^3\right)\right) - \sum_{i=1}^{N} a_i^5 E\left(\epsilon_i^2\right) E\left(\epsilon_i^3\right)\right] + \sum_i a_i^5 E\left(\epsilon_i^5\right) \\
&= 10\left(\sum_{i=1}^{N} a_i^3 E(\epsilon_i^3)\right) + \sum_{i=1}^{N} a_i^5\left[E(\epsilon_i^5) - 10E(\epsilon_i^3)\right]
\end{aligned}$$

$$\begin{aligned}
&\leq 10\sup_i E(\epsilon_i^3)\sum_{i=1}^{N} a_i^3 + \sup_i\left[E(\epsilon_i^5) - 10E(\epsilon_i^3)\right]\sum_{i=1}^{N} a_i^5 \\
&\leq 10\sup_i E(\epsilon_i^3) + \sup_i\left[E(\epsilon_i^5) - 10E(\epsilon_i^3)\right]
\end{aligned}$$

and

$$\begin{aligned}
E(w^6) &= E\left(\sum_{i=1}^{N} a_i\epsilon_i\right)^6 = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}\sum_{n=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N} a_i a_j a_\ell a_n a_p a_q E\left(\epsilon_i\epsilon_j\epsilon_\ell\epsilon_n\epsilon_p\epsilon_q\right) \\
&= 15\sum_{i\neq j\neq \ell} a_i^2 a_j^2 a_\ell^2 E\left(\epsilon_i^2\right)^3 + 10\sum_{i\neq j} a_i^3 a_j^3 E\left(\epsilon_i^3\right)^2 + 15\sum_{i\neq j} a_i^4 a_j^2 E\left(\epsilon_i^4\right) E\left(\epsilon_j^2\right) + \sum_i a_i^6 E\left(\epsilon_i^6\right)
\end{aligned}$$

$$\begin{aligned}
=\ & 15\left\{\left(\sum_{i=1}^{N} a_i^2\right)^3 - 3\left[\left(\sum_{i=1}^{N} a_i^4\right)\left(\sum_{i=1}^{N} a_i^2\right) - \sum_{i=1}^{N} a_i^6\right] - \sum_{i=1}^{N} a_i^6\right\} E\left(\epsilon_i^2\right)^3 \\
&+10\left[\left(\sum_{i=1}^{N} a_i^3\right)^2 - \sum_{i=1}^{N} a_i^6\right] E\left(\epsilon_i^3\right)^2 + 15\left[\left(\sum_{i=1}^{N} a_i^4\right)\left(\sum_{i=1}^{N} a_i^2\right) - \sum_{i=1}^{N} a_i^6\right] E\left(\epsilon_i^4\right) E\left(\epsilon_j^2\right) \\
&+\sum_{i=1}^{N} a_i^6 E(\epsilon_i^6).
\end{aligned}$$

Again noting that $E(\epsilon_i^2) = 1$ and $\sum_{i=1}^{N} a_i^2 = 1$, we have, after some simplifications,

$$\begin{aligned}
E(w^6) &= 15 + 10\left(\sum_{i=1}^{N} a_i^3\left[E(\epsilon_i^3)\right]\right)^2 + 15\sum_{i=1}^{N} a_i^4\left[E(\epsilon_i^4) - 3\right] + \\
&\left[\sum_{i=1}^{N} a_i^6 E(\epsilon_i^6) + 30\sum_{i=1}^{N} a_i^6 - 10\sum_{i=1}^{N} a_i^6\left[E(\epsilon_i^3)\right]^2 - 15\sum_{i=1}^{N} a_i^6 E(\epsilon_i^4)\right]
\end{aligned}$$

$$\begin{aligned}
\leq\ & 15 + 15\sup_i\left[E(\epsilon_i^4) - 3\right]\sum_{i=1}^{N} a_i^4 + 10\sup_i\left[E(\epsilon_i^3)\right]^2\left(\sum_{i=1}^{N} a_i^3\right)^2 + \\
&\sup_i\left[E(\epsilon_i^6) - 10\left[E(\epsilon_i^3)\right]^2 - 15\left[E(\epsilon_i^4) - 3\right] - 15\right]\sum_{i=1}^{N} a_i^6
\end{aligned}$$

$$\leq 15 + 15\sup_i\left[E(\epsilon_i^4) - 3\right] + 10\sup_i\left[E(\epsilon_i^3)\right]^2 + \sup_i\left\{E(\epsilon_i^6) - 10\left[E(\epsilon_i^3)\right]^2 - 15\left[E(\epsilon_i^4) - 3\right] - 15\right\}.$$

The processes can be continued for higher order moments. ■

Lemma 3 *Under Assumptions 1-4,*

(i) $\xi_{it} = u_{it}/\sigma_{ii}^{1/2} \sim IID(0,1)$ *for all* t *and* $E(|\xi_{it}|^r) \leq K < \infty$, *where* u_{it} *is defined by (2) and* $\sigma_{ii} = Var\left(u_{it}\right)$, *and;*

(ii) $\tilde{\eta}_{it} = \eta_{it}/\sigma_{\eta,ii}^{1/2} \sim IID(0,1)$ *for all* t *and* $E(|\tilde{\eta}_{it}|^r) \leq K < \infty$, *where* η_{it} *is defined by (2) and* $\sigma_{\eta,ii} = Var(\eta_{it})$, *for all* i *and* t, $r = 1, 2, ..., 8$.

Proof. We have $u_{it} = \sum_{j=1}^{N} q_{ij}\varepsilon_{jt}$, for $i = 1, 2, ..., N, t = 1, 2, ..., T$, where ε_{jt} is defined by (50), and q_{ij} is the (i,j) element of $\mathbf{Q}$ which is defined by (50). Note that ε_{it} is $IID(0,1)$ across i and t, $E(\varepsilon_{it}^8)$ exists, $\xi_{it} = u_{it}/\sigma_{ii}^{1/2} = \sum_{j=1}^{N} \tilde{q}_{ij}^2 \varepsilon_{jt}$, where $\tilde{q}_{ij} = q_{ij}/\sigma_{ii}^{1/2} = q_{ij}/\left(\sum_{j=1}^{N} q_{ij}^2\right)^{1/2}$, and $\sum_{j=1}^{N} \tilde{q}_{ij}^2 = 1$. Then applying Lemma 2 to $\sum_{j=1}^{N} \tilde{q}_{ij}\varepsilon_{jt}$ yields the required result. For part (ii), a similar discussion for $\tilde{\eta}_{it} = \sum_{j=1}^{N} \tilde{q}_{\eta,ij}\varepsilon_{\eta,jt}$ will lead to the required result, where $\varepsilon_{\eta,jt}$ is defined by (50), $\tilde{q}_{\eta,ij} = \sigma_{\eta,ii}^{1/2} = q_{\eta,ij}/\left(\sum_{j=1}^{N} q_{\eta,ij}^2\right)^{1/2}$, $\sum_{j=1}^{N} \tilde{q}_{\eta,ij}^2 = 1$, $q_{\eta,ij}$ is the (i,j) element of $\mathbf{Q}_\eta$ which is defined by (50). ■

Lemma 4 *Consider the sequences of random variables* $\{X_N\}$ *and* $\{Y_N\}$. *If* $X_N - Y_N \to_p 0$, *and* $Y_N \to_d Z$, *then* $X_N \to_d Z$.

Proof. See Rao (1973, p.122). ■

Lemma 5 *(Lieberman 1994) Let* $\mathbf{\Phi}$ *be a* $T \times T$ *symmetric matrix and* $\mathbf{\Gamma}$ *a positive definite* $T \times T$ *matrix, and suppose that* $\boldsymbol{\xi} \sim IID(\mathbf{0}, \mathbf{I}_T)$, *where* $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_T)'$. *Denote the* p^{th} *cumulant of* $\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi}$ *by* κ_p, *and the* $m+1$ *order,* $m+r$ *degree generalized cumulant of* $(\boldsymbol{\xi}'\mathbf{\Phi}\boldsymbol{\xi})^r(\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi})$ *by* κ_{rm}, *and assume that the following conditions hold:*

- *Condition 1: For* $p = 1, 2, ..., \kappa_p = O(T)$.
- *Condition 2: For* $r = 1, 2, ..., \kappa_{r0} = E(\boldsymbol{\xi}'\mathbf{\Phi}\boldsymbol{\xi})^r = O(T^r)$.
- *Condition 3: For* $r, m = 1, 2, ..., \kappa_{rm} = O(T^\ell)$, *with* $\ell \leq r$.

Then the Laplace approximate expansion for the r^{th} *moment of* $\boldsymbol{\xi}'\mathbf{\Phi}\boldsymbol{\xi}/\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi}$ *is given by*

$$E\left[\left(\frac{\boldsymbol{\xi}'\mathbf{\Phi}\boldsymbol{\xi}}{\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi}}\right)^r\right] = \frac{E[(\boldsymbol{\xi}'\mathbf{\Phi}\boldsymbol{\xi})^r]}{[E(\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi})]^r} + \psi_{rT} + O(T^{-2}), \tag{S.3}$$

where

$$\psi_{rT} = \frac{r(r+1)}{2}\left\{\frac{E[(\boldsymbol{\xi}'\mathbf{\Phi}\boldsymbol{\xi})^r]\kappa_2}{[E(\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi})]^{r+2}}\right\} - r\left\{\frac{\kappa_{r1}}{[E(\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi})]^{r+1}}\right\}, \tag{S.4}$$

and

$$\kappa_{r1} = E[(\boldsymbol{\xi}'\mathbf{\Phi}\boldsymbol{\xi})^r\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi}] - E[(\boldsymbol{\xi}'\mathbf{\Phi}\boldsymbol{\xi})^r]E(\boldsymbol{\xi}'\mathbf{\Gamma}\boldsymbol{\xi}). \tag{S.5}$$

Proof. See Lieberman (1994). ■

Lemma 6 *(Moments of products of quadratic forms under non-Gaussianity): Suppose that* $\boldsymbol{\xi} \sim IID(\mathbf{0}, \mathbf{I}_T)$, *where* $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_T)'$, *with* $\gamma_1 = E(\xi_t^3)$, $\gamma_2 = E(\xi_t^4) - 3$, $\gamma_3 = E(\xi_t^5) - 10\gamma_1$, $\gamma_4 = E\left(\xi_t^6\right) - 15\gamma_2 - 10\gamma_1^2 - 15$ *and* $\gamma_6 = E\left(\xi_t^8\right) - 28\gamma_4 - 56\gamma_3\gamma_1 - 35\gamma_2^2 - 210\gamma_2 - 280\gamma_1^2 - 105$ *for all* $t = 1, 2, ..., T$, *and suppose that* $\mathbf{A}_j$, $j = 1, 2, 3, 4$ *are* $T \times T$ *real symmetric matrices, and* $\boldsymbol{\tau}_T$ *is a* $T \times 1$ *vector of ones. Then*

$$E\left(\boldsymbol{\xi}'\mathbf{A}_1\boldsymbol{\xi}\right) = Tr(\mathbf{A}_1), \tag{S.6}$$

$$E\left(\boldsymbol{\xi}'\mathbf{A}_1\boldsymbol{\xi}\ \boldsymbol{\xi}'\right) = \gamma_1\boldsymbol{\tau}'\left(\mathbf{I} \odot \mathbf{A}_1\right)'$$

$$E\left[\left(\boldsymbol{\xi}'\mathbf{A}_1\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{A}_2\boldsymbol{\xi}\right)\right] = \gamma_2 Tr\left[(\mathbf{A}_1 \odot \mathbf{A}_2)\right] + Tr\left(\mathbf{A}_1\right)Tr(\mathbf{A}_2) + 2Tr\left(\mathbf{A}_1\mathbf{A}_2\right), \tag{S.7}$$

$$\begin{aligned} E\left[\left(\boldsymbol{\xi}'\mathbf{A}_1\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{A}_2\boldsymbol{\xi}\right)\ \boldsymbol{\xi}\right] &= \gamma_3\left(\mathbf{I} \odot \mathbf{A}_1 \odot \mathbf{A}_2\right)\boldsymbol{\tau} + \gamma_1\{4\left[\mathbf{I} \odot (\mathbf{A}_1\mathbf{A}_2)\right]\boldsymbol{\tau} \\ &+ 2\mathbf{A}_1\left(\mathbf{I} \odot \mathbf{A}_2\right)\boldsymbol{\tau} + 2\mathbf{A}_2\left(\mathbf{I} \odot \mathbf{A}_1\right)\boldsymbol{\tau} + Tr\left(\mathbf{A}_1\right)\left(\mathbf{I} \odot \mathbf{A}_2\right)\boldsymbol{\tau} + Tr\left(\mathbf{A}_2\right)\left(\mathbf{I} \odot \mathbf{A}_1\right)\boldsymbol{\tau}\} \end{aligned}$$

$$\begin{aligned} E\left[\left(\boldsymbol{\xi}'\mathbf{A}_1\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{A}_2\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{A}_3\boldsymbol{\xi}\right)\right] &= \gamma_4 Tr\left(\mathbf{A}_1 \odot \mathbf{A}_2 \odot \mathbf{A}_3\right) + \gamma_2 Tr\left(\mathbf{A}_1\right)Tr\left(\mathbf{A}_2 \odot \mathbf{A}_3\right) \\ &+ \gamma_2 Tr\left(\mathbf{A}_2\right)Tr\left(\mathbf{A}_1 \odot \mathbf{A}_3\right) + \gamma_2 Tr\left(\mathbf{A}_3\right)Tr\left(\mathbf{A}_1 \odot \mathbf{A}_2\right) + 4\gamma_2 Tr\left[\mathbf{A}_1 \odot (\mathbf{A}_2\mathbf{A}_3)\right] \\ &+ 4\gamma_2 Tr\left[\mathbf{A}_2 \odot (\mathbf{A}_1\mathbf{A}_3)\right] + 4\gamma_2 Tr\left[\mathbf{A}_3 \odot (\mathbf{A}_1\mathbf{A}_2)\right] + 2\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{A}_1\right)\mathbf{A}_2\left(\mathbf{I}_T \odot \mathbf{A}_3\right)\boldsymbol{\tau}_T\right] \\ &+ 2\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{A}_1\right)\mathbf{A}_3\left(\mathbf{I}_T \odot \mathbf{A}_2\right)\boldsymbol{\tau}_T\right] + 2\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{A}_2\right)\mathbf{A}_1\left(\mathbf{I}_T \odot \mathbf{A}_3\right)\boldsymbol{\tau}_T\right] \\ &+ 4\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{A}_1 \odot \mathbf{A}_2 \odot \mathbf{A}_3\right)\boldsymbol{\tau}_T\right] + Tr\left(\mathbf{A}_1\right)Tr\left(\mathbf{A}_2\right)Tr\left(\mathbf{A}_3\right) + 2Tr\left(\mathbf{A}_1\right)Tr\left(\mathbf{A}_2\mathbf{A}_3\right) \\ &+ 2Tr\left(\mathbf{A}_2\right)Tr\left(\mathbf{A}_1\mathbf{A}_3\right) + 2Tr\left(\mathbf{A}_3\right)Tr\left(\mathbf{A}_1\mathbf{A}_2\right) + 8Tr\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\right), \end{aligned} \tag{S.8}$$

$$E\left[\left(\boldsymbol{\xi}'\mathbf{A}_1\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{A}_2\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{A}_3\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{A}_4\boldsymbol{\xi}\right)\right] = Tr\left(\mathbf{A}_1\right)Tr\left(\mathbf{A}_2\right)Tr\left(\mathbf{A}_3\right)Tr\left(\mathbf{A}_4\right) \tag{S.9}$$
$$+2[Tr\left(\mathbf{A}_1\right)Tr\left(\mathbf{A}_2\right)Tr\left(\mathbf{A}_3\mathbf{A}_4\right)+Tr\left(\mathbf{A}_1\right)Tr\left(\mathbf{A}_3\right)Tr\left(\mathbf{A}_2\mathbf{A}_4\right)+Tr\left(\mathbf{A}_1\right)Tr\left(\mathbf{A}_4\right)Tr\left(\mathbf{A}_2\mathbf{A}_3\right)$$
$$+Tr\left(\mathbf{A}_2\right)Tr\left(\mathbf{A}_3\right)Tr\left(\mathbf{A}_1\mathbf{A}_4\right)+Tr\left(\mathbf{A}_2\right)Tr\left(\mathbf{A}_4\right)Tr\left(\mathbf{A}_1\mathbf{A}_3\right)+Tr\left(\mathbf{A}_3\right)Tr\left(\mathbf{A}_4\right)Tr\left(\mathbf{A}_1\mathbf{A}_2\right)]$$
$$+4[Tr\left(\mathbf{A}_1\mathbf{A}_2\right)Tr\left(\mathbf{A}_3\mathbf{A}_4\right)+Tr\left(\mathbf{A}_1\mathbf{A}_3\right)Tr\left(\mathbf{A}_2\mathbf{A}_4\right)+Tr\left(\mathbf{A}_1\mathbf{A}_4\right)Tr\left(\mathbf{A}_2\mathbf{A}_3\right)]$$
$$+8[Tr\left(\mathbf{A}_1\right)Tr\left(\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4\right)+Tr\left(\mathbf{A}_2\right)Tr\left(\mathbf{A}_1\mathbf{A}_3\mathbf{A}_4\right)+Tr\left(\mathbf{A}_3\right)Tr\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4\right)+Tr\left(\mathbf{A}_4\right)Tr\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\right)]$$
$$+16[Tr\left(\mathbf{A}_1\mathbf{A}_3\mathbf{A}_4\mathbf{A}_2\right)+Tr\left(\mathbf{A}_1\mathbf{A}_4\mathbf{A}_2\mathbf{A}_3\right)+Tr\left(\mathbf{A}_1\mathbf{A}_4\mathbf{A}_3\mathbf{A}_2\right)]$$
$$+\gamma_2 f_{\gamma_2}+\gamma_4 f_{\gamma_4}+\gamma_6 f_{\gamma_6}+\gamma_1^2 f_{\gamma_1^2}+\gamma_2^2 f_{\gamma_2^2}+\gamma_1\gamma_3 f_{\gamma_1\gamma_3}.$$

Expressions for f_{γ_2}, f_{γ_4}, f_{γ_6}, $f_{\gamma_1^2}$, $f_{\gamma_2^2}$ and $f_{\gamma_1\gamma_3}$ are provided in Bao and Ullah (2010).

Proof. For (S.6) and (S.7), see Ullah (2004, Appendix A.5). Result (S.8) was provided to us through a private communication by Yong Bao. Result (S.9) is given in Bao and Ullah (2010). ■

Lemma 7 *Let $\mathbf{A}$ be a real symmetric $T\times T$ matrix. Then $\lambda_{\min}(\mathbf{A})\leq a_{tt}\leq\lambda_{\max}(\mathbf{A})$, where a_{tt} is the t^{th} diagonal element of $\mathbf{A}$.*

Proof. See Theorem 14 in Chapter 11 of Magnus and Neudecker (1999, p.211-212). ■

Lemma 8 *Denote the (t,r) elements of matrices $\mathbf{M}_F$, $\mathbf{M}_G$, and $\mathbf{P}_G$ (defined by (S.2) and (S.1)), by $m_{F,tr}$, m_{tr} and p_{tr}, respectively, and denote t^{th} element of $\mathbf{h}=\mathbf{M}_F\boldsymbol{\tau}_T$ by $h_t=\sum_{r=1}^{T}m_{F,tr}$. Then, under Assumption 1, for all t we have*

$$0\leq m_{F,tt}=\sum_{r=1}^{T}m_{F,tr}^2\leq 1, \tag{S.10}$$
$$0\leq m_{tt}=\sum_{r=1}^{T}m_{tr}^2\leq 1, \tag{S.11}$$
$$0\leq p_{tt}=\sum_{r=1}^{T}p_{tr}^2\leq 1, \tag{S.12}$$
$$\left|\sum_{r=1}^{T}m_{F,tr}\right|=|h_t|\leq K<\infty, \tag{S.13}$$
$$\sum_{r=1}^{T}m_{tr}=0, \tag{S.14}$$

and for any finite p

$$\sum_{t=1}^{T}\left(\sum_{r=1}^{T}m_{F,tr}\right)^p=\sum_{t=1}^{T}h_t^p=O\left(v\right). \tag{S.15}$$

Proof. *(S.10), (S.11) and (S.12) follow immediately using Lemmas 7, since $\mathbf{M}_F$, $\mathbf{M}_G$ and $\mathbf{P}_G$ are idempotent and real symmetric matrices, with eigenvalues that are either one or zero. Next we note that*

$$\mathbf{M}_F\boldsymbol{\tau}_T=\boldsymbol{\tau}_T-\mathbf{F}\left(\frac{\mathbf{F}'\mathbf{F}}{T}\right)^{-1}\frac{\mathbf{F}'\boldsymbol{\tau}_T}{T},$$

where by Assumption 1 all elements of $\left(\frac{\mathbf{F}'\mathbf{F}}{T}\right)^{-1}$ and $\frac{\mathbf{F}'\boldsymbol{\tau}_T}{T}$ are bounded. Let $\mathbf{w}_{F,T}=\left(\frac{\mathbf{F}'\mathbf{F}}{T}\right)^{-1}\frac{\mathbf{F}'\boldsymbol{\tau}_T}{T}$, and note that the m elements of $\mathbf{w}_{F,T}$, being the OLS estimates of the coefficients in the regression of 1 on $\mathbf{f}_t$, are bounded, and hence $\sum_{\ell=1}^{m}|w_{F,T,\ell}|^2\leq K<\infty$, for all T. Then, the t^{th} element of $\mathbf{M}_F\boldsymbol{\tau}_T$ can be written as

$$\sum_{r=1}^{T}m_{F,tr}=1-\mathbf{f}_t'\mathbf{w}_{F,T}=1-\sum_{\ell=1}^{m}f_{t,\ell}w_{F,T,\ell}.$$

$$\left|\sum_{r=1}^{T}m_{F,tr}\right|\leq 1+\left|\sum_{\ell=1}^{m}f_{t,\ell}w_{F,T,\ell}\right|,$$

and by Assumption 1, $\sum_{\ell=1}^{m}|f_{t,\ell}|^2\leq K<\infty$, and hence for all t we have

$$\left|\sum_{\ell=1}^{m}f_{t,\ell}w_{F,T,\ell}\right|\leq\sqrt{\sum_{\ell=1}^{m}|f_{t,\ell}|^2}\sqrt{\sum_{\ell=1}^{m}|w_{F,T,\ell}|^2}\leq K<\infty.$$

Therefore, we have $\left|\sum_{r=1}^{T}m_{F,tr}\right|\leq K<\infty$, as required. (S.14) follows from $\mathbf{M}_G\boldsymbol{\tau}_T=\mathbf{0}$. Finally, (S.15) follows from (S.13) since $\sum_{t=1}^{T}(\sum_{r=1}^{T}m_{F,tr})^p\leq\sum_{t=1}^{T}|\sum_{r=1}^{T}m_{F,tr}|^p\leq\sum_{t=1}^{T}K^p=O(v)$, for p finite. ■

Lemma 9 *Suppose that $\mathbf{A}_j=(a_{j,tr})$, for $j=1,2,3,4$ are $T\times T$ real symmetric matrices, and $\boldsymbol{\tau}_T$ is a $T\times 1$ vector of ones. Then,*

$$Tr\left(\mathbf{A}_1\odot\mathbf{A}_2\odot\mathbf{A}_3\odot\mathbf{A}_4\right)=\sum_{t=1}^{T}a_{1,tt}a_{2,tt}a_{3,tt}a_{4,tt}, \tag{S.16}$$
$$\boldsymbol{\tau}_T'\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\boldsymbol{\tau}_T=\sum_{t=1}^{T}\sum_{r=1}^{T}\sum_{v=1}^{T}\sum_{u=1}^{T}a_{1,tr}a_{2,rv}a_{3,vu}, \tag{S.17}$$

and

$$\boldsymbol{\tau}_T'\left(\mathbf{A}_1\odot\mathbf{A}_2\right)\boldsymbol{\tau}_T=Tr\left(\mathbf{A}_1\mathbf{A}_2'\right)=\sum_{t=1}^{T}\sum_{r=1}^{T}a_{1,tr}a_{2,tr}. \tag{S.18}$$

Proof. (S.16) and (S.17) follow from direct derivations and (S.18) see Magnus and Neudecker (1999; p.46). ■

Lemma 10 *Consider the matrices* $\mathbf{M}_G$, $\mathbf{P}_G$ *and* $\mathbf{H}_F$, *defined by (S.2) and (S.1), and* $v = T - m - 1$. *Then, under Assumption 1 we have*

$$Tr\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G\right) = O\left(v\right), \quad \text{(S.19)}$$

$$Tr\left(\mathbf{H}_F \odot \mathbf{M}_G\right) = O\left(v\right), \quad \text{(S.20)}$$

$$Tr\left(\mathbf{H}_F \odot \mathbf{H}_F\right) = O\left(v\right), \quad \text{(S.21)}$$

$$Tr\left(\mathbf{M}_G \odot \mathbf{M}_G\right) = O\left(v\right), \quad \text{(S.22)}$$

$$Tr\left(\mathbf{P}_G \odot \mathbf{P}_F\right) = O(1), \quad \text{(S.23)}$$

$$Tr\left(\mathbf{P}_G \odot \mathbf{H}_F\right) = O(v^{1/2}), \quad \text{(S.24)}$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O\left(v^2\right), \quad \text{(S.25)}$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O(v^{3/2}), \quad \text{(S.26)}$$

$$\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\ \boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}), \quad \text{(S.27)}$$

$$\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O(v^2),\ \boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = 0,\ \boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = v,$$

$$Tr\left(\mathbf{M}_G \odot \mathbf{H}_F^2\right) = O\left(v^2\right),\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F^2\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O\left(v^2\right),$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = 0,\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = 0 \quad \text{(S.28)}$$

$$Tr\left(\mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right) = O\left(v\right),$$

$$\begin{aligned}\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T &= O\left(v^2\right),\ \boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = 0,\\ \boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T &= 0f,\ \boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O\left(v^2\right),\end{aligned}$$

$$Tr\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right) = O\left(v\right),$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\ ,\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^2),$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O(v^{3/2}),\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^2),$$

$$Tr\left[\mathbf{H}_F^2\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\right] = O(v^{5/2}),\ Tr\left[\mathbf{M}_G\left(\mathbf{H}_F \odot \mathbf{H}_F\right)\right] = O(v^{3/2}),$$

$$\begin{aligned}&\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\\ &\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O(v^{3/2}),\\ &\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{H}_F \odot \mathbf{H}_F\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^2),\\ &\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\end{aligned}$$

$$\begin{aligned}&\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^2),\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\\ &\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^2),\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{H}_F \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O(v^{3/2}),\\ &\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\left(\mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\end{aligned}$$

$$Tr\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{H}_F\right) = O\left(v\right),\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O\left(v^2\right),$$

$$\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O\left(v^2\right),$$

$$Tr\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right) = O(v),\ Tr\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right) = O(v)$$

$$Tr\left[\left(\mathbf{I} \odot \mathbf{M}_G\right)\mathbf{M}_G\right] = O(v),\ Tr\left[\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\mathbf{M}_G\right] = O(v)$$

$$\boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O\left(v\right),\ \boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O\left(v\right)$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\ \boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2})$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v^{3/2}),\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v),$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v),\ \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = O(v).$$

Proof. *Denote the* (t,r) *element of matrices* $\mathbf{M}_F$, $\mathbf{M}_G$ *and* $\mathbf{P}_G$ *by* $m_{F,tr}$, m_{tr} *and* p_{tr}, *respectively, and observe that the* (t,r) *element of* $\mathbf{H}_F = \mathbf{h}\mathbf{h}'$ *is* $\left(\sum_{l=1}^{T} m_{F,tl}\right)\left(\sum_{l=1}^{T} m_{F,rl}\right) = h_t h_r$. *The proofs below follow straightforwardly from application of Lemmas 8 and 9, and making use of Cauchy-Schwarz inequality, and the fact that* $\mathbf{M}_G\mathbf{M}_F = \mathbf{M}_G$, $\mathbf{M}_G\mathbf{H}_F = \mathbf{0}$. *First*

$$Tr\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G\right) = \sum_t h_t^4 m_{tt} \leq \sum_t h_t^4 = O\left(v\right),$$

as $0 \leq m_{tt} \leq 1$ *(by Lemma 8) and* $\sum_t h_t^4 = O(v)$*. Similarly, we have*

$$Tr\left(\mathbf{H}_F \odot \mathbf{M}_G\right) = \sum_t h_t^2 m_{tt} = O\left(v\right),\ Tr\left(\mathbf{H}_F \odot \mathbf{H}_F\right) = \sum_t h_t^4 = O\left(v\right),$$

and

$$Tr\left(\mathbf{M}_G \odot \mathbf{M}_G\right) = \sum_t m_{tt}^2 \leq \sum\nolimits_{t=1}^T m_{tt} = O\left(v\right).$$

Result (S.23) follows since $Tr\left(\mathbf{P}_G \odot \mathbf{P}_F\right) = \sum_{t=1}^T p_{F,tt} p_{tt} \leq \sum_{t=1}^T p_{tt} = m+1$*, recalling that* $0 \leq p_{F,tt} \leq 1$ *by (S.12).*

$$Tr\left(\mathbf{P}_G \odot \mathbf{H}_F\right) = \sum_t p_{tt}^2 h_t^2 \leq \sqrt{\sum\nolimits_{t=1}^T p_{tt}^2}\sqrt{\sum\nolimits_{t=1}^T h_t^4} = O(v^{1/2}),$$

since $0 \leq p_{tt}^2 \leq p_{tt} \leq 1$*, then* $\sum_{t=1}^T p_{tt}^2 \leq \sum_{t=1}^T p_{tt} = m+1$*. Further, using (S.17) in Lemma 9 and results in Lemma 8 we have*

$$\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T\right| \leq \sum_t \left|h_t^3\right| \sum_r \left|h_r m_{rr}\right| = O(v^2).$$

Similarly, noting that $\sum_r m_{tr}^2 = m_{tt}$ *and* $0 \leq m_{tt} \leq 1$ *and that* $0 \leq \sum_r m_{tr}^4 \leq \sum_r m_{tr}^2 \leq 1$*, we have*

$$\begin{aligned}\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T\right| &\leq \sum_t h_t^2 \sum_r \left|m_{tr} h_r^2\right| \leq \sum_t h_t^2 \sqrt{\sum_r m_{tr}^2}\sqrt{\sum_r h_r^4} \qquad \text{(S.29)}\\ &\leq \sum_t h_t^2 \sqrt{\sum_r h_r^4} = O(v^{3/2}),\end{aligned}$$

$$\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T\right| \leq \sum_t \left|m_{tt} h_t\right| \sum_r \left|m_{rr} h_r\right| \leq \sum_t \left|h_t\right| \sum_r \left|h_r\right| = O(v^2)$$

$$\begin{aligned}\left|\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T\right| &\leq \sum_t \sum_r \left|h_t h_r m_{tr}^2\right| \leq \sum_t \left|h_t\right| \sqrt{\sum_r m_{tr}^4}\sqrt{\sum_r h_r^2}\\ &\leq \sum_t \left|h_t\right| \sqrt{\sum_r h_r^2} = O\left(v^{3/2}\right).\end{aligned}$$

Also

$$\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = \boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = O(v^{3/2}). \qquad \text{(S.30)}$$

Using (S.18) we have

$$\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T = Tr\left(\mathbf{H}_F^2\right) = \left[Tr\left(\mathbf{H}_F\right)\right]^2 = O(v^2),$$
$$\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = Tr\left(\mathbf{H}_F \mathbf{M}_G\right) = 0,$$

and

$$\boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = Tr\left(\mathbf{M}_G\right) = v.$$

Also

$$Tr\left(\mathbf{M}_G \odot \mathbf{H}_F^2\right) = Tr\left(\mathbf{H}_F\right) Tr\left(\mathbf{M}_G \odot \mathbf{H}_F\right) = O\left(v^2\right),$$

and

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F^2\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = Tr\left(\mathbf{H}_F\right)\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = Tr\left(\mathbf{H}_F\right) Tr\left(\mathbf{M}_G \odot \mathbf{H}_F\right) = O\left(v^2\right).$$

Since $\sum_r h_r m_{tr} = 0$ *for any* $t \neq r$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = \sum_r \sum_t h_t^3 h_r m_{tr} = 0,$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{H}_F \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T = \sum_r \sum_t m_{tt} h_t h_r m_{tr} = 0.$$

Similarly to the above derivations, we have

$$Tr\left(\mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right) = \sum_t m_{tt}^2 h_t^2 = O\left(v\right),$$

$$\begin{aligned}&\left|\boldsymbol{\tau}_T'\left(\mathbf{H}_F \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T\right| \leq \sum_t \sum_u \sum_r \left|h_t^2 h_u^2 m_{ur} m_{rr}\right|\\ &\leq \sum_t \sum_u h_t^2 h_u^2 \sqrt{\sum_r m_{ur}^2} \leq \sum_t h_t^2 \sum_u h_u^2 = O\left(v^2\right),\end{aligned}$$

and noting $\mathbf{M}_G$ *and* $\mathbf{H}_F$ *are symmetric and* $\mathbf{M}_G\mathbf{H}_F = \mathbf{0}$, $\sum_t h_r h_t m_{tu}$ *for any* $t \neq r$ *and* $t \neq u$

$$\boldsymbol{\tau}_T' \left(\mathbf{H}_F \odot \mathbf{M}_G\right) \mathbf{H}_F \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T = \sum_t \sum_u \sum_r h_t h_u^2 m_{tu} h_r m_{rr} = 0$$

$$\boldsymbol{\tau}_T' \left(\mathbf{H}_F \odot \mathbf{M}_G\right) \mathbf{M}_G \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \boldsymbol{\tau}_T = \sum_t \sum_u \sum_r h_t h_u m_{tu} m_{ur} h_r^2 = 0$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{M}_G \odot \mathbf{M}_G\right) \mathbf{H}_F \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \boldsymbol{\tau}_T\right| \leq \sum_u \sum_t m_{tu}^2 \left|h_u\right| \sum_r \left|h_r^3\right|$$
$$= \sum_u m_{uu} \left|h_u\right| \sum_r \left|h_r^3\right| = O\left(v^2\right),$$

$$Tr\left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right) = \sum_t m_{tt}^2 h_t^4 = O\left(v\right),$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \mathbf{M}_G \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T\right| \leq \sum_t h_t^2 \sum_r \left|m_{tr}\right| m_{rr}$$
$$\leq \quad \sum_t h_t^2 \sqrt{\sum_r m_{tr}^2} \sqrt{\sum_r m_{rr}^2} \leq \sum_t h_t^2 \sqrt{\sum_r m_{rr}} = O\left(v^{3/2}\right),$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \mathbf{H}_F \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T\right| \leq \sum_t \left|m_{tt} h_t\right| \sum_r \left|h_r m_{rr}\right| = O\left(v^2\right),$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \mathbf{M}_G \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \boldsymbol{\tau}_T\right| \leq \sum_t \sum_r h_t^2 \left|m_{tr}\right| h_r^2$$
$$\leq \quad \sum_t h_t^2 \sqrt{\sum_r m_{tr}^2} \sqrt{\sum_r h_r^4} \leq \sum_t h_t^2 \sqrt{\sum_r h_r^4} = O\left(v^{3/2}\right),$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \mathbf{H}_F \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T\right| \leq \sum_t \left|h_t^3\right| \sum_r \left|h_r\right| m_{rr} = O\left(v^2\right),$$

$$Tr\left[\mathbf{H}_F^2 \left(\mathbf{M}_G \odot \mathbf{M}_G\right)\right] = Tr\left(\mathbf{H}_F\right) Tr\left[\mathbf{H}_F \left(\mathbf{M}_G \odot \mathbf{M}_G\right)\right] = Tr\left(\mathbf{H}_F\right) \boldsymbol{\tau}_T' \left(\mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T = O\left(v^{5/2}\right),$$

$$Tr\left[\mathbf{M}_G \left(\mathbf{H}_F \odot \mathbf{H}_F\right)\right] = \boldsymbol{\tau}_T' \left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T = O\left(v^{3/2}\right),$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \left(\mathbf{H}_F \odot \mathbf{M}_G\right) \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T\right| \leq \sum_r \sum_t \left|h_t^3 h_r m_{tr} m_{rr}\right|$$
$$\leq \quad \sum_t \left|h_t^3\right| \sqrt{\sum_r h_r^2} \sqrt{\sum_r m_{tr}^2} = O\left(v^{3/2}\right),$$

$$\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \left(\mathbf{M}_G \odot \mathbf{M}_G\right) \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \boldsymbol{\tau}_T = \sum_r \sum_t h_t^2 m_{tr}^2 h_r^2$$
$$\leq \sum_t h_t^2 \sqrt{\sum_r h_r^4} = O\left(v^{3/2}\right) = O\left(v^{3/2}\right),$$

$$\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \left(\mathbf{H}_F \odot \mathbf{H}_F\right) \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T = \sum_t m_{tt} h_t^2 \sum_r h_r^2 m_{rr} = O\left(v^2\right),$$

$$\boldsymbol{\tau}_T' \left(\mathbf{H}_F \odot \mathbf{H}_F \odot \mathbf{M}_G \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T = \sum_t \sum_r h_t^2 h_r^2 m_{tr}^2 = O\left(v^{3/2}\right)$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \mathbf{H}_F \left(\mathbf{I}_T \odot \mathbf{M}_G \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T\right| \leq \sum_t \left|h_t^3\right| \sum_r h_r^2 = O\left(v^2\right),$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{H}_F\right) \mathbf{M}_G \left(\mathbf{I}_T \odot \mathbf{H}_F \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T\right| \leq \sum_t h_t^2 \sum_r \left|m_{tr} h_r^2 m_{rr}\right|$$
$$\leq \quad \sum_t h_t^2 \sqrt{\sum_r h_r^4} = O\left(v^{3/2}\right),$$

$$\left|\boldsymbol{\tau}_T' \left(\mathbf{I}_T \odot \mathbf{M}_G\right) \mathbf{H}_F \left(\mathbf{I}_T \odot \mathbf{H}_F \odot \mathbf{M}_G\right) \boldsymbol{\tau}_T\right| \leq \sum_t \left|m_{tt} h_t\right| \sum_r \left|h_r^3 m_{rr}\right| = O\left(v^2\right),$$

$$\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{H}_F\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T\right|\leq\sum_t m_{tt}\sum_r\left|m_{tr}h_r^4\right|$$
$$\leq\quad\sum_t m_{tt}\sqrt{\sum_r m_{tr}^2}\sqrt{\sum_r h_r^8}\leq\sum_t m_{tt}\sqrt{\sum_r h_r^8}=O\left(v^{3/2}\right),$$
$$\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\left(\mathbf{H}_F\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right|\leq\sum_t\left|h_t^3\right|\sum_r\left|m_{tr}^2h_r\right|\leq\sum_t\left|h_t^3\right|\sqrt{\sum_r h_r^2}=O\left(v^{3/2}\right),$$
$$\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right|\leq\sum_t m_{tt}h_t^2\sum_r\left|m_{tr}h_r^2\right|\leq\sum_t h_t^2\sqrt{\sum_r h_r^4}=O\left(v^{3/2}\right),$$
$$Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{H}_F\right)=\sum_t h_{tt}^6=O(v)$$
$$\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T\right|\leq\sum_t\left|h_t^3\right|\sum_r\left|h_r^3\right|=O\left(v^2\right),$$
$$\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T=\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T=O\left(v^2\right),$$
$$Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\right)=\sum_t m_{tt}^3=O(v),\;Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\right)=\sum_t m_{tt}^4=O(v)$$
$$Tr\left[\left(\mathbf{I}\odot\mathbf{M}_G\right)\mathbf{M}_G\right]=\sum_t m_{tt}^2=O(v),\;\left|Tr\left[\left(\mathbf{M}_G\odot\mathbf{M}_G\right)\mathbf{M}_G\right]\right|\leq\sum_t\sum_r\left|m_{tr}^3\right|\leq\sum_t m_{tt}=O(v)$$

$$\begin{aligned}\left|\boldsymbol{\tau}_T'\left(\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right| &\leq \sum_t\sum_r\left|m_{tr}^3\right|\leq\sum_t m_{tt}=O\left(v\right),\\ \boldsymbol{\tau}_T'\left(\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T &= \sum_t\sum_r m_{tr}^4\leq\sum_t m_{tt}=O\left(v\right)\end{aligned}$$

$$\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right|\leq\sum_t\sum_r\left|m_{tt}m_{tr}m_{rr}\right|\leq\sum_t\sqrt{m_{tt}}\sqrt{\sum_r m_{rr}}=O(v^{3/2}),$$
$$\left|\boldsymbol{\tau}_T'\left(\mathbf{M}_G\odot\mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right|\leq\sum_t\sum_r\sum_u\left|m_{tu}^2m_{ur}m_{rr}\right|\leq\sum_r\sum_u\left|m_{uu}m_{ur}m_{rr}\right|=O(v^{3/2})$$

$$\begin{aligned}\left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right| &\leq \sum_t\sum_r\left|m_{tt}m_{tr}m_{rr}^2\right|=O(v^{3/2}),\\ \left|\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\left(\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right| &\leq \sum_t\sum_r\left|m_{tt}m_{tr}^3\right|=O(v),\end{aligned}$$

$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\left(\mathbf{M}_G\odot\mathbf{M}_G\right)\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T=\sum_t\sum_r m_{tt}m_{tr}^2m_{rr}\leq\sum_t\sum_r m_{tt}m_{tr}^2=O(v)$$
$$\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T=\sum_t m_{tt}^2=O(v).$$

■

Lemma 11 *Suppose that* $\boldsymbol{\xi}\sim IID(\mathbf{0},\mathbf{I}_T)$, *where* $\boldsymbol{\xi}=(\xi_1,\xi_2,...,\xi_T)'$, *with* $\gamma_1=E(\xi_t^3)$, $\gamma_2=E(\xi_t^4)-3$, $\gamma_3=E(\xi_t^5)-10\gamma_1$, $\gamma_4=E\left(\xi_t^6\right)-15\gamma_2-10\gamma_1^2-15$ *and* $\gamma_6=E\left(\xi_t^8\right)-28\gamma_4-56\gamma_3\gamma_1-35\gamma_2^2-210\gamma_2-280\gamma_1^2-105$ *for all* $t=1,2,...,T$. *Consider the matrices* $\mathbf{M}_G$, $\mathbf{P}_G$ *and* $\mathbf{H}_F=\mathbf{hh}'$, *defined by (S.2) and (S.1),* $w_T=\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T$ *and* $v=T-m-1$. *Then, under Assumptions 1 and 4, we have*

$$E\left(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi}\right)=Tr(\mathbf{H}_F)=w_T,\;E\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)=Tr(\mathbf{M}_G)=v,$$
$$E\left[\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)^2\right]=\gamma_2Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)+v\left(v+2\right)=v\left(v+2\right)+O\left(v\right),$$
$$E\left[\left(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)\right]=\gamma_2Tr\left(\mathbf{M}_G\odot\mathbf{H}_F\right)+v(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T)=vw_T+O\left(v\right),$$
$$E\left[\left(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi}\right)^2\right]=\gamma_2Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\right)+3\left(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T\right)^2=3w_T^2+O\left(v\right),$$

$$\begin{aligned}E\left[\left(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi}\right)^2\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)\right] &= \gamma_4Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\right)+2\gamma_2Tr\left(\mathbf{H}_F\right)Tr\left(\mathbf{H}_F\odot\mathbf{M}_G\right)\\ &+\gamma_2Tr\left(\mathbf{M}_G\right)Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\right)+4\gamma_2Tr\left[\mathbf{M}_G\odot\mathbf{H}_F^2\right]+4\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right]\\ &+2\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T\right]+4\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right]+3\left[Tr\left(\mathbf{H}_F\right)\right]^2Tr\left(\mathbf{M}_G\right)\\ &= 3w_T^2v+O\left(v^2\right),\end{aligned}$$

$$E\left[\left(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi}\right)\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)^2\right]=\gamma_4 Tr\left(\mathbf{H}_F\odot\mathbf{M}_G\odot\mathbf{M}_G\right)+\gamma_2 Tr\left(\mathbf{H}_F\right)Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)$$
$$+2\gamma_2 Tr\left(\mathbf{M}_G\right)Tr\left(\mathbf{H}_F\odot\mathbf{M}_G\right)+4\gamma_2 Tr\left(\mathbf{H}_F\odot\mathbf{M}_G\right)+4\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right]$$
$$+2\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right]+4\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right]$$
$$+Tr\left(\mathbf{H}_F\right)\left[Tr\left(\mathbf{M}_G\right)\right]^2+2Tr\left(\mathbf{H}_F\right)Tr\left(\mathbf{M}_G\right)=w_T v^2+O\left(v^2\right),$$

$$E\left[\left(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi}\right)^3\right]=\gamma_4 Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{H}_F\right)+15\gamma_2 Tr\left(\mathbf{H}_F\right)Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\right)$$
$$+6\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T\right]+4\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T\right]+15\left[Tr\left(\mathbf{H}_F\right)\right]^3$$
$$=15w_T^3+O\left(v^2\right),$$

$$E\left[\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)^3\right]=\gamma_4 Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\right)+3\gamma_2 v Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)$$
$$+12\gamma_2 Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)+6\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right]$$
$$+4\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\right]+v^3+6v^2+8v=v^3+O(v^2)$$

$$E\left[\left(\boldsymbol{\varepsilon}'\mathbf{H}_F\boldsymbol{\varepsilon}\right)^2\left(\boldsymbol{\varepsilon}'\mathbf{M}_G\boldsymbol{\varepsilon}\right)^2\right]=\left[Tr\left(\mathbf{H}_F\right)\right]^2\left[Tr\left(\mathbf{M}_G\right)\right]^2$$
$$+2\left[Tr\left(\mathbf{H}_F\right)\right]^2Tr\left(\mathbf{M}_G\right)+2\left[Tr\left(\mathbf{M}_G\right)\right]^2Tr\left(\mathbf{H}_F^2\right)+4Tr\left(\mathbf{H}_F^2\right)Tr\left(\mathbf{M}_G\right)$$
$$+\gamma_2 f_{\gamma_2}+\gamma_4 f_{\gamma_4}+\gamma_6 f_{\gamma_6}+\gamma_1^2 f_{\gamma_1^2}+\gamma_2^2 f_{\gamma_2^2}+\gamma_1\gamma_3 f_{\gamma_1\gamma_3}$$
$$=3w_T^2v^2+O\left(v^3\right),$$

where

$$\begin{aligned}
f_{\gamma_2} &= \left[Tr\left(\mathbf{H}_F\right)\right]^2Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)+4Tr\left(\mathbf{H}_F\right)Tr\left(\mathbf{M}_G\right)Tr\left(\mathbf{H}_F\odot\mathbf{M}_G\right)+\left[Tr\left(\mathbf{M}_G\right)\right]^2Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\right)\\
&\quad+2\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)+2\boldsymbol{\tau}_T'\left(\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\right)\\
&\quad+8Tr\left(\mathbf{H}_F\right)Tr\left(\mathbf{H}_F\odot\mathbf{M}_G\right)+8Tr\left(\mathbf{M}_G\right)Tr\left(\mathbf{M}_G\odot\mathbf{H}_F^2\right)+16\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F^2\right)\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\\
&= O\left(v^3\right),
\end{aligned}$$

$$\begin{aligned}
f_{\gamma_4} &= 2Tr\left(\mathbf{H}_F\right)Tr\left(\mathbf{H}_F\odot\mathbf{M}_G\odot\mathbf{M}_G\right)+2Tr\left(\mathbf{M}_G\right)Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\right)\\
&\quad+4Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\right)+4Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\odot\mathbf{H}_F^2\right)\\
&= O\left(v^2\right),
\end{aligned}$$

$$f_{\gamma_6}=Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\odot\mathbf{M}_G\right)=O\left(v\right),$$

$$\begin{aligned}
f_{\gamma_1^2} &= 8\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{H}_F\right)+4\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{H}_F\right)\\
&\quad 4\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T Tr\left(\mathbf{M}_G\right)+8\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{M}_G\right)\\
&\quad 8\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T+8\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{H}_F^2\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\\
&\quad 8\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{H}_F\right)+8\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{M}_G\right)\\
&\quad+16\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T+32\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\\
&\quad+32\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T+16\boldsymbol{\tau}_T'\left(\mathbf{M}_G\odot\mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T\\
&\quad+16Tr\left[\mathbf{H}_F^2\left(\mathbf{M}_G\odot\mathbf{M}_G\right)\right]+16Tr\left[\mathbf{M}_G\left(\mathbf{H}_F\odot\mathbf{H}_F\right)\right]\\
&= O\left(v^3\right),
\end{aligned}$$

$$\begin{aligned}
f_{\gamma_2^2} &= Tr\left(\mathbf{H}_F\odot\mathbf{H}_F\right)Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)+2\left[Tr\left(\mathbf{H}_F\odot\mathbf{M}_G\right)\right]^2\\
&\quad+16\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\left(\mathbf{H}_F\odot\mathbf{M}_G\right)\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\\
&\quad+4\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\left(\mathbf{M}_G\odot\mathbf{M}_G\right)\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T\\
&\quad+4\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\left(\mathbf{H}_F\odot\mathbf{H}_F\right)\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\\
&\quad+8\boldsymbol{\tau}_T'\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\\
&= O\left(v^2\right),
\end{aligned}$$

$$\begin{aligned}
f_{\gamma_1\gamma_3} &= 4\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T+8\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{H}_F\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\\
&\quad+8\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{H}_F\left(\mathbf{I}_T\odot\mathbf{H}_F\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T+4\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T\odot\mathbf{H}_F\odot\mathbf{H}_F\right)\boldsymbol{\tau}_T\\
&\quad+16\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{H}_F\right)\left(\mathbf{H}_F\odot\mathbf{M}_G\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T+16\boldsymbol{\tau}_T'\left(\mathbf{I}_T\odot\mathbf{M}_G\right)\left(\mathbf{H}_F\odot\mathbf{H}_F\odot\mathbf{M}_G\right)\boldsymbol{\tau}_T\\
&= O\left(v^2\right),
\end{aligned}$$

and

$$E\left[\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)^4\right] = [Tr\left(\mathbf{M}_G\right)]^4 + 12\left[Tr\left(\mathbf{M}_G\right)\right]^2 Tr\left(\mathbf{M}_G\right) + 12\left[Tr\left(\mathbf{M}_G\right)\right]^2$$
$$+32Tr\left(\mathbf{M}_G\right)Tr\left(\mathbf{M}_G\right) + 48Tr\left(\mathbf{M}_G\right)$$
$$\gamma_2 g_{\gamma_2} + \gamma_4 g_{\gamma_4} + \gamma_6 g_{\gamma_6} + \gamma_1^2 g_{\gamma_1^2} + \gamma_2^2 g_{\gamma_2^2} + \gamma_1\gamma_3 g_{\gamma_1\gamma_3}$$
$$= v^4 + O(v^3),$$

with

$$\begin{aligned} g_{\gamma_2} &= 6\left[Tr(\mathbf{M}_G)\right]^2 Tr\left(\mathbf{M}_G \odot \mathbf{M}_G\right) + 12\boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{M}_G \odot \mathbf{M}_G\right) \\ &\quad +48Tr(\mathbf{M}_G)Tr\left(\mathbf{M}_G \odot \mathbf{M}_G\right) + 96Tr\left[\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{M}_G\right] + 48\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T, \end{aligned}$$

$$g_{\gamma_4} = 4Tr(\mathbf{M}_G)Tr\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right) + 24Tr\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right),$$

$$g_{\gamma_6} = Tr\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right),$$

$$\begin{aligned} g_{\gamma_1^2} &= 24\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{M}_G\right) + 48\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T \\ &\quad +16\boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T Tr\left(\mathbf{M}_G\right) + 96\boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T \\ &\quad +96Tr\left[\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\mathbf{M}_G\right], \end{aligned}$$

$$\begin{aligned} g_{\gamma_2^2} &= 3\left[Tr\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\right]^2 + 24\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{M}_G \odot \mathbf{M}_G\right)\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T \\ &\quad +8\boldsymbol{\tau}_T'\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T, \end{aligned}$$

$$g_{\gamma_1\gamma_3} = 24\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T + 32\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\left(\mathbf{M}_G \odot \mathbf{M}_G \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T.$$

Proof. These results are obtained by using the results established in Lemmas 6 and 10, together with the fact that $E(\xi_t^r)$ for $r = 1, 2, ..., 8$ are time invariant (which is ensured by Assumption 4), and noting that $\mathbf{M}_G\mathbf{H}_F = \mathbf{0}$ (since $\mathbf{M}_F\mathbf{M}_G = \mathbf{M}_G$ and $\mathbf{M}_G\boldsymbol{\tau}_T = \mathbf{0}$), $\mathbf{H}_F^j = \mathbf{H}_F\left[Tr(\mathbf{H}_F)\right]^{j-1}$ for $j > 1$. ■

Lemma 12 *Suppose that $\boldsymbol{\xi} \sim IID(\mathbf{0}, \mathbf{I}_T)$, where $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_T)'$, with $\gamma_1 = E(\xi_t^3)$, $\gamma_2 = E(\xi_t^4) - 3$, $\gamma_3 = E(\xi_t^5) - 10\gamma_1$ and $\gamma_4 = E\left(\xi_t^6\right) - 15\gamma_2 - 10\gamma_1^2 - 15$ for all $t = 1, 2, ..., T$. Consider the matrices $\mathbf{M}_G$, $\mathbf{P}_G$ and $\mathbf{H}_F$, defined by (S.2) and (S.1), and $v = T - m - 1$. Then, under Assumptions 1 and 4 we have*

$$\kappa_2 = E\left[(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi})^2\right] - \left[E(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi})\right]^2 = \gamma_2 Tr\left(\mathbf{M}_G \odot \mathbf{M}_G\right) + 2v = O(v), \tag{S.31}$$

$$\begin{aligned} \kappa_{11} &= E[(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi})\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)] - E(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi})E(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}) \\ &= \gamma_2 Tr\left[\left(\mathbf{M}_G \odot \mathbf{H}_F\right)\right] = O(v), \end{aligned} \tag{S.32}$$

and

$$\begin{aligned} \kappa_{21} &= E\left[\left(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi}\right)^2\left(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}\right)\right] - E[(\boldsymbol{\xi}'\mathbf{H}_F\boldsymbol{\xi})^2]E(\boldsymbol{\xi}'\mathbf{M}_G\boldsymbol{\xi}) \\ &= 6\gamma_2\left(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T\right)Tr\left(\mathbf{M}_G \odot \mathbf{H}_F\right) + 4\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{H}_F\left(\mathbf{I}_T \odot \mathbf{M}_G\right)\boldsymbol{\tau}_T\right] \\ &\quad +6\gamma_1^2\left[\boldsymbol{\tau}_T'\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\mathbf{M}_G\left(\mathbf{I}_T \odot \mathbf{H}_F\right)\boldsymbol{\tau}_T\right] + O(v) = O(v^2). \end{aligned} \tag{S.33}$$

Proof. The results (S.31) and (S.32) follow immediately from Lemmas 11 and 10, together with the fact that $E(\xi_t^r)$ for $r = 1, 2, 3, 4$ are time invariant, which is ensured by Assumption 4. The result (S.33) follows using Lemmas 11 and 10 and the equality (S.30), noting that $Tr\left(\mathbf{H}_F^2\right) = \left[Tr\left(\mathbf{H}_F\right)\right]^2$, and $Tr\left(\mathbf{M}_G \odot \mathbf{H}_F^2\right) = Tr\left(\mathbf{H}_F\right)Tr\left(\mathbf{M}_G \odot \mathbf{H}_F\right)$, since $\mathbf{H}_F^2 = Tr\left(\mathbf{H}_F\right)\mathbf{H}_F$. ■

Lemma 13 *Suppose $\boldsymbol{\varepsilon}_t = (\varepsilon_{it})$, where $\varepsilon_{it} \sim IID\left(0, 1\right)$, with $\gamma_{1,\varepsilon} = E(\varepsilon_{it}^3)$, $\gamma_{2,\varepsilon} = E(\varepsilon_{it}^4) - 3$, $\gamma_{3,\varepsilon} = E(\varepsilon_{it}^5) - 10\gamma_{1,\varepsilon}$ and $\gamma_{4,\varepsilon} = E\left(\varepsilon_{it}^6\right) - 15\gamma_{2,\varepsilon} - 10\gamma_{1,\varepsilon}^2 - 15$, and $\mathbf{q}_i = (q_{i\ell})$. Then,*

$$E\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_i'\boldsymbol{\varepsilon}_t\right) = \sum\nolimits_\ell q_{i\ell}^2,\ E\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_j'\boldsymbol{\varepsilon}_t\right) = \sum\nolimits_\ell q_{i\ell}q_{j\ell}, \tag{S.34}$$

$$E\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_i'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{q}_i\right) = \gamma_{1,\varepsilon}\sum\nolimits_\ell q_{i\ell}^3,\ E\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_j\mathbf{q}_j'\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\mathbf{q}_i\right) = \gamma_{1,\varepsilon}\sum\nolimits_\ell q_{i\ell}q_{j\ell}^2,$$

$$E\left[\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_i'\boldsymbol{\varepsilon}_t\right)^2\right] = \gamma_{2,\varepsilon}\left(\sum\nolimits_\ell q_{i\ell}^4\right) + 3\left(\sum\nolimits_\ell q_{i\ell}^2\right)^2,$$

$$E\left[\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_j'\boldsymbol{\varepsilon}_t\right)^2\right] = \gamma_{2,\varepsilon}\left(\sum\nolimits_\ell q_{i\ell}^2q_{j\ell}^2\right) + \left(\sum\nolimits_\ell q_{i\ell}^2\right)\left(\sum\nolimits_\ell q_{j\ell}^2\right) + 2\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}\right)^2,$$

$$E\left[\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_i'\boldsymbol{\varepsilon}_t\right)\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_j'\boldsymbol{\varepsilon}_t\right)\right] = \gamma_{2,\varepsilon}\left(\sum\nolimits_\ell q_{i\ell}^3q_{j\ell}\right) + 3\left(\sum\nolimits_\ell q_{i\ell}^2\right)\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}\right),$$

$$\begin{aligned} E\left[\mathbf{q}_i'\boldsymbol{\varepsilon}_t\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_i'\boldsymbol{\varepsilon}_t\right)\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_j\mathbf{q}_j'\boldsymbol{\varepsilon}_t\right)\right] &= \gamma_{3,\varepsilon}\sum\nolimits_\ell q_{i\ell}^3q_{j\ell}^2 + \gamma_{1,\varepsilon}\left[6\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}\right)\left(\sum\nolimits_\ell q_{i\ell}^2q_{j\ell}\right)\right. \\ &\quad \left. +3\left(\sum\nolimits_\ell q_{i\ell}^2\right)\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}^2\right) + \left(\sum\nolimits_\ell q_{j\ell}^2\right)\left(\sum\nolimits_\ell q_{i\ell}^3\right)\right], \end{aligned} \tag{S.35}$$

$$E\left[\mathbf{q}_i'\boldsymbol{\varepsilon}_t\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_j\mathbf{q}_j'\boldsymbol{\varepsilon}_t\right)^2\right] = \gamma_{3,\varepsilon}\sum\nolimits_\ell q_{i\ell}q_{j\ell}^4 + \gamma_{1,\varepsilon}\left[4\left(\sum\nolimits_\ell q_{i\ell}\right)\left(\sum\nolimits_\ell q_{j\ell}^4\right)\right.$$
$$\left.+4\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}\right)\left(\sum\nolimits_\ell q_{j\ell}^3\right)+2\left(\sum\nolimits_\ell q_{j\ell}^2\right)\left(\sum\nolimits_\ell q_{j\ell}^2 q_{i\ell}\right)\right], \tag{S.36}$$

$$E\left[\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_i'\boldsymbol{\varepsilon}_t\right)^2\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_j\mathbf{q}_j'\boldsymbol{\varepsilon}_t\right)\right] = \gamma_{4,\varepsilon}\left(\sum\nolimits_\ell q_{i\ell}^4 q_{j\ell}^2\right)+6\gamma_{2,\varepsilon}\left(\sum\nolimits_\ell q_{i\ell}^2\right)\left(\sum\nolimits_\ell q_{i\ell}^2 q_{j\ell}^2\right) \tag{S.37}$$
$$+\gamma_{2,\varepsilon}\left(\sum\nolimits_\ell q_{i\ell}^4\right)\left(\sum\nolimits_\ell q_{j\ell}^2\right)+8\gamma_{2,\varepsilon}\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}\right)\left(\sum\nolimits_\ell q_{i\ell}^3 q_{j\ell}\right)+4\gamma_{1,\varepsilon}^2\left(\sum\nolimits_\ell q_{i\ell}^3\right)\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}^2\right)$$
$$+6\gamma_{1,\varepsilon}^2\left(\sum\nolimits_\ell q_{i\ell}^2 q_{j\ell}\right)^2+3\left(\sum\nolimits_\ell q_{i\ell}^2\right)^2\left(\sum\nolimits_\ell q_{j\ell}^2\right)+12\left(\sum\nolimits_\ell q_{i\ell}^2\right)\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}\right)^2, \tag{S.38}$$

$$E\left[\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_i'\boldsymbol{\varepsilon}_t\right)\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_i\mathbf{q}_j'\boldsymbol{\varepsilon}_t\right)\left(\boldsymbol{\varepsilon}_t'\mathbf{q}_j\mathbf{q}_j'\boldsymbol{\varepsilon}_t\right)\right] = \gamma_{4,\varepsilon}\left(\sum\nolimits_\ell q_{i\ell}^3 q_{j\ell}^3\right)+5\gamma_{2,\varepsilon}\sum\nolimits_\ell q_{i\ell}^2\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}^3\right) \tag{S.39}$$
$$+5\gamma_{2,\varepsilon}\sum\nolimits_\ell q_{i\ell}q_{j\ell}\left(\sum\nolimits_\ell q_{i\ell}^2 q_{j\ell}^2\right)+5\gamma_{2,\varepsilon}\sum\nolimits_\ell q_{j\ell}^2\left(\sum\nolimits_\ell q_{i\ell}^3 q_{j\ell}\right)+2\gamma_{1,\varepsilon}^2\left(\sum\nolimits_\ell q_{i\ell}^3\right)\left(\sum\nolimits_\ell q_{j\ell}^3\right)$$
$$+2\gamma_{1,\varepsilon}^2\left(\sum\nolimits_\ell q_{i\ell}^2 q_{j\ell}\right)\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}^2\right)+2\gamma_{1,\varepsilon}^2\left(\sum\nolimits_\ell q_{i\ell}^2 q_{j\ell}\right)\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}^2\right)$$
$$+4\gamma_{1,\varepsilon}^2\left(\sum\nolimits_\ell q_{i\ell}^2 q_{j\ell}\right)^2+2\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}\right)^3+13\left(\sum\nolimits_\ell q_{i\ell}^2\right)\left(\sum\nolimits_\ell q_{j\ell}^2\right)\left(\sum\nolimits_\ell q_{i\ell}q_{j\ell}\right).$$

Proof. Applying Lemma 6, the results follow. ■

Lemma 14 *Let $\tilde{\gamma}_{is}=\gamma_{is}/\sigma_{ii}^{1/2}$ and $\tilde{q}_{\eta,i\ell}=q_{\eta,i\ell}/\sigma_{\eta,ii}^{1/2}$, where γ_{is} is the s^{th} element of the $k\times 1$ vector of factor loadings, γ_i, defined by (46), $\sigma_{ii}=\gamma_i'\gamma_i+\sigma_{\eta,ii}$, and $q_{\eta,i\ell}$ is the (i,ℓ) element of $\mathbf{Q}_\eta$, where $\mathbf{Q}_\eta$ is defined by (50).*

(a) For any finite M, ν_p and r_p, $p=1,2,...,M$, at least one of ν_p is non-zero and at least one of r_p is non-zero, then

$$\sum_{i=1}^N\sum_{j=1}^N\prod_p^M\left(\sum\nolimits_{s=1}^k\tilde{\gamma}_{is}^{\nu_p}\tilde{\gamma}_{js}^{r_p}\right)=O\left(N^{2\delta_\gamma}\right).$$

(b) Further, for any finite L, ν_h and r_h, $h=1,2,...,L$, where $\nu_h\geq 0$ and $r_h\geq 0$,

$$\sum_{i=1}^N\sum_{j=1}^N\prod_h^L\left(\sum\nolimits_{\ell=1}^N\tilde{q}_{\eta,i\ell}^{\nu_h}\tilde{q}_{\eta,j\ell}^{r_h}\right)\prod_p^M\left(\sum\nolimits_s\tilde{\gamma}_{is}^{\nu_p}\tilde{\gamma}_{js}^{r_p}\right)=O\left(N^{2\delta_\gamma}\right).$$

(c) Further, for any finite $u\geq 1$ and $\nu\geq 1$,

$$\sum_{i=1}^N\sum_{j=1}^N\left(\sum\nolimits_{\ell=1}^N\tilde{q}_{\eta,i\ell}^{u}\tilde{q}_{\eta,j\ell}^{\nu}\right)\prod_h^L\left(\sum\nolimits_{\ell=1}^N\tilde{q}_{\eta,i\ell}^{\nu_h}\tilde{q}_{\eta,j\ell}^{r_h}\right)=O\left(N\right).$$

Proof. Consider part (a) first. Noting that $|\tilde{\gamma}_{is}|\leq 1$ for all i and s, $|\tilde{\gamma}_{is}|^{\nu_p}\leq|\tilde{\gamma}_{is}|$ and $\sup_s\sum_{i=1}^N|\tilde{\gamma}_{is}|=O\left(N^{\delta_\gamma}\right)$ by (47), we have

$$\sum_{i=1}^N\sum_{j=1}^N\prod_p^M\left|\sum\nolimits_s\tilde{\gamma}_{is}^{\nu_p}\tilde{\gamma}_{js}^{r_p}\right|\leq\sum_{i=1}^N\sum_{j=1}^N\prod_p^M\sum\nolimits_s|\tilde{\gamma}_{is}|^{\nu_p}|\tilde{\gamma}_{js}|^{r_p}$$
$$\leq\sum_{i=1}^N\sum_{j=1}^N\prod_p^M\sum\nolimits_s|\tilde{\gamma}_{is}||\tilde{\gamma}_{js}|\leq\sum_{i=1}^N\sum_{j=1}^N\prod_p^M k\left(\sup_s|\tilde{\gamma}_{is}|\sup_s|\tilde{\gamma}_{js}|\right)$$
$$\leq\sum_{i=1}^N\sum_{j=1}^N k^M\left(\sup_s|\tilde{\gamma}_{is}|\sup_s|\tilde{\gamma}_{js}|\right)^M\leq k^M\left(\sup_s\sum_{i=1}^N|\tilde{\gamma}_{is}|\right)\left(\sup_s\sum_{j=1}^N|\tilde{\gamma}_{js}|\right)$$
$$=O\left(N^{2\delta_\gamma}\right),$$

as required. Now consider part (b). By Cauchy-Schwarz

$$\sum_{i=1}^N\sum_{j=1}^N\prod_h^L\left|\sum\nolimits_{\ell=1}^N\tilde{q}_{\eta,i\ell}^{\nu_h}\tilde{q}_{\eta,j\ell}^{r_h}\right|\prod_p^M\left|\sum\nolimits_{s=1}^k\tilde{\gamma}_{is}^{\nu_p}\tilde{\gamma}_{js}^{r_p}\right|$$
$$\leq\sum_{i=1}^N\sum_{j=1}^N\prod_h^L\left|\sqrt{\sum\nolimits_{\ell=1}^N|\tilde{q}_{\eta,i\ell}|^{2\nu_h}}\sqrt{\sum\nolimits_{\ell=1}^N|\tilde{q}_{\eta,j\ell}|^{2r_h}}\right|\prod_p^M\left|\sum\nolimits_{s=1}^k\tilde{\gamma}_{is}^{\nu_p}\tilde{\gamma}_{js}^{r_p}\right|,$$

but, as $\sum_{\ell=1}^{N}|\tilde{q}_{\eta,i\ell}|^{2}=1$, $\sum_{\ell=1}^{N}|\tilde{q}_{\eta,i\ell}|^{2}\geq\sum_{\ell=1}^{N}|\tilde{q}_{\eta,i\ell}|^{r}$ for $r\geq 2$, together with part (a) we have

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N}\prod_{h}^{L}\left|\sqrt{\sum\nolimits_{\ell=1}^{N}|\tilde{q}_{\eta,i\ell}|^{2\nu_h}}\sqrt{\sum\nolimits_{\ell=1}^{N}|\tilde{q}_{\eta,j\ell}|^{2r_h}}\right|\prod_{p}^{M}\left|\sum\nolimits_{s=1}^{k}\tilde{\gamma}_{is}^{\nu_p}\tilde{\gamma}_{js}^{r_p}\right|\\ \leq\ &\sum_{i=1}^{N}\sum_{j=1}^{N}k^{M}\left(\sup_{s}\sum_{i=1}^{N}|\tilde{\gamma}_{is}|\right)\left(\sup_{s}\sum_{j=1}^{N}|\tilde{\gamma}_{js}|\right)=O\left(N^{2\delta_\gamma}\right).\end{aligned}$$

Observe that the result holds when all of ν_h and/or all of r_h are zero. Now consider part (c). Similarly, using Cauchy-Schwarz

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\sum\nolimits_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^{u}\tilde{q}_{\eta,j\ell}^{\nu}\right|\prod_{h}^{L}\left|\sum\nolimits_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^{\nu_h}\tilde{q}_{\eta,j\ell}^{r_h}\right|\\ \leq\ &\sum_{i=1}^{N}\sum_{j=1}^{N}\sum\nolimits_{\ell=1}^{N}|\tilde{q}_{\eta,i\ell}|^{u}|\tilde{q}_{\eta,j\ell}|^{\nu}\prod_{h}^{L}\sqrt{\sum\nolimits_{\ell=1}^{N}|\tilde{q}_{\eta,i\ell}|^{2\nu_h}}\sqrt{\sum\nolimits_{\ell=1}^{N}|\tilde{q}_{\eta,j\ell}|^{2r_h}}\\ \leq\ &\sum\nolimits_{\ell=1}^{N}\sum_{i=1}^{N}|\tilde{q}_{\eta,i\ell}|^{u}\sum_{j=1}^{N}|\tilde{q}_{\eta,j\ell}|^{\nu}\end{aligned}$$

but $\sum_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^{2}=1$ implies $|\tilde{q}_{\eta,i\ell}|\leq 1$, hence, $|\tilde{q}_{\eta,i\ell}|^{r}\leq|\tilde{q}_{\eta,i\ell}|$ for $r\geq 1$, we have

$$\begin{aligned}\sum\nolimits_{\ell=1}^{N}\sum_{i=1}^{N}|\tilde{q}_{\eta,i\ell}|^{u}\sum_{j=1}^{N}|\tilde{q}_{\eta,j\ell}|^{\nu}&\leq\sum\nolimits_{\ell=1}^{N}\sum_{i=1}^{N}|\tilde{q}_{\eta,i\ell}|\sum_{j=1}^{N}|\tilde{q}_{\eta,j\ell}|\\&\leq N\left(\sup_{\ell}\sum_{i=1}^{N}|\tilde{q}_{\eta,i\ell}|\right)\left(\sup_{\ell}\sum_{j=1}^{N}|\tilde{q}_{\eta,j\ell}|\right)=O(N),\end{aligned}$$

as required, where the final line follows from $\sup_{\ell}\sum_{i=1}^{N}|\tilde{q}_{\eta,i\ell}|\leq K$ for all i (by (51)). ■

Lemma 15 *Consider the regression model (2), and suppose that Assumptions 1 and 4 hold. Let* $z_{\eta,i}^{2}=\boldsymbol{\eta}_i'\mathbf{H}_F\boldsymbol{\eta}_i/(w_T\sigma_{\eta,ii})$ *and* $X_{\eta,i}=\boldsymbol{\eta}_i'\mathbf{M}_G\boldsymbol{\eta}_i/(v\sigma_{\eta,ii})$, *where* $\boldsymbol{\eta}_i=(\eta_{i1},\eta_{i2},...,\eta_{iT})'$, $w_T=\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T$, *and* $\mathbf{H}_F=(h_th_{t'})$, $\mathbf{M}_F$ *and* $\mathbf{M}_G$ *are defined by (S.2), and* $v=T-m-1$. *Denote* $\tilde{\eta}_{it}=\eta_{it}/\sigma_{\eta,ii}^{1/2}$, *and set* $\mathbf{D}_{\sigma_\eta}=diag(\sigma_{\eta,ii})$, *so that* $\mathbf{D}_{\sigma_\eta}^{-1/2}\boldsymbol{\eta}_t=\tilde{\boldsymbol{\eta}}_t=\tilde{\mathbf{Q}}_\eta\boldsymbol{\varepsilon}_{\eta,t}$, *where* $\tilde{\mathbf{Q}}_\eta=\mathbf{D}_{\sigma_\eta}^{-1/2}\mathbf{Q}_\eta$, *and* $\tilde{\mathbf{q}}_{\eta,i}'=(\tilde{q}_{\eta,i1},\tilde{q}_{\eta,i2},...,\tilde{q}_{\eta,iN})$ *is the* i^{th} *row of* $\tilde{\mathbf{Q}}_\eta$. *Also, set* $\rho_{\eta,ij}=Cov(\tilde{\eta}_{it},\tilde{\eta}_{jt})$, $\gamma_{1,\varepsilon_\eta}=E(\varepsilon_{\eta,it}^{3})$ *and* $\gamma_{2,\varepsilon_\eta}=E(\varepsilon_{\eta,it}^{4})-3$. *Then we have*

$$E\left(z_{\eta,i}^{2}\right)=1,\ E\left(X_{\eta,i}\right)=1, \tag{S.40}$$

$$\varphi_{\eta,ij}=E\left(\tilde{\eta}_{it}^{2}\tilde{\eta}_{jt}^{2}\right)=1+2\rho_{\eta,ij}^{2}+\gamma_{2,\varepsilon_\eta}\sum_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}^{2}, \tag{S.41}$$

$$E\left(z_{\eta,i}^{2}z_{\eta,j}^{2}\right)=\left(1+2\rho_{\eta,ij}^{2}\right)+\gamma_{2,\varepsilon_\eta}\left(\frac{\sum_t h_t^4}{w_T^2}\right)\left(\sum_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}^{2}\right), \tag{S.42}$$

$$E\left(X_{\eta,i}X_{\eta,j}\right)=1+\frac{2\rho_{\eta,ij}^{2}}{v}+\gamma_{2,\varepsilon_\eta}\left(\frac{\sum_t m_{tt}^2}{v_2}\right)\sum_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}^{2}, \tag{S.43}$$

$$E\left(z_{\eta,i}^{2}X_{\eta,i}\right)=1+\frac{\sum_t h_t^2m_{tt}}{vw_T}\left(\gamma_{2,\varepsilon_\eta}\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{4}\right), \tag{S.44}$$

$$\begin{aligned}E\left(z_{\eta,i}^{2}X_{\eta,i}z_{\eta,j}^{2}\right)=\ &\left(1+2\rho_{\eta,ij}^{2}\right)+\frac{\sum_t h_t^2m_{tt}}{vw_T}\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{4}\right)+\frac{\sum_t h_t^4}{w_T^2}\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}^{2}\right)\\&+\left(\frac{1}{w_T^2v}\sum_t\sum_r h_t^3h_rm_{rr}+3\frac{1}{w_T^2v}\sum_t\sum_r h_t^2h_r^2m_{tr}\right)\gamma_{1,\varepsilon_\eta}^{2}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^{2}\right)\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{3}\right)\\&+2\left(\frac{1}{w_T^2v}\sum_t\sum_r h_t^3h_rm_{rr}+2\frac{1}{w_T^2v}\sum_t\sum_r h_t^2h_r^2m_{tr}\right)\gamma_{1,\varepsilon_\eta}^{2}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}\right)^{2}\\&+\left(\frac{1}{w_Tv}\sum_t h_t^2m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}^{2}\right)\right]\\&+4\rho_{\eta,ij}\left(\frac{1}{w_Tv}\sum_t h_t^2m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{3}\tilde{q}_{\eta,j\ell}\right)\right]+O\left(T^{-2}\right),\end{aligned} \tag{S.45}$$

$$
\begin{aligned}
E\left(z_{\eta,i}^{2}X_{\eta,i}z_{\eta,j}^{2}X_{\eta,j}\right) \;=\;& \left(1+2\rho_{\eta,ij}^{2}\right)+\left(\frac{\sum_t h_t^2 m_{tt}}{vw_T}\right)\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,j\ell}^{4}+\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{4}\right)+\frac{\sum_t h_t^4}{w_T^2}\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}^{2}\right)\\
&+2\rho_{\eta,ij}^{2}\left(-\frac{1}{w_T^2}\sum_t h_t^4-\frac{18}{vw_T}\sum_t h_t^2 m_{tt}-\frac{2}{v^2}\sum_t m_{tt}^2+\frac{1}{v}\right)\\
&+2\rho_{\eta,ij}^{4}\left(\frac{2}{v}-\frac{2}{v^2}\sum_t m_{tt}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{2}{v^2w_T}\sum_t\sum_r h_r h_t m_{rr}m_{tt}+\frac{1}{v^2w_T}\sum_t\sum_r h_t^2 m_{rr}m_{tr}+\frac{2}{vw_T^2}\sum_t\sum_r h_r^3 h_t m_{tt}\right)\\
&\times\gamma_{1,\varepsilon_\eta}^{2}\left[\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}\right)\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,j\ell}^{3}\right)+\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{3}\right)\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^{2}\right)\right]\\
&+\gamma_{1,\varepsilon_\eta}^{2}\rho_{\eta,ij}\left(\frac{4}{v^2w_T}\sum_t\sum_r h_r h_t m_{rr}m_{tt}\right)\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,j\ell}^{3}\right)\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{3}\right)\\
&+\left(4\frac{1}{v^2w_T}\sum_t\sum_r h_r h_t m_{rt}^2+\frac{1}{v^2w_T}\sum_t\sum_r h_t^2 m_{rr}m_{tr}+2\frac{1}{vw_T^2}\sum_t\sum_r h_r^3 h_t m_{tt}\right)\\
&\times\left\{\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^{2}\right)^2+\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}\right)^2\right\}\\
&+\left(4\frac{1}{v^2w_T}\sum_t\sum_r h_r h_t m_{rr}m_{tt}+16\frac{1}{v^2w_T}\sum_t\sum_r h_r h_t m_{rt}^2+8\frac{1}{v^2w_T}\sum_t\sum_r h_t^2 m_{rr}m_{tr}\right)\\
&\times\gamma_{1,\varepsilon_\eta}^{2}\rho_{\eta,ij}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}\right)\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\rho_{\eta,ij}\left(4\frac{1}{vw_T}\sum_t h_t^2 m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{3}\tilde{q}_{\eta,j\ell}\right)+3\rho_{\eta,ij}\right]\\
&+\left(2\frac{1}{vw_T}\sum_t h_t^2 m_{tt}+\frac{1}{v^2}\sum_t m_{tt}^2\right)\\
&\times\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}^{2}\right)+2\rho_{\eta,ij}^{2}\right]+2\rho_{\eta,ij}^{2}\frac{1}{w_T^2}\sum_t h_t^4\\
&+\rho_{\eta,ij}^{2}\left(2\frac{1}{v^2}\sum_t m_{tt}^2\right)\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{2}\tilde{q}_{\eta,j\ell}^{2}\right)+\left(1+2\rho_{\eta,ij}^{2}\right)\right]\\
&+\rho_{\eta,ij}\left(4\frac{1}{vw_T}\sum_t h_t^2 m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_{\ell}\tilde{q}_{\eta,i\ell}^{3}\tilde{q}_{\eta,j\ell}\right)+3\rho_{\eta,ij}\right]+O\left(T^{-2}\right). \qquad \text{(S.46)}
\end{aligned}
$$

Proof. First, $E\left(z_{\eta,i}^{2}\right)=1$ since $E\left(\boldsymbol{\eta}_i'\mathbf{H}_F\boldsymbol{\eta}_i/\sigma_{\eta,ii}\right)=Tr\left(\mathbf{H}_F\right)=w_T$ and $E\left(X_{\eta,i}\right)=1$ since $E\left(\boldsymbol{\eta}_i'\mathbf{M}_G\boldsymbol{\eta}_i/\sigma_{\eta,ii}\right)=Tr\left(\mathbf{M}_G\right)=v$ (see Lemma 11). Noting that $\tilde{\eta}_{it}=\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}$ we have

$$
\varphi_{\eta,ij}=E\left(\tilde{\eta}_{it}^{2}\tilde{\eta}_{jt}^{2}\right)=E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right],
$$

and since $\boldsymbol{\varepsilon}_{\eta,t}\sim IID(\mathbf{0},\mathbf{I}_N)$, then using (S.7) in Lemma 6, and noting that $\sum_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}=\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j}=\rho_{\eta,ij}$, and $\sum_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^{2}=\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,i}=1$, we have

$$
\begin{aligned}
\varphi_{\eta,ij} \;=\;& \gamma_{2,\varepsilon_\eta}Tr\left(\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\odot\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\right)+Tr\left(\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\right)Tr\left(\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\right)\\
&+Tr\left(\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\right),
\end{aligned}
$$

which establishes (S.41). Next, noting $z_{\eta,i}^{2}=\tilde{\boldsymbol{\eta}}_i'\mathbf{H}_F\tilde{\boldsymbol{\eta}}_i=\sum_t\sum_{t'}h_{tt'}\left(\eta_{it}/\sigma_{\eta,ii}^{1/2}\right)\left(\eta_{it'}/\sigma_{\eta,ii}^{1/2}\right)=\sum_t\sum_{t'}h_{tt'}\tilde{\eta}_{it}\tilde{\eta}_{it'}$ and $\tilde{\eta}_{it}=\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}$, we have

$$
E\left(z_{\eta,i}^{2}z_{\eta,j}^{2}\right)=\frac{1}{w_T^2}\sum_t\sum_{t'}\sum_r\sum_{r'}h_t h_{t'}h_r h_{r'}E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t'}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r'}\right)\right],
$$

and note that there are the following combinations of indices $\{t,t',r,r'\}$ to take into account. There is one $t=t'=r=r'$, and three relevant pairs, $t=t'$ and $r=r'$ $(t\neq r)$, $t=r'$ and $t'=r$ $(t\neq r)$, and $t=r$ and

$t' = r'$ $(t \neq t')$. Thus,

$$E\left(z_{\eta,i}^2 z_{\eta,j}^2\right) = \frac{1}{w_T^2}\sum_t h_t^4 E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right] \text{ (for } t = t' = r = r'\text{)}$$

$$+\frac{1}{w_T^2}\sum_{t\neq r} h_t^2 h_r^2 E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right] \text{ (for } t' = t, r' = r,\ t \neq r\text{)}$$

$$+\frac{1}{w_T^2}\sum_{t\neq r} h_t h_r h_r h_t E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right] \text{ (for } r' = t, t' = r,\ t \neq r\text{)}$$

$$+\frac{1}{w_T^2}\sum_{t\neq t'} h_{tt'} h_{tt'} E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t'}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t'}\right)\right] \text{ (for } r = t, r' = t', t \neq t'\text{)}.$$

Hence

$$\begin{aligned} E\left(z_{\eta,i}^2 z_{\eta,j}^2\right) &= \frac{1}{w_T^2}\sum_t h_t^4 E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right] + \frac{1}{w_T^2}\sum_{t\neq r} h_t^2 h_r^2 E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right] \\ &+2\frac{1}{w_T^2}\sum_{t\neq t'} h_t^2 h_{t'}^2 E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{t'}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{t'}\right)\right]. \end{aligned}$$

Observing that the ordering of $h_t h_{t'} h_r h_{r'}$ is arbitrary, we have

$$\begin{aligned} E\left(z_{\eta,i}^2 z_{\eta,j}^2\right) &= \frac{1}{w_T^2}\sum_t h_t^4 E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right] \\ &+\frac{1}{w_T^2}\sum_{t\neq r} h_t^2 h_r^2 \left\{E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right) + 2\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]^2\right\}. \end{aligned}$$

Also note that $E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2$ is given by (S.41), $E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right) = 1$ and $E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right) = \rho_{\eta,ij}$, and $\sum_{t\neq r} h_t^2 h_r^2 = \sum_t\sum_r h_t^2 h_r^2 - \sum_t h_t^4 = w_T^2 - \sum_t h_t^4$. Then, after some simplifications we obtain

$$\begin{aligned} E\left(z_{\eta,i}^2 z_{\eta,j}^2\right) &= \frac{\sum_t h_t^4}{w_T^2}\left(\gamma_{2,\varepsilon_\eta}\sum_{\ell=1}^N \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2 + 1 + 2\rho_{\eta,ij}^2\right) + \frac{\sum_t\sum_r h_t^2 h_r^2 - \sum_t h_t^4}{w_T^2}\left(1 + 2\rho_{\eta,ij}^2\right) \\ &= 1 + 2\rho_{\eta,ij}^2 + \frac{\sum_t h_t^4}{w_T^2}\gamma_{2,\varepsilon_\eta}\left(\sum_{\ell=1}^N \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right), \end{aligned}$$

as required. Next, similarly,

$$\begin{aligned} E\left(X_{\eta,i}X_{\eta,j}\right) &= \frac{1}{v^2}\sum_t\sum_{t'}\sum_r\sum_{r'} m_{tt'}m_{rr'}E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t'}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r'}\right)\right] \\ &= \frac{1}{v^2}\sum_t m_{tt}^2 E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right] \\ &+\frac{1}{v^2}\sum_{t\neq r} m_{tt}m_{rr}E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right) \\ &+\frac{1}{v^2}2\sum_{t\neq r} m_{tr}^2 E\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right]^2 \\ &= 1 + \frac{2\rho_{\eta,ij}^2}{v} + \frac{\sum_t m_{tt}^2}{v_2}\left(\gamma_{2,\varepsilon_\eta}\sum_{\ell=1}^N \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right). \end{aligned}$$

Next consider

$$E\left(z_{\eta,i}^2 X_{\eta,i}\right) = \frac{1}{vw_T}\sum_t\sum_{t'}\sum_r\sum_{r'} h_{tt'}m_{rr'}E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t'}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r'}\right)\right]$$

$$\begin{aligned} &= \frac{1}{vw_T}\sum_t h_t^2 m_{tt}E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right] \\ &+\frac{1}{vw_T}\left(\sum_t\sum_r h_t^2 m_{rr} + 2\sum_t\sum_r h_t h_r m_{tr} - 3\sum_t h_t^2 m_{tt}\right)\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]^2. \end{aligned}$$

But $\sum_t\sum_r h_t h_r m_{tr} = Tr\left(\mathbf{M}_G\mathbf{H}_F\right) = 0$, $\sum_t\sum_r h_t^2 m_{rr} = vw_T$, and $E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right] = \gamma_{2,\varepsilon_\eta}\sum_{\ell=1}^N \tilde{q}_{\eta,i\ell}^4 +$ 3, $E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right) = 1$ by Lemma 13 we have

$$E\left(z_{\eta,i}^2 X_{\eta,i}\right) = 1 + \frac{\sum_t h_t^2 m_{tt}}{vw_T}\gamma_{2,\varepsilon_\eta}\sum_{\ell=1}^N \tilde{q}_{\eta,i\ell}^4.$$

Next, consider

$$E\left(z_{\eta,i}^2X_{\eta,i}z_{\eta,j}^2\right)=w_T^{-2}v^{-1}\sum_t\sum_{t'}\sum_r\sum_{r'}\sum_u\sum_{u'}h_th_{t'}h_rh_{r'}m_{uu'}E\left[\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,t'}\right)\left(\boldsymbol{\varepsilon}'_{\eta,r}\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,r'}\right)\left(\boldsymbol{\varepsilon}'_{\eta,u}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,u'}\right)\right].$$

In addition to the case of $t=t'=r=r'=u=u'$, three combinations of six indices $\{t,t',r,r',u,u'\}$ are to be considered: three pairs, two of threes, and fours and twos, which are with superscripts $(2,2,2)$, $(3,3)$ and $(4,2)$, respectively. As the groups' ordering does not matter when the number of group members are the same, we have $\left(\frac{6!}{2!4!}\right)\left(\frac{4!}{2!2!}\right)\frac{1}{3!}=15$ different combinations of $(2,2,2)$, $\left(\frac{6!}{3!3!}\right)\frac{1}{2!}=10$ of $(3,3)$, and $\frac{6!}{2!4!}=15$ of $(4,2)$. After considering of all the combinations, and observing that the ordering of $h_th_{t'}h_rh_{r'}$ and $\{u,u'\}$ in $m_{uu'}$ is arbitrary (as $\mathbf{M}_G$ is symmetric), after some algebra, we have

$$\begin{aligned}E\left(z_{\eta,i}^2X_{\eta,i}z_{\eta,j}^2\right) &= \left(A_{(2,2,2)}+2B_{(2,2,2)}\right)\left[E\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,t}\right)\right]^2E\left(\boldsymbol{\varepsilon}'_{\eta,r}\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,r}\right)\\
&+2\left(A_{(2,2,2)}+5B_{(2,2,2)}\right)E\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,t}\right)\left[E\left(\boldsymbol{\varepsilon}'_{\eta,r}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,r}\right)\right]^2\\
&+\left(A_{(3,3)}+3B_{(3,3)}\right)E\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,t}\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,j}\right)E\left(\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}'_{\eta,r}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,r}\right)\\
&+2\left(A_{(3,3)}+2B_{(3,3)}\right)\left[E\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,t}\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,j}\right)\right]^2\\
&+\left(A_{(2,4)}+4B_{(2,4)}+C_{(2,4)}\right)E\left[\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,t}\right)\right]E\left(\boldsymbol{\varepsilon}'_{\eta,r}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,r}\right)\\
&+4\left(B_{(2,4)}+C_{(2,4)}\right)E\left[\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,t}\right)\right]E\left(\boldsymbol{\varepsilon}'_{\eta,r}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,r}\right)\\
&+C_{(2,4)}E\left[\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right]E\left(\boldsymbol{\varepsilon}'_{\eta,r}\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,r}\right)\\
&+w_T^{-2}v^{-1}\sum_t h_t^4m_{tt}E\left[\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}'_{\eta,i}\boldsymbol{\varepsilon}_{\eta,t}\right)^2\left(\boldsymbol{\varepsilon}'_{\eta,t}\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}'_{\eta,j}\boldsymbol{\varepsilon}_{\eta,t}\right)\right]\end{aligned}$$

where

$$A_{(2,2,2)}=w_T^{-2}v^{-1}\sum_{t\neq r\neq u}h_t^2h_r^2m_{uu},\ B_{(2,2,2)}=w_T^{-2}v^{-1}\sum_{t\neq r\neq u}h_t^2h_rh_um_{ru},\tag{S.47}$$

$$A_{(3,3)}=w_T^{-2}v^{-1}\sum_{t\neq r}h_t^3h_rm_{rr},\ B_{(3,3)}=w_T^{-2}v^{-1}\sum_{t\neq r}h_t^2h_r^2m_{tr},\tag{S.48}$$

$$A_{(2,4)}=w_T^{-2}v^{-1}\sum_{t\neq r}h_t^4m_{rr},\ B_{(2,4)}=w_T^{-2}v^{-1}\sum_{t\neq r}h_t^3h_rm_{tr},\ C_{(2,4)}=w_T^{-2}v^{-1}\sum_{t\neq r}h_t^2h_r^2m_{tt},\tag{S.49}$$

and noting that $\sum_{t\neq r\neq u}h_t^2h_r^2m_{uu}=\sum_t\sum_r\sum_u h_t^2h_r^2m_{uu}-\sum_t\sum_r h_t^4m_{rr}-\sum_t\sum_r h_t^2h_r^2m_{tt}-\sum_t\sum_r h_t^2h_r^2m_{rr}+2\sum_t h_t^4m_{tt}$,

$$A_{(2,2,2)}=1-w_T^{-2}\sum_t h_t^4-2w_T^{-1}v^{-1}\sum_t h_t^2m_{tt}+O(T^{-2}),$$

since $\sum_t h_t^2=w_T$ and $\sum_t m_{tt}=v$, and $\sum_t h_t^4m_{tt}\leq\sum_t h_t^4=O(T)$, and noting that, as $\mathbf{M}_G$ and $\mathbf{H}_F$ are symmetric and $\mathbf{M}_G\mathbf{H}_F=\mathbf{0}$, $\sum_t h_rh_tm_{tu}$ for any $t\neq r$ and $t\neq u$ we have

$$B_{(2,2,2)}=-w_T^{-1}v^{-1}\sum_t h_t^2m_{tt}+O\left(T^{-2}\right),$$

$$A_{(3,3)}=w_T^{-2}v^{-1}\sum_t\sum_r h_t^3h_rm_{rr}+O\left(T^{-2}\right),\ B_{(3,3)}=w_T^{-2}v^{-1}\sum_t\sum_r h_t^2h_r^2m_{tr}+O\left(T^{-2}\right)$$

$$A_{(2,4)}=w_T^{-2}\sum_r h_t^4+O\left(T^{-2}\right),\ B_{(2,4)}=O\left(T^{-2}\right),\ C_{(2,4)}=w_T^{-1}v^{-1}\sum_t h_t^2m_{tt}+O\left(T^{-2}\right).$$

Using the result in Lemma 13 and noting that $E\left(|\tilde{\eta}_{it}|^8\right)$ is uniformly bounded by Lemma 3, we have

$$\begin{aligned}E\left(z_{\eta,i}^2X_{\eta,i}z_{\eta,j}^2\right) &= 1+2\rho_{\eta,ij}^2+\frac{1}{w_Tv}\sum_t h_t^2m_{tt}\left[\gamma_{2,\varepsilon_\eta}\left(\textstyle\sum_\ell\tilde{q}_{\eta,i\ell}^4\right)\right]\\
&+\left(\frac{1}{w_T^2v}\sum_t\sum_r h_t^3h_rm_{rr}+3\frac{1}{w_T^2v}\sum_t\sum_r h_t^2h_r^2m_{tr}\right)\gamma_{1,\varepsilon_\eta}^2\left(\textstyle\sum_\ell\tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)\left(\textstyle\sum_\ell\tilde{q}_{\eta,i\ell}^3\right)\\
&+2\left(\frac{1}{w_T^2v}\sum_t\sum_r h_t^3h_rm_{rr}+2\frac{1}{w_T^2v}\sum_t\sum_r h_t^2h_r^2m_{tr}\right)\gamma_{1,\varepsilon_\eta}^2\left(\textstyle\sum_\ell\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right)^2\\
&+\left(\frac{1}{w_T^2}\sum_r h_t^4+\frac{1}{w_Tv}\sum_t h_t^2m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}\left(\textstyle\sum_\ell\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right)\right]\\
&+4\rho_{\eta,ij}\left(\frac{1}{w_Tv}\sum_t h_t^2m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}\left(\textstyle\sum_\ell\tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right)\right]\\
&+O\left(T^{-2}\right).\end{aligned}$$

Next consider

$$E\left(z_{\eta,i}^2X_{\eta,i}z_{\eta,j}^2X_{\eta,j}\right) = w_T^{-2}v^{-2}\sum_t\sum_{t'}\sum_r\sum_{r'}\sum_\nu\sum_{\nu'}\sum_u\sum_{u'}h_th_{t'}h_rh_{r'}m_{\nu\nu'}m_{uu'} \times E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t'}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r'}\right)\left(\boldsymbol{\varepsilon}_{\eta,\nu}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,\nu'}\right)\left(\boldsymbol{\varepsilon}_{\eta,u}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,u'}\right)\right].$$

In addition to the case of $t=t'=r=r'=\nu=\nu'=u=u'$, five combinations of eight indices $\{t,t',r,r',\nu,\nu',u,u'\}$ are to be considered, which are subscripted by $(2,6)$, $(3,5)$, $(4,4)$, $(2,3,3)$, $(4,2,2)$, and $(2,2,2,2)$. As the groups' ordering does not matter when the number of group members are the same, we have $\frac{8!}{2!6!}=28$ of different combinations of $(2,6)$, $\frac{8!}{3!5!}=56$ of $(3,5)$, $\frac{8!}{4!4!}\frac{1}{2!}=35$ of $(4,4)$, $\frac{8!}{2!6!}\left(\frac{6!}{3!3!}\frac{1}{2!}\right)=280$ of $(2,3,3)$, $\frac{8!}{4!4!}\left(\frac{4!}{2!2!}\frac{1}{2!}\right)=210$ of $(4,2,2)$, and $\frac{8!}{2!6!}\frac{6!}{2!4!}\frac{4!}{2!2!}\frac{1}{4!}=105$ of $(2,2,2,2)$, respectively. After considering of all the combinations, and observing that the ordering of $h_th_{t'}h_rh_{r'}$ and $\{u,u'\}$ of $m_{uu'}$ are arbitrary, after tedious algebra, we have

$$E\left(z_{\eta,i}^2X_{\eta,i}z_{\eta,j}^2X_{\eta,j}\right)=\left(A_{(2,2,2,2)}+4C_{(2,2,2,2)}+4E_{(2,2,2,2)}\right)\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]^2\left[E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right]^2$$
$$+2\left(A_{(2,2,2,2)}+B_{(2,2,2,2)}+10C_{(2,2,2,2)}+16D_{(2,2,2,2)}+8E_{(2,2,2,2)}\right)\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]^2\left[E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right]^2$$
$$+2\left(2B_{(2,2,2,2)}+8D_{(2,2,2,2)}+2E_{(2,2,2,2)}\right)\left[E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right]^4$$

$$+\left(E_{(2,2,4)}+2G_{(2,2,4)}\right)E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right]\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]^2$$
$$+\left(4C_{(2,2,4)}+8D_{(2,2,4)}+4E_{(2,2,4)}+4F_{(2,2,4)}+8G_{(2,2,4)}+8H_{(2,2,4)}+12I_{(2,2,4)}\right)$$
$$\times E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)$$
$$+\left(A_{(2,2,4)}+8C_{(2,2,4)}+2E_{(2,2,4)}+16H_{(2,2,4)}+8I_{(2,2,4)}+J_{(2,2,4)}\right)$$
$$\times E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)$$
$$+\left(2B_{(2,2,4)}+16D_{(2,2,4)}+16F_{(2,2,4)}+4G_{(2,2,4)}+16H_{(2,2,4)}+16I_{(2,2,4)}+2J_{(2,2,4)}\right)$$
$$\times E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]^2$$
$$+\left(4C_{(2,2,4)}+8D_{(2,2,4)}+4E_{(2,2,4)}+4F_{(2,2,4)}+8G_{(2,2,4)}+8H_{(2,2,4)}+12I_{(2,2,4)}\right)$$
$$\times E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)$$
$$+\left(E_{(2,2,4)}+2G_{(2,2,4)}\right)E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right]\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]^2$$

$$+\left(2A_{(3,3,2)}+C_{(3,3,2)}+9D_{(3,3,2)}+8E_{(3,3,2)}+2G_{(3,3,2)}+2I_{(3,3,2)}\right)$$
$$\times\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\right)E\left(\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,u}\boldsymbol{\varepsilon}_{\eta,u}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,u}\right)\right.$$
$$\left.+E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\right)E\left(\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,u}\boldsymbol{\varepsilon}_{\eta,u}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,u}\right)\right]$$
$$+\left(4A_{(3,3,2)}+8D_{(3,3,2)}+4J_{(3,3,2)}\right)E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\right)E\left(\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,u}\boldsymbol{\varepsilon}_{\eta,u}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,u}\right)$$
$$+\left(4B_{(3,3,2)}+C_{(3,3,2)}+5D_{(3,3,2)}+16E_{(3,3,2)}+4F_{(3,3,2)}+2G_{(3,3,2)}+4I_{(3,3,2)}\right)$$
$$\times\left\{E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left[E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\right)\right]^2+E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left[E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\right)\right]^2\right\}$$
$$+\left(4A_{(3,3,2)}+16B_{(3,3,2)}+8C_{(3,3,2)}+24D_{(3,3,2)}+48E_{(3,3,2)}+8F_{(3,3,2)}+16H_{(3,3,2)}+20J_{(3,3,2)}\right)$$
$$\times E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\right)E\left(\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,u}\boldsymbol{\varepsilon}_{\eta,u}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,u}\right)$$

$$+\left(A_{(2,6)}+4B_{(2,6)}+C_{(2,6)}\right)E\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right]E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\right)$$
$$+4\left(A_{(2,6)}+2B_{(2,6)}+D_{(2,6)}\right)E\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)$$
$$+\left(A_{(2,6)}+4B_{(2,6)}+C_{(2,6)}\right)E\left[E\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)$$

$$+2\left(B_{(3,5)}+C_{(3,5)}\right)E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\right)E\left[\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right]$$
$$+2\left(A_{(3,5)}+5B_{(3,5)}+C_{(3,5)}+4D_{(3,5)}+E_{(3,5)}\right)$$
$$\times E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\right)E\left[\left(\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]$$
$$+2\left(A_{(3,5)}+5B_{(3,5)}+C_{(3,5)}+4D_{(3,5)}+E_{(3,5)}\right)$$
$$\times E\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)E\left[\left(\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]$$
$$+2\left(B_{(3,5)}+C_{(3,5)}\right)E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\right]E\left[\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right]$$

$$+B_{(4,4)}E\left[\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right]E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]$$
$$+4\left(2C_{(4,4)}+D_{(4,4)}+B_{(4,4)}\right)E\left[\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right]E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)\right]$$
$$+\left(A_{(4,4)}+B_{(4,4)}+8C_{(4,4)}+8D_{(4,4)}\right)\left\{E\left[\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,r}\right)\left(\boldsymbol{\varepsilon}_{\eta,r}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,r}\right)\right]\right\}^2$$

$$+v^{-2}w_T^{-2}\sum_t h_t^4 m_{tt}^2 E\left[\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\left(\boldsymbol{\varepsilon}_{\eta,t}'\tilde{\mathbf{q}}_{\eta,j}\tilde{\mathbf{q}}_{\eta,j}'\boldsymbol{\varepsilon}_{\eta,t}\right)^2\right],$$

where

$$\begin{aligned}
A_{(2,2,2,2)} &= w_T^{-2}v^{-2}\sum_{t\neq r\neq \nu\neq u} h_t^2h_r^2 m_{\nu\nu}m_{uu},\ B_{(2,2,2,2)} = w_T^{-2}v^{-2}\sum_{t\neq r\neq \nu\neq u} h_t^2h_r^2 m_{\nu u}^2 \\
C_{(2,2,2,2)} &= w_T^{-2}v^{-2}\sum_{t\neq r\neq \nu\neq u} h_t^2h_rh_\nu m_{r\nu}m_{uu},\ D_{(2,2,2,2)} = w_T^{-2}v^{-2}\sum_{t\neq r\neq \nu\neq u} h_t^2h_rh_\nu m_{\nu u}m_{ur}, \\
E_{(2,2,2,2)} &= w_T^{-2}v^{-2}\sum_{t\neq r\neq \nu\neq u} h_{tu}h_rh_\nu m_{r\nu}m_{tu},
\end{aligned} \tag{S.50}$$

$$\begin{aligned}
A_{(2,2,4)} &= w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_t^4 m_{rr}m_{uu},\ B_{(2,2,4)} = w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_t^4 m_{ru}^2, \\
C_{(2,2,4)} &= w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_t^3h_r m_{tr}m_{uu},\ D_{(2,2,4)} = w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_t^3h_u m_{tr}m_{ru}, \\
E_{(2,2,4)} &= w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_t^2h_r^2 m_{tt}m_{uu},\ F_{(2,2,4)} = w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_t^2h_r^2 m_{tu}^2 \\
G_{(2,2,4)} &= w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_t^2h_rh_u m_{tt}m_{ru},\ H_{(2,2,4)} = w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_t^2h_rh_u m_{tr}m_{tu}, \\
I_{(2,2,4)} &= w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_th_uh_r^2 m_{tt}m_{tu},\ J_{(2,2,4)} = w_T^{-2}v^{-2}\sum_{t\neq r\neq u} h_r^2h_u^2 m_{tt}^2,
\end{aligned}$$

$$\begin{aligned}
A_{(3,3,2)} &= w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_u^2h_rh_t m_{rr}m_{tt},\ B_{(3,3,2)} = w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_u^2h_rh_t m_{rt}^2, \\
C_{(3,3,2)} &= w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_u^2h_t^2 m_{rr}m_{tr},\ D_{(3,3,2)} = w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_r^2h_uh_t m_{ur}m_{tt}, \\
E_{(3,3,2)} &= w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_r^2h_uh_t m_{rt}m_{ut},\ F_{(3,3,2)} = w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_r^3h_u m_{ut}m_{tt}, \\
G_{(3,3,2)} &= w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_r^3h_t m_{uu}m_{tt},\ H_{(3,3,2)} = w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_r^3h_t m_{ut}^2, \\
I_{(3,3,2)} &= w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_r^2h_t^2 m_{uu}m_{tr},\ J_{(3,3,2)} = w_T^{-2}v^{-2}\sum_{u\neq r\neq t} h_r^2h_t^2 m_{tu}m_{ur},
\end{aligned} \tag{S.51}$$

$$\begin{aligned}
A_{(2,6)} &= w_T^{-2}v^{-2}\sum_{t\neq r} h_t^2h_r^2 m_{rr}^2,\ B_{(2,6)} = w_T^{-2}v^{-2}\sum_{t\neq r} h_th_r^3 m_{tr}m_{rr}, \\
C_{(2,6)} &= w_T^{-2}v^{-2}\sum_{t\neq r} h_r^4 m_{tt}m_{rr},\ D_{(2,6)} = w_T^{-2}v^{-2}\sum_{t\neq r} h_r^4 m_{tr}^2,
\end{aligned}$$

$$\begin{aligned}
A_{(3,5)} &= w_T^{-2}v^{-2}\sum_{t\neq r} h_t^3h_r m_{rr}^2,\ B_{(3,5)} = w_T^{-2}v^{-2}\sum_{t\neq r} h_t^2h_r^2 m_{tr}m_{rr}, \\
C_{(3,5)} &= w_T^{-2}v^{-2}\sum_{t\neq r} h_th_r^3 m_{tt}m_{rr},\ D_{(3,5)} = w_T^{-2}v^{-2}\sum_{t\neq r} h_th_r^3 m_{tr}^2, \\
E_{(3,5)} &= w_T^{-2}v^{-2}\sum_{t\neq r} h_r^4 m_{rt}m_{tt},
\end{aligned}$$

$$\begin{aligned}
A_{(4,4)} &= w_T^{-2}v^{-2}\sum_{t\neq r} h_{rr}^4 m_{tt}^2,\ B_{(4,4)} = w_T^{-2}v^{-2}\sum_{t\neq r} h_r^2h_t^2 m_{rr}m_{tt}, \\
C_{(4,4)} &= w_T^{-2}v^{-2}\sum_{t\neq r} h_r^3h_t m_{rt}m_{tt},\ D_{(4,4)} = w_T^{-2}v^{-2}\sum_{t\neq r} h_r^2h_t^2 m_{rt}^2.
\end{aligned} \tag{S.52}$$

But observing that the ordering of indices in $h_th_{t'}h_rh_{r'}$ and $\{u,u'\}$ of $m_{uu'}$ are arbitrary, and noting that as $\mathbf{M}_G$ and $\mathbf{H}_F$ are symmetric and $\mathbf{M}_G\mathbf{H}_F = \mathbf{0}$, $\sum_t\sum_r\sum_u h_rh_tm_{tu}$ for any $t\neq r$ and $t\neq u$, a similar discussion for the proof of Lemma 10 will give

$$A_{(2,2,2,2)} = 1 - \frac{1}{w_T^2}\sum_t h_t^4 - 4\frac{1}{vw_T}\sum_t h_t^2m_{tt} - \frac{1}{v^2}\sum_t m_{tt}^2 + O(T^{-2}), \tag{S.53}$$

$$B_{(2,2,2,2)} = \frac{1}{v} - \frac{1}{v^2}\sum_t m_{tt}^2 + O(T^{-2}),$$
$$C_{(2,2,2,2)} = -\frac{1}{vw_T}\sum_t h_t^2 m_{tt} + O(T^{-2}),$$
$$D_{(2,2,2,2)} = O(T^{-2}),\ E_{(2,2,2,2)} = O(T^{-2}),$$

so that

$$\left(A_{(2,2,2,2)} + 4C_{(2,2,2,2)} + 4E_{(2,2,2,2)}\right) = 1 - \frac{1}{w_T^2}\sum_t h_t^4 - \frac{8}{vw_T}\sum_t h_t^2 m_{tt} - \frac{1}{v^2}\sum_t m_{tt}^2 + O(T^{-2}).$$

Next

$$A_{(3,3,2)} = \frac{1}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rr} m_{tt} + O(T^{-2}),$$
$$B_{(3,3,2)} = \frac{1}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rt}^2 + O(T^{-2}),$$
$$C_{(3,3,2)} = \frac{1}{v^2 w_T}\sum_t\sum_r h_t^2 m_{rr} m_{tr} + O(T^{-2}),$$
$$D_{(3,3,2)} = O(T^{-2}),\ E_{(3,3,2)} = O(T^{-2}),\ F_{(3,3,2)} = O(T^{-2}),$$
$$G_{(3,3,2)} = \frac{1}{vw_T^2}\sum_t\sum_r h_r^3 h_t m_{tt} + O(T^{-2}),$$
$$H_{(3,3,2)} = O(T^{-2}),\ I_{(3,3,2)} = O(T^{-2}),\ J_{(3,3,2)} = O(T^{-2}),$$
$$A_{(2,2,4)} = \frac{1}{w_T^2}\sum_t h_t^4 + O(T^{-2}),$$
$$B_{(2,2,4)} = O(T^{-2}), C_{(2,2,4)} = O(T^{-2}), D_{(2,2,4)} = O(T^{-2}),$$
$$E_{(2,2,4)} = \frac{1}{vw_T}\sum_t h_t^2 m_{tt} + O(T^{-2}),$$
$$F_{(2,2,4)} = O(T^{-2}),\ G_{(2,2,4)} = O(T^{-2}),\ H_{(2,2,4)} = O(T^{-2}),\ I_{(2,2,4)} = O(T^{-2}),$$
$$J_{(2,2,4)} = \frac{1}{v^2}\sum_t m_{tt}^2 + O(T^{-2}). \tag{S.54}$$

Since the functions with subscripts (2, 6), (3, 5) and (4, 4) are all $O(T^{-2})$, and $v^{-2}w_T^{-2}\sum_t h_t^4 m_{tt}^2 \leq v^{-2}w_T^{-2}\sum_t h_t^4 m_{tt}^2 \sum_t h_t^4 = O(T^{-3})$, noting that $E\left(\varepsilon_{\eta,it}^8\right)$ is uniformly bounded, using the results in Lemma 13 we have

$$\begin{aligned}
E\left(z_{\eta,i}^2 X_{\eta,i} z_{\eta,j}^2 X_{\eta,j}\right) \quad = \quad & 1 + 2\rho_{\eta,ij}^2 + \left(\frac{1}{vw_T}\sum_t h_t^2 m_{tt}\right)\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_\ell \tilde{q}_{\eta,j\ell}^4 + \sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^4\right) \\
& + 2\rho_{\eta,ij}^2\left(-\frac{1}{w_T^2}\sum_t h_t^4 - \frac{18}{vw_T}\sum_t h_t^2 m_{tt} - \frac{2}{v^2}\sum_t m_{tt}^2 + \frac{1}{v}\right) \\
& + 2\rho_{\eta,ij}^4\left(\frac{2}{v} - \frac{2}{v^2}\sum_t m_{tt}^2\right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rr} m_{tt} + \frac{1}{v^2 w_T}\sum_t\sum_r h_t^2 m_{rr} m_{tr} + \frac{2}{vw_T^2}\sum_t\sum_r h_r^3 h_t m_{tt}\right) \\
& \times\gamma_{1,\varepsilon_\eta}^2\left[\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2 \tilde{q}_{\eta,j\ell}\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,j\ell}^3\right) + \left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)\right] \\
& + \gamma_{1,\varepsilon_\eta}^2 \rho_{\eta,ij}\left(\frac{4}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rr} m_{tt}\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,j\ell}^3\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\right) \\
& + \left(4\frac{1}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rt}^2 + \frac{1}{v^2 w_T}\sum_t\sum_r h_t^2 m_{rr} m_{tr} + 2\frac{1}{vw_T^2}\sum_t\sum_r h_r^3 h_t m_{tt}\right) \\
& \times\left[\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)^2 + \left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2 \tilde{q}_{\eta,j\ell}\right)^2\right] \\
& + \left(4\frac{1}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rr} m_{tt} + 16\frac{1}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rt}^2 + 8\frac{1}{v^2 w_T}\sum_t\sum_r h_t^2 m_{rr} m_{tr}\right) \\
& \times\gamma_{1,\varepsilon_\eta}^2 \rho_{\eta,ij}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2 \tilde{q}_{\eta,j\ell}\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\rho_{\eta,ij}\left(4\frac{1}{vw_T}\sum_t h_t^2 m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right)+3\rho_{\eta,ij}\right]\\
&+\left(\frac{1}{w_T^2}\sum_t h_t^4+2\frac{1}{vw_T}\sum_t h_t^2 m_{tt}+\frac{1}{v^2}\sum_t m_{tt}^2\right)\\
&\times\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right)+2\rho_{\eta,ij}^2\right]\\
&+\rho_{\eta,ij}^2\left(2\frac{1}{v^2}\sum_t m_{tt}^2\right)\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right)+\left(1+2\rho_{\eta,ij}^2\right)\right]\\
&+\rho_{\eta,ij}\left(4\frac{1}{vw_T}\sum_t h_t^2 m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right)+3\rho_{\eta,ij}\right]\\
&+O\left(T^{-2}\right).
\end{aligned}
$$

■

Lemma 16 *Consider the regression model (2), and suppose that Assumptions 1-4 hold. Let* $z_{\eta,i}^2=\frac{\boldsymbol{\eta}_i'\mathbf{H}_F\boldsymbol{\eta}_i}{\sigma_{\eta,ii}w_T}$ *and* $X_{\eta,i}=\frac{\boldsymbol{\eta}_i'\mathbf{M}_G\boldsymbol{\eta}_i}{\sigma_{\eta,ii}v}$ *where* $w_T=\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T$*, where* $\boldsymbol{\eta}_i=(\eta_{i1},\eta_{i2},...,\eta_{iT})'$*,* $w_T=\mathbf{h}'\mathbf{h}$ *with* $\mathbf{h}=\mathbf{M}_F\boldsymbol{\tau}_T$*, and* $\mathbf{H}_F=\mathbf{h}\mathbf{h}'=(h_th_{t'})$*,* $\mathbf{M}_F=(m_{F,tt'})$*, and* $\mathbf{M}_G=(m_{tt'})$ *are defined by (S.2), and* $v=T-m-1$*. Then we have*

$$
N^{-1}\sum_{i\neq j}Cov\left[z_{\eta,i}^2\left(X_{\eta,i}-1\right),z_{\eta,j}^2\left(X_{\eta,j}-1\right)\right]=O\left(T^{-1}\right)+O\left(\frac{N}{T^2}\right).
$$

Proof. First, consider $N^{-1}\sum_{i\neq j}Cov\left(z_{\eta,i}^2,z_{\eta,j}^2\right)$. Using Lemma 15, we have $E\left(z_{\eta,i}^2\right)=1$ and

$$
E\left(z_{\eta,i}^2z_{\eta,j}^2\right)=1+2\rho_{\eta,ij}^2+\gamma_{2,\varepsilon_\eta}\left(\frac{\sum_t h_t^4}{w_T^2}\right)\left(\sum_{\ell=1}^N\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right),
$$

where $\rho_{\eta,ij}=Cov\left(\tilde{\eta}_{it},\tilde{\eta}_{jt}\right)$, $\gamma_{1,\varepsilon_\eta}=E\left(\varepsilon_{\eta,it}^3\right)$ and $\gamma_{2,\varepsilon_\eta}=E\left(\varepsilon_{\eta,it}^4\right)-3$, $\tilde{\eta}_{it}=\eta_{it}/\sigma_{\eta,ii}^{1/2}$, and $\tilde{\mathbf{q}}_{\eta,i}'$ is the i^{th} row of $\tilde{\mathbf{Q}}_\eta=\mathbf{D}_{\sigma_\eta}^{-1/2}\mathbf{Q}_\eta$, with $\mathbf{D}_{\sigma_\eta}=diag\left(\sigma_{\eta,ii}\right)$. Thus,

$$
N^{-1}\sum_{i\neq j}Cov\left(z_{\eta,i}^2,z_{\eta,j}^2\right)=N^{-1}\sum_{i\neq j}2\rho_{\eta,ij}^2+\frac{\sum_t h_t^4}{w_T^2}\gamma_{2,\varepsilon_\eta}N^{-1}\sum_{i\neq j}\left(\sum\nolimits_\ell\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right),
$$

but, since by Lemma 14 $\sum_{i\neq j}\sum_\ell\left|\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right|=O(N)$, by assumption $\left|\gamma_{2,\varepsilon_\eta}\right|\leq K$, and $\sum_t h_t^4=O(v)$ by Lemma 8, we have

$$
\begin{aligned}
\left|\frac{\sum_t h_t^4}{w_T^2}\gamma_{2,\varepsilon_\eta}N^{-1}\sum_{i\neq j}\left(\sum\nolimits_\ell\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right)\right| &\leq \frac{\sum_t h_t^4}{w_T^2}\left|\gamma_{2,\varepsilon_\eta}\right|N^{-1}\sum_{i\neq j}\sum\nolimits_\ell\left|\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right|\\
&= O\left(T^{-1}\right),
\end{aligned}
$$

and

$$
N^{-1}\sum_{i\neq j}Cov\left(z_{\eta,i}^2,z_{\eta,j}^2\right)=N^{-1}\sum_{i\neq j}2\rho_{\eta,ij}^2+O\left(T^{-1}\right). \tag{S.55}
$$

Next, using Lemma 15 we have

$$
\begin{aligned}
N^{-1}\sum_{i\neq j}Cov\left(z_{\eta,i}^2X_{\eta,i},z_{\eta,j}^2\right) &= N^{-1}\sum_{i\neq j}\left[E\left(z_{\eta,i}^2X_{\eta,i}z_{\eta,j}^2\right)-E\left(z_{\eta,i}^2X_{\eta,i}\right)E\left(z_{\eta,j}^2\right)\right]\\
&= N^{-1}\sum_{i\neq j}2\rho_{\eta,ij}^2+\frac{\sum_t h_t^4}{w_T^2}\gamma_{2,\varepsilon_\eta}N^{-1}\sum_{i\neq j}\left(\sum\nolimits_\ell\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right)\\
&\quad+\gamma_{1,\varepsilon_\eta}^2\left(\frac{1}{w_T^2v}\sum_t\sum_r h_t^3h_rm_{rr}+3\frac{1}{w_T^2v}\sum_t\sum_r h_t^2h_r^2m_{tr}\right)N^{-1}\sum_{i\neq j}\left(\sum\nolimits_\ell\tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)\left(\sum\nolimits_\ell\tilde{q}_{\eta,i\ell}^3\right)\\
&\quad+2\gamma_{1,\varepsilon_\eta}^2\left(\frac{1}{w_T^2v}\sum_t\sum_r h_t^3h_rm_{rr}+2\frac{1}{w_T^2v}\sum_t\sum_r h_t^2h_r^2m_{tr}\right)N^{-1}\sum_{i\neq j}\left(\sum\nolimits_\ell\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right)^2\\
&\quad+\gamma_{2,\varepsilon_\eta}\left(\frac{1}{w_T^2}\sum_r h_t^4+\frac{1}{w_Tv}\sum_t h_t^2m_{tt}\right)N^{-1}\sum_{i\neq j}\left(\sum\nolimits_\ell\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right)\\
&\quad+4\gamma_{2,\varepsilon_\eta}\left(\frac{1}{w_Tv}\sum_t h_t^2m_{tt}\right)N^{-1}\sum_{i\neq j}\left[\rho_{\eta,ij}\left(\sum\nolimits_\ell\tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right)\right]\\
&\quad+O\left(NT^{-2}\right).
\end{aligned}
$$

But the second term is $O(T^{-1})$ as above. Consider the third term. Using Lemma 10 we have

$$\frac{1}{w_T^2 v}\sum_t\sum_r \left|h_t^3 h_r m_{rr}\right| = O\left(T^{-1}\right), \ \frac{1}{w_T^2 v}\sum_t\sum_r \left|h_t^2 h_r^2 m_{tr}\right| = O\left(T^{-3/2}\right),$$

and noting also $\sum_{i\neq j}\left|\sum_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right|\left|\sum_\ell \tilde{q}_{\eta,i\ell}^3\right| = O(N)$ from Lemma 14 and $\gamma_{1,\varepsilon_\eta}^2 \leq K$ by assumption, we have

$$\begin{aligned}
&\left|\gamma_{1,\varepsilon_\eta}^2\left(\frac{1}{w_T^2 v}\sum_t\sum_r h_t^3 h_r m_{rr} + 3\frac{1}{w_T^2 v}\sum_t\sum_r h_t^2 h_r^2 m_{tr}\right) N^{-1}\sum_{i\neq j}\left(\sum_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)\left(\sum_\ell \tilde{q}_{\eta,i\ell}^3\right)\right| \\
\leq \ & \gamma_{1,\varepsilon_\eta}^2\left(\frac{1}{w_T^2 v}\sum_t\sum_r \left|h_t^3 h_r m_{rr}\right| + 3\frac{1}{w_T^2 v}\sum_t\sum_r \left|h_t^2 h_r^2 m_{tr}\right|\right) N^{-1}\sum_{i\neq j}\left|\sum_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right|\left|\sum_\ell \tilde{q}_{\eta,i\ell}^3\right| \\
= \ & O\left(T^{-1}\right) + O\left(T^{-3/2}\right).
\end{aligned}$$

In a similar manner, the fourth term is $O\left(T^{-1}\right) + O\left(T^{-3/2}\right)$, since $\sum_{i\neq j}\left|\sum_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right|^2 = O(N)$ from Lemma 14. Noting that $0 \leq \sum_t h_t^2 m_{tt} \leq \sum_t h_t^2 = w_T$ and $\left|\gamma_{2,\varepsilon_\eta}\right| \leq K$, the fifth term is $O(T^{-1})$. For the sixth term, noting that $\rho_{\eta,ij} = \sum_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}$, we can write $\rho_{\eta,ij}\left(\sum_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right) = \left(\sum_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}\right)\left(\sum_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right)$, so that

$$\begin{aligned}
&\left|4\gamma_{2,\varepsilon_\eta}\left(\frac{1}{w_T v}\sum_t h_t^2 m_{tt}\right) N^{-1}\sum_{i\neq j}\left[\rho_{\eta,ij}\left(\sum_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right)\right]\right| \\
\leq \ & 4\left|\gamma_{2,\varepsilon_\eta}\right|\left(\frac{1}{w_T v}\sum_t h_t^2 m_{tt}\right) N^{-1}\sum_{i\neq j}\left|\sum_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}\right|\left|\sum_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right| \\
= \ & O(T^{-1}),
\end{aligned}$$

because $\sum_{i\neq j}\left|\sum_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}\right|\left|\sum_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right| = O(N)$ from Lemma 14, $\sum_t h_t^2 m_{tt} \leq w_T$, and $\left|\gamma_{2,\varepsilon_\eta}\right| \leq K$ by assumption. All together we have

$$N^{-1}\sum_{i\neq j} Cov\left(z_{\eta,i}^2 X_{\eta,i}, z_{\eta,j}^2\right) = N^{-1}\sum_{i\neq j} 2\rho_{\eta,ij}^2 + O(T^{-1}) + O\left(NT^{-2}\right). \tag{S.56}$$

By symmetry

$$N^{-1}\sum_{i\neq j} Cov\left(z_{\eta,j}^2 X_{\eta,j}, z_{\eta,i}^2\right) = N^{-1}\sum_{i\neq j} 2\rho_{\eta,ij}^2 + O(T^{-1}) + O\left(NT^{-2}\right). \tag{S.57}$$

Next, consider

$$N^{-1}\sum_{i\neq j} Cov\left(z_{\eta,i}^2 X_{\eta,i}, z_{\eta,j}^2 X_{\eta,j}\right) = N^{-1}\sum_{i\neq j}\left[E\left(z_{\eta,i}^2 X_{\eta,i} z_{\eta,j}^2 X_{\eta,j}\right) - E\left(z_{\eta,i}^2 X_{\eta,i}\right) E\left(z_{\eta,j}^2 X_{\eta,j}\right)\right].$$

Since $E\left(z_{\eta,i}^2 X_{\eta,i}\right) = 1 + \frac{\sum_t h_t^2 m_{tt}}{v w_T}\left(\gamma_{2,\varepsilon_\eta}\sum_\ell \tilde{q}_{\eta,i\ell}^4\right)$ from Lemma 15,

$$E\left(z_{\eta,i}^2 X_{\eta,i}\right) E\left(z_{\eta,j}^2 X_{\eta,j}\right) = 1 + \frac{\sum_t h_t^2 m_{tt}}{v w_T}\gamma_{2,\varepsilon_\eta}\left(\sum_\ell \tilde{q}_{\eta,i\ell}^4 + \sum_\ell \tilde{q}_{\eta,i\ell}^4\right) + \left(\frac{\sum_t h_t^2 m_{tt}}{v w_T}\right)^2 \gamma_{2,\varepsilon_\eta}^2\left(\sum_\ell \tilde{q}_{\eta,i\ell}^4\right)\left(\sum_\ell \tilde{q}_{\eta,j\ell}^4\right),$$

and together with (S.46) we have

$$\begin{aligned}
N^{-1}\sum_{i\neq j} Cov\left(z_{\eta,i}^2 X_{\eta,i}, z_{\eta,j}^2 X_{\eta,j}\right) = \ & N^{-1}\sum_{i\neq j} 2\rho_{\eta,ij}^2 + \frac{\sum_t h_t^4}{w_T^2}\gamma_{2,\varepsilon_\eta} N^{-1}\sum_{i\neq j}\left(\sum_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right) \\
& -\left(\frac{\sum_t h_t^2 m_{tt}}{v w_T}\right)^2 \gamma_{2,\varepsilon_\eta}^2 N^{-1}\sum_{i\neq j}\left(\sum_\ell \tilde{q}_{\eta,i\ell}^4\right)\left(\sum_\ell \tilde{q}_{\eta,j\ell}^4\right) \\
& +2\left(-\frac{1}{w_T^2}\sum_t h_t^4 - \frac{18}{v w_T}\sum_t h_t^2 m_{tt} - \frac{2}{v^2}\sum_t m_{tt}^2 + \frac{1}{v}\right) N^{-1}\sum_{i\neq j}\rho_{\eta,ij}^2 \\
& +2\left(\frac{2}{v} - \frac{2}{v^2}\sum_t m_{tt}^2\right) N^{-1}\sum_{i\neq j}\rho_{\eta,ij}^4
\end{aligned}$$

$$+\left(\frac{2}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rr} m_{tt} + \frac{1}{v^2 w_T}\sum_t\sum_r h_t^2 m_{rr} m_{tr} + \frac{2}{v w_T^2}\sum_t\sum_r h_r^3 h_t m_{tt}\right)$$
$$\times\gamma_{1,\varepsilon_\eta}^2 N^{-1}\sum_{i\neq j}\left[\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,j\ell}^3\right)+\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)\right]$$
$$+\gamma_{1,\varepsilon_\eta}^2\left(\frac{4}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rr} m_{tt}\right)N^{-1}\sum_{i\neq j}\rho_{\eta,ij}\left(\sum\nolimits_\ell \tilde{q}_{\eta,j\ell}^3\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\right)$$
$$+\left(4\frac{1}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rt}^2 + \frac{1}{v^2 w_T}\sum_t\sum_r h_t^2 m_{rr} m_{tr} + 2\frac{1}{v w_T^2}\sum_t\sum_r h_r^3 h_t m_{tt}\right)$$
$$\times N^{-1}\sum_{i\neq j}\left[\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)^2+\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right)^2\right]$$
$$+\left(4\frac{1}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rr} m_{tt} + 16\frac{1}{v^2 w_T}\sum_t\sum_r h_r h_t m_{rt}^2 + 8\frac{1}{v^2 w_T}\sum_t\sum_r h_t^2 m_{rr} m_{tr}\right)$$
$$\times\gamma_{1,\varepsilon_\eta}^2 N^{-1}\sum_{i\neq j}\rho_{\eta,ij}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)$$

$$+\left(4\frac{1}{v w_T}\sum_t h_t^2 m_{tt}\right)\left[\gamma_{2,\varepsilon_\eta}N^{-1}\sum_{i\neq j}\rho_{\eta,ij}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right)+3N^{-1}\sum_{i\neq j}\rho_{\eta,ij}^2\right]$$
$$+\left(2\frac{1}{v w_T}\sum_t h_t^2 m_{tt} + \frac{1}{v^2}\sum_t m_{tt}^2\right)$$
$$\times\left[\gamma_{2,\varepsilon_\eta}N^{-1}\sum_{i\neq j}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right)+2N^{-1}\sum_{i\neq j}\rho_{\eta,ij}^2\right]+2\frac{1}{w_T^2}\sum_t h_t^4 N^{-1}\sum_{i\neq j}\rho_{\eta,ij}^2$$
$$+\left(2\frac{1}{v^2}\sum_t m_{tt}^2\right)N^{-1}\sum_{i\neq j}\rho_{\eta,ij}^2\left[\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right)+\left(1+2\rho_{\eta,ij}^2\right)\right]$$
$$+\rho_{\eta,ij}\left(4\frac{1}{v w_T}\sum_t h_t^2 m_{tt}\right)N^{-1}\sum_{i\neq j}\rho_{\eta,ij}\left(\gamma_{2,\varepsilon_\eta}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right)+3\rho_{\eta,ij}\right)$$
$$+O\left(NT^{-2}\right).$$

As established earlier, the second term is $O(T^{-1})$. Noting that $0<\sum_t h_t^2 m_{tt}\leq w_T$, and also $\sum_\ell \tilde{q}_{\eta,i\ell}^4\leq 1$, we have

$$\left|\left(\frac{\sum_t h_t^2 m_{tt}}{v w_T}\right)^2\gamma_{2,\varepsilon_\eta}^2 N^{-1}\sum_{i\neq j}\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^4\right)\left(\sum\nolimits_\ell \tilde{q}_{\eta,j\ell}^4\right)\right|\leq\left(\frac{\sum_t h_t^2 m_{tt}}{v w_T}\right)^2\gamma_{2,\varepsilon_\eta}^2 N = O\left(NT^{-2}\right).$$

In a similar manner, noting that (from Lemma 10)

$$0<\frac{1}{v^2}\sum_t m_{tt}^2 = O\left(T^{-1}\right),\ \frac{1}{v^2 w_T}\sum_t\sum_r\left|h_r h_t m_{rr} m_{tt}\right| = O\left(T^{-1}\right),$$

$$\frac{1}{v^2 w_T}\sum_t\sum_r\left|h_r h_t m_{rt}^2\right| = O\left(T^{-3/2}\right),\ \frac{1}{v^2 w_T}\sum_t\sum_r\left|h_t^2 m_{rr} m_{tr}\right| = O\left(T^{-3/2}\right),\ \frac{2}{v w_T^2}\sum_t\sum_r h_r^3 h_t m_{tt} = O\left(T^{-1}\right),$$

and (from Lemma 14)

$$\sum_{i\neq j}\rho_{\eta,ij}^2 = O(N),\ \sum_{i\neq j}\rho_{\eta,ij}^4 = O(N),\ \sum_{i\neq j}\left|\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right|\left|\sum\nolimits_\ell \tilde{q}_{\eta,j\ell}^3\right| = O(N),\ \sum_{i\neq j}\left|\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right|\left|\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\right| = O(N),$$

$$\sum_{i\neq j}\left|\rho_{\eta,ij}\right|\left|\sum\nolimits_\ell \tilde{q}_{\eta,j\ell}^3\right|\left|\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\right| = O(N),\ \sum_{i\neq j}\left[\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right)^2+\left(\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right)^2\right] = O(N),$$

$$\sum_{i\neq j}\left|\rho_{\eta,ij}\right|\left|\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}\right|\left|\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}\tilde{q}_{\eta,j\ell}^2\right| = O(N),\ \sum_{i\neq j}\left|\rho_{\eta,ij}\right|\left|\sum\nolimits_\ell \tilde{q}_{\eta,i\ell}^3\tilde{q}_{\eta,j\ell}\right| = O(N),$$

and by assumption $\left|\gamma_{1,\varepsilon_\eta}\right|\leq K$ and $\left|\gamma_{2,\varepsilon_\eta}\right|\leq K$, we have

$$N^{-1}\sum_{i\neq j}Cov\left(z_{\eta,i}^2 X_{\eta,i}, z_{\eta,j}^2 X_{\eta,j}\right) = N^{-1}\sum_{i\neq j}2\rho_{\eta,ij}^2 + O\left(T^{-1}\right) + O\left(NT^{-2}\right). \tag{S.58}$$

Using (S.55), (S.56), (S.57), and (S.58), we conclude

$$\begin{aligned} & N^{-1}\sum_{i\neq j} Cov\left[z_{\eta,i}^{2}\left(X_{\eta,i}-1\right), z_{\eta,j}^{2}\left(X_{\eta,j}-1\right)\right] \\ = \; & N^{-1}\sum_{i\neq j} Cov\left(z_{\eta,i}^{2}, z_{\eta,j}^{2}\right) - N^{-1}\sum_{i\neq j} Cov\left(z_{\eta,i}^{2}X_{\eta,i}, z_{\eta,j}^{2}\right) - N^{-1}\sum_{i\neq j} Cov\left(z_{\eta,i}^{2}, X_{\eta,j}z_{\eta,j}^{2}\right) + N^{-1}\sum_{i\neq j} Cov\left(z_{\eta,i}^{2}X_{\eta,i}, X_{\eta,j}z_{\eta,j}^{2}\right) \\ = \; & O\left(T^{-1}\right) + O\left(NT^{-2}\right), \end{aligned}$$

as required, since the terms $N^{-1}\sum_{i\neq j} 2\rho_{\eta,ij}^{2}$ will cancel out. ■

Lemma 17 *Consider the return regressions, (2), and suppose that Assumptions 1-4 hold. Let* $z_i^2 = \boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i/w_T > 0$ *and* $X_i = \boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i/v > 0$*, where* $\mathbf{H}_F = (h_t h_{t'})$ *and* $\mathbf{M}_G = (m_{tt'})$ *are defined by (S.2),* $w_T = \boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T$*,* $v = T - m - 1$*,* $\boldsymbol{\xi}_i = (\xi_{i1}, \xi_{i2}, ..., \xi_{iT})'$*,* $\xi_{it} = u_{it}/\sigma_{ii}^{1/2}$*,* $\sigma_{ij} = E(u_{it}u_{jt})$ *and* $E(\xi_{it}\xi_{jt}) = \rho_{ij}$*. Also let* $z_{\eta,i}^2 = \boldsymbol{\eta}_i'\mathbf{H}_F\boldsymbol{\eta}_i/(w_T\sigma_{\eta,ii}) > 0$*,* $X_{\eta,i} = \boldsymbol{\eta}_i'\mathbf{M}_G\boldsymbol{\eta}_i/(v\sigma_{\eta,ii}) > 0$*. Then,*

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} z_i^2\left(1-X_i\right) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} z_{\eta,i}^2\left(1-X_{\eta,i}\right) + O_p\left(N^{\delta_\gamma - 1/2}\right).$$

Proof. Recalling from (46) that $\mathbf{u}_{i.} = \mathbf{V}\gamma_i + \boldsymbol{\eta}_i = \sum_{s=1}^{k}\mathbf{v}_s\gamma_{is} + \boldsymbol{\eta}_i$, we have

$$z_i^2 = \frac{\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i}{w_T} = \frac{1}{\sigma_{ii}}\frac{\mathbf{u}_{i.}'\mathbf{H}_F\mathbf{u}_{i.}}{w_T} = \left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}z_{\eta,i}^2 + A_i\right), \tag{S.59}$$

where

$$A_i = \frac{\tilde{\gamma}_i'\mathbf{V}'\mathbf{H}_F\mathbf{V}\tilde{\gamma}_i}{w_T} + 2\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\frac{\tilde{\gamma}_i'\mathbf{V}'\mathbf{H}_F\tilde{\boldsymbol{\eta}}_i}{w_T},$$

with $\tilde{\gamma}_i = (\tilde{\gamma}_{i1}, \tilde{\gamma}_{i2}, ..., \tilde{\gamma}_{ik})' = \gamma_i/\sigma_{ii}^{1/2}$, and $\tilde{\boldsymbol{\eta}}_i = \boldsymbol{\eta}_i/\sigma_{\eta,ii}^{1/2}$. Similarly,

$$X_i = \frac{\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i}{v} = \frac{1}{\sigma_{ii}}\frac{\mathbf{u}_{i.}'\mathbf{M}_G\mathbf{u}_{i.}}{v} = \left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}X_{\eta,i} + B_i\right), \tag{S.60}$$

where

$$B_i = \frac{\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\mathbf{V}\tilde{\gamma}_i}{v} + 2\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\frac{\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i}{v}.$$

Using the above results we obtain

$$z_i^2\left(1-X_i\right) = \left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}z_{\eta,i}^2 + A_i\right)\left[1 - X_{\eta,i} + X_{\eta,i}\left(1 - \frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right) - B_i\right],$$

and since $1 - \sigma_{\eta,ii}/\sigma_{ii} = \gamma_i'\gamma_i/\sigma_{ii}$, then (after some algebra) we have

$$\begin{aligned} & \frac{1}{\sqrt{N}}\sum_{i=1}^{N} z_i^2\left(1-X_i\right) - \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\sigma_{\eta,ii}}{\sigma_{ii}}z_{\eta,i}^2\left(1-X_{\eta,i}\right) \\ = \; & \left[\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)z_{\eta,i}^2X_{\eta,i} + A_iX_{\eta,i}\right]\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right) \\ & - \left[A_iB_i + \left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)z_{\eta,i}^2B_i\right] + A_i\left(1-X_{\eta,i}\right) \\ = \; & D_{N,1} + D_{N,2} + D_{N,3}, \end{aligned}$$

where

$$\begin{aligned} D_{N,1} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)z_{\eta,i}^2X_{\eta,i} + A_iX_{\eta,i}\right]\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right), \\ D_{N,2} &= -\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[A_iB_i + \left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)z_{\eta,i}^2B_i\right], \text{ and} \\ D_{N,3} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}A_i\left(1-X_{\eta,i}\right). \end{aligned}$$

Noting that $0 < \frac{\sigma_{\eta,ii}}{\sigma_{ii}} \leq 1$ and $\sup_i|\tilde{\gamma}_{is}| \leq 1$, we have

$$\left|D_{N,1}\right| \leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\left|z_{\eta,i}^2\right| + \left|A_i\right|\right)\left|X_{\eta,i}\right|\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right).$$

Also since $\mathbf{H}_F = \mathbf{h}\mathbf{h}'$, $\mathbf{h} = \mathbf{M}_F \boldsymbol{\tau}_T$, and noting that for any conformable real symmetric positive semi-definite matrices $\mathbf{A}$ and $\mathbf{B}$, $Tr(\mathbf{AB}) \leq Tr(\mathbf{A})\lambda_{\max}(\mathbf{B}) \leq Tr(\mathbf{A})Tr(\mathbf{B})$ (this result is repeatedly used below), we have

$$\begin{aligned} |A_i| &\leq \frac{\tilde{\gamma}_i'\mathbf{V}'\mathbf{H}_F\mathbf{V}\tilde{\gamma}_i}{w_T} + 2\frac{|\tilde{\gamma}_i'\mathbf{V}'\mathbf{h}\mathbf{h}'\tilde{\boldsymbol{\eta}}_i|}{w_T} \\ &\leq \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\lambda_{\max}\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right) + 2\frac{|\tilde{\gamma}_i'\mathbf{V}'\mathbf{h}|\,|\mathbf{h}'\tilde{\boldsymbol{\eta}}_i|}{w_T}, \end{aligned} \tag{S.61}$$

and therefore

$$|D_{N,1}| \leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\left|z_{\eta,i}^2\right| + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\lambda_{\max}\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right) + 2\frac{|\tilde{\gamma}_i'\mathbf{V}'\mathbf{h}|\,|\mathbf{h}'\tilde{\boldsymbol{\eta}}_i|}{w_T}\right]|X_{\eta,i}|\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right),$$

and taking expectations of both sides and noting that $\tilde{\gamma}_i$ and $\mathbf{h}$ are non-stochastic then

$$\begin{aligned} E|D_{N,1}| &\leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left[E\left(z_{\eta,i}^4\right)\right]^{1/2}\left[E\left(X_{\eta,i}^2\right)\right]^{1/2} \\ &+ \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^2 E\left[\lambda_{\max}\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)X_{\eta,i}\right] \\ &+ \frac{2}{\sqrt{N}}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left[E\left(X_{\eta,i}^2\right)\right]^{1/2}\left[\frac{E\left(|\tilde{\gamma}_i'\mathbf{V}'\mathbf{h}|^2|\mathbf{h}'\tilde{\boldsymbol{\eta}}_i|^2\right)}{w_T^2}\right]^{1/2}. \end{aligned}$$

But $E\left(z_{\eta,i}^4\right) < K$, and $E\left(X_{\eta,i}^2\right) < K$ (see Lemma 15), and since $\mathbf{v}_t$ and η_{it} are independently distributed (by assumption), we have

$$\begin{aligned} E|D_{N,1}| &\leq \frac{K}{\sqrt{N}}\sum_{i=1}^{N}\tilde{\gamma}_i'\tilde{\gamma}_i + E\left[\lambda_{\max}\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^2 E\left(X_{\eta,i}\right) \\ &+ \frac{K}{\sqrt{N}}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\frac{E|\tilde{\gamma}_i'\mathbf{V}'\mathbf{h}|^2 E|\mathbf{h}'\tilde{\boldsymbol{\eta}}_i|^2}{w_T^2}\right)^{1/2}. \end{aligned}$$

Further

$$\begin{aligned} w_T^{-1}E\left|\tilde{\gamma}_i'\mathbf{V}'\mathbf{h}\right|^2 &= w_T^{-1}E\left(\tilde{\gamma}_i'\mathbf{V}'\mathbf{h}\mathbf{h}'\mathbf{V}\tilde{\gamma}_i\right) \leq E\left[\lambda_{\max}\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right), \\ w_T^{-1}E\left|\mathbf{h}'\tilde{\boldsymbol{\eta}}_i\right|^2 &= w_T^{-1}E\left(\tilde{\boldsymbol{\eta}}_i'\mathbf{h}\mathbf{h}'\tilde{\boldsymbol{\eta}}_i\right) = w_T^{-1}E\left(\tilde{\boldsymbol{\eta}}_i'\mathbf{H}_F\tilde{\boldsymbol{\eta}}_i\right) = E\left(z_{\eta,i}^2\right) = 1. \end{aligned}$$

Hence, noting that $E(X_{\eta,i}) = 1$ and $\lambda_{\max}\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right) \leq Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)$,

$$\begin{aligned} E|D_{N,1}| &\leq \frac{K}{\sqrt{N}}\left[\sum_{i=1}^{N}\tilde{\gamma}_i'\tilde{\gamma}_i + E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^2\right. \\ &\left. + \left\{E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\right\}^{1/2}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2}\right]. \end{aligned}$$

Also $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$, $\mathbf{v}_s = (v_{s1}, v_{s2}, ..., v_{sT})'$ and by assumption $E\left(\mathbf{v}_s\mathbf{v}_{s'}'\right) = \mathbf{0}$, for $s \neq s'$, and $E\left(\mathbf{v}_s\mathbf{v}_s'\right) = \mathbf{I}_T$. Then $E\left(\mathbf{V}\mathbf{V}'\right) = k\mathbf{I}_T$, and $E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right] = kw_T^{-1}Tr\left(\mathbf{H}_F\right) = k$. Hence

$$E|D_{N,1}| \leq \frac{K}{\sqrt{N}}\left[\sum_{i=1}^{N}\tilde{\gamma}_i'\tilde{\gamma}_i + k\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^2 + k^{1/2}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2}\right].$$

Finally, since $\tilde{\gamma}_i'\tilde{\gamma}_i = \sum_{s=1}^{k}\tilde{\gamma}_{is}^2$, and $|\tilde{\gamma}_{is}| \leq 1$, then

$$\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^2 \leq k\left(\sum_{s=1}^{k}\tilde{\gamma}_{is}^2\right), \quad \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2} \leq k^{1/2}\left(\sum_{s=1}^{k}\tilde{\gamma}_{is}^2\right),$$

and

$$E|D_{N,1}| \leq \frac{K(k^2+k+1)\left(\sum_{i=1}^{N}\tilde{\gamma}_i'\tilde{\gamma}_i\right)}{\sqrt{N}} \leq \frac{K_1}{\sqrt{N}}\sup_s\sum_{i=1}^{N}\tilde{\gamma}_{is}^2 \leq \frac{K_1}{\sqrt{N}}\sup_s\sum_{i=1}^{N}|\tilde{\gamma}_{is}| = O\left(N^{\delta_\gamma - 1/2}\right),$$

and by Markov theorem $D_{N,1} = O_p\left(N^{\delta_\gamma - 1/2}\right)$. Similarly, for $D_{N,2}$, we first note that

$$\begin{aligned} A_i B_i &= \left[\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \mathbf{V} \tilde{\gamma}_i}{w_T} + 2\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i}{w_T}\right] \left[\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \mathbf{V} \tilde{\gamma}_i}{v} + 2\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \tilde{\boldsymbol{\eta}}_i}{v}\right] \\ &= \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \mathbf{V} \tilde{\gamma}_i}{w_T} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \mathbf{V} \tilde{\gamma}_i}{v} + 2\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \mathbf{V} \tilde{\gamma}_i}{w_T} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \tilde{\boldsymbol{\eta}}_i}{v} \\ &\quad + 2\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i}{w_T} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \mathbf{V} \tilde{\gamma}_i}{v} + 4\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right) \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i}{w_T} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \tilde{\boldsymbol{\eta}}_i}{v}. \end{aligned}$$

Also

$$z_{\eta,i}^2 B_i = z_{\eta,i}^2 \left[\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \mathbf{V} \tilde{\gamma}_i}{v} + 2\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2} \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \tilde{\boldsymbol{\eta}}_i}{v}\right],$$

and

$$|D_{N,2}| \leq \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(|A_i B_i| + \left|z_{\eta,i}^2 B_i\right|\right).$$

Consider the terms involving $A_i B_i$. Since $0 < \frac{\sigma_{\eta,ii}}{\sigma_{ii}} \leq 1$, note that

$$\begin{aligned} |A_i B_i| &\leq \left(\tilde{\gamma}_i' \tilde{\gamma}_i\right)^2 \lambda_{\max}\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right) \lambda_{\max}\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right) \\ &\quad + 2\left(\tilde{\gamma}_i' \tilde{\gamma}_i\right) \lambda_{\max}\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \tilde{\boldsymbol{\eta}}_i}{v}\right| \\ &\quad + 2\left(\tilde{\gamma}_i' \tilde{\gamma}_i\right) \lambda_{\max}\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i}{w_T}\right| \\ &\quad + 4 \frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i \tilde{\boldsymbol{\eta}}_i' \mathbf{M}_G \mathbf{V} \tilde{\gamma}_i}{v w_T} \\ &\leq \left(\tilde{\gamma}_i' \tilde{\gamma}_i\right)^2 \lambda_{\max}\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right) \lambda_{\max}\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right) \\ &\quad + 2\left(\tilde{\gamma}_i' \tilde{\gamma}_i\right) \lambda_{\max}\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \tilde{\boldsymbol{\eta}}_i}{v}\right| \\ &\quad + 2\left(\tilde{\gamma}_i' \tilde{\gamma}_i\right) \lambda_{\max}\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i}{w_T}\right| \\ &\quad + \frac{4}{v w_T} \left(\tilde{\gamma}_i' \tilde{\gamma}_i\right) \left(\tilde{\boldsymbol{\eta}}_i' \mathbf{M}_G \mathbf{V} \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i\right), \end{aligned}$$

and hence (again noting that $\tilde{\boldsymbol{\eta}}_i$ and $\mathbf{V}$ are distributed independently and $\mathbf{M}_G \mathbf{H}_F = \mathbf{M}_G \mathbf{M}_F \boldsymbol{\tau}_T \boldsymbol{\tau}_T' \mathbf{M}_F = \mathbf{0}$)

$$\begin{aligned} E\left|A_i B_i\right| &\leq \left(\tilde{\gamma}_i' \tilde{\gamma}_i\right)^2 E\left\{\left[Tr\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right)\right] \left[Tr\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right)\right]\right\} \\ &\quad + 2\left(\tilde{\gamma}_i' \tilde{\gamma}_i\right) E\left[\lambda_{\max}\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \tilde{\boldsymbol{\eta}}_i}{v}\right|\right] \\ &\quad + 2\left(\tilde{\gamma}_i' \tilde{\gamma}_i\right) E\left[\lambda_{\max}\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i}{w_T}\right|\right], \end{aligned}$$

where

$$\begin{aligned} & E\left[\lambda_{\max}\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \tilde{\boldsymbol{\eta}}_i}{v}\right|\right] \\ \leq\; & E\left[\lambda_{\max}\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{M}_G \mathbf{V} \tilde{\gamma}_i}{v}\right|^{1/2} X_{\eta,i}^{1/2}\right] \\ \leq\; & \left(\tilde{\gamma}_i' \tilde{\gamma}_i\right)^{1/2} E\left(X_{\eta,i}^{1/2}\right) E\left[Tr\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right) Tr\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right)^{1/2}\right] \end{aligned}$$

and

$$\begin{aligned} & E\left[\lambda_{\max}\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \tilde{\boldsymbol{\eta}}_i}{w_T}\right|\right] \\ \leq\; & E\left[\lambda_{\max}\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right) \left|\frac{\tilde{\gamma}_i' \mathbf{V}' \mathbf{H}_F \mathbf{V} \tilde{\gamma}_i}{w_T}\right|^{1/2} z_{\eta,i}\right] \\ \leq\; & \left(\tilde{\gamma}_i' \tilde{\gamma}_i\right)^{1/2} E\left(z_{\eta,i}\right) E\left[Tr\left(v^{-1} \mathbf{V}' \mathbf{M}_G \mathbf{V}\right) Tr\left(w_T^{-1} \mathbf{V}' \mathbf{H}_F \mathbf{V}\right)^{1/2}\right], \end{aligned}$$

so that

$$\begin{aligned} E\left|A_iB_i\right| &\leq \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^2 E\left\{\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right]\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\right\} \\ &+2\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2} E\left(X_{\eta,i}^{1/2}\right) E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right) Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)^{1/2}\right] \\ &+2\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2} E\left(z_{\eta,i}\right) E\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right) Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)^{1/2}\right]. \end{aligned}$$

Since

$$Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right) = w_T^{-1}\sum_\ell\sum_t\sum_s h_t h_s v_{t\ell}v_{s\ell},$$

noting that all the elements of $\mathbf{V}$ are independent of each other by assumption, we have

$$\begin{aligned} E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]^2 &= w_T^{-2}\sum_\ell\sum_{\ell'}\sum_t\sum_s\sum_{t'}\sum_{s'} h_t h_s h_{t'} h_{s'} E\left(v_{t\ell}v_{s\ell}v_{t'\ell'}v_{s'\ell'}\right). \\ &= w_T^{-2}k\sum_t h_t^4 E\left(v_{t\ell}^4\right) + w_T^{-2}k^2\sum_t h_t^4\left[E\left(v_{t\ell}^2\right)\right]^2 \\ &\quad + w_T^{-2}k^2\sum_t\sum_s h_t^2h_s^2\left[E\left(v_{t\ell}^2\right)\right]^2 \\ &\quad + w_T^{-2}2k\sum_t\sum_s h_t^2h_s^2\left[E\left(v_{t\ell}^2\right)\right]^2 \\ &= w_T^{-2}\sum_t h_t^4 k\left[E\left(v_{t\ell}^4\right)+k\right] + k\left(k+2\right), \end{aligned} \tag{S.62}$$

since $\sum_t h_t^4 w_T^{-2} = O(T^{-1})$, $E\left(v_{t\ell}^2\right) = 1$, and $w_T^{-2}\sum_t\sum_s h_t^2h_s^2 = 1$, which is bounded as $E\left(v_{s\ell}^4\right) \leq K$ (by assumption). Similarly, as

$$Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right) = v^{-1}\sum_\ell\sum_t\sum_s m_{ts}v_{t\ell}v_{s\ell},$$

we have

$$E\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right] = k,$$

$$\begin{aligned} E\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)^2\right] &= v^{-2}\sum_\ell\sum_{\ell'}\sum_t\sum_s\sum_{t'}\sum_{s'} h_t h_s h_{t'} h_{s'} E\left(v_{t\ell}v_{s\ell}v_{t'\ell'}v_{s'\ell'}\right). \\ &= v^{-2}\sum_t m_{tt}^2 k\left[E\left(v_{t\ell}^4\right)+k\right] + k\left(k+2\right), \end{aligned} \tag{S.63}$$

as $v^{-2}\sum_t m_{tt}^2 \leq v^{-2}\sum_t m_{tt} = v^{-1}$ and $v^{-2}\sum_t\sum_s m_{ts}^2 = v^{-1}$, which is bounded. Using these results, we have

$$\begin{aligned} & E\left\{\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right]\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\right\} \\ \leq & \left(E\left\{\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right]^2\right\}\right)^{1/2}\left(E\left\{\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]^2\right\}\right)^{1/2} \leq K, \end{aligned}$$

$$\begin{aligned} & E\left(X_{\eta,i}^{1/2}\right) E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right) Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)^{1/2}\right] \\ \leq & E\left(X_{\eta,i}^{1/2}\right)\left(E\left\{\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]^2\right\}\right)^{1/2} k^{1/2} \leq K \end{aligned}$$

as $E\left(X_{\eta,i}^{1/2}\right) \leq K$ since $E\left(X_{\eta,i}\right) = 1$,

$$\begin{aligned} & E\left(z_{\eta,i}\right) E\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right) Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)^{1/2}\right] \\ \leq & E\left(z_{\eta,i}\right)\left(E\left\{\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right]^2\right\}\right)^{1/2} k^{1/2} \\ \leq & K \end{aligned}$$

as $E\left(z_{\eta,i}\right) \leq K$ since $E\left(z_{\eta,i}^2\right) = 1$, so that

$$E\left|A_iB_i\right| \leq K\left[\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^2 + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2}\right].$$

Further, as $0 < \frac{\sigma_{\eta,ii}}{\sigma_{ii}} \leq 1$,

$$\begin{aligned} \left|z_{\eta,i}^2 B_i\right| &\leq \left|z_{\eta,i}^2\right|\left|\frac{\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\mathbf{V}\tilde{\gamma}_i}{v}\right| + 2\left|z_{\eta,i}^2\frac{\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i}{v}\right| \\ &\leq \tilde{\gamma}_i'\tilde{\gamma}_i\left|z_{\eta,i}^2\right|\left|\lambda_{\max}\left(\frac{\mathbf{V}'\mathbf{M}_G\mathbf{V}}{v}\right)\right| + 2\left|z_{\eta,i}^2\right|\left|\frac{\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i}{v}\right| \end{aligned}$$

and taking expectation we have

$$\begin{aligned} E\left|z_{\eta,i}^2 B_i\right| &\leq \tilde{\gamma}_i'\tilde{\gamma}_i E\left(z_{\eta,i}^2\right) E\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right] \\ &\quad + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left(E\left|z_{\eta,i}^2\right|^2\right)^{1/2}\left[E\left(v^{-2}\tilde{\boldsymbol{\eta}}_i'\mathbf{M}_G\mathbf{V}\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i\right)\right]^{1/2} \end{aligned}$$

but as $E\left|z_{\eta,i}^2\right|^2$ is bounded (see Lemma 15), $E\left[Tr\left(v^{-1}\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right] = k$,

$$E\left(v^{-2}\tilde{\boldsymbol{\eta}}_i'\mathbf{M}_G\mathbf{V}\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i\right) = v^{-2}Tr\left[E\left(\tilde{\boldsymbol{\eta}}_i\tilde{\boldsymbol{\eta}}_i'\right)\mathbf{M}_G E\left(\mathbf{V}\mathbf{V}'\right)\mathbf{M}_G\right] = v^{-1},$$

we have

$$E\left|z_{\eta,i}^2 B_i\right| \leq K\left[\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right) + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\right].$$

Thus

$$\begin{aligned} \left|D_{N,2}\right| &\leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\left|A_iB_i\right| + \left|z_{\eta,i}^2 B_i\right|\right) \\ &\leq \frac{1}{\sqrt{N}}K\sum_{i=1}^{N}\left[\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^2 + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2} + \tilde{\gamma}_i'\tilde{\gamma}_i + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\right] \\ &= O(N^{\delta_\gamma - 1/2}). \end{aligned}$$

Similarly, for $D_{N,3}$,

$$\left|D_{N,3}\right| \leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left|A_i\left(1 - X_{\eta,i}\right)\right| \leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\left|A_i\right| + \left|A_iX_{\eta,i}\right|\right).$$

Noting $0 < \frac{\sigma_{\eta,ii}}{\sigma_{ii}} \leq 1$ and $\mathbf{H}_F = \mathbf{h}\mathbf{h}'$,

$$\begin{aligned} E\left|A_i\right| &\leq E\left|w_T^{-1}\tilde{\gamma}_i'\mathbf{V}'\mathbf{H}_F\mathbf{V}'\tilde{\gamma}_i\right| + 2E\left|w_T^{-1}\tilde{\gamma}_i'\mathbf{V}'\mathbf{H}_F\tilde{\boldsymbol{\eta}}_i\right| \\ &\leq \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)E\left[\lambda_{\max}\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right] + 2\left[E\left|w_T^{-1}\tilde{\gamma}_i'\mathbf{V}'\mathbf{H}_F\mathbf{V}\tilde{\gamma}_i\right|\right]^{1/2}\left(E\left|z_{\eta,i}^2\right|\right)^{1/2} \\ &\leq \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right] + 2\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left\{E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\right\}^{1/2}\left(E\left|z_{\eta,i}^2\right|\right)^{1/2} \\ &\leq K\left[\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right) + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\right], \end{aligned}$$

as $E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right] = k$ and $E\left|z_{\eta,i}^2\right| = E\left(z_{\eta,i}^2\right) = 1$. Similarly, noting the independence between $\mathbf{V}$ and $\boldsymbol{\eta}_i$,

$$\begin{aligned} E\left|A_iX_{\eta,i}\right| &\leq \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)E\left[Tr\left(w_T^{-1}\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]E\left(X_{\eta,i}\right) \\ &\quad +2\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left[E\left(X_{\eta,i}^2\right)\right]^{1/2}\left\{E\left(w^{-2}\tilde{\boldsymbol{\eta}}_i'\mathbf{H}_F\mathbf{V}\mathbf{V}'\mathbf{H}_F\tilde{\boldsymbol{\eta}}_i\right)\right\}^{1/2} \\ &= K\left[\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right) + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\right], \end{aligned}$$

as $E\left(v^{-2}\tilde{\boldsymbol{\eta}}_i'\mathbf{H}_F\mathbf{V}\mathbf{V}'\mathbf{H}_F\tilde{\boldsymbol{\eta}}_i\right)E\left(X_{\eta,i}^2\right)$ is bounded (by Lemma 15) and

$$E\left(w^{-2}\tilde{\boldsymbol{\eta}}_i'\mathbf{H}_F\mathbf{V}\mathbf{V}'\mathbf{H}_F\tilde{\boldsymbol{\eta}}_i\right) = w^{-2}Tr\left[E\left(\tilde{\boldsymbol{\eta}}_i\tilde{\boldsymbol{\eta}}_i'\right)\mathbf{H}_F E\left(\mathbf{V}\mathbf{V}'\right)\mathbf{H}_F\right] = w^{-2}Tr\left(\mathbf{H}_F^2\right) = 1.$$

Thus,

$$\left|D_{N,3}\right| \leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}K\left[\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right) + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\right] = O(N^{\delta_\gamma - 1/2}).$$

Finally,

$$\begin{aligned} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(1 - \frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)E\left|z_{\eta,i}^2\left(1 - X_{\eta,i}\right)\right| &\leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left[\left(E\left|z_{\eta,i}^2\right|^2\right)^{1/2}\left(E\left|1 - X_{\eta,i}\right|^2\right)^{1/2}\right] \\ &= O\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\right) = O\left(N^{\delta_\gamma - 1/2}\right), \end{aligned}$$

as $E\left|z_{\eta,i}^2\right|^2 \leq K$ and $E\left|X_{\eta,i}\right|^2 \leq K$ from Lemma 15. Therefore, we have

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}z_i^2\left(1 - X_i\right) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}z_{\eta,i}^2\left(1 - X_{\eta,i}\right) + O_p\left(N^{\delta_\gamma - 1/2}\right),$$

as required. ■

Lemma 18 *Consider the regression model (8), and suppose that Assumptions 1-4 hold. Under* $H_0 : \alpha_i = 0$, *in (2) for all* i,

$$\theta_N^2 - (N-1)\rho_N^2 \to 0 \tag{S.64}$$

as N *and* $T \to \infty$, *so long as* $0 < \delta_\gamma < 1/2$, *and* $N/T^2 \to 0$, *where* θ_N^2, ρ_N^2, *and* δ_γ *are defined by (29), (54) and (6), respectively.*

Proof. Theorem 1 ensures that $N^{-1/2}\sum_i \left(z_i^2 - 1\right) / \left[2(1+(N-1)\rho_N^2\right]^{1/2} \to_d N(0,1)$ for $\left[2(1+(N-1)\rho_N^2\right] = O(1)$. Then, Theorem 2 ensures that $N^{-1/2}\sum_i \left(t_i^2 - z_i^2\right) \to_p 0$, so long as $\delta_\gamma < 1/2$ and $N/T^2 \to 0$ as N and $T \to \infty$, which ensures that (from Lemma 21) $Var\left(N^{-1/2}\sum_i t_i^2\right) = \left[\left(\frac{v}{v-2}\right)^2 \frac{2(v-1)}{(v-4)} + O\left(v^{-1}\right)\right]\left(1+\theta_N^2\right) = O(1)$ and $Var\left(N^{-1/2}\sum_i t_i^2\right) - Var\left(N^{-1/2}\sum_i z_i^2\right) \to 0$, since $\left(\frac{v}{v-2}\right)^2 \frac{2(v-1)}{(v-4)} = 2 + O\left(v^{-1}\right)$, which establishes the required result. ■

Lemma 19 *Consider the panel regression model (2), and suppose that Assumptions 1-4 hold. Denote the OLS residuals from the regression of* y_{it} *on* $\mathbf{G} = (\boldsymbol{\tau}_T, \mathbf{F})$ *by* $\hat{\mathbf{u}}_{i.} = (\hat{u}_{i1}, \hat{u}_{i2}, ..., \hat{u}_{iT})'$, *and denote the correlation coefficient of* $\hat{\mathbf{u}}_{i.}$ *and* $\hat{\mathbf{u}}_{j.}$ *by*

$$\hat{\rho}_{ij} = \frac{\hat{\mathbf{u}}_{i.}'\hat{\mathbf{u}}_{j.}}{\left(\hat{\mathbf{u}}_{i.}'\hat{\mathbf{u}}_{i.}\right)^{1/2}\left(\hat{\mathbf{u}}_{j.}'\hat{\mathbf{u}}_{j.}\right)^{1/2}}. \tag{S.65}$$

Then

$$\hat{\rho}_{ij} = \frac{\sum_{t=1}^{v}\zeta_{it}\zeta_{jt}}{\left(\sum_{t=1}^{v}\zeta_{it}^2\right)^{1/2}\left(\sum_{t=1}^{v}\zeta_{jt}^2\right)^{1/2}}, \tag{S.66}$$

where $v = T - m - 1$,

$$\zeta_{it} = \sum_{t'=1}^{T} l_{tt'}\xi_{it'}, \tag{S.67}$$

$\xi_{it} = u_{it}/\sigma_{ii}^{1/2}$, $l_{tt'}$ *is the* (t, t') *element of the* $T \times T$ *orthonormal matrix* $\mathbf{L}$ $(\mathbf{LL}' = \mathbf{I}_T)$, *defined by*

$$\mathbf{LM}_G\mathbf{L}' = \begin{pmatrix} \mathbf{I}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{S.68}$$

Then

$$E\left(\hat{\rho}_{ij}\right) = \rho_{ij} + \frac{a_{ij}}{v} + O\left(v^{-2}\right), \tag{S.69}$$

$$Var\left(\hat{\rho}_{ij}\right) = \frac{b_{ij}}{v} + O\left(v^{-2}\right), \tag{S.70}$$

where $\rho_{ij} = E\left(\zeta_{it}\zeta_{jt}\right) = E\left(\xi_{it}\xi_{jt}\right)$,

$$a_{ij} = -\frac{1}{2}\rho_{ij}(1-\rho_{ij}^2) + \frac{1}{8}\left\{3\rho_{ij}\left[\kappa_{ij}(4,0) + \kappa_{ij}(0,4)\right] - 4\left[\kappa_{ij}(3,1) + \kappa_{ij}(1,3)\right] + 2\rho_{ij}\kappa_{ij}(2,2)\right\}, \tag{S.71}$$

$$b_{ij} = (1-\rho_{ij}^2)^2 + \frac{1}{4}\left\{\rho_{ij}^2\left[\kappa_{ij}(4,0) + \kappa_{ij}(0,4)\right] - 4\rho_{ij}\left[\kappa_{ij}(3,1) + \kappa_{ij}(1,3)\right] + 2(2+\rho_{ij}^2)\kappa_{ij}(2,2)\right\}, \tag{S.72}$$

and

$$\kappa_{ij}(4,0) = E(\zeta_{it}^4) - 3,\ \kappa_{ij}(0,4) = E(\zeta_{it}^4) - 3, \tag{S.73}$$

$$\kappa_{ij}(3,1) = E(\zeta_{it}^3\zeta_{jt}) - 3\rho_{ij},\ \kappa_{ij}(1,3) = E(\zeta_{it}\zeta_{jt}^3) - 3\rho_{ij}, \tag{S.74}$$

$$\kappa_{ij}(2,2) = E(\zeta_{it}^2\zeta_{jt}^2) - 2\rho_{ij} - 1. \tag{S.75}$$

Proof. First note that $\hat{\mathbf{u}}_{i.} = [\mathbf{I}_T - \mathbf{G}\left(\mathbf{G}'\mathbf{G}\right)^{-1}\mathbf{G}]\mathbf{u}_{i.} = \mathbf{M}_G\mathbf{u}_{i.}$, and

$$\hat{\rho}_{ij} = \frac{\hat{\mathbf{u}}_{i.}'\hat{\mathbf{u}}_{j.}}{\left(\hat{\mathbf{u}}_{i.}'\hat{\mathbf{u}}_{i.}\right)^{1/2}\left(\hat{\mathbf{u}}_{j.}'\hat{\mathbf{u}}_{j.}\right)^{1/2}} = \frac{\mathbf{u}_{i.}'\mathbf{M}_G\mathbf{u}_{j.}}{\left(\mathbf{u}_{i.}'\mathbf{M}_G\mathbf{u}_{i.}\right)^{1/2}\left(\mathbf{u}_{j.}'\mathbf{M}_G\mathbf{u}_{j.}\right)^{1/2}}.$$

Also, since $\mathbf{M}_G$ is an $(T \times T)$ idempotent matrix of rank $v = T - m - 1$, there exists an orthogonal $T \times T$ transformation matrix $\mathbf{L}$ $(\mathbf{LL}' = \mathbf{I}_T)$, defined by (S.68). Hence, setting

$$\boldsymbol{\zeta}_{i.} = \sigma_{ii}^{-1/2}\mathbf{L}\mathbf{u}_{i.}, \tag{S.76}$$

then $\hat{\rho}_{ij}$ can be written equivalently in terms of the first v elements of $\zeta_{i.} = (\zeta_{i1}, \zeta_{i2}, ..., \zeta_{iT})'$ as

$$\hat{\rho}_{ij} = \frac{\sum_{t=1}^{v}\zeta_{it}\zeta_{jt}}{\left(\sum_{t=1}^{v}\zeta_{it}^2\right)^{1/2}\left(\sum_{t=1}^{v}\zeta_{jt}^2\right)^{1/2}}.$$

Noting that

$$\zeta_{it}=\sigma_{ii}^{-1/2}\sum_{t'=1}^{T}l_{tt'}u_{it'}=\sum_{t'=1}^{T}l_{tt'}\xi_{it'}, \tag{S.77}$$

it now follows that (under Assumption 4), $E\left(\zeta_{it}\right)=0$ and $E\left(\zeta_{it}^{2}\right)=1$, $\rho_{ij}=E\left(\zeta_{it}\zeta_{jt}\right)$, for all i,j, and t; and for each i, ζ_{it}'s are independently distributed over t. Note that $\sum_{t'=1}^{T}l_{tt'}^{2}=1$, where $l_{tt'}$ is the (t,t') element of $\mathbf{L}$. Now consider

$$E\left(\zeta_{it}^{6}\right)=E\left(\sum_{t'=1}^{T}l_{tt'}\xi_{it'}\right)^{6}, \text{ for } t=1,2,...,v, \tag{S.78}$$

and recall that by Lemma 3, ξ_{it} are independent over t with, $E(\xi_{it})=0$, $E(\xi_{it}^{2})=1$, and $E\left(\xi_{it}^{8}\right)<K<\infty$. Then application of Lemma 2 to (S.78) ensures that $E\left(\zeta_{it}^{6}\right)<K<\infty$, uniformly over i and t, as required. Results (S.69) and (S.70) now follow immediately from Proposition 1 in Bailey, Pesaran and Smith (2016). ■

Lemma 20 *Consider ζ_{it} defined by $\zeta_{it}=\sigma_{ii}^{1/2}\sum_{t'=1}^{T}l_{tt'}u_{it'}$, where $l_{tt'}$ is the (t,t') element of the orthonormal matrix, $\mathbf{L}$, defined by (S.68), and $u_{it}=\gamma_i'\mathbf{v}_t+\eta_{it}$. Let $\gamma_{2,v}=E\left(v_{st}^{4}\right)-3$, and $\gamma_{2,\varepsilon_\eta}=E\left(\varepsilon_{\eta,it}^{4}\right)-3$, and suppose that Assumptions 1-4 hold. Then*

$$\begin{aligned}\sigma_{ii}^{-1}\sigma_{jj}^{-1}E\left(\zeta_{it}^{2}\zeta_{jt}^{2}\right) &= \gamma_{2,v}\left(\sum_{r=1}^{T}l_{tr}^{4}\right)\left(\sum\nolimits_{s}\gamma_{is}^{2}\gamma_{js}^{2}\right)+2\left(\gamma_i'\gamma_j\right)^{2}+\left(\gamma_i'\gamma_i\right)\left(\gamma_j'\gamma_j\right)\\ &\quad+\left(\gamma_i'\gamma_i\right)\sigma_{\eta,jj}+\left(\gamma_j'\gamma_j\right)\sigma_{\eta,ii}+4\left(\gamma_i'\gamma_j\right)\sigma_{\eta,ij}+\\ &\quad+\gamma_{2,\varepsilon_\eta}\left(\sum_{r=1}^{T}l_{tr}^{4}\right)\left(\sum\nolimits_{\ell}q_{\eta,i\ell}^{2}q_{\eta,j\ell}^{2}\right)+2\sigma_{\eta,ij}^{2}+\sigma_{\eta,ii}\sigma_{\eta,jj},\end{aligned} \tag{S.79}$$

and

$$\frac{1}{Nv}\sum\nolimits_{i,j=1}^{N}\left|E(\zeta_{it}^{3}\zeta_{jt})+E(\zeta_{it}\zeta_{jt}^{3})\right|=O\left(v^{-1}N^{2\delta_\gamma-1}\right)+O(v^{-1}). \tag{S.80}$$

Proof. Under Assumption 4, $\tilde{\eta}_{it}=\sigma_{ii}^{-1/2}\eta_{it}=\sigma_{ii}^{-1/2}\mathbf{q}_{\eta,i}'\boldsymbol{\varepsilon}_{\eta,t}$, where $\mathbf{q}_{\eta,i}$ is the i^{th} row of $\mathbf{Q}_\eta$. Also note that $\mathbf{q}_{\eta,i}'\mathbf{q}_{\eta,j}=\sigma_{\eta,ij}$, for all i and j, and $sup_j\sum_{i=1}^{N}|q_{\eta,ij}|<K$. Then using these results in (S.67) we have

$$\zeta_{it}=\sigma_{ii}^{-1/2}\left(\gamma_i'\mathbf{d}_{t,T}+\mathbf{q}_{\eta,i}'\mathbf{g}_{t,T}\right),$$

where $\mathbf{d}_{t,T}=\sum_{t'=1}^{T}l_{tt'}\mathbf{v}_{t'}=(d_{1,t,T},d_{2,t,T},...,d_{k,t,T})'$, and $\mathbf{g}_{t,T}=\sum_{t'=1}^{T}l_{tt'}\boldsymbol{\varepsilon}_{\eta,t'}=(g_{1,t,T},g_{2,t,T},...,g_{N,t,T})'$. But since $\sum_{t'=1}^{T}l_{tt'}^{2}=1$, $\sum_{t'=1}^{T}l_{tt'}l_{st'}=0$ for all $t\neq s$, $\mathbf{v}_t\sim IID(\mathbf{0},\mathbf{I}_k)$ and $\boldsymbol{\varepsilon}_{\eta,t}\sim IID(\mathbf{0},\mathbf{I}_N)$ by assumption, then it follows that $\mathbf{d}_{t,T}\sim IID(\mathbf{0},\mathbf{I}_k)$, and $\mathbf{g}_{t,T}\sim IID(\mathbf{0},\mathbf{I}_N)$. Since v_{st}, for $s=1,2,...,k$ and $\varepsilon_{i,\eta,t}$, for $i=1,2,...,N$ are assumed to have at least finite fourth order moments, then by Lemma 2 we also have $E(d_{i,t,T}^{4})<K$ and $E(g_{i,t,T}^{4})<K$. We now write ζ_{it} as

$$\zeta_{it}=a_{it}+b_{it},$$

where

$$\begin{aligned}a_{it} &= \tilde{\gamma}_i'\mathbf{d}_{t,T}=\sum_{s=1}^{k}\tilde{\gamma}_{is}d_{s,t,T}, \text{ and } b_{it}=\tilde{\mathbf{q}}_{\eta,i}'\mathbf{g}_{t,T},\\ \tilde{\gamma}_i &= \gamma_i/\sigma_{ii}^{1/2},\ \ \tilde{\mathbf{q}}_{\eta,i}=\mathbf{q}_{\eta,i}/\sigma_{ii}^{1/2},\end{aligned}$$

and hence

$$\begin{aligned}\sigma_{ii} &= \gamma_i'\gamma_i+\sigma_{\eta,ii},\ \ \tilde{\sigma}_{\eta,ii}=\sigma_{\eta,ii}/\sigma_{ii}\leq 1,\\ E\left(\zeta_{it}\right) &= 0,\ \ E\left(\zeta_{it}^{2}\right)=1,\ \tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,i}=\tilde{\sigma}_{\eta,ii}\leq 1,\ \tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j}=\sigma_{\eta,ij}/\sigma_{ii}^{1/2}\sigma_{jj}^{1/2}=\tilde{\sigma}_{\eta,ij}.\end{aligned}$$

It is clear that a_{it} and $b_{jt'}$ are distributed independently for all i, j, t and t'. Then

$$\begin{aligned}E\left(\zeta_{it}^{2}\zeta_{jt}^{2}\right) &= E\left[\left(a_{it}+b_{it}\right)^{2}\left(a_{jt}+b_{jt}\right)^{2}\right]\\ &= E\left[\left(a_{it}^{2}+2a_{it}b_{it}+b_{it}^{2}\right)\left(a_{jt}^{2}+2a_{jt}b_{jt}+b_{jt}^{2}\right)\right]\\ &= E\left(a_{it}^{2}a_{jt}^{2}\right)+E\left(a_{it}^{2}\right)E\left(b_{jt}^{2}\right)+4E\left(a_{it}a_{jt}\right)E\left(b_{it}b_{jt}\right)\\ &\quad+E\left(a_{jt}^{2}\right)E\left(b_{it}^{2}\right)+E\left(b_{it}^{2}b_{jt}^{2}\right).\end{aligned}$$

Also (using results in Lemma 6),

$$E\left(a_{it}a_{jt}\right)=\tilde{\gamma}_i'\tilde{\gamma}_j,\ E\left(b_{it}b_{jt}\right)=\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j},$$

$$E\left(a_{it}^2 a_{jt}^2\right) = \gamma_{2,d}\left(\sum\nolimits_{s=1}^{k}\tilde{\gamma}_{is}^2\tilde{\gamma}_{js}^2\right) + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\tilde{\gamma}_j'\tilde{\gamma}_j\right) + 2\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right)^2,$$

$$E\left(b_{it}^2 b_{jt}^2\right) = \gamma_{2,g}\left(\sum\nolimits_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right) + \left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,i}\right)\left(\tilde{\mathbf{q}}_{\eta,j}'\tilde{\mathbf{q}}_{\eta,j}\right) + 2\left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j}\right)^2,$$

where $\gamma_{2,d} = E(d_{s,t,T}^4) - 3$, and $\gamma_{2,g} = E(g_{i,t,T}^4) - 3$. Hence,

$$\begin{aligned} E\left(\zeta_{it}^2\zeta_{jt}^2\right) &= \gamma_{2,d}\left(\sum\nolimits_{s=1}^{k}\tilde{\gamma}_{is}^2\tilde{\gamma}_{js}^2\right) + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\tilde{\gamma}_j'\tilde{\gamma}_j\right) + 2\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right)^2 \\ &\quad + \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\tilde{\mathbf{q}}_{\eta,j}'\tilde{\mathbf{q}}_{\eta,j}\right) + 4\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right)E\left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j}\right) + \left(\tilde{\gamma}_j'\tilde{\gamma}_j\right)\left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,i}\right) \\ &\quad + \gamma_{2,g}\left(\sum\nolimits_{\ell=1}^{N}\tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right) + \left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,i}\right)\left(\tilde{\mathbf{q}}_{\eta,j}'\tilde{\mathbf{q}}_{\eta,j}\right) + 2\left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j}\right)^2, \end{aligned} \tag{S.81}$$

Further we note that

$$\begin{aligned} E(d_{s,t,T}^4) &= E\left(\sum_{r=1}^{T} l_{tr}v_{sr}\right)^4 = \sum_{r=1}^{T}\sum_{r'=1}^{T}\sum_{p=1}^{T}\sum_{p'=1}^{T} l_{tr}l_{tr'}l_{tp}l_{tp'}E(v_{sr}v_{sr'}v_{sp}v_{sp'}) \\ &= \sum_{r=1}^{T} l_{tr}^4 E(v_{sr}^4) + 3\sum_{r\neq p}^{T} l_{tr}^2 l_{tp}^2 E(v_{sr}^2)E(v_{sp}^2) \\ &= \sum_{r=1}^{T} l_{tr}^4 E(v_{sr}^4) + 3\left(\sum_{r=1}^{T} l_{tr}^2\right)\left[E(v_{sr}^2)\right]^2 - 3\sum_{r=1}^{T} l_{tr}^4\left[E(v_{sr}^2)\right]^2 \end{aligned}$$

and since $\sum_{r=1}^{T} l_{tr}^2 = 1$ and $E(v_{sr}^2) = 1$, we have

$$\gamma_{2,d} = E(d_{s,t,T}^4) - 3 = \sum_{r=1}^{T} l_{tr}^4\left[E\left(v_{sr}^4\right) - 3\right] = \left(\sum_{r=1}^{T} l_{tr}^4\right)\gamma_{2,v},$$

where $\gamma_{2,v} = E(v_{sr}^4) - 3$. Similarly, $\gamma_{2,g} = \left(\sum_{r=1}^{T} l_{tr}^4\right)\gamma_{2,\varepsilon_\eta}$, where $\gamma_{2,\varepsilon_\eta} = E\left(\varepsilon_{\eta,it}^4\right) - 3$. Then, the result (S.79) follows by substituting these expressions for $\gamma_{2,d}$ and $\gamma_{2,g}$ in (S.81). Consider now $E\left(\zeta_{it}^3\zeta_{jt}\right)$. Again using results in Lemma 6, we have

$$\begin{aligned} E\left(a_{it}^3 a_{jt}\right) &= E\left[\left(\mathbf{d}_{t,T}'\tilde{\gamma}_i\tilde{\gamma}_i'\mathbf{d}_{t,T}\right)\left(\mathbf{d}_{t,T}'\tilde{\gamma}_i\tilde{\gamma}_j'\mathbf{d}_{t,T}\right)\right] \\ &= \gamma_{2,d}Tr\left[\left(\tilde{\gamma}_i\tilde{\gamma}_i'\right)\odot\left(\tilde{\gamma}_i\tilde{\gamma}_j'\right)\right] + 3\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right) \\ E\left(b_{it}^3 b_{jt}\right) &= E\left[\left(\mathbf{g}_{t,T}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\mathbf{g}_{t,T}\right)\left(\mathbf{g}_{t,T}'\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\mathbf{g}_{t,T}\right)\right] \\ &= \gamma_{2,g}Tr\left[\left(\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,i}'\right)\odot\left(\tilde{\mathbf{q}}_{\eta,i}\tilde{\mathbf{q}}_{\eta,j}'\right)\right] + 3\left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,i}\right)\left(\tilde{\mathbf{q}}_{\eta,j}'\tilde{\mathbf{q}}_{\eta,i}\right) \\ E\left(a_{it}^2\right)E\left(b_{jt}b_{it}\right) &= \left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j};\ E\left(a_{it}a_{jt}\right)E\left(b_{jt}^2\right) = \tilde{\sigma}_{\eta,ii}\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right) \end{aligned}$$

where as before $\gamma_{2,d} = E(d_{i,t,T}^4) - 3$,and $\gamma_{2,g} = E(g_{i,t,T}^4) - 3$. Hence

$$\begin{aligned} E\left(\zeta_{it}^3\zeta_{jt}\right) &= \gamma_{2,d}\sum_{s=1}^{k}\tilde{\gamma}_{is}^3\tilde{\gamma}_{js} + 3\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right) \\ &\quad + \gamma_{2,g}\sum_{s=1}^{N}\tilde{q}_{\eta,is}^3\tilde{q}_{\eta,js} + 3\left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,i}\right)\left(\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j}\right) \\ &\quad + 3\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j} + 3\tilde{\sigma}_{\eta,ii}\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right), \end{aligned}$$

or since $\tilde{\mathbf{q}}_{\eta,i}'\tilde{\mathbf{q}}_{\eta,j} = \tilde{\sigma}_{\eta,ij}$

$$\begin{aligned} E\left(\zeta_{it}^3\zeta_{jt}\right) &= \gamma_{2,d}\sum_{s=1}^{k}\tilde{\gamma}_{is}^3\tilde{\gamma}_{js} + 3\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right) \\ &\quad + \gamma_{2,g}\sum_{s=1}^{N}\tilde{q}_{\eta,is}^3\tilde{q}_{\eta,js} + 3\tilde{\sigma}_{\eta,ii}\tilde{\sigma}_{\eta,ij} \\ &\quad + 3\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\tilde{\sigma}_{\eta,ij} + 3\tilde{\sigma}_{\eta,ii}\left(\tilde{\gamma}_i'\tilde{\gamma}_j\right), \end{aligned}$$

and

$$\begin{aligned} \left|\sum_{i,j} E\left(\zeta_{it}^3\zeta_{jt}\right)\right| &\leq \left|\gamma_{2,d}\right|\sum_{s=1}^{k}\sum_{i,j}\left|\tilde{\gamma}_{is}\right|^3\left|\tilde{\gamma}_{js}\right| + 3\sum_{i,j}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left|\tilde{\gamma}_i'\tilde{\gamma}_j\right| + 3\tilde{\sigma}_{\eta,ii}\sum_{i,j}\left|\tilde{\gamma}_i'\tilde{\gamma}_j\right| \\ &\quad \left|\gamma_{2,g}\right|\sum_{s=1}^{N}\sum_{i,j}\left|\tilde{q}_{\eta,is}\right|^3\left|\tilde{q}_{\eta,js}\right| + 3\sum_{i,j}\tilde{\sigma}_{\eta,ii}\left|\tilde{\sigma}_{\eta,ij}\right| + 3\sum_{i,j}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left|\tilde{\sigma}_{\eta,ij}\right|. \end{aligned}$$

But $\tilde{\gamma}_i'\tilde{\gamma}_j = \sum_{s=1}^{k}\tilde{\gamma}_{is}\tilde{\gamma}_{js}$, and recall that $\left|\gamma_{2,d}\right| < K$, $\left|\gamma_{2,g}\right| < K$, $sup_j\sum_{i=1}^{N}\left|\tilde{q}_{\eta,ij}\right| < K$, $\left|\tilde{\gamma}_{is}\right| \leq 1$, and $\tilde{\sigma}_{\eta,ii} \leq 1$. Also

$$\begin{aligned}
\sum_{s=1}^{k}\sum_{i,j}\left|\tilde{\gamma}_{is}\right|^3\left|\tilde{\gamma}_{js}\right| &\leq \sum_{s=1}^{k}\left(\sum_{i}\left|\tilde{\gamma}_{is}\right|\right)^2 = O\left(N^{2\delta_\gamma}\right), \\
\sum_{i,j}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left|\tilde{\gamma}_i'\tilde{\gamma}_j\right| &\leq \sup_i\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\sum_{s=1}^{k}\sum_{i,j}\left|\tilde{\gamma}_{is}\right|\left|\tilde{\gamma}_{js}\right| = O\left(N^{2\delta_\gamma}\right), \\
\tilde{\sigma}_{\eta,ii}\sum_{i,j}\left|\tilde{\gamma}_i'\tilde{\gamma}_j\right| &\leq \sum_{s=1}^{k}\sum_{i,j}\left|\tilde{\gamma}_{is}\right|\left|\tilde{\gamma}_{js}\right| = \sum_{s=1}^{k}\left(\sum_{i}\left|\tilde{\gamma}_{is}\right|\right)^2 = O\left(N^{2\delta_\gamma}\right),
\end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}_{\eta,ij} &= \left(\sigma_{\eta,ij}/\sigma_{\eta,ii}^{1/2}\sigma_{\eta,jj}^{1/2}\right)\left(\frac{\sigma_{\eta,ii}^{1/2}\sigma_{\eta,jj}^{1/2}}{\sigma_{ii}^{1/2}\sigma_{jj}^{1/2}}\right) = \tilde{\sigma}_{\eta,ii}^{1/2}\tilde{\sigma}_{\eta,jj}^{1/2}\rho_{\eta,ij}, \\
\left|\tilde{\sigma}_{\eta,ij}\right| &\leq \left|\rho_{\eta,ij}\right|, \text{ and by assumption } \sum_{i,j}\left|\rho_{\eta,ij}\right| = O(N).
\end{aligned}$$

$$\begin{aligned}
\sum_{s=1}^{k}\sum_{i,j}\left|\tilde{q}_{\eta,is}\right|^3\left|\tilde{q}_{\eta,js}\right| &\leq \sum_{s=1}^{k}\sum_{i,j}\left|\tilde{q}_{\eta,is}\right|^2\left|\tilde{q}_{\eta,js}\right| \leq \sum_{s=1}^{k}\sum_{i}\left|\tilde{q}_{\eta,js}\right| < K \\
\sum_{i,j}\tilde{\sigma}_{\eta,ii}\left|\tilde{\sigma}_{\eta,ij}\right| &\leq \sum_{i,j}\left|\rho_{\eta,ij}\right| = O(N), \\
\sum_{i,j}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left|\tilde{\sigma}_{\eta,ij}\right| &\leq \sup_i\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\sum_{i,j}\left|\tilde{\sigma}_{\eta,ij}\right| = O(N).
\end{aligned}$$

Hence

$$\left|\sum_{i,j}E\left(\zeta_{it}^3\zeta_{jt}\right)\right| \leq O\left(N^{2\delta_\gamma}\right) + O(N),$$

and

$$N^{-1}\sum_{i,j}E\left(\zeta_{it}^3\zeta_{jt}\right) = O\left(N^{2\delta_\gamma - 1}\right) + O(1).$$

Similarly $N^{-1}\sum_{i,j}E\left(\zeta_{jt}^3\zeta_{it}\right) = O\left(N^{2\delta_\gamma-1}\right)$, and overall

$$\frac{1}{Nv}\sum_{i,j=1}^{N}\left|E(\zeta_{it}^3\zeta_{jt}) + E(\zeta_{it}\zeta_{jt}^3)\right| = O\left(v^{-1}N^{2\delta_\gamma-1}\right) + O(v^{-1}),$$

as required. ■

Lemma 21 *Consider the regression model (8), and suppose that Assumptions 1-4 hold. Then for each i*

$$E\left(t_i^2\right) = \frac{v}{v-2} + O(v^{-3/2}), \tag{S.82}$$

and

$$Var\left(t_i^2\right) = \left(\frac{v}{v-2}\right)^2\frac{2(v-1)}{(v-4)} + O(v^{-1}), \tag{S.83}$$

where t_i^2 is defined by (23), and $v = T - m - 1$.

Proof. Below we use matrices $\mathbf{G}$, $\mathbf{M}_F$, $\mathbf{M}_G$, $\mathbf{P}_G$, $\mathbf{H}_F$, which are defined by (S.2) and (S.1), and also $\gamma_{1,i} = E(\xi_{it}^3)$, $\gamma_{2,i} = E(\xi_{it}^4) - 3$, $\gamma_{3,i} = E(\xi_{it}^5) - 10\gamma_{1,i}$, $\gamma_{4,i} = E(\xi_{it}^6) - 10\gamma_{1,i}^2 - 15\gamma_{2,i} - 15$ for all t, where $\xi_{it} = u_{it}/\sigma_{ii}^{1/2}$, and by assumption $E(\xi_{it}^6) < K$. Furthermore,

$$\left(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T\right)^{-1} = O(v^{-1}). \tag{S.84}$$

Using (23), we can write

$$t_i^2 = \frac{v}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}\left(\frac{\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i}{\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i}\right), \tag{S.85}$$

where $\boldsymbol{\xi}_i = (\xi_{i1}, \xi_{i2}, ..., \xi_{iT})'$, with $\boldsymbol{\xi}_i \sim IID(\mathbf{0}, \mathbf{I}_T)$ for all i (see Lemma 3). Using a slightly extended version of Laplace approximation of moments of the ratio of quadratic forms by Lieberman (1994), that allows $\mathbf{\Gamma}$ defined in Lemma 5 to be a positive semi-definite matrix, and substituting $\mathbf{\Phi} = \mathbf{H}_F$ and $\mathbf{\Gamma} = \mathbf{M}_G$ into Lemma 5, we have (conditional on $\mathbf{F}$)

$$E\left(t_i^2\right) = \frac{v}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}\left[\frac{E\left(\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i\right)}{E\left(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i\right)} + \psi_{i,1v}\right] + O(v^{-2}), \tag{S.86}$$

where

$$\psi_{i,1v} = \left[\frac{E(\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i)\kappa_{i,2}}{[E(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i)]^3}\right] - \left[\frac{\kappa_{i,11}}{[E(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i)]^2}\right],$$

$$\kappa_{i,2} = E\left[(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i)^2\right] - \left[E(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i)\right]^2,$$

and

$$\kappa_{i,11} = E[(\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i)\left(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i\right)] - E(\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i)E(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i).$$

Using Lemmas 11 and 12, it is easily seen that

$$\frac{v}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}\frac{E\left(\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i\right)}{E\left(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i\right)} = 1$$

and

$$\begin{aligned}
\frac{v\psi_{i,1v}}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T} &= \frac{v}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}\left(\frac{E(\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i)\kappa_{i,2}}{[E(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i)]^3} - \frac{\kappa_{i,11}}{[E(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i)]^2}\right) \\
&= \frac{v}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}\left(\frac{\left(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T\right)\left[\gamma_{2,i}Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)+2v\right]}{v^3} - \frac{\gamma_{2,i}Tr\left(\mathbf{M}_G\odot\mathbf{H}_F\right)}{v^2}\right) \\
&= \frac{2}{v} + \gamma_{2,i}K_v,
\end{aligned}$$

where

$$K_v = \frac{1}{v}\left[\frac{Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)}{v} - \frac{Tr\left(\mathbf{M}_G\odot\mathbf{H}_F\right)}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}\right]. \tag{S.87}$$

Noting that $\mathbf{M}_G = \mathbf{I}_T - \mathbf{P}_G$ with $\mathbf{P}_G = \mathbf{G}\left(\mathbf{G}'\mathbf{G}\right)^{-1}\mathbf{G}'$, where $\mathbf{G} = (\mathbf{F}, \boldsymbol{\tau}_T)$, the first term of (S.87) can be written as

$$\begin{aligned}
\frac{Tr\left(\mathbf{M}_G\odot\mathbf{M}_G\right)}{v} &= \frac{1}{v}Tr\left[(\mathbf{I}_T-\mathbf{P}_G)\odot(\mathbf{I}_T-\mathbf{P}_G)\right] \\
&= \frac{1}{v}\left[T - 2Tr\left(\mathbf{P}_G\right) + Tr\left(\mathbf{P}_G\odot\mathbf{P}_G\right)\right] = 1 - \frac{Tr\left(\mathbf{P}_G\right)}{v} + \frac{Tr\left(\mathbf{P}_G\odot\mathbf{P}_G\right)}{v}.
\end{aligned} \tag{S.88}$$

Similarly, for the second term of (S.87) we have

$$\begin{aligned}
\frac{Tr\left(\mathbf{M}_G\odot\mathbf{H}_F\right)}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T} &= \frac{1}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}Tr\left[(\mathbf{I}_T-\mathbf{P}_G)\odot\mathbf{H}_F\right] \\
&= \frac{1}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}\left[Tr\left(\mathbf{H}_F\right) - Tr\left(\mathbf{P}_G\odot\mathbf{H}_F\right)\right] = 1 - \frac{Tr\left(\mathbf{P}_G\odot\mathbf{H}_F\right)}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T}.
\end{aligned} \tag{S.89}$$

Substituting (S.88) and (S.89) into (S.87), then using $Tr\left(\mathbf{P}_G\odot\mathbf{P}_G\right) = O(1)$ and $Tr\left(\mathbf{P}_G\odot\mathbf{H}_F\right) = O(v^{1/2})$, which are established by (S.23) and (S.24) in Lemma 10, we have

$$K_v = \frac{1}{v^{3/2}}\frac{v^{1/2}Tr\left(\mathbf{P}_G\odot\mathbf{H}_F\right)}{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T} + \frac{1}{v^2}Tr\left(\mathbf{P}_G\odot\mathbf{P}_G\right) - \frac{1}{v^2}Tr\left(\mathbf{P}_G\right) = \frac{S_{0v}}{v^{3/2}} + O(v^{-2}),$$

where

$$S_{0v} = \frac{v^{1/2}Tr\left(\mathbf{P}_G\odot\mathbf{H}_F\right)}{\left(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T\right)},$$

which is $O(1)$ by (S.24) and (S.84), so that

$$E\left(t_i^2\right) = 1 + \frac{2}{v} + \gamma_{2,i}\frac{S_{0v}}{v^{3/2}} + O(v^{-2}). \tag{S.90}$$

However, since

$$\frac{v}{v-2} - \left(1+\frac{2}{v}\right) = \frac{4}{v\left(v-2\right)} = O(v^{-2}),$$

and using Lemma 12 ensures that the three conditions in Lieberman's lemma are satisfied. Result in Lieberman (1994; p.683) now implies that the last term can be rewritten as $v^{-2}W_{0,iv}$, where $W_{0,iv}$ is a function of $\gamma_{\ell,i}$, $\mathbf{F}$, and v, for $\ell = 1,2,3,4$. Since under Assumption 4, $\sup_i|\gamma_{\ell,i}| \leq K < \infty$, for $\ell = 1,2,3,4$, all i, then

$$E\left(t_i^2\right) = \frac{v}{v-2} + \gamma_{2,i}\frac{S_{0v}}{v^{3/2}} + \frac{W_{0,iv}}{v^2} = \frac{v}{v-2} + O(v^{-3/2}), \tag{S.91}$$

which establishes (S.82). To prove (S.83), we first note that

$$E\left(t_i^4\right) = \frac{v^2}{\left(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T\right)^2}E\left[\left(\frac{\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i}{\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i}\right)^2\right]. \tag{S.92}$$

But by Lemmas 5 and 11 we have

$$E\left(t_i^4\right) = \frac{v^2}{\left(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T\right)^2}\left\{\frac{E\left[\left(\boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i\right)^2\right]}{\left[E\left(\boldsymbol{\xi}_i'\mathbf{M}_G\boldsymbol{\xi}_i\right)\right]^2} + O(v^{-1})\right\} = 3 + \frac{\gamma_{2,i}Tr\left(\mathbf{H}_F \odot \mathbf{H}_F\right)}{\left(\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T\right)^2} + O(v^{-1}). \tag{S.93}$$

Since $Tr\left(\mathbf{H}_F \odot \mathbf{H}_F\right) = O(v)$ by Lemma 11, Lemma 5 implies that the last two terms can be rewritten as $v^{-1}W_{1,iv}$, where $W_{1,iv}$ is a function of $\gamma_{\ell,i}$, $\mathbf{F}$, and v, with $\ell = 1,2,3,4$. Again under Assumption 2, $\sup_i|\gamma_{\ell,i}| \leq K < \infty$, for $\ell = 1,2,3,4$ and all i, we obtain

$$E\left(t_i^4\right) = 3 + O(v^{-1}). \tag{S.94}$$

Using (S.91) and (S.94), and noting that

$$\left[3 - \left(1 + \frac{2}{v}\right)^2\right] - \left(\frac{v}{v-2}\right)^2\frac{2\left(v-1\right)}{\left(v-4\right)} = O(v^{-1}),$$

then for each i we have

$$Var\left(t_i^2\right) = E\left(t_i^4\right) - \left[E\left(t_i^2\right)\right]^2 = \left(\frac{v}{v-2}\right)^2\frac{2\left(v-1\right)}{\left(v-4\right)} + O(v^{-1}),$$

which completes the proof. ■

Lemma 22 *Consider the regression model (2), and let $z_{i,a}^2 = \hat{\alpha}_i^2 w_T/\sigma_{ii}$, where $w_T = \boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T$, $\mathbf{H}_F$ and $\mathbf{M}_F$ are defined by (S.2), and $\hat{\alpha}_i$ is the OLS estimate of α_i given by (11). Suppose that Assumptions 1-4 hold, and $N^{-1}Tr\left(\mathbf{R}^2\right)$ is bounded in N, where $\mathbf{R} = \left(\rho_{ij}\right)$. Then under the local alternatives defined by (61)*

$$N^{-1/2}\sum_{i=1}^{N}\left(z_{i,a}^2 - 1\right) \rightarrow_d N(\phi^2, 2\omega^2), \tag{S.95}$$

as $N \to \infty$ and $T \to \infty$, jointly, where

$$\phi^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\frac{\varsigma_i^2}{\sigma_{ii}}, \text{ and } \omega^2 = \lim_{N\to\infty}N^{-1}Tr\left(\mathbf{R}^2\right) = 1 + \lim_{N\to\infty}(N-1)\rho_N^2,$$

$\sigma_{ij} = E(u_{it}u_{jt})$, $Corr(u_{it}u_{jt}) = \rho_{ij}$, and ρ_N^2 is defined by (54).

Proof. Using (11) and (12), we first note that

$$z_{i,a}^2 = \left(w_T^{1/2}\tilde{\alpha}_i + w_T^{-1/2}\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\xi}_i\right)^2,$$

where $\boldsymbol{\xi}_i$ is defined by (34), and $\tilde{\alpha}_i = \alpha_i/\sigma_{ii}^{1/2}$, and under (61)

$$\tilde{\alpha}_i = \frac{\tilde{\varsigma}_i}{N^{1/4}v^{1/2}}, \tag{S.96}$$

where $\tilde{\varsigma}_i = \varsigma_i/\sigma_{ii}^{1/2}$ are given and bounded. Then

$$z_{i,a}^2 = z_i^2 + w_T\tilde{\alpha}_i^2 + 2\tilde{\alpha}_i\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\xi}_i, \tag{S.97}$$

where $z_i^2 = \boldsymbol{\xi}_i'\mathbf{H}_F\boldsymbol{\xi}_i/w_T$. Hence

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(z_{i,a}^2 - 1\right) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(z_i^2 - 1\right) + \phi_{NT}^2 + 2b_{NT}, \tag{S.98}$$

where

$$\phi_{NT}^2 = \frac{w_T}{\sqrt{N}}\sum_{i=1}^{N}\tilde{\alpha}_i^2 = \frac{w_T}{v}\left(N^{-1}\sum_{i=1}^{N}\tilde{\varsigma}_i^2\right), \tag{S.99}$$

and

$$b_{NT} = \frac{1}{v^{1/2}N^{3/4}}\sum_{i=1}^{N}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\xi}_i. \tag{S.100}$$

Also, for given values of $|\varsigma_i| < K$, $\phi_{NT}^2 \geq 0$, and we have

$$\lim_{N,T\to\infty}\left(\phi_{NT}^2\right) = \phi^2 = \lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{\varsigma}_i^2\right) \geq \min_i(1/\sigma_{ii})\lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N}\varsigma_i^2\right). \tag{S.101}$$

Since $\sigma_{ii} > 0$, then $\phi^2 > 0$, if $N^{-1}\sum_{i=1}^{N}\varsigma_i^2$ tends to strictly positive limit. Consider now b_{NT}, and note that for given values of ς_i we have[S1]

$$\begin{aligned} b_{NT} &= \frac{1}{v^{1/2}N^{3/4}}\sum_{i=1}^{N}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\xi}_i = \frac{1}{v^{1/2}N^{3/4}}\sum_{i=1}^{N}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\left(\frac{\mathbf{V}\gamma_i+\boldsymbol{\eta}_i}{\sigma_{ii}^{1/2}}\right) \\ &= \frac{1}{v^{1/2}N^{3/4}}\sum_{i=1}^{N}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_i + \frac{1}{v^{1/2}N^{3/4}}\sum_{i=1}^{N}\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\tilde{\boldsymbol{\eta}}_i, \\ & b_{1,NT}+b_{2,NT}, \end{aligned}$$

where $\tilde{\gamma}_i = \gamma_i/\sigma_{ii}^{1/2}$, and $\tilde{\boldsymbol{\eta}}_i = \boldsymbol{\eta}_i/\sigma_{\eta,ii}^{1/2}$. For given values of $\tilde{\varsigma}_i$, it is easily seen that $E\left(b_{1,NT}\right) = 0$, and

$$\begin{aligned} Var\left(b_{1,NT}\right) &= \frac{1}{vN^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{\varsigma}_i\tilde{\varsigma}_j\boldsymbol{\tau}_T'\mathbf{M}_F E\left(\mathbf{V}\tilde{\gamma}_i\tilde{\gamma}_j'\mathbf{V}'\right)\mathbf{M}_F\boldsymbol{\tau}_T, \\ &= \frac{1}{vN^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{\varsigma}_i\tilde{\varsigma}_j\tilde{\gamma}_j'\mathbf{M}_F\boldsymbol{\tau}_T\boldsymbol{\tau}_T'\mathbf{M}_F\tilde{\gamma}_i = \frac{1}{vN^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{\varsigma}_i\tilde{\varsigma}_j\tilde{\gamma}_j'\mathbf{M}_F\boldsymbol{\tau}_T\boldsymbol{\tau}_T'\mathbf{M}_F\tilde{\gamma}_i \\ &\le \frac{\lambda_{\max}\left(\mathbf{M}_F\boldsymbol{\tau}_T\boldsymbol{\tau}_T'\mathbf{M}_F\right)}{vN^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{\varsigma}_i\tilde{\varsigma}_j\tilde{\gamma}_j'\tilde{\gamma}_i \le \left(\frac{w_T}{v}\right)N^{-3/2}\left(\sum_{i=1}^{N}\tilde{\varsigma}_i\tilde{\gamma}_i\right)\left(\sum_{j=1}^{N}\tilde{\varsigma}_j\tilde{\gamma}_j\right)'. \end{aligned}$$

However, $\left|\sum_{i=1}^{N}\tilde{\varsigma}_i\tilde{\gamma}_i\right| \le Kk\sup_s\sum_{i=1}^{N}\left|\tilde{\gamma}_{is}\right| = O\left(N^{\delta_\gamma}\right)$, and since $w_T/v = O(1)$, then $Var\left(b_{1,NT}\right) = O\left(N^{2\delta_\gamma-3/2}\right)$,and $b_{1,NT}\to_p 0$, if $\delta_\gamma < 3/4$. Similarly, $E\left(b_{2,NT}\right) = 0$, and

$$\begin{aligned} Var\left(b_{2,NT}\right) &= \frac{1}{vN^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\frac{\sigma_{\eta,jj}}{\sigma_{jj}}\right)^{1/2}\tilde{\varsigma}_i\tilde{\varsigma}_j\boldsymbol{\tau}_T'\mathbf{M}_F E\left(\tilde{\boldsymbol{\eta}}_i\tilde{\boldsymbol{\eta}}_j'\right)\mathbf{M}_F\boldsymbol{\tau}_T \\ &= \frac{1}{vN^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\frac{\sigma_{\eta,jj}}{\sigma_{jj}}\right)^{1/2}\rho_{\eta,ij}\tilde{\varsigma}_i\tilde{\varsigma}_j\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T \\ &= \left(\frac{w_T}{v}\right)\frac{1}{N^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\frac{\sigma_{\eta,jj}}{\sigma_{jj}}\right)^{1/2}\rho_{\eta,ij}\tilde{\varsigma}_i\tilde{\varsigma}_j \end{aligned}$$

Hence

$$E\left(b_{NT}^2\right) = \frac{\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}}{N^{3/2}v}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\varsigma_i\varsigma_j\rho_{ij}}{\sigma_{ii}^{1/2}\sigma_{jj}^{1/2}}.$$

But since $|\varsigma_i| < K$, and $0 < \sigma_{ii} < K$, for all i, and $\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau} = O(v)$, then

$$Var\left(b_{2,NT}\right) \le K\left(\frac{1}{N^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\rho_{ij}\right|\right) \le K\left(\frac{1}{N^{1/2}}\sup_i\sum_{j=1}^{N}\left|\rho_{ij}\right|\right) = O\left(N^{\delta_\gamma-1/2}\right),$$

and $Var\left(b_{2,NT}\right)\to 0$, if $\delta_\gamma < 1/2$. Hence, $b_{NT}\to_p 0$, and in view of (S.98) $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(z_{i,a}^2-1\right)$ and $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(z_i^2-1\right)+\phi^2$ will have the same asymptotic distributions as N and $T\to\infty$, jointly and $m_N = o(N^{1/2})$. But in view of (53), $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(z_i^2-1\right)\to_d N(0,2\omega^2)$, and therefore it also follows that under local alternatives $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(z_{i,a}^2-1\right)\to_d N(\phi^2,2\omega^2)$. ■

Lemma 23 *Consider the regression model (2), and let* $z_{i,a}^2 = w_T\hat{\alpha}_i^2/\sigma_{ii}$, *where* $w_T = \boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\tau}_T$, $\mathbf{H}_F$ *and* $\mathbf{M}_F$ *are defined by (S.2), and* $\hat{\alpha}_i$ *is the OLS estimate of* α_i *given by (11). Suppose that Assumptions 1-4 hold, and* $N^{-1}Tr\left(\mathbf{R}^2\right)$ *is bounded in* N, *where* $\mathbf{R} = \left(\rho_{ij}\right)$. *Then under the local alternatives defined by (61)*

$$S_{NT} = N^{-1/2}\sum_{i=1}^{N}\left(z_{i,a}^2 - t_i^2\right)\to_p 0,$$

if $N/T^2\to 0$ *and* $0\le\delta_\gamma < 1/2$, *as* $N\to\infty$ *and* $T\to\infty$, *jointly.*

Proof. As with the proof of Theorem 2, we first note that

$$z_{i,a}^2 - t_i^2 = \frac{w_T\hat{\alpha}_i^2}{\sigma_{ii}} - \frac{w_T\hat{\alpha}_i^2}{v^{-1}\mathbf{y}_{i.}'\mathbf{M}_G\mathbf{y}_{i.}'} = z_{i,a}^2\left(1-\frac{1}{X_i}\right),$$

[S1] The same results follow if ς_i are random but distributed independently of $\boldsymbol{\xi}_i$.

where $X_i = \boldsymbol{\xi}_i' \mathbf{M}_G \boldsymbol{\xi}_i / v$, $v = T - m - 1$, $\xi_{it} = u_{it}/\sigma_{ii}^{1/2}$. Using (S.97), we note that

$$z_{i,a}^2 = z_i^2 + g_i,$$
$$g_i = w_T \tilde{\alpha}_i^2 + 2\tilde{\alpha}_i \boldsymbol{\tau}_T' \mathbf{M}_F \boldsymbol{\xi}_i$$

where $\tilde{\alpha}_i = \frac{\tilde{\varsigma}_i}{N^{1/4} v^{1/2}}$, and $\tilde{\varsigma}_i = \varsigma_i / \sigma_{ii}^{1/2}$. Consider

$$S_{NT} = N^{-1/2} \sum_{i=1}^{N} \left[z_{i,a}^2 \left(1 - \frac{1}{\sigma_{ii}^{-1} \tilde{\sigma}_{ii}} \right) \right].$$

Write $X_i = \sigma_{ii}^{-1} \tilde{\sigma}_{ii}$ and note that by assumption $\sigma_{ii} > 0$, and by construction only securities with $\tilde{\sigma}_{ii} > c > 0$ are included in the $\hat{J}_\alpha$ test. Hence, for all $i = 1, 2, ..., N$ we have $X_i > 0$, and (A.18) can be written as

$$\begin{aligned} S_{NT} &= N^{-1/2} \sum_{i=1}^{N} z_{i,a}^2 \left[(1 - X_i) + \frac{(1 - X_i)^2}{X_i} \right] \\ &= S_{1,NT} + S_{2,NT}, \end{aligned}$$

where

$$S_{1,NT} = N^{-1/2} \sum_{i=1}^{N} z_{i,a}^2 (1 - X_i),$$

and

$$S_{2,NT} = N^{-1/2} \sum_{i=1}^{N} \frac{z_{i,a}^2 (1 - X_i)^2}{X_i}.$$

But since $X_i > c > 0$, and $z_{i,a}^2 (1 - X_i)^2 \geq 0$, then

$$|S_{2,NT}| \leq c^{-1} N^{-1/2} \sum_{i=1}^{N} z_{i,a}^2 (1 - X_i)^2,$$

and

$$E|S_{2,NT}| \leq c^{-1} N^{1/2} \sup_i E\left[z_{i,a}^2 (1 - X_i)^2 \right].$$

$$E\left[z_{i,a}^2 (1 - X_i)^2 \right] \leq E\left| z_i^2 (1 - X_i)^2 \right| + E\left| g_i (1 - X_i)^2 \right|. \tag{S.102}$$

From (A.24) we have

$$E\left[z_i^2 (1 - X_i)^2 \right] = O\left(\frac{1}{v} \right), \tag{S.103}$$

uniformly across i. Next,

$$E\left| g_i (1 - X_i)^2 \right| \leq w_T \tilde{\alpha}_i^2 E\left[(1 - X_i)^2 \right] + 2E\left| \tilde{\alpha}_i \boldsymbol{\tau}_T' \mathbf{M}_F \boldsymbol{\xi}_i (1 - X_i)^2 \right|,$$

but by Lemma 11 we have

$$E\left[(1 - X_i)^2 \right] = E\left(X_i^2 \right) - 1 = O(v^{-1}),$$

as $E\left[\left(\boldsymbol{\xi}_i' \mathbf{M}_G \boldsymbol{\xi}_i \right)^2 \right] = v^2 + O(v)$, so that

$$w_T \tilde{\alpha}_i^2 E\left[(1 - X_i)^2 \right] = O(\tilde{\alpha}_i^2).$$

Next

$$\begin{aligned} E\left| \tilde{\alpha}_i \boldsymbol{\tau}_T' \mathbf{M}_F \boldsymbol{\xi}_i (1 - X_i)^2 \right| &\leq |\tilde{\alpha}_i| \left[E\left(\boldsymbol{\xi}_i' \mathbf{H}_F \boldsymbol{\xi}_i \right) \right]^{1/2} \left\{ E\left[(1 - X_i)^4 \right] \right\}^{1/2} \\ &= |\tilde{\alpha}_i| \, w_T^{1/2} \left\{ E\left[(1 - X_i)^4 \right] \right\}^{1/2}. \end{aligned}$$

Noting that, since, by Lemma 11, $E\left[\left(\boldsymbol{\xi}_i' \mathbf{M}_G \boldsymbol{\xi}_i \right)^r \right] = v^r + O\left(v^{r-1} \right)$ and $E\left(\boldsymbol{\xi}_i' \mathbf{M}_G \boldsymbol{\xi}_i \right) = v$, we have $E\left(X_i^r \right) = 1 + O\left(v^{-(r-1)} \right)$ for $r = 2, 3, 4$ and $E\left(X_i \right) = 1$ uniformly over i,

$$E(1 - X_i)^4 = E\left(X_i^4 \right) - 4E\left(X_i^3 \right) + 6E\left(X_i^2 \right) - 4E\left(X_i \right) + 1 = O(v^{-1}).$$

Thus, $E\left| \tilde{\alpha}_i \boldsymbol{\tau}_T' \mathbf{M}_F \boldsymbol{\xi}_i (1 - X_i)^2 \right| = O\left(|\tilde{\alpha}_i| \right) = O\left(N^{-1/4} v^{-1/2} \right)$ and

$$E\left| g_i (1 - X_i)^2 \right| = O\left(|\tilde{\alpha}_i|^2 \right) + O\left(|\tilde{\alpha}_i| \right) = O\left(|\tilde{\alpha}_i| \right) = O\left(N^{-1/4} v^{-1/2} \right). \tag{S.104}$$

Substituting (S.103) and (S.104) into (S.102), we have

$$E\left[z_{i,a}^2 (1 - X_i)^2 \right] = O\left(\frac{1}{v} \right) + O\left(N^{-1/4} v^{-1/2} \right)$$

uniformly across i, so that

$$E\left|S_{2,NT}\right|\leq c^{-1}N^{1/2}\sup_{i}E\left[z_{i,a}^{2}\left(1-X_{i}\right)^{2}\right]=O\left(\frac{\sqrt{N}}{v}\right)+O\left(\frac{N^{1/4}}{v^{1/2}}\right).$$

By Markov inequality we have $S_{2,NT}\rightarrow_{p}0$, so long as $N/T^{2}\rightarrow0$. Therefore, to establish $S_{NT}\rightarrow_{p}0$, it is sufficient to show that $S_{1,NT}\rightarrow_{p}0$. Now

$$\begin{aligned}S_{1,NT} &= N^{-1/2}\sum_{i=1}^{N}z_{i,a}^{2}\left(1-X_{i}\right)\\ &= N^{-1/2}\sum_{i=1}^{N}z_{i}^{2}\left(1-X_{i}\right)-N^{-1/2}\sum_{i=1}^{N}g_{i}\left(X_{i}-1\right).\end{aligned}$$

Consider

$$N^{-1/2}\sum_{i=1}^{N}g_{i}\left(X_{i}-1\right)=\left(\frac{w_{T}}{v}\right)N^{-1}\sum_{i=1}^{N}\tilde{\varsigma}_{i}^{2}\left(X_{i}-1\right)+2v^{-1/2}N^{-3/4}\sum_{i=1}^{N}\tilde{\varsigma}_{i}\boldsymbol{\tau}_{T}^{\prime}\mathbf{M}_{F}\boldsymbol{\xi}_{i}\left(X_{i}-1\right). \tag{S.105}$$

By (S.60), $X_{i}=\frac{\sigma_{\eta,ii}}{\sigma_{ii}}X_{\eta,i}+B_{i}$, where $B_{i}=\frac{\tilde{\boldsymbol{\gamma}}_{i}^{\prime}\mathbf{V}^{\prime}\mathbf{M}_{G}\mathbf{V}\tilde{\boldsymbol{\gamma}}_{i}}{v}+2\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\frac{\tilde{\boldsymbol{\gamma}}_{i}^{\prime}\mathbf{V}^{\prime}\mathbf{M}_{G}\tilde{\boldsymbol{\eta}}_{i}}{v}$, and we have

$$\begin{aligned}N^{-1/2}\sum_{i=1}^{N}\tilde{\varsigma}_{i}^{2}\left(X_{i}-1\right) &= KN^{-1/2}\sum_{i=1}^{N}\tilde{\varsigma}_{i}^{2}\left[X_{\eta,i}-1+\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}-1\right)X_{\eta,i}+B_{i}\right]\\ &= KN^{-1/2}\sum_{i=1}^{N}\tilde{\varsigma}_{i}^{2}\left[\left(X_{\eta,i}-1\right)-\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)X_{\eta,i}+B_{i}\right].\end{aligned}$$

First, as $\sup_{i}\left|\tilde{\varsigma}_{i}\right|\leq K$ and $0<\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\leq1$,

$$N^{-1/2}\sum_{i=1}^{N}E\left|\tilde{\varsigma}_{i}^{2}B_{i}\right|\leq KN^{-1/2}\sum_{i=1}^{N}E\left|B_{i}\right|,$$

but

$$\begin{aligned}N^{-1/2}\sum_{i=1}^{N}E\left|B_{i}\right| &\leq KN^{-1/2}\sum_{i=1}^{N}\left|v^{-1}\tilde{\gamma}_{i}^{\prime}\mathbf{V}^{\prime}\mathbf{M}_{G}\mathbf{V}\tilde{\gamma}_{i}\right|+2KN^{-1/2}\sum_{i=1}^{N}\left|v^{-1}\tilde{\gamma}_{i}^{\prime}\mathbf{V}^{\prime}\mathbf{M}_{G}\tilde{\boldsymbol{\eta}}_{i}\right|\\ &\leq KN^{-1/2}\sum_{i=1}^{N}\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)E\left|Tr\left(v^{-1}\mathbf{V}^{\prime}\mathbf{M}_{G}\mathbf{V}\right)\right|\\ &\quad+2KN^{-1/2}\sum_{i=1}^{N}\left[E\left(v^{-2}\tilde{\gamma}_{i}^{\prime}\mathbf{V}^{\prime}\mathbf{M}_{G}\tilde{\boldsymbol{\eta}}_{i}\tilde{\boldsymbol{\eta}}_{i}^{\prime}\mathbf{M}_{G}\mathbf{V}\tilde{\gamma}_{i}\right)\right]^{1/2}\\ &= KN^{-1/2}\sum_{i=1}^{N}k\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)+2v^{-1}k\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)^{1/2}=O\left(N^{\delta_{\gamma}-1/2}\right),\end{aligned}$$

since $E\left(\mathbf{V}^{\prime}\mathbf{V}\right)=\mathbf{I}_{k}$, $\mathbf{V}$ and $\tilde{\boldsymbol{\eta}}_{i}$ are independent, $E\left|Tr\left(v^{-1}\mathbf{V}^{\prime}\mathbf{M}_{G}\mathbf{V}\right)\right|=k$ and

$$\begin{aligned}E\left(v^{-2}\tilde{\gamma}_{i}^{\prime}\mathbf{V}^{\prime}\mathbf{M}_{G}\tilde{\boldsymbol{\eta}}_{i}\tilde{\boldsymbol{\eta}}_{i}^{\prime}\mathbf{M}_{G}\mathbf{V}\tilde{\gamma}_{i}\right) &\leq v^{-2}\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)Tr\left[E\left(\mathbf{V}^{\prime}\mathbf{M}_{G}\tilde{\boldsymbol{\eta}}_{i}\tilde{\boldsymbol{\eta}}_{i}^{\prime}\mathbf{M}_{G}\mathbf{V}\right)\right]\\ &= v^{-2}\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)Tr\left(\mathbf{M}_{G}\right)=v^{-1}\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right).\end{aligned}$$

Similarly, noting $E\left|X_{\eta,i}\right|=E\left(X_{\eta,i}\right)=1$,

$$\begin{aligned}N^{-1/2}\sum_{i=1}^{N}E\left|\tilde{\varsigma}_{i}^{2}\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)X_{\eta,i}\right| &\leq KN^{-1/2}\sum_{i=1}^{N}\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)E\left|X_{\eta,i}\right|\\ &= KN^{-1/2}\sum_{i=1}^{N}\left(\tilde{\gamma}_{i}^{\prime}\tilde{\gamma}_{i}\right)=O\left(N^{\delta_{\gamma}-1/2}\right).\end{aligned}$$

Hence,

$$KN^{-1/2}\sum_{i=1}^{N}\tilde{\varsigma}_{i}^{2}\left(X_{i}-1\right)=KN^{-1/2}\sum_{i=1}^{N}\tilde{\varsigma}_{i}^{2}\left(X_{\eta,i}-1\right)+O_{p}\left(N^{\delta_{\gamma}-1/2}\right).$$

Next, $E\left[N^{-1/2}\sum_{i=1}^N \tilde{\varsigma}_i^2\left(X_{\eta,i}-1\right)\right]=0$ and

$$E\left\{\left[N^{-1/2}\sum_{i=1}^N \tilde{\varsigma}_i^2\left(X_{\eta,i}-1\right)\right]^2\right\}=N^{-1}\sum_{i=1}^N\sum_{j=1}^N \tilde{\varsigma}_i^2\tilde{\varsigma}_j^2 E\left(X_{\eta,i}X_{\eta,j}-1\right).$$

Noting $E\left(X_{\eta,i}X_{\eta,j}\right)=1+\frac{2\rho_{\eta,ij}^2}{v}+\gamma_{2,\varepsilon_\eta}\left(\frac{\sum_t m_{tt}^2}{v_2}\right)\sum_{\ell=1}^N \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2$ (from (S.43)), we have

$$N^{-1}\sum_{i=1}^N\sum_{j=1}^N \tilde{\varsigma}_i^2\tilde{\varsigma}_j^2\left[\frac{2\rho_{\eta,ij}^2}{v}+\gamma_{2,\varepsilon_\eta}\left(\frac{\sum_t m_{tt}^2}{v^2}\right)\sum_{\ell=1}^N \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\right],$$

but $\sum_{\ell=1}^N \tilde{q}_{\eta,i\ell}^2\tilde{q}_{\eta,j\ell}^2\leq 1$ and $\rho_{\eta,ij}^2\leq 1$, for all i,j, and also $\sum_t m_{tt}^2\leq\sum_t m_{tt}=v$, we have

$$\begin{aligned}E\left\{\left[N^{-1/2}\sum_{i=1}^N \tilde{\varsigma}_i^2\left(X_{\eta,i}-1\right)\right]^2\right\} &\leq N^{-1}\sum_{i=1}^N\sum_{j=1}^N v^{-1}\tilde{\varsigma}_i^2\tilde{\varsigma}_j^2\left(2+\left|\gamma_{2,\varepsilon_\eta}\right|\right)\\ &= O\left(N/v\right).\end{aligned}$$

Therefore, $KN^{-1/2}\sum_{i=1}^N \tilde{\varsigma}_i^2\left(X_{\eta,i}-1\right)=O_p\left(\sqrt{N/v}\right)$. Thus,

$$\left(\frac{w_T}{v}\right)N^{-1}\sum_{i=1}^N \tilde{\varsigma}_i^2\left(X_{\eta,i}-1\right)=O_p\left(N^{\delta_\gamma-1}\right)+O_p\left(v^{-1/2}\right). \qquad \text{(S.106)}$$

Next, using (S.60) and noting $\boldsymbol{\xi}_i=\mathbf{V}\tilde{\gamma}_i+\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\boldsymbol{\eta}_i$ we have

$$\begin{aligned}&N^{-3/4}\sum_{i=1}^N v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\xi}_i\left(X_i-1\right)\\ =\;& N^{-3/4}\sum_{i=1}^N v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\left[\mathbf{V}\tilde{\gamma}_i+\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\tilde{\boldsymbol{\eta}}_i\right]\left[\left(X_{\eta,i}-1\right)-\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)X_{\eta,i}+B_i\right].\end{aligned}$$

Noting $\sup_i\left|\tilde{\varsigma}_i\right|\leq K$, $v^{-1}Tr\left[E\left(\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]=k\left(w_T/v\right)$, $\mathbf{M}_F\boldsymbol{\tau}_T=\mathbf{h}$, $\mathbf{H}_F=\mathbf{h}\mathbf{h}'$ and $E\left|X_{\eta,i}\right|^2\leq K$ by (S.43), we have

$$\begin{aligned}N^{-3/4}\sum_{i=1}^N E\left|v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_i\left(X_{\eta,i}-1\right)\right| &\leq N^{-3/4}K\sum_{i=1}^N E\left|v^{-1/2}\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_i\left(X_{\eta,i}-1\right)\right|\\ &\leq N^{-3/4}K\sum_{i=1}^N\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left\{v^{-1}Tr\left[E\left(\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\right\}^{1/2}\left(E\left|X_{\eta,i}-1\right|^2\right)^{1/2}\\ &\leq KN^{-3/4}\sum_{i=1}^N\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left(\frac{kw_T}{v}\right)^{1/2}=O\left(N^{\delta_\gamma-3/4}\right).\end{aligned}$$

Similarly

$$\begin{aligned}N^{-3/4}\sum_{i=1}^N\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)E\left|v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_iX_{\eta,i}\right| &\leq N^{-3/4}K\sum_{i=1}^N\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2}\left\{v^{-1}Tr\left[E\left(\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right]\right\}^{1/2}\left(E\left|X_{\eta,i}\right|^2\right)^{1/2}\\ &\leq KN^{-3/4}\sum_{i=1}^N\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2}\left(\frac{kw_T}{v}\right)^{1/2}=O\left(N^{\delta_\gamma-3/4}\right).\end{aligned}$$

$$\begin{aligned}N^{-3/4}\sum_{i=1}^N E\left|v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_iB_i\right| &\leq KN^{-3/4}\sum_{i=1}^N E\left|v^{-3/2}\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_i\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\mathbf{V}\tilde{\gamma}_i\right|\\ &\quad+2KN^{-3/4}\sum_{i=1}^N E\left|v^{-3/2}\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_i\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i\right|.\end{aligned}$$

First, by (S.63), noting that $E\left\{\left[v^{-1}Tr\left(\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right]^2\right\} = v^{-2}\sum_t m_{tt}^2 k\left[E\left(v_{t\ell}^4\right)+k\right]+k\left(k+2\right)\leq K$, we have

$$\begin{aligned}
& N^{-3/4}\sum_{i=1}^{N} E\left|v^{-3/2}\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_i\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\mathbf{V}\tilde{\gamma}_i\right| \\
\leq\ & N^{-3/4}\sum_{i=1}^{N}\left\{E\left|v^{-1}\tilde{\gamma}_i'\mathbf{V}'\mathbf{H}_F\mathbf{V}\tilde{\gamma}_i\right|\right\}^{1/2}\left\{E\left|v^{-1}\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\mathbf{V}\tilde{\gamma}_i\right|^2\right\}^{1/2} \\
\leq\ & N^{-3/4}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left\{E\left|v^{-1}Tr\left(\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right|\right\}^{1/2}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left\{E\left(\left[v^{-1}Tr\left(\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right]^2\right)\right\}^{1/2} \\
\leq\ & KN^{-3/4}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{3/2}\left(\frac{kw_T}{v}\right)^{1/2} = O\left(N^{\delta_\gamma-3/4}\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
& N^{-3/4}\sum_{i=1}^{N} E\left|v^{-3/2}\boldsymbol{\tau}_T'\mathbf{M}_F\mathbf{V}\tilde{\gamma}_i\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i\right| \\
\leq\ & N^{-3/4}\sum_{i=1}^{N}\left(E\left|v^{-1}\tilde{\gamma}_i'\mathbf{V}'\mathbf{H}_F\mathbf{V}\tilde{\gamma}_i\right|\right)^{1/2}\left(E\left|v^{-2}\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i\tilde{\boldsymbol{\eta}}_i'\mathbf{M}_G\mathbf{V}'\tilde{\gamma}_i\right|\right)^{1/2} \\
\leq\ & N^{-3/4}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left(E\left|v^{-1}Tr\left(\mathbf{V}'\mathbf{H}_F\mathbf{V}\right)\right|\right)^{1/2}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left\{v^{-2}Tr\left[E\left(\mathbf{V}\mathbf{V}'\right)\mathbf{M}_G E\left(\tilde{\boldsymbol{\eta}}_i\tilde{\boldsymbol{\eta}}_i'\right)\mathbf{M}_G\right]\right\}^{1/2} \\
=\ & N^{-3/4}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left[k\left(\frac{w_T}{v}\right)+v^{-1}\right]^{1/2} = O\left(v^{-1/2}N^{\delta_\gamma-3/4}\right).
\end{aligned}$$

Next, noting that $\left|\tilde{\varsigma}_i\right| < K$, $0 < \frac{\sigma_{\eta,ii}}{\sigma_{ii}} \leq 1$, $E\left|z_{\eta,i}^2\right| = 1$ and $E\left|X_{\eta,i}-1\right|^2 \leq K$, we have

$$\begin{aligned}
N^{-3/4}\sum_{i=1}^{N} E\left|v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\tilde{\boldsymbol{\eta}}_i\left(X_{\eta,i}-1\right)\right| &\leq N^{-3/4}K\sum_{i=1}^{N} E\left|v^{-1/2}\boldsymbol{\tau}_T'\mathbf{M}_F\tilde{\boldsymbol{\eta}}_i\left(X_{\eta,i}-1\right)\right| \\
&\leq N^{-3/4}K\sum_{i=1}^{N}\left\{\left(\frac{w_T}{v}\right)E\left|z_{\eta,i}^2\right|\right\}^{1/2}\left(E\left|X_{\eta,i}-1\right|^2\right)^{1/2} \\
&= O\left(N^{-1/2}\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)E\left|v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\tilde{\boldsymbol{\eta}}_i X_{\eta,i}\right| &\leq N^{-3/4}K\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left[\left(\frac{w_T}{v}\right)E\left|z_{\eta,i}^2\right|\right]^{1/2}\left(E\left|X_{\eta,i}\right|^2\right)^{1/2} \\
&\leq KN^{-3/4}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\frac{w_T}{v}\right)^{1/2} = O\left(N^{\delta_\gamma-3/4}\right).
\end{aligned}$$

$$\begin{aligned}
N^{-3/4}\sum_{i=1}^{N} E\left|v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\left(\frac{\sigma_{\eta,ii}}{\sigma_{ii}}\right)^{1/2}\tilde{\boldsymbol{\eta}}_i B_i\right| &\leq KN^{-3/4}\sum_{i=1}^{N} E\left|v^{-3/2}\boldsymbol{\tau}_T'\mathbf{M}_F\tilde{\boldsymbol{\eta}}_i\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\mathbf{V}\tilde{\gamma}_i\right| \\
&\quad +2KN^{-3/4}\sum_{i=1}^{N} E\left|v^{-3/2}\boldsymbol{\tau}_T'\mathbf{M}_F\tilde{\boldsymbol{\eta}}_i\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i\right|.
\end{aligned}$$

First, by (S.63), noting that $E\left(\left[v^{-1}Tr\left(\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right]^2\right) = v^{-2}\sum_t m_{tt}^2 k\left[E\left(v_{t\ell}^4\right)+k\right]+k\left(k+2\right)\leq K$, we have

$$\begin{aligned}
& N^{-3/4}\sum_{i=1}^{N} E\left|v^{-3/2}\boldsymbol{\tau}_T'\mathbf{M}_F\tilde{\boldsymbol{\eta}}_i\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\mathbf{V}\tilde{\gamma}_i\right| \\
\leq\ & N^{-3/4}\sum_{i=1}^{N}\left[\left(\frac{w_T}{v}\right)E\left|z_{\eta,i}^2\right|\right]^{1/2}\left(E\left|v^{-1}\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\mathbf{V}\tilde{\gamma}_i\right|^2\right)^{1/2} \\
\leq\ & N^{-3/4}\sum_{i=1}^{N}\left[\left(\frac{w_T}{v}\right)E\left|z_{\eta,i}^2\right|\right]^{1/2}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(E\left\{\left[v^{-1}Tr\left(\mathbf{V}'\mathbf{M}_G\mathbf{V}\right)\right]^2\right\}\right)^{1/2} \\
\leq\ & KN^{-3/4}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)\left(\frac{w_T}{v}\right)^{1/2} = O\left(N^{\delta_\gamma-3/4}\right).
\end{aligned}$$

$$
\begin{aligned}
& N^{-3/4}\sum_{i=1}^{N}E\left|v^{-3/2}\boldsymbol{\tau}_T'\mathbf{M}_F\tilde{\boldsymbol{\eta}}_i\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i\right| \\
\leq\; & N^{-3/4}\sum_{i=1}^{N}\left[\left(\frac{w_T}{v}\right)E\left|z_{\eta,i}^2\right|\right]^{1/2}\left(E\left|v^{-2}\tilde{\gamma}_i'\mathbf{V}'\mathbf{M}_G\tilde{\boldsymbol{\eta}}_i\tilde{\boldsymbol{\eta}}_i'\mathbf{M}_G\mathbf{V}'\tilde{\gamma}_i\right|\right)^{1/2} \\
\leq\; & N^{-3/4}\sum_{i=1}^{N}\left\{\left(\frac{w_T}{v}\right)E\left|z_{\eta,i}^2\right|\right\}^{1/2}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left(E\left\{v^{-2}Tr\left[E\left(\mathbf{V}\mathbf{V}'\right)\mathbf{M}_GE\left(\tilde{\boldsymbol{\eta}}_i\tilde{\boldsymbol{\eta}}_i'\right)\mathbf{M}_G\right]\right\}\right)^{1/2} \\
\leq\; & KN^{-3/4}\sum_{i=1}^{N}\left(\tilde{\gamma}_i'\tilde{\gamma}_i\right)^{1/2}\left(\frac{w_T}{v}\right)^{1/2}v^{-1}=O\left(v^{-1/2}N^{\delta_\gamma-3/4}\right).
\end{aligned}
$$

To sum, we have

$$
N^{-3/4}\sum_{i=1}^{N}v^{-1/2}\tilde{\varsigma}_i\boldsymbol{\tau}_T'\mathbf{M}_F\boldsymbol{\xi}_i\left(X_i-1\right)=O\left(N^{\delta_\gamma-3/4}\right)+O\left(N^{-1/2}\right). \tag{S.107}
$$

Substituting the results (S.106) and (S.107) into (S.105),

$$
N^{-1/2}\sum_{i=1}^{N}g_i\left(X_i-1\right)=O\left(N^{\delta_\gamma-3/4}\right)+O\left(N^{-1/2}\right)+O\left(v^{-1/2}\right).
$$

Finally, by applying Theorem 2,

$$
N^{-1/2}\sum_{i=1}^{N}z_i^2\left(1-X_i\right)=O_p\left(N^{\delta_\gamma-1/2}\right)+O_p\left(T^{-1/2}\right)+O_p(\sqrt{N}/T),
$$

thus,

$$
S_{1,NT}=O_p\left(N^{\delta_\gamma-1/2}\right)+O_p(\sqrt{N}/T)+O_p\left(T^{-1/2}\right)+O_p\left(N^{-1/2}\right),
$$

which establishes the required result. ■

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M1 Monte Carlo Supplement

M1.1 Simulating multivariate non-Gaussian random variates

The objective is to generate N random variables u_i, $i = 1, 2, ..., N$ such that (in population) $E(u_i) = 0$, $E(u_i^2) = \sigma_{ii}$, $E(u_i^3) = m_{3i}$, $E(u_i^4) = m_{4i}$ and $E(u_i u_j) = \rho_{ij}, i \neq j$ for $i, j = 1, 2, ..., N$.

The problem of generating multivariate non-normal random variables have been addressed in the literature by Vale and Maurelli (1983) and further discussed by Harwell and Serlin (1989) and Headrick and Sawilowsky (1999). Following Fleishman (1978), Vale and Maurelli (1983, VM) propose generating u_i as,

$$u_i = a_i + b_i \varepsilon_i + c_i \varepsilon_i^2 + d_i \varepsilon_i^3,\ i = 1, 2, ..., N,$$

where $\varepsilon_i \sim IIDN(0, 1)$ and $E(\varepsilon_i \varepsilon_j) = \rho_{\varepsilon,ij}$. The unknown parameters a_i,b_i, c_i, d_i, $\rho_{\varepsilon,ij}$ are obtained using the following relationships (see equations (2)-(5) in VM)

$$a_i + c_i = 0, \tag{M.1}$$

$$b_i^2 + 6b_i d_i + 2c_i^2 + 15d_i^2 = \sigma_{ii}, \tag{M.2}$$

$$2c_i(b_i^2 + 24b_i d_i + 105d_i^2 + 2) = m_{3i}, \tag{M.3}$$

$$24[b_i d_i + c_i^2(1 + b_i^2 + 28b_i d_i) + d_i^2(12 + 48b_i d_i + 141c_i^2 + 225d_i^2)] = m_{4i}, \tag{M.4}$$

for $i = 1, 2, ..., N$, and (see equation (11) in VM)

$$\rho_{ij} = \rho_{\varepsilon,ij}(b_i b_j + 3b_i d_j + 3d_i b_j + 9d_i d_j) + \rho_{\varepsilon,ij}^2(2c_i c_j) + \rho_{\varepsilon,ij}^3(6d_i d_j), \tag{M.5}$$

for $i \neq j = 1, 2, ..., N$.

The VM procedure is shown to work reasonably well for non-extreme values of skewness and kurtosis and when N is small. But even if one follows VM's two step procedure where the equations (M.1)-(M.4) are solved first, the procedure still requires solving a large number of cubic equations, and hoping that the solution of (M.5) for $\rho_{\varepsilon,ij}$ lies in the admissible range of $[-1, 1]$. No proof is provided that such a solution exists.

In what follows we propose a new more compact algorithm for generation of non-normal correlated random variables as a generalization of the standard Cholesky factor approach used routinely to generate correlated normal random variables. Let $\mathbf{u} = (u_1, u_2, ..., u_N)'$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_N)'$, and write each u_i as a linear combination of $\boldsymbol{\varepsilon}$

$$u_i = \sum_{j=1}^{N} q_{ij}\varepsilon_j, \text{for } i = 1, 2, ..., N,$$

or in matrix notation $\mathbf{u} = \mathbf{Q}\boldsymbol{\varepsilon}$, where q_{ij} is the (i, j) element of $\mathbf{Q}$.

We begin by generating ε_j, $j = 1, 2, ..., N$, as independent draws from non-normal distributions with $E(\varepsilon_j) = 0, E(\varepsilon_j^2) = 1, E(\varepsilon_j^3) = m_{\varepsilon,3j}$ and $E(\varepsilon_j^4) = m_{\varepsilon,4j}$. Note also that ρ_{ij} is determined by $\mathbf{Q}$ and is given by the (i, j) element of $\mathbf{QQ}'$ scaled by $\sigma_{ii}^{1/2}\sigma_{jj}^{1/2}$, where $\sigma_{ii} = \sum_{j=1}^{N} q_{ij}^2$. For given values of ρ_{ij} and σ_{ii}, $\mathbf{Q}$ can be obtained as the Cholesky factor of $E(\mathbf{uu}') = \mathbf{V}$. In such a case $\mathbf{Q}$ can be a lower or an upper triangular matrix with strictly positive diagonal elements. It is assumed that $\mathbf{V}$ is non-singular, and as a result $\mathbf{Q}$ will also be non-singular.

Consider now the problem of generating $\varepsilon_j's$ such that $E(u_i^3) = m_{i3}$ and $E(u_i^4) = m_{i4}$. To this end note that

$$\begin{aligned} m_{2i} &= \sigma_{ii} = E(u_i^2) = \sum_{j=1}^{N} q_{ij}^2, \text{ for } i = 1, 2, ..., N, \\ m_{3i} &= E(u_i^3) = E\left(\sum_j \sum_{j'} \sum_{\ell} \sum_{\ell'} q_{ij}q_{ij'}q_{i\ell}\varepsilon_j\varepsilon_{j'}\varepsilon_{\ell}\right) = \sum_{j=1}^{N} q_{ij}^3 m_{\varepsilon,3j}, \text{ for } i = 1, 2, ..., N, \end{aligned}$$

and

$$m_{4i} = E(u_i^4) = E\left(\sum_j \sum_{j'} \sum_{\ell} \sum_{\ell'} q_{ij}q_{ij'}q_{i\ell}q_{i\ell'}\varepsilon_j\varepsilon_{j'}\varepsilon_{\ell}\varepsilon_{\ell'}\right).$$

But since $\varepsilon_j's$ are independent draws with mean 0 and a unit variance we have

$$\begin{aligned} E\left(\varepsilon_j \varepsilon_{j'} \varepsilon_\ell \varepsilon_{\ell'}\right) &= m_{\varepsilon,4j}, \text{ if } j=j'=\ell=\ell' \\ &= 1, \text{ if } j=j' \text{ and } \ell=\ell' \text{ or if } j=\ell \text{ and } j'=\ell' \text{ or if } j=\ell' \text{ and } j'=\ell \\ &= 0 \text{ otherwise.} \end{aligned}$$

Hence, it readily follows that

$$m_{4i} = \sum_{j=1}^{N} q_{ij}^4 m_{\varepsilon,4j} + 3\sum_{j\neq\ell} q_{ij}^2 q_{i\ell}^2. \tag{M.6}$$

But

$$\sum_{j\neq\ell} q_{ij}^2 q_{i\ell}^2 = \sum_{j=1}^{N}\sum_{\ell=1}^{N} q_{ij}^2 q_{i\ell}^2 - \sum_{j=1}^{N} q_{ij}^4 = \left(\sum_{j=1}^{N} q_{ij}^2\right)^2 - \sum_{j=1}^{N} q_{ij}^4 = \sigma_i^4 - \sum_{j=1}^{N} q_{ij}^4.$$

Therefore, (M.6) can be written as

$$m_{4i} - 3\sigma_{ii}^2 = \sum\nolimits_{j=1}^{N} q_{ij}^4 \left(m_{\varepsilon,4j} - 3\right).$$

Let $\kappa_{\varepsilon j} = m_{\varepsilon,4j} - 3$ and $\kappa_i = m_{4i} - 3\sigma_i^4$, and write the above relations in matrix notation, namely

$$\boldsymbol{\kappa}_u = \mathbf{Q}_{(4)}\boldsymbol{\kappa}_\varepsilon,$$

where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, ..., \kappa_{\mathrm{N}})'$, $\boldsymbol{\kappa}_\varepsilon = (\kappa_{\varepsilon 1}, \kappa_{\varepsilon 2}, ..., \kappa_{\varepsilon N})'$ and $\mathbf{Q}_{(4)} = \mathbf{Q}\odot\mathbf{Q}\odot\mathbf{Q}\odot\mathbf{Q}$, where $\odot$ is the Hadamard matrix operator (or element-wise operator). Similarly, for the third moments we have

$$\mathbf{m}_3 = \mathbf{Q}_{(3)}\mathbf{m}_{\varepsilon,3},$$

where $\mathbf{m}_3 = (m_{3,1}, m_{3,2}, ..., m_{3,N})$, and $\mathbf{m}_{\varepsilon,3} = (m_{\varepsilon,3,1}, m_{\varepsilon,3,2}, ..., m_{\varepsilon,3,N})$. Since $\mathbf{Q}$ is a triangular matrix with strictly positive diagonal elements it follows that $\mathbf{Q}_{(3)}$ and $\mathbf{Q}_{(4)}$ are also non-singular and hence invertible. Thus

$$\mathbf{m}_{\varepsilon,3} = \mathbf{Q}_{(3)}^{-1}\mathbf{m}_3 \tag{M.7}$$

$$\boldsymbol{\kappa}_\varepsilon = \mathbf{Q}_{(4)}^{-1}\boldsymbol{\kappa}_u. \tag{M.8}$$

Denoting $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, ..., \sigma_{NN})'$ we also have $\boldsymbol{\sigma} = \mathbf{Q}_{(2)}\boldsymbol{\tau}_N$.

Having computed $m_{\varepsilon,3i}$ and $m_{\varepsilon,4i}$ we can now generate ε_i as

$$\varepsilon_i = a_i + b_i\upsilon_i + c_i\upsilon_i^2 + d_i\upsilon_i^3,\ i = 1, 2, ..., N, \tag{M.9}$$

where $\upsilon_i \sim IIDN(0,1)$ and the coefficients a_i, b_i, c_i and d_i are determined so that $E(\varepsilon_i) = 0, E(\varepsilon_i^2) = 1$, $E(\varepsilon_i^3) = m_{\varepsilon,3i}$ and $E(\varepsilon_i^4) = m_{\varepsilon,4i}$, using Fleishman's formula

$$a_i + c_i = 0, \tag{M.10}$$

$$b_i^2 + 6b_id_i + 2c_i^2 + 15d_i^2 = 1, \tag{M.11}$$

$$2c_i(b_i^2 + 24b_id_i + 105d_i^2 + 2) = m_{\varepsilon,3i}, \tag{M.12}$$

$$24[b_id_i + c_i^2(1 + b_i^2 + 28b_id_i) + d_i^2(12 + 48b_id_i + 141c_i^2 + 225d_i^2)] = \kappa_{\varepsilon i}. \tag{M.13}$$

Accordingly, in order to mimic as far as possible the main characteristics of observed security returns, for each replication, r, we generate $\sigma_{ii}^{(r)}$, $\gamma_{1,i}^{(r)}$, $\gamma_{2,i}^{(r)}$, $\left\{\beta_{\ell,i}^{(r)}, \text{ for } \ell = 1,2,3\right\}$, as random draws from their respective empirical distributions. For example, to generate $\sigma_{ii}^{(r)}$ over r and i, we first place the estimates $\hat{\sigma}_{ii,\tau}$, for $i = 1, 2, ..., N_\tau$, and $\tau = 1, 2, ..., 265$, that lie in the 2.5% to 97.5% quantile range, into 10 bins and then randomly select a bin with probability equal to the proportion of the estimates in each bin, and then draw randomly a value for $\sigma_{ii}^{(r)}$ from the selected bin. This procedure is repeated over $i = 1, 2, ..., N$ and replications $r = 1, 2, ..., R$.

M1.2 Details of the test statistics considered in the MC experiments in Section 5

Standardised Wald tests, SW_{LW} and SW_{POET}

First we present how to compute the estimates of $N \times N$ variance matrix $\mathbf{V}$ which is used to construct the feasible versions of the Standardised Wald statistic defined by (17). We considered two estimates, proposed by Ledoit and Wolf (2004), and the POET estimates of Fan et al (2013, FLM).

Ledoit and Wolf (2004, LW) considered a shrinkage estimator for regularisation which is based on a linear combination of the covariance matrix, $\hat{\boldsymbol{V}}$, and an identity matrix $\boldsymbol{I}_N$, and provide formulae for the appropriate weights. The LW shrinkage is expressed as

$$\hat{\boldsymbol{V}}_{LW} = \hat{\rho}_1 \boldsymbol{I}_N + \hat{\rho}_2 \hat{\boldsymbol{V}}, \tag{M.14}$$

with the estimated weights given by

$$\hat{\rho}_1 = m_T b_T^2 / d_T^2, \;\; \hat{\rho}_2 = a_T^2 / d_T^2$$

where

$$\begin{aligned} m_T &= N^{-1} \operatorname{tr}\left(\hat{\boldsymbol{V}}\right), \; d_T^2 = N^{-1} \operatorname{tr}\left(\hat{\boldsymbol{V}}^2\right) - m_T^2, \\ a_T^2 &= d_T^2 - b_T^2, \; b_T^2 = \min(\bar{b}_T^2, d_T^2), \end{aligned}$$

and

$$\bar{b}_T^2 = \frac{1}{NT^2} \sum_{t=1}^{T} \left\| \hat{\boldsymbol{u}}_t \hat{\boldsymbol{u}}_t' - \hat{\boldsymbol{V}} \right\|_F^2 = \frac{1}{NT^2} \sum_{t=1}^{T} \operatorname{tr}\left[\left(\hat{\boldsymbol{u}}_t \hat{\boldsymbol{u}}_t' \right) \left(\hat{\boldsymbol{u}}_t \hat{\boldsymbol{u}}_t' \right) \right] - \frac{2}{NT^2} \sum_{t=1}^{T} \operatorname{tr}\left(\hat{\boldsymbol{u}}_t' \hat{\boldsymbol{V}} \hat{\boldsymbol{u}}_t \right) + \frac{1}{NT} \operatorname{tr}\left(\hat{\boldsymbol{V}}^2 \right),$$

and noting that $\sum_{t=1}^{T} \operatorname{tr}\left(\hat{\boldsymbol{u}}_t' \hat{\boldsymbol{V}} \hat{\boldsymbol{u}}_t \right) = T \sum_{t=1}^{T} \operatorname{tr}\left(\hat{\boldsymbol{V}}^2 \right)$, we have

$$\bar{b}_T^2 = \frac{1}{NT^2} \sum_{t=1}^{T} \left(\sum_{i=1}^{N} \hat{u}_{it}^2 \right)^2 - \frac{1}{NT} \operatorname{tr}\left(\hat{\boldsymbol{V}}^2 \right),$$

with $\hat{u}_t = (\hat{u}_{1t}, \hat{u}_{2t}, \ldots, \hat{u}_{Nt})'$. $\hat{\mathbf{V}}_{LW}$ is positive definite by construction. Thus, the inverse $\hat{\mathbf{V}}_{LW}^{-1}$ exists and is well conditioned.

Extending the CL approach, FLM propose the POET estimator

$$\hat{\mathbf{V}}_{POET} = \left(\hat{\sigma}_{ij} s_{\tau_{ij}} \left[|\hat{\sigma}_{ij}| \geq \tau_{ij} \right] \right), \; i = 1, 2, \ldots, N-1, \; j = i+1, i+2, \ldots, N, \tag{M.15}$$

where $\tau_{ij} > 0$ is an entry-dependent adaptive threshold such that $\tau_{ij} = \sqrt{\hat{\varphi}_{ij}} \hat{\omega}_T$, with $\hat{\varphi}_{ij}^2 = T^{-1} \sum_{i=1}^{T} (\hat{u}_{it} \hat{u}_{jt} - \hat{\sigma}_{ij})^2$ and $\hat{\omega}_T = \hat{C} \sqrt{\log(N)/T}$, for some constant $\hat{C} > 0$, setting a lower bound on the cross-validation grid when searching for C such that the minimum eigenvalue of their threshold estimator is positive, $\lambda_{\min}\left(\hat{\mathbf{V}}_{POET} \right) > 0$. The consistency rate of the CL estimator is $C_0 m_N \sqrt{\log(N)/T}$ under the spectral norm of the error matrix $\left(\hat{\mathbf{V}}_{POET} - \mathbf{V} \right)$.

We perform a grid search for the choice of C over a specified range: $C = \{c : C_{\min} \leq c \leq C_{\max}\}$. We set $C_{\min} = 0$ and $C_{\max} = 4$, and impose increments of c/N. In each point of this range, c, we use $\hat{u}_{it}$, $i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$ and select the $N \times 1$ column vectors $\hat{\mathbf{u}}_t = (\hat{u}_{1t}, \hat{u}_{2t}, \ldots, \hat{u}_{Nt})'$, $t = 1, 2, \ldots, T$ which we randomly reshuffle over the t-dimension. This gives rise to a new set of $N \times 1$ column vectors $\hat{\mathbf{u}}_t^{(s)} = \left(\hat{u}_{1t}^{(s)}, \hat{u}_{2t}^{(s)}, \ldots, \hat{u}_{Nt}^{(s)} \right)'$ for the first shuffle $s = 1$. We repeat this reshuffling S times in total where we set $S = 20$ (as suggested by FLM). We consider this to be sufficiently large. In each shuffle $s = 1, 2, \ldots, S$, we divide $\hat{\mathbf{u}}^{(s)} = \left(\hat{\boldsymbol{u}}_1^{(s)}, \hat{\boldsymbol{u}}_2^{(s)}, \ldots, \hat{\boldsymbol{u}}_T^{(s)} \right)$ into two subsamples of size $N \times T_1$ and $N \times T_2$, where $T_2 = T - T_1$ where we set $T_1 = \frac{2T}{3}$ and $T_2 = \frac{T}{3}$. Let $\hat{\mathbf{V}}_{POET1}^{(s)} = \left(\hat{\sigma}_{1,ij}^{(s)} \right)$, with elements $\hat{\sigma}_{1,ij}^{(s)} = T_1^{-1} \sum_{t=1}^{T_1} \hat{u}_{it}^{(s)} \hat{u}_{jt}^{(s)}$, and $\hat{\mathbf{V}}_2^{(s)} = \left(\hat{\sigma}_{2,ij}^{(s)} \right)$ with elements $\hat{\sigma}_{2,ij}^{(s)} = T_2^{-1} \sum_{t=T_1+1}^{T} \hat{u}_{it}^{(s)} \hat{u}_{jt}^{(s)}$, $i, j = 1, 2, \ldots, N$, denote the sample covariance matrices generated using T_1 and T_2 respectively, for each split s. We threshold $\hat{\mathbf{V}}_{POET1}^{(s)}$ as in (M.15) using $I(.)$ as the thresholding function, where both $\hat{\varphi}_{ij}$ and ω_T are adjusted to

$$\hat{\varphi}_{1,ij}^{(s)} = \frac{1}{T_1} \sum_{t=1}^{T_1} (\hat{u}_{it}^{(s)} \hat{u}_{jt}^{(s)} - \hat{\sigma}_{1,ij}^{(s)})^2,$$

and

$$\omega_{T_1}(c) = c\sqrt{\frac{\log(N)}{T_1}}.$$

Then (M.15) becomes

$$\hat{\mathbf{V}}^{(s)}_{POET1}(c) = \left(\hat{\sigma}^{(s)}_{1,ij} I\left[\left|\hat{\sigma}^{(s)}_{1,ij}\right| \geq \tau^{(s)}_{1,ij}(c)\right]\right)$$

for each c, where

$$\tau^{(s)}_{1,ij}(c) = \sqrt{\hat{\varphi}^{(s)}_{1,ij}}\omega_{T_1}(c) > 0,$$

and $\hat{\varphi}^{(s)}_{1,ij}$ and $\omega_{T_1}(c)$ are defined above.

The following is then computed

$$\hat{G}(c) = \frac{1}{S}\sum_{s=1}^{S}\left\|\hat{\mathbf{V}}^{(s)}_{POET1}(c) - \hat{\mathbf{V}}^{(s)}_{POET2}\right\|^2_F, \tag{M.16}$$

for each c, and

$$\hat{C} = \arg\min_{C_{pd}+\epsilon\leq c\leq C_{\max}} \hat{G}(c), \tag{M.17}$$

where C_{pd} is the lowest c such that $\lambda_{\min}\left(\hat{\mathbf{V}}_{POET}(C_{pd})\right) > 0$ (To ensure that the threshold estimator is positive definite) and ϵ is a small positive constant. We do not conduct thresholding on the diagonal elements of the covariance matrices which remain intact.

Gungor and Luger (2009) SS and WS tests

These tests allow the error distribution to be non-normal but require it to be conditionally symmetric around zero.[M1] These tests are relatively easy to compute and are applicable even when $N > T$. However, they are constructed for models with a single factor and their validity is established only under $N < T$.

The SS test is based on the sign statistic

$$SS_N = \sum\nolimits_{i=1}^{N} S_i^2, \tag{M.18}$$

where

$$S_i = \frac{\left[\sum_{t=1}^{\mathcal{T}} I(z_{it} > 0)\right] - \mathcal{T}/2}{\sqrt{\mathcal{T}/4}},$$

$I(A)$ is the indicator function as defined by (56),

$$z_{it} = \left(\frac{y_{i,t+\mathcal{T}}}{f_{t+\mathcal{T}}} - \frac{y_{it}}{f_t}\right)\left(\frac{f_t - f_{t+\mathcal{T}}}{f_t f_{t+\mathcal{T}}}\right),\ t = 1, 2, ..., \mathcal{T},$$

$\mathcal{T}$ is the nearest integer part of $T/2$. The WS test is based on the Wilcoxon signed rank statistic

$$WS_N = \sum\nolimits_{i=1}^{N} \mathcal{W}_i^2, \tag{M.19}$$

where

$$\mathcal{W}_i = \frac{\left[\sum_{t=1}^{\mathcal{T}} I(z_{it} > 0)\, Rank(|z_{it}|)\right] - \mathcal{T}(\mathcal{T}+1)/4}{\sqrt{\mathcal{T}(\mathcal{T}+1)(2\mathcal{T}+1)/24}},$$

$Rank(|z_{it}|)$ is the rank (natural number) of $|z_{it}|$ when $|z_{i1}|, |z_{i2}|, ..., |z_{i\mathcal{T}}|$ are placed in an ascending order of magnitude. Gungor and Luger (2009) show that under the null hypothesis, $\alpha_i = 0$ for all i, both S_i and W_i statistics have limiting (as $T \to \infty$) standard normal distributions. Under the additional assumption that the errors in the CAPM regressions are cross-sectionally independent, conditional on the values of the single factor $(f_1, f_2, ..., f_T)$, SS_N and WS_N follow χ^2_N distributions.

Gungor and Luger (2016) $F_{\max}$ test

[M1] See equation (13) in Gungor and Luger (2009) for the definition of SS and WS test statistics.

Their test is based on the F-statistic

$$F_i = \frac{RRSS_i - URSS_i}{URSS_i/\left(T-m-1\right)},$$

where $RRSS_i$ and $URSS_i$ are restricted (imposing $\alpha_i = 0$ for all i) and unrestricted sum of squared residuals of the i^{th} regression. They consider various versions of the test, and recommend the use of the maximum test

$$F_{\max} = \max_{1 \le i \le N} F_i,$$

which we will consider in our Monte Carlo exercise.[M2] They claim that their resampling test procedure is robust against non-normality and cross-sectional dependence in specific errors. Their test is effectively based on wild bootstrap resampling in such a way that the sample residual cross-sectional correlation will be preserved, and unconsidered nuisance parameters are dealt with introduction of bounds test. Their test procedure is computable where $N > T$ and it allows the error distribution to be non-normal.

Specifically, their test procedure is as follows:

1. Obtain the $N \times 1$ b^{th} bootstrap error vector $\mathbf{u}_t^{(b)} = \tilde{\mathbf{u}}_t \chi_t$, where $\tilde{\mathbf{u}}_t = \left(\tilde{u}_{1t}, \tilde{u}_{2t}, ..., \tilde{u}_{Nt}\right)'$ is the residual vector consisting of the restricted regression (imposing no intercept), $y_{it} = \mathbf{f}_t' \tilde{\boldsymbol{\beta}}_i + \tilde{u}_{it}$, and χ_t is IID random variable over t which takes +1 or -1 with 1/2 chance, $b = 1, 2, ..., B-1$. Then, obtain the bootstrap sample using $\mathbf{y}_t^{(b)} = \mathbf{f}_t' \tilde{\boldsymbol{\beta}}_i + \mathbf{u}_t^{(b)}$.

2. Compute the liberal p-value (p^L) and the conservative p-value (p^C), where $p^C = \frac{B-R^C+1}{B}$ and $p^L = \frac{B-R^L+1}{B}$ with $R^C = 1 + \sum_{b=1}^{B-1} I\left[F_{\max} > F_{C\max}^{(b)}\right] + \sum_{b=1}^{B-1} I\left[F_{\max} = F_{C\max}^{(b)}\right] \times I\left[U_B > U_b\right]$, $R^L = 1 + \sum_{b=1}^{B-1} I\left[F_{\max} > F_{L\max}^{(b)}\right] + \sum_{b=1}^{B-1} I\left[F_{\max} = F_{L\max}^{(b)}\right] \times I\left[U_B > U_b\right]$, where $U_b \sim i.i.d.Uniform[0,1]$, $b = 1, 2, ..., B$, $F_{C\max}^{(b)} = \max_{1\le i \le N} F_{i,C}^{(b)}$, with $F_{i,C}^{(b)} = \frac{RRSS_i - URSS_i^{(b)}}{URSS_i^{(b)}/(T-m-1)}$, $F_{L\max}^{(b)} = \max_{1 \le i \le N} F_{i,L}^{(b)}$ with $F_{i,L}^{(b)} = \frac{RRSS_i^{(b)} - URSS_i^{(b)}}{URSS_i^{(b)}/(T-m-1)}$, $RRSS_i = \sum_{t=1}^{T} \tilde{u}_{it}^2$, $RRSS^{(b)}$ and $URSS^{(b)}$ are bootstrap restricted and unrestricted sum of squared residuals.

3. Follow the bounds test procedure: "Reject" H_0 if conservative bootstrap p-value, $p^C \le \alpha$, "accept" H_0 if liberal bootstrap p-value, $p^L > \alpha$, otherwise "inconclusive", where α is the significance level.

[M2] We are grateful to Richard Luger for sharing the code to compute the resampling test discussed in Gungor and Luger (2016).

M1.3 Supplementary Monte Carlo results

Table M1: Frequencies of Inconclusive Results of Gungor and Luger (2016) test for Table 2

Panel A: Normal Errors													
		$\delta_\gamma = 1/4$				$\delta_\gamma = 1/2$				$\delta_\gamma = 3/5$			
	(T,N)	50	100	200	500	50	100	200	500	50	100	200	500
Size: $\alpha_i = 0$ for all i													
$F_{\max}$	60	3.3	3.1	4.6	2.7	3.2	3.7	4.3	3.5	4.2	3.0	3.4	3.7
$(Inconclusive)$	100	4.2	3.8	4.0	3.9	3.6	3.9	3.9	3.8	3.7	3.8	4.3	3.3
Power: $\alpha_i \sim IIDN(0,1)$ for $i = 1, 2, ..., N_\alpha$ with $N_\alpha = \lfloor N^{\lambda_\alpha} \rfloor$, $\lambda_\alpha = 0.8$ otherwise $\alpha_i = 0$													
$F_{\max}$	60	29.3	35.9	40.3	45.5	30.6	34.1	39.6	44.5	27.4	36.3	38.9	46.0
$(Inconclusive)$	100	39.0	40.0	36.7	29.1	36.8	39.0	37.7	29.4	37.0	39.9	35.8	29.3
Panel B: Non-normal Errors													
		$\delta_\gamma = 1/4$				$\delta_\gamma = 1/2$				$\delta_\gamma = 3/5$			
	(T,N)	50	100	200	500	50	100	200	500	50	100	200	500
Size: $\alpha_i = 0$ for all i													
$F_{\max}$	60	4.2	3.7	4.8	5.2	4.5	4.8	4.0	4.9	4.3	3.8	4.8	5.1
$(Inconclusive)$	100	4.4	3.6	5.0	3.8	4.3	4.0	4.4	5.0	4.5	3.9	4.8	5.0
Power: $\alpha_i \sim IIDN(0,1)$ for $i = 1, 2, ..., N_\alpha$ with $N_\alpha = \lfloor N^{\lambda_\alpha} \rfloor$, $\lambda_\alpha = 0.8$ otherwise $\alpha_i = 0$													
$F_{\max}$	60	31.1	35.8	40.1	46.0	30.7	34.9	39.8	46.5	28.6	34.5	39.6	45.5
$(Inconclusive)$	100	37.3	39.1	37.7	28.6	39.0	38.8	35.8	27.9	37.5	38.9	36.1	31.7

See notes to Table 2 in the body paper.

Table M2: Size of the $\hat{J}_\alpha$ test **using the estimator of** $(N-1)\rho^2_{N,T}$ **based on the elements in** $\hat{\mathbf{V}}_{POET}$

This table summarises the size of the $\hat{J}_\alpha$ test using the estimator of $(N-1)\rho^2_{N,T}$ based on the elements in POET estimator of $\mathbf{V}$ proposed by FLM. Specifically, the test statistic is defined by $N^{-1/2}\sum_{i=1}^{N}\left(t_i^2-\frac{v}{v-2}\right)/\{\left(\frac{v}{v-2}\right)\sqrt{\frac{2(v-1)}{(v-4)}\left[1+(N-1)\hat{\rho}^2_{POET}\right]}\}$, where $\hat{\rho}^2_{POET}=\frac{2}{N(N-1)}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\hat{\rho}^2_{POET,ij}$ with $\hat{\rho}_{POET,ij}=\frac{\hat{\sigma}_{POET,ij}}{\sqrt{\hat{\sigma}_{POET,ii}}\sqrt{\hat{\sigma}_{POET,jj}}}$ where $\hat{\mathbf{V}}_{POET}=\{\hat{\sigma}_{POET,ij}\}$. The data is generated as described in the notes to Table 2. Values of the tests are compared to a positive one-sided critical value of the standard normal distribution. The test is conducted at the 5% significance level. Experiments are based on 2,000 replications.

	$\delta_\gamma=1/4$				$\delta_\gamma=1/2$				$\delta_\gamma=3/5$			
(T,N)	50	100	200	500	50	100	200	500	50	100	200	500
	Normal Errors											
$T=60$	7.6	5.6	6.2	5.3	10.3	9.5	9.4	10.1	12.5	12.2	15.0	17.1
$T=100$	6.8	5.3	5.5	5.6	6.8	9.5	9.3	9.7	9.0	14.0	15.7	15.7
	Non-normal Errors											
$T=60$	6.7	7.0	6.1	6.9	10.4	10.9	11.6	11.8	13.6	15.0	14.6	18.1
$T=100$	5.8	6.9	6.7	7.5	8.2	10.2	11.3	12.6	11.9	14.5	15.3	16.2

Table M3: Size of the $\hat{J}_\alpha$ test using the mean 1 in the place of $v/(v-2)$ to standardise t_i^2

This table summarises the size of $\hat{J}_\alpha$ test using the mean 1 to standardise. Specifically, the test statistic is defined by $N^{-1/2}\sum_{i=1}^{N}\left(t_i^2-1\right)/\{\left(\frac{v}{v-2}\right)\sqrt{\frac{2(v-1)}{(v-4)}\left[1+(N-1)\tilde{\rho}^2_{N,T}\right]}\}$. The data is generated as described in the notes to Table 2. Values of the tests are compared to a positive one-sided critical value of the standard normal distribution. The test is conducted at the 5% significance level. Experiments are based on 2,000 replications.

	$\delta_\gamma=1/4$				$\delta_\gamma=1/2$				$\delta_\gamma=3/5$			
(T,N)	50	100	200	500	50	100	200	500	50	100	200	500
	Normal Errors											
$T=60$	8.4	8.8	9.9	14.8	7.5	8.4	9.5	11.7	8.0	8.0	8.6	8.8
$T=100$	7.4	7.6	8.5	10.3	7.7	8.2	8.2	7.8	6.9	7.7	7.5	8.4
	Non-normal Errors											
$T=60$	7.4	9.0	10.3	15.1	8.2	8.1	9.0	13.1	7.5	8.7	8.8	10.1
$T=100$	7.9	7.9	8.5	10.2	6.9	7.0	8.7	8.1	7.1	8.1	7.7	7.3

Table M4: Size and power of SS and WS tests in the case of models with a single factor

The data is generated as $y_{it} = \alpha_i + \beta_{1i} f_{1t} + u_{it}, i = 1,2,..,N; t = 1,2,...,T,$ $f_{1t} = \mu_{f1} + \rho_{f1} f_{1,t-1} + \sqrt{h_{1t}}\,\zeta_{1t}$, $h_{1t} = \mu_{h1} + \rho_{1h1} h_{1,t-1} + \rho_{2h1}\zeta^2_{1,t-1}$, $\zeta_{1t} \sim IIDN(0,1)$, $t = -49,...,0,1,...,T$ with $f_{1,-50} = h_{1,-50} = 0$, $\mu_{f1} = 0.53$, $\rho_{f1} = 0.06$, $\mu_{h1} = 0.89$, $\rho_{1h1} = 0.85$, $\rho_{2h1} = 0.11$. For the size of the test, $\alpha_i = 0$ for all i, and for the power of the test, $\alpha_i \sim IIDN(0,1)$ for $i = 1,2,...,N_\alpha$ with $N_\alpha = \lfloor N^{\lambda_\alpha} \rfloor$, $\lambda_\alpha = 0.8$, otherwise $\alpha_i = 0$, where $\lfloor A \rfloor$ is the largest integer part of A. We generate the idiosyncratic errors, $\mathbf{u}_t = (u_{1t}, u_{2t}, ..., u_{Nt})'$, according to $\mathbf{u}_t = \mathbf{Q}\boldsymbol{\varepsilon}_t$, where $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t}, ..., \varepsilon_{Nt})'$, and $\mathbf{Q} = \mathbf{D}^{1/2}\mathbf{P}$ with $\mathbf{D} = diag(\sigma_1^2, \sigma_2^2, ..., \sigma_N^2)'$ and $\mathbf{P}$ being a Cholesky factor of correlation matrix of $\mathbf{u}_t$, $\mathbf{R}$, which is an $N \times N$ matrix used to calibrate the cross correlation of returns. $\mathbf{R} = \mathbf{I}_N + \mathbf{b}\mathbf{b}' - \check{\mathbf{B}}^2$,where $\mathbf{b} = (b_1, b_2, ..., b_N)'$, $\check{\mathbf{B}} = diag(\mathbf{b})$, we draw the first and the last N_γ $(< N)$ elements of $\mathbf{b}$ as $Uniform(0.7, 0.9)$, and set the remaining middle elements to 0. We set $N_\gamma = \lfloor N^{\delta_\gamma} \rfloor$. We examine $\delta_\gamma = 1/4, 1/2$ and $3/5$. For non-normal case, u_{it} are generated following steps 1-4 of the procedure in Appendix B. SS and WS are the signed and singed rank tests of Gungor and Luger (2009), which are distributed as χ^2_N and applicable for one-factor model (see Section M1.2 for more details) All tests are conducted at the 5% significance level. Experiments are based on 2,000 replications.

		$\delta_\gamma = 1/4$				$\delta_\gamma = 1/2$				$\delta_\gamma = 3/5$			
	(T,N)	50	100	200	500	50	100	200	500	50	100	200	500
Panel A: With Single Factor, Normal Errors													
Size: $\alpha_i = 0$ for all i													
SS	60	4.3	5.2	4.3	5.1	7.0	7.7	8.5	7.8	9.1	9.7	12.6	12.4
	100	4.5	4.7	5.3	5.1	7.4	7.9	8.3	7.7	10.5	10.0	11.5	12.2
WS	60	4.3	4.8	4.4	4.6	7.6	8.2	9.0	8.6	9.8	9.9	13.1	13.2
	100	3.8	5.3	5.2	5.1	7.9	8.1	8.1	7.8	10.4	11.4	12.9	13.4
Power: $\alpha_i \sim IIDN(0,1)$ for $i = 1,2,...,N_\alpha$ with $N_\alpha = \lfloor N^{\lambda_\alpha} \rfloor$, $\lambda_\alpha = 0.8$ otherwise $\alpha_i = 0$.													
SS	60	20.8	26.2	34.9	47.9	22.2	25.5	35.2	48.9	21.1	28.2	35.4	45.7
	100	36.6	47.0	62.8	80.7	35.1	45.6	59.9	77.9	35.3	44.5	56.8	72.6
WS	60	23.4	32.3	43.0	59.2	25.4	30.8	40.4	58.2	25.5	32.4	41.3	52.1
	100	44.3	58.7	74.0	90.3	42.0	55.3	70.9	87.6	41.5	51.9	67.2	83.3
Panel B: With Single Factor, Non-normal Errors													
Size: $\alpha_i = 0$ for all i													
SS	60	10.3	13.8	19.9	33.4	11.8	14.0	18.5	33.4	11.8	17.4	22.8	32.2
	100	16.3	23.7	35.2	63.3	15.5	21.3	33.8	57.2	18.4	24.5	32.6	49.9
WS	60	8.3	11.5	16.5	24.9	12.7	12.7	16.9	26.8	13.1	16.5	19.1	28.7
	100	14.0	18.3	27.1	51.6	16.0	18.6	28.2	44.1	17.2	20.8	28.3	39.0
Power: $\alpha_i \sim IIDN(0,1)$ for $i = 1,2,...,N_\alpha$ with $N_\alpha = \lfloor N^{\lambda_\alpha} \rfloor$, $\lambda_\alpha = 0.8$ otherwise $\alpha_i = 0$.													
SS	60	31.8	43.5	57.7	83.2	30.6	42.1	57.0	79.8	29.2	41.0	54.8	74.1
	100	55.9	73.6	90.6	99.2	51.5	67.1	88.0	98.8	50.6	64.7	81.8	97.5
WS	60	33.3	46.2	62.6	87.1	32.2	44.6	61.2	81.5	32.3	43.3	55.8	76.1
	100	59.1	77.2	92.6	99.6	55.4	70.5	90.7	99.3	52.5	68.3	84.6	98.0

Table M5: Size and power of $\hat{J}_\alpha$ test with mixed spatial-factor models with the value of spatial parameter $\rho_\varepsilon = 0.8$

DGP is identical to that for the results reported in Table 5 except $\rho_\varepsilon = 0.8$. Also see notes to Table 2.

Panel A: Normal Errors with $\rho_\varepsilon = 0.8$															
		Size							Power						
	(T,N)	50	100	200	500	1000	2000	5000	50	100	200	500	1000	2000	5000
Pure spatial models ($\boldsymbol{\gamma} = \mathbf{0}$)															
$\hat{J}_\alpha$	60	6.6	7.0	7.3	7.8	7.5	6.6	7.3	38.6	52.1	68.9	86.8	96.5	99.2	99.8
	100	7.0	7.1	6.9	6.4	5.5	5.6	5.7	68.1	82.8	94.5	99.5	100.0	100.0	100.0
$J_\alpha(0)$	60	15.8	18.5	17.8	19.1	18.4	16.5	19.0	61.4	73.6	87.6	95.1	99.2	99.8	99.9
	100	18.3	17.4	16.7	17.1	16.7	16.5	17.6	84.9	94.3	98.5	100.0	100.0	100.0	100.0
Mixed spatial-factor models ($\delta_\gamma = 1/4$)															
$\hat{J}_\alpha$	60	5.8	6.0	6.5	7.0	5.7	7.3	6.6	39.4	51.3	67.5	87.4	96.4	99.5	100.0
	100	7.0	7.8	6.7	7.1	5.4	6.0	6.1	66.6	81.6	94.8	99.4	100.0	100.0	100.0
$J_\alpha(0)$	60	16.3	16.4	16.3	17.7	16.5	16.9	16.8	61.8	72.4	84.7	95.6	98.6	100.0	100.0
	100	17.2	18.9	17.6	17.4	15.3	18.1	17.8	84.8	93.5	98.8	100.0	100.0	100.0	100.0
Mixed spatial-factor models ($\delta_\gamma = 1/2$)															
$\hat{J}_\alpha$	60	6.6	7.6	6.9	7.1	6.0	6.7	5.8	39.1	50.7	66.6	85.8	95.6	98.8	100.0
	100	6.8	6.1	7.2	6.7	6.1	6.9	6.3	66.4	83.1	94.4	99.6	100.0	100.0	100.0
$J_\alpha(0)$	60	17.2	17.9	16.8	18.9	18.0	17.7	16.5	60.0	72.9	86.1	95.2	99.4	99.8	100.0
	100	17.5	17.6	17.6	19.4	17.0	18.9	18.6	85.3	94.5	98.6	100.0	100.0	100.0	100.0
Mixed spatial-factor models ($\delta_\gamma = 3/5$)															
$\hat{J}_\alpha$	60	6.4	7.5	5.8	7.6	7.8	7.9	7.5	38.2	51.3	67.5	85.2	96.2	99.3	99.9
	100	6.8	6.4	7.0	7.0	5.5	6.4	5.9	67.9	82.4	94.3	99.7	100.0	100.0	100.0
$J_\alpha(0)$	60	15.7	18.7	16.8	19.5	17.3	19.1	18.3	60.0	74.1	85.6	95.4	99.1	99.9	100.0
	100	17.5	17.3	18.2	17.3	17.7	17.7	18.1	86.2	93.5	98.8	100.0	100.0	100.0	100.0

Table M5 —*Continued*

Panel B: Non-normal Errors with $\rho_\varepsilon = 0.8$															
		Size							Power						
	(T,N)	50	100	200	500	1000	2000	5000	50	100	200	500	1000	2000	5000
Pure spatial models ($\boldsymbol{\gamma} = \mathbf{0}$)															
$\hat{J}_\alpha$	60	8.9	7.5	7.5	6.9	8.1	8.0	8.6	35.5	45.3	60.0	78.7	91.4	97.0	99.7
	100	7.3	6.0	7.0	6.4	7.1	6.4	6.4	57.8	72.1	89.2	97.8	99.8	100.0	100.0
$J_\alpha(0)$	60	18.7	18.2	18.4	18.3	18.1	20.3	20.2	57.1	66.0	79.0	91.9	97.1	99.5	99.8
	100	16.6	17.1	18.5	18.9	18.8	20.2	17.9	78.9	88.7	96.5	99.7	100.0	100.0	100.0
Mixed spatial-factor models ($\delta_\gamma = 1/4$)															
$\hat{J}_\alpha$	60	7.4	6.4	8.4	7.1	7.0	7.4	7.5	35.9	43.0	58.7	77.5	89.3	97.0	99.7
	100	6.3	6.3	7.1	5.4	6.2	7.1	6.9	58.3	73.6	87.5	98.4	99.6	100.0	100.0
$J_\alpha(0)$	60	16.5	16.2	19.6	18.1	18.0	19.1	19.2	56.4	65.0	79.8	92.3	96.9	99.4	99.9
	100	16.3	16.6	17.7	17.5	19.0	18.8	19.0	77.2	88.4	96.4	99.7	100.0	100.0	100.0
Mixed spatial-factor models ($\delta_\gamma = 1/2$)															
$\hat{J}_\alpha$	60	8.2	6.9	7.3	7.0	7.0	8.3	7.6	32.9	43.3	57.7	77.8	90.9	97.1	99.7
	100	6.8	6.7	7.0	7.1	6.5	7.1	7.0	55.7	73.5	88.1	98.2	99.8	100.0	100.0
$J_\alpha(0)$	60	16.7	16.8	18.8	18.8	21.2	20.5	20.1	54.5	66.1	78.0	91.0	97.2	99.3	100.0
	100	17.8	17.0	18.3	18.8	19.9	19.1	20.5	76.9	89.5	97.0	99.8	100.0	100.0	100.0
Mixed spatial-factor models ($\delta_\gamma = 3/5$)															
$\hat{J}_\alpha$	60	7.2	7.9	6.4	6.4	8.4	7.4	7.8	31.8	44.0	58.1	76.9	89.8	96.9	99.6
	100	7.2	6.6	7.9	6.6	6.9	7.0	6.7	58.0	73.0	86.7	98.5	99.7	100.0	100.0
$J_\alpha(0)$	60	16.7	18.0	18.0	18.9	20.9	18.6	19.9	54.5	67.0	79.2	91.0	96.5	99.0	100.0
	100	17.7	16.4	18.7	18.1	19.2	19.3	18.6	77.9	88.9	96.0	99.8	100.0	100.0	100.0

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