Measurement of Factor Strength: Theory and Practice*

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Abstract

This paper proposes an estimator of factor strength and establishes its consistency and asymptotic distribution. The proposed estimator is based on the number of statistically significant factor loadings, taking account of the multiple testing problem. We focus on the case where the factors are observed which is of primary interest in many applications in macroeconomics and finance. We also consider using cross section averages as a proxy in the case of unobserved common factors. We face a fundamental factor identification issue when there are more than one unobserved common factors. We investigate the small sample properties of the proposed estimator by means of Monte Carlo experiments under a variety of scenarios. In general, we find that the estimator, and the associated inference, perform well. The test is conservative under the null hypothesis, but, nevertheless, has excellent power properties, especially when the factor strength is sufficiently high. Application of the proposed estimation strategy to factor models of asset returns shows that out of 146 factors recently considered in the finance literature, only the market factor is truly strong, while all other factors are at best semi-strong, with their strength varying considerably over time. Similarly, we only find evidence of semi-strong factors in an updated version of the Stock and Watson (2012) macroeconomic dataset.

Keywords: Factor models, factor strength, measures of pervasiveness, cross-sectional dependence, market factor

JEL Classifications: C38, E20, G20

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1 Introduction

Interest in the analysis of cross-sectional dependence (CSD), applied to households, firms, markets, regional and national economies, has become prominent over the past decade, especially so in the aftermath of the latest financial crisis and its important implications for the global economy. Researchers in many fields have turned to network theory, spatial and factor models, to obtain a better understanding of the extent, nature and strength of such cross dependencies. Bailey et al. (2016) (BKP hereafter) give a thorough account of the rationale and motivation behind the need for determining the extent of CSD, be it in finance, micro or macroeconomics. To estimate the degree of CSD in a panel dataset, BKP analyse the rate at which the variance of the cross section average of observations in that panel tends to zero and show that it depends on the degree or exponent of CSD which they denote by α. They explore a latent factor model setting as a vehicle for characterising strong and semi-strong covariance structures as defined in Chudik et al. (2011). They relate these to the degree of pervasiveness of factors in unobserved factor models often used in the literature to model CSD. In a follow up paper to BKP, Bailey et al. (2019) extend their analysis in two respects. First, they consider a more generic setting which does not require a common factor representation and holds more generally for both moderate and sizeable CSD. They achieve this by directly considering the significance of individual pair-wise correlations and base the estimation of α on the proportion of statistically significant correlations. Second, they show that their estimator also applies to the residuals obtained from panel data regressions.

The estimators developed in Bailey et al. (2016, 2019) are helpful as overall measures of CSD, but they do not provide information on the strength of individual factors which is of interest, for example, in the pricing of risk in empirical finance and in identifying dominant factors in macroeconomic fluctuations. This paper relates the estimation of factor strength to the degree to which the factor in question has pervasive effects on all the units in the dataset. As a simple example consider the following single factor model

$$x_{it} = c_i + \gamma_i f_t + u_{it},\ i = 1, 2, \dots, n;\ t = 1, 2, \dots, T, \tag{1}$$

where f_t, is a known factor, c_i is the unit-specific effect, $u_{it} \sim IID(0, \sigma_i^2)$ is an idiosyncratic error, and γ_i is the factor loading for unit i. The strength of f_t can be characterised by the degree of pervasiveness of its effects (i.e. the number of non-zero factor loadings), and measured by the rate at which $\sum_{i=1}^{n} \gamma_i^2$ rises with n. Denoting this rate by α, the standard large n and T latent factor models assume that $\alpha = 1$, as required, for example, by Assumption B in Bai and Ng (2002) and Bai (2003). At the other extreme, a factor is deemed to be weak if $0 \leq \alpha < 0.5$. This case is studied in Onatski (2012). Similar notions of factor strength are also used in recent financial studies by Lettau and Pelger (2018) and Anatolyev and Mikusheva (2019). In most empirical applications, the value of α is unknown. Incorrectly setting it to $\alpha = 1$ can result in misleading inference. Also, as we shall see, it is not possible to identify α when the factor in question is weak and therefore, in effect, can be absorbed into the error term with little consequence for the analysis of CSD. In most empirical applications in finance and macroeconomics, the values of α that are of interest and of consequence, are within the range $\alpha \in (0.5, 1]$. As recently shown by Pesaran and Smith (2019), factor strengths play a crucial role in the identification of risk premia in arbitrage asset pricing models, and determine the rates at which risk premia can be estimated. The strength of macroeconomic shocks is also of special interest, as its value has important bearing on forecasting and policy analysis. Contributions in terms of factor selection and factor model estimation when $\alpha \in (0.5, 1]$ include Freyaldenhoven (2019) and Uematsu and Yamagata (2019).

In this paper we propose an estimator of factor strength and establish its consistency and asymptotic distribution when $\alpha > 1/2$. The proposed estimator is based on the number of statistically significant factor loadings, taking account of the multiple tests being carried out. We find that it is a powerful and highly accurate estimator, especially for higher levels of factor strength. Despite its simplicity, the distribution of the estimator, being based on sums of random variables that follow, potentially heterogeneous, Bernoulli distributions, is quite complicated and non-standard. While the parameters of these distributions are hard to pin down, they can be bounded in such a way as to provide both grounds for the validity of a central limit theorem for the asymptotically dominant part of the estimator and an upper bound for the asymptotic variance. These two elements allow for the construction of asymptotically conservative test statistics.

We focus on the case where the factors are observed, which is of primary interest in tackling the empirical examples mentioned earlier, among many others. We also consider using cross section averages as a proxy in the case of unobserved common factors. In practice, we face a significant factor identification issue when there are more than one unobserved common factors. In the case of multiple unobserved factor models, our contribution is best viewed as providing inferential information about the exponent of the strongest factor, shared amongst the cross section units.

We investigate the small sample properties of the proposed estimator by means of Monte Carlo experiments under a variety of scenarios. In general, we find that the estimator, and the associated inference, perform well. The test is conservative under the null hypothesis, but, nevertheless, has excellent power properties, especially when α is close to unity, even for moderate sample sizes.

We illustrate the relevance of our proposed estimator by means of two empirical applications, using well known datasets in finance and macroeconomics. First, we consider a large number of factors proposed in the finance literature for asset pricing. For example, Harvey and Liu (2019) document over 400 such factors, and Feng et al. (2020) consider the problem of factor selection using penalised regressions. In view of recent theoretical results in Pesaran and Smith (2019), our empirical contribution focuses on the estimation of factor strengths, since factor selection is only meaningful for asset pricing if the factors under consideration are sufficiently strong. We compute 10-year rolling estimates of α (together with their standard error bands) for the excess market return (as a measure of the market factor), and the remaining 145 factors considered by Feng et al. (2020). Out of the 146 factors considered, we find that only the market factor is sufficiently strong over all rolling windows, with its average strength estimated to be around 0.99 over the full sample (from September 1989 to December 2017). In contrast, none of the other factors achieve strengths exceeding 0.90 over the full sample, but over the sub-sample that includes the recent financial crisis as many as 48 (out of 145) have average strength estimated to lie between 0.9 and 0.94. Remarkably, the well-known size and value factors introduced in Fama and French (1993) are not particularly prominent as compared to cash and leverage factors. Further, of special interest is the high degree of time variation in the estimates of factor strengths, which cannot be attributed to sampling variation, considering the high precision with which the factor strengths are estimated, particularly when the true factor strength is close to unity.

Our second empirical application considers an unobserved factor model and asks if there exists any strong latent factor shared by the set of macroeconomic variables originally investigated by Stock and Watson (2012). In particular, we consider an updated version of Stock and Watson (SW) dataset covering 187 variables over the period 1988Q1-2019Q2. Although it is not possible to separately identify the strengths of individual latent factors, we are able to show that the strength of the strongest of the latent factors in the updated SW data set is around 0.94 which is sufficiently high for the factor to be important for macroeconomic analysis, but yet statistically different from 1, usually assumed in the literature.

The rest of the paper is organised as follows: Section 2 introduces our proposed measure of factor strength and develops the estimation and inference theory for the single factor case. A general multi-factor set up is then considered in Section 3 which includes the main theoretical results of the paper. Section 4 discusses the case of unobserved factors, and after highlighting the identification problem involved, considers the estimation of the strength of the strongest factor implied by the model. Sections 5 and 6 provide extensive simulation and empirical evidence of the performance of our estimator. Section 7 provides some concluding remarks. Mathematical proofs are contained in an appendix at the end of the document and further simulation and empirical results are provided in an online supplement.

Notation: Generic positive finite constants are denoted by C_i, for $i = 1, 2, \ldots$. They can take different values at different instances. If $\{f_n\}_{n=1}^{\infty}$ is a real sequence and $\{g_n\}_{n=1}^{\infty}$ is a sequence of positive numbers, then $f_n = O(g_n)$, if there exists a positive finite constant C_0 such that $|f_n|/g_n \leq C_0$ for all n. $f_n = o(g_n)$ if $f_n/g_n \to 0$ as $n \to \infty$. If $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are both positive sequences of real numbers, then $f_n = \ominus(g_n)$ if there exist $N_0 \geq 1$ and positive finite constants C_0 and C_1, such that $\inf_{n \geq N_0}(f_n/g_n) \geq C_0$, and $\sup_{n \geq N_0}(f_n/g_n) \leq C_1$. $\to_d$ denotes convergence in distribution as $n, T \to \infty$.

2 Estimation strategy

To illustrate the basic idea behind our estimation strategy we begin with a single factor model where the factor is observed, and turn subsequently to the cases of multiple, observed or unobserved factors. Suppose that T observations are given, on n cross section units, namely $\{x_{it}, i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, T\}$, and follow the single factor model (1), repeated here for convenience:

$$x_{it} = c_i + \gamma_i f_t + u_{it}, \tag{2}$$

where f_t, $t = 1, 2, \ldots, T$ is a known factor, c_i is the unit-specific effect, $u_{it} \sim IID(0, \sigma_i^2)$ is an idiosyncratic error, and γ_i is the factor loading for unit i. The factor loadings are assumed to be non-zero for the first $[n^{\alpha}]$ units, and zero for the rest, where $[\cdot]$ denotes the integer part function. More specifically, suppose that, for some $c > 0$,

$$\begin{aligned} &|\gamma_i| > c \text{ a.s. for } i = 1, 2, \ldots, [n^{\alpha}], \\ &|\gamma_i| = 0 \text{ a.s. for } i = [n^{\alpha}] + 1, [n^{\alpha}] + 2, \ldots, n, \end{aligned} \tag{3}$$

where α is the *exponent of cross section dependence* discussed in BKP.[1] The exponent α measures the degree of pervasiveness or strength of the factor. It is important to reiterate that BKP focus on estimating an overall measure of cross-sectional dependence in x_{it}, without particular reference to a single specific factor. They base their estimator on the variance of the cross-sectional average, while noting the pros and cons of alternative approaches, based on other characteristics of x_{it}, such as, e.g., the maximum eigenvalue of the covariance of x_{it}. Given the prominence of this maximum eigenvalue as a basis for characterising CSD, they note existing work, as well as reasons for which a formal eigenvalue analysis may not be promising for this purpose.

[1] More generally, we can have $|\gamma_i| = c_1 \gamma^{i-[n^{\alpha}]}$, with $|\gamma| < 1$ and $c_1 > 0$, for $i = [n^{\alpha}] + 1, [n^{\alpha}] + 2, \ldots, n$, in (3). But for simplicity of exposition, we opt for $|\gamma_i| = 0$ a.s. instead.

As we noted above our aim is different. We wish to determine the strength of pervasiveness of particular factors and use α, as defined through (3), as a tool for that purpose. To estimate α we begin by running the least squares regressions of $\{x_{it}\}_{t=1}^{T}$ for each $i = 1, 2, \ldots, n$ on an intercept and f_t to obtain

$$x_{it} = \hat{c}_{iT} + \hat{\gamma}_{iT} f_t + \hat{\nu}_{it}, \quad t = 1, 2, \ldots, T$$

where $\hat{c}_{iT}$ and $\hat{\gamma}_{iT}$ are the Ordinary Least Squares (OLS) estimates of this regression. Denote by $t_{iT} = \hat{\gamma}_{iT} \,/\, \text{s.e.}\,(\hat{\gamma}_{iT})$ the t-statistic corresponding to γ_i:

$$t_{iT} = \frac{(\mathbf{f}'\mathbf{M}_\tau \mathbf{f})^{1/2}\, \hat{\gamma}_{iT}}{\hat{\sigma}_{iT}} = \frac{(\mathbf{f}'\mathbf{M}_\tau \mathbf{f})^{-1/2}\, (\mathbf{f}'\mathbf{M}_\tau \mathbf{x}_i)}{\hat{\sigma}_{iT}}, \tag{4}$$

where $\mathbf{M}_\tau = \mathbf{I}_T - T^{-1}\boldsymbol{\tau}\boldsymbol{\tau}'$, $\boldsymbol{\tau}$ is a $T \times 1$ vector of ones, $\mathbf{f} = (f_1, f_2, \ldots, f_T)'$, $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iT})'$, and $\hat{\sigma}_{iT}^2 = T^{-1} \sum_{t=1}^{T} \hat{\nu}_{it}^2$. Also assume that, for some $c > 0$, $T^{-1}\mathbf{f}'\mathbf{M}_\tau \mathbf{f} > c$, which is necessary for identification of γ_i. Consider the proportion of regressions with statistically significant coefficients γ_i:

$$\hat{\pi}_{nT} = n^{-1} \sum_{i=1}^{n} \hat{d}_{i,nT}, \tag{5}$$

where $\hat{d}_{i,nT} = \mathbf{1}\left[|t_{iT}| > c_p(n)\right]$, $\mathbf{1}(A) = 1$ if $A > 0$, and zero otherwise, and the critical value function, $c_p(n)$, is given by

$$c_p(n) = \Phi^{-1}\left(1 - \frac{p}{2n^\delta}\right). \tag{6}$$

Here p is the nominal size of the individual tests, $\delta > 0$ is the critical value exponent and $\Phi^{-1}(\cdot)$ denotes the inverse cumulative distribution function of the standard normal distribution.

Suppose that $\hat{\pi}_{nT} > 0$, and consider the following estimator of α

$$\tilde{\alpha} = 1 + \frac{\ln \hat{\pi}_{nT}}{\ln n}.$$

In the rare case where $\hat{\pi}_{nT} = 0$, we then set $\tilde{\alpha} = 0$. Overall

$$\hat{\alpha} = \begin{cases} \tilde{\alpha}, & \text{if } \hat{\pi}_{nT} > 0, \\ 0, & \text{if } \hat{\pi}_{nT} = 0. \end{cases} \tag{7}$$

Clearly $\hat{\alpha} \in [0, 1]$ a.s.; also, $\hat{\alpha}$ and $\tilde{\alpha}$ are asymptotically equivalent since for $\alpha > 0$ then $\mathbb{P}(n\ \hat{\pi}_{nT} = 0) \to 0$ as $n \to \infty$.

Remark 1 *It is tempting to argue in favour of using the proportion of non-zero loadings, π, instead of the exponent α. The two measures are clearly related - $\pi = n^{\alpha-1}$, and coincide only when $\alpha = 1$. But when $\alpha < 1$, π becomes smaller and smaller as $n \to \infty$, and eventually tends to 0, for all values of $\alpha < 1$. The rate at which π tends to zero with n is determined by α, and hence α is a more discriminating measure of pervasiveness than π. It is also unclear how a particular value of π should be chosen as a measure of pervasiveness. Unlike α which can be chosen to be fixed in n, any choice of π which is fixed in n requires $\alpha \to 1$ as $n \to \infty$, albeit at the very slow $ln(n)$ rate. Note that when π is set to $\pi^0 > 0$, a fixed value, then $\alpha = 1 + ln(\pi^0)/ln(n)$, and $\alpha \to 1$ if π^0 is fixed in n.*

2.1 Asymptotic distribution

Denote the true α by α_0, let $d_i^0 = \mathbf{1}(\gamma_i \neq 0)$ and note that $D_n^0 = \sum_{i=1}^n d_i^0 = n^{\alpha_0}$ (the integer part symbol is dropped for simplicity). Let

$$\hat{D}_{nT} = n\hat{\pi}_{nT} = \sum_{i=1}^{n} \hat{d}_{i,nT}, \tag{8}$$

and note that $\hat{D}_{nT}/D_n^0 = n^{\hat{\alpha}-\alpha_0}$. Taking logs, we obtain

$$\begin{aligned}(\ln n)(\hat{\alpha}-\alpha_0) &= \ln\left(\frac{\hat{D}_{nT}}{D_n^0}\right) = \ln\left(1+\frac{\hat{D}_{nT}-n^{\alpha_0}}{n^{\alpha_0}}\right) \\ &= \ln(1+A_{nT}+B_{nT}) \\ &= A_{nT}+B_{nT}+O_p\left(A_{nT}^2\right)+O\left(B_{nT}^2\right)+O_p\left(A_{nT}B_{nT}\right)+\ldots, \end{aligned} \tag{9}$$

where

$$A_{nT} = \frac{\sum_{i=1}^n \left[\hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)\right]}{n^{\alpha_0}}, \tag{10}$$

$$B_{nT} = \frac{\sum_{i=1}^n E\left(\hat{d}_{i,nT}\right) - n^{\alpha_0}}{n^{\alpha_0}}. \tag{11}$$

To motivate the proposed estimator and to simplify the derivations, here we assume σ_i is known and u_{it} is Gaussian, and turn to the more general multi-factor case with non-Gaussian errors in Section 3. In this simple case we have the following lemmas proven in Appendix A.

Lemma 1 *Let the model be given by (2) where (3) holds, σ_i is known and u_{it} is a Gaussian martingale difference process for all i. Then, for some $C_1 > 0$,*

$$B_{nT} = \frac{p\left(n-n^{\alpha_0}\right)}{n^{\delta+\alpha_0}} + O\left[\exp\left(-T^{C_1}\right)\right], \tag{12}$$

where p is the nominal size of the individual tests, and δ is the exponent of the critical value function defined in (6).

Lemma 2 *Let the model be given by (2) where (3) holds, σ_i is known and u_{it} is a Gaussian martingale difference process for all i. Then, in the case where $\alpha_0 < 1$, for some $C_1 > 0$,*

$$Var\left(A_{nT}\right) = \psi_n(\alpha_0) + O\left[n^{-\alpha_0/2}\exp\left(-T^{C_1}\right)\right], \tag{13}$$

where

$$\psi_n(\alpha_0) = p\left(n-n^{a_0}\right)n^{-\delta-2\alpha_0}\left(1-\frac{p}{n^\delta}\right). \tag{14}$$

If $\alpha_0 = 1$, for some $C_1 > 0$,

$$Var\left(A_{nT}\right) = O\left[\exp\left(-T^{C_1}\right)\right]. \tag{15}$$

As we note from the above lemmas, we need to distinguish between the two cases where $\alpha_0 = 1$ and where $\alpha_0 < 1$. In the former case, $A_{nT} \to_p 0$ exponentially fast in T, and overall

$$(\ln n)(\hat{\alpha}-1) = O_p\left[n^{-1}\exp\left(-C_2 T\right)\right] + O\left[\exp\left(-C_1 T\right)\right],$$

for some positive constants C_1 and C_2. Furthermore, in the case where $\alpha_0 < 1$, using (13) and (14), it follows that

$$\begin{aligned} A_{nT} &= O_p\left[\psi_n(\alpha_0)^{1/2}\right] + O\left[n^{-\alpha_0/2}\exp\left(-C_1T/2\right)\right] \\ &= O_p\left(n^{1/2-\delta/2-\alpha_0}\right) + O\left[n^{-\alpha_0/2}\exp\left(-C_1T/2\right)\right]. \end{aligned}$$

Therefore, $A_{nT} = o_p(1)$ if $\delta > 1 - 2\alpha_0$, which is in turn met if $\delta > 0$, for all values of $\alpha_0 > 1/2$.

Remark 2 *It is clear that the distribution of $\hat{\alpha}$ experiences a form of degeneracy when $\alpha_0 = 1$, and $\hat{\alpha}$ tends to its true value of 1 exponentially fast. We refer to this property as ultraconsistency to distinguish it from the more usual terminology of superconsistency that refers to rates of convergence that are faster than the usual one of the square root of the sample size. Usually faster rates are polynomial in the sample size and not exponential, and therefore the new term reflects this important difference.*

The above results suggest the following scaling of $\hat{\alpha}$ when $\alpha_0 < 1$:

$$\psi_n^{-1/2}\left(\ln n\right)\left(\hat{\alpha} - \alpha_0\right) = \psi_n^{-1/2}A_{nT} + \psi_n^{-1/2}B_{nT} + o_p(1).$$

Also, using (A.6) from Appendix A, we have

$$B_{nT} = \frac{\sum_{i=1}^{n} E\left(\hat{d}_{i,nT}\right) - n^{\alpha_0}}{n^{\alpha_0}} = \frac{p\left(n - n^{\alpha_0}\right)}{n^{\delta+\alpha_0}} + O\left[\exp\left(-C_1T\right)\right].$$

It is also easily seen that $B_{nT} = o(1)$ if $\delta > 1 - \alpha_0$.

Remark 3 *Since $1/2 < \alpha_0 < 1$ (recall that the case of $\alpha_0 = 1$ is treated separately), then for values of α_0 close to unity (from below) it is sufficient that $\delta > 0$, and for values of α_0 close to $1/2$, we need $\delta > 1/2$. In the absence of a priori knowledge of α_0, it is sufficient to set $\delta = 1/2$. In practice, factors that are sufficiently strong with α_0 falling in the range $[2/3, 1]$ are likely to be of greater interest, and for precise estimation of such factors it would be sufficient to set $\delta = 1/4$. Our Monte Carlo results show that the estimates of factor strength are reasonably robust to the choice of δ, so long as it is not too small and lies in the range $1/4 - 1/2$. Alternatively, one can consider various cross-validation methods to calibrate δ.*

Also, since $\left[\psi_n(\alpha_0)\right]^{-1/2} A_{nT} = O_p(1)$, then $\left[\psi_n(\alpha_0)\right]^{-1/2} A_{nT}^2 = O_p\left(A_{nT}\right) = o(1)$. Using these results we can now write

$$\left[\psi_n(\alpha_0)\right]^{-1/2}\left(\ln n\right)\left(\hat{\alpha} - \alpha_0 - \zeta_n\right) = \left[\psi_n(\alpha_0)\right]^{-1/2} A_{nT} + o_p(1),$$

where

$$\zeta_n\left(\alpha_0\right) = \frac{p\left(n - n^{\alpha_0}\right)}{\left(\ln n\right) n^{\delta+\alpha_0}}.$$

Finally, since u_{it} are independent across i, and $\hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)$ have zero means, then by a standard martingale difference central limit theorem, we have (as n and $T \to \infty$)

$$\left[\psi_n(\alpha_0)\right]^{-1/2} A_{nT} = \left[\psi_n(\alpha_0)\right]^{-1/2} \frac{1}{n^{\alpha_0}} \sum_{i=1}^{n}\left[\hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)\right] \to_d N(0,1).$$

Hence,

$$[\psi_n(\alpha_0)]^{-1/2} (\ln n) [\hat{\alpha} - \alpha_0 - \zeta_n(\alpha_0)] \to_d N(0,1), \tag{16}$$

where

$$\zeta_n(\alpha_0) = \frac{p(n - n^{\alpha_0})}{(\ln n) n^{\delta+\alpha_0}}. \tag{17}$$

To test $H_0 : \alpha = \alpha_0$, we utilise the following score statistics where α_0 in the normalisation part of the test is replaced by its estimator, $\hat{\alpha}$:

$$z_{\hat{\alpha}:\alpha_0} = \frac{(\ln n)(\hat{\alpha} - \alpha_0) - p\left(n - n^{\hat{\alpha}}\right) n^{-\delta-\hat{\alpha}}}{\left[p\left(n - n^{\hat{a}}\right) n^{-\delta-2\hat{\alpha}} \left(1 - \frac{p}{n^\delta}\right)\right]^{1/2}}. \tag{18}$$

The null will be rejected if $|z_\alpha| > cv$, where cv is the critical value of the standard normal distribution at the desired significance level (which need not be the same as p). For a two sided test at 5% level, $cv = 1.96$.

3 A general treatment with a multi-factor model

As a generalisation of the above set up consider the multi-factor regressions

$$x_{it} = c_i + \sum_{j=1}^{m} \gamma_{ij} f_{jt} + u_{it} = c_i + \boldsymbol{\gamma}_i' \mathbf{f}_t + u_{it}, \text{ for } i = 1, 2, \ldots, n \text{ and } t = 1, 2, \ldots, T \tag{19}$$

where $\boldsymbol{\gamma}_i = (\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{im})'$, and we assume that the m-dimensional vector, $\mathbf{f}_t = (f_{1t}, f_{2t}, \ldots, f_{mt})'$, is observed. We also assume that, for some unknown ordering of units over i,

$$|\gamma_{ij}| > 0 \text{ a.s. for } i = 1, 2, \ldots, [n^{\alpha_{j0}}],$$
$$|\gamma_{ij}| = 0 \text{ a.s. for } i = [n^{\alpha_{j0}}] + 1, [n^{\alpha_{j0}}] + 2, \ldots, n.$$

Then the following strategy may be employed to provide inference on α_{j0}, for $j = 1, 2, \ldots, m$. For a given unit i, consider the least squares regression of $\{x_{it}\}_{t=1}^{T}$ on the intercept and $\mathbf{f}_t$. $\hat{c}_{iT}$ and $\hat{\boldsymbol{\gamma}}_{iT}$ are the OLS estimates of this regression. Denote by $t_{ijT} = \hat{\gamma}_{ijT} / \text{s.e.}(\hat{\gamma}_{ijT})$, the t-statistic corresponding to γ_{ij}:

$$t_{ijT} = \frac{\left(\mathbf{f}_{j\circ}' \mathbf{M}_{F_{-j}} \mathbf{f}_{j\circ}\right)^{-1/2} \left(\mathbf{f}_{j\circ}' \mathbf{M}_{F_{-j}} \mathbf{x}_i\right)}{\hat{\sigma}_{iT}}, \; j = 1, 2, \ldots, m; \; i = 1, 2, \ldots, n,$$

$\mathbf{f}_{j\circ} = (f_{j1}, f_{j2}, \ldots, f_{jT})'$, $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iT})'$, $\mathbf{M}_{F_{-j}} = \mathbf{I} - \mathbf{F}_{-j}\left(\mathbf{F}_{-j}'\mathbf{F}_{-j}\right)^{-1}\mathbf{F}_{-j}'$,

$\mathbf{F}_{-j} = (\mathbf{f}_{1\circ}, \ldots, \mathbf{f}_{j-1\circ}, \mathbf{f}_{j+1\circ}, \ldots, \mathbf{f}_{m\circ})'$, $\hat{\sigma}_{iT}^2 = T^{-1}\sum_{t=1}^{T} \hat{u}_{it}^2$, and $\hat{u}_{it} = x_{it} - \hat{c}_{iT} - \hat{\boldsymbol{\gamma}}_{iT}'\mathbf{f}_t$.

Consider the total number of factor loadings of factor j, γ_{ij}, that are statistically significant over $i = 1, 2, \ldots, n$:

$$\hat{D}_{nT,j} = \sum_{i=1}^{n} \hat{d}_{ij,nT} = \sum_{i=1}^{n} \mathbf{1}\left[|t_{ijT}| > c_p(n)\right],$$

where $\mathbf{1}(A) = 1$ if $A > 0$, and zero otherwise, and the critical value function that allows for the multiple testing nature of the problem, $c_p(n)$, is given by

$$c_p(n) = \Phi^{-1}\left(1 - \frac{p}{2n^\delta}\right).$$

As before, p is the nominal size, $\delta > 0$ is the critical value exponent and $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard normal distribution. Let $\hat{\pi}_{nT,j}$ be the fraction of significant loadings of factor j, and note that $\hat{\pi}_{nT,j} = \hat{D}_{nT,j}/n$. As in the single factor case, we consider the following estimator of α_j, for $j = 1, 2, \ldots, m$

$$\hat{\alpha}_j = \begin{cases} 1 + \frac{\ln \hat{\pi}_{nT,j}}{\ln n}, \text{ if } \hat{\pi}_{nT,j} > 0, \\ 0, \text{ if } \hat{\pi}_{nT,j} = 0. \end{cases} \tag{20}$$

We make the following assumptions:

Assumption 1 *The error terms, u_{it}, and demeaned factors $\mathbf{f}_t - E(\mathbf{f}_t)$, are martingale difference processes with respect to $\mathcal{F}_{t-1}^{u_i} = \sigma(u_{i,t-1}, u_{i,t-2}, \ldots)$ and $\mathcal{F}_{t-1}^{f} = \sigma(\mathbf{f}_t, \mathbf{f}_{t-1}, \ldots)$, respectively, and $E\left\{[\mathbf{f}_t - E(\mathbf{f}_t)][\mathbf{f}_t - E(\mathbf{f}_t)]'\right\} = \boldsymbol{I}_m$. u_{it} are independent over i, and of $\mathbf{f}_t$, and have constant variances, $0 < \sigma_i^2 < C < \infty$.*

Assumption 2 *There exist sufficiently large positive constants C_0, C_1, and $s > 0$ such that*

$$\sup_{i,t} \Pr(|x_{it}| > \nu) \leq C_0 \exp(-C_1 \nu^s), \textit{ for all } \nu > 0, \tag{21}$$

$$\sup_{j,t} \Pr(|f_{jt}| > \nu) \leq C_0 \exp(-C_1 \nu^s), \textit{ for all } \nu > 0. \tag{22}$$

Then, we have the following theorem:

Theorem 1 *Consider model (19) with m observed factors and let Assumptions 1 and 2 hold. Then, for any $\alpha_{j0} < 1$, $j = 1, 2, \ldots, m$,*

$$\psi_n(\alpha_{j0})^{-1/2} (\ln n) (\hat{\alpha}_j - \alpha_{j0}) \to_d N(0, C) \tag{23}$$

for some $C < 1$, where

$$\psi_n(\alpha_{j0}) = p(n - n^{\alpha_{j0}}) n^{-\delta - 2\alpha_{j0}} \left(1 - \frac{p}{n^\delta}\right). \tag{24}$$

The above theorem provides the inferential basis for testing hypotheses on the true value of α_j. The proof of the theorem is provided in Appendix B. In the remarks below we discuss operational matters concerning the above result and how to relax some of the assumptions of Theorem 1.

Remark 4 *A test based on $\psi_n(\alpha_{j0})^{-1/2} (\ln n) (\hat{\alpha}_j - \alpha_{j0})$ will be conservative, in the sense that the rejection probability under the null hypothesis will be bounded from above by the significance level. The reason is that in general we cannot get an asymptotic approximation for the variance of $\hat{\alpha}_j - \alpha_{j0}$ but only an upper bound resulting in a conservative test.*

Remark 5 *Assumptions 1 and 2 can be relaxed. Rather than independence over i for u_{it} in Assumption 1, one can assume some spatial mixing condition, which would still allow the central limit theorem underlying (23), to hold. Further, the thin probability tails in Assumption 2 can be replaced with a suitable moment condition in order to derive the variance bound needed to construct a test statistic. We abstract from such complications by maintaining Assumption 2. The martingale difference assumption for $\mathbf{f}_t$ simplifies the analysis and allows the use of the theory in the main part of Chudik et al. (2018). Relaxing this to a mixing assumption is possible at the expense of further mathematical complexity using, e.g., the results in the online appendix of Chudik et al. (2018).*

Remark 6 *Our distributional result is stated only for $\alpha_{j0} < 1$. Similar arguments would apply for the variance $\hat{\alpha}_j - \alpha_{j0}$ when $\alpha_{j0} = 1$. But the upper bound for the variance of $\hat{\alpha}_j - \alpha_{j0}$ would be a function of nuisance parameters including γ_{ij}. This is the case since the dominant term in the variance is the one relating to units not affected by $\mathbf{f}_t$, when $\alpha_{j0} < 1$, and for these units, $\gamma_{ij} = 0$. But when $\alpha_{j0} = 1$, the probability bounds that are used to derive the variance bound will not have such a dominant term, and the remaining terms will contain γ_{ij}. However, testing under the null hypothesis that $\alpha_{j0} = 1$ is further complicated by the fact that $\alpha_{j0} = 1$ is at the boundary of the parameter space for α_{j0}. It is well known (see, e.g., Andrews (2001)) that such cases cannot be handled using standard asymptotic inference, and therefore this case is beyond the scope of the present paper. Nevertheless, it is clear from Remark 2 that estimation when $\alpha_0 = 1$ has some very desirable properties, such as a very fast rate of convergence, which we have referred to as ultraconsistency. We conjecture that in the case where $\alpha_{j0} = 1$ for some values of j, and $\alpha_{j0} < 1$ for some values of j, the distributional results presented in Theorem 1 hold for factors for which $\alpha_{j0} < 1$.*

4 Case of unobserved factors

When the factors are unobserved we can only provide practical guidance on the strength of the strongest factor or factors, and estimating the strength of other factors encounters a significant identification problem. This is related to the known fact that latent factors are identified only up to a non-singular $m \times m$ rotation matrix, $\mathbf{Q} = (q_{ij})$, where m is the assumed number of factors.

It is instructive to review this fact. Consider the multi-factor model (19) with $\mathbf{f}_t$ unobserved. Without loss of generality suppose that $m = 2$ and assume that factors, $\mathbf{f}_t = (f_{1t}, f_{2t})'$, are unobserved with strengths $\alpha_1 > 1/2$ and $\alpha_2 > 1/2$. Denote the principal component (PC) estimates of these factors by $\hat{\mathbf{g}}_t = (\hat{g}_{1t}, \hat{g}_{2t})'$, and note that under standard regularity conditions in the literature (as n and $T \to \infty$)

$$f_{1t} = q_{11}\hat{g}_{1t} + q_{12}\hat{g}_{2t} + o_p(1), \tag{25}$$

$$f_{2t} = q_{21}\hat{g}_{1t} + q_{22}\hat{g}_{2t} + o_p(1). \tag{26}$$

Then the estimates of the loadings associated with these PCs are given by

$$\tilde{\gamma}_i = \begin{pmatrix} \tilde{\gamma}_{i1} \\ \tilde{\gamma}_{i2} \end{pmatrix} = \left(\hat{\mathbf{G}}'\mathbf{M}_\tau\hat{\mathbf{G}}\right)^{-1}\hat{\mathbf{G}}'\mathbf{M}_\tau\mathbf{x}_i = \left(\hat{\mathbf{G}}'\mathbf{M}_\tau\hat{\mathbf{G}}\right)^{-1}\hat{\mathbf{G}}'\mathbf{M}_\tau\mathbf{F}\boldsymbol{\gamma}_i + \left(\hat{\mathbf{G}}'\mathbf{M}_\tau\hat{\mathbf{G}}\right)^{-1}\hat{\mathbf{G}}'\mathbf{M}_\tau\mathbf{u}_i,$$

where $\hat{\mathbf{G}} = (\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \ldots, \hat{\mathbf{g}}_T)'$. Also since $\mathbf{Q}$ is non-singular, $\hat{\mathbf{G}} \to_p \mathbf{F}\mathbf{Q}^{-1}$, and using the above we have $\tilde{\boldsymbol{\gamma}}_i \to_p \mathbf{Q}\boldsymbol{\gamma}_i$. It is now easily seen that the strength of f_{1t} (or f_{2t}) computed using the estimates, $\tilde{\gamma}_{i1}$, $i = 1, 2, \ldots, n$ may not provide consistent estimates of the associated factor strengths. To see this write the result $\tilde{\boldsymbol{\gamma}}_i \to_p \mathbf{Q}\boldsymbol{\gamma}_i$ in an expanded format as

$$\tilde{\gamma}_{i1} = q_{11}\gamma_{i1} + q_{12}\gamma_{i2} + o_p(1),$$
$$\tilde{\gamma}_{i2} = q_{21}\gamma_{i1} + q_{22}\gamma_{i2} + o_p(1).$$

Squaring both sides and summing over i we have

$$\sum_{i=1}^{n}\tilde{\gamma}_{i1}^2 = q_{11}^2\sum_{i=1}^{n}\gamma_{i1}^2 + q_{12}^2\sum_{i=1}^{n}\gamma_{i2}^2 + 2q_{11}q_{12}\sum_{i=1}^{n}\gamma_{i1}\gamma_{i2} + o_p(1),$$
$$\sum_{i=1}^{n}\tilde{\gamma}_{i2}^2 = q_{21}^2\sum_{i=1}^{n}\gamma_{i1}^2 + q_{22}^2\sum_{i=1}^{n}\gamma_{i2}^2 + 2q_{21}q_{22}\sum_{i=1}^{n}\gamma_{i1}\gamma_{i2} + o_p(1).$$

Now using the definition of factor strength in (3) and assuming that $\alpha_1 > \alpha_2$, in general we have[2]

$$\sum_{i=1}^{n} \tilde{\gamma}_{i1}^2 = \ominus(n^{\alpha_1}),\ \sum_{i=1}^{n} \tilde{\gamma}_{i2}^2 = \ominus(n^{\alpha_1}),$$

namely, using the estimated loadings of the principal components does not allow us to distinguish between the strength of the two factors, and only the strength of the strongest factor can be identified. When $\alpha_1 > \alpha_2$, identification of α_2 requires setting $q_{21} = 0$, and conversely to identify α_1 when $\alpha_1 < \alpha_2$ requires setting $q_{12} = 0$. It is worth noting that using covariance eigenvalues does not help resolve this problem. There are two separate issues – ordering eigenvalues and how to identify the factors associated with ordered eigenvalues. The eigenvectors associated with the largest eigenvalues are not uniquely determined and therefore the identification issue remains.

Therefore, in general, we focus on identifying $\alpha = \max(\alpha_{1,}\alpha_2)$. The exponent α can be estimated using the estimators proposed in Bailey et al. (2016) and Bailey et al. (2019). The approach of this paper can also be used to estimate α by computing the strength of the first PC, or that of the simple cross section average, namely $\bar{x}_t = n^{-1} \sum_{i=1}^n x_{it}$. One can also use the weighted cross section average $\bar{x}_{t,\gamma} = \sum_{i=1}^n \hat{w}_i x_{it}$, where $\hat{w}_i$ is estimated as the slope of $\bar{x}_t$ in the OLS regression of x_{it} on an intercept and $\bar{x}_t$.[3]

Accordingly, in the rest of this section we assume that the m unobserved factors are strong and/or semi-strong with $1/2 < \alpha_j \leq 1$, and focus on estimation of $\alpha = \max_j(\alpha_j)$. At the end of the Section we provide a remark on how to identify, in theory, the strengths of weaker factors. Reintroducing a subscript 0 to denote true parameters, we assume that $\{x_{it},\ i = 1, 2, \ldots, n;\ t = 1, 2, \ldots, T\}$ are generated from the multi-factor model (19) where the factors are unobserved with strengths $\alpha_{10} > \alpha_{20} \geq \alpha_{30} \geq \cdots \geq \alpha_{m0} > 1/2$. Clearly $\alpha_0 = \alpha_{10}$. To emphasize the focus on the factor with the largest α, we recast the model as follows:

$$x_{it} = c_i + \gamma_i f_t + v_{it},\ \text{for } i = 1, 2, \ldots, n \text{ and } t = 1, 2, \ldots, T \tag{27}$$

$$v_{it} = \sum_{j=2}^{m} \gamma_{ij} f_{jt} + u_{it}, \tag{28}$$

where the strongest factor f_t has strength α while the rest of the factors have strengths $\alpha_{20} \geq \alpha_{30} \geq \cdots \geq \alpha_{m0} > 1/2$. We assume that the m-dimensional vector, $\mathbf{f}_t = (f_t, f_{2t}, \ldots, f_{mt})'$, is unobserved. We also assume that, for some unknown ordering of units over i,

$$\begin{aligned} |\gamma_i| &> 0 \text{ a.s. for } i = 1, 2, \ldots, [n^{\alpha_0}], \\ |\gamma_i| &= 0 \text{ a.s. for } i = [n^{\alpha_0}] + 1, [n^{\alpha_0}] + 2, \ldots, n. \end{aligned} \tag{29}$$

$$\begin{aligned} |\gamma_{ij}| &> 0 \text{ a.s. for } i = 1, 2, \ldots, [n^{\alpha_{j0}}],\ j = 2, \ldots, m \\ |\gamma_{ij}| &= 0 \text{ a.s. for } i = [n^{\alpha_{j0}}] + 1, [n^{\alpha_{j0}}] + 2, \ldots, n,\ j = 2, \ldots, m. \end{aligned} \tag{30}$$

In what follows, we continue to consider that Assumptions 1 and 2 hold for the above representation, and use the simple cross section average, $\bar{x}_t$ to consistently estimate $\alpha_0 = \alpha_{10}$. Taking the

[2]Note that $\left|\sum_{i=1}^n \gamma_{i1}\gamma_{i2}\right| < \sup_i |\gamma_{i1}| \left(\sum_{i=1}^n |\gamma_{i2}|\right) = \ominus(n^{\alpha_2})$.

[3]In most applications, α can be estimated consistently using the simple average. But as shown in Pesaran (2015), pp. 452-454, the weighted average is more appropriate when the loadings of the strong factors have zero means. Also note that by construction $\sum_{i=1}^n \hat{w}_i = 1$.

first factor to be the strongest is made for convenience (with $\alpha_0 - \alpha_{j0} > 0$, for $j = 2, 3, \ldots, m$). The strength of the strongest factor, α_0, is defined by (with γ_i denoting the associated loadings)

$$\sum_{i=1}^{n} |\gamma_i| \ = \ominus \left(n^{\alpha_0}\right),$$

and the strengths of the remaining factors by

$$\sum_{i=1}^{n} |\gamma_{ij}| \ = \ominus \left(n^{\alpha_{j0}}\right), \text{ for } j = 2, 3, \ldots, m.$$

In addition, we assume that the non-zero factor loadings have non-zero means, namely

$$\lim_{n \to \infty} n^{-\alpha_0} \sum_{i=1}^{n} \gamma_i \neq 0, \text{ and } \lim_{n \to \infty} n^{-\alpha_{j0}} \sum_{i=1}^{n} \gamma_{ij} \neq 0,$$

and hence,

$$\bar{\gamma} = \bar{\gamma}_1 = n^{-1} \sum_{i=1}^{n} \gamma_i = \ominus \left(n^{\alpha_0 - 1}\right),$$

$$\bar{\gamma}_j = n^{-1} \sum_{i=1}^{n} \gamma_{ij} = \ominus \left(n^{\alpha_{j0} - 1}\right), \text{ for } j = 2, \ldots, m.$$

Note that we do not assume any ordering of the zero loadings across the units.

For each i, consider the least squares regression of $\{x_{it}\}_{t=1}^{T}$ on an intercept and the cross section average of x_{it}, $\bar{x}_t$, and denote the resulting estimators by $\hat{c}_{iT}$ and $\hat{\beta}_{iT}$, respectively. As in the single factor case, $\alpha_0 = \max_j(\alpha_{j0})$ is estimated by (7), except that when computing the t-statistics, t_{iT}, defined by (4), $\mathbf{f}$ is replaced by $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_T)'$. Denote by $\bar{t}_{iT} = \hat{\beta}_{iT} / \text{s.e.}\left(\hat{\beta}_{iT}\right)$, the t-statistic corresponding to γ_i:

$$\bar{t}_{iT} = \frac{\left(\bar{\mathbf{x}}' \mathbf{M}_{\tau} \bar{\mathbf{x}}\right)^{-1/2} \left(\bar{\mathbf{x}}' \mathbf{M}_{\tau} \mathbf{x}_i\right)}{\hat{\sigma}_{iT}},$$

$\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iT})'$, and $\hat{\sigma}_{iT}^2 = T^{-1} \mathbf{x}_i' \mathbf{M}_{\bar{H}} \mathbf{x}_i$, where $\mathbf{M}_{\bar{H}} = \mathbf{I}_T - \bar{\mathbf{H}} \left(\bar{\mathbf{H}}' \bar{\mathbf{H}}\right)^{-1} \bar{\mathbf{H}}'$, with $\bar{\mathbf{H}} = (\boldsymbol{\tau}_T, \bar{\mathbf{x}})$.

As before, consider the number of regressions with significant slope coefficients:

$$\bar{D}_{nT} = \sum_{i=1}^{n} \bar{d}_{i,nT} = \sum_{i=1}^{n} \mathbf{1} \left[|\bar{t}_{iT}| > c_p(n) \right],$$

where the critical value function, $c_p(n)$, is as specified earlier. Then, setting $\bar{\pi}_{nT} = \bar{D}_{nT}/n$, we have

$$\hat{\alpha} = \begin{cases} 1 + \frac{\ln \bar{\pi}_{nT}}{\ln n}, & \text{if } \bar{\pi}_{nT} > 0, \\ 0, & \text{if } \bar{\pi}_{nT} = 0. \end{cases}$$

To investigate the limiting properties of $\hat{\alpha}$ we first consider the value of $\bar{t}_{iT}$ under (19) and note that

$$\bar{\mathbf{x}} = \bar{c} \boldsymbol{\tau} + \mathbf{F} \bar{\boldsymbol{\gamma}} + \bar{\mathbf{u}}, \text{ and } \mathbf{x}_i = c_i \boldsymbol{\tau}_T + \mathbf{F} \boldsymbol{\gamma}_i + \mathbf{u}_i,$$

where $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_T)'$, $\boldsymbol{\gamma}_i = (\gamma_i, \gamma_{i2}, \ldots, \gamma_{im})'$, $\bar{\boldsymbol{\gamma}} = n^{-1} \sum_{i=1}^{n} \boldsymbol{\gamma}_i$, $\mathbf{u}_i = (u_{i1}, u_{i2}, \ldots, u_{iT})'$ and $\bar{\mathbf{u}} = n^{-1} \sum_{i=1}^{n} u_i$. Using these results we have

$$\bar{t}_{iT} = \frac{T^{-1/2} \left(\bar{\mathbf{x}}' \mathbf{M}_{\tau} \mathbf{x}_i\right)}{\hat{\sigma}_{iT} \left(T^{-1} \bar{\mathbf{x}}' \mathbf{M}_{\tau} \bar{\mathbf{x}}\right)^{1/2}} = \frac{T^{-1/2} (\mathbf{F} \bar{\boldsymbol{\gamma}} + \bar{\mathbf{u}})' \mathbf{M}_{\tau} \left(\mathbf{F} \boldsymbol{\gamma}_i + \mathbf{u}_i\right)}{\hat{\sigma}_{iT} \left[T^{-1} (\mathbf{F} \bar{\boldsymbol{\gamma}} + \bar{\mathbf{u}})' \mathbf{M}_{\tau} (\mathbf{F} \bar{\boldsymbol{\gamma}} + \bar{\mathbf{u}})\right]^{1/2}}, \tag{31}$$

and

$$\hat{\sigma}_{iT}^2 = T^{-1}\left(\mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i\right)' \mathbf{M}_{\bar{H}}\left(\mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i\right). \tag{32}$$

The following lemmas, which are of fundamental importance, are proven in Appendix A. The first of the lemmas is auxiliary and technical in nature. It presents rates in probability and probability tail bounds for the constituent parts of $\bar{t}_{iT}$. These results are then used in Lemma 4 to provide probability bounds for $\bar{t}_{iT}$.

Lemma 3 *Consider model (27)-(28) with factor loadings given by (29)-(30), where $\mathbf{f}_t$ is an $m \times 1$ vector of unobserved factors, and let Assumptions 1 and 2 hold. Then,*

$$\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} = \ominus_p\left(\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'\boldsymbol{\gamma}_i}{\left(\bar{\boldsymbol{\gamma}}'\bar{\boldsymbol{\gamma}}\right)^{1/2}}\right), \tag{33}$$

$$\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} = O_p\left(n^{1/2-\alpha_0}\right), \tag{34}$$

$$\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} = O_p\left(n^{1/2-\alpha_0}\right), \tag{35}$$

$$\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} \to_d N(0, \sigma_i^2). \tag{36}$$

Further, for some $C, C_0, C_1 > 0$,

$$\Pr\left(\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} > c_p(n)\right) \le \frac{Cp}{n^\delta},\ \text{if } \gamma_{i1} = 0,\ \text{and } T^{1/2} = o(n^{\alpha_{20}-\alpha_0}), \tag{37}$$

$$\Pr\left(\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} > c_p(n)\right) \le \frac{Cp}{n^\delta},\ \text{if } \gamma_{i1} \neq 0,\ \text{or } n^{\alpha_{20}-\alpha_0} = o(T^{1/2}), \tag{38}$$

$$\Pr\left(\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} > c_p(n)\right) \le \exp\left[-C_0 T^{C_1}\right], \tag{39}$$

$$\Pr\left(\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} > c_p(n)\right) \le \exp\left[-C_0 T^{C_1}\right], \tag{40}$$

$$\Pr\left(\frac{T^{-1/2}\bar{\boldsymbol{\gamma}}'\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} > c_p(n)\right) \le \frac{Cp}{n^\delta}. \tag{41}$$

Lemma 4 *Consider model (27)-(28) with factor loadings given by (29)-(30), where $\mathbf{f}_t$ is unobserved, and let Assumptions 1 and 2 hold. Then, as long as $\sqrt{T}n^{(\alpha_{20}-\alpha_0)} \to 0$, for some $C > 0$,*

$$\Pr\left[|\bar{t}_{iT}| > c_p(n) | \gamma_i \neq 0\right] > 1 - O\left[\exp(-T^C)\right], \tag{42}$$

and

$$\Pr\left[|\bar{t}_{iT}| > c_p(n) | \gamma_i = 0\right] \le \frac{Cp}{n^\delta}. \tag{43}$$

Equations (42) and (43) provide the crucial ingredients for the main result given below, as (42) ensures that the t-statistic rejects with high probability when a unit contains a factor, while (43) ensures that the probability of rejection for a unit that does not contain a factor, is small.

Overall, we have the following theorem, proven in Appendix B, justifying the proposed method for unobserved factors.

Theorem 2 *Consider model (27)-(28) with factor loadings given by (29)-(30), where* $\mathbf{f}_t$ *is unobserved, let Assumptions 1 and 2 hold and denote by* α_0 *the true value of* α*. Then, as long as* $\sqrt{T}n^{(\alpha_{20}-\alpha_0)} \to 0$*, for any* $\alpha_0 < 1$*,*

$$\psi_n(\alpha_0)^{-1/2}\,(\ln n)\,(\hat{\alpha}-\alpha_0) \to_d N(0,C)$$

for some $C < 1$*, where* α_{20} *denotes the strength of the second strongest factor, and*

$$\psi_n(\alpha_0) = p\,(n - n^{\alpha_0})\,n^{-\delta-2\alpha_0}\left(1 - \frac{p}{n^{\delta}}\right).$$

The above theorem provides the inferential basis for testing hypotheses on the true value of α, in the case of unobserved factors.

Remark 7 *While our exposition is based on unobserved factors, it is clear that it can be extended to a mixed case where there are observed and unobserved factors. In that case,* x_{it} *denotes a residual from a panel regression of the form (19). The initial regression model is used to carry out inference on the observed factors and, again, a model of the form (27)-(28) is used for inference on the unobserved factors, where the cross section average is computed using the residuals of first-stage regressions.*

Remark 8 *The above analysis readily extends to the case where two or more of the unobserved factors have the same strength. For example, suppose that* $\alpha_0 = max_j(\alpha_{j0}) = \alpha_{10} = \alpha_{20} > \alpha_{30} \geq \alpha_{40} \geq .... \geq \alpha_{m0}$*. Then it is easily seen that* α *is consistently estimated by* $\hat{\alpha}$*, even though* $\alpha_{10} = \alpha_{20}$*. What matters for identification of* α_0 *in this case is that* $\sqrt{T}n^{(\alpha_{30}-\alpha_0)} \to 0$*. This case is further investigated below using Monte Carlo techniques.*

Remark 9 *Our analysis focuses on* $\alpha_0 = \alpha_{10} = max_j(\alpha_{j0})$*. A possible way to provide some information on* α_{j0}*,* $j > 1$*, may be based on a sequential application of weighted cross section averages. In particular, once the least squares regression of* $\{x_{it}\}_{t=1}^{T}$ *on an intercept and the cross section average of* x_{it}*,* $\bar{x}_t$*, has been fitted, residuals can be obtained. Simple cross section averages of these residuals are easily seen to be identically equal to zero. However, weighted cross section averages can be constructed, along the lines discussed in Pesaran (2015), pp. 452-454, and the t-statistics of the relevant loadings can be used, in a similar way to that discussed above, to construct estimators for* α_{20} *and, sequentially via the construction of further sets of residuals, for* α_{j0}*,* $j > 2$*. It is possible to show that, if* $\sqrt{T}n^{(\alpha_{j+1,0}-\alpha_{j0})} \to 0$*,* $j > 1$*, a result similar to that of Theorem 2 holds for* α_{j0}*,* $j > 1$*. However, this result clearly requires considerable differences between the* α*'s and/or very large values for* n*. Further, Monte Carlo evidence suggests that the estimators perform very poorly for relevant sample sizes. Therefore, we do not pursue this analysis further as it is very clear that it is not practically relevant.*

5 Monte Carlo study

5.1 Design

We investigate the small sample properties of the proposed estimator of α under both observed and unobserved factors using a number of Monte Carlo simulations. We consider the following two-factor data generating process (DGP):

$$x_{it} = c_i + \gamma_{1i}f_{1t} + \gamma_{2i}f_{2t} + u_{it}, \tag{44}$$

for $i = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, T$. We generate the unit specific effects as $c_i \sim IIDN(0, 1)$, for $i = 1, 2, \ldots, n$. The factors, $\mathbf{f}_t = (f_{1t}, f_{2t})'$, are generated as multivariate normal: $\mathbf{f}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_f)$, where

$$\boldsymbol{\Sigma}_f = \begin{pmatrix} \sigma^2_{f_1} & \rho_{12}\sigma_{f_1}\sigma_{f2} \\ \rho_{12}\sigma_{f_1}\sigma_{f2} & \sigma^2_{f2} \end{pmatrix},$$

with $\sigma_{f_1} = \sigma_{f2} = 1$, and $\rho_{12} = corr(f_{1t}, f_{2t})$, using the values $\rho_{12} = 0.0, 0.3$. The factors are generated as autoregressive processes (considering both stationary and unit root cases):

$$f_{jt} = \begin{cases} \rho_{f_j} f_{j,t-1} + \sqrt{1-\rho^2_{f_j}}\varepsilon_{jt}, & \text{if } |\rho_{f_j}| < 1 \\ f_{j,t-1} + \varepsilon_{jt}, & \text{if } \rho_{f_j} = 1 \end{cases}, \text{ for } t = -49, -48, \ldots, 1, \ldots, T$$

with $f_{j,-50} = 0$ and $\varepsilon_{jt} \sim i.i.d.N(0, 1)$, $j = 1, 2$. In the stationary case, we set $\rho_{f_1} = \rho_{f_2} = 0.5$.

For the innovations, u_{it} we consider two cases: (i) Gaussian, where $u_{it} \sim IIDN(0, \sigma_i^2)$ for $i = 1, 2, \ldots, n$; (ii) non-Gaussian, where the errors are generated as $u_{it} = \frac{\sigma_i}{2}\left(\chi^2_{2,it} - 2\right)$, where $\chi^2_{2,it}$ for $i = 1, 2, \ldots, n$ are independent draws from a chi-squared distribution with 2 degrees of freedom, and σ_i^2 are generated as $IID(1 + \chi^2_{2,i})/3$.

In terms of the factor loadings, γ_{i1} and γ_{i2}, first we generate $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, for $i = 1, 2, \ldots, n$ and $j = 1, 2$ (such that $E(v_{ij}) = \mu_{v_j}$). Next, we randomly assign $[n^{\alpha_{10}}]$ and $[n^{\alpha_{20}}]$ of these random variables as elements of vectors $\boldsymbol{\gamma}_j = (\gamma_{2j}, \gamma_{2j}, \ldots, \gamma_{nj})'$, $j = 1, 2$, respectively, where $[.]$ denotes the integer part operator.[4] For α_{10} and α_{20}, we consider values of $(\alpha_{10}, \alpha_{20})$ starting with 0.75 and rising to 1 at 0.05 increments, namely $0.75, 0.80, \ldots, 0.95, 1.00$, comprising of 36 experiments for all combinations of a_{10} and a_{20} in the range $[0.75, 1.00]$.[5] We set $\mu_{v_1} = \mu_{v_2} = 0.71$ so that both means are sufficiently different from zero. We then select the error variances, σ_i^2, so as to achieve an average fit across all units of around $\bar{R}^2_n = n^{-1}\sum_{i=1}^n R_i^2 \approx 0.34$. This coincides with the average fits of regressions from our finance application. Scaling σ_i^2 by 3/4 achieves $\bar{R}^2_n \approx 0.41$. To this end, we note that:

$$R_i^2 = \frac{\gamma^2_{i1} + \gamma^2_{i2}}{\gamma^2_{i1} + \gamma^2_{i2} + \sigma_i^2} = \frac{\varpi^2_{i1} + \varpi^2_{i2}}{1 + \varpi^2_{i1} + \varpi^2_{i2}}, \text{ if for the } i^{th} \text{ unit: both } \gamma_{i1} \neq 0 \text{ and } \gamma_{i2} \neq 0,$$

where $\varpi^2_{ij} = \gamma^2_{ij}/\sigma_i^2$, for $j = 1, 2$. Similarly, $R_i^2 = \varpi^2_{i1}/\left(1 + \varpi^2_{i1}\right)$, if $\gamma_{i1} \neq 0$ and $\gamma_{i2} = 0$, $R_i^2 = \varpi^2_{i2}/\left(1 + \varpi^2_{i2}\right)$, $\gamma_{i2} \neq 0$ and $\gamma_{i1} = 0$, and clearly $R_i^2 = 0$, if $\gamma_{i1} = \gamma_{i2} = 0$.

We consider the following experiments:

EXP 1A: (observed single factor - Gaussian errors): Using (44) with $\gamma_{i2} = 0$, for all i, and Gaussian errors.

EXP 1B: (observed single factor - non-Gaussian errors): Using (44) with $\gamma_{i2} = 0$, for all i, and non-Gaussian errors.

EXP 2A: (two observed factors - Gaussian errors) A two-factor model with correlated observed factors ($\rho_{12} = 0.3$) and Gaussian errors.

EXP 2B: (two observed factors - non-Gaussian errors) A two-factor model with correlated observed factors ($\rho_{12} = 0.3$) and non-Gaussian errors.

[4] The randomisation of loadings becomes important when analysing the case of unobserved factors, as discussed in Section 4.

[5] Results for combinations of α_{10} and α_{20} below 0.75 are available upon request.

EXP 3A: (unobserved single factor - non-Gaussian errors) Using (44) subject to $\gamma_{i2} = 0$, for all i, and non-Gaussian errors with $\alpha_0 = \alpha_{10}$ computed using the simple cross section average $\bar{x}_t = n^{-1} \sum_{i=1}^{n} x_{it}$.

EXP 3B: (two unobserved factors - non-Gaussian errors) Using (44) with $\rho_{12} = 0.3$ and non-Gaussian errors, $\alpha_{10} = 0.95, 1.00$, and $\alpha_{20} = 0.51, 0.75, 0.95, 1.00$. In this case $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated using the simple cross section average $\bar{x}_t = n^{-1} \sum_{i=1}^{n} x_{it}$.

Further, we consider the following additional experiment that assumes a misspecified observed factor model that mirrors the analysis of our empirical finance example in Section 6.1:

EXP 4: (observed misspecified single factor - Gaussian errors) A misspecified single observed factor model, where the DGP is a two-factor model with correlated factors ($\rho_{12} = 0.3$) and Gaussian errors in (44), $\alpha_{10} = 1$, and $\alpha_{20} = 0.75, 0.80, \ldots,\ 0.95, 1.00$. For this experiment we report the estimates of α_{10} computed based on the misspecified single factor model $x_{it} = c_i + \beta_i f_{1t} + e_{it}$.

The factor strengths are estimated using (7), with the nominal size of the associated multiple tests set to $p = 0.10$, and the critical value exponent to $\delta = 1/4$.[6]

For all experiments we report bias and RMSE of $\hat{\alpha}_j$, size and power of tests of $H_0 : \alpha_j = \alpha_{j0}$ against $\alpha_j = \alpha_{ja}$, $j = 1, 2$, using the test statistic given by

$$z_{\hat{\alpha}_j:\alpha_{j0}} = \frac{(\ln n)\left(\hat{\alpha}_j - \alpha_{j0}\right) - p\left(n - n^{\hat{\alpha}_j}\right) n^{-\delta-\hat{\alpha}_j}}{\left[p\left(n - n^{\hat{\alpha}_j}\right) n^{-\delta-2\hat{\alpha}_j}\left(1 - \frac{p}{n^{\delta}}\right)\right]^{1/2}},\ j = 1, 2. \tag{45}$$

We consider two-sided tests throughout. Empirical size is computed as

$$size_R = R^{-1} \sum_{r=1}^{R} I\left(\left|z_{\hat{\alpha}_j:\alpha_{j0}}\right| > c\,|H_0\right),\ j = 1, 2.$$

The empirical power of the tests of $H_0 : \alpha_j = \alpha_{j0}$ against the alternative $H_1 : \alpha_j = \alpha_{ja}$, are obtained for $\alpha_{ja} = \alpha_{j0} + \kappa$, $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$ (20 alternatives) for all values of $\alpha_{j0} \in [0.75, 1.00)$. Here, DGP (44) is generated under H_1 and the rejection frequency is computed as

$$power_R = R^{-1} \sum_{r=1}^{R} I\left(\left|z_{\hat{\alpha}_j:\alpha_{j0}}\right| > c\,|H_1\right),\ j = 1, 2.$$

For both size and power, $z_{\hat{\alpha}_j:\alpha_{j0}}$ is given by (45). We do not compute size and power when α_{j0} and/or α_{ja} is equal to unity, since in this case the distribution of the estimator that we propose is degenerate and the estimator is ultraconsistent.

For all experiments we consider all combinations of $n = \{100, 200, 500, 1000\}$ and $T = \{120, 200, 500,\ 1,000\}$, and set the number of replications per experiment to $R = 2,000$. The values of c_i and γ_{ij} are redrawn at each replication.

[6] We also consider other values of p and δ, namely $p = 0.05$ and $\delta = 1/3$ or $1/2$, and found the results to be qualitatively very similar to those obtained when $p = 0.10$ and $\delta = 1/4$.

5.2 MC findings

We start with the single factor model where the factor is observed and report the results in Tables 1 and 2 for experiments 1A and 1B. These tables show bias, RMSE and size for the estimator of the strength of factor f_1, namely $\hat{\alpha}_1$, for different values of α_{10}, and different (n, T) combinations. Table 1 gives the results for Gaussian errors, and Table 2 when the errors are non-Gaussian. Overall, the outcomes are very similar when the model is generated under normal and non-normal errors. In both tables, bias and RMSE are universally low and gradually decrease as n, T and α_{10} rise, as to be expected. Especially when $\alpha_{10} = 1$, bias and RMSE are negligible even when $T = 120$. Moving on to the rejection probabilities under the null hypothesis, we note that given that our estimator has low variance, the rejection probabilities are sensitive to the bias of $\hat{\alpha}_1$. Hence, for smaller values of α_{10} the test is considerably oversized, which is expected. However, as the sample size and α_{10} increase, the size distortion reduces considerably, resulting in a well behaved test under the null hypothesis. For $\alpha_{10} = 0.95$ correct empirical size is achieved even for moderate values of T, while, as mentioned earlier, when α_{10} equals to unity our estimator has an exponential rate of convergence, with the distribution of the estimator collapsing to its true value at 1. Next, we turn to the power of the test and consider the rejection probabilities under a sequence of alternative hypotheses. Figures 1 and 2 depict power functions corresponding to the strength of factor f_1 under Gaussian and non-Gaussian errors, respectively, for values of $\alpha_{10} = 0.80, 0.85, 0.90$ and 0.95 when $T = 200$ and as n increases from 100 to $1,000$. These figures clearly show that the proposed estimator is very precisely estimated for all values of α_{10} considered, and for all (n, T) combinations. Also as α_{10} rises towards unity the power approaches 1 even for very small deviations from the null. We do not report power results for $\alpha_{10} = 1$, due to the ultraconsistency of the estimator in this case.

The above findings continue to hold when we consider models with two observed factors (experiments 2A and 2B), irrespective of whether the factors are orthogonal ($\rho_{12} = 0$) or moderately correlated ($\rho_{12} = 0.3$), or whether the errors are Gaussian. To save space we only give the results for the non-Gaussian errors when $\rho_{12} = 0.3$ in Table 3. The remaining results are provided in the online supplement. Corresponding power functions are shown in Figure 3, and give a similar picture as the one we discussed for the single factor case.

Consider now the experiments where at the estimation stage the number or the identity of factors are assumed unknown. In the case of experiment 3A, the DGP is generated with a single factor, whilst in the case of experiment 3B the DGP is generated with two correlated factors. In both of these experiments the factor strength $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is computed with respect to the pervasiveness of the simple cross section average, $\bar{x}_t$. This case is analysed in Section 4. The results for this case when errors are non-Gaussian are summarised in Table 4 with the associated power functions in Figure 4. As can be seen, the small sample performance of the estimator of factor strength deteriorates somewhat as compared to when the factor is known, particularly for values of α_0 that are not sufficiently close to unity. The empirical size is particularly elevated for values of $\alpha_0 \leq 0.9$ when compared to the case of observed factors. However, for large sample sizes and values for α_0 close to unity, the proposed estimator seems to be reasonably well behaved. Similar conclusions are obtained for Gaussian errors. (see Table S14 and Figure S12 in the online supplement).

In the case of two unobserved factors (experiment 3B), we estimate the strength $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$, using the simple cross section average, $\bar{x}_t$, first when $\alpha_{10} = 1$ and $\alpha_{20} = 0.51, 0.75, 0.95, 1$. As shown in Table 5 under non-Gaussian errors, when α_{20} is set to the lower bound ($= 0.51$), then bias and RMSE results are again universally very low and reflect those of the case of one unobserved factor, which is expected. A slight deterioration in results can be detected as α_{20} is increased towards unity, for small values of T, e.g. $T = 120$, but the size distortions vanish as T increases. The ultraconsistency of our estimator when $\alpha_{10} = 1$ is evident by the values for both bias and RMSE measures which are so small that we have scaled them by 10,000 in Table 5. When $\alpha_{10}, \alpha_{20} < 1$, estimating α_0 becomes more challenging. This is clear from the bias and RMSE results shown in Table 6, when $\alpha_{10} = 0.95$ and α_{20} is set to the same values as before (here the scaling of all bias and RMSE values is returned to 100). In line with the conditions of Theorem 2, namely $\sqrt{T}n^{(\alpha_{20}-\alpha_{10})} \to 0$, results worsen for values of α_{10} relatively close to α_{20} but improve as the distance between α_{10} and α_{20} widens, for any given value of n and T. When $\alpha_{20} = 1$, then the estimate of $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ becomes ultraconsistent, as shown in Table 5.[7] As before, similar conclusions are obtained for Gaussian errors. (see Tables S15 and S16 in the online supplement).

Finally, consider experiment 4 designed to reflect the setting of the finance empirical application presented in subsection 6.1. Here we focus on a DGP with two factors that are correlated, but a single observed factor model is used for estimating the strength of the first factor, f_1. The results for $\alpha_{10} = 1$ are shown in Table 7, and as can be seen, omitting a second relevant and correlated factor in this case does not unduly affect the performance of the estimator of the strength of the first factor.[8] This seems to be the case for all (n, T) combinations and different values of α_{20}.[9] However, misspecification is likely to be consequential if the first factor is not sufficiently strong.

6 Empirical applications to finance and macroeconomics

6.1 Identifying risk factors in asset pricing models

The asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965), and its multi-factor extension in the context of the Arbitrage Pricing Theory (APT) developed by Ross (1976) are the leading theoretical contributions implemented widely in modern empirical finance to analyse the cross-sectional differences in expected returns. Both approaches imply that expected returns are linear in asset betas with respect to fundamental economic aggregates, and the Fama-MacBeth two-pass procedure (Fama and MacBeth (1973)) is one of the most broadly used methodologies to assess these linear pricing relationships. The first stage in this approach entails choosing the risk factors to be included in the asset pricing model. Given the upsurge in the number of factors deemed relevant to asset pricing in the past few years, a rapidly growing area of the finance literature has been concerned with evaluating the contribution of potential factors to these models. Harvey and Liu (2019) document over 400 such factors published in top ranking academic journals. The primary focus of this literature has been on factor selection on the basis of performance metrics such as the Gibbons,Ross and Shanken statistic of Gibbons et al. (1989), or the maximum squared Sharpe ratio of Fama and French (2018) among many others. More recent contributions further allow for the possibility of false discovery when the number of potential factors is large and multiple testing issues arise - see Feng et al. (2020).

Our application focuses on determining the strength of these factors as a means of evaluating whether their risk can be priced correctly and abstracts from the question of factor selection as

[7] Using the first principal component (PC) of x_{it} instead of the cross section average (CSA) produces similar results when $\alpha_{j0} = 1.00$, $j = 1, 2$, but underperforms in comparison to CSA when $\alpha_{j0} < 1.00$, as shown in Tables S17 and S18 of the online supplement. See Section 19.5.1 of Pesaran (2015) where the asymptotic properties of cross section average and the first PC are compared.

[8] The bias and RMSE values for this experiment are negligible so that in Table 7 they are reported after scaling them up by the factor of 10,000.

[9] Corresponding results for the case of uncorrelated factors ($\rho_{12} = 0.0$) are shown in Table S19 in the online supplement.

such. As shown by Pesaran and Smith (2019), the APT theory requires that risk factors should be sufficiently strong if their associated risk premium is to be estimated consistently. The risk premium of a factor with strength α can be estimated at the rate of $n^{-a/2}$, where n is the number of individual securities under consideration. As a result, $\sqrt{n}$ consistent estimation of the risk premium of a given factor requires the factor in question to be strong with its α equal to unity. Factors with strength less than 0.5 cannot be priced and are absorbed in pricing errors. But in principle, it should be possible to identify the risk premium of semi-strong factors (factors whose α lies in the range $1 > \alpha > 1/2$), but very large number of securities are needed for this purpose. In practice, where n is not sufficiently large, at best only factors with strength sufficiently close to unity can be priced.[10] As an illustration of their theoretical results, Pesaran and Smith (2019) consider the widely used Fama and French (1993) three-factor model applied to the constituents of the S&P500 index and assess the strength of each of the factors included in the model, namely the market, size and value factors. In what follows we carry out a more comprehensive investigation of this topic, by assessing the strength of a total of 146 factors.

6.1.1 Data

We consider monthly excess returns of the securities included in the S&P 500 index over the period from September 1989 to December 2017. Since the composition of the index changes over time, we compiled returns on all 500 securities at the end of each month and included in our analysis only those securities that had at least 10 years of history in the month under consideration. On average, we ended up with $n = 442$ securities at the end of each month. The one-month US treasury bill rate (in percent) was chosen as the risk-free rate (r_{ft}), and excess returns were computed as $\tilde{r}_{it} = r_{it} - r_{ft}$, where r_{it} is the return on the i^{th} security between months $t-1$ and t in the sample, inclusive of dividend payments (if any).[11] In addition to the market factor (measured as the excess market return) we consider the 145 factors considered by Feng et al. (2020), which are largely constructed as long/short portfolios capturing a number of different characteristics.[12] In order to account for time variations in factor strength, we use rolling samples (340 in total) of 120 months (10 years). The choice of the rolling window is guided by the balance between T and n, and follows the usual practice in the finance literature.[13]

[10] In an early critique of tests of asset pricing theory, Roll (1977) argued that for a test to be valid, it is required that all assets traded in the economy are included in the empirical analysis. In effect requiring n to be very large, and much larger than the number of securities traded on exchanges.

[11] Further details relating to the construction of this dataset can be found in the online supplement and in Bailey et al. (2016, 2019).

[12] The authors would like to thank Dacheng Xiu for providing the dataset that covers all the 146 factors, inclusive of the market factor. Apart from 15 factors obtained from specific websites, the remaining factors are constructed using only stocks for companies listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11. Moreover, financial firms and firms with negative book equity are excluded. For each characteristic, stocks are sorted using NYSE breakpoints based on their previous year-end values, then long-short value-weighted portfolios (top 30% - bottom 30% or 1-0 dummy difference) are built and rebalanced every June for a 12-month holding period. Further details about the construction of this dataset can be found in Feng et al. (2020).

[13] We also consider rolling samples of size 60 months (5 years). Results are available upon request.

6.1.2 Factor models for individual securities

We commence with the following regressions:

$$r_{it} - r_{ft} = a_i + \beta_{im}\left(r_{mt} - r_{ft}\right) + \sum_{j=1}^{k} \beta_{ij} f_{jt} + u_{it}, \text{ for } i = 1, 2, \ldots, n_\tau, \tag{46}$$

where n_τ are the number of securities in 10-year rolling samples from September 1989 to December 2017, with $\tau = 1, 2, \ldots, 340$. r_{mt} denotes the return on investing in the market portfolio, which here is approximated by a value weighted average of all CRSP firms incorporated in the US and listed on the NYSE, AMEX, or NASDAQ that have data for month t. As such, this definition of the market portfolio is wider than one which assumes an average of the 440 or so S&P500 securities considered in this study. The excess market return, $(r_{mt} - r_{ft})$, then approximates the market factor. f_{jt} for $j = 1, 2, \ldots, 145$ represent the potential risk factors in the active set under consideration. As explained in Section 5 of Pesaran and Smith (2019), the strength of factor j is defined by $\sum_{i=1}^{n}\left(\beta_{ij} - \bar{\beta}_j\right)^2 = \ominus\left(n^{\alpha_j}\right)$, and once the market factor is included in (46), it is the case that the coefficients are expressed as deviations of the factor loadings from their means, as required.

Initially, we set $k = 0$ and consider the original CAPM specification of Sharpe (1964) and Lintner (1965),

$$r_{it} - r_{ft} = a_{im} + \beta_{im}\left(r_{mt} - r_{ft}\right) + u_{it,m}. \tag{47}$$

We apply our estimator (7) to the loadings β_{im}, $i = 1, 2, \ldots, n_\tau$, and obtain estimates of the strength of the market factor across the rolling windows, $\hat{\alpha}_{m,\tau}$, $\tau = 1, 2, \ldots, 340$.

Next, in order to assess the effect on the market factor strength estimates of adding more factors to (47), as well as to quantify the strength of these additional factors, we add the 145 factors to the CAPM regression, (47), one at a time; namely we run the regressions

$$r_{it} - r_{ft} = a_{is} + \beta_{im|s}\left(r_{mt} - r_{ft}\right) + \beta_{is} f_{st} + u_{it,s}, \; i = 1, 2, \ldots, n_\tau \tag{48}$$

for each $s = 1, 2, \ldots, 145$, and each rolling window $\tau = 1, 2, \ldots, 340$. Our choice of model is motivated by the fact that once we have conditioned on the market factor, we can use the One Covariate at the time Multiple Testing (OCMT) methodology of Chudik et al. (2018) as an additional step for selecting the factors that ought to be included in our final asset pricing model. Again, we compute the strength of the market factor, $\hat{\alpha}_{m,\tau|s}$, with the s^{th} factor included, as well as the strength of each of the additional factors, $\hat{\alpha}_{s,\tau}$, for all 340 rolling windows. As with the Monte Carlo experiments, in the computation of factor strength we set the nominal size of the associated multiple tests to $p = 0.10$, and the critical value exponent to $\delta = 1/4$.

6.1.3 Estimates of factor strengths

First, we consider the rolling estimates obtained for the strength of market factor, α_m, when using the CAPM and the augmented CAPM specifications given by (47) and (48). Figure 5 displays $\hat{\alpha}_{m,\tau}$, $\tau = 1, 2, \ldots, 340$; the 10-year rolling estimates obtained using the CAPM regressions over the period September 1989 to December 2017. As can be seen, all $\hat{\alpha}_{m,\tau}$ are quite close to unity, and it can be safely concluded that the market factor is strong and its risk premium can be estimated consistently at the usual rate of $\sqrt{n}$. There is some evidence of departure from unity over the period between December 1999 to January 2011 which saw a number of sizeable financial events such as the Long-Term Capital Management (LTCM) crisis, the burst of the dot-com bubble and, more recently, the global financial crisis. $\hat{\alpha}_{m,\tau}$ records its minimum value of 0.958 in August 2008, around the time of the Lehman Brothers collapse. As implied by our theoretical results of Section 3, standard errors around these estimates are extremely tight and hard to distinguish graphically from the point estimates.[14] It is also interesting that the estimates of market factor strength are generally unaffected if we consider the augmented CAPM regressions. For each rolling window we now obtain 145 estimates of α_m, denoted by $\hat{\alpha}_{m,\tau|s}$ for $s = 1, 2, \ldots, 145$. We display the average of these estimates, namely, $\overline{\hat{\alpha}}_{m,\tau} = (1/145)\sum_{s=1}^{145} \hat{\alpha}_{m,\tau|s}$, in Figure 5. It is clear that $\overline{\hat{\alpha}}_{m,\tau}$ closely track $\hat{\alpha}_{m,\tau}$. The two series are almost identical during the periods September 1989 to December 1999 and January 2011 to December 2017. There are some minor deviations between $\hat{\alpha}_{m,\tau|s}$ and $\hat{\alpha}_{m,\tau}$ during the period December 1999 to January 2011, when they both deviate marginally from unity, with a maximum deviation of 0.011 in September 2008. The average estimates of $\alpha_{m,\tau}$ also have very narrow confidence bands, with an average standard error of 0.0038 over the full sample, taking its maximum value of 0.0099 in September 2008. Overall, it is evident that the inclusion of an additional factor in (48) has little effect on estimates of the market factor strength, which is in line with the Monte Carlo evidence for experiment 4 summarised in the previous Section.

We can safely conclude that the market factor is strong with the exception of a short period during the recent financial crisis. We now consider the 10-year rolling estimates of the strength of the remaining factors, denoted by $\alpha_{s,\tau}$, using the augmented CAPM regressions. These estimates together with their 90% confidence bands are shown in Figures S13 to S22 of the online supplement. They show considerable time variation, especially during December 1999 to January 2011. However, at no point during the full sample (September 1989 to December 2017) do any of these factors become strong in the sense that $\hat{\alpha}_{s,\tau}$ is clearly below 1, for all s and τ. The market factor dominates all other factors in strength. Indeed, in Figure 6 we observe that the proportion of factors (out of the 145 in total) whose strength exceeds the threshold values of $0.85, 0.90$ and 0.95 in each rolling window progressively drops so that there are no factors left whose strength exceeds 0.95 throughout our sample period. This suggests that only the market factor can be considered to be a risk factor whose risk premium can be estimated consistently at a rate of $n^{1/2}$. The role of the remaining 145 factors in the asset pricing models (48) could be to filter out the effects of any additional semi-strong cross-dependence in asset returns in order to achieve weak enough cross-sectional dependence in the errors u_{it}, required for consistent estimation of market risk premia.

Next, we rank the 145 factors (plus the market factor) from strongest to weakest in terms of the percentage of months in our sample period (340 in total) that their strength exceeds the threshold of 0.90. Table 8 displays the identities of the 65 factors that meet this criterion. As expected, the market factor ranks first with an average estimated strength of 0.99, followed by factors associated with leverage, and the ratios of sales to cash, cash flow to price, net debt to price and earnings to price. The second ranking factor, leverage, has average strength of 0.827, with only 37.9% of the time being above 0.9. Interestingly, the Fama French value factor (high minus low) ranks 34th in our table while the size factor (small minus big) does not even enter our ranking, recording values of $\hat{\alpha}$ below 0.90 across all rolling windows. For completeness, Table 8 also includes time averages of each factor strength over the full sample (September 1989 - December 2017), and the three sub-samples: September 1989 - August 1999, September 1999 - August 2009, and September 2009 - December 2017. While on average, the strengths of these factors are around 0.80 in the first and the last decade in our sample, in the period between September 1999 to August 2009, the strength of many factors rises to around 0.91. This rise could be due to non-fundamental factors gaining importance

[14] The corresponding plot of $\hat{\alpha}_{m,\tau}$ estimates under (47) which includes its standard errors is shown at the top left corner of Figure S13 in the online supplement.

over the fundamental factors during the recent financial crisis, and can be viewed as evidence of market decoupling.[15]

Finally, it is of interest to investigate whether the strength of the strongest latent factor implied by the panel of S&P500 securities' excess returns coincides with that of the market risk factor, which we identified as the strongest observed factor under our previous analysis. In line with the discussion of Section 4, the strength of the strongest unobserved factor will be captured by the strength of the cross section average of the excess returns in each rolling window. Figure 7 plots the 10-year rolling $\hat{\alpha}_{csa,\tau}$ estimates implied by the cross section average of excess returns against the 10-year rolling $\hat{\alpha}_{m,\tau}$ estimates implied by the simple CAPM regression (47). It is evident that the two series are almost identical throughout our sample period except for the period between September 1999 to January 2011 where they deviate from each other to some extent. The average correlation between $\hat{\alpha}_{csa,\tau}$ and $\hat{\alpha}_{m,\tau}$ over $\tau = 1, 2, \ldots, 340$ stands at 0.93. On this basis, we also computed the rolling correlation coefficients between the cross section average of individual securities' excess returns and the observed market risk factor again over the rolling windows $\tau = 1, 2, \ldots, 340$. These are consistently close to unity apart for the period between September 1999 to January 2011 where they drop slightly towards 0.85.[16] The average correlation coefficient across rolling windows stands at 0.95.

6.2 Strength of common macroeconomic shocks

Similar considerations apply to macroeconomic shocks and their pervasive effects on different parts of the macroeconomy. As discussed in Giannone et al. (2017) and references therein, the advent of 'high-dimensional' datasets has led to the development of predictive models that are either based on shrinkage of useful information inherent across the whole set of data into a finite number of latent factors (e.g. Stock and Watson (2015) and references therein), or assume that all relevant information for prediction is captured by a small subset of variables from the larger pool of regressors implied by these data (e.g. Hastie et al. (2015), Belloni et al. (2011) among others). Such methods are appealing in macroeconomics since they tend to provide more reliable impulse responses and forecasts over traditional models, when used for macroeconomic policy analysis and forecasting. However, as argued in Giannone et al. (2017), it is not evident that either approach is always clearly supported by the (unknown) structure of the given data and that model averaging might be preferable.

To measure the pervasiveness of the macroeconomic shocks, we make use of an updated version of the macroeconomic dataset compiled originally by Stock and Watson (2012) and subsequently extended by McCracken and Ng (2016). Here, we assume that the macroeconomic shocks are unobserved and estimate the strength of the strongest of such shocks from the updated dataset which consists of balanced quarterly observations over the period 1988Q1-2019Q2 ($T = 126$) on $n = 187$ out of the 200 macroeconomic variables used in Stock and Watson (2012).[17] Ten out of the 200 macroeconomic variables used in Stock and Watson (2012) are no longer available in the updated version of the dataset.[18] Further details on this dataset can be found in the online supplement.

[15] The ranking of all 145 factors and their average strengths over different sub-samples are given in Table S20 of the online supplement.

[16] Rolling correlation coefficients between the market risk factor and the cross section average of S&P500 securities' excess returns in shown in Figure S23 of the online supplement.

[17] The raw data, which include both high-level economic and financial aggregates as well as disaggregated components, are updated regularly and can be found on the Federal Reserve Bank of St Louis website at: https://research.stlouisfed.org/econ/mccracken/static.html. All variables were screened for outliers and transformed as required to achieve stationarity. Details about variable definitions, descriptions and transformations can be found in the accompanying FRED-QD appendix to McCracken and Ng (2016) which links to Stock and Watson (2012) and is downloadable from the aforementioned website.

[18] These are: (1) Construction contracts, (2) Manufacturing and trade inventories, (3) Index of sensitive materials

6.2.1 Strength estimates of strongest unobserved common shock

As discussed in Section 4, identifying and estimating the strengths of unobserved factors of varying strengths becomes challenging due to the fact that, in general, factors are identified only up to a non-singular rotation matrix. However, as argued above we are still able to identify and estimate the strength (α) of the strongest shock using the cross section average of the variables in the dataset. We computed estimates of α for the pre-crisis period, 1988Q1 to 2007Q4, as well as for the full sample period ending on 2019Q2. The factor strength estimates are shown in Table 9. They are clustered around 0.94, and are quite robust to the choice of the parameters p and δ in the critical value function (6), as well as to the time period considered. These estimates are consistently below 1, and suggest that whilst there exist strong macroeconomic shocks, the effects of such shocks are not nearly as pervasive as have been assumed in the factor literature applied to macro variables. This finding is further corroborated by the estimates of the exponent of cross-sectional dependence (CSD) of BKP, also shown in Table 9.[19]

7 Conclusions

Recent work by Bailey et al. (2016, 2019) has focused on the rationale and motivation behind the need for determining the extent of cross-sectional dependence (CSD), be it in finance or macroeconomics, and has provided a conceptual framework and tools for estimating the strength of such interdependencies in economic and financial systems. However, this literature does not address the problem of estimating the strength of individual factors that underlie such cross dependencies, which can be of interest, for example, for pricing of risk in empirical finance, or for quantifying the pervasiveness of macroeconomic shocks.

The current paper addresses this gap. It proposes a novel estimator of factor strength based on the number of statistically significant t-statistics in a regression of each unit in the panel dataset on the factor under consideration, and provide inferential theory for the proposed estimator. Detailed and extensive Monte Carlo and empirical analyses showcase the potential of the proposed method.

The current paper considers estimation and inference when the panel regressions are based on a finite number of observed factors. Some theoretical evidence is also provided for the case when the model contains unobserved factors. Further research is required to link our analysis to the problem of factor selection discussed by Feng et al. (2020). Also, it would be of interest to address the identification problem when there are multiple unobserved factors. One possibility would be to exploit the approach recently developed in Kapetanios et al. (2019) to see whether the unobserved factors can be associated with dominant units or some other observable components.

prices (disc), (4) Spot market price index BLS&CRB: all commodities, (5) NAPM commodity price index, (6) 3m Eurodollar deposit rate, (7) MED3-TB3MS, (8) GZ-spread, (9) GZ Excess bond premium, and (10) DJIA.

[19] Using the Sequential Multiple Testing (SMT) detection procedure developed in Kapetanios et al. (2019), we also checked to see if any of the unit(s) in the macro dataset can be viewed as pervasive, namely sufficiently influential to affect all other variables. The SMT procedure could not detect any such variables for all choices $p_{\max} = 0, 1, \ldots, 6$, where p_{max} denotes the assumed maximum number of potential factors in the dataset.

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Table 1: Bias, RMSE and Size (×100) of estimating different factor strengths in the case of experiment 1A (observed single factor - Gaussian errors)

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$											
100	1.23	1.16	1.08	1.07	1.58	1.51	1.46	1.43	3.70	3.10	2.60	2.00
200	1.44	1.40	1.31	1.29	1.60	1.55	1.47	1.44	9.80	8.15	7.45	6.00
500	1.30	1.23	1.14	1.14	1.38	1.30	1.21	1.21	14.00	10.55	7.35	7.65
1000	1.26	1.21	1.12	1.11	1.31	1.25	1.16	1.15	15.90	12.40	7.25	6.95
	$\alpha_{10} = 0.80$											
100	0.73	0.68	0.64	0.62	1.06	1.01	0.99	0.95	18.35	17.25	19.55	18.55
200	0.95	0.93	0.87	0.85	1.09	1.05	1.00	0.98	12.65	11.20	9.80	9.00
500	0.91	0.86	0.81	0.81	0.97	0.91	0.86	0.86	12.35	8.60	6.40	6.20
1000	0.85	0.82	0.76	0.76	0.88	0.85	0.79	0.78	16.65	13.50	8.40	8.25
	$\alpha_{10} = 0.85$											
100	0.70	0.68	0.65	0.64	0.89	0.87	0.84	0.83	10.75	8.95	8.50	7.60
200	0.60	0.59	0.54	0.54	0.71	0.69	0.65	0.64	5.70	4.00	3.70	2.90
500	0.52	0.50	0.46	0.46	0.57	0.54	0.50	0.50	11.05	8.35	6.85	7.10
1000	0.50	0.48	0.45	0.44	0.52	0.50	0.46	0.46	10.05	8.05	5.15	5.15
	$\alpha_{10} = 0.90$											
100	0.41	0.40	0.39	0.38	0.57	0.55	0.54	0.53	5.00	3.70	3.60	3.55
200	0.27	0.28	0.24	0.24	0.37	0.37	0.34	0.34	14.15	12.40	12.20	12.85
500	0.29	0.28	0.26	0.26	0.33	0.31	0.29	0.29	9.70	8.00	7.40	6.35
1000	0.28	0.28	0.26	0.26	0.30	0.29	0.27	0.27	10.60	9.45	6.55	6.60
	$\alpha_{10} = 0.95$											
100	0.08	0.08	0.08	0.07	0.26	0.24	0.24	0.24	5.40	3.40	3.65	2.40
200	0.11	0.12	0.10	0.10	0.19	0.18	0.17	0.17	7.40	5.40	4.10	4.25
500	0.12	0.12	0.11	0.11	0.14	0.14	0.13	0.13	12.25	7.45	8.00	7.25
1000	0.09	0.10	0.09	0.09	0.11	0.11	0.10	0.10	8.55	6.10	5.10	4.35
	$\alpha_{10} = 1.00$											
100	-0.01	0.00	0.00	0.00	0.04	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

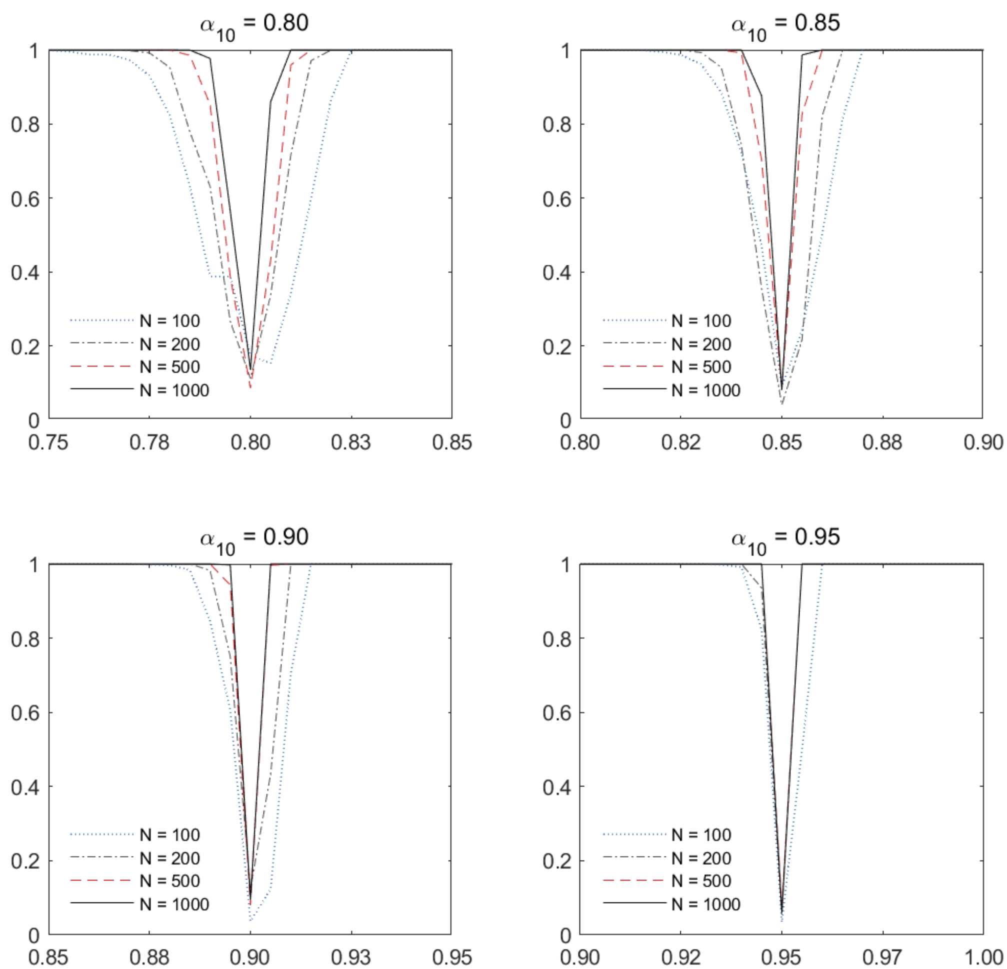
Notes: Parameters of DGP (44) are generated as follows: for unit specific effects, $c_i \sim IIDN\,(0,1)$, for $i = 1, 2, \dots, n$. The factor, f_{1t}, is normally distributed with variance $\sigma^2_{f_1} = 1$. The factor assumes an autoregressive process with correlation coefficient $\rho_{f_1} = 0.5$. The factor loadings are generated as $v_{i1} \sim IIDU(\mu_{v_1} - 0.2, \mu_{v_1} + 0.2)$, for $[n^{\alpha_{10}}]$ units, and zero otherwise. $v_{i2} = 0$, for all i. We set $\mu_{v_1} = 0.71$. α_{10} ranges between $[0.75, 1.00]$ with 0.05 increments. The innovations u_{it} are Gaussian, such that $u_{it} \sim IIDN(0, \sigma_i^2)$, with $\sigma_i^2 \sim IID(1 + \chi^2_{2,i})/3$, for $i = 1, 2, \dots, n$. In the computation of $\hat{\alpha}_1$ we use $p = 0.10$ and $\delta = 1/4$ when setting the critical value. Size is computed under H_0: α_1=α_{10}, using a two-sided alternative. When $\alpha_{10} = 1.00$, our estimator is ultraconsistent, hence size results for this case are not meaningful. The number of replications is set to $R = 2000$.

Table 2: Bias, RMSE and Size (×100) of estimating different factor strengths in the case of experiment 1B (observed single factor - non-Gaussian errors)

	Bias (×100)				RMSE (×100)				Size (×100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$											
100	1.21	1.17	1.06	1.04	1.57	1.52	1.44	1.39	3.75	3.20	2.65	1.70
200	1.47	1.40	1.34	1.30	1.62	1.56	1.49	1.45	9.80	9.50	6.45	6.50
500	1.32	1.24	1.14	1.15	1.40	1.31	1.21	1.22	15.25	11.00	8.20	8.15
1000	1.27	1.19	1.13	1.11	1.31	1.23	1.17	1.15	17.00	11.30	8.80	7.55
	$\alpha_{10} = 0.80$											
100	0.71	0.69	0.62	0.60	1.04	1.02	0.97	0.93	17.60	18.00	19.60	18.15
200	0.98	0.93	0.89	0.86	1.10	1.07	1.01	0.99	12.45	12.00	9.80	9.05
500	0.92	0.87	0.80	0.81	0.98	0.92	0.86	0.86	13.90	8.90	6.95	7.55
1000	0.86	0.81	0.77	0.76	0.89	0.84	0.79	0.79	18.35	12.75	9.90	8.70
	$\alpha_{10} = 0.85$											
100	0.69	0.69	0.64	0.63	0.88	0.88	0.83	0.82	10.15	10.35	8.25	7.80
200	0.62	0.59	0.56	0.54	0.72	0.69	0.66	0.64	5.50	4.60	3.50	2.45
500	0.53	0.51	0.46	0.46	0.58	0.55	0.51	0.50	12.65	9.30	8.95	7.65
1000	0.50	0.48	0.45	0.44	0.53	0.50	0.47	0.46	12.35	7.55	6.55	5.50
	$\alpha_{10} = 0.90$											
100	0.40	0.40	0.37	0.37	0.55	0.55	0.52	0.52	4.55	3.90	3.30	2.75
200	0.28	0.27	0.25	0.24	0.38	0.36	0.34	0.33	12.35	11.30	11.25	13.80
500	0.30	0.29	0.26	0.26	0.34	0.32	0.30	0.30	10.95	9.15	7.90	7.35
1000	0.28	0.27	0.26	0.26	0.30	0.29	0.27	0.27	12.55	8.95	7.20	5.80
	$\alpha_{10} = 0.95$											
100	0.07	0.08	0.06	0.07	0.24	0.24	0.23	0.23	4.90	4.05	2.85	2.30
200	0.12	0.11	0.10	0.10	0.19	0.18	0.17	0.17	7.10	4.60	3.55	4.00
500	0.12	0.12	0.11	0.11	0.14	0.14	0.13	0.13	12.15	8.85	8.45	8.20
1000	0.09	0.10	0.09	0.09	0.11	0.11	0.10	0.10	9.10	4.95	5.35	5.05
	$\alpha_{10} = 1.00$											
100	-0.01	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.04	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

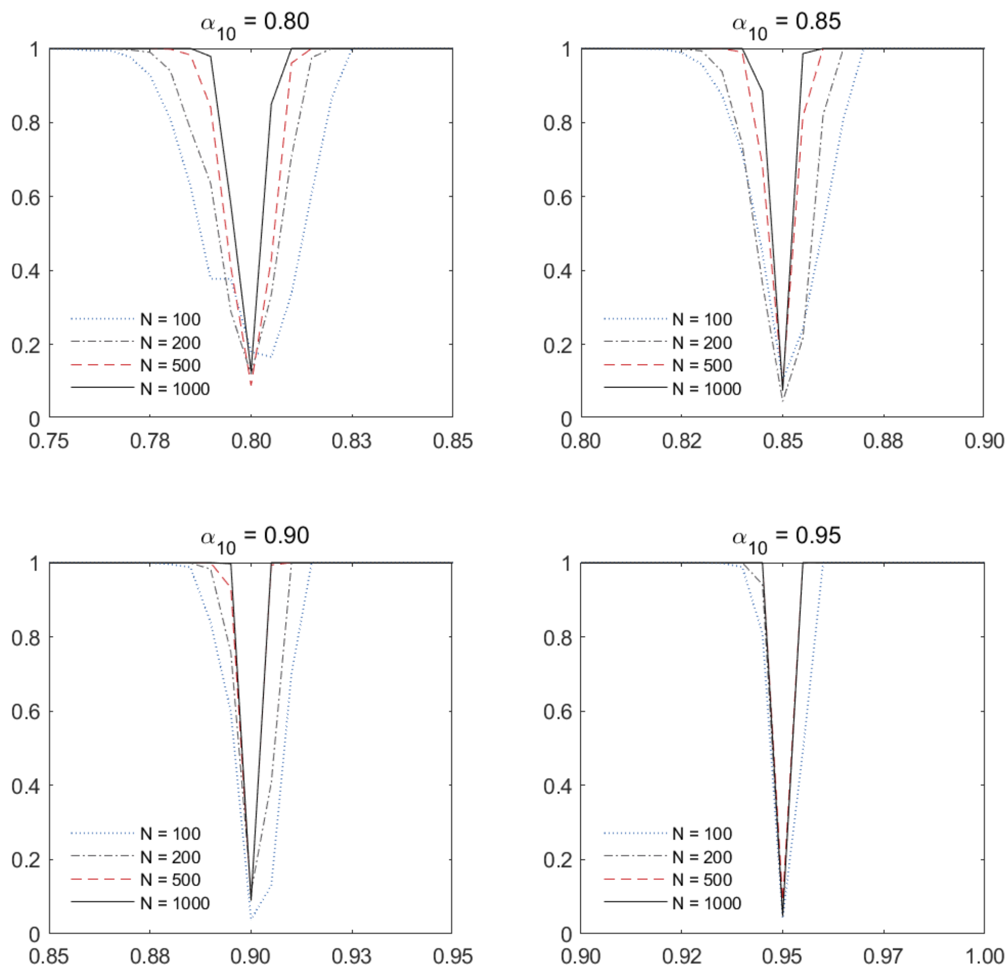
Notes: Parameters of DGP (44) are generated as described in Table 1. The innovations u_{it} are non-Gaussian, such that $u_{it} = \frac{\sigma_i}{2}\left(\chi^2_{2,it} - 2\right)$, with $\sigma_i^2 \sim IID(1 + \chi^2_{2,i})/3$, for $i = 1, 2, \ldots, n$.

Figure 1: Empirical power functions associated with testing different factor strengths in the case of experiment 1A (observed single factor - Gaussian errors), when $n = 100, 200, 500, 1000$ and $T = 200$



Notes: See the notes to Table 1 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Figure 2: Empirical power functions associated with testing different factor strengths in the case of experiment 1B (observed single factor - non-Gaussian errors), when $n = 100, 200, 500, 1000$ and $T = 200$



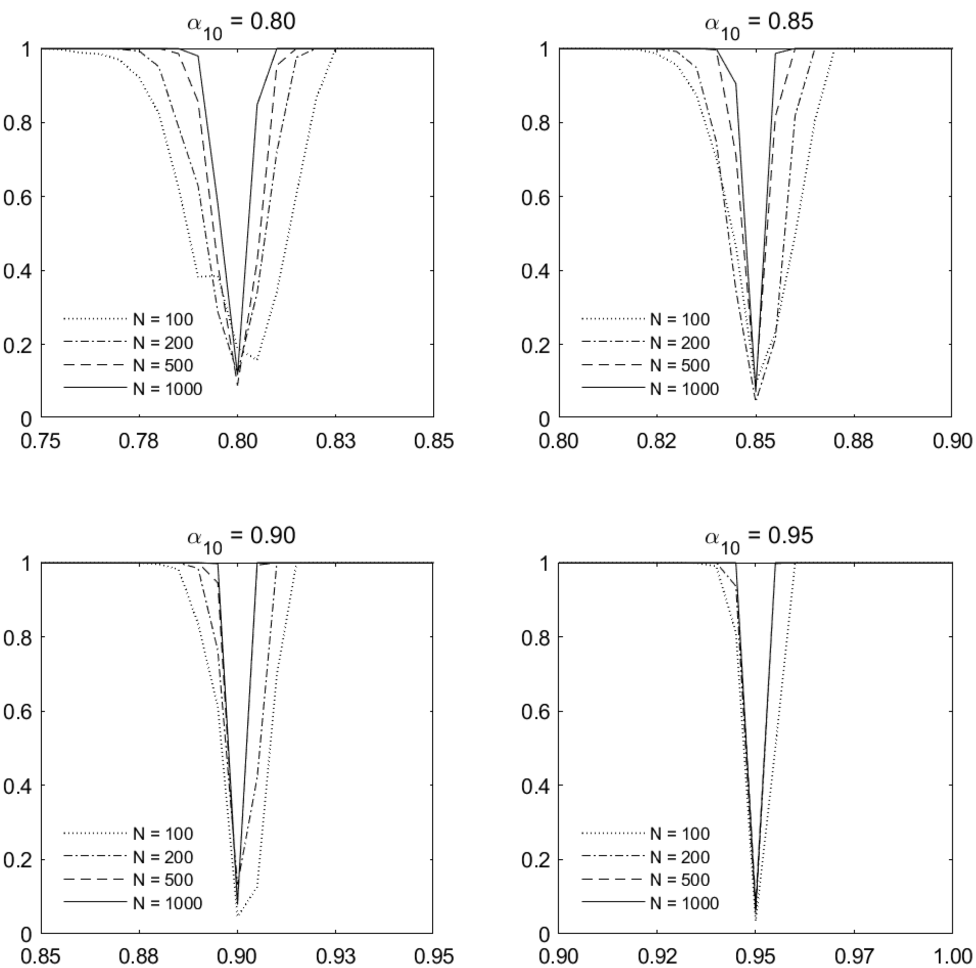
Notes: See the notes to Table 2 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table 3: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.85

	Bias (×100)				RMSE (×100)				Size (×100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10}=0.75$, $\alpha_{20}=0.85$											
100	1.20	1.13	1.08	1.04	1.58	1.50	1.46	1.41	4.35	2.80	3.25	2.30
200	1.43	1.39	1.30	1.31	1.59	1.54	1.45	1.47	10.00	7.65	7.00	7.55
500	1.30	1.23	1.17	1.14	1.38	1.30	1.24	1.21	13.55	10.25	8.20	7.20
1000	1.27	1.18	1.12	1.11	1.31	1.22	1.16	1.15	17.25	11.05	7.45	7.60
	$\alpha_{10}=0.80$, $\alpha_{20}=0.85$											
100	0.71	0.66	0.63	0.61	1.03	1.00	0.96	0.95	17.75	18.80	18.15	19.45
200	0.95	0.93	0.85	0.86	1.09	1.05	0.97	0.99	13.10	11.35	9.20	9.75
500	0.91	0.86	0.82	0.80	0.96	0.92	0.87	0.86	11.80	9.25	5.95	7.40
1000	0.85	0.81	0.76	0.76	0.88	0.83	0.79	0.78	18.10	11.00	8.85	7.80
	$\alpha_{10}=0.85$, $\alpha_{20}=0.85$											
100	0.68	0.67	0.64	0.62	0.87	0.86	0.83	0.81	9.70	9.40	7.70	7.35
200	0.61	0.59	0.54	0.54	0.72	0.69	0.65	0.65	5.95	3.90	4.10	3.05
500	0.51	0.50	0.47	0.46	0.56	0.54	0.51	0.51	10.80	7.70	7.35	7.75
1000	0.50	0.47	0.45	0.44	0.52	0.49	0.47	0.46	12.45	8.45	5.45	5.40
	$\alpha_{10}=0.90$, $\alpha_{20}=0.85$											
100	0.40	0.40	0.38	0.36	0.56	0.55	0.53	0.51	5.35	3.75	3.55	3.05
200	0.27	0.26	0.23	0.24	0.38	0.36	0.33	0.34	14.95	12.45	13.20	13.35
500	0.28	0.29	0.27	0.26	0.32	0.32	0.30	0.29	9.85	8.35	6.85	6.20
1000	0.28	0.27	0.26	0.25	0.30	0.28	0.27	0.27	12.60	8.25	6.50	6.05
	$\alpha_{10}=0.95$, $\alpha_{20}=0.85$											
100	0.06	0.07	0.07	0.06	0.25	0.24	0.24	0.22	6.70	3.45	3.40	2.75
200	0.10	0.11	0.10	0.10	0.18	0.18	0.17	0.17	9.15	4.05	3.85	4.45
500	0.10	0.11	0.11	0.11	0.13	0.14	0.13	0.13	13.35	8.75	8.85	7.40
1000	0.09	0.09	0.09	0.09	0.11	0.11	0.10	0.10	11.50	5.75	5.65	5.05
	$\alpha_{10}=1.00$, $\alpha_{20}=0.85$											
100	-0.02	0.00	0.00	0.00	0.07	0.02	0.00	0.00	-	-	-	-
200	-0.02	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
500	-0.02	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
1000	-0.02	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as follows: for unit specific effects, $c_i \sim IIDN\,(0,1)$, for $i = 1, 2, \ldots, n$. The factors, (f_{1t}, f_{2t}), are multivariate normal with variances $\sigma^2_{f_1} = \sigma^2_{f2} = 1$ and correlation given by $\rho_{12} = corr(f_1, f_2) = 0.3$. Each factor assumes an autoregressive process with correlation coefficients $\rho_{f_j} = 0.5$, $j = 1, 2$. The factor loadings are generated as $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, for $[n^{\alpha_{j0}}]$ units, $j = 1, 2$, respectively, and zero otherwise. We set $\mu_{v_1} = \mu_{v_2} = 0.71$. Both α_{10} and α_{20} range between $[0.75, 1.00]$ with 0.05 increments. The innovations u_{it} are non-Gaussian, such that $u_{it} = \frac{\sigma_i}{2}\left(\chi^2_{2,it} - 2\right)$, with $\sigma^2_i \sim IID(1 + \chi^2_{2,i})/3$, for $i = 1, 2, \ldots, n$. In the computation of $\hat{\alpha}_j$, $j = 1, 2$, we use $p = 0.10$ and $\delta = 1/4$ when setting the critical value. Size is computed under H_0: α_j=α_{j0}, for $j = 1, 2$, using a two-sided alternative. When $\alpha_{10} = 1.00$, our estimator is ultra consistent, hence size results for this case are not meaningful. The number of replications is set to $R = 2000$.

Figure 3: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.85, $n = 100, 200, 500, 1000$ and $T = 200$



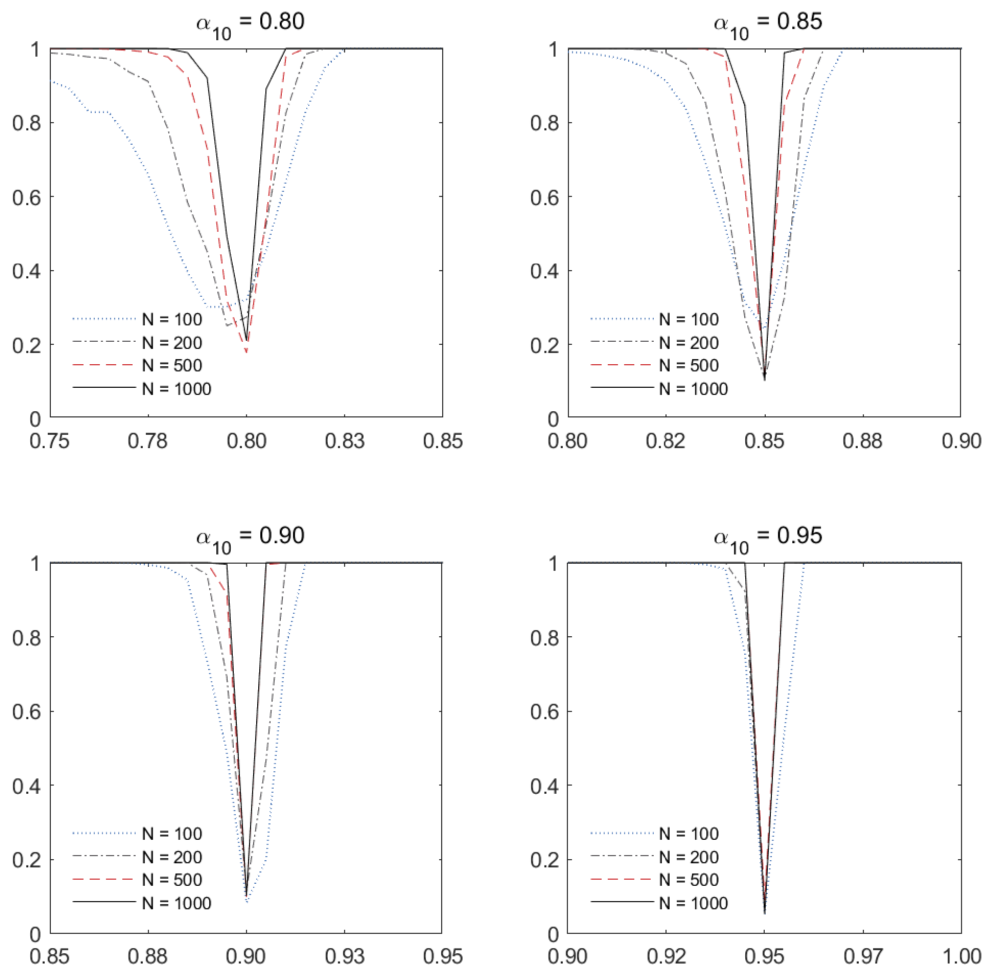
Notes: See the notes to Table 3 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table 4: Bias, RMSE and Size (×100) of estimating the strength of strongest factor in the case of experiment 3A (unobserved single factor - non-Gaussian errors) using cross section average

	Bias (×100)				RMSE (×100)				Size (×100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$											
100	2.40	2.79	4.25	6.73	2.84	3.22	4.60	7.02	27.05	37.35	73.75	98.40
200	2.09	2.12	2.60	3.47	2.32	2.35	2.82	3.67	30.20	34.05	51.60	81.60
500	1.62	1.55	1.59	1.81	1.77	1.68	1.71	1.92	30.95	26.65	29.85	44.80
1000	1.46	1.39	1.38	1.41	1.54	1.46	1.44	1.47	31.95	26.00	27.70	28.00
	$\alpha_{10} = 0.80$											
100	1.26	1.47	2.14	3.42	1.61	1.81	2.43	3.66	28.05	32.20	55.35	87.45
200	1.24	1.24	1.42	1.76	1.39	1.40	1.57	1.90	24.75	27.35	35.40	54.80
500	1.03	0.98	0.97	1.04	1.11	1.05	1.03	1.10	21.80	17.75	15.40	21.95
1000	0.92	0.88	0.85	0.86	0.96	0.92	0.89	0.89	27.00	21.00	18.10	16.95
	$\alpha_{10} = 0.85$											
100	0.91	1.00	1.24	1.78	1.11	1.19	1.44	1.96	19.30	24.15	37.10	64.75
200	0.72	0.71	0.76	0.88	0.83	0.82	0.88	0.99	10.30	10.70	12.75	20.35
500	0.57	0.54	0.52	0.54	0.63	0.59	0.56	0.59	15.25	12.35	10.30	11.90
1000	0.52	0.50	0.48	0.47	0.55	0.52	0.50	0.50	15.50	10.15	8.75	7.45
	$\alpha_{10} = 0.90$											
100	0.49	0.51	0.60	0.79	0.63	0.67	0.76	0.95	6.75	8.40	12.50	22.90
200	0.32	0.31	0.32	0.35	0.42	0.40	0.42	0.44	13.75	11.50	13.50	13.40
500	0.31	0.30	0.28	0.29	0.35	0.33	0.31	0.32	12.50	9.95	8.85	8.50
1000	0.29	0.28	0.27	0.27	0.31	0.30	0.28	0.28	14.50	10.30	8.25	6.95
	$\alpha_{10} = 0.95$											
100	0.10	0.12	0.13	0.18	0.26	0.27	0.29	0.34	5.40	5.25	6.55	11.05
200	0.13	0.12	0.12	0.13	0.20	0.19	0.18	0.20	7.85	5.65	5.35	6.50
500	0.12	0.12	0.11	0.11	0.15	0.15	0.14	0.14	12.40	9.10	7.80	8.75
1000	0.10	0.10	0.09	0.09	0.11	0.11	0.11	0.10	8.75	5.35	4.90	4.90
	$\alpha_{10} = 1.00$											
100	-0.01	0.00	0.00	0.00	0.04	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table 2. $\alpha_0 = \alpha_{10}$ is estimated by regressing observations, x_{it}, on an intercept and the cross section average of x_{it}, $\bar{x}_t = n^{-1} \sum_{i=1}^{n} x_{it}$, for $t = 1, 2, \dots, T$.

Figure 4: Empirical power functions associated with testing different strengths of strongest factor in the case of experiment 3A (unobserved single factor - non-Gaussian errors) using cross section average, when $n = 100, 200, 500, 1000$ and $T = 200$



Notes: See the notes to Table 4 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table 5: Bias and RMSE ($\times 10,000$) of estimating the strength of strongest factor in the case of experiment 3B (two unobserved factors - non-Gaussian errors) using cross section average, when $\alpha_{10} = 1.00$

	Bias ($\times 10,000$)				RMSE ($\times 10,000$)			
$n\backslash$T	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.51$							
100	-0.76	-0.05	0.00	0.00	4.20	1.09	0.00	0.00
200	-0.95	-0.05	0.00	0.00	3.14	0.67	0.00	0.00
500	-1.12	-0.07	0.00	0.00	2.39	0.46	0.00	0.00
1000	-1.25	-0.08	0.00	0.00	2.04	0.37	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.75$							
100	-0.92	-0.04	0.00	0.00	4.58	0.98	0.00	0.00
200	-0.94	-0.03	0.00	0.00	3.11	0.52	0.00	0.00
500	-1.12	-0.06	0.00	0.00	2.39	0.45	0.00	0.00
1000	-1.26	-0.09	0.00	0.00	2.08	0.38	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.95$							
100	-1.44	-0.15	0.00	0.00	5.78	1.83	0.00	0.00
200	-2.05	-0.19	0.00	0.00	5.31	1.37	0.00	0.00
500	-2.08	-0.19	0.00	0.00	3.99	0.82	0.00	0.00
1000	-2.27	-0.23	0.00	0.00	3.99	0.82	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 1.00$							
100	-0.02	0.00	0.00	0.00	0.69	0.00	0.00	0.00
200	-0.01	0.00	0.00	0.00	0.30	0.00	0.00	0.00
500	-0.02	0.00	0.00	0.00	0.30	0.00	0.00	0.00
1000	-0.02	0.00	0.00	0.00	0.19	0.00	0.00	0.00

Notes: Parameters of DGP (44) are generated as described in Table 3. $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated by regressing observations, x_{it}, on an intercept and the cross section average of x_{it}, $\bar{x}_t = n^{-1}\sum_{i=1}^{n} x_{it}$, for $t = 1, 2, \ldots, T$.

Table 6: Bias and RMSE ($\times 10,000$) of estimating the strength of strongest factor in the case of experiment 3B (two unobserved factors - non-Gaussian errors) using cross section average, when $\alpha_{10} = 0.95$

	Bias ($\times 100$)				RMSE ($\times 100$)			
n\T	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.51$							
100	0.18	0.21	0.40	0.57	0.36	0.38	0.55	0.70
200	0.16	0.16	0.22	0.30	0.23	0.24	0.29	0.38
500	0.13	0.13	0.15	0.17	0.16	0.16	0.18	0.20
1000	0.10	0.10	0.10	0.11	0.12	0.12	0.12	0.13
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.75$							
100	1.27	1.56	1.73	1.79	1.41	1.65	1.80	1.85
200	0.98	1.24	1.52	1.54	1.10	1.31	1.55	1.57
500	0.61	0.86	1.19	1.26	0.72	0.92	1.21	1.27
1000	0.42	0.59	0.95	1.07	0.51	0.67	0.97	1.08
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.95$							
100	3.98	4.03	4.04	4.05	4.00	4.05	4.06	4.07
200	3.87	3.95	3.95	3.96	3.88	3.96	3.96	3.97
500	3.74	3.82	3.84	3.83	3.74	3.82	3.84	3.83
1000	3.62	3.71	3.72	3.73	3.63	3.72	3.72	3.73
	$\alpha_{10} = 0.95$, $\alpha_{20} = 1.00$							
100	-0.02	0.00	0.00	0.00	0.07	0.02	0.00	0.00
200	-0.02	0.00	0.00	0.00	0.05	0.01	0.00	0.00
500	-0.02	0.00	0.00	0.00	0.04	0.01	0.00	0.00
1000	-0.02	0.00	0.00	0.00	0.04	0.01	0.00	0.00

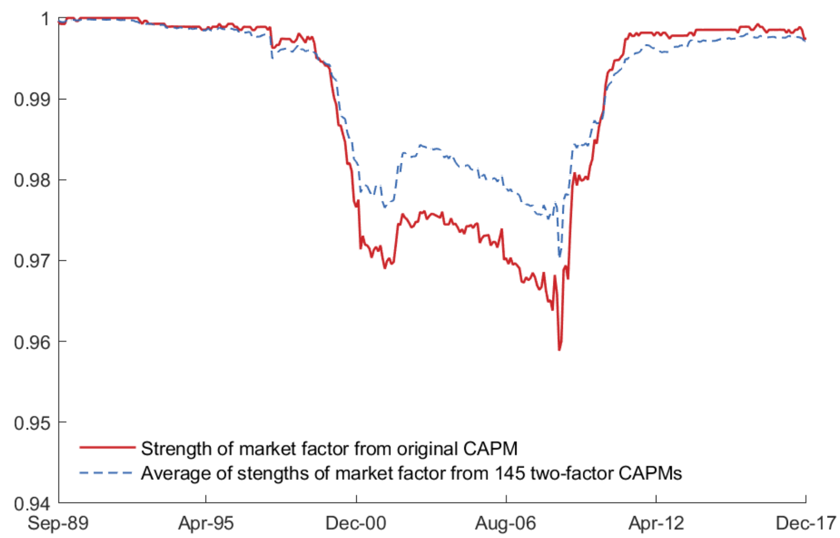
Notes: Parameters of DGP (44) are generated as described in Table 3. $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated by regressing observations, x_{it}, on an intercept and the cross section average of x_{it}, $\bar{x}_t = n^{-1}\sum_{i=1}^{n} x_{it}$, for $t = 1, 2, \ldots, T$.

Table 7: Bias and RMSE ($\times 10,000$) of estimating factor strength in the case of experiment 4 (observed misspecified single factor - Gaussian errors) when set to 1.00

	Bias ($\times 10,000$)				RMSE ($\times 10,000$)			
n\T	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.75$							
100	-0.68	-0.01	0.00	0.00	4.03	0.49	0.00	0.00
200	-0.62	-0.03	0.00	0.00	2.55	0.52	0.00	0.00
500	-0.76	-0.04	0.00	0.00	1.86	0.38	0.00	0.00
1000	-0.76	-0.05	0.00	0.00	1.42	0.27	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.80$							
100	-0.70	-0.01	0.00	0.00	4.03	0.49	0.00	0.00
200	-0.54	-0.02	0.00	0.00	2.38	0.47	0.00	0.00
500	-0.72	-0.04	0.00	0.00	1.82	0.35	0.00	0.00
1000	-0.71	-0.04	0.00	0.00	1.37	0.26	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.85$							
100	-0.61	-0.01	0.00	0.00	3.78	0.49	0.00	0.00
200	-0.45	-0.01	0.00	0.00	2.15	0.37	0.00	0.00
500	-0.62	-0.04	0.00	0.00	1.64	0.35	0.00	0.00
1000	-0.65	-0.04	0.00	0.00	1.27	0.24	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.90$							
100	-0.47	0.00	0.00	0.00	3.28	0.00	0.00	0.00
200	-0.39	-0.01	0.00	0.00	2.02	0.30	0.00	0.00
500	-0.48	-0.02	0.00	0.00	1.42	0.25	0.00	0.00
1000	-0.51	-0.03	0.00	0.00	1.11	0.22	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.95$							
100	-0.35	0.00	0.00	0.00	2.85	0.00	0.00	0.00
200	-0.31	-0.01	0.00	0.00	1.80	0.30	0.00	0.00
500	-0.34	-0.01	0.00	0.00	1.24	0.16	0.00	0.00
1000	-0.35	-0.02	0.00	0.00	0.88	0.18	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 1.00$							
100	-0.16	0.00	0.00	0.00	2.01	0.00	0.00	0.00
200	-0.13	0.00	0.00	0.00	1.10	0.00	0.00	0.00
500	-0.15	0.00	0.00	0.00	1.01	0.07	0.00	0.00
1000	-0.13	0.00	0.00	0.00	0.57	0.06	0.00	0.00

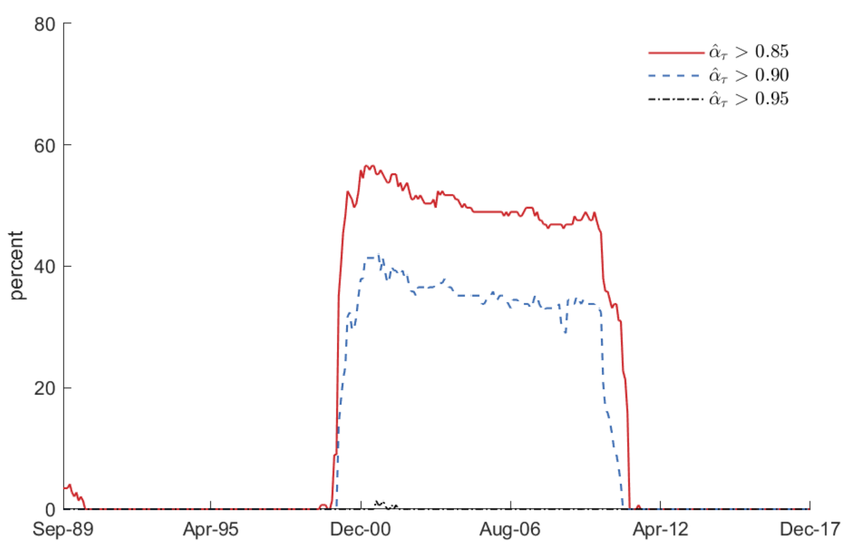
Notes: The parameters of the true DGP, (44), are generated as described in Table S3. We set $\alpha_{10} = 1$ and α_{20} in the range $[0.75, 1.00]$ with 0.05 increments. The misspecified model assumes the existence of factor f_1 only.

Figure 5: Comparison of the market factor strength estimates obtained from the original single factor CAPM ($\hat{\alpha}_{m,\tau}$) and the average estimates of its strength when computed using 145 two-factor asset pricing models ($\overline{\hat{\alpha}}_{m,\tau}$), over 10-year rolling windows



Notes: The market factor strength rolling estimates are computed using (7). The market factor strength average estimates produced from the 145 two-factor CAPMs are computed as $\overline{\hat{\alpha}}_{m,\tau} = (1/145)\sum_{s=1}^{145}(\hat{\alpha}_{s,\tau})$, for $\tau = 1, 2, \ldots, 340$ rolling windows.

Figure 6: Percentage of factors (out of 145) whose estimated strength ($\hat{\alpha}_{s,\tau}$), $\tau = 1, 2, \ldots, 340$ exceeds the thresholds of 0.85, 0.90 and 0.95, in each 10-year rolling window



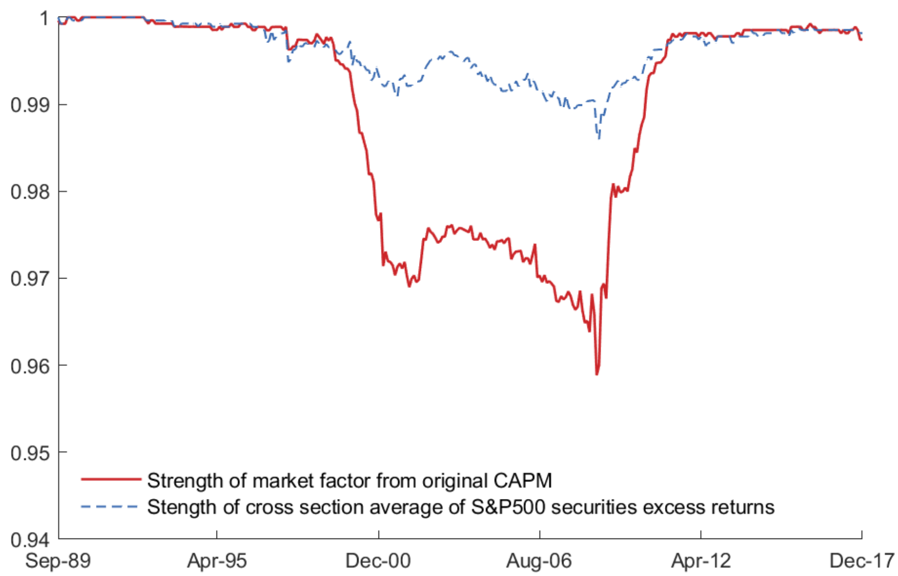
Notes: The 145 factor strength estimates, $\hat{\alpha}_{s,\tau}$, $s = 1, 2, \ldots, 145$, are computed using (7).

Table 8: Ranking of 65 factors in terms of the % of months their estimated strengths exceed the threshold of 0.90 during the full sample period of September 1989 to December 2017 and corresponding time averages of $\hat{\alpha}_{s,\tau}$, $s = 1, 2, \ldots, 65$, over different subsamples

	% of months when $\hat{\alpha}_{s,\tau} > 0.90$ over:	Time averages of $\hat{\alpha}_{s,\tau}$ over:			
Factor	Full sample	Full sample	September 1989 - August 1999	September 1999 - August 2009	September 2009 - December 2017
Market	100.0	0.990	0.999	0.974	0.997
Leverage	37.9	0.827	0.739	0.932	0.808
Sales to cash	37.9	0.817	0.716	0.936	0.793
Cash flow-to-price	37.9	0.832	0.765	0.933	0.792
Net debt-to-price	37.9	0.838	0.753	0.936	0.823
Earnings to price	37.9	0.811	0.743	0.935	0.745
Net payout yield	37.6	0.844	0.769	0.932	0.829
Years since first Compustat coverage	37.6	0.828	0.724	0.935	0.823
Cash flow to price ratio	37.6	0.818	0.737	0.934	0.775
Quick ratio	37.4	0.835	0.782	0.936	0.777
Altman's Z-score	37.4	0.828	0.740	0.931	0.808
Payout yield	37.1	0.851	0.785	0.932	0.831
Earnings volatility	37.1	0.852	0.779	0.936	0.840
Change in shares outstanding	37.1	0.805	0.671	0.932	0.815
Enterprise book-to-price	36.8	0.830	0.741	0.933	0.812
Cash holdings	36.8	0.826	0.740	0.935	0.797
Dividend to price	36.5	0.846	0.789	0.932	0.811
Depreciation / PP&E	36.5	0.851	0.813	0.930	0.801
Kaplan-Zingales Index	36.2	0.822	0.731	0.930	0.801
R&D-to-sales	36.2	0.815	0.731	0.923	0.786
Cash flow volatility	36.2	0.783	0.617	0.924	0.812
Accrual volatility	36.2	0.779	0.613	0.926	0.803
Current ratio	35.9	0.846	0.815	0.926	0.785
Idiosyncratic return volatility	35.6	0.851	0.799	0.923	0.828
Debt capacity/firm tangibility	35.6	0.829	0.735	0.920	0.832
Maximum daily return	35.3	0.838	0.764	0.927	0.821
Bid-ask spread	35.3	0.847	0.786	0.931	0.821
Cash productivity	35.3	0.819	0.751	0.911	0.789
Return volatility	34.7	0.844	0.786	0.922	0.820
Robust Minus Weak	34.7	0.773	0.694	0.910	0.705
Whited-Wu Index	34.7	0.781	0.697	0.913	0.724
New equity issue	34.7	0.756	0.620	0.912	0.732
Sales to price	34.7	0.832	0.768	0.919	0.804
High Minus Low	34.4	0.830	0.757	0.926	0.802
Vol. of liquidity (share turnover)	34.4	0.846	0.786	0.920	0.830
Market Beta	34.1	0.859	0.824	0.921	0.828
Zero trading days	34.1	0.855	0.808	0.918	0.836
Share turnover	34.1	0.857	0.815	0.917	0.834
Advertising Expense-to-market	34.1	0.810	0.707	0.914	0.809
Net equity finance	34.1	0.841	0.797	0.916	0.803
Asset turnover	34.1	0.788	0.643	0.911	0.815
Net external finance	32.1	0.827	0.781	0.900	0.793
Absolute accruals	31.8	0.818	0.750	0.903	0.799
Growth in long-term debt	31.5	0.767	0.678	0.902	0.711
Industry-adjusted book to market	30.9	0.810	0.771	0.901	0.748
Working capital accruals	30.6	0.812	0.748	0.900	0.783
HML Devil	30.3	0.820	0.747	0.905	0.805
Change in Net Financial Assets	29.4	0.697	0.581	0.907	0.583
Chg in Current Oper. Liabilities	28.2	0.773	0.710	0.904	0.690
Sin stocks	27.6	0.749	0.603	0.884	0.762
Sales to receivables	27.4	0.820	0.781	0.896	0.777
Employee growth rate	22.6	0.773	0.710	0.898	0.699
Net Operating Assets	16.8	0.778	0.664	0.900	0.767
HXZ Investment	13.2	0.797	0.739	0.892	0.753
Chg in Net Non-current Oper. Assets	8.2	0.791	0.729	0.886	0.753
Financial statements score	7.9	0.738	0.700	0.885	0.605
R&D Expense-to-market	7.6	0.804	0.770	0.883	0.751
R&D increase	5.3	0.742	0.676	0.873	0.664
Industry momentum	2.9	0.772	0.748	0.840	0.721
Abnormal Corporate Investment	2.9	0.674	0.497	0.866	0.654
Sales growth	2.4	0.761	0.706	0.876	0.690
Conservative Minus Aggressive	1.8	0.766	0.716	0.860	0.714
Momentum	1.2	0.755	0.715	0.793	0.758
Change in Short- term Investments	0.3	0.625	0.377	0.801	0.712
Return on net operating assets	0.3	0.764	0.645	0.877	0.773

Notes: Factor strength estimates, $\hat{\alpha}_{s,\tau}$, where $s = 1, 2, \ldots, 65$, are computed using (7) for 10-year rolling windows $\tau = 1, 2, \ldots, 340$. Remaining factors whose estimated strength resides below 0.9 throughout the sample period can be found in Table S20 of the online supplement.

Figure 7: Comparison of the market factor strength estimates obtained from the original single factor CAPM ($\hat{\alpha}_{m,\tau}$) and those from using the cross section average (CSA) of S&P500 securities' excess returns ($\hat{\alpha}_{csa,\tau}$), over 10-year rolling windows



Notes: The market factor and CSA of S&P500 securities' excess returns strength estimates over $\tau = 1, 2, \ldots, 340$ rolling windows are computed using (7).

Table 9: Strength estimates of the strongest unobserved factor using the cross section average (CSA) of the Stock and Watson (2012) dataset ($n = 187$ variables) and the corresponding exponent of cross section dependence (CSD)

	Q1 1988 - Q4 2007 ($T = 80$)			Q1 1988 - Q2 2019 ($T = 126$)		
	$\hat{\alpha}^*_{0.05}$	$\hat{\alpha}$	$\hat{\alpha}^*_{0.95}$	$\hat{\alpha}^*_{0.05}$	$\hat{\alpha}$	$\hat{\alpha}^*_{0.95}$
			$p = 0.10$			
Strength of CSA ($\delta = 1/4$)	0.962	0.964	0.966	0.928	0.930	0.933
Strength of CSA ($\delta = 1/2$)	0.957	0.958	0.959	0.918	0.920	0.922
Exponent of CSD	0.833	0.873	0.913	0.858	0.920	0.981
			$p = 0.05$			
Strength of CSA ($\delta = 1/4$)	0.962	0.964	0.966	0.927	0.929	0.931
Strength of CSA ($\delta = 1/2$)	0.957	0.958	0.959	0.912	0.914	0.915
Exponent of CSD	0.833	0.873	0.913	0.856	0.918	0.979

Notes: *90% confidence bands. In the computation of the strength of CSA, parameters p and δ are used when setting the critical value (6).
The exponent of CSD corresponds to the most robust estimator of cross-sectional dependence proposed in Bailey et al. (2016) and corrects for both serial correlation in the factors and weak cross-sectional dependence in the error terms.

Appendix A Proofs of Lemmas

Proof of Lemma 1

We have that

$$E\left(\hat{d}_{i,nT}\right) = \pi_{i,nT} = \Pr\left[|t_{iT}| > c_p(n)\right]$$
$$= \Phi\left(-c_p(n) + \sqrt{T}\theta_{iT}\right) + \Phi\left(-c_p(n) - \sqrt{T}\theta_{iT}\right),$$

and

$$\pi_{i,nT} = 1 - \Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right) + \Phi\left(-c_p(n) - \sqrt{T}\theta_{iT}\right) \tag{A.1}$$

where

$$\theta_{iT} = (\gamma_i/\sigma_i)\left(T^{-1}\mathbf{f}'\mathbf{M}_\tau\mathbf{f}\right)^{1/2}. \tag{A.2}$$

Then,

$$\sum_{i=1}^{n} E\left(\hat{d}_{i,nT}\right) = \sum_{i=1}^{n} \mathbf{1}(\gamma_i \neq 0)\left[1 - \Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right) + \Phi\left(-\sqrt{T}\theta_{iT} - c_p(n)\right)\right]$$
$$+ (n - n^{\alpha_0})\left[2\Phi\left(-c_p(n)\right)\right].$$

Note that

$$\left[\Phi\left(-c_p(n)\right)\right] = 1 - \left[\Phi\left(c_p(n)\right)\right] = 1 - \Phi\left[\Phi^{-1}\left(1 - \frac{p}{2n^\delta}\right)\right] \tag{A.3}$$
$$= 1 - \left(1 - \frac{p}{2n^\delta}\right) = \frac{p}{2n^\delta}.$$

Hence,

$$\sum_{i=1}^{n} E\left(\hat{d}_{i,nT}\right) = n^{\alpha_0} + \sum_{i=1}^{n} \mathbf{1}(\theta_{iT} \neq 0)\left[\Phi\left(-\sqrt{T}\theta_{iT} - c_p(n)\right) - \Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right)\right]$$
$$+ \frac{p\left(n - n^{\alpha_0}\right)}{n^\delta},$$

where θ_{iT} is defined by (A.2). Note also that

$$\Phi\left(-\sqrt{T}\theta_{iT} - c_p(n)\right) - \Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right)$$
$$= \left[1 - \Phi\left(\sqrt{T}\theta_{iT} + c_p(n)\right)\right] - \left[1 - \Phi\left(\sqrt{T}\theta_{iT} - c_p(n)\right)\right]$$
$$= \Phi\left(\sqrt{T}\theta_{iT} - c_p(n)\right) - \Phi\left(\sqrt{T}\theta_{iT} + c_p(n)\right).$$

Hence

$$\Phi\left(-\sqrt{T}\theta_{iT} - c_p(n)\right) - \Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right)$$
$$= \Phi\left(-\sqrt{T}\left|\theta_{iT}\right| - c_p(n)\right) - \Phi\left(-\sqrt{T}\left|\theta_{iT}\right| + c_p(n)\right). \tag{A.4}$$

Also since $c_p(n) > 0$, for small p and $\delta > 0$, then $\Phi\left(-\sqrt{T}\left|\theta_{iT}\right| - c_p(n)\right) < \Phi\left(-\sqrt{T}\left|\theta_{iT}\right| + c_p(n)\right)$, and we have

$$\sum_{i=1}^{n} \mathbf{1}(\theta_{iT} \neq 0)\left[\Phi\left(-\sqrt{T}\theta_{iT} - c_p(n)\right) - \Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right)\right]$$
$$= \sum_{i=1}^{n} \mathbf{1}(\theta_{iT} \neq 0)\left[\Phi\left(-\sqrt{T}\left|\theta_{iT}\right| - c_p(n)\right) - \Phi\left(-\sqrt{T}\left|\theta_{iT}\right| + c_p(n)\right)\right].$$

Suppose now that there exists T_0 such that for all $T > T_0$, and some i, $|\theta_{iT}| > 0$, we have $-\sqrt{T}\,|\theta_{iT}|+c_p(n) < 0$. Such a T_0 exists since $c_p(n)/\sqrt{T} \to 0$ as $n, T \to \infty$, jointly, for $\delta \geq 0$ - for a proof see result (a) in Lemma 2 of the supplement to Bailey et al. (2019). Also

$$\Phi\left(-\sqrt{T}\,|\theta_{iT}| + c_p(n)\right) \leq (1/2)\exp\left\{\frac{-1}{2}\left[\sqrt{T}\theta_{iT} - c_p(n)\right]^2\right\} = (1/2)\exp\left\{\frac{-T\theta_{iT}^2}{2}\left[1 - \frac{c_p(n)}{\sqrt{T}\theta_{iT}}\right]^2\right\}, \tag{A.5}$$

and

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\mathbf{1}(\gamma_i \neq 0)\left[\Phi\left(-\sqrt{T}\theta_{iT} - c_p(n)\right) - \Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right)\right]\right| \\
&\leq \sum_{i=1}^{n}\mathbf{1}(\theta_{iT} \neq 0)\left[\Phi\left(-\sqrt{T}\,|\theta_{iT}| - c_p(n)\right) + \Phi\left(-\sqrt{T}\,|\theta_{iT}| + c_p(n)\right)\right] \leq n^{\alpha_0}\sup_i \exp\left\{\frac{-T\theta_{iT}^2}{2}\left[1 - \frac{c_p(n)}{\sqrt{T}\,|\theta_{iT}|}\right]^2\right\}.
\end{aligned}$$

Overall,

$$B_{nT} = \frac{\sum_{i=1}^{n} E\left(\hat{d}_{i,nT}\right) - n^{\alpha_0}}{n^{\alpha_0}} = C_0 \sup_i \exp\left\{\frac{-T\theta_{iT}^2}{2}\left[1 - \frac{c_p(n)}{\sqrt{T}\,|\theta_{iT}|}\right]^2\right\} + \frac{p\,(n - n^{\alpha_0})}{n^{\delta+\alpha_0}}. \tag{A.6}$$

Proof of Lemma 2

Consider the first term of (9) and note that

$$A_{nT} = \frac{1}{n^{\alpha_0}}\sum_{i=1}^{n}\left[\hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)\right].$$

Under the assumption that u_{it} are cross-sectionally independently distributed, $z_{i,nT} = \hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)$ are uncorrelated across i and

$$Var\left(z_{i,nT}\right) = Var\left(\hat{d}_{i,nT}\right) = \pi_{i,nT}(1 - \pi_{i,nT}) \leq 1/2,$$

where $\pi_{i,nT}$ is defined by (A.1). Then

$$Var\left(A_{nT}\right) = \frac{1}{n^{2\alpha_0}}\sum_{i=1}^{n}\pi_{i,nT}(1 - \pi_{i,nT}).$$

Now, using (A.1), first we note that (using (A.4))

$$\begin{aligned}
|1 - \pi_{i,nT}| &= \left|\Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right) - \Phi\left(-c_p(n) - \sqrt{T}\theta_{iT}\right)\right| \\
&= \left|\Phi\left(-c_p(n) - \sqrt{T}\theta_{iT}\right) - \Phi\left(-\sqrt{T}\theta_{iT} + c_p(n)\right)\right| \\
&= \left|\Phi\left(-\sqrt{T}\,|\theta_{iT}| - c_p(n)\right) - \Phi\left(-\sqrt{T}\,|\theta_{iT}| + c_p(n)\right)\right| \\
&\leq \Phi\left(-\sqrt{T}\,|\theta_{iT}| - c_p(n)\right) + \Phi\left(-\sqrt{T}\,|\theta_{iT}| + c_p(n)\right) \leq 2\Phi\left(-\sqrt{T}\,|\theta_{iT}| + c_p(n)\right),
\end{aligned}$$

and hence using (A.5) we have

$$|1 - \pi_{i,nT}| = O\left[\exp\left(-C_1 T\right)\right], \text{ if } |\theta_{iT}| > 0,$$

and when $\theta_{iT}=0$, using (A.3), we have

$$\begin{aligned}1-\pi_{i,nT} &= \Phi\left[c_p(n)\right]-\Phi\left[-c_p(n)\right]\\ &= 1-2\Phi\left[-c_p(n)\right]=1-\frac{p}{n^{\delta}}.\end{aligned}$$

Overall,

$$Var\left(A_{nT}\right)=\frac{1}{n^{2\alpha_0}}\left\{\left(n-n^{\alpha_0}\right)\frac{p}{n^{\delta}}\left(1-\frac{p}{n^{\delta}}\right)+n^{\alpha_0}O\left[\exp\left(-C_2T\right)\right]\right\}. \tag{A.7}$$

Proof of Lemma 3

We note that $\frac{\sqrt{n}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{F}\boldsymbol{\gamma}_i}{\sqrt{T}}=O_p(1)$, $\frac{\sqrt{n}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{F}\mathbf{u}_i}{\sqrt{T}}=O_p(1)$, and $E\left[\mathbf{f}_t-E\left(\mathbf{f}_t\right)\right]\left[\mathbf{f}_t-E\left(\mathbf{f}_t\right)\right]'=\boldsymbol{I}_m$. Then, the orders of the four terms given by $\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F})\boldsymbol{\gamma}_i}{[\bar{\boldsymbol{\gamma}}'(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F})\bar{\boldsymbol{\gamma}}]^{1/2}}$, $\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{F}\gamma_i}{[\bar{\gamma}'(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F})\bar{\gamma}]^{1/2}}$, $\frac{T^{-1/2}\bar{\boldsymbol{\gamma}}'\mathbf{F}'\mathbf{M}_{\tau}\mathbf{u}_i}{[\bar{\gamma}'(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F})\bar{\gamma}]^{1/2}}$ and $\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{F}\mathbf{u}_i}{[\bar{\boldsymbol{\gamma}}'(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F})\bar{\boldsymbol{\gamma}}]^{1/2}}$, in the statement of the lemma, are as follows,

$$\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}=\ominus_p\left(\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'\boldsymbol{\gamma}_i}{\left(\bar{\boldsymbol{\gamma}}'\bar{\boldsymbol{\gamma}}\right)^{1/2}}\right), \tag{A.8}$$

$$\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{F}\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}=\frac{n^{-1/2}\left(\frac{\sqrt{n}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{F}\boldsymbol{\gamma}_i}{\sqrt{T}}\right)}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}=O_p\left(\frac{n^{-1/2}}{\left(\bar{\boldsymbol{\gamma}}'\bar{\boldsymbol{\gamma}}\right)^{1/2}}\right)=O_p\left(n^{1/2-\alpha_0}\right), \tag{A.9}$$

$$\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}=\frac{n^{-1/2}\left(\frac{\sqrt{n}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{u}_i}{\sqrt{T}}\right)}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}=O_p\left(n^{1/2-\alpha_0}\right). \tag{A.10}$$

Also, since by Assumption 1, $\mathbf{u}_i$ is distributed independently of $\boldsymbol{\gamma}_i$ and $\mathbf{F}$, we also have

$$E\left(\frac{T^{-1/2}\bar{\boldsymbol{\gamma}}'\mathbf{F}'\mathbf{M}_{\tau}\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}\right)=0,\ Var\left(\frac{T^{-1/2}\bar{\boldsymbol{\gamma}}'\mathbf{F}'\mathbf{M}_{\tau}\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}\left|\mathbf{F},\bar{\boldsymbol{\gamma}}\right.\right)=\sigma_i^2. \tag{A.11}$$

It readily follows that so long as $\alpha=\alpha_1>1/2$ then

$$\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_{\tau}\mathbf{F}\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}\rightarrow_d N(0,\sigma_i^2).$$

proving the first set of probability order results.

Next we provide a more refined analysis to obtain exponential probability inequalities for each of (A.8)-(A.11). We start with (A.8). First we handle the denominator. Let $\bar{\gamma}_{\alpha,j}=\frac{1}{n^{\alpha_{j0}}}\sum_{i=1}^{n}\gamma_{ij}$, $j=1,2,\ldots,m$,

$$\mathbf{F}\bar{\boldsymbol{\gamma}}=\frac{1}{n^{1-\alpha_0}}(\mathbf{f}_1\bar{\gamma}_{\alpha,1},\frac{1}{n^{\alpha_0-\alpha_{20}}}\mathbf{f}_2\bar{\gamma}_{\alpha,2},\ldots,\frac{1}{n^{\alpha_0-\alpha_{m0}}}\mathbf{f}_m\bar{\gamma}_{\alpha,m})=\frac{1}{n^{1-\alpha_0}}(\mathbf{f}_{\alpha,1},\frac{1}{n^{\alpha_0-\alpha_{20}}}\mathbf{f}_{\alpha,2},\ldots,\frac{1}{n^{\alpha_0-\alpha_{m0}}}\mathbf{f}_{\alpha,m})$$

where $\mathbf{f}_{\alpha,j}=(f_{1,\alpha,j},\ldots,f_{T,\alpha,j})'$. Note now that $f_{t,\alpha,j}$ are covariance stationary, martingale difference processes with non-zero, finite second moment, $\sigma^2_{f\gamma,j}$. Then by Lemma A9 of Chudik et al. (2018),

$$\Pr\left(\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}}>c_p(n)\right)\leq\Pr\left(\frac{n^{1-\alpha_1}\sqrt{T}\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_{\tau}\mathbf{F}\right)\boldsymbol{\gamma}_i}{\sigma_{f\gamma,1}}>c_p(n)\right)+\exp\left(-C_0T^{C_1}\right). \tag{A.12}$$

A similar result holds for (A.9)-(A.11). We proceed to analyse the first term on the RHS of (A.12). For some $0 < \pi < 1$, it follows that

$$\Pr\left(\frac{n^{1-\alpha_0}\sqrt{T}\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\boldsymbol{\gamma}_i}{\sigma_{f\gamma,1}} > c_p(n)\right) \tag{A.13}$$

$$\leq \sum_{r=1}^{m}\sum_{s=1}^{m}\Pr\left(\frac{T^{-1/2}n^{\alpha_{r0}-\alpha_0}\sum_{t=1}^{T}(f_{t,\alpha,r}-\bar{f}_{\alpha,r})(f_{t,s}\gamma_{is}-\bar{f}_s\gamma_{is})-E\left[(f_{t,\alpha,r}-\bar{f}_{\alpha,r})(f_{t,s}\gamma_{is}-\bar{f}_s\gamma_{is})\right]}{\sigma_{f\gamma,1}} > \pi c_p(n)\right) \tag{A.14}$$

$$+\sum_{r=1}^{m}\sum_{s=1}^{m}\Pr\left(\frac{[T^{1/2}n^{\alpha_{r0}-\alpha_0-1}\sum_{j=1}^{n}\gamma_{jr}\gamma_{is}}{\sigma_{f\gamma,1}} > (1-\pi)c_p(n)\right), \tag{A.15}$$

where $\bar{f}_{\alpha,r}$ and $\bar{f}_s$ are the sample averages of $f_{t,\alpha,r}$ and $f_{t,s}$ respectively. By Lemma A10 of Chudik et al. (2018),

$$\sum_{r=1}^{m}\sum_{s=1}^{m}\Pr\left(\frac{T^{-1/2}n^{\alpha_{r0}-\alpha_0}\sum_{t=1}^{T}(f_{t,\alpha,r}-\bar{f}_{\alpha,r}))(f_{t,s}\gamma_{is}-\bar{f}_s\gamma_{is})-E((f_{t,\alpha,r}-\bar{f}_{\alpha,r}))(f_{t,s}\gamma_{is}-\bar{f}_s\gamma_{is}))}{\sigma_{f\gamma,1}} > \pi c_p(n)\right)$$
$$\leq \frac{Cp}{n^{\delta}}.$$

For (A.15), we consider two cases - $\gamma_i = 0$, and $\gamma_i \neq 0$. If $\gamma_{i1} = 0$, (A.15) is bounded from below by $1-\exp\left[-C_0T^{C_1}\right]$, if $n^{\alpha_0-\alpha_{20}} = o(T^{1/2})$, and bounded from above by $\exp\left(-C_0T^{C_1}\right)$ if not. If $\gamma_i \neq 0$, (A.15) is bounded from below by $1-\exp\left(-C_0T^{C_1}\right)$, in any case.

Identical arguments can be used to show that

$$\Pr\left(\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} > c_p(n)\right) \leq \exp\left(-C_0T^{C_1}\right),$$

and

$$\Pr\left(\frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} > c_p(n)\right) \leq \exp\left(-C_0T^{C_1}\right).$$

Finally, and again using similar arguments,

$$\Pr\left(\frac{T^{-1/2}\bar{\boldsymbol{\gamma}}'\mathbf{F}'\mathbf{M}_\tau\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}} > c_p(n)\right) \leq \frac{Cp}{n^{\delta}},$$

completing the proof of the lemma.

Proof of Lemma 4

We proceed by considering $\bar{t}_{iT}$ under (27) and note that

$$\bar{\mathbf{x}} = \bar{c}\boldsymbol{\tau}_T + \mathbf{F}\bar{\boldsymbol{\gamma}} + \bar{\mathbf{u}}, \text{ and } \mathbf{x}_i = c_i\boldsymbol{\tau}_T + \mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i.$$

Then,

$$\bar{t}_{iT} = \frac{T^{-1/2}\left(\bar{\mathbf{x}}'\mathbf{M}_\tau\mathbf{x}_i\right)}{\hat{\sigma}_{iT}\left(T^{-1}\bar{\mathbf{x}}'\mathbf{M}_\tau\bar{\mathbf{x}}\right)^{1/2}} = \frac{T^{-1/2}(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})'\mathbf{M}_\tau\left(\mathbf{F}\boldsymbol{\gamma}_i+\mathbf{u}_i\right)}{\hat{\sigma}_{iT}\left[T^{-1}(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})'\mathbf{M}_\tau(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})\right]^{1/2}}, \tag{A.16}$$

and

$$\hat{\sigma}_{iT}^2 = T^{-1}\left(\mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i\right)' \mathbf{M}_{\bar{H}}\left(\mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i\right).$$

Consider first the denominator of (A.16) and note that

$$T^{-1}(\mathbf{F}\bar{\boldsymbol{\gamma}} + \bar{\mathbf{u}})'\mathbf{M}_\tau\left(\mathbf{F}\bar{\boldsymbol{\gamma}} + \bar{\mathbf{u}}\right) = \bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}} + 2T^{-1}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\bar{\boldsymbol{\gamma}} + T^{-1}\bar{\mathbf{u}}'\mathbf{M}_\tau\bar{\mathbf{u}}.$$

Under our assumptions, $\hat{\boldsymbol{\Sigma}}_f = T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}$ is a positive definite matrix and

$$\lambda_{\min}\left(\hat{\boldsymbol{\Sigma}}_f\right)\left(\bar{\boldsymbol{\gamma}}'\bar{\boldsymbol{\gamma}}\right) \leq \bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}} \leq \left(\bar{\boldsymbol{\gamma}}'\bar{\boldsymbol{\gamma}}\right)\lambda_{\max}\left(\hat{\boldsymbol{\Sigma}}_f\right).$$

Since $0 < \lambda_{\min}\left(\hat{\boldsymbol{\Sigma}}_f\right) < \lambda_{\max}\left(\hat{\boldsymbol{\Sigma}}_f\right) < C$, it follows that $\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}$ and $\bar{\boldsymbol{\gamma}}'\bar{\boldsymbol{\gamma}}$ have the same order in n. Recalling that $\alpha_0 > \alpha_{20}\ldots$,

$$\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}} = \ominus_p\left(n^{2(\alpha_0-1)}\right).$$

Also using results from Pesaran (2006) we have $T^{-1}\bar{\mathbf{u}}'\mathbf{M}_\tau\bar{\mathbf{u}} = O_p(n^{-1})$, and $T^{-1}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\bar{\boldsymbol{\gamma}} = O_p(n^{-1/2+\alpha_0-1})$. Therefore, overall

$$\left[T^{-1}(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})'\mathbf{M}_\tau(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})\right]^{1/2} = \left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}\left\{1 + O_p(n^{1/2-\alpha_0}) + O_p(n^{1-2\alpha_0})\right\}^{1/2}.$$

But since $a_0 > 1/2$ then

$$\left[T^{-1}(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})'\mathbf{M}_\tau(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})\right]^{1/2} = \left[\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\bar{\boldsymbol{\gamma}}\right]^{1/2}\left[1 + o_p(1)\right].$$

Using this result in (A.16) we now have

$$\begin{aligned}
\bar{t}_{iT} &= \frac{T^{-1/2}(\mathbf{F}\bar{\boldsymbol{\gamma}}+\bar{\mathbf{u}})'\mathbf{M}_\tau\left(\mathbf{F}\boldsymbol{\gamma}_i + \mathbf{u}_i\right)}{\hat{\sigma}_{iT}} \\
&= \frac{\frac{\sqrt{T}\bar{\boldsymbol{\gamma}}'\left(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F}\right)\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F})\bar{\boldsymbol{\gamma}}\right]^{1/2}} + \frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\boldsymbol{\gamma}_i}{\left[\bar{\boldsymbol{\gamma}}'(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F})\bar{\boldsymbol{\gamma}}\right]^{1/2}} + \frac{T^{-1/2}\bar{\boldsymbol{\gamma}}'\mathbf{F}'\mathbf{M}_\tau\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F})\bar{\boldsymbol{\gamma}}\right]^{1/2}} + \frac{T^{-1/2}\bar{\mathbf{u}}'\mathbf{M}_\tau\mathbf{F}\mathbf{u}_i}{\left[\bar{\boldsymbol{\gamma}}'(T^{-1}\mathbf{F}'\mathbf{M}_\tau\mathbf{F})\bar{\boldsymbol{\gamma}}\right]^{1/2}}}{\hat{\sigma}_{iT}}.
\end{aligned}$$

Then the result of the lemma follows by Lemma 3.

Appendix B Proofs of Theorems

Proof of Theorem 1

We abstract from the subscript j in what follows. We consider the following relations

$$\begin{aligned}
(\ln n)\left(\hat{\alpha} - \alpha_0\right) = \ln\left(\frac{\hat{D}_{nT}}{D_n^0}\right) &= \ln\left(1 + \frac{\hat{D}_{nT} - n^{\alpha_0}}{n^{\alpha_0}}\right) \\
&= \ln\left(1 + A_{nT} + B_{nT}\right) \\
&= A_{nT} + B_{nT} + O_p\left(A_{nT}^2\right) + O\left(B_{nT}^2\right) + O_p\left(A_{nT}B_{nT}\right) + \ldots,
\end{aligned}$$

where

$$\begin{aligned}
A_{nT} &= \frac{\sum_{i=1}^n\left[\hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)\right]}{n^{\alpha_0}}, \\
B_{nT} &= \frac{\sum_{i=1}^n E\left(\hat{d}_{i,nT}\right) - n^{\alpha_0}}{n^{\alpha_0}}, \text{ with } \hat{d}_{i,nT} = \mathbf{1}\left[|t_{iT}| > c_p(n)\right].
\end{aligned}$$

Note that $E\left(\hat{d}_{i,nT}\right) = \pi_{i,nT} = \Pr\left[|t_{iT}| > c_p(n)\right]$. Then, we wish to determine

$$B_{nT} = \frac{\sum_{i=1}^{n} \Pr\left[|t_{iT}| > c_p(n)\right] - n^{\alpha_0}}{n^{\alpha_0}} = \frac{\sum_{i=1}^{[n^{\alpha_0}]} \Pr\left[|t_{iT}| > c_p(n)|\gamma_i \neq 0\right] - n^{\alpha_0}}{n^{\alpha_0}} + \frac{\sum_{i=[n^{\alpha_0}]+1}^{n} \Pr\left[|t_{iT}| > c_p(n)|\gamma_i = 0\right]}{n^{\alpha_0}}.$$

Under regularity conditions and by Lemma A.10 of Chudik et al. (2018),

$$\Pr\left[|t_{iT}| > c_p(n)|\gamma_i \neq 0\right] > 1 - O\left[\exp(-T^C)\right], \text{ for some } C > 0.$$

So

$$\frac{\sum_{i=1}^{[n^{\alpha_0}]} \Pr\left[|t_{iT}| > c_p(n)|\gamma_i \neq 0\right] - n^{\alpha_0}}{n^{\alpha_0}} = O\left[\exp(-T^C)\right].$$

Again by Lemma A.10 of Chudik et al. (2018),

$$\Pr\left[|t_{iT}| > c_p(n)|\gamma_i = 0\right] \leq \frac{Cp}{n^{\delta}}.$$

So, for some $C > 0$,

$$\frac{\sum_{i=[n^{\alpha_0}]+1}^{n} \Pr\left[|t_{iT}| > c_p(n)|\gamma_i = 0\right]}{n^{\alpha_0}} \leq \frac{Cp\left(n - n^{\alpha_0}\right)}{n^{\delta+\alpha_0}} = O\left(n^{1-\delta-\alpha_0}\right).$$

Overall,

$$B_{nT} = O\left(n^{1-\delta-\alpha_0}\right) + O\left[\exp(-T^C)\right].$$

Next, note that

$$A_{nT} = \frac{1}{n^{\alpha_0}} \sum_{i=1}^{n} \left[\hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)\right].$$

Under the assumption that u_{it} are cross-sectionally independently distributed, a martingale difference central limit theorem holds for $z_{i,nT} = \hat{d}_{i,nT} - E\left(\hat{d}_{i,nT}\right)$ and further

$$Var\left(z_{i,nT}\right) = Var\left(\hat{d}_{i,nT}\right) = \pi_{i,nT}(1 - \pi_{i,nT}).$$

Then,

$$Var\left(A_{nT}\right) = \frac{1}{n^{2\alpha_0}} \sum_{i=1}^{n} \pi_{i,nT}(1 - \pi_{i,nT}) \leq \frac{1}{n^{2\alpha_0}} \sum_{i=1}^{n} \pi_{i,nT} = \frac{1}{n^{2\alpha_0}} \sum_{i=1}^{[n^{\alpha_0}]} \pi_{i,nT} + \frac{1}{n^{2\alpha_0}} \sum_{i=[n^{\alpha_0}]+1}^{n} \pi_{i,nT} = O\left[\exp(-T^C)\right] + O\left(n^{1-\delta-2\alpha_0}\right).$$

So, $A_{nT} = O_p\left(n^{1/2-\delta/2-\alpha_0}\right)$, and further $\psi_n(\alpha_0)^{-1/2} A_{nT} \to_d N(0, C)$, for some $C < 1$, where $\psi_n(\alpha_0) = p\left(n - n^{a_0}\right) n^{-\delta-2\alpha_0}\left(1 - \frac{p}{n^{\delta}}\right)$.

Proof of Theorem 2

To prove this theorem it is sufficient to retrace the proof of Theorem 1 using

$$\Pr\left[|\bar{t}_{iT}| > c_p(n)|\gamma_i \neq 0\right] > 1 - O\left[\exp(-T^C)\right], \text{ for some } C > 0, \tag{B.17}$$

and

$$\Pr\left[|\bar{t}_{iT}| > c_p(n)|\gamma_i = 0\right] \leq \frac{Cp}{n^{\delta}}. \tag{B.18}$$

Both (B.17) and (B.18) follow from Lemmas 3 and 4, proving the result.

Online Supplement
for

Measurement of Factor Strength: Theory and Practice

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Introduction

This online supplement is composed of two subsections which provide additional Monte Carlo and empirical results.

Additional Monte Carlo results

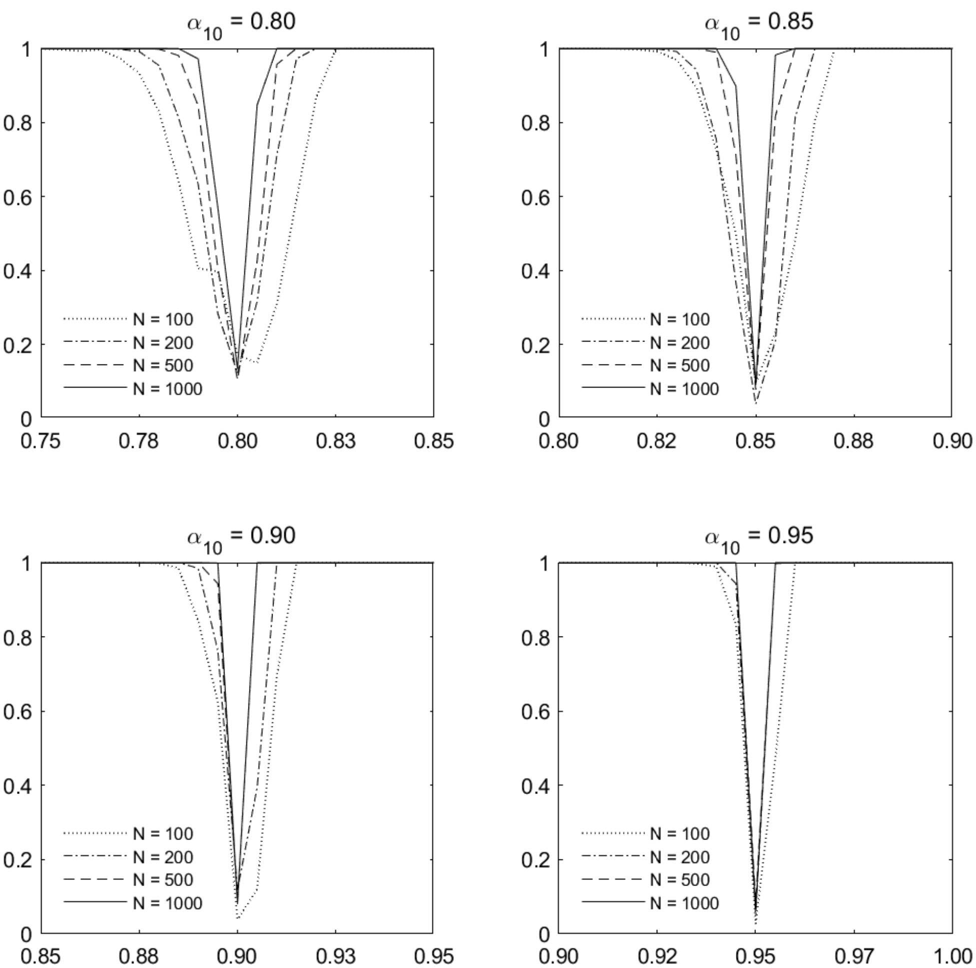
The Monte Carlo results provided in the tables and plots below are based on the designs set out in Section 5 of the paper.

Table S1: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.75

	Bias (×100)				RMSE (×100)				Size (×100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10}=0.75$, $\alpha_{20}=0.75$											
100	1.18	1.19	1.08	1.03	1.56	1.56	1.42	1.39	3.35	3.85	2.35	2.05
200	1.43	1.36	1.32	1.31	1.59	1.51	1.47	1.46	9.30	8.15	6.10	6.25
500	1.31	1.24	1.16	1.14	1.39	1.31	1.23	1.21	13.65	11.00	8.25	7.70
1000	1.26	1.19	1.13	1.11	1.31	1.23	1.17	1.15	16.35	11.05	7.95	6.90
	$\alpha_{10}=0.80$, $\alpha_{20}=0.75$											
100	0.71	0.69	0.62	0.61	1.04	1.02	0.95	0.93	18.30	18.65	17.10	17.10
200	0.96	0.91	0.88	0.87	1.09	1.04	1.01	0.99	13.15	11.20	10.10	9.45
500	0.92	0.87	0.82	0.80	0.98	0.92	0.87	0.86	13.20	9.25	7.05	6.40
1000	0.86	0.81	0.77	0.75	0.89	0.84	0.80	0.78	17.40	12.45	9.15	8.15
	$\alpha_{10}=0.85$, $\alpha_{20}=0.75$											
100	0.71	0.69	0.64	0.61	0.91	0.89	0.83	0.80	11.65	11.10	8.00	6.70
200	0.61	0.58	0.55	0.55	0.71	0.68	0.65	0.65	5.35	4.00	3.20	3.60
500	0.53	0.50	0.46	0.46	0.58	0.54	0.51	0.50	12.00	8.60	6.95	7.80
1000	0.50	0.47	0.45	0.44	0.52	0.50	0.47	0.46	9.95	8.05	5.75	5.70
	$\alpha_{10}=0.90$, $\alpha_{20}=0.75$											
100	0.40	0.40	0.38	0.36	0.56	0.55	0.53	0.51	4.80	4.00	3.25	2.70
200	0.27	0.26	0.25	0.24	0.38	0.36	0.34	0.33	13.80	12.60	12.10	12.90
500	0.30	0.29	0.27	0.26	0.33	0.32	0.30	0.29	8.80	8.40	6.10	6.00
1000	0.28	0.27	0.26	0.25	0.30	0.29	0.27	0.27	11.80	8.65	5.90	6.60
	$\alpha_{10}=0.95$, $\alpha_{20}=0.75$											
100	0.07	0.08	0.07	0.06	0.25	0.24	0.23	0.23	5.25	3.55	2.60	2.50
200	0.11	0.11	0.11	0.10	0.18	0.18	0.17	0.17	6.45	4.65	4.10	4.00
500	0.12	0.12	0.11	0.10	0.14	0.14	0.13	0.13	10.95	9.25	8.10	7.70
1000	0.10	0.10	0.09	0.09	0.11	0.11	0.10	0.10	9.20	6.50	5.30	4.15
	$\alpha_{10}=1.00$, $\alpha_{20}=0.75$											
100	-0.01	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) of the main paper are generated as follows: for unit specific effects, $c_i \sim IIDN(0,1)$, for $i = 1, 2, \ldots, n$. The factors, (f_{1t}, f_{2t}), are multivariate normal with variances $\sigma^2_{f_1} = \sigma^2_{f2} = 1$ and correlation given by $\rho_{12} = corr(f_1, f_2) = 0.3$. Each factor assumes an autoregressive process with correlation coefficients $\rho_{f_j} = 0.5$, $j = 1, 2$. The factor loadings are generated as $v_{ij} \sim IIDU(\mu_{v_j} - 0.2, \mu_{v_j} + 0.2)$, for $[n^{\alpha_{j0}}]$ units, $j = 1, 2$, respectively, and zero otherwise. We set $\mu_{v_1} = \mu_{v_2} = 0.71$. Both α_{10} and α_{20} range between $[0.75, 1.00]$ with 0.05 increments. The innovations u_{it} are Gaussian, such that $u_{it} \sim IIDN(0, \sigma^2_i)$, with $\sigma^2_i \sim IID(1 + \chi^2_{2,i})/3$, for $i = 1, 2, \ldots, n$. In the computation of $\hat{\alpha}_j$, $j = 1, 2$, we use $p = 0.10$ and $\delta = 1/4$ when setting the critical value. Size is computed under H_0: $\alpha_j=\alpha_{j0}$, for $j = 1, 2$, using a two-sided alternative. When $\alpha_{10} = 1.00$, our estimator is ultra consistent, hence size results for this case are not meaningful. The number of replications is set to $R = 2000$.

Figure S1: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.75, $n = 100, 200, 500, 1000$ and $T = 200$



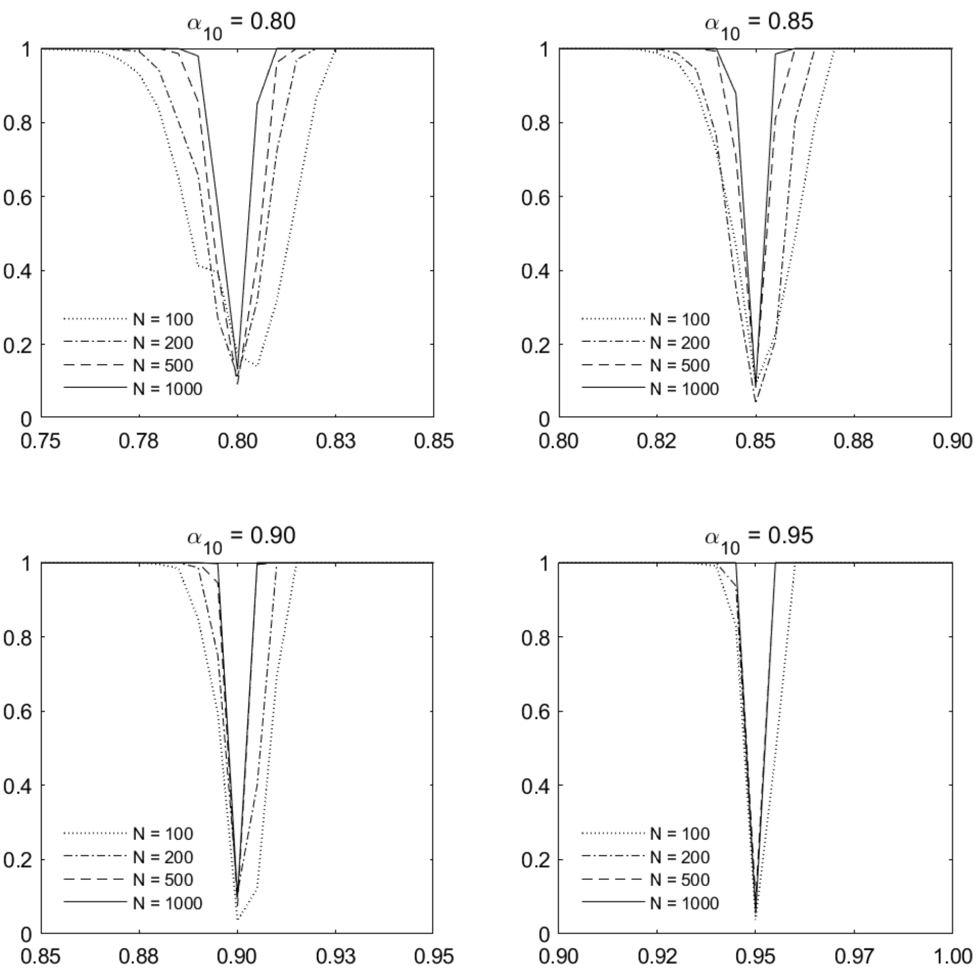
Notes: See the notes to Table S1 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S2: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.80

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.80$											
100	1.20	1.18	1.08	1.03	1.58	1.54	1.44	1.39	4.55	3.45	2.05	2.00
200	1.45	1.36	1.34	1.30	1.61	1.52	1.49	1.46	10.30	8.10	6.85	7.05
500	1.31	1.24	1.15	1.14	1.39	1.31	1.22	1.22	14.55	10.50	8.90	8.75
1000	1.26	1.19	1.13	1.12	1.31	1.24	1.17	1.15	16.30	12.35	7.90	7.35
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.80$											
100	0.72	0.71	0.63	0.61	1.06	1.04	0.95	0.94	19.05	18.70	17.30	17.70
200	0.96	0.91	0.89	0.87	1.10	1.04	1.01	1.00	12.95	10.60	9.00	9.45
500	0.91	0.87	0.82	0.80	0.97	0.92	0.87	0.86	12.00	9.60	6.70	6.80
1000	0.85	0.81	0.77	0.76	0.88	0.84	0.80	0.78	17.05	12.00	8.15	7.95
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.80$											
100	0.69	0.71	0.64	0.63	0.89	0.90	0.82	0.82	10.80	10.60	6.85	7.60
200	0.60	0.57	0.55	0.53	0.70	0.67	0.65	0.64	5.10	4.40	3.10	2.90
500	0.53	0.50	0.46	0.46	0.58	0.54	0.51	0.50	10.55	8.50	7.55	7.80
1000	0.50	0.48	0.45	0.44	0.52	0.50	0.47	0.46	10.90	7.75	5.25	4.75
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.80$											
100	0.40	0.40	0.37	0.36	0.55	0.56	0.51	0.50	4.80	4.80	2.75	2.60
200	0.28	0.26	0.24	0.25	0.38	0.35	0.33	0.34	13.45	11.70	12.00	11.35
500	0.30	0.28	0.26	0.26	0.33	0.31	0.30	0.30	9.85	7.85	6.85	7.00
1000	0.28	0.27	0.26	0.26	0.30	0.29	0.27	0.27	11.80	8.15	6.50	6.35
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.80$											
100	0.08	0.08	0.07	0.07	0.25	0.25	0.23	0.23	5.95	4.00	2.90	2.70
200	0.12	0.11	0.10	0.10	0.19	0.18	0.18	0.17	6.95	4.65	4.90	3.85
500	0.12	0.12	0.11	0.11	0.14	0.14	0.13	0.13	11.45	8.60	7.65	7.30
1000	0.10	0.10	0.09	0.09	0.11	0.11	0.10	0.10	7.90	5.45	5.95	5.40
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.80$											
100	-0.01	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table S1.

Figure S2: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.80, $n = 100, 200, 500, 1000$ and $T = 200$



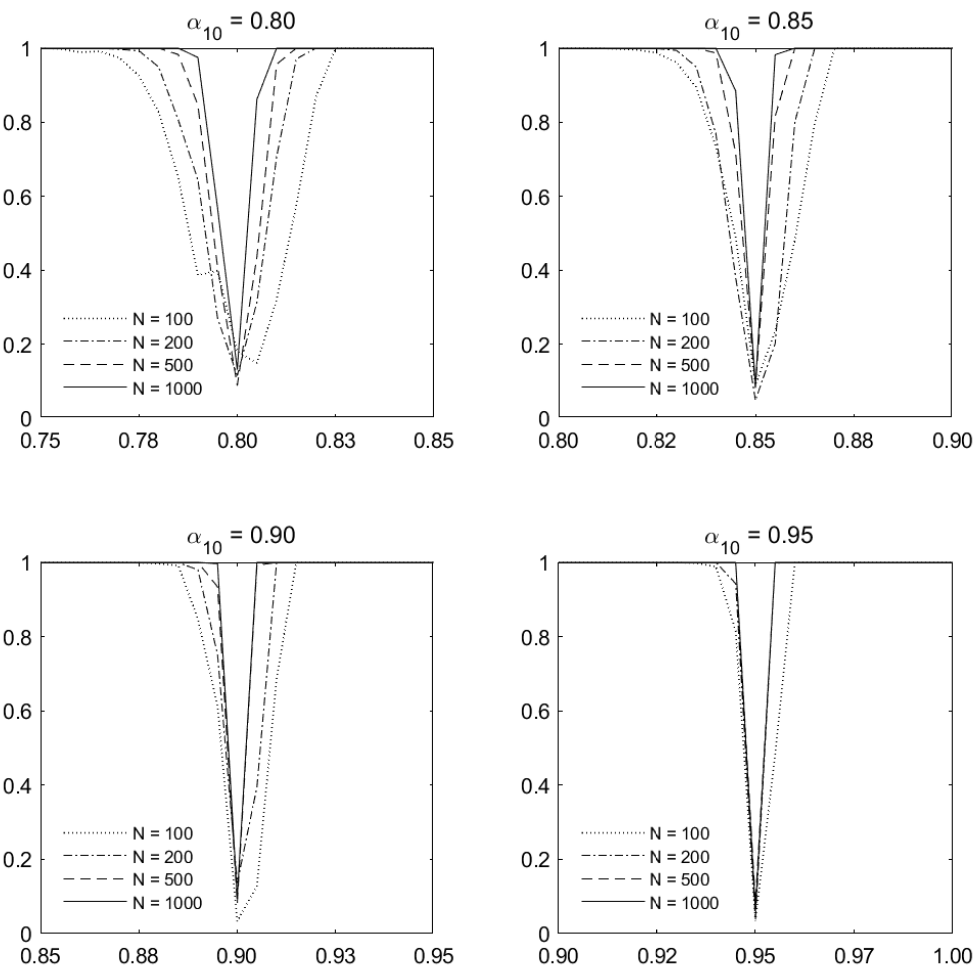
Notes: See the notes to Table S1 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S3: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.85

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.85$											
100	1.20	1.17	1.07	1.02	1.59	1.53	1.44	1.38	4.85	3.35	2.70	2.30
200	1.44	1.37	1.33	1.31	1.60	1.52	1.48	1.46	9.50	7.70	7.45	6.65
500	1.31	1.23	1.16	1.13	1.39	1.30	1.22	1.21	13.95	11.30	8.10	9.20
1000	1.26	1.20	1.13	1.11	1.31	1.24	1.17	1.15	16.50	11.75	8.30	7.15
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.85$											
100	0.70	0.70	0.62	0.58	1.04	1.03	0.95	0.91	18.20	17.80	16.90	18.00
200	0.96	0.90	0.87	0.86	1.09	1.03	0.99	0.98	14.20	10.65	9.15	9.30
500	0.91	0.87	0.82	0.80	0.97	0.92	0.87	0.86	13.10	9.10	7.20	6.70
1000	0.85	0.81	0.77	0.76	0.88	0.84	0.79	0.78	18.10	13.00	7.95	7.65
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.85$											
100	0.69	0.69	0.64	0.63	0.89	0.89	0.82	0.81	11.30	10.65	7.85	7.70
200	0.61	0.57	0.57	0.54	0.72	0.68	0.67	0.65	6.45	4.20	3.35	3.55
500	0.53	0.50	0.47	0.46	0.58	0.55	0.51	0.50	12.30	9.60	6.60	7.95
1000	0.49	0.47	0.45	0.44	0.52	0.50	0.47	0.46	10.95	8.05	5.95	5.10
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.85$											
100	0.41	0.40	0.40	0.37	0.56	0.54	0.54	0.51	5.20	3.20	3.30	2.60
200	0.27	0.26	0.25	0.24	0.37	0.36	0.35	0.34	13.55	13.15	12.60	12.60
500	0.30	0.29	0.27	0.26	0.33	0.32	0.30	0.30	9.70	8.45	7.10	7.85
1000	0.28	0.28	0.26	0.26	0.30	0.29	0.28	0.27	10.75	9.45	6.65	6.00
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.85$											
100	0.07	0.08	0.07	0.07	0.26	0.24	0.23	0.23	6.20	3.95	2.80	3.05
200	0.11	0.11	0.11	0.10	0.19	0.18	0.17	0.17	8.15	5.50	4.00	4.10
500	0.11	0.12	0.11	0.11	0.14	0.14	0.13	0.13	13.20	8.05	7.45	7.70
1000	0.09	0.10	0.09	0.09	0.10	0.11	0.10	0.10	9.80	6.70	5.30	5.10
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.85$											
100	-0.01	0.00	0.00	0.00	0.06	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.04	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table S1.

Figure S3: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.85, $n = 100, 200, 500, 1000$ and $T = 200$



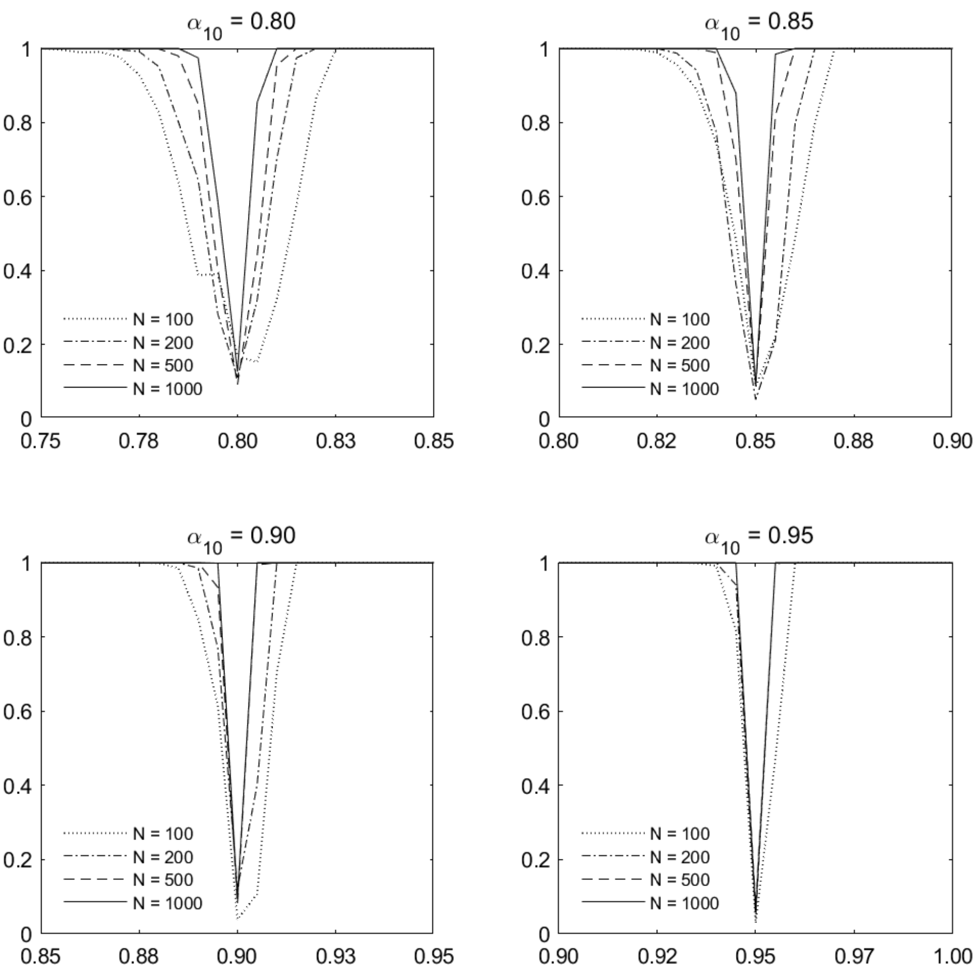
Notes: See the notes to Table S1 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S4: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.90

	Bias (×100)				RMSE (×100)				Size (×100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.90$											
100	1.19	1.18	1.05	1.02	1.58	1.54	1.41	1.39	4.55	3.70	2.30	2.45
200	1.44	1.36	1.32	1.30	1.60	1.51	1.47	1.45	9.25	8.20	6.75	6.50
500	1.31	1.24	1.15	1.14	1.39	1.31	1.22	1.21	14.35	11.20	7.50	8.90
1000	1.27	1.19	1.13	1.11	1.31	1.23	1.17	1.15	16.15	11.40	7.50	6.75
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.90$											
100	0.71	0.69	0.61	0.61	1.05	1.02	0.95	0.94	19.15	18.80	18.55	18.85
200	0.95	0.90	0.88	0.87	1.08	1.03	1.01	1.00	12.75	10.55	9.60	11.55
500	0.92	0.87	0.81	0.81	0.98	0.92	0.86	0.86	13.50	9.20	6.95	6.60
1000	0.86	0.81	0.77	0.76	0.89	0.84	0.80	0.79	18.35	12.70	8.90	8.60
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.90$											
100	0.68	0.69	0.64	0.61	0.89	0.88	0.83	0.80	11.10	9.50	8.30	6.30
200	0.61	0.57	0.55	0.54	0.72	0.68	0.65	0.64	6.65	3.90	2.95	2.50
500	0.53	0.50	0.47	0.45	0.57	0.54	0.51	0.50	10.35	9.60	7.00	7.35
1000	0.50	0.48	0.45	0.44	0.52	0.50	0.47	0.46	11.15	7.85	5.80	5.20
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.90$											
100	0.41	0.40	0.38	0.37	0.57	0.55	0.53	0.52	5.80	3.50	3.40	3.45
200	0.28	0.26	0.24	0.24	0.38	0.35	0.33	0.33	13.15	12.70	12.15	12.55
500	0.30	0.29	0.27	0.27	0.33	0.32	0.30	0.30	10.60	7.95	6.75	7.05
1000	0.28	0.27	0.26	0.25	0.30	0.29	0.27	0.27	11.75	8.70	7.70	5.50
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.90$											
100	0.08	0.08	0.07	0.07	0.25	0.24	0.24	0.23	5.20	3.70	3.25	3.35
200	0.11	0.11	0.10	0.10	0.19	0.18	0.17	0.17	7.85	5.05	4.20	3.95
500	0.11	0.12	0.11	0.11	0.14	0.14	0.13	0.13	12.75	7.90	8.60	7.40
1000	0.10	0.10	0.09	0.09	0.11	0.11	0.10	0.10	9.00	5.25	3.90	4.80
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.90$											
100	-0.01	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table S1.

Figure S4: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.90, $n = 100, 200, 500, 1000$ and $T = 200$



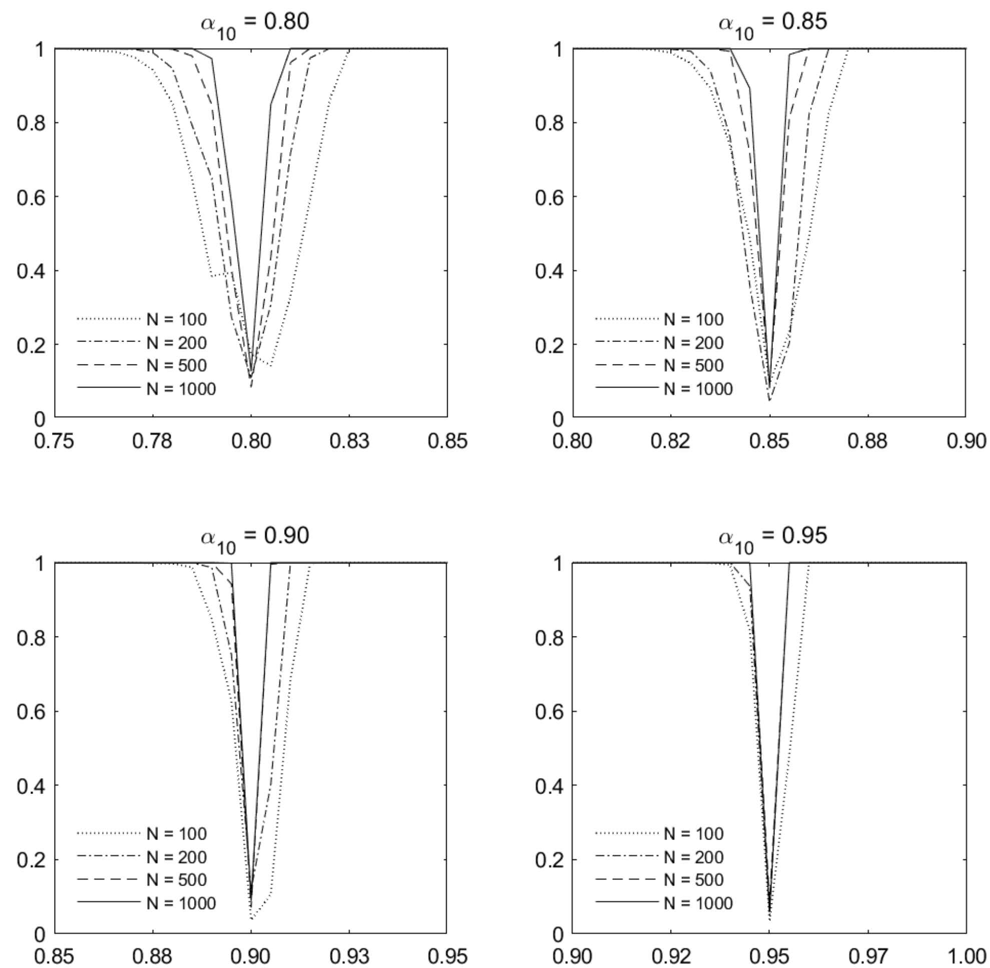
Notes: See the notes to Table S1 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S5: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.95

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.95$											
100	1.19	1.17	1.06	1.01	1.56	1.53	1.42	1.37	3.75	3.25	1.95	2.00
200	1.45	1.35	1.32	1.30	1.60	1.51	1.47	1.45	10.55	8.10	6.70	6.65
500	1.31	1.24	1.15	1.13	1.39	1.31	1.22	1.21	14.45	10.95	7.95	8.50
1000	1.26	1.20	1.13	1.11	1.31	1.24	1.17	1.15	16.70	11.35	7.40	6.45
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.95$											
100	0.71	0.70	0.61	0.60	1.06	1.04	0.94	0.93	18.20	19.50	18.35	17.30
200	0.96	0.92	0.88	0.87	1.09	1.05	1.01	0.99	13.10	10.95	9.85	9.00
500	0.91	0.87	0.81	0.81	0.97	0.93	0.86	0.86	12.90	9.40	6.90	7.40
1000	0.85	0.81	0.77	0.75	0.89	0.84	0.79	0.78	17.40	11.95	8.35	8.05
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.95$											
100	0.71	0.69	0.64	0.62	0.91	0.88	0.82	0.80	11.40	9.40	7.95	7.70
200	0.60	0.57	0.56	0.54	0.71	0.67	0.66	0.64	6.05	3.60	3.45	2.90
500	0.53	0.50	0.47	0.45	0.58	0.54	0.51	0.50	10.85	8.10	8.05	6.90
1000	0.50	0.48	0.45	0.44	0.52	0.50	0.47	0.46	10.75	9.00	5.45	5.65
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.95$											
100	0.40	0.41	0.38	0.36	0.55	0.55	0.53	0.51	5.10	3.80	3.50	2.60
200	0.28	0.25	0.25	0.24	0.37	0.35	0.34	0.34	12.40	11.70	11.65	12.65
500	0.30	0.29	0.27	0.26	0.34	0.31	0.30	0.29	11.85	7.20	6.90	7.40
1000	0.28	0.28	0.26	0.26	0.30	0.29	0.27	0.27	11.55	9.95	6.95	6.85
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.95$											
100	0.07	0.08	0.07	0.06	0.24	0.25	0.23	0.23	4.35	4.35	2.55	2.40
200	0.11	0.11	0.10	0.10	0.18	0.18	0.17	0.17	7.55	4.85	4.05	3.60
500	0.12	0.12	0.11	0.11	0.14	0.14	0.13	0.13	11.05	7.60	7.00	8.05
1000	0.09	0.10	0.09	0.09	0.11	0.11	0.10	0.10	9.05	6.55	4.60	5.40
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.95$											
100	-0.01	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table S1.

Figure S5: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 0.95, $n = 100, 200, 500, 1000$ and $T = 200$



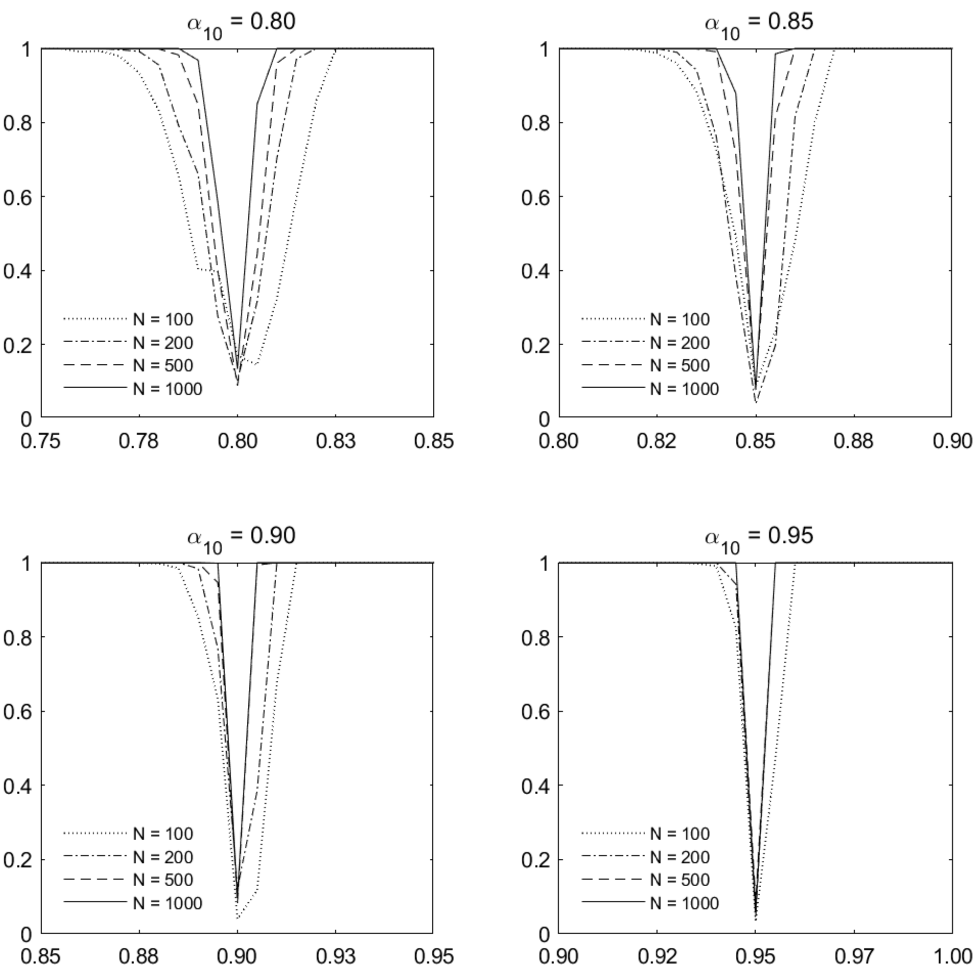
Notes: See the notes to Table S1 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S6: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 1.00

	Bias (×100)				RMSE (×100)				Size (×100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 1.00$											
100	1.20	1.18	1.06	1.03	1.58	1.56	1.43	1.38	4.15	3.90	2.55	1.65
200	1.45	1.35	1.33	1.31	1.61	1.51	1.48	1.47	9.90	8.30	6.35	6.45
500	1.31	1.24	1.16	1.14	1.39	1.31	1.23	1.21	13.50	10.95	7.40	8.35
1000	1.27	1.20	1.13	1.11	1.31	1.24	1.17	1.15	16.45	11.40	7.05	6.65
	$\alpha_{10} = 0.80$, $\alpha_{20} = 1.00$											
100	0.70	0.68	0.61	0.60	1.06	1.01	0.96	0.95	20.15	17.45	18.80	19.25
200	0.96	0.91	0.89	0.86	1.08	1.04	1.01	0.98	12.30	11.00	10.75	9.65
500	0.92	0.87	0.82	0.81	0.98	0.92	0.87	0.86	12.80	8.85	7.35	7.45
1000	0.85	0.81	0.77	0.75	0.89	0.84	0.79	0.78	17.45	12.30	8.90	7.90
	$\alpha_{10} = 0.85$, $\alpha_{20} = 1.00$											
100	0.70	0.69	0.65	0.63	0.90	0.87	0.83	0.81	11.45	9.10	8.20	7.25
200	0.62	0.57	0.56	0.54	0.72	0.67	0.65	0.64	5.70	3.90	3.50	3.00
500	0.54	0.50	0.47	0.46	0.58	0.54	0.51	0.50	10.60	8.90	7.55	7.90
1000	0.50	0.48	0.45	0.44	0.52	0.50	0.47	0.46	10.20	7.90	5.15	4.85
	$\alpha_{10} = 0.90$, $\alpha_{20} = 1.00$											
100	0.41	0.40	0.37	0.36	0.56	0.55	0.51	0.50	4.65	4.00	2.80	2.30
200	0.28	0.26	0.25	0.24	0.38	0.36	0.35	0.33	12.40	12.00	13.05	12.15
500	0.30	0.28	0.27	0.26	0.33	0.32	0.30	0.29	10.15	8.50	6.10	6.95
1000	0.28	0.27	0.26	0.26	0.30	0.29	0.28	0.27	12.65	9.45	6.50	6.55
	$\alpha_{10} = 0.95$, $\alpha_{20} = 1.00$											
100	0.08	0.07	0.06	0.06	0.25	0.24	0.22	0.22	6.00	3.15	2.30	2.50
200	0.10	0.11	0.10	0.10	0.18	0.18	0.17	0.17	6.95	4.65	4.20	3.35
500	0.12	0.12	0.11	0.11	0.14	0.14	0.13	0.13	11.25	8.95	7.40	7.80
1000	0.10	0.10	0.09	0.09	0.11	0.11	0.10	0.10	9.55	6.30	4.75	5.15
	$\alpha_{10} = 1.00$, $\alpha_{20} = 1.00$											
100	-0.01	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table S1.

Figure S6: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2A (two observed factors - Gaussian errors), when the strength of the second factor is set to 1.00, $n = 100, 200, 500, 1000$ and $T = 200$



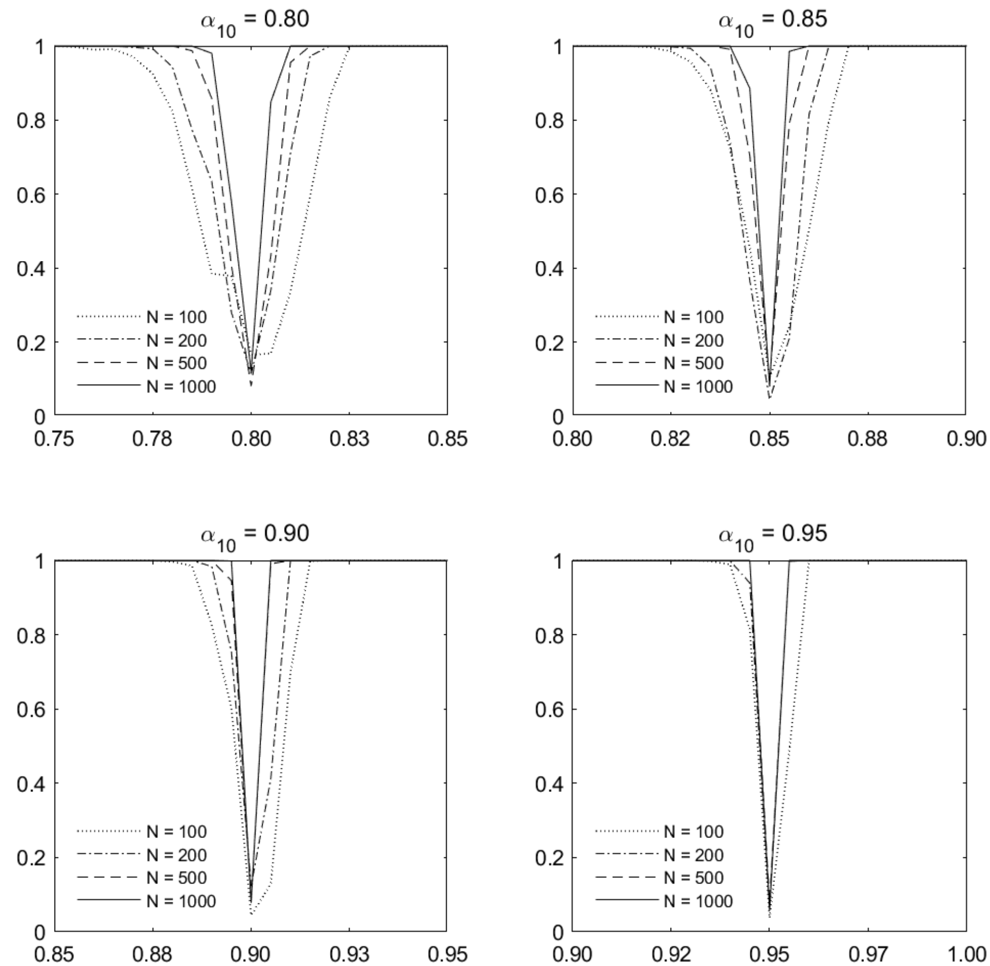
Notes: See the notes to Table S1 for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S7: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.75

	Bias (×100)				RMSE (×100)				Size (×100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.75$											
100	1.16	1.13	1.08	1.02	1.53	1.51	1.46	1.39	4.10	3.55	3.00	2.45
200	1.45	1.39	1.29	1.30	1.61	1.54	1.45	1.46	10.10	8.00	7.05	7.20
500	1.29	1.23	1.17	1.14	1.37	1.31	1.24	1.21	12.65	9.75	8.05	6.85
1000	1.26	1.19	1.12	1.11	1.30	1.23	1.16	1.15	16.55	10.25	7.45	6.85
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.75$											
100	0.68	0.67	0.61	0.59	1.03	1.01	0.96	0.93	19.85	19.00	19.55	18.15
200	0.95	0.92	0.86	0.86	1.09	1.04	0.99	0.98	13.85	10.80	10.45	10.15
500	0.90	0.86	0.82	0.81	0.96	0.92	0.88	0.86	11.80	8.80	7.25	6.05
1000	0.86	0.81	0.76	0.76	0.89	0.83	0.79	0.78	17.80	12.05	8.45	8.50
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.75$											
100	0.68	0.68	0.65	0.62	0.88	0.87	0.84	0.82	10.95	10.15	8.10	8.65
200	0.60	0.59	0.54	0.55	0.71	0.69	0.64	0.65	5.95	3.75	3.05	3.45
500	0.52	0.50	0.47	0.46	0.56	0.54	0.51	0.50	10.65	8.65	6.80	7.00
1000	0.49	0.47	0.44	0.44	0.52	0.49	0.46	0.46	10.30	7.75	5.35	5.25
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.75$											
100	0.39	0.40	0.38	0.36	0.55	0.55	0.53	0.51	5.05	3.60	2.95	2.60
200	0.27	0.26	0.23	0.24	0.38	0.35	0.33	0.34	15.30	10.70	12.90	12.90
500	0.29	0.28	0.27	0.26	0.32	0.31	0.30	0.29	11.20	7.65	6.80	7.35
1000	0.28	0.27	0.26	0.26	0.29	0.29	0.27	0.27	13.00	8.20	7.25	6.10
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.75$											
100	0.06	0.07	0.06	0.06	0.24	0.23	0.22	0.23	6.30	3.40	2.95	3.10
200	0.11	0.11	0.10	0.10	0.18	0.18	0.17	0.17	8.80	5.10	3.25	3.75
500	0.11	0.11	0.11	0.11	0.14	0.13	0.13	0.13	14.50	9.40	8.80	8.10
1000	0.09	0.10	0.09	0.09	0.10	0.11	0.10	0.10	9.50	4.95	4.35	5.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.75$											
100	-0.02	0.00	0.00	0.00	0.07	0.02	0.00	0.00	-	-	-	-
200	-0.02	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
500	-0.02	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
1000	-0.02	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table 3 of the main paper.

Figure S7: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.75, $n = 100, 200, 500, 1000$ and $T = 200$



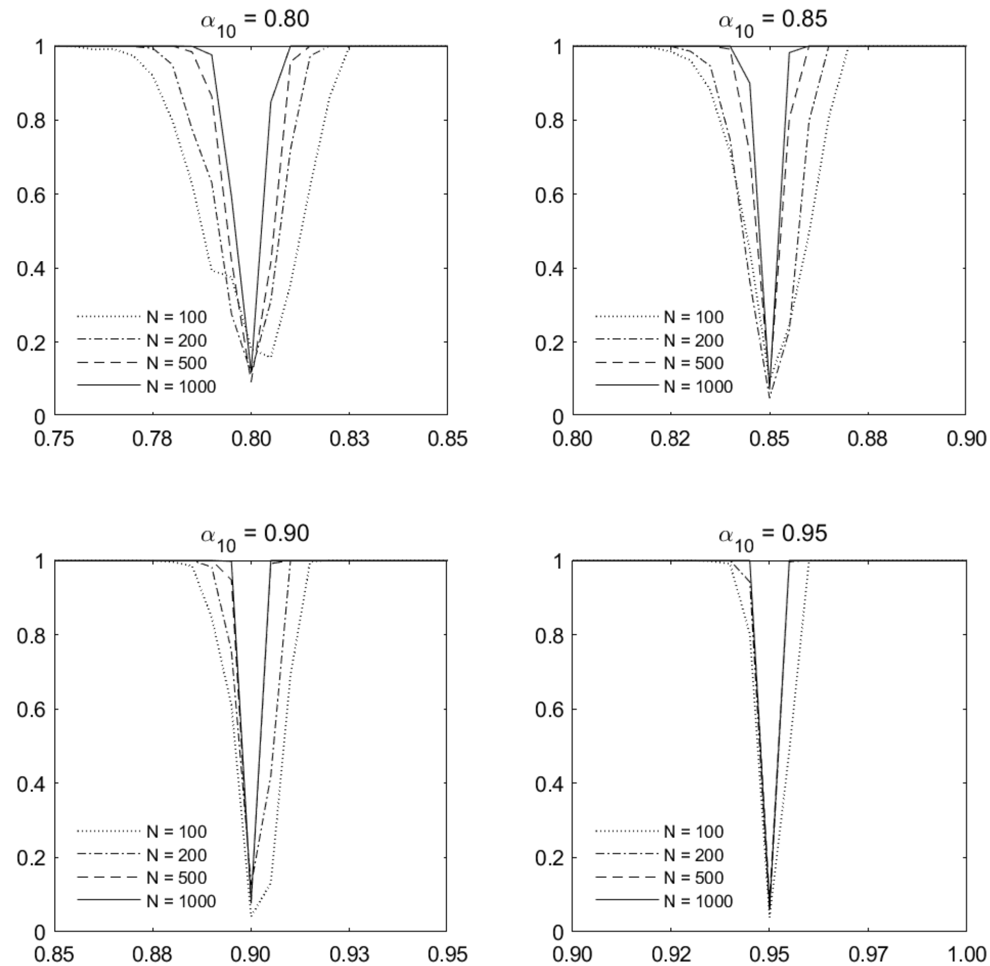
Notes: See the notes to Table 3 of the main paper for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S8: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.80

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.80$											
100	1.17	1.14	1.07	1.03	1.52	1.50	1.45	1.41	3.05	3.35	2.70	2.50
200	1.46	1.39	1.28	1.28	1.62	1.55	1.44	1.44	10.35	8.85	6.60	6.65
500	1.29	1.23	1.17	1.14	1.37	1.31	1.24	1.21	12.35	9.95	8.45	7.85
1000	1.27	1.18	1.12	1.11	1.32	1.22	1.16	1.15	17.10	10.95	6.90	7.05
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.80$											
100	0.70	0.67	0.62	0.58	1.04	1.00	0.97	0.93	18.25	18.40	19.45	18.85
200	0.96	0.92	0.86	0.86	1.09	1.05	0.99	0.99	14.15	12.20	9.10	10.50
500	0.90	0.87	0.83	0.80	0.96	0.92	0.88	0.85	12.10	8.75	7.25	6.55
1000	0.85	0.80	0.77	0.76	0.88	0.83	0.79	0.78	18.30	11.95	9.65	7.70
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.80$											
100	0.68	0.67	0.63	0.63	0.89	0.86	0.83	0.82	11.20	9.35	8.30	7.60
200	0.60	0.59	0.55	0.54	0.70	0.68	0.65	0.65	4.95	3.60	3.85	3.20
500	0.51	0.50	0.47	0.46	0.56	0.55	0.51	0.50	10.95	8.50	7.35	7.20
1000	0.50	0.47	0.44	0.44	0.52	0.49	0.46	0.46	12.30	8.90	5.65	5.40
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.80$											
100	0.38	0.39	0.38	0.36	0.54	0.54	0.52	0.51	5.00	3.55	3.10	2.75
200	0.27	0.27	0.24	0.24	0.37	0.36	0.33	0.34	14.00	12.30	12.95	12.50
500	0.29	0.28	0.27	0.26	0.32	0.32	0.30	0.29	11.20	8.95	7.00	7.05
1000	0.28	0.27	0.26	0.25	0.29	0.29	0.27	0.27	12.00	8.05	6.50	6.45
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.80$											
100	0.07	0.07	0.07	0.07	0.25	0.23	0.24	0.23	6.75	3.30	3.20	2.90
200	0.11	0.11	0.10	0.10	0.19	0.18	0.17	0.17	9.45	4.80	4.00	4.05
500	0.11	0.12	0.11	0.11	0.14	0.14	0.13	0.13	13.95	8.90	8.45	8.80
1000	0.09	0.10	0.09	0.09	0.10	0.11	0.10	0.10	11.55	5.80	4.20	4.45
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.80$											
100	-0.02	0.00	0.00	0.00	0.07	0.01	0.00	0.00	-	-	-	-
200	-0.02	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
500	-0.02	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
1000	-0.02	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table 3 of the main paper.

Figure S8: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.80, $n = 100, 200, 500, 1000$ and $T = 200$



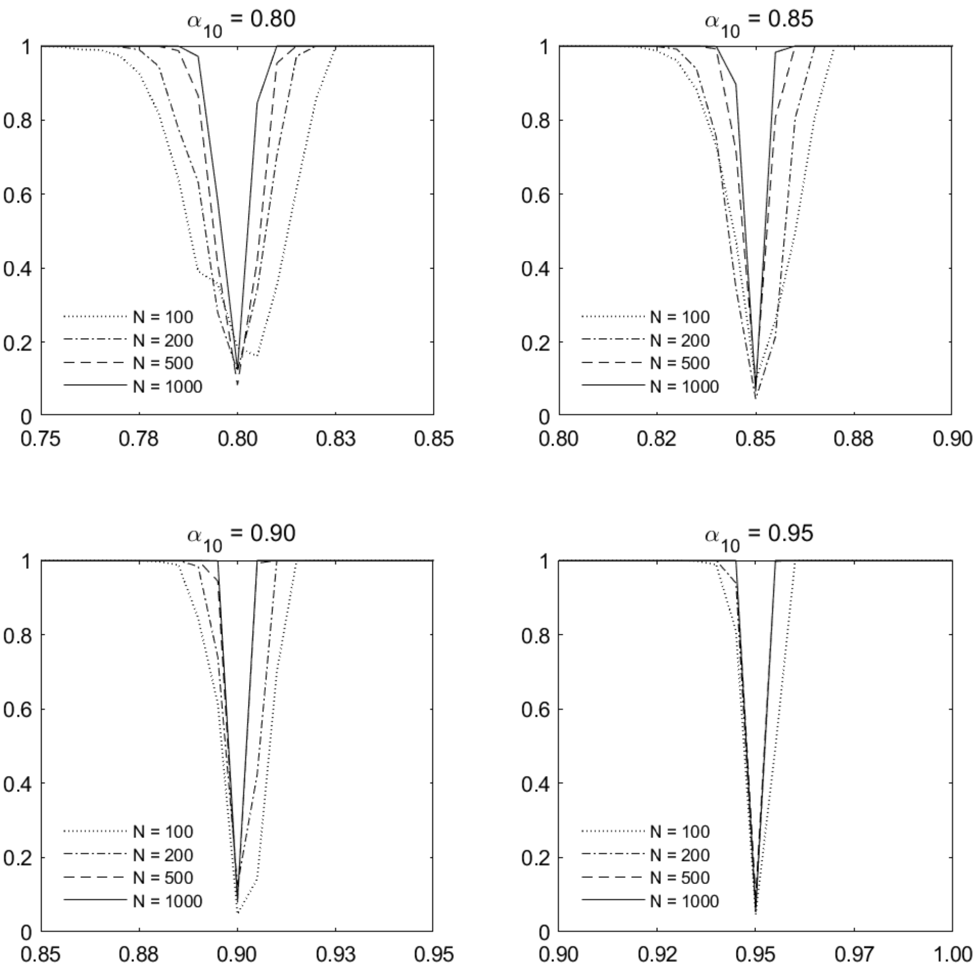
Notes: See the notes to Table 3 of the main paper for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S9: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.90

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.90$											
100	1.18	1.12	1.08	1.04	1.54	1.49	1.44	1.41	3.60	3.30	2.70	2.40
200	1.43	1.41	1.29	1.28	1.60	1.56	1.44	1.44	10.15	8.40	7.65	6.00
500	1.29	1.23	1.17	1.13	1.37	1.31	1.24	1.20	14.05	10.60	7.15	7.90
1000	1.27	1.18	1.13	1.11	1.31	1.22	1.17	1.15	17.55	10.80	7.70	6.55
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.90$											
100	0.69	0.64	0.62	0.59	1.03	0.98	0.96	0.93	18.70	18.45	18.35	18.40
200	0.96	0.92	0.85	0.85	1.09	1.04	0.98	0.98	13.30	9.05	9.90	9.05
500	0.90	0.87	0.83	0.81	0.96	0.92	0.87	0.86	11.60	9.90	6.90	6.30
1000	0.86	0.81	0.76	0.75	0.89	0.84	0.79	0.78	18.35	12.60	7.95	7.40
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.90$											
100	0.68	0.65	0.63	0.62	0.89	0.84	0.83	0.82	11.45	8.30	8.40	7.20
200	0.60	0.59	0.54	0.54	0.71	0.69	0.64	0.65	5.95	4.20	3.40	3.65
500	0.52	0.50	0.47	0.46	0.57	0.55	0.51	0.51	10.85	8.65	6.75	7.85
1000	0.49	0.47	0.45	0.44	0.52	0.49	0.47	0.46	11.55	7.20	5.50	5.65
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.90$											
100	0.39	0.40	0.39	0.37	0.55	0.55	0.54	0.52	5.25	4.00	4.00	3.40
200	0.28	0.26	0.24	0.24	0.38	0.35	0.34	0.33	12.85	12.95	13.60	12.55
500	0.29	0.28	0.27	0.26	0.32	0.32	0.30	0.30	10.65	8.35	6.30	7.95
1000	0.28	0.27	0.26	0.26	0.30	0.29	0.27	0.27	12.60	7.75	7.15	6.35
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.90$											
100	0.07	0.08	0.07	0.05	0.25	0.24	0.23	0.22	7.05	3.45	3.05	2.35
200	0.11	0.11	0.10	0.10	0.19	0.18	0.17	0.17	8.85	5.15	3.70	4.10
500	0.11	0.11	0.11	0.10	0.14	0.14	0.13	0.13	13.15	9.55	7.85	7.65
1000	0.09	0.09	0.09	0.09	0.10	0.11	0.10	0.10	10.85	4.90	4.75	5.35
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.90$											
100	-0.02	0.00	0.00	0.00	0.07	0.01	0.00	0.00	-	-	-	-
200	-0.02	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
500	-0.02	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
1000	-0.02	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table 3 of the main paper.

Figure S9: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.90, $n = 100, 200, 500, 1000$ and $T = 200$



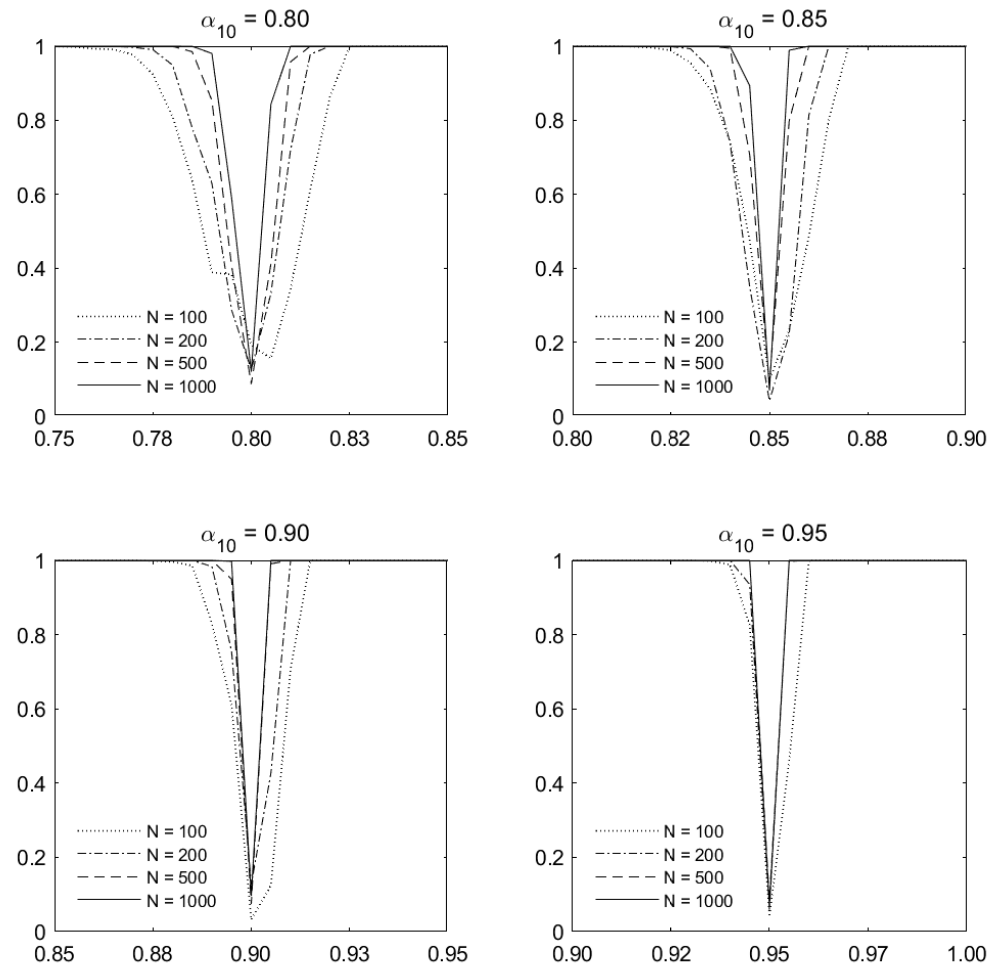
Notes: See the notes to Table 3 of the main paper for details of the data generating process. Power is computed under H_1: $\alpha_{1a}=\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S10: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.95

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10}=0.75$, $\alpha_{20}=0.95$											
100	1.18	1.14	1.08	1.03	1.54	1.50	1.46	1.41	4.00	3.10	2.60	2.55
200	1.44	1.38	1.30	1.29	1.61	1.53	1.45	1.45	10.75	7.60	6.55	7.00
500	1.30	1.23	1.16	1.14	1.37	1.30	1.23	1.21	13.45	10.40	8.25	8.20
1000	1.26	1.18	1.12	1.11	1.31	1.22	1.16	1.15	17.90	11.30	7.10	7.40
	$\alpha_{10}=0.80$, $\alpha_{20}=0.95$											
100	0.69	0.65	0.63	0.60	1.03	0.99	0.97	0.94	18.80	18.20	18.70	19.40
200	0.96	0.93	0.85	0.86	1.09	1.05	0.98	0.98	13.75	11.15	9.35	9.45
500	0.91	0.86	0.82	0.81	0.96	0.91	0.87	0.86	11.45	8.85	6.60	6.65
1000	0.85	0.81	0.77	0.76	0.88	0.83	0.79	0.78	17.05	13.00	8.70	8.65
	$\alpha_{10}=0.85$, $\alpha_{20}=0.95$											
100	0.68	0.67	0.64	0.61	0.87	0.86	0.83	0.80	9.45	9.35	8.05	7.20
200	0.60	0.59	0.54	0.54	0.71	0.69	0.64	0.64	5.60	4.20	3.00	3.20
500	0.52	0.50	0.47	0.46	0.57	0.54	0.51	0.50	11.20	9.00	8.25	7.25
1000	0.50	0.47	0.44	0.44	0.52	0.49	0.46	0.46	11.55	8.25	5.30	6.35
	$\alpha_{10}=0.90$, $\alpha_{20}=0.95$											
100	0.40	0.38	0.37	0.37	0.54	0.52	0.52	0.52	4.80	3.20	3.40	3.00
200	0.27	0.26	0.23	0.24	0.37	0.36	0.33	0.33	14.60	11.60	12.85	12.80
500	0.29	0.28	0.27	0.27	0.32	0.32	0.30	0.30	10.80	7.65	6.75	7.00
1000	0.27	0.27	0.26	0.25	0.29	0.29	0.27	0.27	12.25	8.95	7.85	6.80
	$\alpha_{10}=0.95$, $\alpha_{20}=0.95$											
100	0.07	0.07	0.07	0.07	0.25	0.23	0.23	0.23	6.80	3.30	3.35	2.80
200	0.11	0.10	0.11	0.10	0.19	0.17	0.17	0.17	8.55	4.00	3.70	3.30
500	0.11	0.12	0.11	0.11	0.14	0.14	0.13	0.13	14.80	8.95	6.70	7.35
1000	0.09	0.10	0.09	0.09	0.11	0.11	0.10	0.10	11.75	5.75	4.80	4.35
	$\alpha_{10}=1.00$, $\alpha_{20}=0.95$											
100	-0.02	0.00	0.00	0.00	0.07	0.01	0.00	0.00	-	-	-	-
200	-0.02	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
500	-0.02	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
1000	-0.02	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table 3 of the main paper.

Figure S10: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 0.95, $n = 100, 200, 500, 1000$ and $T = 200$



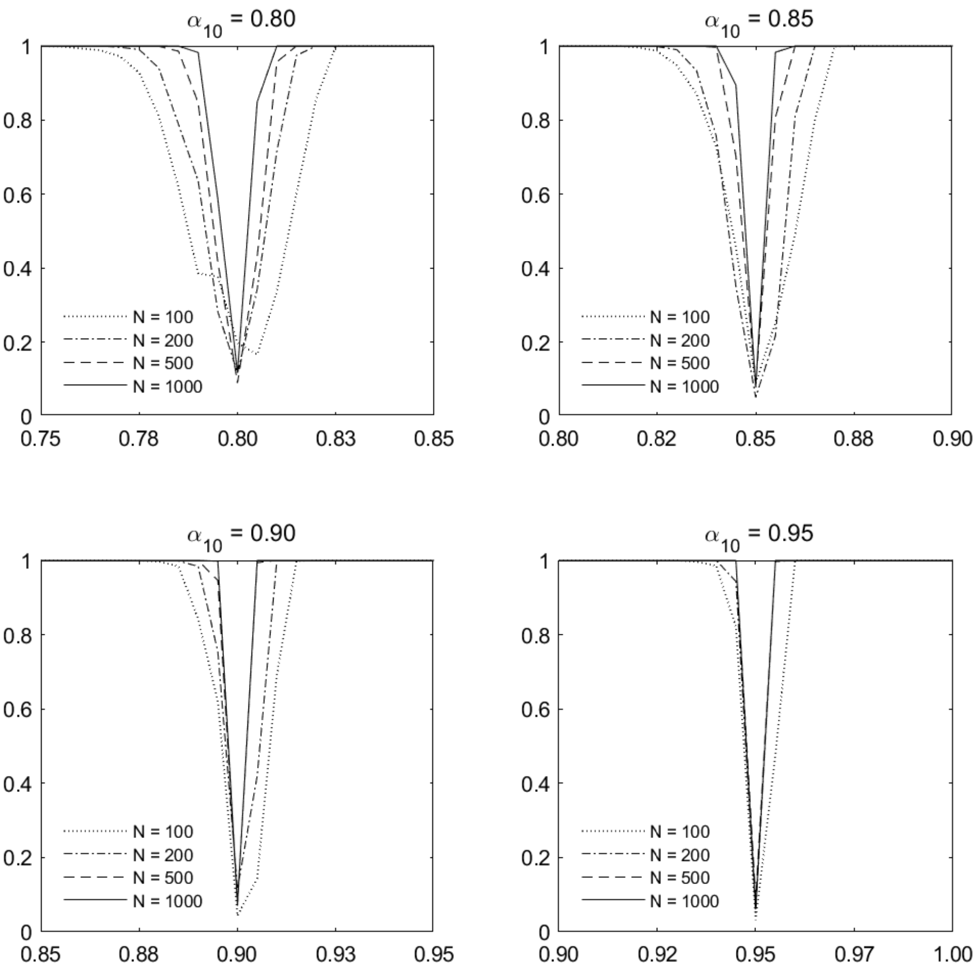
Notes: See the notes to Table 3 of the main paper for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S11: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 1.00

	Bias (×100)				RMSE (×100)				Size (×100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 1.00$											
100	1.17	1.12	1.06	1.03	1.55	1.49	1.43	1.41	4.15	3.10	2.30	2.45
200	1.46	1.38	1.29	1.29	1.62	1.54	1.44	1.45	9.65	7.95	7.55	7.10
500	1.29	1.23	1.16	1.13	1.37	1.30	1.23	1.20	13.75	10.25	7.35	7.75
1000	1.27	1.18	1.12	1.11	1.31	1.22	1.16	1.15	17.75	11.15	7.65	7.20
	$\alpha_{10} = 0.80$, $\alpha_{20} = 1.00$											
100	0.67	0.66	0.62	0.58	1.01	1.00	0.98	0.94	19.20	18.10	19.10	20.15
200	0.96	0.94	0.86	0.86	1.10	1.06	0.99	0.99	13.25	11.60	10.95	10.60
500	0.90	0.87	0.82	0.80	0.96	0.93	0.88	0.85	12.30	10.10	7.70	6.70
1000	0.85	0.81	0.77	0.76	0.88	0.84	0.79	0.79	16.75	12.35	8.30	8.10
	$\alpha_{10} = 0.85$, $\alpha_{20} = 1.00$											
100	0.67	0.65	0.64	0.62	0.87	0.84	0.83	0.81	10.60	8.50	8.35	8.00
200	0.60	0.59	0.54	0.55	0.71	0.69	0.65	0.65	5.60	4.55	3.55	3.30
500	0.51	0.50	0.47	0.46	0.56	0.54	0.51	0.50	10.95	8.90	8.10	7.05
1000	0.50	0.47	0.45	0.44	0.52	0.49	0.47	0.46	11.75	7.20	5.75	6.20
	$\alpha_{10} = 0.90$, $\alpha_{20} = 1.00$											
100	0.39	0.39	0.37	0.35	0.55	0.54	0.52	0.50	5.65	3.65	2.75	2.80
200	0.27	0.26	0.24	0.23	0.37	0.36	0.33	0.33	14.10	12.75	12.50	12.65
500	0.28	0.28	0.27	0.26	0.32	0.31	0.30	0.29	9.20	7.95	6.35	6.50
1000	0.28	0.27	0.26	0.26	0.30	0.29	0.27	0.27	12.25	8.50	6.60	6.80
	$\alpha_{10} = 0.95$, $\alpha_{20} = 1.00$											
100	0.07	0.07	0.07	0.06	0.26	0.24	0.23	0.23	7.40	3.60	3.15	3.35
200	0.10	0.11	0.10	0.10	0.19	0.18	0.17	0.16	9.20	5.05	3.75	3.50
500	0.11	0.12	0.11	0.11	0.13	0.14	0.13	0.13	14.55	10.20	8.40	8.10
1000	0.09	0.09	0.09	0.09	0.11	0.11	0.10	0.10	11.40	5.80	5.25	5.15
	$\alpha_{10} = 1.00$, $\alpha_{20} = 1.00$											
100	-0.02	0.00	0.00	0.00	0.07	0.01	0.00	0.00	-	-	-	-
200	-0.02	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
500	-0.02	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
1000	-0.02	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table 3 of the main paper.

Figure S11: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2B (two observed factors - non-Gaussian errors), when the strength of the second factor is set to 1.00, $n = 100, 200, 500, 1000$ and $T = 200$



Notes: See the notes to Table 3 of the main paper for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S12: Bias, RMSE and Size (×100) of estimating different strengths of first factor in the case of experiment 2A (two observed (uncorrelated) factors - Gaussian errors), when the strength of the second factor is set to 0.85

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.85$											
100	1.20	1.17	1.07	1.02	1.59	1.53	1.44	1.38	4.75	3.35	2.70	2.30
200	1.45	1.37	1.33	1.31	1.61	1.52	1.48	1.46	9.40	7.70	7.45	6.65
500	1.32	1.23	1.16	1.13	1.39	1.31	1.22	1.21	14.05	11.35	8.10	9.20
1000	1.27	1.20	1.13	1.11	1.31	1.24	1.17	1.15	16.90	11.75	8.30	7.15
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.85$											
100	0.71	0.70	0.62	0.58	1.04	1.03	0.95	0.91	17.90	17.75	16.90	18.00
200	0.97	0.90	0.87	0.86	1.10	1.03	0.99	0.98	14.20	10.65	9.15	9.30
500	0.92	0.87	0.82	0.80	0.98	0.92	0.87	0.86	13.05	9.10	7.20	6.70
1000	0.85	0.81	0.77	0.76	0.89	0.84	0.79	0.78	18.70	13.00	7.95	7.65
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.85$											
100	0.69	0.69	0.64	0.63	0.89	0.89	0.82	0.81	11.20	10.65	7.85	7.70
200	0.61	0.57	0.57	0.54	0.72	0.68	0.67	0.65	6.55	4.20	3.35	3.55
500	0.53	0.50	0.47	0.46	0.58	0.55	0.51	0.50	12.30	9.55	6.60	7.95
1000	0.50	0.47	0.45	0.44	0.52	0.50	0.47	0.46	11.25	8.10	5.95	5.10
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.85$											
100	0.41	0.40	0.40	0.37	0.56	0.54	0.54	0.51	5.00	3.20	3.30	2.60
200	0.28	0.26	0.25	0.24	0.38	0.36	0.35	0.34	13.05	13.15	12.60	12.60
500	0.30	0.29	0.27	0.26	0.33	0.32	0.30	0.30	9.50	8.40	7.10	7.85
1000	0.28	0.28	0.26	0.26	0.30	0.29	0.28	0.27	11.10	9.45	6.65	6.00
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.85$											
100	0.08	0.08	0.07	0.07	0.25	0.24	0.23	0.23	5.30	3.90	2.80	3.05
200	0.11	0.11	0.11	0.10	0.19	0.18	0.17	0.17	7.25	5.50	4.00	4.10
500	0.12	0.12	0.11	0.11	0.14	0.14	0.13	0.13	12.15	8.15	7.45	7.70
1000	0.09	0.10	0.09	0.09	0.11	0.11	0.10	0.10	8.05	6.55	5.30	5.10
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.85$											
100	-0.01	0.00	0.00	0.00	0.05	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table S3 of the main paper. The factors, f_1, f_2, have correlation given by $\rho_{12} = corr(f_1, f_2) = 0.0$.

Table S13: Empirical power functions associated with testing different strengths of first factor in the case of experiment 2B (two observed (uncorrelated) factors - non-Gaussian errors), when the strength of the second factor is set to 0.85, $n = 100, 200, 500, 1000$ and $T = 200$

	Bias ($\times$100)				RMSE ($\times$100)				Size ($\times$100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$, $\alpha_{20} = 0.85$											
100	1.20	1.13	1.08	1.04	1.58	1.50	1.46	1.41	4.25	2.80	3.25	2.30
200	1.43	1.39	1.30	1.31	1.60	1.55	1.45	1.47	9.85	7.70	7.00	7.55
500	1.31	1.23	1.17	1.14	1.38	1.30	1.24	1.21	13.50	10.25	8.20	7.20
1000	1.27	1.18	1.12	1.11	1.32	1.22	1.16	1.15	17.40	11.05	7.45	7.60
	$\alpha_{10} = 0.80$, $\alpha_{20} = 0.85$											
100	0.71	0.66	0.63	0.61	1.03	1.00	0.96	0.95	17.60	18.75	18.15	19.45
200	0.96	0.93	0.85	0.86	1.09	1.05	0.97	0.99	13.05	11.35	9.20	9.75
500	0.91	0.87	0.82	0.80	0.97	0.92	0.87	0.86	12.00	9.25	5.95	7.40
1000	0.86	0.81	0.76	0.76	0.89	0.83	0.79	0.78	18.50	11.00	8.85	7.80
	$\alpha_{10} = 0.85$, $\alpha_{20} = 0.85$											
100	0.69	0.67	0.64	0.62	0.88	0.86	0.83	0.81	9.65	9.40	7.70	7.35
200	0.62	0.59	0.54	0.54	0.72	0.69	0.65	0.65	5.95	3.90	4.10	3.05
500	0.52	0.50	0.47	0.46	0.56	0.54	0.51	0.51	10.60	7.70	7.35	7.75
1000	0.50	0.47	0.45	0.44	0.53	0.49	0.47	0.46	12.45	8.40	5.45	5.40
	$\alpha_{10} = 0.90$, $\alpha_{20} = 0.85$											
100	0.40	0.40	0.38	0.36	0.56	0.55	0.53	0.51	5.10	3.70	3.55	3.05
200	0.28	0.26	0.23	0.24	0.39	0.36	0.33	0.34	14.55	12.45	13.20	13.35
500	0.29	0.29	0.27	0.26	0.32	0.32	0.30	0.29	9.45	8.35	6.85	6.20
1000	0.28	0.27	0.26	0.25	0.30	0.28	0.27	0.27	12.90	8.25	6.50	6.05
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.85$											
100	0.07	0.08	0.07	0.06	0.25	0.24	0.24	0.22	5.70	3.40	3.40	2.75
200	0.11	0.11	0.10	0.10	0.18	0.18	0.17	0.17	8.30	4.00	3.85	4.45
500	0.11	0.12	0.11	0.11	0.14	0.14	0.13	0.13	11.90	8.60	8.85	7.40
1000	0.10	0.10	0.09	0.09	0.11	0.11	0.10	0.10	9.65	5.55	5.65	5.05
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.85$											
100	-0.01	0.00	0.00	0.00	0.06	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.04	0.01	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	

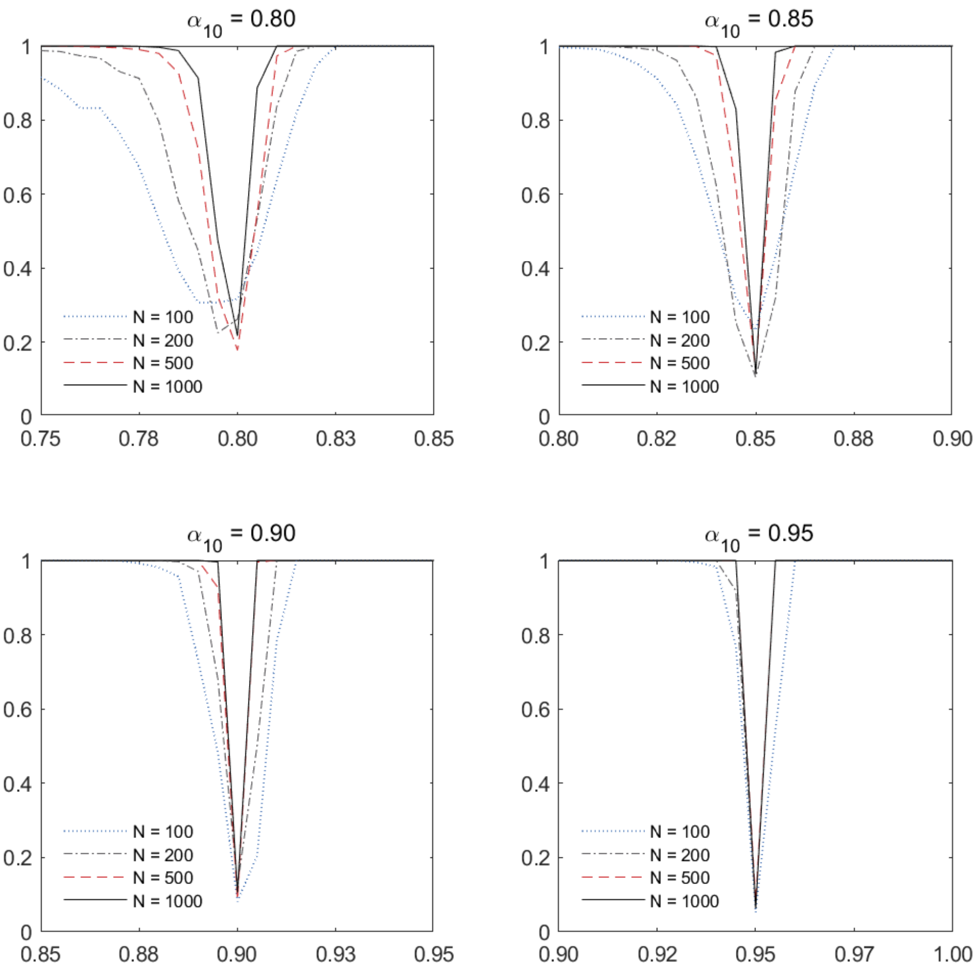
Notes: Parameters of DGP (44) are generated as described in Table 3 of the main paper. The factors, f_1, f_2, have correlation given by $\rho_{12} = corr(f_1, f_2) = 0.0$.

Table S14: Bias, RMSE and Size (×100) of estimating the strength of strongest factor in the case of experiment 3A (unobserved single factor - with Gaussian errors instead) using cross section average

	Bias (×100)				RMSE (×100)				Size (×100)			
n\T	120	200	500	1000	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.75$											
100	2.35	2.75	4.29	6.62	2.80	3.17	4.66	6.90	26.05	35.45	71.55	97.70
200	2.04	2.14	2.59	3.48	2.29	2.37	2.81	3.68	29.70	33.55	51.10	82.55
500	1.61	1.55	1.58	1.82	1.75	1.68	1.69	1.93	30.70	28.60	29.25	44.95
1000	1.47	1.41	1.36	1.42	1.57	1.48	1.42	1.48	31.60	27.70	24.85	29.30
	$\alpha_{10} = 0.80$											
100	1.27	1.44	2.16	3.30	1.63	1.77	2.48	3.54	28.25	31.65	55.20	86.05
200	1.21	1.25	1.41	1.78	1.38	1.40	1.56	1.91	25.05	26.20	35.15	55.40
500	1.02	0.98	0.97	1.05	1.10	1.05	1.03	1.11	20.50	17.80	15.60	22.45
1000	0.93	0.89	0.85	0.86	0.97	0.92	0.88	0.89	25.05	21.70	15.55	18.35
	$\alpha_{10} = 0.85$											
100	0.92	0.99	1.25	1.71	1.12	1.19	1.45	1.89	19.25	23.20	38.35	61.50
200	0.71	0.72	0.75	0.89	0.83	0.83	0.86	1.00	10.55	10.55	11.50	19.95
500	0.56	0.54	0.52	0.54	0.61	0.59	0.56	0.58	14.60	11.85	8.65	9.85
1000	0.52	0.50	0.47	0.48	0.55	0.53	0.49	0.50	14.30	11.40	6.20	7.55
	$\alpha_{10} = 0.90$											
100	0.50	0.52	0.62	0.77	0.65	0.67	0.77	0.91	7.70	8.15	13.50	21.55
200	0.31	0.32	0.32	0.36	0.41	0.42	0.40	0.45	13.85	12.50	10.85	13.05
500	0.31	0.30	0.28	0.29	0.34	0.33	0.31	0.31	11.70	8.60	6.55	7.10
1000	0.29	0.29	0.27	0.27	0.31	0.30	0.28	0.28	12.55	10.60	7.90	7.65
	$\alpha_{10} = 0.95$											
100	0.11	0.11	0.15	0.18	0.27	0.27	0.30	0.33	6.05	5.25	7.30	9.25
200	0.13	0.13	0.12	0.13	0.20	0.20	0.19	0.20	7.80	6.40	5.95	6.80
500	0.12	0.12	0.11	0.11	0.15	0.14	0.13	0.13	12.45	8.00	7.80	7.85
1000	0.10	0.10	0.09	0.09	0.11	0.11	0.10	0.10	8.20	6.30	4.75	4.40
	$\alpha_{10} = 1.00$											
100	0.00	0.00	0.00	0.00	0.03	0.01	0.00	0.00	-	-	-	-
200	-0.01	0.00	0.00	0.00	0.03	0.00	0.00	0.00	-	-	-	-
500	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-
1000	-0.01	0.00	0.00	0.00	0.02	0.00	0.00	0.00	-	-	-	-

Notes: Parameters of DGP (44) are generated as described in Table 1. $\alpha_0 = \alpha_{10}$ is estimated by regressing observations, x_{it}, on an intercept and the cross section averages of x_{it}, $\bar{x}_t = n^{-1}\sum_{i=1}^{n} x_{it}$, for $t = 1, 2, \ldots, T$.

Figure S12: Empirical power functions associated with testing different strengths of strongest factor in the case of experiment 3A (unobserved single factor - with Gaussian errors instead) using cross section average, when $n = 100, 200, 500, 1000$ and $T = 200$



Notes: See the notes to Table 1 of the main paper for details of the data generating process. Power is computed under H_1: α_{1a}=$\alpha_{10} + \kappa$, where $\kappa = -0.05, -0.045, \ldots, 0.045, 0.05$. The number of replications is set to $R = 2000$.

Table S15: Bias and RMSE ($\times 10,000$) of estimating the strength of strongest factor in the case of experiment 3B (two unobserved factors - with Gaussian errors instead) using cross section average, when $\alpha_{10} = 1.00$

	Bias ($\times 10,000$)				RMSE ($\times 10,000$)			
$n\backslash$T	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.51$							
100	-0.58	-0.02	0.00	0.00	3.55	0.69	0.00	0.00
200	-0.88	-0.03	0.00	0.00	3.16	0.52	0.00	0.00
500	-0.84	-0.05	0.00	0.00	2.00	0.42	0.00	0.00
1000	-0.89	-0.05	0.00	0.00	1.67	0.26	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.75$							
100	-0.71	-0.08	0.00	0.00	4.00	1.29	0.00	0.00
200	-0.90	-0.02	0.00	0.00	3.18	0.47	0.00	0.00
500	-0.84	-0.06	0.00	0.00	2.00	0.44	0.00	0.00
1000	-0.88	-0.05	0.00	0.00	1.63	0.27	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.95$							
100	-1.45	-0.16	0.00	0.00	5.96	1.89	0.00	0.00
200	-1.74	-0.15	0.00	0.00	4.89	1.20	0.00	0.00
500	-1.91	-0.17	0.00	0.00	3.72	0.83	0.00	0.00
1000	-1.86	-0.18	0.00	0.00	3.72	0.83	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 1.00$							
100	-0.01	0.00	0.00	0.00	0.49	0.00	0.00	0.00
200	-0.01	0.00	0.00	0.00	0.30	0.00	0.00	0.00
500	-0.01	0.00	0.00	0.00	0.14	0.00	0.00	0.00
1000	-0.01	0.00	0.00	0.00	0.13	0.00	0.00	0.00

Notes: Parameters of DGP (44) are generated as described in Table S3. $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated by regressing observations, x_{it}, on an intercept and the cross section average of x_{it}, $\bar{x}_t = n^{-1}\sum_{i=1}^{n} x_{it}$, for $t = 1, 2, \ldots, T$.

Table S16: Bias and RMSE ($\times 10,000$) of estimating the strength of strongest factor in the case of experiment 3B (two unobserved factors - with Gaussian errors instead) using cross section average, when $\alpha_{10} = 0.95$

	Bias ($\times$100)				RMSE ($\times$100)			
$n\backslash$T	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.51$							
100	0.18	0.23	0.38	0.58	0.35	0.40	0.54	0.71
200	0.15	0.17	0.22	0.31	0.24	0.25	0.30	0.38
500	0.13	0.13	0.14	0.16	0.16	0.16	0.17	0.20
1000	0.11	0.11	0.10	0.11	0.12	0.12	0.12	0.13
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.75$							
100	1.28	1.53	1.74	1.78	1.42	1.62	1.81	1.85
200	0.98	1.26	1.51	1.55	1.10	1.33	1.54	1.58
500	0.61	0.83	1.19	1.27	0.71	0.91	1.21	1.28
1000	0.41	0.60	0.95	1.07	0.51	0.67	0.97	1.08
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.95$							
100	3.99	4.05	4.04	4.06	4.01	4.06	4.06	4.08
200	3.88	3.94	3.95	3.96	3.89	3.95	3.96	3.97
500	3.74	3.82	3.83	3.83	3.74	3.82	3.83	3.83
1000	3.63	3.71	3.73	3.72	3.63	3.72	3.73	3.73
	$\alpha_{10} = 0.95$, $\alpha_{20} = 1.00$							
100	-0.02	0.00	0.00	0.00	0.06	0.02	0.00	0.00
200	-0.02	0.00	0.00	0.00	0.05	0.01	0.00	0.00
500	-0.02	0.00	0.00	0.00	0.04	0.01	0.00	0.00
1000	-0.02	0.00	0.00	0.00	0.03	0.01	0.00	0.00

Notes: Parameters of DGP (44) are generated as described in Table S3. $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated by regressing observations, x_{it}, on an intercept and the cross section averages of x_{it}, $\bar{x}_t = n^{-1} \sum_{i=1}^{n} x_{it}$, for $t = 1, 2, \dots, T$.

Table S17: Bias and RMSE ($\times 10,000$) of estimating the strength of strongest factor in the case of experiment 3B (two unobserved factors - non-Gaussian errors) using first principal component, when $\alpha_{10} = 1.00$

	Bias ($\times 10,000$)				RMSE ($\times 10,000$)			
$n \backslash$T	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.51$							
100	-11.87	-8.37	-5.52	-4.55	55.32	41.65	28.74	23.02
200	-5.53	-4.60	-3.53	-2.85	30.40	26.61	21.33	16.85
500	-1.86	-2.60	-1.38	-1.14	14.03	16.97	10.93	9.44
1000	-0.97	-0.65	-0.66	-0.74	8.56	6.59	6.82	6.32
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.75$							
100	-9.97	-7.43	-4.27	-3.26	45.05	39.65	24.70	20.09
200	-5.93	-3.70	-2.84	-1.80	32.39	22.91	17.11	12.37
500	-2.28	-2.15	-1.50	-1.07	14.43	15.43	10.93	8.00
1000	-1.01	-1.11	-0.67	-0.41	8.14	8.54	6.04	4.27
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.95$							
100	-2.20	-0.44	-0.01	0.00	13.29	6.06	0.49	0.00
200	-1.93	-0.34	-0.04	0.00	6.36	3.26	1.94	0.00
500	-1.67	-0.21	-0.03	-0.01	5.01	2.00	1.39	0.36
1000	-1.61	-0.23	0.00	0.00	5.01	2.00	1.39	0.36
	$\alpha_{10} = 1.00$, $\alpha_{20} = 1.00$							
100	-0.57	0.00	0.00	0.00	15.11	0.00	0.00	0.00
200	-0.03	0.00	0.00	0.00	1.11	0.00	0.00	0.00
500	-0.02	0.00	0.00	0.00	0.23	0.00	0.00	0.00
1000	-0.02	0.00	0.00	0.00	0.16	0.00	0.00	0.00

Notes: Parameters of DGP (44) are generated as described in Table 3. $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated by regressing observations, x_{it}, on an intercept and the first principal component of x_{it}, $i = 1, 2, \ldots, n$, $t = 1, 2, \ldots, T$.

Table S18: Bias and RMSE ($\times 10,000$) of estimating the strength of strongest factor in the case of experiment 3B (two unobserved factors - non-Gaussian errors) using first principal component, when $\alpha_{10} = 0.95$

	Bias ($\times 100$)				RMSE ($\times 100$)			
$n \backslash$T	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.51$							
100	3.15	3.34	3.72	3.94	3.32	3.49	3.83	4.03
200	3.50	3.65	3.94	4.14	3.59	3.73	4.00	4.18
500	3.70	3.90	4.17	4.33	3.78	3.96	4.20	4.36
1000	3.86	4.02	4.28	4.43	3.93	4.07	4.31	4.45
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.75$							
100	3.29	3.46	3.78	3.98	3.42	3.57	3.86	4.05
200	3.54	3.70	4.01	4.19	3.62	3.77	4.05	4.22
500	3.77	3.97	4.22	4.38	3.83	4.01	4.25	4.40
1000	3.89	4.07	4.34	4.48	3.95	4.10	4.35	4.49
	$\alpha_{10} = 0.95$, $\alpha_{20} = 0.95$							
100	4.15	4.17	4.16	4.17	4.17	4.19	4.18	4.20
200	4.12	4.14	4.13	4.13	4.14	4.15	4.14	4.14
500	4.12	4.12	4.15	4.17	4.13	4.14	4.17	4.19
1000	4.11	4.15	4.20	4.26	4.12	4.17	4.22	4.27
	$\alpha_{10} = 0.95$, $\alpha_{20} = 1.00$							
100	-0.02	0.00	0.00	0.00	0.16	0.05	0.00	0.00
200	-0.02	0.00	0.00	0.00	0.06	0.02	0.00	0.00
500	-0.02	0.00	0.00	0.00	0.04	0.04	0.00	0.00
1000	-0.02	0.00	0.00	0.00	0.03	0.02	0.00	0.00

Notes: Parameters of DGP (44) are generated as described in Table 3. $\alpha_0 = \max(\alpha_{10}, \alpha_{20})$ is estimated by regressing observations, x_{it}, on an intercept and the first principal component of x_{it}, $i = 1, 2, \ldots, n$, $t = 1, 2, \ldots, T$.

Table S19: Bias and RMSE ($\times 10,000$) of estimating factor strength in the case of experiment 4 (observed misspecified single factor - Gaussian errors) when set to 1.00 and true DGP contains two uncorrelated factors

	Bias ($\times 10,000$)				RMSE ($\times 10,000$)			
$n\backslash$T	120	200	500	1000	120	200	500	1000
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.75$							
100	-1.88	-0.10	-0.01	0.00	7.53	1.46	0.49	0.00
200	-1.61	-0.09	0.00	0.00	4.82	0.99	0.00	0.00
500	-1.51	-0.09	0.00	0.00	3.94	0.62	0.07	0.00
1000	-1.48	-0.09	0.00	0.00	2.88	0.38	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.80$							
100	-2.14	-0.12	0.00	0.00	8.37	1.62	0.00	0.00
200	-1.83	-0.14	0.00	0.00	5.43	1.20	0.00	0.00
500	-1.71	-0.11	0.00	0.00	4.81	0.65	0.00	0.00
1000	-1.71	-0.11	0.00	0.00	3.65	0.45	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.85$							
100	-2.67	-0.14	-0.01	0.00	10.25	1.76	0.49	0.00
200	-2.20	-0.14	0.00	0.00	6.41	1.36	0.00	0.00
500	-2.12	-0.15	0.00	0.00	6.36	0.86	0.07	0.00
1000	-2.05	-0.12	0.00	0.00	4.83	0.52	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.90$							
100	-2.90	-0.13	0.00	0.00	10.30	1.69	0.00	0.00
200	-2.37	-0.17	0.00	0.00	7.26	1.37	0.00	0.00
500	-2.46	-0.15	0.00	0.00	7.97	1.00	0.07	0.00
1000	-2.52	-0.14	0.00	0.00	6.40	0.56	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 0.95$							
100	-3.32	-0.17	-0.01	0.00	11.90	1.95	0.49	0.00
200	-2.99	-0.21	0.00	0.00	9.07	1.60	0.00	0.00
500	-3.00	-0.20	0.00	0.00	10.90	1.23	0.00	0.00
1000	-3.22	-0.19	0.00	0.00	9.01	0.76	0.00	0.00
	$\alpha_{10} = 1.00$, $\alpha_{20} = 1.00$							
100	-3.93	-0.26	-0.01	0.00	13.64	2.39	0.49	0.00
200	-3.72	-0.23	0.00	0.00	11.47	1.72	0.00	0.00
500	-3.87	-0.24	0.00	0.00	15.24	1.28	0.00	0.00
1000	-4.13	-0.25	0.00	0.00	12.18	0.98	0.00	0.00

Notes: The parameters of the true DGP, (44), are generated as described in Table S3 of the main paper. The factors, f_1, f_2, have correlation given by $\rho_{12} = corr(f_1, f_2) = 0.0$. We set $\alpha_{10} = 1$ and α_{20} in the range $[0.75, 1.00]$ with 0.05 increments. The misspecified model assumes the existence of factor f_1 only.

Data construction and additional empirical results

S&P500 security returns

As reference country for this study we pick the United States and as equity market index of preference we opt for the *Standard & Poor's (S&P) 500* index. In this respect, we consider the distinct monthly composites of the *S&P500* index from September 1989 to December 2017. Our analysis is based on a rolling window sample scheme. We work with security returns defined as

$$r_{it} = 100\left(\frac{P_{it} - P_{i,t-1}}{P_{it-1}}\right) + \frac{DY_{it}}{12}, \quad \text{for } i = 1, 2, \ldots, n_\tau \text{ and } t = 1, 2, \ldots, T,$$

where P_{it} and DY_{it} stand for the price and dividend yield of security i at time t, and $\tau = 1, 2, \ldots, 340$ denote the 10-year rolling samples of security returns.

Historical end-of month security price and dividend yield data, P_{it} and DY_{it}, for $i = 1, 2, \ldots, n_\tau$ and $t = 1, 2, \ldots, T$, are obtained from Thompson Reuters Datastream. We are grateful to Takashi Yamagata for providing part of the constructed dataset which is used in Pesaran and Yamagata (2017). n_τ represents all 500 stocks per monthly composition of the S&P500 from 09/1989 to 12/2017 as displayed at the end of each month and T expands from 31/01/1950 to 31/12/2017. For example, code LS&PCOMP1210 will give the 500 constituents of the S&P500 index as of December 2010. P_{it} is the price of security i at the market close of the last day of the month (t), adjusted for subsequent capital actions. DY_{it} is the dividend per share as a percentage of the share price based on an anticipated annual dividend and excludes special or one-off dividends. Both P_{it} and DY_{it}, for $i = 1, 2, \ldots, n_\tau$, $t = 1, 2, \ldots, T$ and $\tau = 1, 2, \ldots, 340$ are obtained at the default 4 decimal places for the US market. The codes used are DPL#(CFM#(x(P#S),VAL),4) and DPL#(CFM#(x(DY#S),VAL),4) for price and dividend yield respectively. Note that 499 securities were downloaded for November 20, 1999 and September 30, 2008. It is confirmed on Standard & Poor's website that the S&P 500 index on these days was based on 499 securities.

SW macroeconomic dataset

The SW macroeconomic dataset that we use extends from 1959Q1-2019Q2 and is an updated version of the dataset compiled originally by Stock and Watson (2012). We opted for a time dimension commencing in 1988Q1 in order to obtain a balanced panel. We excluded three variables as they recorded missing values beyond 1988Q1. These are: (1) Manufacturers' new orders, consumer goods and materials, (2) Case-Shiller 10 City average deflated by PCEPILFE, and (3) Case-Shiller 20 City average deflated by PCEPILFE.

Additional empirical results

The table and graphs that follow show estimates of factor strengths associated with the asset pricing models considered in Section 6 of the main paper:

Table S20: Ranking of all 145 factors (plus the market factor) in terms of the % of months their estimated strengths exceed the threshold of 0.90 during the full sample period of September 1989 to December 2017 and corresponding time averages of $\hat{\alpha}_{s,\tau}$, $s = 1, 2, \ldots, 145$, over different subsamples

	% of months when $\hat{\alpha}_{s,\tau} > 0.90$ over:	Time averages of $\hat{\alpha}_{s,\tau}$ over:			
Factor	Full sample	Full sample	September 1989 - August 1999	September 1999 - August 2009	September 2009 - December 2017
Market	100.0	0.990	0.999	0.974	0.997
Leverage	37.9	0.827	0.739	0.932	0.808
Sales to cash	37.9	0.817	0.716	0.936	0.793
Cash flow-to-price	37.9	0.832	0.765	0.933	0.792
Net debt-to-price	37.9	0.838	0.753	0.936	0.823
Earnings to price	37.9	0.811	0.743	0.935	0.745
Net payout yield	37.6	0.844	0.769	0.932	0.829
Years since first Compustat coverage	37.6	0.828	0.724	0.935	0.823
Cash flow to price ratio	37.6	0.818	0.737	0.934	0.775
Quick ratio	37.4	0.835	0.782	0.936	0.777
Altman's Z-score	37.4	0.828	0.740	0.931	0.808
Payout yield	37.1	0.851	0.785	0.932	0.831
Earnings volatility	37.1	0.852	0.779	0.936	0.840
Change in shares outstanding	37.1	0.805	0.671	0.932	0.815
Enterprise book-to-price	36.8	0.830	0.741	0.933	0.812
Cash holdings	36.8	0.826	0.740	0.935	0.797
Dividend to price	36.5	0.846	0.789	0.932	0.811
Depreciation / PP&E	36.5	0.851	0.813	0.930	0.801
Kaplan-Zingales Index	36.2	0.822	0.731	0.930	0.801
R&D-to-sales	36.2	0.815	0.731	0.923	0.786
Cash flow volatility	36.2	0.783	0.617	0.924	0.812
Accrual volatility	36.2	0.779	0.613	0.926	0.803
Current ratio	35.9	0.846	0.815	0.926	0.785
Idiosyncratic return volatility	35.6	0.851	0.799	0.923	0.828
Debt capacity/firm tangibility	35.6	0.829	0.735	0.920	0.832
Maximum daily return	35.3	0.838	0.764	0.927	0.821
Bid-ask spread	35.3	0.847	0.786	0.931	0.821
Cash productivity	35.3	0.819	0.751	0.911	0.789
Return volatility	34.7	0.844	0.786	0.922	0.820
Robust Minus Weak	34.7	0.773	0.694	0.910	0.705
Whited-Wu Index	34.7	0.781	0.697	0.913	0.724
New equity issue	34.7	0.756	0.620	0.912	0.732
Sales to price	34.7	0.832	0.768	0.919	0.804
High Minus Low	34.4	0.830	0.757	0.926	0.802
Vol. of liquidity (share turnover)	34.4	0.846	0.786	0.920	0.830
Market Beta	34.1	0.859	0.824	0.921	0.828
Zero trading days	34.1	0.855	0.808	0.918	0.836
Share turnover	34.1	0.857	0.815	0.917	0.834
Advertising Expense-to-market	34.1	0.810	0.707	0.914	0.809
Net equity finance	34.1	0.841	0.797	0.916	0.803
Asset turnover	34.1	0.788	0.643	0.911	0.815
Net external finance	32.1	0.827	0.781	0.900	0.793
Absolute accruals	31.8	0.818	0.750	0.903	0.799
Growth in long-term debt	31.5	0.767	0.678	0.902	0.711
Industry-adjusted book to market	30.9	0.810	0.771	0.901	0.748
Working capital accruals	30.6	0.812	0.748	0.900	0.783
HML Devil	30.3	0.820	0.747	0.905	0.805
Change in Net Financial Assets	29.4	0.697	0.581	0.907	0.583
Chg in Current Oper. Liabilities	28.2	0.773	0.710	0.904	0.690
Sin stocks	27.6	0.749	0.603	0.884	0.762
Sales to receivables	27.4	0.820	0.781	0.896	0.777
Employee growth rate	22.6	0.773	0.710	0.898	0.699
Net Operating Assets	16.8	0.778	0.664	0.900	0.767
HXZ Investment	13.2	0.797	0.739	0.892	0.753
Chg in Net Non-current Oper. Assets	8.2	0.791	0.729	0.886	0.753
Financial statements score	7.9	0.738	0.700	0.885	0.605
R&D Expense-to-market	7.6	0.804	0.770	0.883	0.751
R&D increase	5.3	0.742	0.676	0.873	0.664
Industry momentum	2.9	0.772	0.748	0.840	0.721
Abnormal Corporate Investment	2.9	0.674	0.497	0.866	0.654
Sales growth	2.4	0.761	0.706	0.876	0.690
Conservative Minus Aggressive	1.8	0.766	0.716	0.860	0.714
Momentum	1.2	0.755	0.715	0.793	0.758

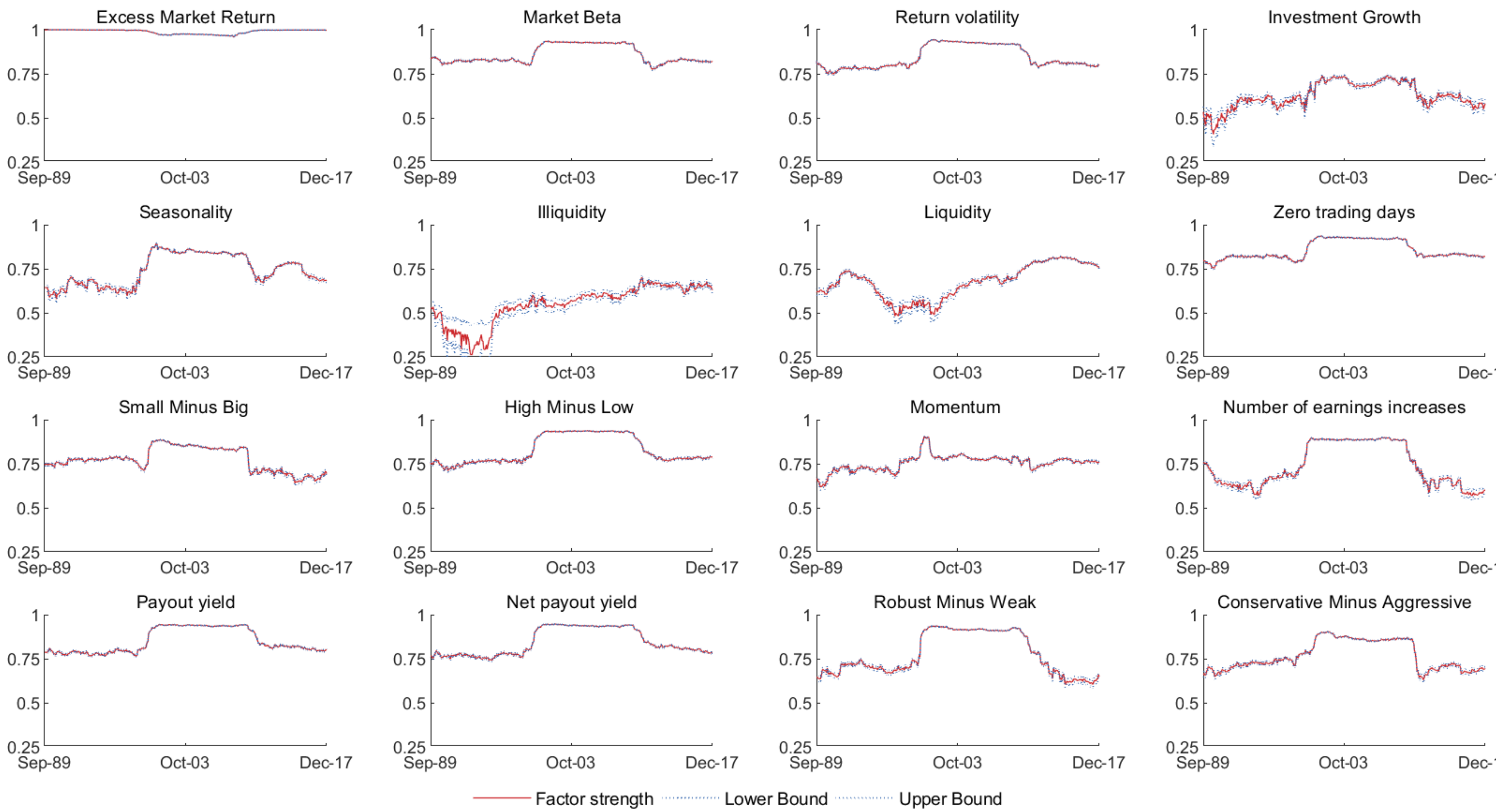
Factor	% of months when $\hat{\alpha}_{s,\tau} > 0.90$ over: Full sample	Time averages of $\hat{\alpha}_{s,\tau}$ over: Full sample	September 1989 - August 1999	September 1999 - August 2009	September 2009 - December 2017
Change in Short- term Investments	0.3	0.625	0.377	0.801	0.712
Return on net operating assets	0.3	0.764	0.645	0.877	0.773
Investment Growth	0.0	0.627	0.565	0.698	0.617
Seasonality	0.0	0.743	0.648	0.844	0.735
Illiquidity	0.0	0.549	0.433	0.578	0.652
Liquidity	0.0	0.674	0.624	0.632	0.787
Small Minus Big	0.0	0.774	0.766	0.846	0.697
Number of earnings increases	0.0	0.738	0.658	0.883	0.659
HXZ Profitability	0.0	0.778	0.748	0.835	0.746
Share price	0.0	0.706	0.721	0.673	0.727
Industry-adj. cash flow to price ratio	0.0	0.672	0.592	0.766	0.655
Industry-adjust. chg in employees	0.0	0.626	0.599	0.684	0.588
Change in 6-month momentum	0.0	0.642	0.654	0.602	0.676
Earnings announcement return	0.0	0.514	0.511	0.556	0.468
Revenue surprise	0.0	0.702	0.654	0.818	0.620
Return on assets	0.0	0.691	0.699	0.764	0.594
Betting Against Beta	0.0	0.767	0.645	0.872	0.787
Quality Minus Junk	0.0	0.793	0.774	0.855	0.740
Dollar trading volume	0.0	0.723	0.611	0.864	0.688
Vol. of liquidity (dollar trading volume)	0.0	0.619	0.580	0.647	0.632
Price delay	0.0	0.763	0.770	0.778	0.737
Book Asset Liquidity	0.0	0.833	0.811	0.866	0.821
Abnormal earnings announc. volume	0.0	0.763	0.728	0.806	0.751
Unexpected quarterly earnings	0.0	0.632	0.636	0.619	0.641
Cash flow to debt	0.0	0.690	0.645	0.747	0.674
% change in current ratio	0.0	0.606	0.448	0.817	0.541
% change in quick ratio	0.0	0.595	0.445	0.783	0.549
% change sales-to-inventory	0.0	0.574	0.500	0.756	0.446
Sales to inventory	0.0	0.770	0.832	0.728	0.746
% change in depreciation	0.0	0.647	0.430	0.834	0.683
Capital turnover	0.0	0.773	0.795	0.771	0.749
% chg in gross margin - % chg in sales	0.0	0.581	0.534	0.626	0.585
% chg in sales - % chg in inventory	0.0	0.571	0.536	0.704	0.453
% chg in sales - % chg in A/R	0.0	0.621	0.568	0.749	0.529
% chg in sales - % chg in SG&A	0.0	0.579	0.522	0.663	0.545
Effective Tax Rate	0.0	0.531	0.551	0.478	0.569
Labor Force Efficiency	0.0	0.568	0.517	0.573	0.623
Ohlson's O-score	0.0	0.690	0.645	0.733	0.694
Industry adjg % chg in capital expend.	0.0	0.642	0.467	0.786	0.678
Change in inventory	0.0	0.677	0.744	0.684	0.588
Change in tax expense	0.0	0.670	0.640	0.708	0.659
Growth in long term net oper. assets	0.0	0.659	0.589	0.645	0.759
Order backlog	0.0	0.783	0.717	0.831	0.806
Chg in Long-term Net Operating Assets	0.0	0.731	0.639	0.834	0.719
Corporate investment	0.0	0.710	0.650	0.803	0.672
Changes in Net Operating Assets	0.0	0.529	0.481	0.577	0.528
Tax income to book income	0.0	0.693	0.544	0.848	0.686
Growth in common shareholder equity	0.0	0.757	0.697	0.814	0.761
Chg in Current Operating Assets	0.0	0.723	0.760	0.802	0.584
Chg in Net Non-cash Working Capital	0.0	0.639	0.691	0.671	0.536
Chg in Non-current Operating Assets	0.0	0.725	0.651	0.811	0.709
Chg in Non-current Oper. Liabilities	0.0	0.711	0.638	0.768	0.732
Total accruals	0.0	0.659	0.585	0.769	0.616
Change in Financial Liabilities	0.0	0.648	0.604	0.797	0.523
Change in Book Equity	0.0	0.776	0.706	0.857	0.764
Financial statements score	0.0	0.729	0.681	0.759	0.751
Growth in capital expenditures	0.0	0.622	0.566	0.602	0.713
Three-year Investment Growth	0.0	0.749	0.664	0.819	0.766
Composite Equity Issuance	0.0	0.784	0.774	0.833	0.737
Net debt finance	0.0	0.668	0.603	0.844	0.535
Revenue Surprises	0.0	0.622	0.692	0.583	0.584
Industry Concentration	0.0	0.821	0.820	0.870	0.763
Return on invested capital	0.0	0.734	0.754	0.827	0.600
Chg in PPE and Inventory-to-assets	0.0	0.697	0.663	0.675	0.763
Composite Debt Issuance	0.0	0.696	0.738	0.735	0.597
Profit margin	0.0	0.773	0.798	0.761	0.758

Table S20 continued from previous page

	% of months when $\hat{\alpha}_{s,\tau} > 0.90$ over:	Time averages of $\hat{\alpha}_{s,\tau}$ over:			
Factor	Full sample	Full sample	September 1989 - August 1999	September 1999 - August 2009	September 2009 - December 2017
Industry-adj. change in asset turnover	0.0	0.616	0.650	0.618	0.573
Industry-adj. change in profit margin	0.0	0.521	0.427	0.583	0.559
Capital expenditures and inventory	0.0	0.702	0.664	0.688	0.765
Industry-adj. Real Estate Ratio	0.0	0.810	0.751	0.872	0.807
Percent accruals	0.0	0.727	0.704	0.792	0.678
Operating Leverage	0.0	0.801	0.784	0.817	0.803
Inventory Growth	0.0	0.626	0.714	0.552	0.608
Percent Operating Accruals	0.0	0.755	0.726	0.824	0.707
Enterprise multiple	0.0	0.722	0.742	0.704	0.719
Gross profitability	0.0	0.774	0.792	0.774	0.754
Organizational Capital	0.0	0.787	0.785	0.784	0.791
Convertible debt indicator	0.0	0.767	0.798	0.809	0.680
Long-Term Reversal	0.0	0.565	0.518	0.590	0.591
1-month momentum	0.0	0.714	0.767	0.647	0.732
6-month momentum	0.0	0.646	0.515	0.727	0.706
36-month momentum	0.0	0.732	0.726	0.798	0.660
Growth in advertising expense	0.0	0.622	0.501	0.812	0.540

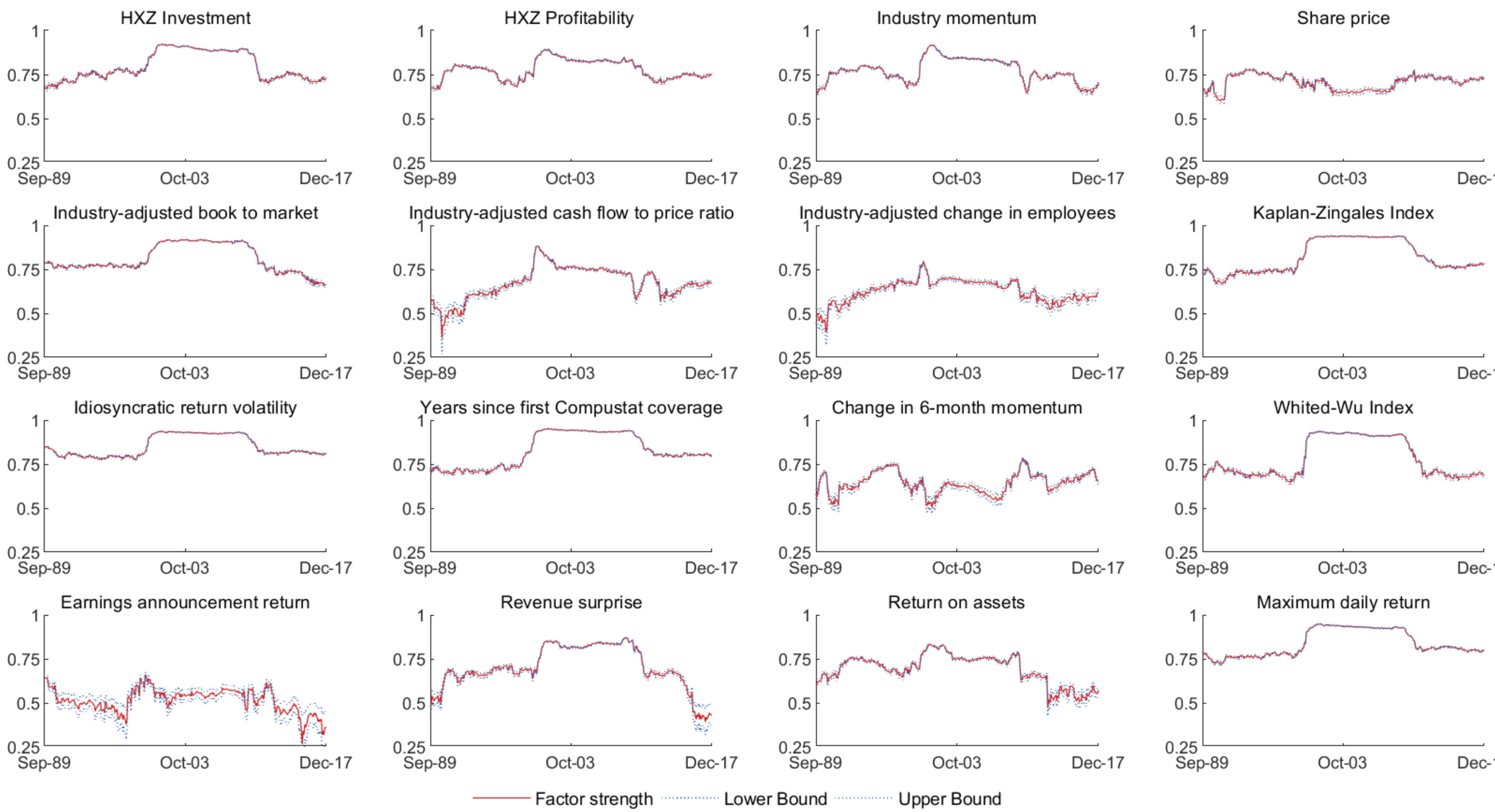
Notes: All factor strength estimates, $\hat{\alpha}_{s,\tau}$, where $s = 1, 2, \ldots, 145$, are computed using (7) for 10-year rolling windows $\tau = 1, 2, \ldots, 340$.

Figure S13: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



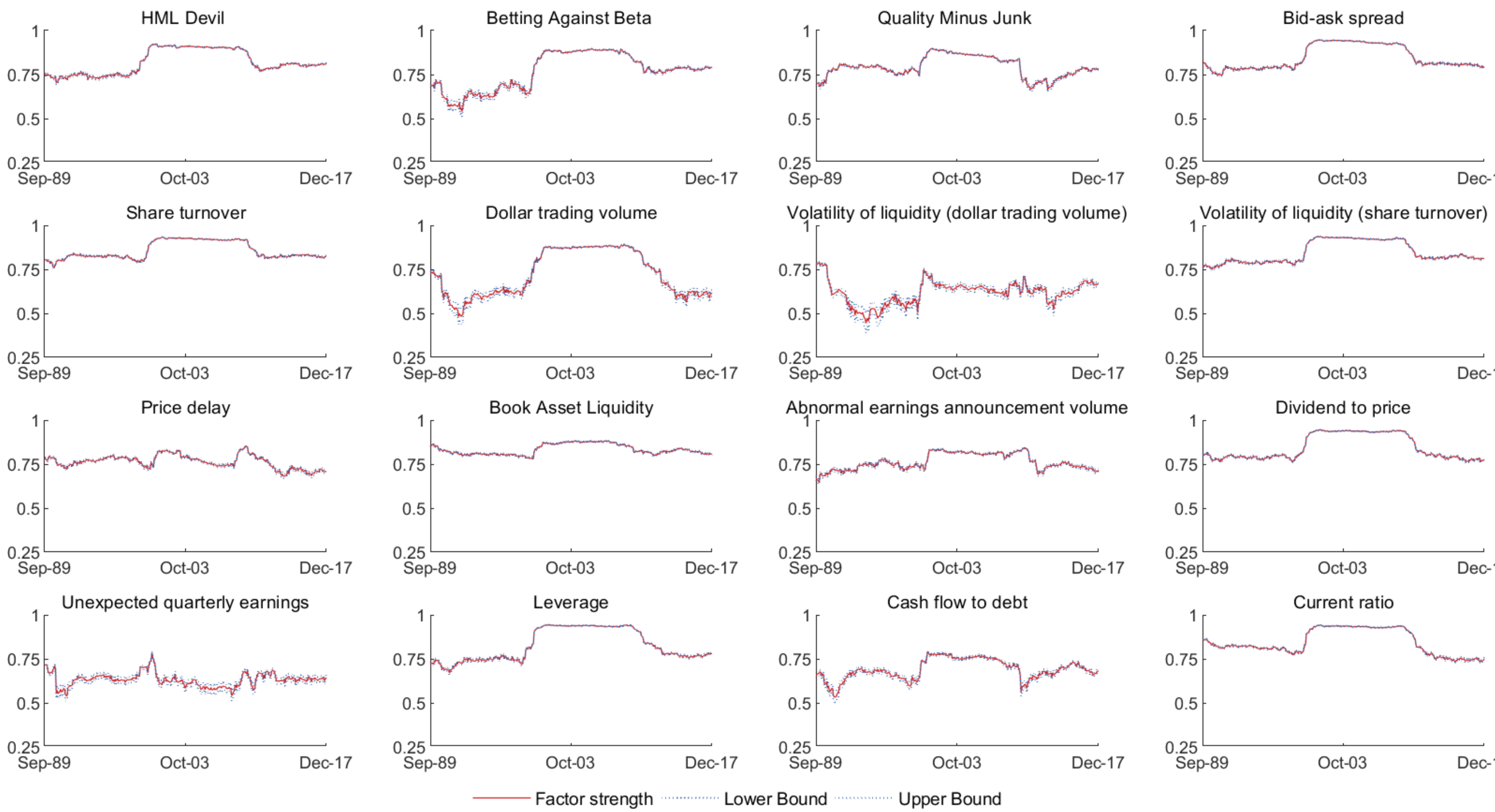
Notes: All factor strength rolling estimates are computed using (7). Only the market factor strength estimates are generated from the original CAPM single factor representation. The remaining factor strength estimates are produced from capital asset pricing models which include the market factor in addition to one extra factor (145 in total). The 90% confidence bands are computed using (24).

Figure S14: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



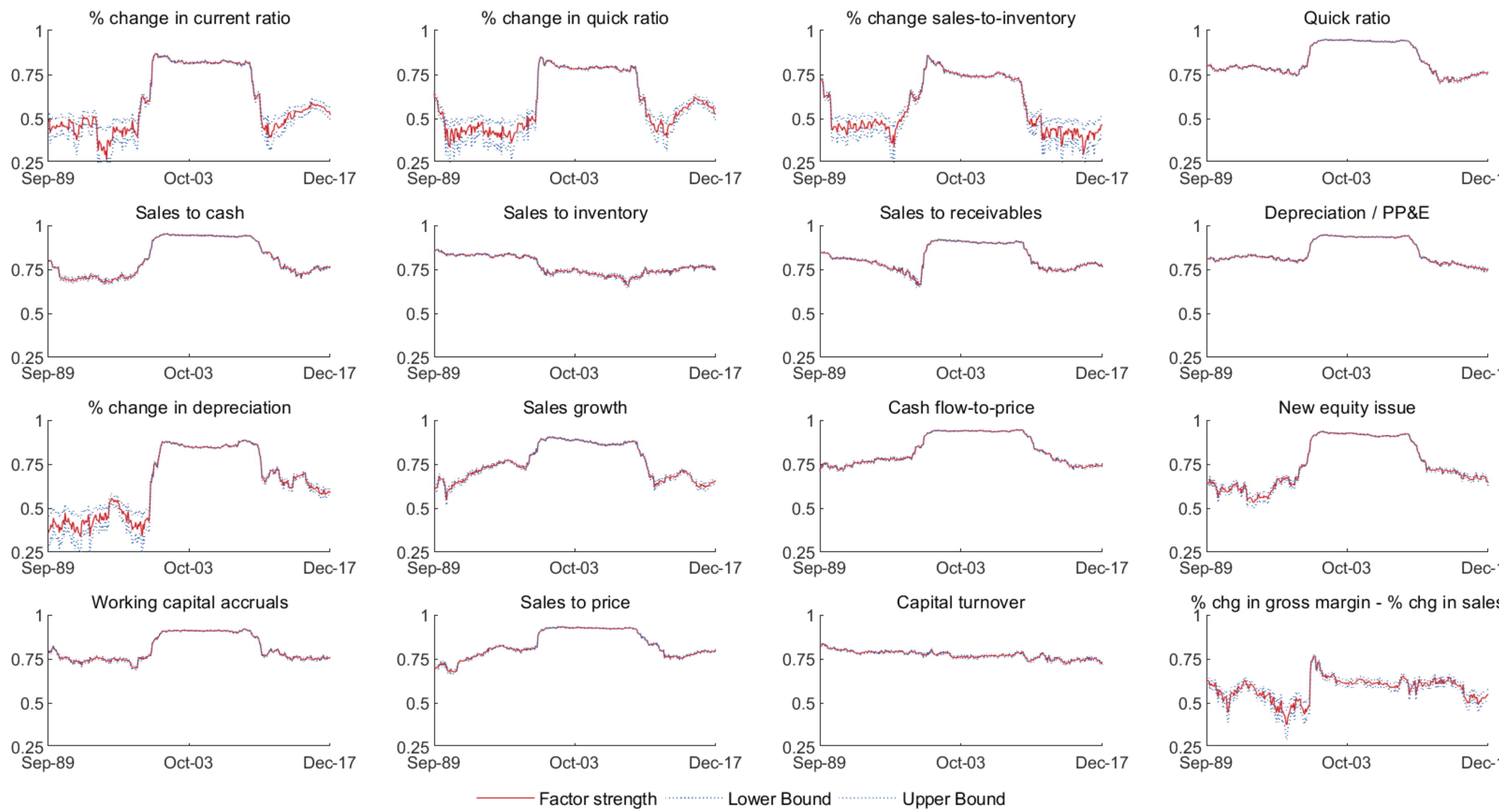
Notes: See notes to Figure S13.

Figure S15: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



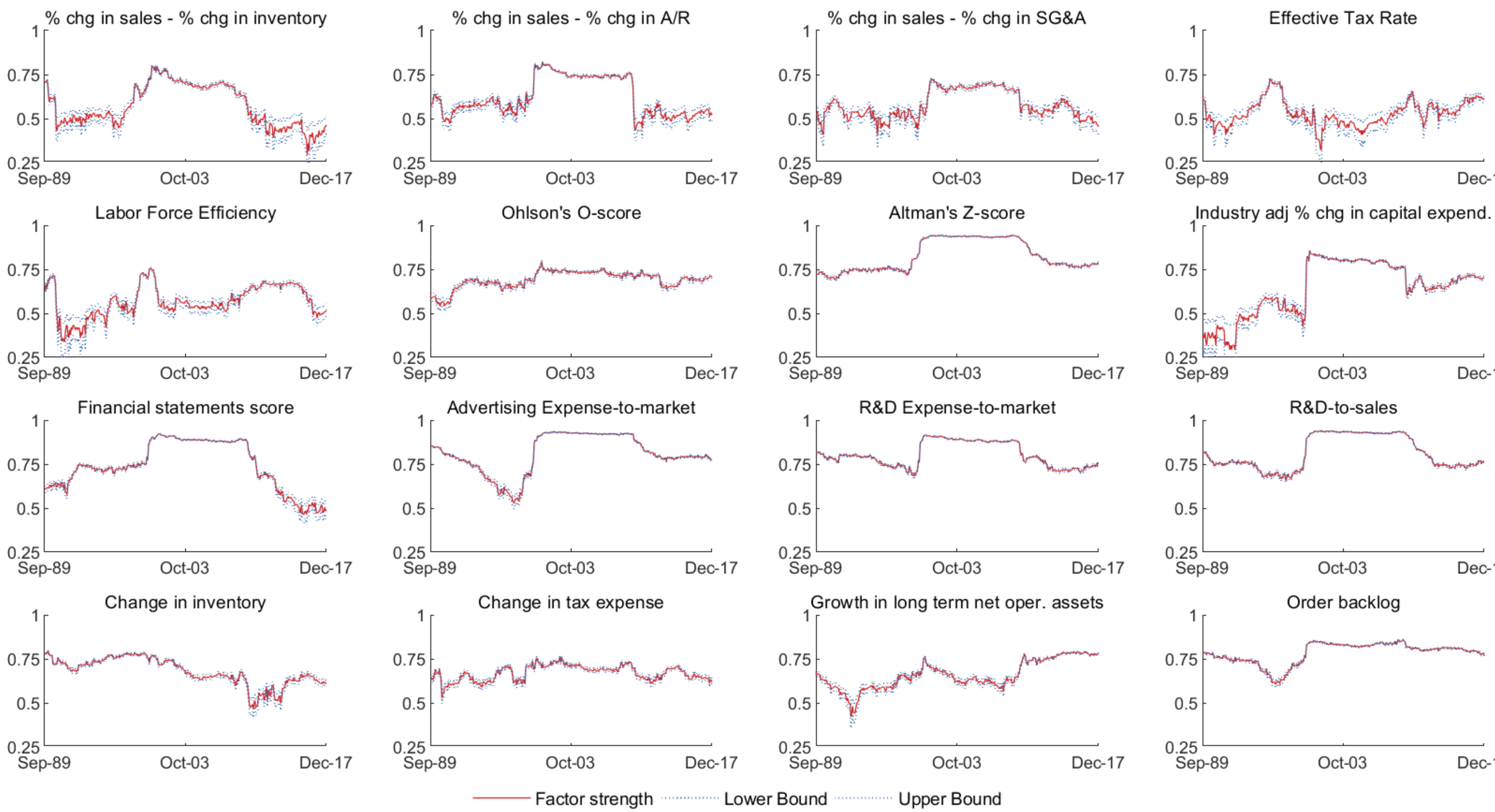
Notes: See notes to Figure S13.

Figure S16: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



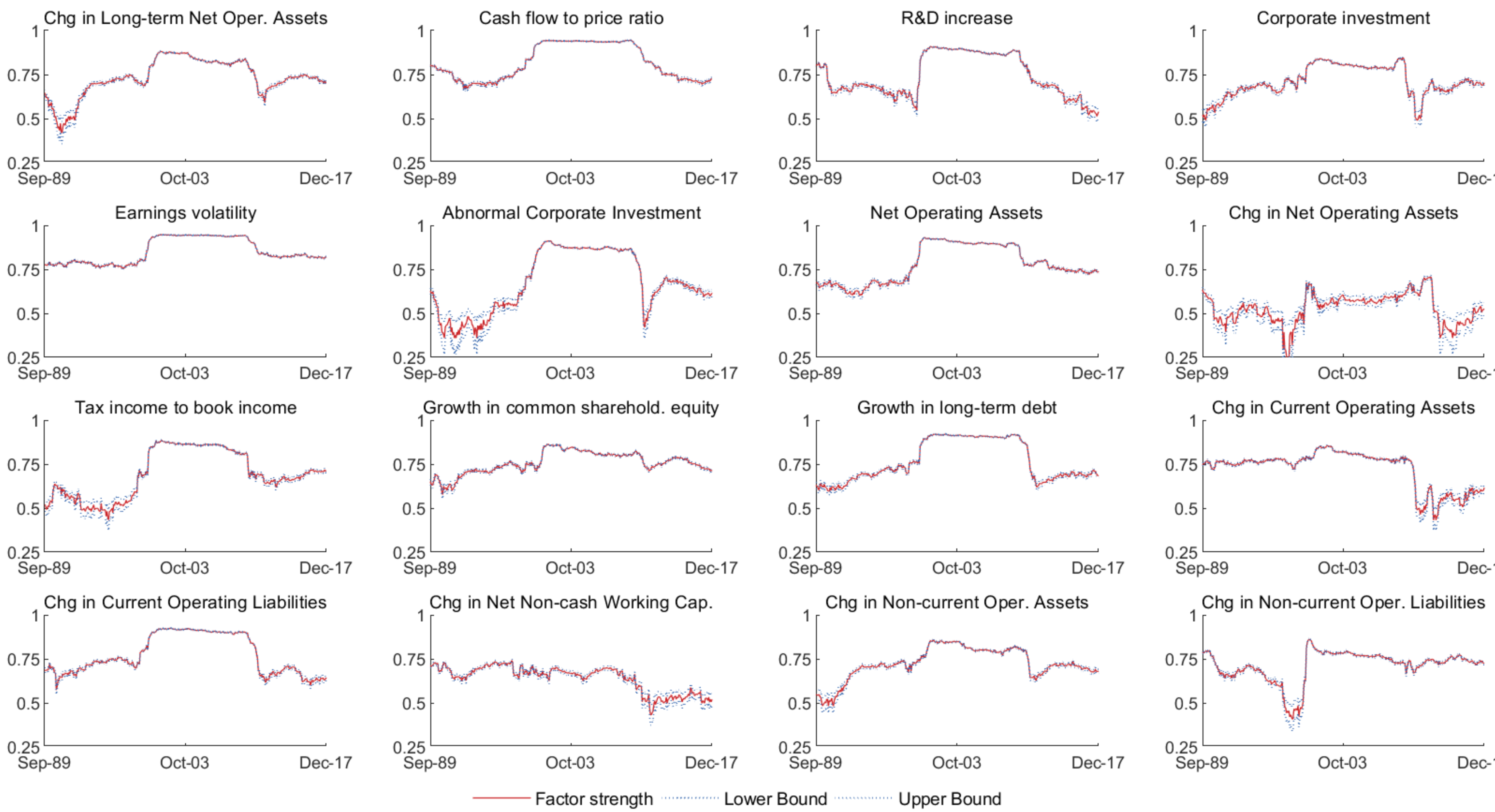
Notes: See notes to Figure S13.

Figure S17: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



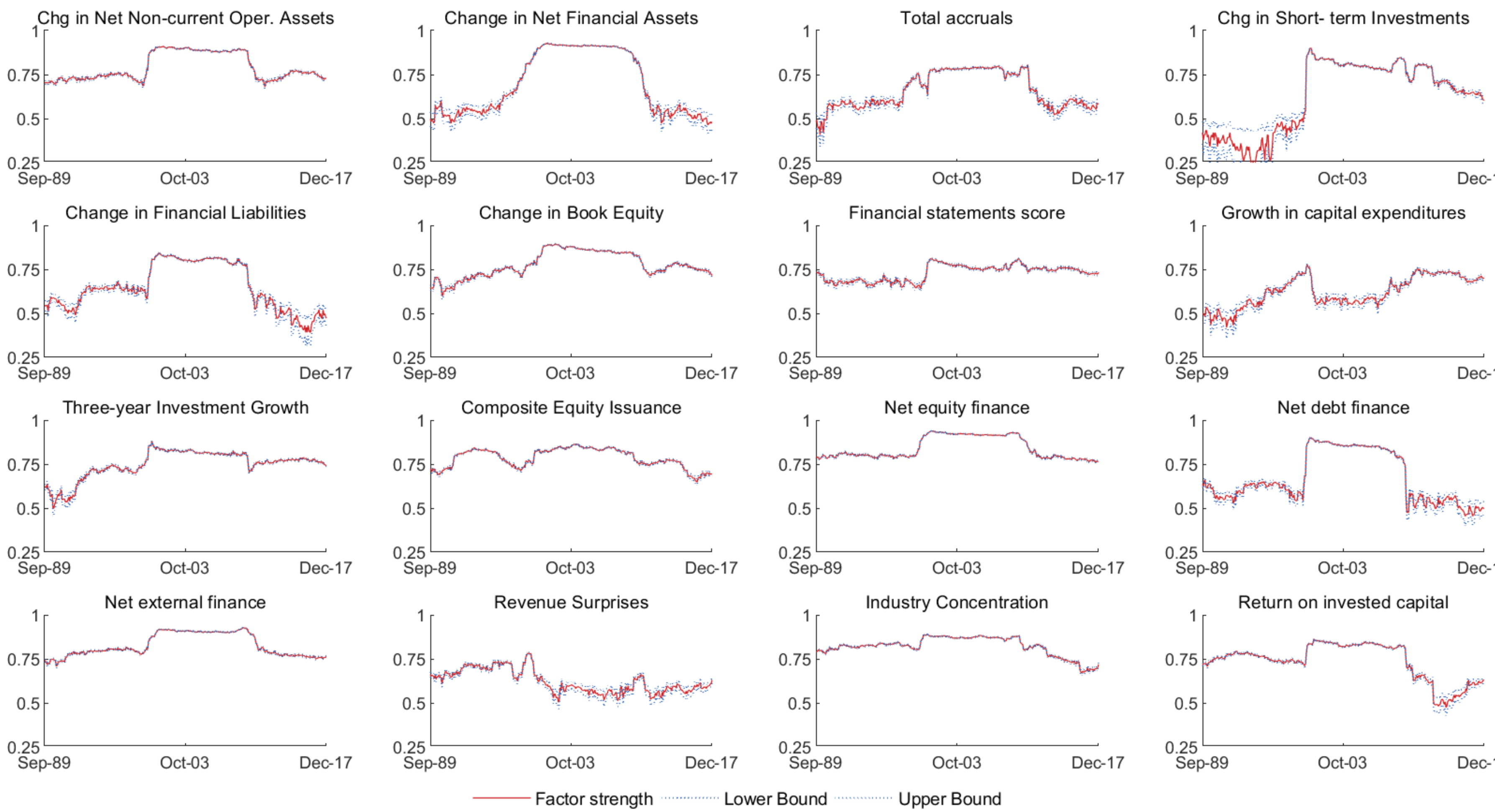
Notes: See notes to Figure S13.

Figure S18: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



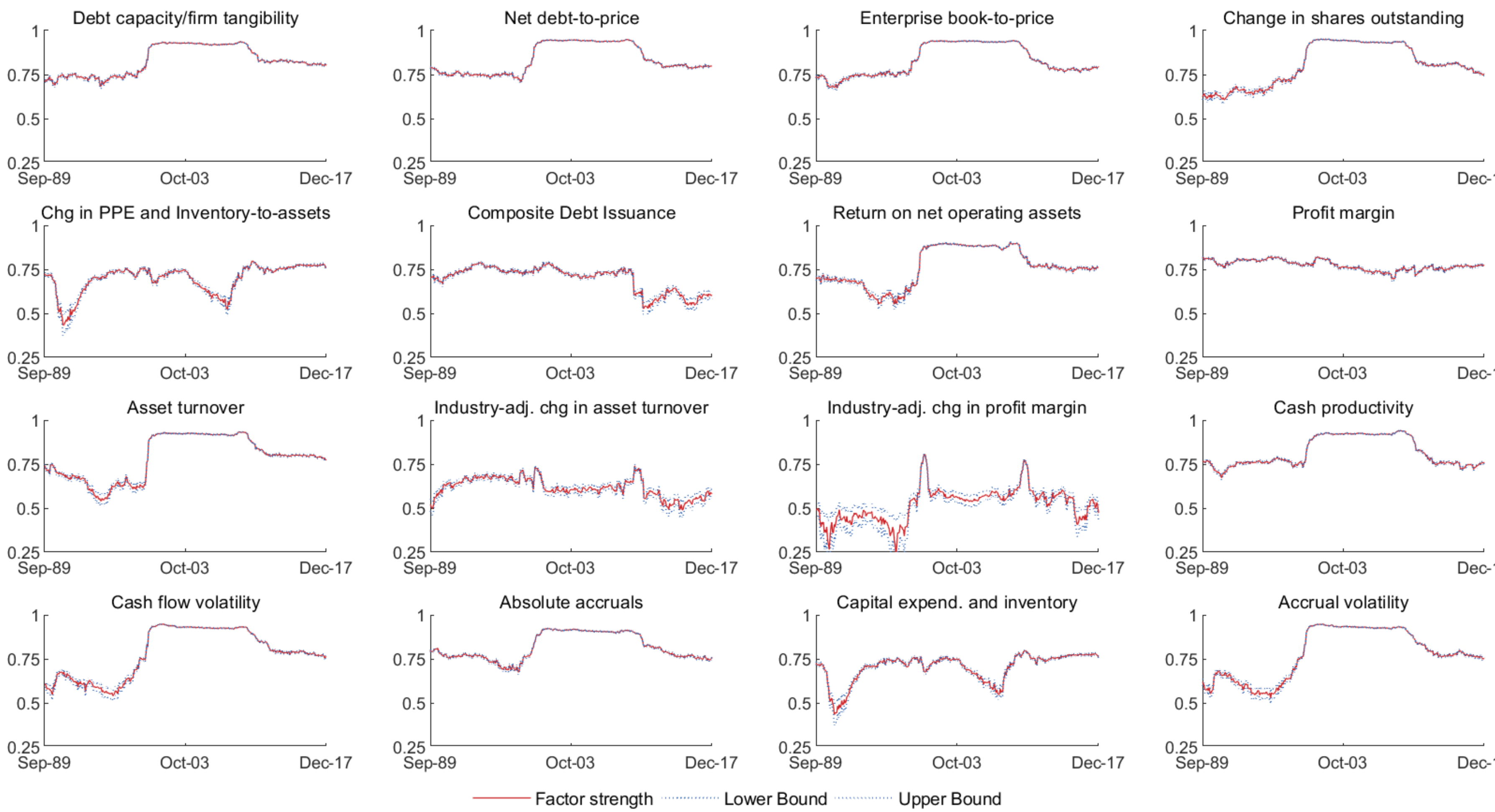
Notes: See notes to Figure S13.

Figure S19: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



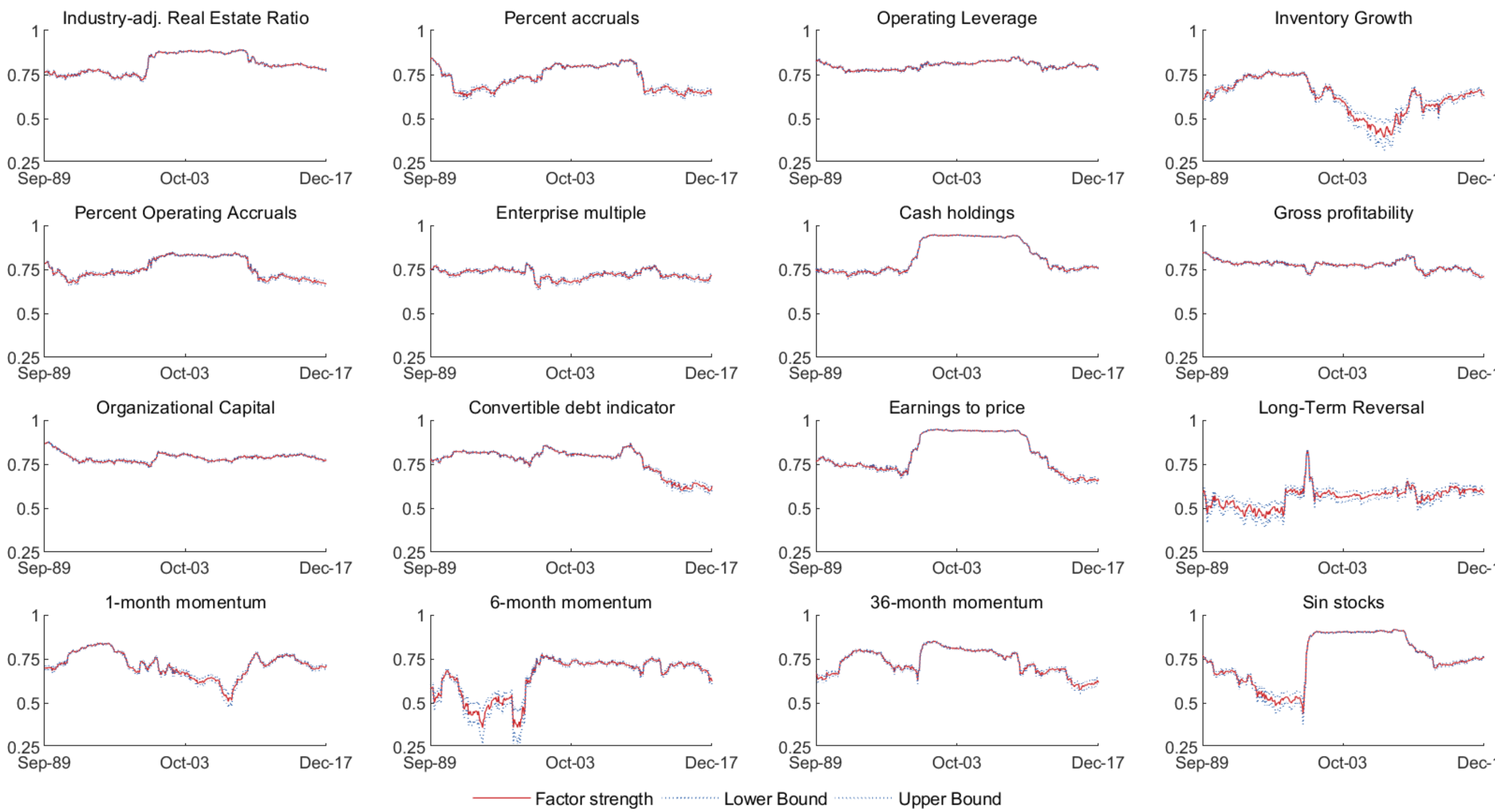
Notes: See notes to Figure S13.

Figure S20: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



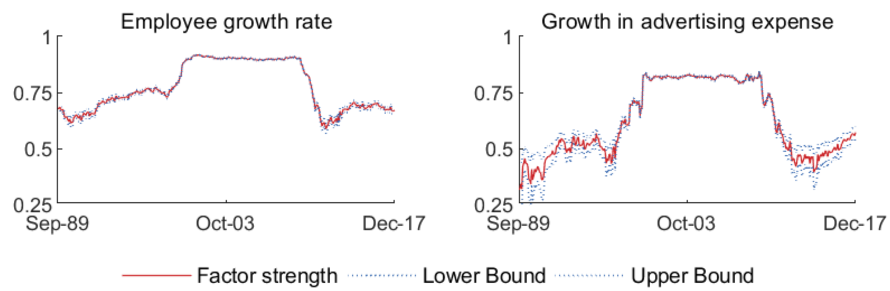
Notes: See notes to Figure S13.

Figure S21: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



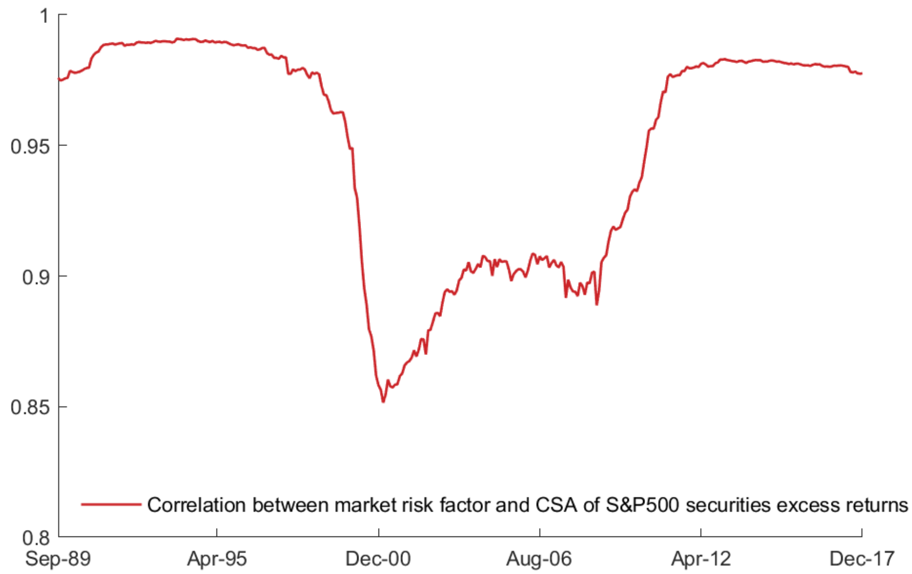
Notes: See notes to Figure S13.

Figure S22: Strength estimates of 145 observed factors used in asset pricing models which include the market factor plus one additional factor in their specification, over 10-year rolling samples



Notes: See notes to Figure S13.

Figure S23: Correlation coefficients between the market risk factor and the cross section average of S&P500 securities' excess returns over 10-year rolling windows



Notes: The correlation coefficients are computed over $\tau = 1, 2, \ldots, 340$ rolling windows.

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