Local Asymptotic Normality of the spectrum of high-dimensional spiked F-ratios

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Abstract

We consider two types of *spiked* multivariate F distributions: a scaled distribution with the scale matrix equal to a rank-k perturbation of the identity, and a distribution with trivial scale, but rank-k non-centrality. The eigenvalues of the rank-r matrix (*spikes*) parameterize the joint distribution of the eigenvalues of the corresponding F matrix. We show that, for the *spikes* located above a *phase transition* threshold, the asymptotic behavior of the log ratio of the joint density of the eigenvalues of the F matrix to their joint density under a local deviation from these values depends only on the k of the largest eigenvalues $\lambda_1, ..., \lambda_k$. Furthermore, we show that $\lambda_1, ..., \lambda_k$ are asymptotically jointly normal, and the *statistical experiment* of observing all the eigenvalues of the F matrix converges in the Le Cam sense to a *Gaussian shift experiment* that depends on the asymptotic means and variances of $\lambda_1, ..., \lambda_k$. In particular, the best statistical inference about sufficiently large *spikes* in the local asymptotic regime is based on the k of the largest eigenvalues only.

Key words: Spiked F-ratio, Local Asymptotic Normality, multivariate F distribution, phase transition, super-critical regime, asymptotic normality of eigenvalues, limits of statistical experiments.

1 Introduction

In this paper we establish the *Local Asymptotic Normality* (LAN) of the *statistical experiments* of observing the eigenvalues of the F-ratio, $B^{-1}A$, of two high-dimensional independent *Wishart* matrices, A and B. We consider two situations.

First, both A and B are central Wisharts with dimensionality and degrees of freedom that grow proportionally, and with the covariance parameters that differ by a matrix of fixed rank k. Second, A and B have the same covariance parameter, but A is a non-central Wishart with the non-centrality parameter of rank k. In both cases, the joint distribution of the eigenvalues of $B^{-1}A$ depends on the eigenvalues of the rank-k matrix, which we call the *spikes*. We find that the considered *statistical experiments* are *LAN* under a local parameterization of the *spikes* when the localities are above a *phase transition* threshold.

Many classical multivariate statistical tests are based on the eigenvalues of F-ratio matrices. For example, all tests of the equality of two covariance matrices and of the *general linear hypothesis* in the *Multivariate Linear Model* described in Muirhead's (1982) chapters 8 and 10 are of this form. Contemporaneous statistical applications often require the dimensionality of the F-ratio and its degrees of freedom be large and comparable. Therefore, we consider the asymptotic regime where the dimensionality and the degrees of freedom diverge to infinity at the same rate.

Our requirement that the parameters of the two Wisharts differ by a rank-k matrix corresponds to situations where the data that generate A contain k factors or signals, which are absent from the data that generate B. Inference conditional on factors requires considering non-central F-ratios, whereas the unconditional inference leads to F-ratios with unequal covariances.

The main result of this paper can be summarized as follows. We show that the asymptotic behavior of the log ratio of the joint density of the eigenvalues of $B^{-1}A$, which corresponds to sufficiently large values of the *spikes*, to their joint density under a local deviation from these values depends only on the k of the largest eigenvalues $\lambda_1 \geq ... \geq \lambda_k$. Furthermore, $\lambda_1, ..., \lambda_k$ are asymptotically jointly normal, and the *statistical experiment* of observing all the eigenvalues of $B^{-1}A$ converges in the Le Cam sense to a *Gaussian shift experiment* that depends on the asymptotic means and variances of $(\lambda_1, ..., \lambda_k)$. In particular, the best statistical inference about k sufficiently large *spikes* in the local asymptotic regime is based on the k of the largest eigenvalues *only*. Such an optimality result is new, and it is the most important contribution of this paper.

We derive an explicit formula for the *phase transition* threshold demarcating the area of the sufficiently large *spikes*. We show that, if the parameters of A and B differ by a rank-k matrix Δ, and the norm of Δ is below the threshold, any finite number of the largest eigenvalues of $B^{-1}A$ almost surely converge to the upper boundary of the support of the limiting spectral distribution of $B^{-1}A$, derived by Wachter (1980) and Silverstein (1985). In contrast, when m of the largest eigenvalues of Δ are above the threshold, we find that the m of the largest eigenvalues of $B^{-1}A$ almost surely converge to locations strictly above the upper boundary of the Wachter-Silverstein distribution, and that their local fluctuations about these limits are asymptotically jointly normal.

In a setting of two independent and not necessarily normal samples, the *phase transition* phenomenon has been studied in Nadakuditi and Silverstein (2010). They obtain a formula for the threshold, and establish the almost sure limits of the m largest eigenvalues for the case where Δ describes the difference between covariance matrices of the two samples. The limiting distribution of fluctuations above the threshold is described in their paper as an open problem. Our paper solves this problem for the case of two normal samples.

The *phase transition* phenomenon for a single Wishart matrix has also been a subject of active recent research. Baik et al (2005) study the joint distributions of a few of the largest eigenvalues of *complex Wisharts* with spiked covariance parameters. They derive the asymptotic distributions of a few of the largest eigenvalues, which turn out to be different depending on whether the sizes of the corresponding spikes are below, at, or above a *phase transition* threshold, the situations often referred to as the *sub-critical*, *critical*, and *super-critical regimes.*

Similar transition takes place for real Wisharts. Paul (2007) establishes asymptotic normality of the fluctuations of a few of the largest eigenvalues in the *super-critical regime* of the real case. Féral and Péché (2009), Benaych-Georges et al (2011) and Bao et al (2014a) show that the fluctuations in the *sub-critical* real case have the *Tracy-Widom distribution*, while Mo (2012) and Bloemendal and Viràg (2011, 2013) establish the asymptotic distribution of a different type in the critical regime. In a setting of two normal samples, Bao et al (2014b) study the almost sure limits of the sample canonical correlations when the population canonical correlations are below and when they are above a *phase transition* threshold.

Our results on the joint asymptotic normality of the largest eigenvalues in the *super-critical regime* for F-ratios can be used to make statistical inference about the eigenvalues of the "ratio" of the population covariances of A and B, or the eigenvalues of the non-centrality parameter of A. The estimates of these eigenvalues play important role in MANOVA and the discriminant analysis, and can also be used in constructing modified model selection criteria as discussed in Sheena et al (2004). Further, they may be important in as diverse applications as constructing genetic selection indices and describing a degree of financial turbulence (see Hayes and Hill (1981), and Kritzman and Li (2010)).

We expect that our asymptotic normality results can be extended to the case of the "ratio" of two sample covariance matrices constructed from non-normal samples. In the one-sample case, an extension of Paul's (2007) asymptotic normality results has been done in Bai and Yao (2008). In this paper, we focus on normal data. This focus is dictated by our main goal: establishing the *LAN* property of the statistical experiments of observing the eigenvalues of $B^{-1}A$. To reach this goal, we derive an asymptotic approximation to a log likelihood process by representing it in the form of a multiple contour integral, and applying the *Laplace approximation* method. The explicit form of the joint distribution of the eigenvalues of $B^{-1}A$ is known only in the normal case, and we need such an explicit form for our analysis.

A decision-theoretic approach to the finite sample estimation of the eigenvalues of the "ratio" of the population covariances of A and B, or the eigenvalues of the non-centrality parameter of A was taken in many previous studies (see Sheena et al (2004), Bilodeau and Srivastava (1992), and references therein). In one of the first such studies, Muirhead and Verathaworn (1985) explain that the ideal decision-theoretic approach that directly analyzes expected loss with respect to the joint distribution of the eigenvalues of $B^{-1}A$ "does not seem feasible due primarily to the complexity of the distribution of the ordered latent roots..." Instead, they focus on deriving an optimal estimator from a particular class.

Our *LAN* result makes possible an asymptotic implementation of the ideal decision-theoretic approach. We overcome the complexity of the joint distribution of the eigenvalues by using a tractable multiple contour integral representation of the log likelihood process, which follows from the multiple contour integral representation of hypergeometric functions of two matrix arguments, established in Onatski (2013), Dharmawansa and Johnstone (2014), and Passemier et al (2014).

It is interesting to contrast the *LAN* result in the *super-critical* regime with the asymptotic behavior of the log likelihood ratio in the case of a *sub-critical* spike. In a separate research effort, we follow Onatski et al (2013), who analyze the log likelihood ratio in the *sub-critical* regime for the case of a single Wishart matrix, to show that the experiment of observing the eigenvalues of $B^{-1}A$ in the *sub-critical* regime is not of the *LAN* type. Furthermore, the log-likelihood process turns out to depend only on a smooth functional of the empirical distribution of all the eigenvalues of $B^{-1}A$, so that asymptotically efficient inference procedures may ignore the information contained in $\lambda_1, ..., \lambda_k$ altogether. The results of this *sub-critical* analysis will be published elsewhere.

The rest of the paper is structured as follows. In the next section, we describe our setting. In Section 3, we derive the almost sure limits of a few of the largest eigenvalues of the F-ratio. In Section 4, we establish the asymptotic normality of the eigenvalue fluctuations in the *super-critical* regime. In Section 5, we derive an asymptotic approximation to the joint distribution of the eigenvalues of $B^{-1}A$ for the case of k *super-critical* spikes. In Section 6, we show that the likelihood ratio in the local parameter space is asymptotically equivalent to a linear combination of k of the largest eigenvalues, and establish the LAN property. Section 7 concludes.

2 Setup

Suppose that

$$A \sim W_p\left(n_1 + k, \Sigma_1, \Omega_1\right) \qquad \text{and} \qquad B \sim W_p\left(n_2, \Sigma_2\right)$$

are independent non-central and central Wishart matrices respectively. For the non-centrality parameter Ω_1, we use a symmetric version of the definition in Muirhead (1982, p. 442). That is, if Z is an $n \times p$ matrix distributed as $N\left(M, I_n \otimes \Sigma\right)$, then $Z'Z \sim W_p\left(n, \Sigma, \Omega\right)$ with the non-centrality parameter $\Omega = \Sigma^{-1/2}M'M\Sigma^{-1/2}$. We will consider two different settings for the parameters Σ_1, Σ_2, and Ω_1.

Setting 1 (Spiked covariance) $\Sigma_2 = \Sigma$, $\Sigma_1 = \Sigma^{1/2}\left(I_p + VhV'\right)\Sigma^{1/2}$, and $\Omega_1 = 0$. Here $\Sigma^{1/2}$ is the symmetric square root of a positive definite matrix Σ; V in a $p \times k$ matrix of nuisance parameters with orthonormal columns, and $h = \text{diag}\left\{h_1, ..., h_k\right\}$ is the diagonal matrix of the "covariance spikes" with $h_1 > ... > h_k$.

Setting 2 (Spiked non-centrality) $\Sigma_2 = \Sigma$, $\Sigma_1 = \Sigma$, and $\Omega_1 = \left(n_1 + k\right)VhV'$, where Σ, V, and h are as defined above, but h_j with $j = 1, ..., k$ are interpreted as "non-centrality spikes."

We are interested in the behavior of the eigenvalues of

$$\mathbf{F} \equiv \left(B/n_2\right)^{-1} A/n_A,$$

where

$$n_A = n_1 + k,$$

as n_1, n_2, and p grow so that $p/n_1 \to c_1$ and $p/n_2 \to c_2$ with $0 < c_i < 1$, while k, the number of spikes, remains fixed. In what follows, we will assume that $\Sigma = I_p$. This assumption is without loss of generality because the eigenvalues of $\mathbf{F}$ do not change under the transformation $A \mapsto \Sigma^{-1/2} A \Sigma^{-1/2}$, $B \mapsto \Sigma^{-1/2} B \Sigma^{-1/2}$.

It is convenient to think of A/n_A as a sample covariance matrix XX'/n_A of the sample X having the factor structure

$$X = V\mathcal{F}' + \varepsilon \tag{1}$$

with $V, \mathcal{F}$, and ε playing the roles of the factor loadings, factors, and idiosyncratic terms, respectively. Matrices $\mathcal{F}$ and ε are mutually independent, and independent from B. The distribution of ε is $N\left(0, I_p \otimes I_{n_A}\right)$, and the distribution of $\mathcal{F}$ depends on the setting. For Setting 1, $\mathcal{F} \sim N\left(0, I_p \otimes h\right)$, whereas for Setting 2, $\mathcal{F}$ is a deterministic matrix such that $\mathcal{F}'\mathcal{F}/n_A = h$. With this interpretation, Settings 1 and 2 describe, respectively, distributions which are unconditional and conditional on the factors. In both cases the spike parameters h_j, $j = 1, ..., k$, measure the factors' variability.

We would like to introduce a convenient representation of the eigenvalues of $\mathbf{F}$, that we will denote as $\lambda_1 \geq ... \geq \lambda_p$. First, note that λ_j, $j = 1, ..., p$, are invariant with respect to the simultaneous transformations

$$A \mapsto UAU' \equiv n_A \tilde{H} \qquad \text{and} \qquad B \mapsto UBU' \equiv n_2 E, \tag{2}$$

where U is a random matrix uniformly distributed over the orthogonal group $\mathcal{O}(p)$. Under the assumption that $\Sigma = I_p$, matrix $n_2 E$ is distributed as $W_p(n_2, I_p)$ and is independent from $\tilde{H}$. Matrix $\tilde{H}$ has the form $\tilde{X}\tilde{X}'/n_A$, where

$$\tilde{X} = \tilde{V}\mathcal{F}' + \tilde{\varepsilon}$$

with $\tilde{\varepsilon} \sim N\left(0, I_p \otimes I_{n_A}\right)$ independent from $\tilde{V}$, and $\tilde{V}$ being a random $p \times k$ matrix uniformly distributed on the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^p$. We can think of $\tilde{V}$ as having the form

$$\tilde{V} = v\left(v'v\right)^{-1/2} \equiv v W_v^{-1/2},$$

where $v \sim N(0, I_p \otimes I_k)$ and $W_v \equiv v'v$ is Wishart $W_k(p, I_k)$.

Further, let $O_{\mathcal{F}} \in \mathcal{O}(n_A)$ be such that the submatrix of its first k columns equals $\mathcal{F}(\mathcal{F}'\mathcal{F})^{-1/2}$, and let $\hat{X} = \tilde{X}O_{\mathcal{F}}$. Clearly,

$$\tilde{H} = \tilde{X}\tilde{X}'/n_A = \hat{X}\hat{X}'/n_A, \tag{3}$$

and matrix $\hat{X}$ has the form

$$\hat{X} = vW_v^{-1/2}h^{1/2}W_{\mathcal{F}}^{1/2} + \hat{\varepsilon},$$

where $v, W_{\mathcal{F}}$ and $\hat{\varepsilon}$ are mutually independent and independent from E; $\hat{\varepsilon} \sim N(0, I_p \otimes I_{n_A})$; and the distribution of $W_{\mathcal{F}}$ depends on the setting. For Setting 1, $W_{\mathcal{F}} \sim W_k(n_A, I_k)$, whereas for Setting 2, $W_{\mathcal{F}} = n_A I_k$.

Finally, let us denote the submatrix of the first k columns of $\hat{\varepsilon}$ as u. Then

$$\hat{X}\hat{X}' = \xi\xi' + n_1 H, \tag{4}$$

where $n_1 H \sim W_p(n_1, I_p)$, H and $\xi\xi'$ are mutually independent, and independent from E, and

$$\xi = vW_v^{-1/2}h^{1/2}W_{\mathcal{F}}^{1/2} + u. \tag{5}$$

Using (2), (3), and (4), we obtain the convenient representation for the eigenvalues, announced above. Let $\hat{x}_1 \geq ... \geq \hat{x}_p$ be the roots of the equation

$$\det(\xi\xi'/n_1 + H - xE) = 0. \tag{6}$$

Then

$$\lambda_j = n_1\hat{x}_j/(n_1 + k). \tag{7}$$

This representation is convenient because the roots of (6) can be viewed and analyzed as perturbations of the roots of equation $\det(H - xE) = 0$ caused by adding the low-rank matrix $\xi\xi'/n_1$ to H.

If $x \in \mathbb{R}$ is such that $H - xE$ is invertible, then

$$(\xi\xi'/n_1 + H - xE)^{-1} = S - S\xi(I_k + \xi'S\xi/n_1)^{-1}\xi'S/n_1,$$

where $S \equiv (H - xE)^{-1}$. Therefore, if x is a root of the equation

$$\det\left(I_k + \xi'(H - xE)^{-1}\xi/n_1\right) = 0, \tag{8}$$

then it also solves (6), and hence, the asymptotic behavior of the roots of (6) can be inferred from that of the random matrix-valued function

$$M(x) = \xi' (H - xE)^{-1} \xi / n_1. \tag{9}$$

This is the main idea of the analysis in the next section of the paper.

3 Almost sure limits of the largest eigenvalues

Let $\mathbf{n} \equiv (n_1, n_2)$ and $\mathbf{c} \equiv (c_1, c_2)$. We will denote the asymptotic regime where n_1, n_2, and p grow so that $p/n_1 \to c_1$ and $p/n_2 \to c_2$ with $c_j \in (0,1)$ as $p, \mathbf{n} \to_{\mathbf{c}} \infty$. As follows from Wachter's (1980) work (see also Yin et al. (1983) and Silverstein (1985)), as $p, \mathbf{n} \to_{\mathbf{c}} \infty$, the empirical distribution of the eigenvalues of $E^{-1}H$ converges in probability to the distribution with density

$$\frac{1 - c_2}{2\pi} \frac{\sqrt{(b_+ - \lambda)(\lambda - b_-)}}{\lambda (c_1 + c_2 \lambda)} \mathbf{1}\{b_- \leq \lambda \leq b_+\}. \tag{10}$$

The upper and the lower boundaries of the support of this density are

$$b_{\pm} = \left(\frac{1 \pm r}{1 - c_2} \right)^2, \text{ where } r = \sqrt{c_1 + c_2 - c_1 c_2}.$$

The results of Silverstein and Bai (1995) and Silverstein (1995) show that the empirical distribution converges not only in probability, but also almost surely (a.s.). Furthermore, as follows from Theorem 1.1 of Bai and Silverstein (1998), the largest eigenvalue of $E^{-1}H$ a.s. converges to b_+. [**That theorem does not seem to cover nonrandom E. We might need to provide an argument along the lines of Remark 6.5 on p.125 of the Bai-Silverstein book (2nd edn)**]

The latter convergence, together with (7) and Weyl's inequalities for the eigenvalues of a sum of two Hermitian matrices (see Theorem 4.3.7 in Horn and Johnson (1985)), imply that the $k+1$-th largest eigenvalue of $\mathbf{F}$, λ_{k+1}, a.s. converges to b_+. Those of the k largest eigenvalues that remain separated from b_+ as $p, \mathbf{n} \to_{\mathbf{c}} \infty$, must correspond to solutions of (8). Below, we study these solutions in detail.

Lemma 1 *For any $x > b_+$, as $p, \mathbf{n} \to_{\mathbf{c}} \infty$,*

$$\frac{1}{p}\operatorname{tr}\left[(H - xE)^{-1}\right] \overset{a.s.}{\to} m_x(0) \text{ and} \tag{11}$$

$$\frac{1}{p}\operatorname{tr}\left[\frac{\mathrm{d}}{\mathrm{d}x}(H - xE)^{-1}\right] \overset{a.s.}{\to} \frac{\mathrm{d}}{\mathrm{d}x} m_x(0), \tag{12}$$

where $m_x(0) = \lim_{z\to 0} m_x(z)$, and $m_x(z) \in \mathbb{C}^+$ is analytic in $z \in \mathbb{C}^+$, and satisfies equation

$$z - \frac{1}{1 + c_1 m_x(z)} = -\frac{1}{m_x(z)} - \frac{x}{1 - c_2 x m_x(z)}. \tag{13}$$

Proof: Let $x \in \mathbb{R}$ be such that $x > b_+$, and let $F_x(\lambda)$ be the empirical distribution function of the eigenvalues of $H - xE$. For any $z \in \mathbb{C}^+$, let

$$\hat{m}_x(z) = \int (\lambda - z)^{-1} \mathrm{d}F_x(\lambda)$$

be the Stieltjes transform of $F_x(\lambda)$. Note that matrix $H - xE$ can be represented in the form YTY'/p, where $Y \sim N(0, I_p \otimes I_{n_1+n_2})$ and T is a diagonal matrix with the first n_1 and the last n_2 diagonal elements equal to p/n_1 and $-xp/n_2$, respectively. Therefore, by Theorem 1.1 of Silverstein and Bai (1995), for any $z \in \mathbb{C}^+$, $\hat{m}_x(z)$ a.s. converges to $m_x(z) \in \mathbb{C}^+$, which is an analytic function in the domain $z \in \mathbb{C}^+$ that solves the functional equation (13).

By Theorem 1.1 of Bai and Silverstein (1998), the largest eigenvalue of $E^{-1}H$ a.s. converges to b_+. [**need to justify extension to random E**] Therefore, for any $x > b_+$, the largest eigenvalue of $H - xE$ is a.s. asymptotically bounded away from the positive semi-axis. Hence, $\hat{m}_x(z)$ is analytic and bounded in a small disc D around $z = 0$ for all sufficiently large p and $\mathbf{n}$, a.s. By Vitali's theorem (see Titchmarsh (1960), p.168), $\hat{m}_x(z)$ is a.s. converging to an analytic function in D. Since, in $D \cap \mathbb{C}^+$, the limiting function is $m_x(z)$, we have

$$\frac{1}{p}\operatorname{tr}\left[(H - xE)^{-1}\right] = \hat{m}_x(0) \overset{a.s.}{\to} m_x(0),$$

where $m_x(0) = \lim_{z\to 0} m_x(z)$. Further, $\frac{1}{p}\operatorname{tr}\left[(H - \zeta E)^{-1}\right]$ is an analytic bounded function of ζ in a small disk D_x around x, for all sufficiently large p and $\mathbf{n}$, a.s. Therefore, by Vitali's theorem its a.s. limit $f(\zeta)$ is analytic in D_x, and

$$\frac{1}{p}\operatorname{tr}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}(H - \zeta E)^{-1}\right] \overset{a.s.}{\to} \frac{\mathrm{d}}{\mathrm{d}\zeta} f(\zeta)$$

in D_x. On the other hand, we know that $f(\zeta) = m_{\operatorname{Re}\zeta}(0)$ for ζ from D_x. Therefore, we have (12).□

Lemma 2 *For any $x > b_+$, as $p, \mathbf{n} \to_{\mathbf{c}} \infty$,*

$$\left\| M\left(x\right) - \left(h + c_1 I_k\right) \frac{1}{p} \operatorname{tr}\left[\left(H - xE\right)^{-1}\right] \right\| \overset{a.s.}{\to} 0 \text{ and}$$
$$\left\| \frac{\mathrm{d}}{\mathrm{d}x} M\left(x\right) - \left(h + c_1 I_k\right) \frac{1}{p} \operatorname{tr}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(H - xE\right)^{-1}\right] \right\| \overset{a.s.}{\to} 0,$$

where $\|\cdot\|$ denotes the spectral norm.

Proof: This convergences follow from (5), (9), and Lemma 3 stated below.□

Lemma 3 *Let C be a random $p \times p$ matrix, independent from u and v, which are as defined in Section 2, and such that $p\,\|C\|$ is bounded for all sufficiently large p, a.s. Then, as $p \to \infty$,*

$$\|v'Cv - (\operatorname{tr} C)\, I_k\| \overset{a.s.}{\to} 0 \text{ and } \|v'Cu\| \overset{a.s.}{\to} 0.$$

Proof: This lemma follows from the Borel-Cantelli lemma, and the upper bounds on the fourth moments of the entries $v'Cv - (\operatorname{tr} C)\, I_k$ and $v'Cu$ established by Lemma 2.7 of Bai and Silverstein (1998).□

Lemma 4 *(i) For any $\varepsilon > 0$, the k eigenvalues of $M(x)$ are strictly increasing functions of $x \in (b_+ + \varepsilon, \infty)$ for sufficiently large p and $\mathbf{n}$, a.s.; (ii) $m_x(0)$ is a strictly increasing, continuous function of $x \in (b_+, \infty)$; (iii) $\lim_{x\to\infty} m_x(0) = 0$, and $\lim_{x\downarrow b_+} m_x(0)\,(h_i + c_1) < -1$ if and only if $h_i > \bar{h}$, where*

$$\bar{h} = \frac{c_2 + r}{1 - c_2}.$$

Proof: Let $\mu_1 \in (0, \infty)$ be the largest eigenvalue of $E^{-1}H$. For any $x_1 > x_2 > \mu_1$, matrix $(x_1 E - H)^{-1} - (x_2 E - H)^{-1}$ is negative definite, a.s. Part (i) follows from this, from the definition (9), and from the fact that $\mu_1 \overset{a.s.}{\to} b_+$. Part (i) together with Lemmas 1 and 2 imply that $m_x(0)$ is increasing on (b_+, ∞). It is strictly increasing because, otherwise, (13) would not be satisfied for some $z \in \mathbb{C}^+$ that are sufficiently close to zero. The continuity follows from the analyticity of $m_x(0)$ established in the proof of Lemma 1. Finally, $\lim_{x\to\infty} m_x(0) = 0$ is implied by (ii) and (11). Equation (13) implies that

$$\lim_{x \downarrow b_+} m_x(0) = \frac{c_2 - 1}{(r+1)\, r},$$

which, in its turn, implies the second statement of (iii).□

Let $\hat{x}_1 \geq ... \geq \hat{x}_k$ be the solutions of equation (8). By Lemmas 1, 2, and 4, if

$$h_1 > ... > h_m > \bar{h} > h_{m+1} > ... > h_k, \tag{14}$$

then $\hat{x}_i \overset{a.s.}{\to} x_i$, where x_i, $i = 1, ..., m$, are such that

$$1 + (h_i + c_1)\, m_{x_i}(0) = 0 \tag{15}$$

and $m_{x_i}(0)$ satisfies (13) with x replaced by x_i. In particular,

$$\frac{1}{1 + c_1 m_{x_i}(0)} - \frac{1}{m_{x_i}(0)} - \frac{x_i}{1 - c_2 x_i m_{x_i}(0)} = 0. \tag{16}$$

Combining (15) and (16), we obtain

$$\frac{1}{h_i} + 1 - \frac{x_i}{h_i + c_1 + c_2 x_i} = 0,$$

which implies that

$$x_i = \frac{(h_i + c_1)\,(h_i + 1)}{h_i - c_2\,(h_i + 1)}. \tag{17}$$

By (7), $n_1 \hat{x}_i / (n_1 + k)$, $i = 1, ..., m$, must be the m largest eigenvalues of $\mathbf{F}$, and thus, x_i, $i = 1, ..., m$, describe their a.s. limits. Since there are only m roots of (8) that are asymptotically separated from b_+ and are located above b_+, the other $k - r$ of the largest eigenvalues of $\mathbf{F}$ must a.s. converge to b_+. To summarize, the following proposition holds.

Proposition 5 *Suppose that $h_1 > ... > h_m > \bar{h} > h_{m+1} > ... > h_k$. Then, for $i \leq m$, the i-th largest eigenvalue of $\mathbf{F}$ a.s. converges to x_i defined in (17). For $m < i \leq k$, the i-th largest eigenvalue a.s. converges to b_+.*

As follows from Proposition 5, $\bar{h} = (c_2 + r) / (1 - c_2)$ is the phase transition threshold for the eigenvalues of the spiked F-ratio $\mathbf{F}$. The value of this threshold diverges to infinity when $c_2 \to 1$. Note that, when c_2 is close to one, the smallest eigenvalue of B/n_2 is close to zero, which makes $(B/n_2)^{-1}$ a particularly bad estimator of the inverse of the population covariance, Σ^{-1}. When $c_2 \to 0$, the phase transition converges to $\sqrt{c_1}$, which is the phase transition threshold for the eigenvalues of a single spiked Wishart matrix. In such a case, x_i converges to $(h_i + c_1)(h_i + 1)/h_i$, which is the a.s. limit of the i-th largest eigenvalue of the spiked Wishart when the i-th spike h_i is above $\sqrt{c_1}$.

4 Asymptotic normality

In what follows, we will assume that (14) holds, so that only m eigenvalues of $\mathbf{F}$ separate from the bulk asymptotically. We would like to study their fluctuations around the corresponding a.s. limits. Proposition 5 shows that the limits x_i depend on c_1 and c_2. Because of this dependence, the rate of the convergence has to depend on the rates of the convergences $p/n_1 \to c_1$ and $p/n_2 \to c_2$. However, as will be shown below, the latter rates do not affect the fluctuations of λ_i around

$$x_{pi} = \frac{(h_i + c_{p1})(h_i + 1)}{h_i - c_{p2}(h_i + 1)},$$

which are obtained from (17) by replacing c_1 and c_2 by $c_{p1} = p/n_1$ and $c_{p2} = p/n_2$.

Similar to x_i, which are linked to the Stieltjes transform of the limiting spectral distribution of $xE - H$ via (15), x_{pi} also can be linked to the limiting Stieltjes transform, albeit under a slightly different asymptotic regime. Precisely, let $m_{px}(z)$ be the Stieltjes transform of the limiting spectral distribution of $xE - H$ as n_1, n_2, and p grow so that p/n_1 and p/n_2 *remain fixed*. Then, similarly to (15), we have

$$1 + (h_i + c_{p1})\, m_{px_{pi}}(0) = 0. \tag{18}$$

This link will be useful in our analysis below, where we maintain the assumption that p/n_1 and p/n_2 are not necessarily fixed, but converge to c_1 and c_2, respectively.

Recall that, by (7), $\lambda_i = n_1 \hat{x}_i/(n_1 + k)$, where $\hat{x}_i$, $i = 1, ..., m$, satisfy (**8**). Clearly, the asymptotic distributions of $\sqrt{p}(\lambda_i - x_{pi})$ and $\sqrt{p}(\hat{x}_i - x_{pi})$, $i = 1, ..., m$, coincide. Therefore, below we will study the asymptotic behavior of $\sqrt{p}(\hat{x}_i - x_{pi})$, $i = 1, ..., m$. By the standard Taylor expansion argument,

$$\sqrt{p}(\hat{x}_i - x_{pi}) = -\frac{\sqrt{p}\det \mathcal{M}(x_{pi})}{\frac{\mathrm{d}}{\mathrm{d}x}\det \mathcal{M}(x_{pi}) + \frac{1}{2}(\hat{x}_i - x_{pi})\frac{\mathrm{d}^2}{\mathrm{d}x^2}\det \mathcal{M}(\tilde{x}_{pi})}, \tag{19}$$

$i = 1, ..., m,$ where $\mathcal{M}(x) = I_k + M(x)$, and $\tilde{x}_{pi} \in [x_{pi}, \hat{x}_i]$. We have

$$\frac{\mathrm{d}}{\mathrm{d}x} \det \mathcal{M}(x_{pi}) = \det \mathcal{M}(x_{pi}) \operatorname{tr} S(x_{pi}),$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \det \mathcal{M}(x_{pi}) = \det \mathcal{M}(x_{pi}) \left\{ \operatorname{tr} R(x_{pi}) + (\operatorname{tr} S(x_{pi}))^2 - \operatorname{tr}\left[S^2(x_{pi})\right] \right\},$$

where

$$S(x) = \mathcal{M}(x)^{-1} \frac{\mathrm{d}}{\mathrm{d}x} M(x), \text{ and } R(x) = \mathcal{M}(x)^{-1} \frac{\mathrm{d}^2}{\mathrm{d}x^2} M(x).$$

Since the event

$$\det \mathcal{M}(x_{pi}) = 0 \text{ or } 1 + M_{ii}(x_{pi}) = 0 \text{ for some } i = 1, ..., m$$

happens with probability zero, we can simultaneously multiply the numerator and denominator of (19) by $(1 + M_{ii}(x_{pi})) / \det \mathcal{M}(x_{pi})$ to obtain

$$\sqrt{p}(\hat{x}_i - x_{pi}) = -\frac{\sqrt{p}(1 + M_{ii}(x_{pi}))}{s(x_{pi}) + \frac{1}{2}(\hat{x}_i - x_{pi})\delta(x_{pi})}, \tag{20}$$

where

$$s(x_{pi}) = (1 + M_{ii}(x_{pi})) \operatorname{tr} S(x_{pi}),$$

and

$$\delta(x_{pi}) = (1 + M_{ii}(x_{pi})) \left\{ \operatorname{tr} R(x_{pi}) + (\operatorname{tr} S(x_{pi}))^2 - \operatorname{tr}\left[S^2(x_{pi})\right] \right\}.$$

Lemma 6 *For any $i = 1, ..., m$, we have: (i) $s(x_{pi}) \xrightarrow{\mathrm{P}} (h_i + c_1) \frac{\mathrm{d}}{\mathrm{d}x} m_{x_i}(0)$; (ii) $\delta(x_{pi}) = O(1)$ a.s.*

Proof: By Lemmas 1 and 2,

$$\frac{\mathrm{d}}{\mathrm{d}x} M(x_{pi}) \xrightarrow{a.s.} (h + c_1 I_k) \frac{\mathrm{d}}{\mathrm{d}x} m_{x_i}(0). \tag{21}$$

Further,

$$(1 + M_{ii}(x_{pi}))(I_k + M(x_{pi}))^{-1} \xrightarrow{a.s.} \operatorname{diag}\{0, ..., 0, 1, 0, ..., 0\} \tag{22}$$

with 1 at the i-th place on the diagonal. The latter convergence follows from the fact that $I_k + M(x_{pi})$ can be viewed as a small perturbation of a diagonal matrix

$$I_k + (h + c_1 I_k)\, m_{x_i}(0),$$

which has non-zero diagonal elements, except at the i-th position. The eigenvalue perturbation formulae (see, for example, (2.33) on p.79 of Kato (1980)) will then lead to (22). Combining (21) and (22), and using the definition of $s(x_{pi})$, we obtain (i).

To establish (ii), we note that $(1 + M_{ii}(x_{pi}))\, \mathrm{tr}\, R\,(x_{pi}) = O_{\mathrm{P}}(1)$ by an argument similar to that used to establish (i). Further, $(\mathrm{tr}\, S\,(x_{pi}))^2 - \mathrm{tr}\,[S^2\,(x_{pi})]$ is a linear function of the only eigenvalue of $S\,(x_{pi})$ that diverges to infinity. By the eigenvalue perturbation formulae, such an eigenvalue equals $(1 + M_{ii}(x_{pi}))^{-1}\, O(1)$ a.s. Therefore,

$$(1 + M_{ii}(x_{pi}))\left((\mathrm{tr}\, S\,(x_{pi}))^2 - \mathrm{tr}\left[S^2\,(x_{pi})\right]\right) = O(1),$$

which concludes the proof of (ii).□

Equation (20), Lemma 6, and the Slutsky theorem imply that, for the purpose of establishing convergence in distribution of $\sqrt{p}\,(\hat{x}_i - x_{pi})$, $i = 1, ..., m$, we may focus on the numerator of (20)

$$Z_{ii}(x_{pi}) \equiv \sqrt{p}\,(1 + M_{ii}(x_{pi})) = \sqrt{p}\left(M_{ii}(x_{pi}) - (h_i + c_{p1})\, m_{px_{pi}}(0)\right),$$

where the last equality follows from (18).

The random variable Z_{ii} is the entry of the matrix

$$Z(x_{pi}) = \sqrt{p}\left(M(x_{pi}) - (h + c_{p1} I_k)\, m_{px_{pi}}(0)\right)$$

that belongs to the i-th row and the i-th column. Let us now introduce new notations. Let

$$\begin{aligned} D &= (W_{\mathcal{F}}/n_1)^{1/2}\, h^{1/2}\, (W_v/p)^{-1/2}, \\ G &= (H - x_{pi} E)^{-1}/p, \\ \Delta_{\mathcal{F}} &= \sqrt{n_1}\left((W_{\mathcal{F}}/n_1)^{1/2} - I_k\right), \text{ and} \\ \Delta_v &= \sqrt{p}\,(W_v/p - I_k). \end{aligned}$$

Then, using equations (9) and (5), we obtain the following decomposition.

$$Z(x_{pi}) = \sum\nolimits_{v=1}^{6} Z^{(v)},$$

where

$$Z^{(1)} = D\sqrt{p}\left(v'Gv - I_k \operatorname{tr} G\right) D',$$

$$Z^{(2)} = (\operatorname{tr} G)\, D \left(W_v/p\right)^{-1/2} h^{1/2} \sqrt{c_{p1}} \Delta_{\mathcal{F}},$$

$$Z^{(3)} = \operatorname{tr} G \sqrt{c_{p1}} \Delta_{\mathcal{F}} h^{1/2} \left(W_v/p\right)^{-1} h^{1/2},$$

$$Z^{(4)} = -\left(\operatorname{tr} G\right) h^{1/2} \Delta_v \left(W_v/p\right)^{-1} h^{1/2},$$

$$Z^{(5)} = \sqrt{c_{p1}}\sqrt{p}\left(Dv'Gu + u'GvD'\right),$$

$$Z^{(6)} = c_{p1}\sqrt{p}\left(u'Gu - I_k \operatorname{tr} G\right),$$

and

$$Z^{(7)} = \left(h + c_{p1} I_k\right)\sqrt{p}\left(\operatorname{tr} G - m_{px_{pi}}(0)\right).$$

For the last term, $Z^{(7)}$, we prove the following lemma.

Lemma 7 $Z^{(7)} \xrightarrow{\mathrm{P}} 0$.

Proof: The proof of this lemma will appear in a separate work. Had x_{pi} been negative, $H - x_{pi}E$ would have been having the form YTY' with $Y \sim N\left(0, I_p \otimes I_{n_1+n_2}\right)$ and a positive definite diagonal T with converging spectral distribution. The Lemma would have been following then from the results of Bai and Silverstein (2004). Our proof extends Bai and Silverstein's (2004) arguments to the case of negative x_{pi}.□

Further, the asymptotic behavior of the terms $Z^{(2)}$ and $Z^{(3)}$ differ depending on the setting. Recall that for Setting 1, $W_{\mathcal{F}} \sim W_k\left(n_A, I_k\right)$. Then, since

$$\Delta_{\mathcal{F}} = \sqrt{n_1}\left(W_{\mathcal{F}}/n_1 - I_k\right)/2 + o_{\mathrm{P}}(1),$$

a standard CLT together with Lemma 1 imply that

$$Z^{(2)} + Z^{(3)} \xrightarrow{d} N\left(0, 2c_1 m_{x_i}^2(0) h^2\right). \tag{23}$$

The latter limit is independent from the limits of $Z^{(j)}$, $j \neq 2, 3$, because $W_{\mathcal{F}}$ is independent from u and v.

In contrast, for Setting 2, we have $W_{\mathcal{F}} = n_A I_k$, and $\Delta_{\mathcal{F}} = o(1)$. Therefore,

$$Z^{(2)} + Z^{(3)} \xrightarrow{\mathrm{P}} 0. \tag{24}$$

Let us now establish the convergence of $Z^{(j)}$, $j \leq 6$ such that $j \neq 2, 3$. Let l_i and L_i be such that $[l_i, L_i]$ includes the support of the limiting spectral distribution, G_{x_i}, of $H - x_{pi}E$. Moreover, let $[l_i, L_i]$ be such that none of the eigenvalues $\lambda_{p1}^{(i)}, ..., \lambda_{pp}^{(i)}$ of $H - x_{pi}E$ lies outside $[l_i, L_i]$ for sufficiently large p, a.s. Further, let g_q with $q = 1, ..., Q$, where Q is an arbitrary positive integer, be functions which are continuous on $[l_i, L_i]$ and let ζ denote a $p \times m$ matrix with i.i.d. $N(0, 1)$ entries. Finally, let

$$\Theta = \{(q, s, t) : q = 1, ..., Q; 1 \leq s \leq t \leq m\}.$$

The following Lemma is a slight modification of Lemma 13 of the Supplementary Appendix in Onatski (2012).

Lemma 8 *The joint distribution of random variables*

$$\left\{ \frac{1}{\sqrt{p}} \sum\nolimits_{j=1}^{p} g_q\left(\lambda_{pj}^{(i)}\right) (\zeta_{js}\zeta_{jt} - \delta_{st}), (q, s, t) \in \Theta \right\}$$

weakly converges to a multivariate normal. The covariance between components (q, s, t) *and* (q_1, s_1, t_1) *of the limiting distribution is equal to* 0 *when* $(s, t) \neq (s_1, t_1)$, *and to* $(1 + \delta_{st}) \int g_q(\lambda) g_{q_1}(\lambda) \, \mathrm{d}G_{x_i}(\lambda)$ *when* $(s, t) = (s_1, t_1)$.

Proof: For readers' convenience, we provide a proof of this Lemma in the Appendix.□

Note that all entries of $Z^{(j)}$, $j \leq 6$ such that $j \neq 2, 3$, are linear combinations of the terms having the form considered in Lemma 8, with weights converging in probability to finite constants. Take, for example $Z^{(1)}$. Its entries are linear combinations of the entries of

$$\frac{1}{\sqrt{p}} v' (H - x_{pi}E)^{-1} v - I_k \frac{1}{\sqrt{p}} \operatorname{tr} (H - x_{pi}E)^{-1},$$

which, in turn, can be represented in the form $\frac{1}{\sqrt{p}} \sum\nolimits_{j=1}^{p} \left(\lambda_{pj}^{(i)}\right)^{-1} (\zeta_{js}\zeta_{jt} - \delta_{st})$. The matrix ζ is obtained by multiplying $[u, v]$ from the left by the eigenvector matrix of $H - x_{pi}E$.

Lemma 8 implies that vector $\left(Z_{ii}^{(1)}, Z_{ii}^{(4)}, Z_{ii}^{(5)}, Z_{ii}^{(6)}\right)$ converges in distribution to a four-dimensional normal vector with zero mean and the following covariance matrix

$$\begin{pmatrix} 2h_i^2 m'_{x_i}(0) & -2h_i^2 m_{x_i}^2(0) & 0 & 0 \\ -2h_i^2 m_{x_i}^2(0) & 2h_i^2 m_{x_i}^2(0) & 0 & 0 \\ 0 & 0 & 4c_1 h_i m'_{x_i}(0) & 0 \\ 0 & 0 & 0 & 2c_1^2 m'_{x_i}(0) \end{pmatrix}.$$

Combining this result with Lemma 7, and convergencies (23), and (24), we obtain, for Setting 1,

$$Z_{ii}(x_{pi}) \xrightarrow{d} N\left(0, 2\left(h_i + c_1\right)^2 m'_{x_i}(0) - 2h_i^2\left(1 - c_1\right) m_{x_i}^2(0)\right), \tag{25}$$

and, for Setting 2,

$$Z_{ii}(x_{pi}) \xrightarrow{d} N\left(0, 2\left(h_i + c_1\right)^2 m'_{x_i}(0) - 2h_i^2 m_{x_i}^2(0)\right). \tag{26}$$

To establish the joint convergence of $Z_{ii}(x_{pi})$, $i = 1, ..., m$, we need another lemma. For each $i = 1, ..., m$, let $g_q^{(i)}$, with $q = 1, ..., Q$, be functions continuous on $[l_i, L_i]$.

Lemma 9 *For any set of pairs $\{(s_i, t_i) : i = 1, ..., m\}$ such that $(s_{i_1}, t_{i_1}) \neq (s_{i_2}, t_{i_2})$ for any $i_1 \neq i_2$, the joint distribution of random variables*

$$\left\{\frac{1}{\sqrt{p}} \sum_{j=1}^{p} g_q^{(i)}\left(\lambda_{pj}^{(i)}\right)\left(\zeta_{js_i}\zeta_{jt_i} - \delta_{s_i t_i}\right), i = 1, ..., m\right\}$$

weakly converges to a multivariate normal. The covariance between components i_1 and i_2 of the limiting distribution is equal to 0 when $i_1 \neq i_2$.

The proof of this lemma is very similar to that of Lemma 8, and we omit it to save space. Lemma 9 implies that $Z_{ii}(x_{pi})$, $i = 1, ..., m$ jointly converge to an m-dimensional normal vector with a diagonal covariance matrix. This result, together with equation (20), Lemma 6, and convergences (25, 26) establish the following Lemma.

Lemma 10 *The joint asymptotic distribution of $\sqrt{p}\left(\lambda_i - x_{pi}\right), i = 1, ..., m$ is normal, with diagonal covariance matrix. For Setting 1, the i-th diagonal element of the covariance matrix equals*

$$\frac{2\left(h_i + c_1\right)^2 m'_{x_i}(0) - 2h_i^2\left(1 - c_1\right) m_{x_i}^2(0)}{\left(h_i + c_1\right)^2\left(\frac{\mathrm{d}}{\mathrm{d}x} m_{x_i}(0)\right)^2}. \tag{27}$$

For Setting 2, it equals

$$\frac{2\left(h_i+c_1\right)^2 m'_{x_i}(0)-2h_i^2 m_{x_i}^2(0)}{\left(h_i+c_1\right)^2\left(\frac{\mathrm{d}}{\mathrm{d}x}m_{x_i}(0)\right)^2}. \tag{28}$$

In the Appendix, we establish the following explicit expressions for $m_{x_i}^2(0)$, $m'_{x_i}(0)$, and $\frac{\mathrm{d}}{\mathrm{d}x}m_{x_i}(0)$:

$$m_{x_i}^2(0)=\left(h_i+c_1\right)^{-2}, \tag{29}$$

$$m'_{x_i}(0)=-\frac{h_i^2}{\left(h_i+c_1\right)^2\left(c_1+c_2\left(1+h_i\right)^2-h_i^2\right)}, \tag{30}$$

$$\mathrm{d}m_{x_i}(0)/\mathrm{d}x=\frac{-\left(c_2\left(1+h_i\right)-h_i\right)^2}{\left(h_i+c_1\right)^2\left(c_1+c_2\left(1+h_i\right)^2-h_i^2\right)}. \tag{31}$$

Using (29), (30), and (31) in (27) and (28), we obtain

Proposition 11 *For any $h_1>...>h_m>\bar{h}\equiv\left(c_2+r\right)/\left(1-c_2\right)$, the joint asymptotic distribution of $\sqrt{p}\left(\lambda_i-x_{pi}\right)$, $i=1,...,m$ is normal with diagonal covariance matrix. For Setting 1,*

$$\sqrt{p}\left(\lambda_i-x_{pi}\right)\xrightarrow{d}N\left(0,2r^2\sigma_i^2\right), \tag{32}$$

whereas for Setting 2,

$$\sqrt{p}\left(\lambda_i-x_{pi}\right)\xrightarrow{d}N\left(0,2t_i^2\sigma_i^2\right). \tag{33}$$

Here

$$r^2=c_1+c_2-c_1c_2,\quad t_i^2=c_1+c_2-\frac{c_1\left(h_i^2-c_1\right)}{\left(1+h_i\right)^2},$$
$$\sigma_i^2=\frac{h_i^2\left(h_i+1\right)^2\left(h_i^2-c_2\left(h_i+1\right)^2-c_1\right)}{\left(c_2-h_i+c_2h_i\right)^4}$$

and

$$x_{pi}=\frac{\left(h_i+p/n_1\right)\left(h_i+1\right)}{h_i-\left(h_i+1\right)p/n_2}.$$

Remark 12 *It is straightforward to verify that $t_i^2<r^2$ as long as $h_i>\bar{h}$. Therefore, the asymptotic variance of λ_i is smaller for Setting 2 than for Setting 1. This accords with intuition because, as discussed above, Setting 2 corresponds to the asymptotic analysis conditional on factors $\mathcal{F}$, whereas Setting 1 corresponds to the* *unconditional asymptotic analysis. The factors' variance adds to the asymptotic variance of λ_i.*

Remark 13 *For Setting 1, when $c_2 \to 0$, the asymptotic variance of λ_i converges to the correct asymptotic variance*

$$2c_1 (h_i + 1)^2 \left(h_i^2 - c_1\right) / h_i^2$$

of the largest eigenvalue of the spiked Wishart model. Non-centrality spikes in Wishart distribution were considered in Onatski (2007). The limit of the asymptotic variance in (33) when $c_2 \to 0$ coincides with the formula for the asymptotic variance derived there.

[**Lemmas 8 and 9 were used in my old work. It might be possible to modernize the arguments, which may lead to some saving of space**]

5 Analysis of the joint density of eigenvalues

In the rest of the paper we study the statistical experiment of observing the eigenvalues of $\mathbf{F}$ when the k spikes are local to some fixed points $h_{01} > ... > h_{0k}$ above the phase transition threshold $\bar{h}$. The asymptotics of such an experiment can be characterized by that of the likelihood ratio corresponding to the null and alternative hypotheses

$$H_0 : h = h_0 \text{ and } H_1 : h = h_p \equiv h_0 + \gamma/\sqrt{p},$$

where $h_0 = \operatorname{diag}\{h_{01}, ..., h_{0k}\}$, and $\gamma = \operatorname{diag}\{\gamma_1, ..., \gamma_k\}$ is the diagonal matrix of local parameters $\gamma_j \in \mathbb{R}$.

Following James (1964), Khatri (1967), and Muirhead (1982), pp. 312-314, we write the joint density of the eigenvalues of $\mathbf{F}$, evaluated at the observed values of these eigenvalues, as follows. For Setting 1 we have

$$f_1(\tilde{\Lambda}; h) = \frac{Z_{p,\mathbf{n},1}(\tilde{\Lambda})}{\det(I_k + h)^{n_A/2}} \, {}_1F_0 \left(n/2; Vh(I_k + h)^{-1}V', \tilde{\Lambda}\right), \tag{34}$$

whereas for Setting 2 we have

$$f_2(\tilde{\Lambda}; h) = \frac{Z_{p,\mathbf{n},2}(\tilde{\Lambda})}{\operatorname{etr}\{n_A h/2\}} \, {}_1F_1 \left(n/2, n_A/2; n_A V h V'/2, \tilde{\Lambda}\right), \tag{35}$$

where ${}_1F_0$ and ${}_1F_1$ are the hypergeometric functions of two matrix arguments; $\tilde{\Lambda} = \text{diag}\left\{\tilde{\lambda}_1, \cdots, \tilde{\lambda}_p\right\}$ with

$$\tilde{\lambda}_j = \alpha_{\mathbf{n}}\lambda_j / (1 + \alpha_{\mathbf{n}}\lambda_j) \text{ and } \alpha_{\mathbf{n}} = n_A/n_2;$$

$n = n_A + n_2$; and $Z_{p,\mathbf{n},j}(\tilde{\Lambda})$ with $j = 1, 2$ depend on n_A, n_2, p and $\tilde{\Lambda}$, but not on h. We would like to study the asymptotic behavior, under the null hypothesis, of the likelihood ratios

$$f_j(\tilde{\Lambda}; h_p)/f_j(\tilde{\Lambda}; h_0)$$

with $j = 1, 2$ as $p, \mathbf{n} \to_{\mathbf{c}} \infty$. Note that $\tilde{\lambda}_j$ are the eigenvalues of the multivariate Beta matrix $(A + B)^{-1} A$. For the purpose of the analysis of the likelihood ratios, we find it more convenient to work with $\tilde{\lambda}_j$ rather than with λ_j.

First, we use Lemma 1 of Passemier et al (2014) to rewrite $f_1(\tilde{\Lambda}; h)$ and $f_2(\tilde{\Lambda}; h)$ in the form of repeated contour integrals that involve hypergeometric functions of two matrix arguments of fixed dimension $k \times k$. Let Z be a $k \times k$ diagonal matrix with complex variables z_j along the diagonal, and let

$$\omega\left(Z, \tilde{\Lambda}\right) = \prod_{j \neq i}^{k} \left(1 - z_i z_j^{-1}\right)^{1/2} \prod_{j=1}^{k} \left[z_j^{-(p-k+1)/2} \prod_{s=1}^{p} \left(1 - \tilde{\lambda}_s z_j^{-1}\right)^{-1/2} \right], \tag{36}$$

where the principal branches of all the fractional powers are taken. Finally, let $C_{p,\mathbf{n},1}(\tilde{\Lambda})$ and $C_{p,\mathbf{n},2}(\tilde{\Lambda})$ be some real quantities that depend on n_A, n_2, p and $\tilde{\Lambda}$, but not on h; and let

$$k_{p,\mathbf{n},1}(h) = [\det(I_k + h)]^{\frac{p-k-n_A-1}{2}} [\det h]^{-\frac{p-k-1}{2}}, \tag{37}$$

and

$$k_{p,\mathbf{n},2}(h) = \text{etr}\left\{-n_A h/2\right\} [\det h]^{-\frac{p-k-1}{2}}. \tag{38}$$

Lemma 14 *Let $\tilde{\mathcal{K}}$ be a counter-clockwise oriented contour in the complex plane that encircles zero and $\tilde{\lambda}_j$, $j = 1, ..., p$, and intersects each of the rays $\{z : \arg z = \varphi\}$, $\varphi \in (-\pi, \pi]$ only once. Then, for even $p - k + 1$, we have*

$$f_j(\tilde{\Lambda}; h) = \frac{C_{p,\mathbf{n},j}(\tilde{\Lambda}) k_{p,\mathbf{n},j}(h)}{(2\mathrm{i})^k} \int_{\tilde{\mathcal{K}}} \ldots \int_{\tilde{\mathcal{K}}} \omega\left(Z, \tilde{\Lambda}\right) \mathcal{F}_{p,\mathbf{n},j}\left(h, Z\right) \prod_{i=1}^{k} \mathrm{d}z_i, \tag{39}$$

where i *is the imaginary unit,*

$$\mathcal{F}_{p,\mathbf{n},1}(h,Z) = {}_1F_0\left(\frac{n-p+k+1}{2}; h(I_k+h)^{-1}, Z\right), \tag{40}$$

and

$$\mathcal{F}_{p,\mathbf{n},2}(h,Z) = {}_1F_1\left(\frac{n-p+k+1}{2}, \frac{n_A-p+k+1}{2}; \frac{n_A}{2}h, Z\right). \tag{41}$$

The lemma is a direct corollary of Lemma 1 of Passemier et al (2014). The requirement that $\tilde{\mathcal{K}}$ intersects each of the rays emanating from $z=0$ only once ensures that the branches of the fractional powers in $\omega\left(Z,\tilde{\Lambda}\right)$ are principal. Indeed, Onatski's (2013) Lemma 1, which Lemma 1 of Passemier et al (2014) is based on, is proven first under the assumption that $\tilde{\mathcal{K}}$ is the unit circle and the principal branches of the fractional powers in $\omega\left(Z,\tilde{\Lambda}\right)$ are used. Then the contour is deformed without changing the value of the integrals. When $\tilde{\mathcal{K}}$ is deformed so that the rays $\{z : \arg z = \varphi\}, \varphi \in (-\pi, \pi]$ are intersected by $\tilde{\mathcal{K}}$ only once, the arguments of the fractional power functions in $\omega\left(Z,\tilde{\Lambda}\right)$ never hit the negative semi-axis (note that $z_j^{-(p-k+1)/2}$ is not a fractional power when $p-k+1$ is even), and therefore, the principal branches of the fractional powers should still be used after the deformation of $\tilde{\mathcal{K}}$. We now turn to a derivation of the asymptotic approximation to the contour integrals in (39).

5.1 Contour deformation

Let us deform the contour of integration $\tilde{\mathcal{K}} \mapsto \mathcal{K}$ as shown on Figure 1. Parts $\mathcal{K}_j^+$ and $\mathcal{K}_j^-$, $j = 1, ..., k$, of $\mathcal{K}$ are shown non-overlaping with the real axis to enhance visibility. In fact, these parts coincide with the axis. The position $\tilde{x}_0$ of the kinks in $\mathcal{K}$ is fixed so that

$$\alpha b_+/\left(1+\alpha b_+\right) < \tilde{x}_0 < \alpha x_k/\left(1+\alpha x_k\right)$$

with $\alpha = \lim \alpha_{\mathbf{n}} = c_2/c_1$, and

$$x_j = \lim x_{pj} = \frac{(h_{0j}+c_1)(h_{0j}+1)}{h_{0j}-(h_{0j}+1)c_2}, \qquad j = 1, ..., k. \tag{42}$$

As follows from our results in the previous section,

$$\tilde{\lambda}_k \overset{a.s.}{\to} \alpha x_k/\left(1+\alpha x_k\right),$$

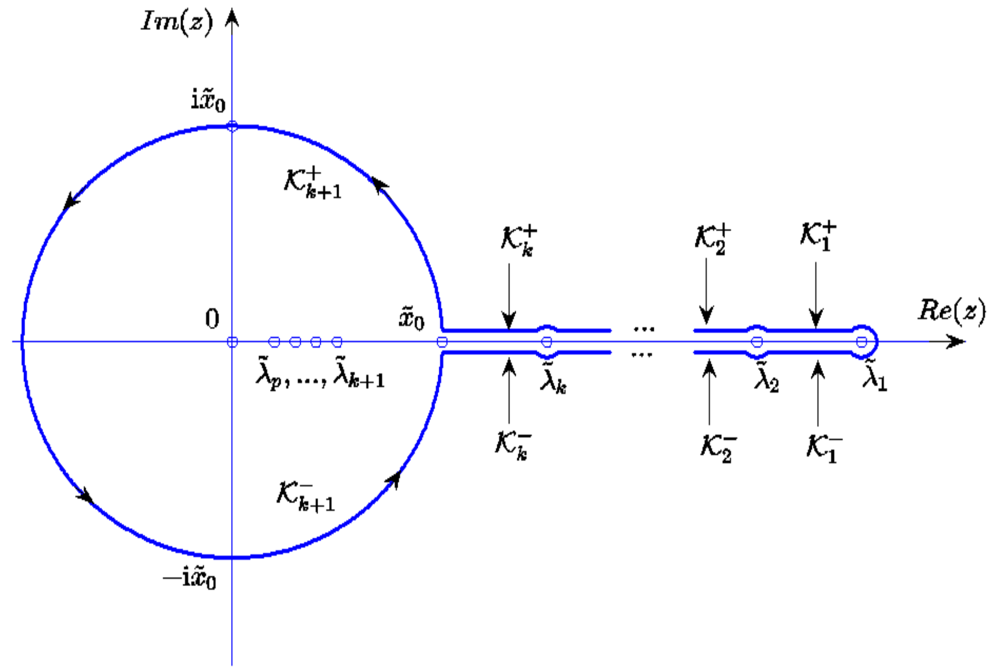

Figure 1: Deformed contour $\mathcal{K}$.

and

$$\tilde{\lambda}_{k+1} \stackrel{a.s.}{\to} \alpha b_+ / (1 + \alpha b_+),$$

so $\tilde{x}_0 \in \left(\tilde{\lambda}_{k+1}, \tilde{\lambda}_k\right)$ for sufficiently large p and $\mathbf{n}$, a.s.

The radius of the circles around $\tilde{\lambda}_j$ with $j = 1, \dots, k$ can be chosen arbitrarily small. Since, as can be seen from (36), the singularities of the integrand at $\tilde{\lambda}_j$ are of the inverse square-root-type, the contribution of the circles to the integral disappear in the limit when the radius tends to zero. Below, we will consider this limiting version of $\mathcal{K}$, that is, the contour with the horizontal part given by the two differently oriented copies of $[\tilde{x}_0, \tilde{\lambda}_1]$, where the points $\tilde{\lambda}_1, \dots, \tilde{\lambda}_k$ are excluded.

Since the deformed contour has common intervals with the ray $\{z : \arg z = 0\}$, some of the arguments of the fractional power functions involved in $\omega\left(Z, \tilde{\Lambda}\right)$ are real and negative. Therefore, care should be taken to identify the branches used. To determine the branches, we shall view the part of $\mathcal{K}$ on the real axis as the limit of a wedge-like contour

$$\mathcal{W} = (\tilde{x}_0 + \mathrm{i}\varepsilon, \tilde{\lambda}_1) \cup (\tilde{x}_0 - \mathrm{i}\varepsilon, \tilde{\lambda}_1)$$

as $\varepsilon \downarrow 0$, where i is the imaginary unit. Contour $\mathcal{W}$ intersects with each of the rays $\{z : \arg z = \varphi\}$, $\varphi \in (-\pi, \pi]$ no more than once, and therefore, the branches of all the fractional powers in $\omega\left(Z, \tilde{\Lambda}\right)$ must be principal as discussed above. As $\varepsilon \downarrow 0$, we identify the branches by continuity as follows.

Suppose that

$$z_{j_1}, ..., z_{j_r} \in [\tilde{x}_0, \tilde{\lambda}_1] \backslash \{\tilde{\lambda}_1, ..., \tilde{\lambda}_k\},$$

where $r \leq k$ and $z_{j_1} < ... < z_{j_r}$, and let all z_j with $j \notin \{j_1, ..., j_r\}$ belong to $\mathcal{K} \backslash [\tilde{x}_0, \tilde{\lambda}_1]$. To simplify notation, we may assume that $j_s = s$. Since $\omega\left(Z, \tilde{\Lambda}\right)$ is symmetric in $z_1, ..., z_k$, this assumption is without loss of generality. Then the parts of $\omega\left(Z, \tilde{\Lambda}\right)$ that need the branch identification are

$$\prod_{j>i}^{r} \left(1 - z_j z_i^{-1}\right)^{1/2} \text{ and } \left(1 - \lambda_s z_i^{-1}\right)^{-1/2} \text{ for } \lambda_s > z_i.$$

The situation will depend on which of $z_1, \ldots, z_r$ belong to the "upper" and which of them belong to the "lower" parts of $\mathcal{K} \cap [\tilde{x}_0, \tilde{\lambda}_1]$, that is the parts that are oriented from $\tilde{\lambda}_1$ to $\tilde{x}_0$, and from $\tilde{x}_0$ to $\tilde{\lambda}_1$, respectively.

There are 2^r possible scenarios: $(s_1 = \pm 1, ..., s_r = \pm 1)$, where $s_j = +1$ means that z_j belongs to the "upper" part, and $s_j = -1$ means that z_j belongs to the "lower" part of $\mathcal{K} \cap [\tilde{x}_0, \tilde{\lambda}_1]$. Consider a particular scenario $(s_1, .., s_r)$. Deforming $\mathcal{K} \cap [\tilde{x}_0, \tilde{\lambda}_1]$ to the wedge-like contour $\mathcal{W}$, we move z_j to

$$z_{j\varepsilon} = z_j + \mathrm{i} s_j \frac{\tilde{\lambda}_1 - z_j}{\tilde{\lambda}_1 - \bar{\lambda}} \varepsilon.$$

Since on $\mathcal{W}$, the principal branches of fractional powers are taken, the sign of the imaginary part of $\left(1 - z_{j\varepsilon} z_{i\varepsilon}^{-1}\right)^{1/2}$ for $j > i$ must be equal to s_i. Therefore, for $j > i$,

$$\operatorname{sgn} \operatorname{Im} \left(1 - z_j z_i^{-1}\right)^{1/2} = \lim_{\varepsilon \downarrow 0} \operatorname{sgn} \operatorname{Im} \left(1 - z_{j\varepsilon} z_{i\varepsilon}^{-1}\right)^{1/2} = s_i.$$

Similarly, for λ_s and z_i such that $\lambda_s > z_i$, we have

$$\operatorname{sgn} \operatorname{Im} \left(1 - \lambda_s z_i^{-1}\right)^{-1/2} = \lim_{\varepsilon \downarrow 0} \operatorname{sgn} \operatorname{Im} \left(1 - \lambda_s z_{i\varepsilon}^{-1}\right)^{-1/2} = -s_i.$$

Thus, we have the following lemma.

Lemma 15 *Suppose that $z_1, ..., z_r \in \mathcal{K} \cap [\tilde{x}_0, \tilde{\lambda}_1]$ are such that $z_1 < ... < z_r$, and let $s_i = +1$ if z_i belongs to the "upper" portion of $\mathcal{K} \cap [\tilde{x}_0, \tilde{\lambda}_1]$, that is, the* *portion oriented from $\tilde{\lambda}_1$ to $\tilde{x}_0$, and $s_i = -1$ if z_i belongs to the "lower" portion of $\mathcal{K} \cap [\tilde{x}_0, \tilde{\lambda}_1]$, that is, the portion oriented from $\tilde{x}_0$ to $\tilde{\lambda}_1$. Then, for $j > i$, we have*

$$\left(1 - z_j z_i^{-1}\right)^{1/2} = \mathrm{i} \times s_i \left|1 - z_j z_i^{-1}\right|^{1/2}.$$

Similarly, for $\lambda_s > z_i$, we have

$$\left(1 - \lambda_s z_i^{-1}\right)^{-1/2} = -\mathrm{i} \times s_i \left|1 - \lambda_s z_i^{-1}\right|^{-1/2}.$$

5.2 Decomposition of the contour integral

Let us split $\mathcal{K}$ into $2 \times (k+1)$ parts

$$\mathcal{K} = \bigcup_{i=1}^{k+1} \left\{\mathcal{K}_i^+ \cup \mathcal{K}_i^-\right\}$$

as shown on Figure 1, and let $\mathcal{K}_i = \mathcal{K}_i^+ \cup \mathcal{K}_i^-$. For any $\sigma = (\sigma_1, ..., \sigma_k)$ with $\sigma_i \in \{1, ..., k+1\}$, let

$$\mathcal{I}_{\sigma,l} = \frac{1}{(2\mathrm{i})^k} \int_{\mathcal{K}_{\sigma_k}} ... \int_{\mathcal{K}_{\sigma_1}} \omega\left(Z, \tilde{\Lambda}\right) \mathcal{F}_{p,\mathbf{n},l}(h, Z) \prod_{i=1}^{r} \mathrm{d}z_i$$

with $l = 1, 2$. Suppose that there exist $i \neq j$ such that $\sigma_i = \sigma_j \leq k$. Let us show that then $\mathcal{I}_{\sigma,l} = 0$.

Without loss of generality, let us assume that $\sigma_1 = \sigma_2 \leq k$. Consider the inner double integral part of $\mathcal{I}_{\sigma,l}$

$$\mathcal{I}_{\sigma,l}^{in} = \int_{\mathcal{K}_{\sigma_1}} \int_{\mathcal{K}_{\sigma_1}} \omega\left(Z, \tilde{\Lambda}\right) \mathcal{F}_{p,\mathbf{n},l}(h, Z)\, \mathrm{d}z_1 \mathrm{d}z_2,$$

and let

$$\tilde{\omega} = \omega\left(Z, \tilde{\Lambda}\right) \frac{\left|1 - z_1 z_2^{-1}\right|^{1/2}}{\left(1 - z_1 z_2^{-1}\right)^{1/2}} \frac{\left|1 - z_2 z_1^{-1}\right|^{1/2}}{\left(1 - z_2 z_1^{-1}\right)^{1/2}}.$$

Then, by Lemma 15, we have

$$\mathcal{I}_{\sigma,l}^{in} = \mathrm{i} \int_{\mathcal{K}_{\sigma_1}^+} \int_{\mathcal{K}_{\sigma_1}^+} \tilde{\omega} \mathcal{F}_{p,\mathbf{n},l}(h, Z)\, \mathrm{d}z_1 \mathrm{d}z_2 - \mathrm{i} \int_{\mathcal{K}_{\sigma_1}^-} \int_{\mathcal{K}_{\sigma_1}^-} \tilde{\omega} \mathcal{F}_{p,\mathbf{n},l}(h, Z)\, \mathrm{d}z_1 \mathrm{d}z_2 \qquad (43)$$
$$+ \mathcal{I}_{\sigma,l}^{in,1} + \mathcal{I}_{\sigma,l}^{in,2},$$

where

$$\mathcal{I}_{\sigma,l}^{in,1} = \int_{\mathcal{K}_{\sigma_1}^{+}} \int_{\mathcal{K}_{\sigma_1}^{-}} \omega\left(Z, \tilde{\Lambda}\right) \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \mathrm{d}z_1 \mathrm{d}z_2,$$

and

$$\mathcal{I}_{\sigma,l}^{in,2} = \int_{\mathcal{K}_{\sigma_1}^{-}} \int_{\mathcal{K}_{\sigma_1}^{+}} \omega\left(Z, \tilde{\Lambda}\right) \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \mathrm{d}z_1 \mathrm{d}z_2.$$

The first two terms on the right hand side of (43) cancel out. For $\mathcal{I}_{\sigma,l}^{in,1}$ we have from Lemma 15,

$$\mathcal{I}_{\sigma,l}^{in,1} = -\mathrm{i}\int_{\tilde{\lambda}_1}^{\tilde{\lambda}_2} \int_{\tilde{\lambda}_2}^{z_2} \tilde{\omega} \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \mathrm{d}z_1 \mathrm{d}z_2 + \mathrm{i}\int_{\tilde{\lambda}_1}^{\tilde{\lambda}_2} \int_{z_2}^{\tilde{\lambda}_1} \tilde{\omega} \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \mathrm{d}z_1 \mathrm{d}z_2.$$

Similarly, for $\mathcal{I}_{\sigma,l}^{in,2}$ we have

$$\mathcal{I}_{\sigma,l}^{in,2} = \mathrm{i}\int_{\tilde{\lambda}_2}^{\tilde{\lambda}_1} \int_{z_2}^{\tilde{\lambda}_2} \tilde{\omega} \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \mathrm{d}z_1 \mathrm{d}z_2 - \mathrm{i}\int_{\tilde{\lambda}_2}^{\tilde{\lambda}_1} \int_{\tilde{\lambda}_1}^{z_2} \tilde{\omega} \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \mathrm{d}z_1 \mathrm{d}z_2.$$

Therefore, the last two terms on the right hand side of (43) cancel out as well, and we have $\mathcal{I}_{\sigma,l}^{in}$ and $\mathcal{I}_{\sigma,l}$ equal to zero.

Let τ be any subset of $\{1, 2, ..., k\}$, and let $\sigma_\tau = (\sigma_{1\tau}, ..., \sigma_{k\tau})$, where

$$\sigma_{j\tau} = \begin{cases} k+1 & \text{if } j \in \tau \\ j & \text{if } j \notin \tau \end{cases}.$$

The above analysis implies the following lemma.

Lemma 16 *Let T be the set of all the subsets of $\{1, 2, ..., k\}$. Then,*

$$\mathcal{I}_l \equiv \frac{1}{(2\mathrm{i})^k} \int_{\mathcal{K}} ... \int_{\mathcal{K}} \omega\left(Z, \tilde{\Lambda}\right) \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \prod_{j=1}^{k} \mathrm{d}z_j = \sum_{\tau \in T} \frac{k!}{|\tau|!} \mathcal{I}_{\sigma_\tau,l}. \tag{44}$$

Remark 17 *The multiplier $k!/|\tau|!$ in the latter expression counts the number of integrals $\mathcal{I}_{\sigma,l}$, which are different from $\mathcal{I}_{\sigma_\tau,l}$ only by permutation of the variables of integration, $z_1, ..., z_k$.*

Below, we will show that, asymptotically as $p, \mathbf{n} \to_{\mathbf{c}} \infty$, all integrals $\mathcal{I}_{\sigma_\tau,l}$ are dominated by

$$\mathcal{I}_{\sigma_\varnothing,l} = \frac{1}{(2\mathrm{i})^k} \int_{\mathcal{K}_k} ... \int_{\mathcal{K}_1} \omega\left(Z, \tilde{\Lambda}\right) \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \prod_{j=1}^{k} \mathrm{d}z_j$$

so that $\mathcal{I}_l$ is asymptotically equivalent to $k!\mathcal{I}_{\sigma_\varnothing,l}$. Using Lemma 15, it is straightforward to verify that

$$\mathcal{I}_{\sigma_\varnothing,l} = \int_{\tilde{x}_0}^{\tilde{\lambda}_k} \int_{\tilde{\lambda}_k}^{\tilde{\lambda}_{k-1}} \dots \int_{\tilde{\lambda}_2}^{\tilde{\lambda}_1} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \prod_{j=1}^{k} \mathrm{d}z_j \tag{45}$$

Note that the constant $(2\mathrm{i})^k$ in the denominator has canceled out. To study the asymptotics of $\mathcal{I}_{\sigma_\varnothing,l}$, we will use Laplace approximation to the integrals involved in the above expression.

5.3 Asymptotic analysis of $\mathcal{I}_{\sigma_\varnothing,l}$

To apply the Laplace approximation to integrals in (45), we will replace $\mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right)$ by their asymptotic approximations. Chang (1970) studies the asymptotic behavior of ${}_1F_0\left(n/2; -A, L\right)$ as $n \to \infty$, where $A = \operatorname{diag}\left\{a_1, ..., a_k\right\}$ with $0 < a_1 < ... < a_k$, and $L = \operatorname{diag}\left\{l_1, ..., l_k\right\}$ with $l_1 > ... > l_k > 0$. Slightly modifying his analysis, we obtain the following lemma.

Lemma 18 *Let*

$$c_{ij} = \left(a_j - a_i\right)\left(l_i - l_j\right) / \left\{\left(1 + a_i l_i\right)\left(1 + a_j l_j\right)\right\}.$$

Then, as $n \to \infty$,

$$\begin{aligned} {}_1F_0\left(n/2; -A, L\right) = \Gamma_k\left(k/2\right) \pi^{-k(k+1)/4} \prod_{i=1}^{k} \left(1 + a_i l_i\right)^{-n/2} \\ \times \prod_{i<j}^{k} \left(n c_{ij}/2\right)^{-1/2} \left(1 + o(1)\right), \end{aligned} \tag{46}$$

where $o(1)$ *is uniform on any set of* a_i*'s and* l_i*'s such that the* a_i*'s are bounded, and strictly bounded away from* 0 *and one another and the* l_i*'s are similarly bounded.*

Here

$$\Gamma_k\left(x\right) = \pi^{k(k-1)/4} \prod_{i=1}^{k} \Gamma\left(x - \left(i-1\right)/2\right)$$

is the multivariate gamma function. The above lemma is a modification of Chang's (1970) Theorem 1, which establishes (46)[1] for fixed A and L. Chang's proof relies on Hsu's (1948) Lemma 1. Using Glynn's (1980, Theorem 2.1) extension of Hsu's lemma in Chang's proof, establishes the uniformity of $o(1)$ in (46).

Note that

$$_1F_0\left(m; h(I_k+h)^{-1}, Z\right) = \left[\det\left(I-Z\right)\right]^{-m} {}_1F_0\left(m; -A, L\right),$$

where

$$A = \left(I_k+h\right)^{-1} \text{ and } L = Z\left(I_k-Z\right)^{-1}.$$

Since

$$\mathcal{F}_{p,\mathbf{n},1}\left(h,Z\right) = {}_1F_0\left(m; h(I_k+h)^{-1}, Z\right)$$

with $m = \left(n-p+k+1\right)/2$, we have the following corollary.

Corollary 19 *For $Z = \operatorname{diag}\left\{z_1,...,z_k\right\}$ such that $1 > z_1 > ... > z_k > 0$, and for $h = \operatorname{diag}\left\{h_1,...,h_k\right\}$ such that $h_1 > ... > h_k > 0$, as $p, \mathbf{n} \to_{\mathbf{c}} \infty$, we have*

$$\mathcal{F}_{p,\mathbf{n},1}\left(h,Z\right) = \Gamma_k\left(k/2\right)\pi^{-k(k+1)/4}D^{-m}\prod_{i<j}^{k}\left(mc_{ij}\right)^{-1/2}\left(1+o(1)\right), \tag{47}$$

where $m = \left(n-p+k+1\right)/2$,

$$D = \det\left(I_k - h(I_k+h)^{-1}Z\right),$$

$$c_{ij} = \frac{\left(h_i-h_j\right)\left(z_i-z_j\right)}{\left(1+h_i-h_iz_i\right)\left(1+h_j-h_jz_j\right)},$$

and $o(1)$ is uniform on any set of h_i's and z_i's such that the h_i's are bounded, and strictly bounded away from 0 and one another and the z_i's are strictly bounded away from $1, 0$ and one another.

For $\mathcal{F}_{p,\mathbf{n},2}\left(h,Z\right)$, we need to analyze the asymptotic behavior of the confluent hypergeometric function ${}_1F_1$. The asymptotics of ${}_1F_1\left(a,b;A,B\right)$ where a and A diverge to ∞ at the same rate was studied in Glynn (1980). We however need the asymptotics of this function when not only a and A, but also b diverge to infinity. Following Glynn's (1980) strategy of proof, we derive the following result. Its proof is available from the authors upon request.

[1] In Chang's (1970) Theorem 1, both sides of (46) are divided by the volume of the orthogonal group $\mathcal{O}\left(k\right)$.

Lemma 20 *Consider function* $\mathcal{F} \equiv {}_1F_1\left(Na + \frac{k+1}{2}, Nb + \frac{k+1}{2}, \frac{N}{2}h, Z\right)$, *where* $a \in (1/2, \infty)$ *remains separated away from* $1/2$ *and* ∞ *as* $N \to \infty$; $b \in (0, 1/2)$ *remains separated from* 0 *and* $1/2$ *as* $n \to \infty$; $h = \operatorname{diag}(h_1, ..., h_k)$ *with* $h_1 > ... > h_k > 0$ *that are bounded from above and are separated from one another and from zero; and* $Z = \operatorname{diag}(z_1, ..., z_k)$ *with real* $z_1 > ... > z_k > 0$ *that are bounded from above and are separated from one another and from zero. As* $N \to \infty$, *we have*

$$\begin{aligned} \mathcal{F} \sim & \left(\frac{N}{2}\right)^{-k(k-1)/4} \frac{\Gamma_k(k/2)}{\pi^{k(k+1)/4}} \frac{b^{kNb+k(k+1)/4}}{a^{kNa+k(k+1)/4}} \\ & \times \prod_{j=1}^{k} e^{Nz_{j+}} \frac{(z_{j+}+a)^{aN}}{(z_{j+}+b)^{bN}} \left(\frac{z_{j+}(z_{j+}+a)}{z_{j+}^2 + a\zeta_j}\right)^{\frac{1}{2}} \\ & \times \prod_{i<j} (z_i - z_j)^{-1/2} (h_i - h_j)^{-1/2}, \end{aligned} \tag{48}$$

where

$$z_{j+} = \frac{1}{2}\left\{\zeta_j - b + \sqrt{(b-\zeta_j)^2 + 4a\zeta_j}\right\} \text{ with } \zeta_j = z_j h_j/2,$$

and $\sim$ *denotes the asymptotic equivalence in the sense that the ratio of the asymptotically equivalent terms converges to one. The asymptotic approximation (48) is uniform in* u, v, h *and* Z *that satisfy the above requirements.*

Note that the asymptotic approximations (47) and (48) do not hold for z_i, $i = 1, ..., k$, that may approach one another. Therefore, we shall, first, analyze a multiple integral with trimmed integration domains

$$\overline{\mathcal{I}}_{\sigma_\varnothing, l} = \int_{\tilde{x}_0}^{\tilde{\lambda}_k} \int_{\tilde{\lambda}_k+\varepsilon}^{\tilde{\lambda}_{k-1}} ... \int_{\tilde{\lambda}_2+\varepsilon}^{\tilde{\lambda}_1} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \mathcal{F}_{p,\mathbf{n},l}(h, Z) \prod_{j=1}^{k} \mathrm{d}z_j, \tag{49}$$

where ε is a fixed small positive number.

Assume that the null hypothesis holds, that is,

$$H_0 : h_j = h_{0j} \text{ with } j = 1, ..., k,$$

where $h_{01} > ... > h_{0k} > \bar{h}$. Then, for any $j = 1, ..., k$,

$$\tilde{\lambda}_j \overset{a.s.}{\to} \tilde{x}_j \equiv \alpha x_j / (1 + \alpha x_j),$$

where x_j are as defined in (42), and

$$\tilde{\lambda}_{k+1} \overset{a.s.}{\to} \alpha b_+ / (1 + \alpha b_+).$$

Let ε' and ε'' be small positive numbers. We shall assume that p and $\mathbf{n}$ are so large that, with probability larger than $1 - \varepsilon'$,

$$\left| \tilde{\lambda}_j - \tilde{x}_j \right| < \varepsilon'' \text{ for all } j = 1, ..., k+1. \tag{50}$$

In what follows, we will condition our analysis on this event. Furthermore, we shall assume that

$$|h_j - h_{0j}| < \varepsilon'' \text{ for all } j = 1, ..., k. \tag{51}$$

Note that such an assumption is consistent with both null and the local alternative

$$H_1 : h_j = h_{pj} \equiv h_{0j} + \gamma_j / \sqrt{p} \text{ with } j = 1, ..., k,$$

at least for sufficiently large p.

Choosing ε' and ε'' sufficiently small, assuming that (51) holds, and conditioning on the event (50), we shall analyze (49) using Laplace approximations to the inner-most, the second inner-most, etc. integrals. The strategy of the analysis will be the same for $\overline{\mathcal{I}}_{\sigma_\varnothing, l}$ with $l = 1, 2$. For $l = 1$, we use Corollary 19, to obtain

$$\overline{\mathcal{I}}_{\sigma_\varnothing, 1} = b_1(h) \int_{\tilde{x}_0}^{\tilde{\lambda}_k} \int_{\tilde{\lambda}_k + \varepsilon}^{\tilde{\lambda}_{k-1}} ... \int_{\tilde{\lambda}_2 + \varepsilon}^{\tilde{\lambda}_1} G_1\left(Z, \tilde{\Lambda}\right)(1 + o(1)) \prod_{j=1}^{k} \mathrm{d}z_j, \tag{52}$$

where

$$G_1\left(Z, \tilde{\Lambda}\right) = \left|\omega\left(Z, \tilde{\Lambda}\right)\right| D^{-\frac{n-p+2}{2}} \prod_{i<j}^{k} (z_i - z_j)^{-1/2},$$

$$b_1(h) = \frac{\Gamma_k(k/2)}{\pi^{k(k+1)/4} m^{k(k-1)/4}} \prod_{i<j}^{k} \left(\frac{(1 + h_i)(1 + h_j)}{h_i - h_j} \right)^{1/2}, \tag{53}$$

and $o(1)$ converges to zero as $p, \mathbf{n} \to_{\mathbf{c}} \infty$, uniformly over h such that (51) holds, and over Z such that $(z_1, ..., z_r)$ belongs to the trimmed domain of integration.

Consider the inner-most integrals in (52),

$$\overline{\mathcal{I}}^{in}_{\sigma_\varnothing, 1} = \int_{\tilde{\lambda}_2 + \varepsilon}^{\tilde{\lambda}_1} G_1\left(Z, \tilde{\Lambda}\right)(1 + o(1))\, \mathrm{d}z_1.$$

Using the definition of $\omega\left(Z,\tilde{\Lambda}\right)$, we can rewrite this integral in the following form

$$\overline{\mathcal{I}}^{in}_{\sigma_\varnothing,1}=G_{1,-1}\left(Z,\tilde{\Lambda}\right)\int_{\tilde{\lambda}_2+\varepsilon}^{\tilde{\lambda}_1}e^{-pf_1(z_1)}g_1\left(z_1\right)\left(1+o\left(1\right)\right)\mathrm{d}z_1 \tag{54}$$

where

$$G_{1,-1}\left(Z,\tilde{\Lambda}\right)=\left|\omega_{-1}\left(Z,\lambda\right)\right|D_{-1}^{-\frac{n-p+2}{2}}\prod_{i,j=2,\dots,k;i<j}\left(z_i-z_j\right)^{-1/2},$$

$$\omega_{-1}\left(Z,\lambda\right)=\prod_{i,j=2,\dots,k;j\neq i}\left(1-z_iz_j^{-1}\right)^{1/2}\prod_{j=2}^{k}\left[z_j^{-(p-k+2)/2}\prod_{s=1}^{p}\left(1-\tilde{\lambda}_sz_j^{-1}\right)^{-1/2}\right],$$

$$D_{-1}=\det\left(I_{k-1}-h_{-1}(I_{k-1}+h_{-1})^{-1}Z_{-1}\right),$$

$$Z_{-1}=\operatorname{diag}\left\{z_2,\dots,z_k\right\},$$

$$h_{-1}=\operatorname{diag}\left\{h_2,\dots,h_k\right\},$$

$$f_1\left(z_1\right)=\frac{n-p+2}{2p}\ln\left(1-\frac{h_1z_1}{1+h_1}\right)+\frac{1}{2p}\sum_{s=k+1}^{p}\ln\left(z_1-\tilde{\lambda}_s\right), \tag{55}$$

and

$$g_1\left(z_1\right)=\left(\tilde{\lambda}_1-z_1\right)^{-1/2}\prod_{j=2}^{k}\left(\frac{z_1-z_j}{z_1-\tilde{\lambda}_j}\right)^{1/2}.$$

Similarly, for $l=2$ we use Lemma 20, to obtain

$$\overline{\mathcal{I}}_{\sigma_\varnothing,2}=b_2(h)\int_{\tilde{x}_0}^{\tilde{\lambda}_k}\int_{\tilde{\lambda}_k+\varepsilon}^{\tilde{\lambda}_{k-1}}\dots\int_{\tilde{\lambda}_2+\varepsilon}^{\tilde{\lambda}_1}G_2\left(Z,\tilde{\Lambda}\right)\left(1+o(1)\right)\prod_{j=1}^{k}\mathrm{d}z_j, \tag{56}$$

where

$$G_2\left(Z,\tilde{\Lambda}\right)=\left|\omega\left(Z,\tilde{\Lambda}\right)\right|\Delta\prod_{i<j}\left(z_i-z_j\right)^{-1/2},$$

$$\Delta=\prod_{j=1}^{k}e^{n_Az_{j+}}\frac{\left(z_{j+}+a\right)^{an_A}}{\left(z_{j+}+b\right)^{bn_A}}\left(\frac{z_{j+}\left(z_{j+}+a\right)}{z_{j+}^2+a\zeta_j}\right)^{\frac{1}{2}},$$

$$a=\frac{1}{2}\frac{n-p}{n_A},\quad b=\frac{1}{2}\frac{n_A-p}{n_A},$$

and

$$b_2(h)=\left(\frac{n_A}{2}\right)^{-k(k-1)/4}\frac{\Gamma_k\left(k/2\right)}{\pi^{k(k+1)/4}}\frac{b^{kn_Ab+k(k+1)/4}}{a^{kn_Aa+k(k+1)/4}}\prod_{i<j}\left(h_i-h_j\right)^{-1/2}. \tag{57}$$

The same uniformity properties of $o\left(1\right)$ as in the case of (52) apply.

Consider the inner-most integrals in (56),

$$\overline{\mathcal{I}}^{in}_{\sigma_\varnothing,2} = \int_{\tilde{\lambda}_2+\varepsilon}^{\tilde{\lambda}_1} G_2\left(Z,\tilde{\Lambda}\right)(1+o(1))\,\mathrm{d}z_1.$$

Using the definition of $\omega\left(Z,\tilde{\Lambda}\right)$, we can rewrite this integral in the following form

$$\overline{\mathcal{I}}^{in}_{\sigma_\varnothing,2} = G_{2,-1}\left(Z,\tilde{\Lambda}\right)\int_{\tilde{\lambda}_2+\varepsilon}^{\tilde{\lambda}_1} e^{-n_A f_2(z_1)} g_2(z_1)(1+o(1))\,\mathrm{d}z_1, \tag{58}$$

where

$$G_{2,-1}\left(Z,\tilde{\Lambda}\right) = |\omega_{-1}(Z,\lambda)|\,\Delta_{-1}\prod_{i,j=2,\ldots,k;i<j}(z_i-z_j)^{-1/2},$$

$$\Delta_{-1} = \prod_{j=2}^{k} e^{n_A z_{j+}}\frac{(z_{j+}+a)^{an_A}}{(z_{j+}+b)^{bn_A}}\left(\frac{z_{j+}(z_{j+}+a)}{z_{j+}^2+a\zeta_j}\right)^{\frac{1}{2}},$$

$$f_2(z_1) = -z_{1+} - a\ln(z_{1+}+a) + b\ln(z_{1+}+b) + \frac{1}{2n_A}\sum_{s=k+1}^{p}\ln\left(z_1-\tilde{\lambda}_s\right), \tag{59}$$

and

$$g_2(z_1) = \left(\frac{z_{1+}(z_{1+}+a)}{z_{1+}^2+a\zeta_1}\right)^{\frac{1}{2}}\left(\tilde{\lambda}_1-z_1\right)^{-1/2}\prod_{j=2}^{k}\left(\frac{z_1-z_j}{z_1-\tilde{\lambda}_j}\right)^{1/2}.$$

The integrals in (54) and (58) can now be analyzed using standard Laplace approximation steps (see Olver (1997), pp. 81-82). The most important of these steps is verifying that the derivatives $\frac{\mathrm{d}}{\mathrm{d}z_1}f_l(z_1)$, $l=1,2$, are continuous and negative on $z_1\in[\tilde{\lambda}_2+\varepsilon,\tilde{\lambda}_1]$, for sufficiently large p and $\mathbf{n}$. Such a verification is done in the Appendix, where we also show that

$$-\frac{\mathrm{d}}{\mathrm{d}z_1}f_1(z_1)\Big|_{z_1=\tilde{\lambda}_1} \xrightarrow{a.s.} H_{11} \text{ and } -\frac{\mathrm{d}}{\mathrm{d}z_1}f_2(z_1)\Big|_{z_1=\tilde{\lambda}_1} \xrightarrow{a.s.} H_{21}.$$

Here H_{01} depends only on h_{01}, c_1, and c_2, and $H_{21}=c_1H_{11}$. An explicit form of H_{11} is given in (82), but it is not essential for the further analysis. The Laplace approximation yields the following result.

Lemma 21 *Let $p_1=p$ and $p_2=n_A$. Then, conditionally on (50), we have as*

$p, \mathbf{n} \to_{\mathbf{c}} \infty$

$$\int_{\tilde{\lambda}_2+\varepsilon}^{\tilde{\lambda}_1} e^{-p_l f_l(z_1)} g_l(z_1) \, \mathrm{d}z_1 = e^{-p_l f_l(\tilde{\lambda}_1)} \left(\frac{\pi}{p_l H_{l1}} \right)^{1/2} \bar{g}_l (1 + o(1)),$$

where

$$\bar{g}_l = \lim_{z_1 \to \tilde{\lambda}_1} g_l(z_1) \left(\tilde{\lambda}_1 - z_1 \right)^{1/2},$$

and $o(1) \to 0$ *uniformly over* $\tilde{\lambda}_j$, $j = 1, ..., k$, *that satisfy (50), over* h_j, $j = 1, ..., k$, *that satisfy (51), and over* $z_2, ..., z_k$ *that belong to the (trimmed) domain of integration in (49).*

More specifically,

$$\bar{g}_1 = \prod_{j=2}^{k} \left(\frac{\tilde{\lambda}_1 - z_j}{\tilde{\lambda}_1 - \tilde{\lambda}_j} \right)^{1/2},$$

and

$$\bar{g}_2 = \frac{\bar{g}_1 (c_1 + c_2 + c_2 h_{01}) (1 + o(1))}{\left(c_1 + c_2 (h_{01} + 1)^2 + c_1^2 + 2 c_1 h_{01} \right)^{1/2} c_2^{1/2}}.$$

The latter formula is derived by direct computation of the limit of the right hand side of the identity

$$\frac{g_2(z_1)}{g_1(z_1)} = \left(\frac{z_{1+} (z_{1+} + a)}{z_{1+}^2 + a \zeta_1} \right)^{\frac{1}{2}}$$

as $p, \mathbf{n} \to_{\mathbf{c}} \infty$, when $z_1 = \tilde{\lambda}_1$. Using the definition of z_{1+}, it can, for example, be verified that the limit of z_{1+} equals $\frac{1}{2}(h_{01} + c_1)$. The above formula for $\bar{g}_2$ follows from this after some algebra.

Repeating the above analysis for the second, third, etc. to the inner-most integral in (49) and combining the results, we obtain the following lemma.

Lemma 22 *As* $p, \mathbf{n} \to_{\mathbf{c}} \infty$, *conditionally on (50), we have*

$$\overline{\mathcal{I}}_{\sigma_\varnothing, l} = b_l(h) \prod_{j=1}^{k} \left[\Omega_{jl} \left(\frac{\pi}{p H_{j1}} \right)^{1/2} \prod_{s=k+1}^{p} \left(\tilde{\lambda}_j - \tilde{\lambda}_s \right)^{-1/2} \right] (1 + o(1)),$$

where

$$\Omega_{j1} = \left(1 - \frac{h_j}{1 + h_j} \tilde{\lambda}_j \right)^{-\frac{n-p+2}{2}},$$

$$\Omega_{j2} = e^{n_A \bar{z}_{j+}} \frac{(\bar{z}_{j+} + a)^{a n_A}}{(\bar{z}_{j+} + b)^{b n_A}} \gamma_j,$$

$$\gamma_j = \frac{(c_1 + c_2 + c_2 h_{0j})}{\left(c_1 + c_2 (h_{0j}+1)^2 + c_1^2 + 2c_1 h_{0j}\right)^{1/2} c_2^{1/2}},$$

and $\bar{z}_{j+}$ *is the value of* z_{j+} *that corresponds to* $\zeta_j = \frac{h_j}{2}\tilde{\lambda}_j$. *The quantities* H_{j1} *are given by (82). The* $o(1)$ *converges to* 0 *uniformly over* $\tilde{\lambda}_j$, $j = 1, ..., k$, *that satisfy (50), and over* h_j, $j = 1, ..., k$, *that satisfy (51).*

Now, let us show that $\overline{\mathcal{I}}_{\sigma_\varnothing,l}$ asymptotically dominates $\mathcal{I}_{\sigma_\varnothing,l} - \overline{\mathcal{I}}_{\sigma_\varnothing,l}$. The latter difference can be separated into a sum

$$\mathcal{I}_{\sigma_\varnothing} - \overline{\mathcal{I}}_{\sigma_\varnothing} = \sum_{\mathcal{D}} \int_{\tilde{x}_0}^{\tilde{\lambda}_k} \int_{\mathcal{D}_{k-1}} ... \int_{\mathcal{D}_1} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \mathcal{F}_{p,\mathbf{n},l}(h, Z) \prod_{j=1}^{k} \mathrm{d}z_j$$

where all $\mathcal{D}_j$ are represented by either $[\tilde{\lambda}_{j+1} + \varepsilon, \tilde{\lambda}_j]$ or $[\tilde{\lambda}_{j+1}, \tilde{\lambda}_{j+1} + \varepsilon]$, and at least one $\mathcal{D}_j$, $j = 1, ..., k-1$, is represented by $[\tilde{\lambda}_{j+1}, \tilde{\lambda}_{j+1} + \varepsilon]$. All the terms in the above sum cam be analyzed similarly. Therefore, we shall give details of the analysis only for the term

$$J_l \equiv \int_{\tilde{x}_0}^{\tilde{\lambda}_k} \int_{\tilde{\lambda}_k+\varepsilon}^{\tilde{\lambda}_{k-1}} ... \int_{\tilde{\lambda}_3+\varepsilon}^{\tilde{\lambda}_2} \int_{\tilde{\lambda}_2}^{\tilde{\lambda}_2+\varepsilon} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \mathcal{F}_{p,\mathbf{n},l}(h, Z) \prod_{j=1}^{k} \mathrm{d}z_j.$$

Note that $\mathcal{F}_{p,\mathbf{n},l}(h, Z)$ is monotonically increasing in each of $0 < z_j < 1$, $j = 1, ..., k$, when h_j, $j = 1, ..., k$, satisfy (51). This follows from the representation of ${}_1F_0$ and ${}_1F_1$ in the series of zonal polynomials and from the monotonicity of the zonal polynomials of Z in each of $0 < z_j < 1$, $j = 1, ..., k$. Such a monotonicity follows from the fact that zonal polynomials are linear combinations of monomial symmetric functions of z_j with positive coefficients (see Chattopahyay and Pillai (1970), Lemma 2). The same fact implies that

$$|\mathcal{F}_{p,\mathbf{n},l}(h, Z)| \leq \mathcal{F}_{p,\mathbf{n},l}(h, |Z|) \tag{60}$$

when h_j with $j = 1, ..., k$ are positive. We will use this inequality in a different context below.

The monotonicity of $\mathcal{F}_{p,\mathbf{n},l}(h, Z)$ implies that

$$J_l \leq \int_{\tilde{x}_0}^{\tilde{\lambda}_k} \int_{\tilde{\lambda}_k+\varepsilon}^{\tilde{\lambda}_{k-1}} ... \int_{\tilde{\lambda}_3+\varepsilon}^{\tilde{\lambda}_2} \int_{\tilde{\lambda}_2}^{\tilde{\lambda}_2+\varepsilon} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \mathcal{F}_{p,\mathbf{n},l}\left(h, \tilde{Z}_1\right) \prod_{j=1}^{k} \mathrm{d}z_j, \tag{61}$$

where $\tilde{Z}_1 = \text{diag}\left\{\tilde{\lambda}_2 + \varepsilon, z_2, ..., z_k\right\}$. Approximating $\mathcal{F}_{p,\mathbf{n},l}\left(h, \tilde{Z}_1\right)$ using Corollary 19 for the case $l = 1$ and Lemma 20 for the case $l = 2$, and obtaining an upper bound on the approximation using the fact that $f_l\left(z_j\right)$ are monotonically decreasing on $z_j \in [\tilde{\lambda}_{j+1}, \tilde{\lambda}_j]$, $j = 2, ..., k$, we obtain the following inequalities

$$|J_1| \leq C_1 \left(1 - \frac{h_1\left(\tilde{\lambda}_2 + \varepsilon\right)}{1 + h_1}\right)^{-\frac{n-p+2}{2}} \prod_{s=k+1}^{p} \left(\tilde{\lambda}_2 - \tilde{\lambda}_s\right)^{-1/2} \prod_{j=2}^{k} \exp\left\{-p f_1\left(\tilde{\lambda}_j\right)\right\}, \tag{62}$$

and

$$|J_2| \leq C_2 \exp\left\{-n_A \varphi\left(\tilde{\lambda}_2 + \varepsilon\right)\right\} \prod_{s=k+1}^{p} \left(\tilde{\lambda}_2 - \tilde{\lambda}_s\right)^{-1/2} \prod_{j=2}^{k} \exp\left\{-n_A f_2\left(\tilde{\lambda}_j\right)\right\}, \tag{63}$$

where C_1 and C_2 are some positive constant that may depend on h and $\tilde{\lambda}_j$, $j = 1, ..., k$, and

$$\varphi\left(z_j\right) \equiv -z_{j+} - a \ln\left(z_{j+} + a\right) + b \ln\left(z_{j+} + b\right).$$

But, for sufficiently small $\eta > 0$ and $\varepsilon > 0$, and for large enough p and $\mathbf{n}$, we have

$$\begin{aligned} &-\frac{n-p+2}{2p} \ln\left(1 - \frac{h_1\left(\tilde{\lambda}_2 + \varepsilon\right)}{1 + h_1}\right) - \frac{1}{2p} \sum_{s=k+1}^{p} \ln\left(\tilde{\lambda}_2 - \tilde{\lambda}_s\right) \\ &< -\frac{n-p+2}{2p} \ln\left(1 - \frac{h_1 \tilde{\lambda}_1}{1 + h_1}\right) - \frac{1}{2p} \sum_{s=k+1}^{p} \ln\left(\tilde{\lambda}_1 - \tilde{\lambda}_s\right) - \eta. \end{aligned}$$

This follows from the fact that $f_1\left(z_1\right)$ is monotonically decreasing on $z_1 \in [\tilde{\lambda}_2, \tilde{\lambda}_1]$. Similarly, from the fact that $f_2\left(z_1\right)$ is monotonically decreasing on $z_2 \in [\tilde{\lambda}_2, \tilde{\lambda}_1]$, we have

$$\begin{aligned} &- n_A \varphi\left(\tilde{\lambda}_2 + \varepsilon\right) - \frac{1}{2n_A} \sum_{s=k+1}^{p} \ln\left(\tilde{\lambda}_2 - \tilde{\lambda}_s\right) \\ &< -n_A \varphi\left(\tilde{\lambda}_1\right) - \frac{1}{2n_A} \sum_{s=k+1}^{p} \ln\left(\tilde{\lambda}_1 - \tilde{\lambda}_s\right) - \eta. \end{aligned}$$

Combining the latter two inequalities with (62) and (63), we obtain

$$|J_l| \leq C_l \exp\{-p_l \eta\} \prod_{j=1}^{k} \exp\left\{-p_l f_l\left(\tilde{\lambda}_j\right)\right\}, \tag{64}$$

where $p_1 = p$ and $p_2 = n_A$. The right hand side of (64) is asymptotically exponentially smaller than $\overline{\mathcal{I}}_{\sigma_\varnothing,l}$, which implies that $\overline{\mathcal{I}}_{\sigma_\varnothing,l}$ indeed asymptotically dominates $\mathcal{I}_{\sigma_\varnothing,l} - \overline{\mathcal{I}}_{\sigma_\varnothing,l}$, and we have the following lemma.

Lemma 23 *As $p, \mathbf{n} \to_{\mathbf{c}} \infty$, conditionally on (50), we have*

$$\mathcal{I}_{\sigma_\varnothing,l} = \overline{\mathcal{I}}_{\sigma_\varnothing,l}\left(1 + o(1)\right),\ \ l = 1, 2,$$

where $o(1) \to 0$ uniformly over λ_j, $j = 1, ..., k$, that satisfy (50), and over h_j, $j = 1, ..., k$, that satisfy (51).

5.4 Asymptotic analysis of $\mathcal{I}_{\sigma_\tau,l}$ with $\tau \neq \varnothing$.

Now, we would like to show that multiple integrals $\mathcal{I}_{\sigma_\tau,l}$ with $\tau \neq \varnothing$ are asymptotically dominated by $\mathcal{I}_{\sigma_\varnothing,l}$. This can be proven similarly to the fact that $\overline{\mathcal{I}}_{\sigma_\varnothing,l}$ asymptotically dominates $\mathcal{I}_{\sigma_\varnothing,l} - \overline{\mathcal{I}}_{\sigma_\varnothing,l}$. Below, we provide details for one specific τ, $\tau = \{1, 2, ..., k\}$. The proofs for the other subsets $\tau \subseteq \{1, ..., k\}$ are similar, and we omit them to save space.

For $\tau = \{1, 2, ..., k\}$, we have

$$\mathcal{I}_{\sigma_\tau,l} = \frac{1}{(2\mathrm{i})^k} \int_{\mathcal{K}_{k+1}} \cdots \int_{\mathcal{K}_{k+1}} \omega\left(Z, \tilde{\Lambda}\right) \mathcal{F}_{p,\mathbf{n},l}\left(h, Z\right) \prod_{j=1}^{k} \mathrm{d}z_j.$$

Using inequality (60), we obtain

$$|\mathcal{I}_{\sigma_\tau,l}| = \frac{1}{2^k} \int_{\mathcal{K}_{k+1}} \cdots \int_{\mathcal{K}_{k+1}} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \mathcal{F}_{p,\mathbf{n},l}\left(h, \tilde{x}_0 I_k\right) \prod_{j=1}^{k} |\mathrm{d}z_j|.$$

From the monotonicity property of zonal polynomials, discussed above, we have

$$|\mathcal{F}_{p,\mathbf{n},l}\left(h, \tilde{x}_0 I_k\right)| \leq |\mathcal{F}_{p,\mathbf{n},l}\left(h, Z_\eta\right)|,$$

where

$$Z_\eta = \mathrm{diag}\left\{\tilde{x}_0 + k\eta, ..., \tilde{x}_0 + 2\eta, \tilde{x}_0 + \eta\right\}$$

and η is a fixed small positive number. Using Corollary 19 for the case $l = 1$ and Lemma 20 for the case $l = 2$, we obtain

$$\left|\mathcal{I}_{\sigma_\tau,1}\right| \leq C_1 \prod_{j=1}^{k} \left(1 - \frac{h_j}{1+h_j}\left(\tilde{x}_0 + j\eta\right)\right)^{-\frac{n-p+2}{2}} \times \int_{\mathcal{K}_{k+1}} \cdots \int_{\mathcal{K}_{k+1}} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \prod_{j=1}^{k} \left|\mathrm{d}z_j\right|, \tag{65}$$

and

$$\left|\mathcal{I}_{\sigma_\tau,2}\right| \leq C_2 \prod_{j=1}^{k} \exp\left\{-n_A \varphi\left(\tilde{x}_0 + j\eta\right)\right\} \times \int_{\mathcal{K}_{k+1}} \cdots \int_{\mathcal{K}_{k+1}} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \prod_{j=1}^{k} \left|\mathrm{d}z_j\right|, \tag{66}$$

where C_1 and C_2 are some constant that may depend on k and η.

Using the definitions of $\omega\left(Z, \tilde{\Lambda}\right)$ and of $\mathcal{K}_{k+1}$, we have

$$\int_{\mathcal{K}_{k+1}} \cdots \int_{\mathcal{K}_{k+1}} \left|\omega\left(Z, \tilde{\Lambda}\right)\right| \prod_{j=1}^{k} \left|\mathrm{d}z_j\right| < C_3 \prod_{j=1}^{k} \prod_{s=k+1}^{p} \left(\tilde{x}_0 - \tilde{\lambda}_s\right)^{-1/2},$$

where C_3 is a constant that may depend on k, $\tilde{x}_0$, and $\tilde{\lambda}_j$, $j = 1, ..., k$ so hat it remains bounded conditionally on (50). Combining the latter inequality with (65) and (66), we obtain

$$\left|\mathcal{I}_{\sigma_\tau,1}\right| \leq \tilde{C}_1 \prod_{j=1}^{k} \left[\left(1 - \frac{h_j}{1+h_j}\left(\tilde{x}_0 + j\eta\right)\right)^{-\frac{n-p+2}{2}} \prod_{s=k+1}^{p} \left(\tilde{x}_0 - \tilde{\lambda}_s\right)^{-1/2}\right],$$

$$\left|\mathcal{I}_{\sigma_\tau,2}\right| \leq \tilde{C}_2 \prod_{j=1}^{k} \left[\exp\left\{-n_A \varphi\left(\tilde{x}_0 + j\eta\right)\right\} \prod_{s=k+1}^{p} \left(\tilde{x}_0 - \tilde{\lambda}_s\right)^{-1/2}\right],$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are some quantities that remain bounded as $p, \mathbf{n} \to_{\mathbf{c}} \infty$, conditionally on (50). Now, using the fact that $f_l(z)$ with $l = 1, 2$ are decreasing on $z \in [\tilde{x}_0, \tilde{\lambda}_1]$, and choosing η sufficiently small, we obtain, for some positive $\tilde{\eta}$,

$$\left|\mathcal{I}_{\sigma_\tau,l}\right| \leq C \exp\left\{-p_l \tilde{\eta}\right\} \prod_{j=1}^{k} \exp\left\{-p_l f_l\left(\tilde{\lambda}_j\right)\right\},$$

which is asymptotically exponentially smaller than $\mathcal{I}_{\sigma_\varnothing,l}$.

The following lemma summarizes the results of all the subsections of this section of the paper.

Lemma 24 *As $p,\mathbf{n}\rightarrow_{\mathbf{c}}\infty$ so that $p-k+1$ remains even, we have*

$$f_l(\tilde{\Lambda};h)=C_{p,\mathbf{n},l}(\tilde{\Lambda})k_{p,\mathbf{n},l}(h)k!b_l(h)$$
$$\times\prod_{j=1}^{k}\left[\Omega_{jl}\left(\frac{\pi}{pH_{j1}}\right)^{1/2}\prod_{s=k+1}^{p}\left(\tilde{\lambda}_j-\tilde{\lambda}_s\right)^{-1/2}\right](1+o(1)),$$

where

$$k_{p,\mathbf{n},1}(h)=[\det(I_k+h)]^{\frac{p-k-n_A-1}{2}}[\det h]^{-\frac{p-k-1}{2}},$$

$$k_{p,\mathbf{n},2}(h)=\operatorname{etr}\{-n_Ah/2\}[\det h]^{-\frac{p-k-1}{2}},$$

$$b_1(h)=\frac{\Gamma_k(k/2)}{\pi^{k(k+1)/4}m^{k(k-1)/4}}\prod_{i<j}^{k}\left(\frac{(1+h_i)(1+h_j)}{h_i-h_j}\right)^{1/2},$$

$$b_2(h)=\left(\frac{n_A}{2}\right)^{-k(k-1)/4}\frac{\Gamma_k(k/2)}{\pi^{k(k+1)/4}}\frac{b^{kn_Ab+k(k+1)/4}}{a^{kn_Aa+k(k+1)/4}}\prod_{i<j}(h_i-h_j)^{-1/2},$$

$$\Omega_{j1}=\left(1-\frac{h_j}{1+h_j}\tilde{\lambda}_j\right)^{-\frac{n-p+2}{2}},$$

$$\Omega_{j2}=e^{n_A\bar{z}_{j+}}\frac{(\bar{z}_{j+}+a)^{an_A}}{(\bar{z}_{j+}+b)^{bn_A}}\gamma_j,$$

$$\gamma_j=\frac{(c_1+c_2+c_2h_{0j})}{\left(c_1+c_2(h_{0j}+1)^2+c_1^2+2c_1h_{0j}\right)^{1/2}c_2^{1/2}},$$

and $\bar{z}_{j+}$ is the value of z_{j+} that corresponds to $\zeta_j=\frac{h_j}{2}\tilde{\lambda}_j$. The quantities H_{j1} are given by (82). The quantities $C_{p,\mathbf{n},1}(\tilde{\Lambda})$ and $C_{p,\mathbf{n},2}(\tilde{\Lambda})$ depend on n_A,n_2,p and $\tilde{\Lambda}$, but not on h. The $o(1)$ converges to 0 uniformly over $\tilde{\lambda}_j$, $j=1,...,k$, that satisfy (50), and over h_j, $j=1,...,k$, that satisfy (51).

6 Local Asymptotic Normality

6.1 Analysis for Setting 1

Let us denote the likelihood ratio by

$$L_{p1}(\gamma,\Lambda)=\frac{f_1(\tilde{\Lambda};h_p)}{f_1(\tilde{\Lambda};h_0)}. \tag{67}$$

Using Lemma 24, we obtain

$$L_{p1}(\gamma,\Lambda)\overset{\mathrm{P}}{\sim}\frac{k_{p1}(h_p)}{k_{p1}(h_0)}\prod_{j=1}^{k}\left(\frac{1-\frac{h_{pj}}{1+h_{pj}}\tilde{\lambda}_{pj}}{1-\frac{h_{0j}}{1+h_{0j}}\tilde{\lambda}_{pj}}\right)^{\frac{p-n-2}{2}}. \tag{68}$$

The right hand side does not depend on $b_1(h_p)/b_1(h_0)$ because this ratio is asymptotically equivalent to one. Consider new local parameters

$$\theta_{j1}=\gamma_j/\omega_1(h_{0j}),$$

where

$$\omega_1(h_{0j})=\frac{2h_{0j}^2(1+h_{0j})^2r^2}{(h_{0j}-c_2(1+h_{0j}))^2}.$$

We have the following lemma.

Lemma 25 *Under the null hypothesis that $h=h_0$, uniformly in θ_{j1}, $j=1,...,k$, from any compact subset of $\mathbb{R}^k$, as $p,\mathbf{n}\to_{\mathbf{c}}\infty$ so that $p-k+1$ remains even*

$$\ln L_{p1}(\gamma,\Lambda)=\sum_{j=1}^{k}\left\{\theta_{j1}\sqrt{p}\,(\lambda_j-x_{pj})-\frac{1}{2}\theta_{j1}^2\tau_1^2(h_{0j})\right\}+o_{\mathrm{P}}(1),$$

where

$$x_{pj}=\frac{(h_{0j}+p/n_1)(h_{0j}+1)}{h_{0j}-(h_{0j}+1)p/n_2},\ \textit{and}$$
$$\tau_1^2(h_{0j})=2r^2\frac{h_{0j}^2(h_{0j}+1)^2\left(h_{0j}^2-c_2(h_{0j}+1)^2-c_1\right)}{(c_2-h_{0j}+c_2h_{0j})^4}.$$

Proof: Taking the logarithm of (68) yields

$$\ln L_{p1}(\gamma,\Lambda) = \frac{n+2-p}{2}\sum_{j=1}^{k}\left(\ln\left(1-\frac{\tilde{\lambda}_{pj}h_{0j}}{1+h_{0j}}\right)-\ln\left(1-\frac{\tilde{\lambda}_{pj}h_{pj}}{1+h_{pj}}\right)\right)$$
$$-\left(\frac{p-2}{2}\right)\sum_{j=1}^{k}\ln\frac{h_{pj}}{h_{0j}}+\left(\frac{p-n_1-2}{2}\right)\sum_{j=1}^{k}\ln\frac{1+h_{pj}}{1+h_{0j}}+o_{\mathrm{P}}(1). \quad (69)$$

Moreover, we have the following expansions

$$\ln\left(1-\frac{\tilde{\lambda}_{pj}h_{0j}}{1+h_{0j}}\right)-\ln\left(1-\frac{\tilde{\lambda}_{pj}h_{pj}}{1+h_{pj}}\right)$$
$$= p^{-\frac{1}{2}}\gamma_j\frac{\tilde{\lambda}_{pj}}{(1+h_{0j})\left(1+h_{0j}(1-\tilde{\lambda}_{pj})\right)}$$
$$- p^{-1}\gamma_j^2\frac{\tilde{\lambda}_{pj}}{(1+h_{0j})^2(1+h_{0j}(1-\tilde{\lambda}_{pj}))} \quad (70)$$
$$+ p^{-1}\gamma_j^2\frac{\tilde{\lambda}_{pj}^2}{2(1+h_{0j})^2(1+h_{0j}(1-\tilde{\lambda}_{pj}))^2}+o_{\mathrm{P}}(p^{-1}),$$

$$\ln\frac{1+h_{pj}}{1+h_{0j}} = p^{-\frac{1}{2}}\gamma_j\frac{1}{1+h_{0j}}-p^{-1}\gamma_j^2\frac{1}{2(1+h_{0j})^2}+o(p^{-1}), \quad (71)$$

and

$$\ln\frac{h_{pj}}{h_{0j}} = p^{-\frac{1}{2}}\gamma_j h_{0j}^{-1}-\frac{1}{2}p^{-1}\gamma_j^2 h_{0j}^{-2}+o(p^{-1}). \quad (72)$$

Finally, using (70), (71), and (72) in (69) and noting the fact that $\lambda_1-x_{p1}\overset{a.s.}{\to}0$, we obtain the statement of the lemma by straightforward algebraic manipulations.□

Lemma 25 together with the joint asymptotic normality of $\sqrt{p}\,(\lambda_j-x_{pj})\,,\ j=1,...,k$, established in Proposition 11 imply, via Le Cam's First Lemma (see van der Vaart (1998), p.88), that the sequences of the probability measures $\{\mathbb{P}_{h_0,p}\}$ and $\{\mathbb{P}_{h_0+\gamma/\sqrt{p},p}\}$ describing the joint distribution of the eigenvalues of $\mathbf{F}$ under the null $H_0: h=h_0$ and under the local alternative $H_1: h=h_p\equiv h_0+\gamma/\sqrt{p}$ are mutually contiguous. Moreover, denote $\mathbb{P}_{h_0+\gamma/\sqrt{p},p}$ as $\mathbb{P}_{\theta_1,p}$, where

$$\theta_1=(\theta_{11},...,\theta_{k1})',\ \text{and}\ \theta_{j1}=\gamma_j\omega_1^{-1}(h_{0j})\,,\ j=1,...,k.$$

Further, let

$$\mu_1(\theta_1)=\left(\theta_{11}\tau_1^2(h_{01}),...,\theta_{1k}\tau_1^2(h_{0k})\right),$$

and

$$\mathcal{T}_1 = \operatorname{diag}\left\{\tau_1^2(h_{01}), ..., \tau_1^2(h_{0k})\right\}.$$

Then the experiments $\left(\mathbb{P}_{\theta_1,p} : \theta_1 \in \mathbb{R}^k\right)$ converge to the Gaussian shift experiment $\left(N\left(\mu_1\left(\theta_1\right), \mathcal{T}_1\right) : \theta_1 \in \mathbb{R}^k\right)$. In particular, these experiments are *LAN*.

6.2 Analysis for Setting 2

Let us denote the likelihood ratio by

$$L_{p2}(\gamma, \Lambda) = \frac{f_2(\tilde{\Lambda}; h_p)}{f_2(\tilde{\Lambda}; h_0)}. \tag{73}$$

Using Lemma 24, and the identities

$$\zeta_j = \frac{z_{j+}\left(z_{j+} + b\right)}{z_{j+} + a}, \ j = 1, ..., k,$$

and

$$z_{j+} + b = \frac{\zeta_j\left(a - b\right)}{z_{j+} - \zeta_j}, \ j = 1, ..., k,$$

where $\zeta_j = h_j z_j/2$, we obtain

$$L_{p2}(\gamma, \Lambda) \overset{\mathrm{P}}{\sim} \exp\left[-n_A \sum_{j=1}^{k} \sum_{s=1}^{4} \left(a_s\left(h_{pj}\right) - a_s\left(h_{0j}\right)\right)\right], \tag{74}$$

where

$$a_1\left(h_j\right) = \frac{h_j + \ln h_j}{2},$$

$$a_2(h_j) = -\frac{1}{2}\left(\frac{h_j}{2}\tilde{\lambda}_j - b + R_j\right),$$

$$a_3(h_j) = -a \ln\left[\frac{1}{2}\left(\frac{h_j}{2}\tilde{\lambda}_j - b + R_j\right)\right],$$

$$a_4\left(h_j\right) = (a - b)\ln\left[\frac{1}{2}\left(-\frac{h_j}{2}\tilde{\lambda}_j - b + R_j\right)\right],$$

and

$$R_j = \sqrt{\left(\frac{h_j}{2}\tilde{\lambda}_j - b\right)^2 + 4a\frac{h_j}{2}\tilde{\lambda}_j}.$$

First, to expand $a_s\left(h_{pj}\right) - a_s(h_{0j})$, with $s = 1, ..., 4$, in the power series of

$\gamma_j/\sqrt{p}$ up to, and including, the terms of order $O_{\mathrm{P}}\left(\frac{1}{p}\right)$. Then, we define $\Delta_j = \sqrt{p}\left(\lambda_j - x_{pj}\right)$ and expand the coefficients in the obtained expansions of $a_s\left(h_{pj}\right) - a_s(h_{0j})$ into power series of $\Delta_j/\sqrt{p}$ up to the linear terms only. We do these expansions using Maple symbolic algebra software. Combining the results, we obtain

$$\ln L_{p2}(\gamma,\Lambda) \overset{\mathrm{P}}{\sim} \sum_{j=1}^{k}\frac{1}{2}\frac{\left(c_2 - h_{0j} + c_2 h_{0j}\right)^2}{h_{0j}^2\left(c_1 + c_2 + c_2 h_{0j}^2 + c_1^2 + 2c_1 h_{0j} + 2c_2 h_{0j}\right)}\gamma_j\Delta_j \tag{75}$$
$$+\sum_{j=1}^{k}\frac{1}{4}\frac{c_1 + c_2 + c_2 h_{0j}^2 - h_{0j}^2 + 2c_2 h_{0j}}{h_{0j}^2\left(c_1 + c_2 + c_2 h_{0j}^2 + c_1^2 + 2c_1 h_{0j} + 2c_2 h_{0j}\right)}\gamma_j^2.$$

Consider a different local parameter

$$\theta_{j2} = \gamma_j/\omega_2\left(h_{0j}\right),$$

where

$$\omega_2\left(h_{0j}\right) = \frac{2h_{0j}^2\left(c_1 + c_2 + c_2 h_{0j}^2 + c_1^2 + 2c_1 h_{0j} + 2c_2 h_{0j}\right)}{\left(h_{0j} - c_2\left(1 + h_{0j}\right)\right)^2}.$$

Asymptotic approximation (75) implies the following lemma.

Lemma 26 *Under the null hypothesis that $h = h_0$, uniformly in $\theta_2 = (\theta_{12}, ..., \theta_{k2})'$ from any compact subset of $\mathbb{R}^k$, as $p, \mathbf{n} \to_{\mathbf{c}} \infty$ so that $p - k + 1$ remains even,*

$$\ln L_{p2}(\gamma,\Lambda) = \sum_{k=1}^{k}\left\{\theta_{j2}\sqrt{p}\left(\lambda_j - x_{pj}\right) - \frac{1}{2}\theta_{j2}^2\tau_2^2(h_{0j})\right\} + o_{\mathrm{P}}(1)$$

where

$$x_{pj} = \frac{\left(h_{0j} + p/n_1\right)\left(h_{0j} + 1\right)}{h_{0j} - \left(h_{0j} + 1\right)p/n_2}, \ \textit{and}$$
$$\tau_2^2(h_{0j}) = \frac{2h_{0j}^2\left(h_{0j}^2 - c_2\left(1 + h_{0j}\right)^2 - c_1\right)}{\left(c_2 - h_{0j} + c_2 h_{0j}\right)^4} \times\left(\left(c_1 + c_2\right)\left(1 + h_{0j}\right)^2 - c_1\left(h_{0j}^2 - c_1\right)\right).$$

Similarly to the case of Setting 1, Lemma 26 together with the joint asymptotic normality of $\sqrt{p}\left(\lambda_j - x_{pj}\right)$, $j = 1, ..., k$, established in Proposition 11 imply, via Le Cam's First Lemma (see van der Vaart (1998), p.88), that the sequences of the probability measures $\{\mathbb{P}_{h_0,p}\}$ and $\left\{\mathbb{P}_{h_0+\gamma/\sqrt{p},p}\right\}$ describing the joint distribution of the eigenvalues of $\mathbf{F}$ under the null $H_0 : h = h_0$ and under the local alternative $H_1 : h = h_p \equiv h_0 + \gamma/\sqrt{p}$ are mutually contiguous. Moreover, denote $\mathbb{P}_{h_0+\gamma/\sqrt{p},p}$ as $\mathbb{P}_{\theta_2,p}$, where

$$\theta_2 = (\theta_{12}, ..., \theta_{k2})', \text{ and } \theta_{j2} = \gamma_j \omega_2^{-1} (h_{0j}), \; j = 1, ..., k.$$

Further, let

$$\mu_2 (\theta_2) = \left(\theta_{12}\tau_2^2(h_{01}), ..., \theta_{k2}\tau_2^2(h_{0k})\right),$$

and

$$\mathcal{T}_2 = \text{diag} \left\{\tau_2^2(h_{01}), ..., \tau_2^2(h_{0k})\right\}.$$

Then the experiments $\left(\mathbb{P}_{\theta_2,p} : \theta_2 \in \mathbb{R}^k\right)$ converge to the Gaussian shift experiment $\left(N (\mu_2 (\theta_2), \mathcal{T}_2) : \theta_2 \in \mathbb{R}^k\right)$. In particular, these experiments are *LAN*.

7 Conclusion

In this paper, we establish the Local Asymptotic Normality of the experiments of observing the eigenvalues of the F-ratio $\mathbf{F} \equiv (B/n_2)^{-1} A/n_A$ of two large-dimensional Wishart matrices. The experiments are parameterized by the values of a finite number k of spikes that describe the "ratio" of the covariance parameters of A and B, or, in the case of equal covariance parameters, the non-centrality parameter of A. We find that the asymptotic behavior of the log ratio of the joint density of the eigenvalues of $\mathbf{F}$, which corresponds to super-critical spikes, to their joint density under a local deviation from these values depends only on the k of the largest eigenvalues $\lambda_1, ..., \lambda_k$. This implies, in particular, that the best statistical inference about k super-critical spikes in the local asymptotic regime is based on the k largest eigenvalue only.

As a by-product of our analysis, we establish the joint asymptotic normality of a few of the largest eigenvalues of $\mathbf{F}$ that correspond to the super-critical spikes. We derive an explicit formulas for the almost sure limits of these eigenvalues, and for the asymptotic variances of their fluctuations around these limits.

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9 Appendix

9.1 Proof of Lemma 8

We will need the following two lemmas.

Lemma 27 *(McLeish 1974) Let $\{X_{pj}, \mathcal{G}_{pj}, j = 1, ..., p\}$ be a martingale difference array on the probability triple $(\Omega, \mathcal{G}, P)$. If the following conditions are satisfied: a) Lindeberg's condition: for all $\varepsilon > 0$, $\sum_j \int_{|X_{pj}|>\varepsilon} X_{pj}^2 \mathrm{d}P \to 0$ as $p \to \infty$; b) $\sum_j X_{pj}^2 \xrightarrow{\mathrm{P}} 1$, then $\sum_j X_{pj} \xrightarrow{d} N(0,1)$.*

Proof: This is a consequence of Theorem (2.3) of McLeish (1974). Two conditions of the theorem: i) $\max_{j\leq p}|X_{pj}|$ is uniformly bounded in L_2 norm, and ii) $\max_{j\leq p}|X_{pj}| \xrightarrow{\mathrm{P}} 0$, are replaced here by the Lindeberg condition.□

Lemma 28 *(Hall and Heyde) Let $\{X_{pj}, \mathcal{G}_{pj}, j = 1, ..., p\}$ be a martingale difference array, and define $V_{pJ}^2 = \sum_{j=1}^J E\left(X_{pj}^2|\mathcal{G}_{p,j-1}\right)$ and $U_{pJ}^2 = \sum_{j=1}^J X_{pj}^2$ for $J = 1, ..., p$. Suppose that the conditional variances V_{pp}^2 are tight, that is $\sup_p P\left(V_{pp}^2 > \varepsilon\right) \to 0$ as $\varepsilon \to \infty$, and that the conditional Lindeberg condition holds, that is, for all $\varepsilon > 0$, $\sum_j E\left[X_{pj}^2 \mathbf{1}\{|X_{pj}| > \varepsilon\}|\mathcal{G}_{p,j-1}\right] \xrightarrow{\mathrm{P}} 0$. Then $\max_J \left|U_{pJ}^2 - V_{pJ}^2\right| \xrightarrow{\mathrm{P}} 0$.*

Proof: This is a shortened version of Theorem 2.23 in Hall and Heyde (1980).□

Let $f_q(\lambda)$, $q = 1, ..., Q$, be such that $f_q(\lambda) = g_q(\lambda)$ for $\lambda \in [l_i, L_i]$ and $f_q(\lambda) = 0$ otherwise. Consider random variables

$$X_{pj} = \frac{1}{\sqrt{p}} \sum_{(q,s,t)\in\Theta} \gamma_{qst} f_q\left(\lambda_{pj}^{(i)}\right)(\zeta_{js}\zeta_{jt} - \delta_{st}),$$

where γ_{qst} are some constants. Let $\mathcal{G}_{pJ}$ be the σ-algebra generated by $\lambda_{p1}^{(i)}, ..., \lambda_{pp}^{(i)}$ and ζ_{js} with $j = 1, ..., J$; $s = 1, ..., m$. Clearly, $\{X_{pj}, \mathcal{G}_{pj}, j = 1, ..., p\}$ form a martingale difference array. Let K be the number of different triples $(q, s, t) \in \Theta$. Consider an arbitrary order in Θ. In Hölder's inequality

$$\sum_{a=1}^K y_a z_a \leq \left(\sum_{a=1}^K (y_a)^b\right)^{1/b} \left(\sum_{a=1}^K (z_a)^c\right)^{1/c},$$

which holds for $y_a > 0$, $z_a > 0$, $b > 1$, $c > 1$, and $1/b + 1/c = 1$, take

$$y_a = \left| \frac{1}{\sqrt{p}} \gamma_{qst} f_q \left(\lambda_{pj}^{(i)} \right) \left(\zeta_{js} \zeta_{jt} - \delta_{st} \right) \right|,$$

where (q, s, t) is the a-th triple in Θ, $z_a = 1$, and $b = 2 + \delta$ for some $\delta > 0$. Then, the inequality implies that

$$|X_{pj}|^{2+\delta} \leq K^{1+\delta} R_i^{2+\delta} \sum\nolimits_{(q,s,t)\in\Theta} \left| \frac{1}{\sqrt{p}} \gamma_{qst} \left(\zeta_{js} \zeta_{jt} - \delta_{st} \right) \right|^{2+\delta}, \tag{76}$$

where

$$R_i = \max_{q=1,\dots,Q} \sup_{\lambda \in [l_i, L_i]} |g_q(\lambda)|.$$

Since ζ_{js} are i.i.d. $N(0,1)$, (76) implies that $\sum\nolimits_{j=1}^{p} E|X_{pj}|^{2+\delta} \to 0$ as $p \to \infty$, which means that the Lyapunov condition holds for X_{pj}. As is well known, Lyapunov's condition implies Lindeberg's condition. Hence, condition a) of Lemma 27 is satisfied for X_{pj}.

Let us consider $\sum\nolimits_{j=1}^{p} X_{pj}^2$. Since the convergence in mean implies the convergence in probability, the conditional Lindeberg condition is satisfied for X_{pj} because the unconditional Lindeberg condition is satisfied as checked above. Further, in notations of Lemma 28, it is easy to see that

$$V_{pp}^2 = \sum\nolimits_{q,q_1} \left[\left(\sum\nolimits_{1\leq s\leq t\leq m} \gamma_{qst} \gamma_{q_1 st} (1 + \delta_{st}) \right) \frac{1}{p} \sum\nolimits_{j=1}^{p} f_q \left(\lambda_{pj}^{(i)} \right) f_{q_1} \left(\lambda_{pj}^{(i)} \right) \right].$$

The convergence of the empirical distribution of $\lambda_{p1}^{(i)}, ..., \lambda_{pp}^{(i)}$ to G_{x_i} and the equality of g_q and f_q on the support of G_{x_i} implies that

$$V_{pp}^2 \xrightarrow{\mathrm{P}} \Sigma \equiv \sum\nolimits_{q,q_1} \left[\left(\sum\nolimits_{1\leq s\leq t\leq m} \gamma_{qst} \gamma_{q_1 st} (1 + \delta_{st}) \right) \int g_q(\lambda) g_{q_1}(\lambda) \, \mathrm{d}G_{x_i} \right].$$

In particular, V_{pp}^2 is tight and Lemma 28 applies. Therefore, $\sum\nolimits_{j=1}^{p} X_{pj}^2$ converges to the same limit as V_{pp}^2. Thus, by Lemma 27, we get $\sum\nolimits_{j=1}^{p} X_{pj} \xrightarrow{d} N(0, \Sigma)$.

Finally, let

$$Y_{pj} = \frac{1}{\sqrt{p}} \sum\nolimits_{(q,s,t)\in\Theta} \gamma_{qst} g_q \left(\lambda_{pj}^{(i)} \right) \left(\zeta_{js} \zeta_{jt} - \delta_{st} \right).$$

Since

$$\Pr\left(\sum\nolimits_{j=1}^{p} X_{pj} \neq \sum\nolimits_{j=1}^{p} Y_{pj}\right) \to 0$$

as $p \to \infty$, we have $\sum\nolimits_{j=1}^{p} Y_{pj} \xrightarrow{d} N(0, \Sigma)$. Lemma 8 follows from this convergence via the Cramer-Wold device.□

9.2 Derivation of (29), (30), and (31)

Expression (29) immediately follows from (15). Next, differentiating identity (13) with respect to z, we obtain

$$1 + \frac{c_1 m_x'(z)}{(1 + c_1 m_x(z))^2} = \frac{m_x'(z)}{m_x^2(z)} + \frac{-x^2 c_2 m_x'(z)}{(1 - c_2 x m_x(z))^2}.$$

Setting $z = 0$ and $x = x_i$, and using the fact that

$$m_{x_i}(0) = -(h_i + c_1)^{-1}, \tag{77}$$

which follows from (15), we obtain

$$1 + \frac{c_1 m_{x_i}'(0)}{\left(1 - c_1 (h_i + c_1)^{-1}\right)^2} = \frac{m_{x_i}'(0)}{(h_i + c_1)^{-2}} + \frac{-x_i^2 c_2 m_{x_i}'(0)}{\left(1 + c_2 x_i (h_i + c_1)^{-1}\right)^2}.$$

Using the definition (17) of x_i, we obtain

$$\begin{aligned} 1 + \frac{c_1 m_{x_i}'(0)}{\left(1 - c_1 (h_i + c_1)^{-1}\right)^2} &= \frac{m_{x_i}'(0)}{(h_i + c_1)^{-2}} \\ &\quad - \frac{(h_i + c_1)^2 (h_i + 1)^2 c_2 m_{x_0}'(0)}{h_i^2}, \end{aligned}$$

which implies (30). Finally, differentiating identity (13) with respect to x, we obtain

$$\begin{aligned} \frac{c_1 \mathrm{d} m_x(z)/\mathrm{d}x}{(1 + c_1 m_x(z))^2} &= \frac{\mathrm{d} m_x(z)/\mathrm{d}x}{(m_x(z))^2} \\ &\quad + \frac{-1 + c_2 x m_x(z) - x\left(c_2 m_x(z) + c_2 x \mathrm{d} m_x(z)/\mathrm{d}x\right)}{(1 - c_2 x m_x(z))^2}. \end{aligned}$$

Setting $z = 0$ and $x = x_i$, we obtain

$$\frac{c_1 \mathrm{d} m_{x_i}(0)/\mathrm{d}x}{(1 + c_1 m_{x_i}(0))^2} = \frac{\mathrm{d} m_{x_i}(0)/\mathrm{d}x}{(m_{x_i}(0))^2} + \frac{-1 - c_2 x_i^2 \mathrm{d} m_{x_i}(0)/\mathrm{d}x}{(1 - c_2 x_i m_{x_i}(0))^2}.$$

This equality, the definition (17) of x_i, and equation (77) imply (31).

9.3 Analysis of the derivatives of f_l

Let us show that the derivatives $\frac{\mathrm{d}}{\mathrm{d}x} f_l(x)$, $l = 1, 2$, are continuous and negative on $x \in [\tilde{\lambda}_2 + \varepsilon, \tilde{\lambda}_1]$, for sufficiently large p and $\mathbf{n}$. Recall that we condition our analysis on the event (50). Unconditionally, the statements below hold with probability arbitrarily close to one for sufficiently large p and $\mathbf{n}$.

For $l = 1$, the continuity follows from the explicit form of $\frac{\mathrm{d}}{\mathrm{d}x} f_1(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x} f_1(x) = -\frac{n - p + 2}{2p} \frac{h_1}{1 + h_1 - h_1 x} + \frac{1}{2p} \sum_{s=k+1}^{p} \left(x - \tilde{\lambda}_s\right)^{-1}. \tag{78}$$

The negativity follows from the fact that, under the null,

$$-\frac{\mathrm{d}}{\mathrm{d}x} f_1(x) \overset{a.s.}{\to} \frac{r^2 h_{01}}{2 c_1 c_2 (1 + h_{01} - h_{01} x)} - \frac{1}{2} \tilde{m}(x), \tag{79}$$

where, $\tilde{m}(\cdot)$ is the Stieltjes transform of the limiting empirical distribution of the eigenvalues $\tilde{\lambda}_j$, $j = 1, ..., p$, of $(A + B)^{-1} A$ as $p, \mathbf{n} \to_{\mathbf{c}} \infty$.

Indeed, the monotonicity of $\frac{\mathrm{d}}{\mathrm{d}x} f_1(x)$ on $x \in [\tilde{\lambda}_2 + \varepsilon, \tilde{\lambda}_1]$ implies that the convergence (79) is uniform over this interval. Therefore, the minimum of $-\frac{\mathrm{d}}{\mathrm{d}x} f_1(x)$ on $x \in [\tilde{\lambda}_2 + \varepsilon, \tilde{\lambda}_1]$, which equals

$$\frac{n - p + 2}{2p} \frac{h_1}{1 + h_1 - h_1 \left(\tilde{\lambda}_2 + \varepsilon\right)} - \frac{1}{2p} \sum_{s=k+1}^{p} \left(\varepsilon + \tilde{\lambda}_2 - \tilde{\lambda}_s\right)^{-1}$$

almost surely converges to

$$\frac{r^2 h_{01}}{2 c_1 c_2 \left(1 + h_{01} - h_{01} \left(\tilde{\lambda}_{02} + \varepsilon\right)\right)} + \frac{1}{2} \tilde{m}\left(\varepsilon + \tilde{\lambda}_{02}\right),$$

where

$$\tilde{\lambda}_{0j} = \lim_{p, \mathbf{n} \to_{\mathbf{c}} \infty} \tilde{\lambda}_j = \frac{c_2 (h_{0j} + c_1)(h_{0j} + 1)}{h_{0j} (c_1 + c_2 + h_{0j} c_2)}, \; j = 1, ..., k.$$

Note that the function

$$\Psi(y;h_{01})=\frac{r^2h_{01}}{2c_1c_2\left(1+h_{01}-h_{01}y\right)}+\frac{1}{2}\tilde{m}\left(y\right)$$

is increasing on $y\in\left(\frac{\alpha b_+}{1+\alpha b_+},\infty\right)$, and $\Psi(\frac{\alpha b_+}{1+\alpha b_+};h_{01})$ is an increasing function of $h_{01}>\bar{h}\equiv\left(c_2+r\right)/\left(1-c_2\right)$. On the other hand, a direct computation shows that $\Psi(\frac{\alpha b_+}{1+\alpha b_+};\bar{h})=0$. This implies the positivity of $-\frac{\mathrm{d}}{\mathrm{d}x}f_1\left(x\right)$ and thus, the negativity of $\frac{\mathrm{d}}{\mathrm{d}x}f_1\left(x\right)$, on $x\in[\tilde{\lambda}_2+\varepsilon,\tilde{\lambda}_1]$.

To find the a.s. limit of $-\frac{\mathrm{d}}{\mathrm{d}x}f_1\left(x\right)$ at $x=\tilde{\lambda}_1$. Note that

$$\tilde{m}\left(x\right)=\left(\alpha y+1\right)+\alpha^{-1}\left(\alpha y+1\right)^2m(y),\tag{80}$$

where $m(\cdot)$ is the Stieltjes transform of the limiting empirical distribution of the eigenvalues λ_j, and $x=\alpha y/\left(1+\alpha y\right)$. A direct evaluation shows that

$$\lim_{\lambda_1\to x_1}m(\lambda_1)=-(1+h_{01})/\left(x_1h_{01}\right).\tag{81}$$

Using this and (80) in the right hand side of (79) shows that

$$-\frac{\mathrm{d}}{\mathrm{d}x}f_1\left(x\right)\bigg|_{x=\tilde{\lambda}_1}\overset{a.s.}{\to}H_{11},$$

where

$$H_{1j}=\frac{h_{0j}(1-c_2)(1+h_{0j}-\sqrt{b_+})(1+h_{0j}-\sqrt{b_-})(c_1+c_2+c_2h_{0j})}{2c_1c_2\left(h_{0j}-c_2-c_2h_{0j}\right)\left(1+h_{0j}\right)\left(c_1+h_{0j}\right)}\tag{82}$$

with $j=1,...,k$.

For $l=2$, recall the definition of z_{1+},

$$z_{1+}=\frac{1}{2}\left\{\zeta_1-b+\sqrt{\left(b-\zeta_1\right)^2+4a\zeta_1}\right\}\text{ with }\zeta_1=z_1h_1/2.$$

It is straightforward to verify that

$$\zeta_1=\frac{z_{1+}\left(z_{1+}+b\right)}{z_{1+}+a}.\tag{83}$$

Therefore $z_{1+} > \zeta_1$ for positive ζ_1, and

$$\frac{\mathrm{d}}{\mathrm{d}z_1} z_{1+} = \frac{h_1}{2} \frac{z_{1+} + a}{2z_{1+} + b - \zeta_1} = \frac{h_1}{2} \frac{(a + z_{1+})^2}{ab + 2az_{1+} + z_{1+}^2} > 0. \tag{84}$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}z_{1+}} \left(-z_{1+} - a \ln (z_{1+} + a) + b \ln (z_{1+} + b)\right) = -\frac{ab + 2az_{1+} + z_{1+}^2}{(b + z_{1+})(a + z_{1+})} < 0$$

Thus,

$$\varphi (z_1) \equiv -z_{1+} - a \ln (z_{1+} + a) + b \ln (z_{1+} + b)$$

is a strictly decreasing function of z_1. Furthermore, it is a convex function of $z_1 > 0$. Indeed,

$$\frac{\mathrm{d}^2}{\mathrm{d}z_+^2} \varphi (z_1) = -\frac{b}{(z_{1+} + b)^2} + \frac{a}{(z_{1+} + a)^2} = \frac{(a - b)\left(z_{1+}^2 - ab\right)}{(z_{1+} + b)^2 (z_{1+} + a)^2},$$

and, using (83) and (84), we also have

$$\frac{\mathrm{d}^2}{\mathrm{d}z_1^2} z_{+1} = -\frac{h_1^2}{2} a (a + z_{1+})^3 \frac{a - b}{\left(ab + 2az_{1+} + z_{1+}^2\right)^3}.$$

Therefore, we obtain

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z_1^2} \varphi (z_1) &= \frac{\mathrm{d}^2}{\mathrm{d}z_{1+}^2} \varphi (z_1) \left(\frac{\mathrm{d}}{\mathrm{d}z_1} z_{1+}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}z_{1+}} \varphi (z_1) \frac{\mathrm{d}^2}{\mathrm{d}z_1^2} z_{1+} \\ &= \frac{h_1^2}{4} \frac{(a + z_{1+})^2 (a - b)}{(b + z_{1+})^2 \left(ab + 2az_{1+} + z_{1+}^2\right)} > 0. \end{aligned}$$

Therefore, $\varphi (z_1)$ is, indeed, convex for positive z_1, and has a continuous derivative.

Further, since

$$w (z_1) \equiv \frac{1}{2n_A} \sum_{s=k+1}^{p} \ln \left(z_1 - \tilde{\lambda}_s\right)$$

is a strictly increasing concave function of $z_1 > \tilde{\lambda}_{k+1}$, we have

$$\begin{aligned}\max_{z_1\in\left[\tilde{\lambda}_2+\varepsilon,\tilde{\lambda}_1\right]} \frac{\mathrm{d}}{\mathrm{d}z_1} f_2\left(z_1\right) < \; & \frac{\mathrm{d}}{\mathrm{d}z_1}\varphi\left(z_1\right)\Big|_{z_1=\tilde{\lambda}_1} + \frac{\mathrm{d}}{\mathrm{d}z_1} w\left(z_1\right)\Big|_{z_1=\tilde{\lambda}_2+\varepsilon} \\ = & -\frac{ab+2az_{1+}+z_{1+}^2}{\left(b+z_{1+}\right)\left(a+z_{1+}\right)}\left(1+\frac{a(a-b)}{2z_{1+}\left(a+z_{1+}\right)}\right)\Bigg|_{z_1=\tilde{\lambda}_1} \\ & -\frac{2p}{h_1 n_A}\frac{p-1}{2p}(1+\alpha y) \\ & -\frac{2p}{h_1 n_A}\frac{1}{2\alpha_{\mathbf{n}} p}(1+\alpha y)^2 \sum_{j=k+1}^{p}\left(\lambda_j-\alpha y/\alpha_{\mathbf{n}}\right)^{-1},\end{aligned}$$

where

$$y=\frac{\tilde{\lambda}_2+\varepsilon}{\alpha\left(1-\tilde{\lambda}_2-\varepsilon\right)}.$$

The right hand side of the latter equality a.s. converges to

$$\Pi(y,h_{01})=-\frac{c_1}{c_2\left(h_{01}+1\right)}-1-\frac{c_1}{h_{01}}(1+\alpha y)-\frac{c_1}{h_{01}\alpha}(1+\alpha y)^2 m\left(y\right).$$

Since $m(y)$ is an increasing function of $y > b_+$,

$$\Pi(y,h_{01}) < \lim_{y\downarrow b_+}\Pi(y,h_{01}).$$

On the other hand, a direct evaluation shows that

$$m(y)\to -1/(b_+-\sqrt{b_+})$$

as $y\downarrow b_+$. Using this fact, we obtain

$$\lim_{y\downarrow b_+}\Pi(y,h_{01})=-\frac{c_1}{c_2\left(h_{01}+1\right)}-1+r\frac{\left(r+c_2\right)^2}{c_2 h_{01}\left(1-c_2\right)\left(r+1\right)}. \tag{85}$$

Note that, considered as a function of $h_{01} > \bar{h}$, $\lim_{y\downarrow b_+}\Pi(y,h_{01})$ may have positive derivative only when $\lim_{y\downarrow b_+}\Pi(y,h_{01}) < 0$. Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}h_{01}}\lim_{y\downarrow b_+}\Pi(y,h_{01}) = \; & \frac{c_1}{c_2\left(h_{01}+1\right)^2}-r\frac{\left(r+c_2\right)^2}{c_2 h_{01}^2\left(1-c_2\right)\left(r+1\right)} \\ < \; & \frac{1}{h_{01}}\left(\frac{c_1}{c_2\left(h_{01}+1\right)}-r\frac{\left(r+c_2\right)^2}{c_2 h_{01}\left(1-c_2\right)\left(r+1\right)}\right)\end{aligned}$$

If the latter expression is positive for $h_{01} > \bar{h} > 0$, then $\lim_{y \downarrow b_+} \Pi(y, h_{01})$ is clearly negative. Therefore,

$$\lim_{y \downarrow b_+} \Pi(y, h_{01}) < \max\left\{0, \lim_{y \downarrow b_+} \Pi(y, \bar{h})\right\}.$$

But, using the definition $\bar{h} = (c_2 + r) / (1 - c_2)$ in (85), we obtain

$$\lim_{y \downarrow b_+} \Pi(y, \bar{h}) = -\frac{c_1 (1 - c_2)}{c_2 (1 + r)} - 1 + r\frac{(r + c_2)}{c_2 (r + 1)} = 0.$$

This implies that $\max_{z_1 \in [\tilde{\lambda}_2 + \varepsilon, \tilde{\lambda}_1]} \frac{\mathrm{d}}{\mathrm{d}z_1} f_2 (z_1)$ is a.s. negative for sufficiently large p and $\mathbf{n}$.

To find the a.s. limit of $-\frac{\mathrm{d}}{\mathrm{d}z_1} f_2 (z_1)$ at $z_1 = \tilde{\lambda}_1$, note that

$$-\frac{\mathrm{d}}{\mathrm{d}z_1} f_2 (z_1) \stackrel{a.s.}{\to} \frac{ab + 2az_{1+} + z_{1+}^2}{(b + z_{1+}) (a + z_{1+})} \left(1 + \frac{a(a-b)}{2z_{1+} (a + z_{1+})}\right)\Bigg|_{z_1 = \tilde{\lambda}_{01}} + \frac{c_1}{2} \tilde{m} \left(\tilde{\lambda}_{01}\right).$$

Using (80), (81), and the definition of z_{1+}, we conclude, after some algebra, that

$$-\frac{\mathrm{d}}{\mathrm{d}z_1} f_2 (z_1)\Bigg|_{z_1 = \tilde{\lambda}_1} \stackrel{a.s.}{\to} H_{12},$$

where $H_{12} = c_1 H_{11}$.

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