Appendix to the paper "Determining the Number of Factors from Empirical Distribution of Eigenvalues"

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December 31, 2008

Abstract

This technical appendix contains a proof of Lemma 3 of the paper.

To prove Lemma 3, we will need the following Lemmas A1-A4.

Lemma A1. Let F be the cdf of a non-negative random variable with a finite upper boundary of support $u(F)$ and a finite positive expectation E_F. Let $m(z)$ be the Stieltjes transform of F. Then, for any $z \in \mathbb{C}^+$ such that $|z| > u(F)$ we have: $|zm(z) + 1| \leq \frac{u(F)}{|z|-u(F)}$; and for any $z \in \mathbb{C}^+$ such that $z = x + iy$, where $|x| > u(F) + 3y$ we have: $|zm(z) + 1| \geq \frac{E_F}{4(|z|+u(F))}$.

Proof: For $|z| > u(F)$, we have: $|zm(z) + 1| = \left|\int \left(\frac{z}{\lambda - z} + 1\right) dF(\lambda)\right| = \left|\int \frac{\lambda}{\lambda - z} dF(\lambda)\right| \leq \int \left|\frac{\lambda}{\lambda - z}\right| dF(\lambda) \leq \int \frac{\lambda}{|z| - \lambda} dF(\lambda) \leq \frac{u(F)}{|z|-u(F)}$, which proves one of the lemma's inequalities. Further, write:

$$zm(z) + 1 = \int \frac{\lambda}{\lambda - z} dF(\lambda) = \int \frac{\lambda(\lambda - x)}{|\lambda - z|^2} dF(\lambda) + i\int \frac{\lambda y}{|\lambda - z|^2} dF(\lambda). \tag{1}$$

Since for any real a and b, $|a + ib|^2 = a^2 + b^2 \geq \frac{1}{2}(a + b)^2$, we have: $|zm(z) + 1| \geq$

$\frac{1}{\sqrt{2}}\left|\int \frac{\lambda(\lambda-x+y)}{|\lambda-z|^2}dF(\lambda)\right|$. If $x<0$, then, since for any real and positive a and b, $\frac{a+b}{a^2+b^2} \geq (a^2+b^2)^{-1/2}$, we have $\frac{\lambda(\lambda-x+y)}{|\lambda-z|^2} \geq \frac{\lambda}{|\lambda-z|}$, and therefore: $|zm(z)+1| \geq \frac{1}{\sqrt{2}}\int \frac{\lambda}{|\lambda-z|}dF(\lambda) \geq \frac{1}{\sqrt{2}}\int \frac{\lambda}{|z|+\lambda}dF(\lambda) \geq \frac{1}{\sqrt{2}}\int \frac{\lambda}{|z|+u(F)}dF(\lambda) = \frac{E_F}{\sqrt{2}(|z|+u(F))}$. If $x>0$, then by assumption of the lemma, $x>u(F)+3y$, which implies that $\frac{1}{2}(x-\lambda-y)>y$ and hence, $x-\lambda > y+\frac{1}{2}(x-\lambda+y) \geq y+\frac{1}{2}|\lambda-z|$ so that $x-\lambda+y \geq |\lambda-z|$. Therefore: $|zm(z)+1| \geq \frac{1}{\sqrt{2}}\int \frac{\lambda(x-\lambda-y)}{|\lambda-z|^2}dF(\lambda) \geq \frac{1}{2\sqrt{2}}\int \frac{\lambda}{|\lambda-z|}dF(\lambda) \geq \frac{E_F}{2\sqrt{2}(|z|+u(F))}$.¤

Lemma A2. Suppose that $u(z), v(z) \in \mathbb{C}^+$ are analytic functions satisfying Zhang's system:

$$\begin{cases} zm(z)+1 = u(z)\, m_A(u(z))+1 \\ zm(z)+1 = c^{-1}\left[v(z)\, m_B(v(z))+1\right] \\ zm(z)+1 = -c^{-1}\frac{z}{u(z)v(z)} \end{cases} \quad (2)$$

Let $U=\{z=x+iy:\ x>\underline{x} \text{ and } 0<y<\overline{y}\}$. Then, for any $\underline{x}>u\left(\mathcal{F}^{c,A,B}\right)$, there exists $\overline{y}>0$ such that for any z from U : $\operatorname{Re} u(z) > u\left(\mathcal{F}^{A}\right)$ and $\operatorname{Re} v(z) > u\left(\mathcal{F}^{B}\right)$.

Proof: The idea of the proof is as follows. First, using Lemma A1 we prove that for any $\underline{x}>u(F)$ and $\overline{y}>0$, there exists $z_1 \in U$ such that $\operatorname{Re} u(z_1) > u\left(\mathcal{F}^{A}\right)$ and $\operatorname{Re} v(z_1) > u\left(\mathcal{F}^{B}\right)$. Then, we assume that Lemma A2 does not hold so that for some $\underline{x} > u(F)$ and any $\overline{y}>0$ there exists $z_2 \in U$ such that $\operatorname{Re} u(z_2) \leq u\left(\mathcal{F}^{A}\right)$ or/and $\operatorname{Re} v(z_2) \leq u\left(\mathcal{F}^{B}\right)$. Connecting z_1 and z_2 by a continuous path $z(t) \in U$, we establish the existence of $z_3 \in U$ such that $\operatorname{Re} u(z_3) = u\left(\mathcal{F}^{A}\right)$ or/and $\operatorname{Re} v(z_3) = u\left(\mathcal{F}^{B}\right)$. Then, we show that for small enough $\overline{y}$, $\operatorname{Im}(z_3 m(z_3)+1)$ must be smaller than $\operatorname{Im}(u(z_3)\, m_A(u(z_3))+1)$ or than $c^{-1}\operatorname{Im}(v(z_3)\, m_B(v(z_3))+1)$, which contradicts the assumption that $u(z), v(z)$ satisfy Zhang's system.

First, we prove the existence of z_1. The last equation of Zhang's system and the first inequality of Lemma A1 imply that $\left|\frac{z}{uv}\right| \to 0$ as $|z| \to \infty$. Hence, as $|z| \to \infty$, $\max\{|u|, |v|\} \to \infty$. Suppose without loss of generality that $|u| \to \infty$. Let us show that also $|v| \to \infty$. Indeed, from the first equation of Zhang's system $|zm(z)+1| =$

$|um_A(u) + 1|$. Therefore, for $z \in U$ with large enough $|z|$:

$$\frac{E_{\mathcal{F}^{c,A,B}}}{4\left(|z| + u(\mathcal{F}^{c,A,B})\right)} \leq |zm(z) + 1| = |um_A(u) + 1| \leq \frac{u(\mathcal{F}^A)}{|u| - u(\mathcal{F}^A)}. \tag{3}$$

where the latter inequality is obtained from Lemma A1 applied to $um_A(u) + 1$. This implies that $\liminf_{|z|\to\infty} \left|\frac{z}{u}\right| > 0$, for $z \in U$. However, since $\left|\frac{z}{uv}\right| = c|zm(z) + 1| \leq \frac{cu\left(\mathcal{F}^{c,A,B}\right)}{(|z| - u(\mathcal{F}^{c,A,B}))} \to 0$, we must have $|v| \to \infty$. Hence, as $|z| \to \infty$ so that z remains in U, both $|u| \to \infty$ and $|v| \to \infty$. Let us prove that $\operatorname{Re} u \to \infty$ and $\operatorname{Re} v \to \infty$.

First, notice that for $z \in U$:

$$\operatorname{Im}(zm(z) + 1) < \overline{y} \int \frac{\lambda}{|\lambda - z|^2} d\mathcal{F}^{c,A,B}(\lambda) \leq \frac{\overline{y} E_{\mathcal{F}^{c,A,B}}}{(x - u(\mathcal{F}^{c,A,B}))^2}. \tag{4}$$

Further, $\operatorname{Im}\left(-\frac{z}{uv}\right) = \frac{x \operatorname{Im}(uv) - y \operatorname{Re}(uv)}{|uv|^2}$. Therefore, since Zhang's third equation is $zm(z) + 1 = -c^{-1}\frac{z}{uv}$, we have: $\frac{x \operatorname{Im}(uv) - y \operatorname{Re}(uv)}{|uv|^2} \leq c \frac{\overline{y} E_{\mathcal{F}^{c,A,B}}}{(x - u(\mathcal{F}^{c,A,B}))^2}$. Hence, for $z \in U$, where $\overline{y} \leq \frac{\underline{x} - u\left(\mathcal{F}^{c,A,B}\right)}{3}$, we have:

$$\frac{\operatorname{Im}(uv)}{|uv|} \leq \frac{|uv|}{x} \frac{c\overline{y} E_{\mathcal{F}^{c,A,B}}}{(x - u(\mathcal{F}^{c,A,B}))^2} + \frac{y \operatorname{Re}(uv)}{x|uv|} \leq \frac{\overline{y}}{u(\mathcal{F}^{c,A,B})}\left(\left|\frac{cuv}{z^2}\right| \frac{E_{\mathcal{F}^{c,A,B}}(x^2 + \overline{y}^2)}{(x - u(\mathcal{F}^{c,A,B}))^2} + 1\right).$$

Now, the third equation of (2) and the second inequality of Lemma A1 imply that $\left|\frac{cuv}{z^2}\right| = \left|\frac{1}{z(zm(z)+1)}\right| \leq \frac{4\left(|z| + u(\mathcal{F}^{c,A,B})\right)}{|z| E_{\mathcal{F}^{c,A,B}}} \leq \frac{8}{E_{\mathcal{F}^{c,A,B}}}$. Therefore, $\frac{\operatorname{Im}(uv)}{|uv|} \leq \frac{\overline{y}}{u(\mathcal{F}^{c,A,B})}\left(\frac{\underline{x}^2 + \overline{y}^2}{8(\underline{x} - u(\mathcal{F}^{c,A,B}))^2} + 1\right)$. Noting that $\frac{\operatorname{Im} u}{|u|} \leq \frac{\operatorname{Im}(uv)}{|uv|}$, we have:

$$\frac{\operatorname{Im} u}{|u|} \leq \frac{\overline{y}}{u(\mathcal{F}^{c,A,B})}\left(\frac{(\underline{x}^2 + \overline{y}^2)}{8(\underline{x} - u(\mathcal{F}^{c,A,B}))^2} + 1\right) \tag{5}$$

for $z \in U$, where $\overline{y} \leq \frac{\underline{x} - u\left(\mathcal{F}^{c,A,B}\right)}{3}$. The same inequality also holds for $\frac{\operatorname{Im} v}{|v|}$.

Inequality (5) and the fact that $|u| \to \infty$ imply that $|\operatorname{Re} u| \to \infty$ as $|z| \to \infty$ while z remains in U. Similarly, $|\operatorname{Re} v| \to \infty$. But $\operatorname{Re} u$ and $\operatorname{Re} v$ must be positive for $z \in U$ when

$|z|$ is large enough. Indeed, (1) implies that $\mathrm{Re}\,(zm(z)+1)<0$. Hence, from the first equation of Zhang, $\mathrm{Re}\,(um_A(u)+1)<0$. But from (1) applied to $um_A(u)+1$ and the fact that $|\mathrm{Re}\,u|\to\infty$, $\mathrm{Re}\,(um_A(u)+1)$ must be of the same sign as $-\mathrm{Re}\,u$ for $|z|$ large enough. Hence, $\mathrm{Re}\,u\to+\infty$ as $|z|\to\infty$ while z remains in U. Similarly, $\mathrm{Re}\,v\to+\infty$ as $|z|\to\infty$ while z remains in U. This proves the existence of $z_1\in U$ with properties outlined above.

Assuming that Lemma A2 does not hold, the existence of z_3 follows from the fact that $u(z)$ and $v(z)$ are analytic, and hence continuous, functions of z. Suppose without loss of generality that $\mathrm{Re}\,u(z_3)=u\left(\mathcal{F}^A\right)$. Let us finish the proof of the lemma by comparing $\mathrm{Im}\,(z_3m(z_3)+1)$ with $\mathrm{Im}\,(u(z_3)\,m_A(u(z_3))+1)$ when $\bar{y}$ is small. By assumption that $\liminf_{\delta\to0}\frac{1}{\delta}\int_{|\lambda-u(\mathcal{F}^A)|\le\delta}\lambda d\mathcal{F}^A(\lambda)=k^A>0$, for $u(z)$ such that $\mathrm{Re}\,u=u\left(\mathcal{F}^A\right)$ and $\mathrm{Im}\,u$ is small enough, we have:

$$\begin{aligned}\mathrm{Im}(um_A(u)+1) &= \int\frac{\lambda\,\mathrm{Im}\,u}{\left(\lambda-u\left(\mathcal{F}^A\right)\right)^2+(\mathrm{Im}\,u)^2}d\mathcal{F}^A(\lambda)\\ &\ge \frac{1}{2\,\mathrm{Im}\,u}\int_{|\lambda-u(\mathcal{F}^A)|\le\mathrm{Im}\,u}\lambda d\mathcal{F}^A(\lambda)\ge\frac{k^A}{2}>0. \qquad (6)\end{aligned}$$

From (6) and (5), we can choose $\bar{y}$ small enough so that $\mathrm{Im}(u(z_3)\,m_A(u(z_3))+1)\ge\frac{k^A}{2}$. On the other hand, from the first equation of Zhang, $u(z_3)\,m_A(u(z_3))+1=z_3m(z_3)+1$ and hence $\mathrm{Im}\,(z_3m(z_3)+1)\ge\frac{k^A}{2}$. But from (4) we know that for small enough $\bar{y}$, $\mathrm{Im}\,(z_3m(z_3)+1)$ must be smaller than $\frac{k^A}{2}$. We have got a contradiction, which implies that the statement of Lemma A2 holds. ¤

Lemma A3. For any real $x>u\left(\mathcal{F}^{c,A,B}\right)$, there exist real limits $u(x)\equiv\lim_{z\in\mathbb{C}^+,z\to x}u(z)$ and $v(x)\equiv\lim_{z\in\mathbb{C}^+,z\to x}v(z)$. Functions $u(x)$ and $v(x)$ satisfy the limit version of Zhang's system:

$$\begin{cases} xm(x) + 1 = um_A(u) + 1 \\ xm(x) + 1 = c^{-1}(vm_B(v) + 1) \\ xm(x) + 1 = -c^{-1}\frac{x}{uv} \end{cases}, \tag{7}$$

are analytic and such that $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = \infty$.

Proof: Let $G = \left\{z \in \mathbb{C}^+ : u(\mathcal{F}^{c,A,B}) < \underline{x} \leq \text{Re}\, z \leq \overline{x} < \infty, 0 < \text{Im}\, z < \overline{y} < \infty\right\}$. Then $\sup_{z\in G} \max(|u(z)|, |v(z)|) < \infty$. Had this been not true, there would have existed a sequence $\{z_n\} \in G$ such that $|u(z_n)| \to \infty$ or $|v(z_n)| \to \infty$. Without loss of generality, let $|u(z_n)| \to \infty$. Lemma A1 then would imply that $|u(z_n) m_A(u(z_n)) + 1| \to 0$, and hence, from Zhang's first equation, $|z_n m(z_n) + 1| \to 0$. But, as follows from (1), $|\text{Re}(z_n m(z_n) + 1)| \geq \frac{E_F(\underline{x} - u(F))}{\overline{x}^2 + \overline{y}^2} > 0$, which gives a contradiction.

Since $\sup_{z\in G} \max(|u(z)|, |v(z)|) < \infty$, inequality (5) and a similar inequality for $\frac{\text{Im}\, v}{|v|}$ imply that for any sequence $\{z_n\} \in G$ such that $z_n \to x \in R$ the concentration points of $\{u(z_n)\}$ and $\{v\{z_n\}\}$ must be real. Suppose that there exist subsequences of $z_n, \{z_i\}$ and $\{z_j\}$, such that $u(z_i) \to u_1 \in R$ and $u(z_j) \to u_2 \in R$ and $u_1 \neq u_2$. By Lemma A2, $u_1 \geq u(\mathcal{F}^A)$ and $u_2 \geq u(\mathcal{F}^A)$. If $u_1 = u(\mathcal{F}^A)$, then using inequalities similar to (6), we find that $\text{Im}(u(z_i) m_A(u(z_i)) + 1) \geq \frac{kA}{2}$ for large enough i, which cannot be the case because $\text{Im}(u(z_i) m_A(u(z_i)) + 1) = \text{Im}(z_i m(z_i) + 1) \to 0$ as $i \to \infty$. Hence, $u_1 > u(\mathcal{F}^A)$. Similarly, $u_2 > u(\mathcal{F}^A)$.

Since $m(x)$ exists and is continuous for $x > u(\mathcal{F}^{c,A,B})$, we have:
$\lim_{z_n\to x}(z_n m(z_n) + 1) = xm(x) + 1$. Since $m_A(u)$ exists and is continuous for $u > u(\mathcal{F}^A)$, we have: $\lim_{z_i\to x}(u(z_i) m_A(u(z_i)) + 1) = u_1 m_A(u_1) + 1$ and $\lim_{z_j\to x}(u(z_j) m_A(u(z_j)) + 1) = u_2 m_A(u_2) + 1$. The first equation of Zhang's system implies that we must have:

$$xm(x) + 1 = u_1 m_A(u_1) + 1 = u_2 m_A(u_2) + 1.$$

But this is not possible with $u_1 \neq u_2$ such that $u_1 > u\left(\mathcal{F}^A\right)$ and $u_2 > u\left(\mathcal{F}^A\right)$ because function $um_A(u) + 1$ is strictly increasing for $u > u\left(\mathcal{F}^A\right)$. Hence, there exists only one concentration point of $\{u(z_n)\}$, that is there exists a real limit $u(x) \equiv \lim_{z\in \mathbb{C}^+, z\to x} u(z)$. Similarly, there exists a real limit $v(x) \equiv \lim_{z\in \mathbb{C}^+, z\to x} v(z)$.

That $u(x)$ and $v(x)$ satisfy the limit version of Zhang's system follows from the existence and continuity of $m_A(u)$ for $|u| > u\left(\mathcal{F}^A\right)$ and from the existence and continuity of $m_B(v)$ for $|v| > u\left(\mathcal{F}^B\right)$. The analyticity of $u(x)$ follows from the analyticity of $F(x,u) \equiv xm(x) + 1 - (um_A(u) + 1)$ for $x > u\left(\mathcal{F}^{c,A,B}\right)$ and $u > u\left(\mathcal{F}^A\right)$ and from the implicit function theorem. Similarly, the analyticity of $v(x)$ follows from the analyticity of $F_1(x,v) \equiv xm(x) + 1 - c^{-1}(vm_B(v) + 1)$ for $x > u\left(\mathcal{F}^{c,A,B}\right)$ and $v > u\left(\mathcal{F}^B\right)$ and from the implicit function theorem. Finally, (7) implies that as $x \to \infty$, $um_A(u) + 1 \to 0$ and $vm_B(v) + 1 \to 0$, which can be the case only when $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = \infty$. ¤

Lemma A4. For $x > u(\mathcal{F}^{c,A,B})$, the following system

$$\begin{cases} v = x\left(c\int \frac{\lambda u}{u-\lambda} d\mathcal{F}_A(\lambda)\right)^{-1} \\ u = x\left(\int \frac{\lambda v}{v-\lambda} d\mathcal{F}_B(\lambda)\right)^{-1} \end{cases} \tag{8}$$

has exactly two solutions (u_1, v_1) and (u_2, v_2) such that $u_i > u\left(\mathcal{F}^A\right)$ and $v_i > u\left(\mathcal{F}^B\right)$ for $i = 1, 2$. For $x = u\left(\mathcal{F}^{c,A,B}\right)$ and for $x < u(\mathcal{F}^{c,A,B})$, the system has only one such solution and no such solutions, respectively

Proof: For any $x > u\left(\mathcal{F}^{c,A,B}\right)$, one solution to (8) satisfying $u(x) > u\left(\mathcal{F}^A\right)$ and $v(x) > u\left(\mathcal{F}^B\right)$ is given by $u(x)$ and $v(x)$ defined in Lemma A3. That such $u(x)$ and $v(x)$ indeed provide a solution to (8) follows from the fact that (8) can be obtained from (7) by substituting the third equation into the first two. Let us now show that for $x > u(\mathcal{F}^{c,A,B})$, there exists another solution to (8).

First, note that $x\left(c\int \frac{\lambda u}{u-\lambda} dF_A(\lambda)\right)^{-1}$ as a function of $u > u\left(\mathcal{F}^A\right)$ is concave, tends to zero as $u \downarrow u(\mathcal{F}_A)$ and to $x(cE_A)^{-1}$ as $u \to \infty$. The concavity follows from the expression

$\frac{d^2}{du^2}x\left(c\int\frac{\lambda u}{u-\lambda}d\mathcal{F}_A(\lambda)\right)^{-1} = 2xc^{-1}\left(\int\frac{\lambda u}{u-\lambda}d\mathcal{F}_A(\lambda)\right)^{-3}\cdot\left(\left(\int\frac{\lambda^2}{(u-\lambda)^2}d\mathcal{F}_A(\lambda)\right)^2 - \left(\int\frac{\lambda^2}{(u-\lambda)^3}d\mathcal{F}_A(\lambda)\right)\left(E_A + \int\frac{\lambda^2}{u-\lambda}d\mathcal{F}_A(\lambda)\right)\right)$ and the Cauchy inequality $\int\frac{\lambda}{(u-\lambda)^{3/2}}\frac{\lambda}{(u-\lambda)^{1/2}}d\mathcal{F}_A(\lambda) \leq \left(\int\frac{\lambda^2}{(u-\lambda)^3}d\mathcal{F}_A(\lambda)\right)^{1/2}\left(\int\frac{\lambda^2}{u-\lambda}d\mathcal{F}_A(\lambda)\right)^{1/2}$. The tendency to zero follows from the fact that $c\int\frac{\lambda u}{u-\lambda}d\mathcal{F}_A(\lambda)\to\infty$ as $u\downarrow u(\mathcal{F}_A)$, which is easy to show using the monotone convergence theorem and assumption $\liminf_{\delta\to0}\frac{1}{\delta}\int_{|\lambda-u(\mathcal{F}^A)|\leq\delta}\lambda d\mathcal{F}^A(\lambda) = k^A > 0$. Finally, the convergence to $x(cE_A)^{-1}$ as $u\to\infty$ is obvious. Similarly, $x\left(\int\frac{\lambda v}{v-\lambda}d\mathcal{F}_B(\lambda)\right)^{-1}$ as a function of $v > u\left(\mathcal{F}^B\right)$ is concave, tends to zero as $v\downarrow u(\mathcal{F}_B)$ and to $x(cE_B)^{-1}$ as $v\to\infty$.

The above properties of $x\left(c\int\frac{\lambda u}{u-\lambda}d\mathcal{F}_A(\lambda)\right)^{-1}$ and $x\left(\int\frac{\lambda v}{v-\lambda}d\mathcal{F}_B(\lambda)\right)^{-1}$ imply that the curves in the $\{u > u(\mathcal{F}_A), v > v(\mathcal{F}_B)\}$ subset of the (u,v)-plane defined by (8) are either intersecting at two points, touching at a single point, or having no common points. Since there exists a solution to (8) for any $x > u\left(\mathcal{F}^{c,A,B}\right)$ and since $x\left(c\int\frac{\lambda u}{u-\lambda}d\mathcal{F}_A(\lambda)\right)^{-1}$ and $x\left(\int\frac{\lambda v}{v-\lambda}d\mathcal{F}_B(\lambda)\right)^{-1}$ are monotone increasing in x, the curves must intersect at two points for any $x > u\left(\mathcal{F}^{c,A,B}\right)$. Let us show that the curves are touching at a single point when $x = u\left(\mathcal{F}^{c,A,B}\right)$.

Suppose the curves intersect at two points (u_1, v_1) and (u_2, v_2) when $x = u\left(\mathcal{F}^{c,A,B}\right)$. Let $u_2 > u_1$ and $v_2 > v_1$. Define $f_1(x,u,v) = x + cuv(um_A(u) + 1)$ and $f_2(x,u,v) = x + uv(vm_B(v) + 1)$. Note that system (8) is equivalent to $f_i(x,u,v) = 0$ for $i = 1,2$. It is straightforward to check that the assumption of the proper intersection of the curves (not just a tangency at one point) is equivalent to $\det\begin{pmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{f_2}{\partial v}\end{pmatrix} \neq 0$ at any of the two intersection points. Then the implicit function theorem (see Krantz (1992), Theorem 1.4.11) implies that there exist holomorphic functions $u(z), v(z)$ defined in an open neighborhood of $z = u\left(\mathcal{F}^{c,A,B}\right)$ in C, which satisfy $f_i(z,u,v) = 0$ for $i = 1,2$. To each of the two intersection points, there will correspond its own set of holomorphic functions $u(z), v(z)$. We will consider the functions $u(z)$ and $v(z)$ corresponding to (u_2, v_2). For such a choice, it is straightforward to check that $\frac{d}{d(\mathrm{Re}\,z)}\mathrm{Re}\,u(z) > 0$ and $\frac{d}{d(\mathrm{Re}\,z)}\mathrm{Re}\,v(z) > 0$

at $z = u\left(\mathcal{F}^{c,A,B}\right)$.

Furthermore, using identities $f_i\left(z, u(z), v(z)\right) = 0$ for $i = 1, 2$ it is straightforward to check that in a small enough neighborhood of $z = u\left(\mathcal{F}^{c,A,B}\right)$ in C, $\mathrm{Im}\, z > 0$ implies that $\mathrm{Im}\, u(z)$ and $\mathrm{Im}\, v(z)$ are of the same sign and are not equal to zero. Cauchy-Riemann equations for holomorphic functions imply that $\frac{d}{d(\mathrm{Im}\, z)} \mathrm{Im}\, u(z) = \frac{d}{d(\mathrm{Re}\, z)} \mathrm{Re}\, u(z) > 0$ and $\frac{d}{d(\mathrm{Im}\, z)} \mathrm{Im}\, v(z) = \frac{d}{d(\mathrm{Re}\, z)} \mathrm{Re}\, v(z) > 0$ at $z = u\left(\mathcal{F}^{c,A,B}\right)$. Hence, $\mathrm{Im}\, u(z)$ and $\mathrm{Im}\, v(z)$ are positive when $\mathrm{Im}\, z$ is positive and z lies in a small enough complex neighborhood of $u\left(\mathcal{F}^{c,A,B}\right)$. Let us define $m(z) = -\frac{c^{-1}}{u(z)v(z)} - \frac{1}{z}$. Clearly, for z in the small complex neighborhood of $u\left(\mathcal{F}^{c,A,B}\right)$, $\mathrm{Im}\, m(z) > 0$.

Zhang shows that for any $z \in \mathrm{C}^+$, there is only one solution to (2) such that m, u and v belong to C^+. Hence, $u(z), v(z)$, and $m(z)$ defined above constitute the solution to Zhang's system (2) for z in a small neighborhood of $u\left(\mathcal{F}^{c,A,B}\right)$ and such that $\mathrm{Im}\, z > 0$. Finally, for any real x which belongs to the neighborhood of $u(\mathcal{F}^{c,A,B})$, we have: $\lim_{z\to x} \mathrm{Im}\, m(z) = \lim_{z\to x} \mathrm{Im}\left(-\frac{c^{-1}}{u(z)v(z)} - \frac{1}{z}\right) = 0$. Thus, using the Frobenius-Perron inversion formula, we get $\int_{u(\mathcal{F}^{c,A,B})-\delta}^{u(\mathcal{F}^{c,A,B})} dF(\lambda) = 0$ for small positive δ, which is impossible by definition of $u(\mathcal{F}^{c,A,B})$. Hence, the curves are touching at a single point when $x = u\left(\mathcal{F}^{c,A,B}\right)$. This implies that they do not intersect when $x < u(\mathcal{F}^{c,A,B})$. ¤

Now we are ready to prove Lemma 3.

Proof of Lemma 3: Recall that by assumption, $\mathcal{F}^{AA'}$ almost surely weakly converges to $\mathcal{F}_A$ and $u\left(\mathcal{F}^{AA'}\right) \to u\left(\mathcal{F}_A\right)$. Similarly, $\mathcal{F}^{BB'}$ almost surely weakly converges to $\mathcal{F}_B$ and $u\left(\mathcal{F}^{BB'}\right) \to u\left(\mathcal{F}_B\right)$. These facts imply that if the curves in the $\{u > u\left(\mathcal{F}_A\right), v > v\left(\mathcal{F}_B\right)\}$ subset of the (u, v)-plane defined by (8) intersect at zero, or at two points, then the curves in the $\left\{u > u\left(\mathcal{F}^{AA'}\right), v > v\left(\mathcal{F}^{BB'}\right)\right\}$ subset of the (u, v)-plane defined by

$$\begin{cases} v = x\left(c_n \int \frac{\lambda u}{u-\lambda} d\mathcal{F}^{AA'}(\lambda)\right)^{-1} \\ u = x\left(\int \frac{\lambda v}{v-\lambda} d\mathcal{F}^{BB'}(\lambda)\right)^{-1} \end{cases}$$

also intersect at zero, or at two points for large enough n. Therefore, by Lemma A4, $u\left(\mathcal{F}^{c_n,A_n,B_n}\right)$ converges to $u\left(\mathcal{F}^{c,A,B}\right)$ and Lemma 3 follows from Lemma 2. ¤

References

[1] Krantz, S. G. (1992) Function theory of several complex variables. Second Edition. AMS Chelsea Publishing. Providence, Rhode Island.