Detection of weak signals in high-dimensional complex-valued data

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Abstract

This paper considers the problem of detecting a few signals in high-dimensional complex-valued Gaussian data satisfying Johnstone's (2001) *spiked covariance model*. We focus on the difficult case where signals are weak in the sense that the sizes of the corresponding covariance spikes are below the *phase transition threshold* studied in Baik et al (2005). We derive a simple analytical expression for the maximal possible asymptotic probability of correct detection holding the asymptotic probability of false detection fixed. To accomplish this derivation, we establish what we believe to be a new formula for the *Harish-Chandra/Itzykson-Zuber (HCIZ) integral* $\int_{\mathcal{U}(p)} e^{\mathrm{tr}\left(AUB\bar{U}'\right)} (dU)$, where A has a deficient rank $r < p$. The formula links the HCIZ integral over $\mathcal{U}(p)$ to an HCIZ integral over a potentially much smaller unitary group $\mathcal{U}(r)$. We show that the formula generalizes to the integrals over orthogonal and symplectic groups. In the most general form, it expresses the hypergeometric function $_0F_0^{(\alpha)}$ of two $p \times p$ matrix arguments as a repeated contour integral of the hypergeometric function $_0F_0^{(\alpha)}$ of two $r \times r$ matrix arguments.

1   Introduction

Much contemporary research in statistics concerns with situations where the dimensionality of data is large and comparable to the number of observations (see special issues of the *Philosophical Transactions of the Royal Society* (2009) **367** and *Annals of Statistics* (2008) **36**). Often, the goal is to estimate or detect a few signals contaminated by high-dimensional noise. One general conclusion that seems to emerge from this research is that, in the absence of *a priori* sparsity assumptions about signals, there is a lower limit for the signal-to-noise ratio below which statistical inference about the signals completely fails (Johnstone and Titterington, 2009, Nadakuditi and Edelman, 2008, Nadakuditi and Silverstein, 2010). This limit equals the *phase transition threshold* studied in Baik et al (2005). In a recent paper, Onatski et al (2012) show that not all is lost below the threshold. They consider the case of a single non-sparse signal in high-dimensional noisy data and establish sharp non-trivial limits for the asymptotic power, as both the data dimensionality and the number of observations go to infinity, of statistical tests for signal detection when the signal may be arbitrarily weak.

This paper extends Onatski et al (2012) to the case of multiple non-sparse arbitrarily weak signals when the data are complex-valued. Complex-valued data are of interest in signal processing (Schreier and Scharf, 2010), wireless communication (Telatar, 1999, Tulino and Verdu, 2004), and the spectral analysis of economic and financial time series (Onatski, 2009). Considering the case of multiple signals is important for applied work because the constraint that there is no more than one

signal can rarely be justified in practice. We derive a simple analytical expression for the maximal possible asymptotic probability of correct detection, based on the sample covariance eigenvalues of the data, holding the asymptotic probability of false detection fixed.

We find that the asymptotic probability of detection may be close to one even in cases where the strength of all signals is substantially below the phase transition threshold. This finding is, perhaps, surprising in light of the fact (Péché, 2003) that in such cases, sometimes referred to as the sub-critical regime, the asymptotic behavior of any finite number of the largest sample covariance eigenvalues is not different from their behavior when the data are pure noise. We show that in these difficult cases, the detection power lies not in the different behavior of a few of the largest eigenvalues, but in the small deviations of the empirical distribution of all the eigenvalues from the *Marchenko-Pastur limit* (Marchenko and Pastur, 1967).

Let us discuss our findings in more detail. We assume that data consist of n independent observations of p-dimensional complex-valued Gaussian vectors X_t with mean zero and covariance matrix $\sigma^2 \left(I_p + V H \bar{V}' \right)$, where I_p is the p-dimensional identity matrix, σ is a real scalar, H is an $r \times r$ real diagonal matrix with elements $h_j \geq 0$ along the diagonal, and V is a $(p \times r)$-dimensional complex parameter normalized so that $\bar{V}'V = I_r$. Such a *spiked covariance model* was proposed by Johnstone (2001) as a simple model of a situation, often observed in applications, where a few eigenvalues of the sample covariance matrix, corresponding to signals, are relatively large, whereas the rest of the eigenvalues are relatively small and tightly clustered. In our notation, the size of the spikes is regulated by the values of h_j, and the signal space is spanned by the columns of matrix V.

Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p$ be the ordered eigenvalues of XX'/n, where $X = [X_1, ..., X_n]$, and let $\lambda = (\lambda_1, ..., \lambda_m)$, where $m = \min \{n, p\}$. We are interested in the asymptotic power of tests for signal detection based on the information

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contained in λ when $p, n \to \infty$ so that $p/n \to c$ with $0 < c < \infty$. Our null hypothesis is $H_0 : h_1 = ... = h_r = 0$ (no signals), and our alternative is $H_1 : h_i > 0$ for some $i = 1, ..., r$. The matrix V is left as an unspecified nuisance parameter. In this framework, signal detection tests can also be interpreted as tests of sphericity.

We consider both cases of specified and unspecified σ^2. For the purpose of brevity, in the introduction, we will discuss only the case of specified $\sigma^2 = 1$. First, we study the likelihood ratio $L(h; \lambda)$, defined as the ratio of the densities of λ corresponding to unrestricted h and restricted $h = 0$, the densities being evaluated at the observed value of λ. We show that $L(h; \lambda)$ can be represented in the form of the determinant of an $r \times r$ matrix with entries equal to contour integrals of elementary functions. We use Laplace approximations to these contour integrals to show that for any $\bar{h}$ such that $0 < \bar{h} < \sqrt{c}$, with $\sqrt{c}$ being the value of the phase transition threshold, the sequence of log-likelihood processes $\{\ln L(h; \lambda); h \in [0, \bar{h}]^r\}$ converges weakly to a Gaussian process[1] $\{\mathcal{L}_\lambda(h); h \in [0, \bar{h}]^r\}$ under the null hypothesis as $n, p \to \infty$. The limiting process has mean $\mathrm{E}[\mathcal{L}_\lambda(h)] = \frac{1}{2} \sum_{i,j=1}^r \ln(1 - h_i h_j / c)$ and autocovariance function $\mathrm{Cov}\left(\mathcal{L}_\lambda(h), \mathcal{L}_\lambda(\tilde{h})\right) = -\sum_{i,j=1}^r \ln\left(1 - h_i \tilde{h}_j / c\right)$. The established weak convergence of statistical experiments implies, via Le Cam's first lemma (see van der Vaart 1998, p.88), that the joint distributions of the sample covariance eigenvalues under the null and under alternatives associated with $h \in [0, \sqrt{c})^r$ are mutually contiguous.

An asymptotic power envelope for eigenvalue-based tests of H_0 against H_1 can be constructed using the Neyman-Pearson lemma and Le Cam's third lemma. We show that, for tests of size α, the maximum achievable asymptotic power against a point alternative $h = (h_1, ..., h_r)$ equals $1 - \Phi\left[\Phi^{-1}(1 - \alpha) - \sqrt{W}\right]$, where Φ is the standard normal distribution function and $W = -\sum_{i,j=1}^r \ln(1 - h_i h_j / c)$. A

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[1]Here the index λ in the notation $\mathcal{L}_\lambda(h)$ is used to distinguish the limiting log-likelihood process in the case of specified $\sigma^2 = 1$, from that in the case of unspecified σ^2, which we denote by $\mathcal{L}_\mu(h)$.

preliminary analysis indicates that the asymptotic power of the likelihood ratio test based on the information contained in λ is close to the asymptotic power envelope. In contrast, we find that the asymptotic powers of various previously proposed tests are well below the envelope.

The central technical result of this paper is the contour integral representation of the likelihood ratio. To derive such a representation, we establish what we believe to be a novel formula for the hypergeometric functions of two matrix arguments $_0F_0^{(\alpha)}(A,B)$, where a $p \times p$ matrix A has rank $r < p$, so that, without loss of generality, only its upper-left $r \times r$ block $\mathcal{A}$ is non-zero. Such functions appear as a key term in the explicit expressions for the joint density of the eigenvalues of Wishart matrices with spiked covariance parameter. In Lemma 1, we show that

$$_0F_0^{(\alpha)}(A,B) = \frac{1}{r!\,(2\pi\mathrm{i})^r} \oint_{\mathcal{K}} \cdots \oint_{\mathcal{K}} {}_0F_0^{(\alpha)}(\mathcal{A},\mathcal{Z})\,\omega^{(\alpha)}(A,B,\mathcal{Z}) \prod_{j=1}^{r} \mathrm{d}z_j, \qquad (1)$$

where $\mathcal{Z} = \mathrm{diag}(z_1,...,z_r)$ is an auxiliary matrix, and $\omega^{(\alpha)}(\cdot)$ is a simple function of A, B, and $\mathcal{Z}$. This formula expresses the hypergeometric function of high-dimensional arguments as a repeated contour integral of a hypergeometric function of low-dimensional arguments, which is convenient for analysis.

For the special case $r = 1$, (1) reduces to the formula that has been recently derived in Mo (2011) and, independently, in Wang (2012) and Onatski et al (2012) (see also Forrester, 2011 for a short derivation). Our method of proof is different from the methods used by these authors. It is based on the orthogonality of Jack polynomials with respect to the torus scalar product (Macdonald (1995), Chapter VI, §10).

Although our analysis of signal detection in complex data requires only the formula for $_0F_0^{(1)}(A,B)$, we establish (1) for all $\alpha = 2/\beta$, where β is a positive

integer.[2] The importance of finding "serviceable approximations" to $_0F_0^{(2)}(A, B)$ has been recently emphasized by Johnstone (2007, p.322). Since in applications that rely on the spiked covariance matrix framework r is typically much smaller than p, analyzing $_0F_0^{(2)}(\mathcal{A}, \mathcal{Z})$ is much easier than analyzing $_0F_0^{(2)}(A, B)$, and the established contour integral representation of the latter may be of the welcomed service to practitioners.

For $\alpha = 2, 1$ and $1/2$, function $_0F_0^{(\alpha)}(A, B)$ has an integral representation $\int_{\mathcal{G}^{(\alpha)}(p)} e^{\mathrm{tr}\left(AGBG^{-1}\right)}(\mathrm{d}G)$, where $\mathcal{G}^{(\alpha)}(p)$ is the orthogonal group $\mathcal{O}(p)$ for $\alpha = 2$, the unitary group $\mathcal{U}(p)$ for $\alpha = 1$, and the compact symplectic group $\mathcal{S}p(p)$ for $\alpha = 1/2$, and where $(\mathrm{d}G)$ is the normalized Haar measure over $\mathcal{G}^{(\alpha)}(p)$. Such integrals have various important applications in mathematics and physics, where they are referred to as Harish-Chandra/Itzykson-Zuber (HCIZ) integrals (Zinn-Justin and Zuber, 2003). The HCIZ integrals with rank-deficient A have been used in the analysis of spin glasses (Marinari et al, 1994), wireless communication systems (Muller et al, 2008), statistical tests for signal detection (Bianchi et al, 2010, and Onatski et al, 2012), distribution of the largest sample covariance eigenvalue (Mo, 2011, and Wang, 2012), and spiked Wishart β-ensembles (Forrester, 2011). Their asymptotic behavior as $p \to \infty$ has been studied in Guionnet and Maïda (2005) and Collins and Śniady (2007). We hope that the reduction of HCIZ integrals over large group $\mathcal{G}^{(\alpha)}(p)$ to those over smaller group $\mathcal{G}^{(\alpha)}(r)$ that follows from (1) will be useful in a wide spectrum of applications.

The rest of this paper is organized as follows. In Section 2, we derive explicit formulae for the likelihood ratios. Section 3 establishes relationship (1). Section 4 uses (1) to derive contour integral representations for the likelihood ratios. Section 5 applies Laplace approximations to the contour integrals in the derived represen-

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[2]For cases where β is odd, we require that $p - r + 1$ be even. For even β, such a requirement is not needed.

tation to obtain the asymptotics of the likelihood ratio process. This asymptotics is then used along with the Neyman-Pearson lemma and Le Cam's third lemma to establish a simple analytical formula for the maximal possible asymptotic probability of correct signal detection holding the asymptotic probability of false detection fixed. Section 6 concludes. All proofs are relegated to the Appendix.

2   Likelihood ratios

As mentioned above, we assume that data consist of n independent observations of p-dimensional complex-valued Gaussian vectors $X_t \sim N_{\mathbb{C}}(0, \Sigma)$. This means that $X_t = Y_t + \mathrm{i}Z_t$, where i denotes the imaginary unit, and the joint density of (Y_t, Z_t) at (y, z) equals $\frac{1}{(2\pi)^p \det \Sigma} \exp\left\{ -\operatorname{tr}\left[\Sigma^{-1} (y + \mathrm{i}z)(y - \mathrm{i}z)' \right] \right\}$ (see, for example, Goodman, 1963). Further, we assume that the covariance matrix Σ equals $\sigma^2 \left(I_p + V H \bar{V}' \right)$, where $H = \operatorname{diag}(h_1, ..., h_r)$ quantifies the sizes of the covariance spikes. Our goal is to study the asymptotic power of tests of $H_0 : h_1 = ... = h_r = 0$ against $H_1 : h_i > 0$ for some $i = 1, ..., r$.

If σ^2 is specified, the model is invariant with respect to unitary transformations and the maximal invariant statistic is λ, the vector of the first $m = \min\{n, p\}$ eigenvalues of XX'/n, where $X = [X_1, ..., X_n]$. Therefore, we consider tests based on λ. If σ^2 is unspecified, the model is invariant with respect to the unitary transformations and multiplications by non-zero scalars, and the maximal invariant is the vector of normalized eigenvalues $\mu = (\mu_1, ..., \mu_{m-1})$, where $\mu_j = \lambda_j / (\lambda_1 + ... + \lambda_p)$. Hence, we consider tests based on μ. Note that the distribution of μ does not depend on σ^2, whereas if σ^2 is specified, we can always normalize λ dividing it by σ^2. Therefore, in what follows, we will assume without loss of generality that $\sigma^2 = 1$.

Let $h = (h_1, ..., h_r)$, and let us denote the joint density of $\lambda_1, ..., \lambda_m$ as $p_\lambda(x; h)$, $x = (x_1, ..., x_m) \in (\mathbb{R}^+)^m$ and that of $\mu_1, ..., \mu_{m-1}$ as $p_\mu(y; h)$, $y = (y_1, ..., y_{m-1}) \in$

$\left(\mathbb{R}^{+}\right)^{m-1}$. We have

$$p_{\lambda}\left(x;h\right) = \tilde{\gamma}\frac{\prod_{i=1}^{m} x_i^{|p-n|} \prod_{i<j}^{m}\left(x_i - x_j\right)^2}{\prod_{i=1}^{r}\left(1 + h_i\right)^n} \int_{\mathcal{U}(p)} e^{-n\operatorname{tr}\left(\Pi G \mathcal{X} G^{-1}\right)}\left(\mathrm{d}G\right), \qquad (2)$$

where $\tilde{\gamma}$ depends only on n and p; $\Pi = \operatorname{diag}\left(\left(1 + h_1\right)^{-1}, ..., \left(1 + h_r\right)^{-1}, 1, ..., 1\right)$; $\mathcal{X} = \operatorname{diag}\left(x_1, ..., x_m, 0, ..., 0\right)$ is a $\left(p \times p\right)$ diagonal matrix, so that there are no zeros along the diagonal if $m = p$; $\mathcal{U}\left(p\right)$ is the set of all $p \times p$ unitary matrices; and $\left(\mathrm{d}G\right)$ is the invariant measure on the unitary group $\mathcal{U}\left(p\right)$ normalized to make the total measure unity. Formula (2) is a special case of the densities given in James (1964, p.489) for $n \geq p$ and in Ratnarajah and Vaillancourt (2005) for $n < p$.

Let $s = x_1 + ... + x_m$ and let $y_j = x_j/s$. Note that the Jacobian of the coordinate change from $\left(x_1, ..., x_m\right)$ to $\left(y_1, ..., y_{m-1}, s\right)$ equals s^{m-1}. Changing variables in (2) and integrating s out, we obtain

$$p_{\mu}\left(y;h\right) = \tilde{\gamma}\frac{\prod_{i=1}^{m} y_i^{|p-n|} \prod_{i<j}^{m}\left(y_i - y_j\right)^2}{\prod_{i=1}^{r}\left(1 + h_i\right)^n} \int_0^{\infty} s^{np-1} \int_{\mathcal{U}(p)} e^{-ns\operatorname{tr}\left(\Pi G \mathcal{Y} G^{-1}\right)}\left(\mathrm{d}G\right)\mathrm{d}s, \quad (3)$$

where $\mathcal{Y} = \operatorname{diag}\left(y_1, ..., y_{m-1}, 0, ..., 0\right)$ is a $\left(p \times p\right)$ diagonal matrix.

Consider the likelihood ratios: $L\left(h;\lambda\right) = p_{\lambda}\left(\lambda;h\right)/p_{\lambda}\left(\lambda;0\right)$ and $L\left(h;\mu\right) = p_{\mu}\left(\mu;h\right)/p_{\mu}\left(\mu;0\right)$. Formulae (2) and (3) imply the following Proposition.

Proposition 1 *Let $\mathcal{U}\left(p\right)$ be the set of all $p \times p$ unitary matrices. Denote by $\left(\mathrm{d}G\right)$ the invariant measure on the unitary group $\mathcal{U}\left(p\right)$ normalized to make the total measure unity. Further, let $\Lambda = \operatorname{diag}\left(\lambda_1, ..., \lambda_p\right)$ and $M = \operatorname{diag}\left(\mu_1, ..., \mu_p\right)$. Then*

$$L\left(h;\lambda\right) = \prod_{i=1}^{r}\left(1 + h_i\right)^{-n} \int_{\mathcal{U}(p)} e^{-n\operatorname{tr}\left(\left(\Pi - I\right)G\Lambda G^{-1}\right)}\left(\mathrm{d}G\right) \quad and \qquad (4)$$

$$L\left(h;\mu\right) = \frac{\prod_{i=1}^{r}\left(1 + h_i\right)^{-n} n^{np}}{\Gamma\left(np\right)} \int_0^{\infty} s^{np-1} e^{-ns} \int_{\mathcal{U}(p)} e^{-ns\operatorname{tr}\left(\left(\Pi - I\right)GMG^{-1}\right)}\left(\mathrm{d}G\right)\mathrm{d}s. (5)$$

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Our analysis of the asymptotic power of tests for signal detection is based on a study of the asymptotic properties of the likelihood ratio processes $\{L(h; \lambda) ; h \in (R^+)^r\}$ and $\{L(h; \mu) ; h \in (R^+)^r\}$. First, we will focus on the key terms in the expressions (4) and (5), which are the integrals over the unitary group. These integrals are special cases of the complex hypergeometric function $_0F_0^{(1)}(A, B) = \int_{\mathcal{U}(p)} e^{\mathrm{tr}(AGBG^{-1})}(\mathrm{d}G)$, where A and, possibly, B are rank-deficient. In the next section, we derive a formula for $_0F_0^{(\alpha)}(A, B)$ with rank-deficient A and B that links this function to a hypergeometric functions of full-rank matrix arguments of lower dimensions. We do not restrict attention to the case $\alpha = 1$ because, as discussed in the introduction, other cases constitute independent interest.

3　Contour integral representation for $_0F_0^{(\alpha)}(A, B)$

Let us first provide a necessary background on hypergeometric functions. Let A and B be Hermitian $p \times p$ matrices over real, complex, or quaternion division algebra. The eigenvalues of such matrices are real and we will denote them as $a = (a_1, ..., a_p)$ and $b = (b_1, ..., b_p)$. The hypergeometric function $_0F_0^{(\alpha)}(A, B)$ is defined as (see, for example, Koev and Edelman, 2006)

$$_0F_0^{(\alpha)}(A, B) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{1}{k!} \frac{C_\kappa^{(\alpha)}(A) C_\kappa^{(\alpha)}(B)}{C_\kappa^{(\alpha)}(I_p)}, \tag{6}$$

where $C_\kappa^{(\alpha)}(A) = C_\kappa^{(\alpha)}(a)$, $C_\kappa^{(\alpha)}(B) = C_\kappa^{(\alpha)}(b)$ and $C_\kappa^{(\alpha)}(I_p) = C_\kappa^{(\alpha)}(1, ..., 1)$ are normalized Jack polynomials (Macdonald, 1995, chapter VI, §10), and the inner sum runs over all partitions κ of k, that is over all non-increasing sequences of non-negative integers $\kappa = (\kappa_1, \kappa_2, ...)$ such that $\kappa_1 + \kappa_2 + ... = k$. The normalization of

$C_\kappa^{(\alpha)}(x_1, ..., x_p)$ is chosen so that

$$(x_1 + ... + x_p)^k = \sum_{\kappa \vdash k} C_\kappa^{(\alpha)}(x_1, ..., x_p). \tag{7}$$

Note that $_0F_0^{(\alpha)}(A, B)$ depends on A and B only through a and b. Therefore, in what follows, without loss of generality, we will consider only diagonal matrices $A = \text{diag}(a_1, ..., a_p)$ and $B = \text{diag}(b_1, ..., b_p)$. We will allow a_j and b_j to be complex, thus extending definition (6) to complex diagonal matrices A and B.

As was mentioned in the introduction, for $\alpha = 2, 1$ and $1/2$, hypergeometric functions $_0F_0^{(\alpha)}(A, B)$ admit the integral representation

$$_0F_0^{(\alpha)}(A, B) = \int_{\mathcal{G}^{(\alpha)}(p)} e^{\text{tr}\left(AGBG^{-1}\right)} (\mathrm{d}G), \tag{8}$$

where $\mathcal{G}^{(\alpha)}(p)$ is the orthogonal group $\mathcal{O}(p)$ for $\alpha = 2$, the unitary group $\mathcal{U}(p)$ for $\alpha = 1$, and the compact symplectic group $\mathcal{S}p(p)$ for $\alpha = 1/2$. For real diagonal A and B, such a representation follows from the fact that $\int_{\mathcal{G}^{(\alpha)}(p)} C_\kappa^{(\alpha)}(AGBG^{-1})(\mathrm{d}G) = \frac{C_\kappa^{(\alpha)}(A)C_\kappa^{(\alpha)}(B)}{C_\kappa^{(\alpha)}(I_p)}$ (see Proposition 5.5 of Gross and Richards, 1987), and the fact that $\sum_{k=0}^\infty \sum_{\kappa \vdash k} \frac{1}{k!} C_\kappa^{(\alpha)}(AGBG^{-1}) = e^{tr\left(AGBG^{-1}\right)}$, which follows from (7). For complex diagonal A and B, the representation holds by the analytic continuation because both parts of equality (8) are complex analytic functions of the diagonal elements of A and B.

The main result of this section is as follows.

Lemma 1 *Let $A = \text{diag}(a_1, ..., a_p)$ and $B = \text{diag}(b_1, ..., b_p)$, where a_j and b_j are real or complex numbers. Assume that $a_j \neq 0$ for $1 \leq j \leq r$ and $a_j = 0$ for $r < j \leq p$, and denote the upper left block of A, $\text{diag}(a_1, ..., a_r)$, as $\mathcal{A}$. Further, let $\mathcal{Z} = \text{diag}(z_1, ..., z_r)$, where z_j are complex variables, and let $\mathcal{K}$ be a contour in the complex plane that encircles $b_1, ..., b_p$ counter-clockwise. Finally, let $\alpha = 2/\beta$,*

where β is a positive integer. Then, assuming that $p - r + 1$ is an even integer in cases where β is odd, and without this additional assumption in cases where β is even, we have

$$_0F_0^{(\alpha)}(A, B) = \frac{1}{r!\,(2\pi\mathrm{i})^r} \oint_{\mathcal{K}} \cdots \oint_{\mathcal{K}} {}_0F_0^{(\alpha)}(\mathcal{A}, \mathcal{Z})\,\omega^{(\alpha)}(A, B, \mathcal{Z}) \prod_{j=1}^{r} \mathrm{d}z_j, \qquad (1)$$

where

$$\begin{aligned}
\omega^{(\alpha)}(A, B, \mathcal{Z}) &= (-1)^{r(r-1)/(2\alpha)} \prod_{j=1}^{r} \left[\frac{\Gamma\left((p+1-j)/\alpha\right)\Gamma\left(1/\alpha\right)}{\Gamma\left((r+1-j)/\alpha\right)} \right] \times \\
&\quad \prod_{j>i}^{r} (z_j - z_i)^{2/\alpha} \prod_{j=1}^{r} \left[a_j^{1-(p-r+1)/\alpha} \prod_{s=1}^{p} (z_j - b_s)^{-1/\alpha} \right]
\end{aligned}$$

The proposition reduces $_0F_0^{(\alpha)}(A, B)$, a hypergeometric function with potentially high-dimensional matrix arguments, to a repeated contour integral of $_0F_0^{(\alpha)}(\mathcal{A}, \mathcal{Z})$, a hypergeometric function with matrix arguments of possibly much lower dimensions. In the special case where $r = 1$, $_0F_0^{(\alpha)}(\mathcal{A}, \mathcal{Z}) = e^{a_1 z_1}$ and (1) becomes

$$_0F_0^{(\alpha)}(A, B) = \Gamma\left(p/\alpha\right) a_1^{1-p/\alpha} \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{K}} e^{a_1 z_1} \prod_{s=1}^{p} (z_1 - b_s)^{-\frac{1}{\alpha}} \mathrm{d}z_1. \qquad (9)$$

For $\alpha = 2$, this formula has been established by Mo (2011), who used it to analyze the asymptotic behavior of the largest eigenvalue of a rank-one perturbation of a real Wishart matrix. He gives two proofs of the formula. One of the proofs uses Jack polynomial expansions and requires that p be an even integer (consistent with our requirement that $p - r + 1$ is even). The other proof, which Mo (2011) calls geometric, allows for odd p.

Similar to the first proof of Mo, our proof of Lemma 1 uses Jack polynomial expansions. In contrast to that proof, we do not rely on the simplification of the Jack polynomials for top-order partitions, but use Jack polynomials' orthogonality

with respect to the torus scalar product (Macdonald, chapter VI, §10). It is likely that our requirement that $p - r + 1$ is even in cases where $\beta = 2/\alpha$ is odd can be lifted without affecting relationship (1). This would require a different proof of the proposition, which is left for future research.

For $\alpha = 2$ and $\alpha = 2/\beta$ with even β, formula (9) has been independently established by Wang (2012). He uses the formula to study the asymptotic distribution of the largest eigenvalue of the real, complex and quaternionic Wishart matrices perturbed by matrices of rank one. Wang's proof is similar to the first proof of Mo (2011) (see Forrester, 2011, for an alternative proof). For $\alpha = 2$, formula (9) has also been independently established by Onatski et al (2012). Their proof is based on the properties of the so-called Lauricella function.

In contrast to (9), the general relationship (1) contains special functions on both left- and right-hand sides. However, for $\alpha = 1$, it is possible to further simplify the right-hand side of (1) using Harish-Chandra/Itzykson-Zuber formula (see Harish-Chandra, 1957, and Itzykson and Zuber, 1980)

$$
{}_0F_0^{(1)}\left(\mathcal{A}, \mathcal{Z}\right) = \frac{\prod_{j=1}^{r-1} j!}{V_r\left(\mathcal{A}\right) V_r\left(\mathcal{Z}\right)} \det_{1 \leq i,j \leq r}\left(e^{a_i z_j}\right), \tag{10}
$$

where $V_r\left(\mathcal{A}\right) = \prod_{j>i}^{r}\left(a_j - a_i\right)$ and $V_r\left(\mathcal{Z}\right) = \prod_{j>i}^{r}\left(z_j - z_i\right)$ are the Vandermonde determinants associated with the diagonal elements $a_1, ..., a_r$ of $\mathcal{A}$ and the diagonal elements $z_1, ..., z_r$ of $\mathcal{Z}$, respectively. Using (10) in (1), noting that one of the terms in the definition of $\omega^{(1)}\left(A, B, \mathcal{Z}\right)$ equals $V_r\left(\mathcal{Z}\right)^2$, and applying Andreief's identity (Andreief, 1883)

$$
\det_{1 \leq i,j \leq r}\left(\int f_i\left(x\right) g_j(x) \mathrm{d}\mu\left(x\right)\right) = \frac{1}{r!} \int \cdots \int \det_{1 \leq i,j \leq r}\left(f_i\left(x_j\right)\right) \det_{1 \leq i,j \leq r}\left(g_i\left(x_j\right)\right) \prod_j \mathrm{d}\mu\left(x_j\right),
$$

we obtain the following Corollary.

Corollary 1 *Under assumptions of Lemma 1,*

$$_0F_0^{(1)}\left(A,B\right)=\frac{(-1)^{r(r-1)/2}}{V_r\left(\mathcal{A}\right)}\prod_{j=1}^{r}\frac{(p-j)!}{a_j^{p-r}}\det_{1\le i,j\le r}\left(\frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{K}}\frac{e^{a_iz}z^{j-1}\mathrm{d}z}{\prod_{s=1}^{p}\left(z-b_s\right)}\right). \tag{11}$$

An alternative way of deriving (11) is to apply l'Hôpital's rule to the Harish-Chandra/Itzykson-Zuber determinantal formula

$$_0F_0^{(1)}\left(A,B\right)=\frac{\prod_{j=1}^{p-1}j!}{V_p\left(A\right)V_p\left(B\right)}\det_{1\le i,j\le p}\left(e^{a_ib_j}\right), \tag{12}$$

the right-hand side of which is degenerate because A is rank-deficient. We include a proof of (11) that uses this approach in the Supplementary Appendix. The proof is elementary in the sense that it does not rely on properties of Jack polynomials.

4 Likelihood ratios as contour integrals

Combining Proposition 1 and Corollary 1 leads to useful contour integral representations of the likelihood ratios $L\left(h;\lambda\right)$ and $L\left(h;\mu\right)$. We now introduce new notation to express such representations in a convenient form. For any $z\in\mathcal{K}$, let us define a random variable

$$\Delta_p\left(z\right)=\sum_{j=1}^{p}\ln\left(z-\lambda_j\right)-p\int\ln\left(z-\lambda\right)\mathrm{d}\mathcal{F}_p\left(\lambda\right), \tag{13}$$

where $\mathcal{F}_p\left(\lambda\right)$ is the cumulative distribution function of the Marchenko-Pastur distribution with a point mass of $\max\left(0,1-c_p^{-1}\right)$ at zero, where $c_p=p/n$, and density

$$\psi_p\left(x\right)=\frac{1}{2\pi c_px}\sqrt{\left(\bar{b}_p-x\right)\left(x-\bar{a}_p\right)}, \tag{14}$$

where $\bar{a}_p = \left(1 - \sqrt{c_p}\right)^2$ and $\bar{b}_p = \left(1 + \sqrt{c_p}\right)^2$. Further, let

$$f_i(z) = -\left(\frac{h_i}{1 + h_i} z - c_p \int \ln(z - \lambda) \, d\mathcal{F}_p(\lambda)\right) \text{ and} \tag{15}$$

$$g_j(z) = z^{j-1} \exp\{-\Delta_p(z)\}. \tag{16}$$

Finally, for any permutation ρ of the sequence $(1, 2, ..., r)$ and any vector $\mathbf{z} = (z_1, ..., z_r)$, let

$$q_\rho(\mathbf{z}) = \left(1 - \sum_{j=1}^{r} \frac{h_{\rho(j)}}{1 + h_{\rho(j)}} \frac{z_j}{S}\right)^{-p(n-r)-r(r+1)/2} \exp\left\{-\sum_{j=1}^{r} \frac{n h_{\rho(j)} z_j}{1 + h_{\rho(j)}}\right\}, \tag{17}$$

where $S = \lambda_1 + ... + \lambda_p$.

Proposition 2 *Let the contour $\mathcal{K}$ that encircles $\lambda_1, ..., \lambda_p$ counter-clockwise be chosen so that for any $z \in \mathcal{K}$, $\mathrm{Re}\, z < \left(\sum_{j=1}^{r} \frac{h_j}{1+h_j}\right)^{-1} S$. Then*

$$L(h; \lambda) = k_1 \det_{1 \leq i,j \leq r} \left(\frac{1}{2\pi i} \oint_{\mathcal{K}} e^{-n f_i(z)} g_j(z) \, dz\right) \text{ and} \tag{18}$$

$$L(h; \mu) = k_2 \sum_\rho \frac{\mathrm{sgn}\, \rho}{(2\pi i)^r} \oint_{\mathcal{K}} \cdots \oint_{\mathcal{K}} q_\rho(\mathbf{z}) \prod_{j=1}^{r} \left\{e^{-n f_{\rho(j)}(z_j)} g_j(z_j)\right\} dz_r ... dz_1, \tag{19}$$

where i denotes the imaginary unit, the summation in (19) is over all permutations ρ of the sequence $(1, 2, ..., r)$,

$$k_1 = (-1)^{r(r-1)/2} n^{-pr+r(r+1)/2} \prod_{i>j} (h_i - h_j)^{-1} \prod_{t=1}^{r} \left[h_t^{r-p} (1 + h_t)^{p-n-1} (p - t)!\right], \text{ and}$$

$$k_2 = k_1 (nS)^{pr-r(r+1)/2} \Gamma(p(n-r) + r(r+1)/2) [\Gamma(np)]^{-1}.$$

In the next section, we perform the asymptotic analysis of $L(h; \lambda)$ and $L(h; \mu)$ that relies on the Laplace approximations of the contour integrals in (18) and (19) after the contours are suitably deformed without changing the value of the integrals.
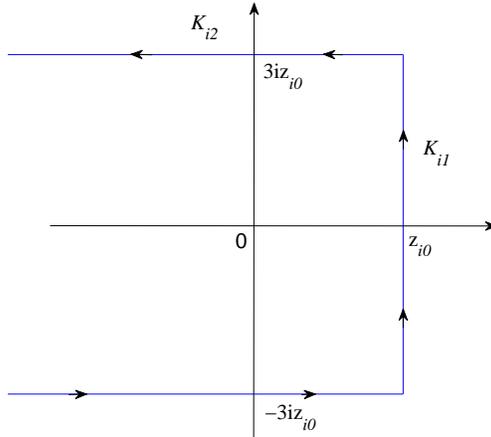
Figure 1: Contour $\mathcal{K}_i$ (see (20)-(21)).

5 Asymptotic analysis

Consider contours $\mathcal{K}_i$ with $i = 1, ..., r$ which are obtained by deforming the contour $\mathcal{K}$ defined in Proposition 2 so that $\mathcal{K}_i$ passes through

$$z_{i0} = \frac{(1 + h_i)(c_p + h_i)}{h_i}.$$

Precisely, we define $\mathcal{K}_i$ as $\mathcal{K}_i = \mathcal{K}_{i+} \cup \mathcal{K}_{i-}$, where $\mathcal{K}_{i-}$ is the complex conjugate of $\mathcal{K}_{i+}$ and $\mathcal{K}_{i+} = \mathcal{K}_{i1} \cup \mathcal{K}_{i2}$ with

$$\mathcal{K}_{i1} = \{z_{i0} + \mathrm{i}t : 0 \le t \le 3z_{i0}\} \text{ and} \tag{20}$$

$$\mathcal{K}_{i2} = \{x + 3\mathrm{i}z_{i0} : -\infty < x \le z_{i0}\}. \tag{21}$$

Figure 1 illustrates the choice of $\mathcal{K}_i$.

It is possible to verify that, when $0 < h_i < \sqrt{c_p}$, the derivative of $f_i(z)$ equals zero at z_{i0}. Therefore, choosing contours of integration so they pass through z_{i0}

allows us to use the method of steepest descent in the asymptotic analysis of the corresponding integrals in (18) and (19). The next lemma shows that the change of contours in (18) and (19) does not lead to a change in the value of the corresponding integrals.

Lemma 2 *Suppose that the null hypothesis is true, and let $\bar{h}$ be an arbitrary number such that $0 < \bar{h} < \sqrt{c}$. Suppose further that $h_i \leq \bar{h}$ for all $i = 1, ..., r$. Then, as $n, p \to \infty$ so that $c_p \to c \in (0, +\infty)$,*

$$\left(\oint_{\mathcal{K}} - \oint_{\mathcal{K}_i} \right) e^{-nf_i(z)} g_j(z) \, \mathrm{d}z = 0 \ \text{ and }$$

$$\left(\oint_{\mathcal{K}} \cdots \oint_{\mathcal{K}} - \oint_{\mathcal{K}_{\rho(1)}} \cdots \oint_{\mathcal{K}_{\rho(r)}} \right) q_\rho(\mathbf{z}) \prod_{j=1}^{r} \left\{ e^{-nf_{\rho(j)}(z_j)} g_j(z_j) \right\} \mathrm{d}z_r...\mathrm{d}z_1 = 0.$$

Our next lemma establishes Laplace approximations to the contour integrals in (18) and (19) after the change of the contours. The lemma uses some new notation that we introduce now. When $f_i(z)$ is analytic at z_{i0}, let f_{is} with $s = 0, 1, ...$ be the coefficients in the power series representation

$$f_i(z) = \sum_{s=0}^{\infty} f_{is}(z - z_{i0})^s. \tag{22}$$

When $f_i(z)$ is not analytic at z_{i0}, let the coefficients f_{is} be arbitrary numbers for all $s \in \mathbb{N}$.

Lemma 3 *Under the conditions of Lemma 2,*

$$\oint_{\mathcal{K}_i} e^{-nf_i(z)} g_j(z) \mathrm{d}z = e^{-nf_{i0}} \left[\frac{g_j(z_{i0}) \pi^{1/2}}{f_{i2}^{1/2} n^{1/2}} + \frac{O_p(1)}{h_i^j n^{3/2}} \right] \ \text{ and } \tag{23}$$

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$$\oint_{\mathcal{K}_{\rho(1)}} \cdots \oint_{\mathcal{K}_{\rho(r)}} q_\rho(\mathbf{z}) \prod_{j=1}^{r} \left\{ e^{-n f_{\rho(j)}(z_j)} g_j(z_j) \right\} dz_r \ldots dz_1$$

$$= q_\rho(\mathbf{z}_0) \prod_{j=1}^{r} e^{-n f_{\rho(j)0}} \frac{g_j(z_{\rho(j)0}) \pi^{1/2}}{f_{\rho(j)2}^{1/2} n^{1/2}} + \frac{O_p(1)}{n} \prod_{j=1}^{r} \frac{e^{-n f_{\rho(j)0}}}{h_{\rho(j)}^{j} n^{1/2}}, \quad (24)$$

where $O_p(1)$ is uniform in $h_1, \ldots, h_r \in \left[0, \overline{h}\right]$, and $\mathbf{z}_0 = \left(z_{\rho(1)0}, \ldots, z_{\rho(r)0}\right)$. The branch of the square root in formulae (23) and (24) is chosen so that $(-1)^{1/2} = -\mathrm{i}$.

Using Lemma 3, we establish the following theorem.

Theorem 1 *Suppose that the null hypothesis is true ($h = 0$). Let $\overline{h}$ be any fixed number such that $0 < \overline{h} < \sqrt{c}$ and let $C\left[0, \overline{h}\right]^r$ be the space of real-valued contin-uous functions on $\left[0, \overline{h}\right]^r$ equipped with the supremum norm. Then, as $n, p \to \infty$ so that $p/n = c_p \to c \in (0, +\infty)$, we have*

$$L(h; \lambda) = \exp\left\{ -\sum_{i=1}^{r} \Delta_p(z_{i0}) + \frac{1}{2} \sum_{i,j=1}^{r} \ln\left(1 - \frac{h_i h_j}{c_p}\right) \right\} + O_p\left(\frac{1}{n}\right) \quad and \quad (25)$$

$$L(h; \mu) = \exp\left\{ -\sum_{i=1}^{r} \Delta_p(z_{i0}) + \frac{1}{2} \sum_{i,j=1}^{r} \left(\ln\left(1 - \frac{h_i h_j}{c_p}\right) + \frac{h_i h_j}{c_p} \right) - \frac{S - p}{c_p} \sum_{j=1}^{r} h_j \right\} + O_p\left(\frac{1}{n}\right) (26)$$

where the $O_p(n^{-1})$ terms are uniform in $h \in \left(0, \overline{h}\right]^r$. Furthermore, $\ln L(h; \lambda)$ and $\ln L(h; \mu)$, viewed as random elements of $C\left[0, \overline{h}\right]^r$, converge weakly to $\mathcal{L}_\lambda(h)$ and $\mathcal{L}_\mu(h)$ with Gaussian finite-dimensional distributions such that, for any $h, \tilde{h} \in$

$$\left[0, \overline{h}\right]^r,$$

$$E\left(\mathcal{L}_\lambda\left(h\right)\right) = \frac{1}{2} \sum_{i,j=1}^{r} \ln\left(1 - \frac{h_i h_j}{c}\right), \tag{27}$$

$$\mathrm{Cov}\left(\mathcal{L}_\lambda\left(h\right), \mathcal{L}_\lambda\left(\tilde{h}\right)\right) = -\sum_{i,j=1}^{r} \ln\left(1 - \frac{h_i \tilde{h}_j}{c}\right), \tag{28}$$

$$E\left(\mathcal{L}_\mu\left(h\right)\right) = \frac{1}{2} \sum_{i,j=1}^{r} \left(\ln\left(1 - \frac{h_i h_j}{c}\right) + \frac{h_i h_j}{c}\right), \quad and \tag{29}$$

$$\mathrm{Cov}\left(\mathcal{L}_\mu\left(h\right), \mathcal{L}_\mu\left(\tilde{h}\right)\right) = -\sum_{i,j=1}^{r} \left(\ln\left(1 - \frac{h_i \tilde{h}_j}{c}\right) + \frac{h_i \tilde{h}_j}{c}\right). \tag{30}$$

Theorem 1 and Le Cam's first lemma (van der Vaart (1998), p.88) imply that the joint distributions of $\lambda_1, ..., \lambda_m$ (as well as those of $\mu_1, ..., \mu_{m-1}$) under the null and under the alternative are mutually contiguous for any $h \in [0, \sqrt{c})^r$. Along with Le Cam's third lemma (van der Vaart (1998), p.90), this can be used to study the "local" powers of tests detecting signals in noise.

Let $\beta_\lambda\left(h\right)$ and $\beta_\mu\left(h\right)$ be the asymptotic powers of the asymptotically most powerful λ- and μ-based tests of size α of the null $h = 0$ against a point alternative $h = (h_1, ..., h_r)$ with $h_j < \sqrt{c}$, $j = 1, ..., r$. We have

Theorem 2 *Let Φ denote the standard normal distribution function. Then,*

$$\beta_\lambda\left(h\right) = 1 - \Phi\left[\Phi^{-1}\left(1 - \alpha\right) - \sqrt{-\sum_{i,j=1}^{r} \ln\left(1 - \frac{h_i h_j}{c}\right)}\right] \quad and \tag{31}$$

$$\beta_\mu\left(h\right) = 1 - \Phi\left[\Phi^{-1}\left(1 - \alpha\right) - \sqrt{-\sum_{i,j=1}^{r} \left(\ln\left(1 - \frac{h_i h_j}{c}\right) + \frac{h_i h_j}{c}\right)}\right]. \tag{32}$$

The theorem implies in particular that detection of signals corresponding to covariance spikes of sizes well below the phase transition threshold is possible with high probability. Consider for example the case where the number of observations equals the dimensionality of data so that $c = 1$, the number of signals under the

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alternative equals five, and the signals have equal but rather weak strengths $h_1 = \ldots = h_5 = 0.5$. Then the best possible λ-based procedure for detecting such signals with the asymptotic probability of false detection fixed at 0.05 has asymptotic probability of correct detection $1 - \Phi\left[\Phi^{-1}(0.95) - \sqrt{-25\ln(1 - 0.25)}\right] \approx 0.85$.

Unfortunately, constructing testing procedures with uniformly optimal power is hard because the log-likelihood process established in Theorem 1 is not of the Gaussian shift type, so that the statistical experiments we study are not locally asymptotically normal (LAN) ones. For the case of real-valued data and $r = 1$, Onatski et al (2012) use numerical simulations to show that the asymptotic powers of the likelihood ratio (LR) tests based on λ and on μ are close to the respective asymptotic power envelopes $\beta_\lambda(h)$ and $\beta_\mu(h)$. The λ- and μ-based LR tests of $h = 0$ against the alternative $h \in (0, \bar{h})^r$ reject the null if and only if , respectively, $2\sup_{h\in(0,\bar{h})} \ln L(h; \lambda)$ and $2\sup_{h\in(0,\bar{h})} \ln L(h; \mu)$ are sufficiently large. As r grows, it becomes increasingly difficult to find the asymptotic critical values for the LR tests by simulation. This requires simulating an r-dimensional Gaussian random field with the covariance function and the mean function described in Theorem 1, which, for relatively large r, is computationally expensive.

For $r = 2$, Figure 2 shows the contour plots of the power envelope $\beta_\lambda(h)$ (left panel) and of the asymptotic power of the likelihood ratio test based on λ. We chose parameter $\bar{h}$ so that it is very close to the threshold $\sqrt{c}$, precisely $\bar{h} = \sqrt{c(1 - e^{-36})}$. We see that the contours of $\beta_\lambda(h)$ and of the asymptotic power of the λ-based LR test corresponding to the same value of these functions are relatively close to each other, which suggests that the LR test has good asymptotic power properties. More detailed analysis of the asymptotic and finite sample power of the LR test is, however, beyond the scope of this paper, and is left for future research.

In contrast to the LR test, the popular signal detection procedures based on the information in a few of the largest eigenvalues of XX'/n (see, for example,
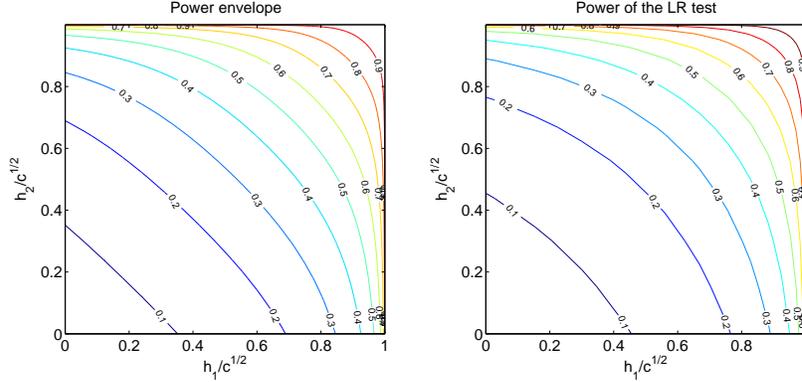
Figure 2: The asymptotic power envelope $\beta_\lambda(h)$ and the asymptotic power of the LR test based on λ; $r = 2$, asymptotic size is 0.05.

Krichman and Nadler (2009), Nadakuditi and Silverstein (2010), Onatski (2009), Patterson et al (2006), Perry and Wolf (2010), and Tracy and Widom (2009)), have trivial asymptotic power (that is, the asymptotic power, which equals the asymptotic size) in the region $h \in [0, \sqrt{c})^r$. It is because the asymptotic behavior of any finite number of the largest sample covariance eigenvalues when $h \in [0, \sqrt{c})^r$ is not different from their behavior when the data are pure noise (Péché, 2003).

As was mentioned above, signal detection tests can be interpreted as tests of sphericity. Vice versa, previously proposed sphericity tests, can, in principle, be used for signal detection. In the Supplementary Appendix, we use Theorem 1 along with Le Cam's third lemma to derive asymptotic powers of several such tests against "spiked covariance" alternatives. The derived asymptotic powers turn out to be much lower than the asymptotic power envelopes $\beta_\lambda(h)$ and $\beta_\mu(h)$. However, we feel that this comparison is somewhat unfair to the sphericity tests because they are typically designed against general alternatives, as opposed to "the spiked covariance" alternatives. Therefore, and to save space, we do not report these results here.

6   Conclusion

This paper studies the asymptotic power of the signal detection tests in complex-valued Gaussian data as both the number of observations and data dimensionality go to infinity. Contrary to the conventional wisdom that detection of signals becomes nearly impossible when their strength, measured by the size of the covariance spikes, is below the phase transition threshold, we find that detection of such signals may be possible with high probability. The detection power lies not in the different behavior of a few of the largest sample covariance eigenvalues under the null and the alternative, which is exploited by the popular signal detection tests, but in small deviations of the empirical distribution of all the eigenvalues from the Marchenko-Pastur limit.

To derive our results, we consider the ratio of the densities of the sample covariance eigenvalues under the null and under the alternative hypothesis. We establish a contour integral representation of this likelihood ratio, and use the Laplace approximation to derive its asymptotic limit. Our analysis of the limiting log-likelihood ratio process shows that the sub-critical region, where the sizes of the covariance spikes are below the phase transition threshold, is the region of mutual contiguity of the joint densities of the sample covariance eigenvalues under the null and the alternative. We use the derived limiting log-likelihood process along with Le Cam's third lemma and the Neyman-Pearson lemma to obtain the asymptotic power envelope for the signal detection tests. Preliminary analysis indicates that the asymptotic power of the likelihood ratio test based on the sample covariance eigenvalues is close to the asymptotic power envelope.

Our technical analysis is based on what we believe to be a novel representation of the Harish-Chandra/Itzykson-Zuber integral with one of the $p \times p$ matrices being of reduced rank r in the form of an $r \times r$ matrix of contour integrals. We obtain

such a representation as a corollary to a much more general result established in Lemma 1. This result expresses the hypergeometric function $_0F_0^{(\alpha)}$ of two $p \times p$ matrix arguments, one of which has rank r, as a repeated contour integral of the hypergeometric function $_0F_0^{(\alpha)}$ of two $r \times r$ matrix arguments. As discussed in the introduction, the established dimension reduction for the hypergeometric function may be important in various applied and theoretical fields of study. In particular, for $\alpha = 2$, it can, potentially, be used to extend the analysis of this paper to the case of real-valued data. Such an extension is currently under investigation.

7 Appendix

PROOF OF LEMMA 1.

Let $f(\mathcal{Z})$ and $g(\mathcal{Z})$ be functions defined on the r-dimensional torus $\{|z_j| = 1, \text{ for } j = 1, ..., r\}$. Consider the scalar product, sometimes called the torus scalar product,

$$\langle f, g \rangle_\alpha = \frac{1}{r! \, (2\pi i)^r} \oint \cdots \oint f(\mathcal{Z}) \, \overline{g(\mathcal{Z})} \prod_{i \neq j} \left(1 - z_i z_j^{-1}\right)^{1/\alpha} \prod_{j=1}^r \frac{\mathrm{d}z_j}{z_j}, \qquad (33)$$

where the contours of integration are the unit circles in the complex plane. Our proof relies on the orthogonality property of Jack polynomials: $\left\langle C_\kappa^{(\alpha)}, C_\tau^{(\alpha)} \right\rangle_\alpha = 0$ for $\kappa \neq \tau$ (Macdonald, chapter VI, §10).

Let us, first, introduce a few definitions (following Macdonald, 1995, chapter I, §1, and Dumitriu et al, 2007): The non-zero κ_j in the partition $\kappa = [\kappa_1, \kappa_2, ...]$ are called the parts of κ. The number of parts is the length of κ, denoted as $l(\kappa)$. The sum of the parts is the weight of κ, denoted as $|\kappa|$. We will identify partition κ with its Ferrers diagram, defined as an arrangement of $|\kappa|$ boxes in $l(\kappa)$ left-justified rows, the number of boxes in row i being the same as κ_i (see Figure 3). For each square s in the Ferrers diagram, let $l'(s), l(s), a(s),$ and $a'(s)$ be respectively the numbers of squares in the diagram to the north, south, east, and west of the square
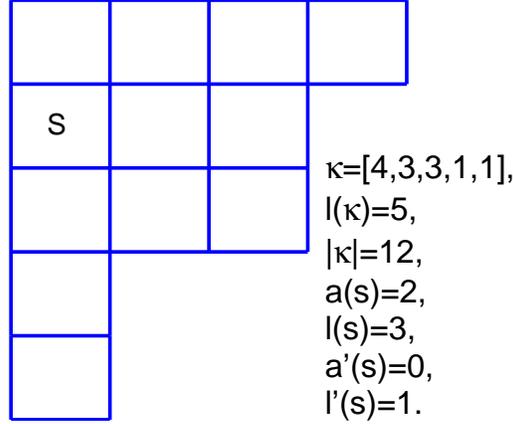
Figure 3: The Ferrers diagram of partition $[4, 3, 3, 1, 1]$.

s. Further, let $h^*(s) = l(s) + \alpha(1 + a(s))$ and $h_*(s) = l(s) + 1 + \alpha a(s)$. Finally, let $c(\kappa, \alpha) = \prod_{s \in \kappa} h_*(s)$, $c'(\kappa, \alpha) = \prod_{s \in \kappa} h^*(s)$, and $w(\kappa, \alpha) = c(\kappa, \alpha) c'(\kappa, \alpha)$.

We will need the following lemmata.

Lemma A1. *For the torus scalar product of $C_\kappa^{(\alpha)}$ with itself, we have*

$$\left\langle C_\kappa^{(\alpha)}, C_\kappa^{(\alpha)} \right\rangle_\alpha = \frac{\left(\alpha^{|\kappa|} |\kappa|!\right)^2}{w(\kappa, \alpha)} \prod_{j=1}^{r} \left(\frac{\Gamma\left(\frac{r-j+1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right) \Gamma\left(1 + \frac{r-j}{\alpha}\right)} \right) \prod_{s \in \kappa} \frac{r + a'(s)\alpha - l'(s)}{r + (a'(s) + 1)\alpha - l'(s) - 1}. \tag{34}$$

PROOF: Macdonald (1995, Chapter VI, §10) establishes the orthogonality of "P" normalizations of Jack polynomials, $P_\kappa^{(\alpha)}$, with respect to the torus scalar product. His formula (10.37) gives an explicit expression (up to a constant that can be evaluated using (10.38)) for $\left\langle P_\kappa^{(\alpha)}, Q_\kappa^{(\alpha)} \right\rangle_\alpha$, where $Q_\kappa^{(\alpha)} = \frac{c(\kappa, \alpha)}{c'(\kappa, \alpha)} P_\kappa^{(\alpha)}$ (see (10.16)). On the other hand,

$$P_\kappa^{(\alpha)} = \frac{c'(\kappa, \alpha)}{\alpha^{|\kappa|} |\kappa|!} C_\kappa^{(\alpha)} \tag{35}$$

(see, for example, Table 6 of Dumitriu et al, 2007). Substituting this expression in Macdonald's formulae, we get (34). □

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Lemma A2. *Let $\mathcal{Z} = \mathrm{diag}\,(z_1, ..., z_r)$, where $z_1, ..., z_r$ are complex variables, and let $b_1, ..., b_p$ be complex constants. Then*

$$\prod_{j=1}^{r}\prod_{s=1}^{p}(1-b_s z_j)^{-1/\alpha} = \sum_{k=0}^{\infty}\sum_{\kappa\vdash k}\frac{w\,(\kappa, \alpha)}{(\alpha^{|\kappa|}\,|\kappa|!)^2}C_\kappa^{(\alpha)}\,(B)\,C_\kappa^{(\alpha)}\,(\mathcal{Z})\,. \tag{36}$$

The series on the right-hand side of this equality converges uniformly over $\Omega_\rho = \{\mathcal{Z} : \max_{j\leq r}|z_j| \leq \rho^{-1}\}$, for any $\rho > \max_{s\leq p}|b_s|$.

PROOF: Macdonald (1995, Chapter VI, §10) shows that $\prod_{j=1}^{r}\prod_{s=1}^{p}(1-b_s z_j)^{-1/\alpha} = \sum_{k=0}^{\infty}\sum_{\kappa\vdash k}P_\kappa^{(\alpha)}\,(B)\,Q_\kappa^{(\alpha)}\,(\mathcal{Z})$, where $Q_\kappa^{(\alpha)} = \frac{c(\kappa,\alpha)}{c'(\kappa,\alpha)}P_\kappa^{(\alpha)}$. This result together with (35) imply (36). The uniform convergence in (36) follows from the fact that function $\prod_{j=1}^{r}\prod_{s=1}^{p}(1-b_s z_j)^{-1/\alpha}$ is analytic in an open region that includes Ω_ρ.$\square$

We are now ready to prove Lemma 1. Consider the right-hand side of (1), which we will denote as RHS. We will assume that $\max_{s\leq p}|b_s| < 1$ and that the contour $\mathcal{K}$ is the unit circle in the complex plane. That these assumptions are without loss of generality follows from the fact that the value of RHS does not change under the transformation $\mathcal{Z} \to \varphi\mathcal{Z}$, $B \to \varphi B$, and $A \to \varphi^{-1}A$, where φ is any positive number, and under a deformation of $\mathcal{K}$ into the unit circle (because such a deformation leaves the contour in the region of the analyticity of the integrand). With these assumptions, and noting that the component $\prod_{j>i}^{r}(z_j-z_i)^{2/\alpha}$ of $\omega^{(\alpha)}\,(A, B, \mathcal{Z})$ equals $(-1)^{r(r-1)/(2\alpha)}\prod_{j=1}^{r}z_j^{(r-1)/\alpha}\prod_{j\neq i}^{r}(1-z_i z_j^{-1})^{1/\alpha}$, we can rewrite RHS for $\alpha = 2/\beta$, where β is any positive integer, in the form of the torus scalar product

$$RHS = \gamma^{(\alpha)}\left\langle {}_0F_0^{(\alpha)}(\mathcal{A}, \mathcal{Z})\,,\prod_{j=1}^{r}(z_j/a_j)^{(p-r+1)/\alpha-1}\prod_{j=1}^{r}\prod_{s=1}^{p}(1-b_s z_j)^{-1/\alpha}\right\rangle_\alpha\,,$$

where

$$\gamma^{(\alpha)} = \prod_{j=1}^{r}\left[\frac{\Gamma\,((p+1-j)/\alpha)\,\Gamma\,(1/\alpha)}{\Gamma\,((r+1-j)/\alpha)}\right]\,. \tag{37}$$

Substituting $_0F_0^{(\alpha)}(\mathcal{A}, \mathcal{Z})$ and $\prod_{j=1}^{r} \prod_{s=1}^{p} (1 - b_s z_j)^{-1/\alpha}$ in the above formula by their expansions (6) and (36) in the series of Jack polynomials, and interchanging the order of integration and summation, which is possible because the series converge uniformly over the unit torus, we obtain

$$
\begin{aligned}
RHS \;=\; & \gamma^{(\alpha)} \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \sum_{t=0}^{\infty} \sum_{\tau \vdash t} \frac{w(\tau, \alpha)}{k!\,(\alpha^t t!)^2} \frac{C_\kappa^{(\alpha)}(\mathcal{A})\,C_\tau^{(\alpha)}(B)}{C_\kappa^{(\alpha)}(I_r)} \times \\
& \left\langle C_\kappa^{(\alpha)}(\mathcal{Z}), \prod_{j=1}^{r} (z_j/a_j)^{(p-r+1)/\alpha-1} C_\tau^{(\alpha)}(\mathcal{Z}) \right\rangle_\alpha .
\end{aligned}
$$

But $\prod_{j=1}^{r} (z_j)^{(p-r+1)/\alpha-1} C_\tau^{(\alpha)}(\mathcal{Z}) = C_{\tilde\tau}^{(\alpha)}(\mathcal{Z})$, where $\tilde\tau$ denotes partition $[\tau_1 + \frac{p-r+1-\alpha}{\alpha}, ..., \tau_r + \frac{p-r+1-\alpha}{\alpha}]$. Note that $\tilde\tau$ is well defined for $\alpha = 2/\beta$, where β is an even integer. If β is an odd integer, we need to assume that $p - r + 1$ is even. Therefore, using the orthogonality of the Jack polynomials with respect to the torus scalar product, we have

$$
RHS = \gamma^{(\alpha)} \prod_{j=1}^{r} a_j^{-(p-r+1)/\alpha+1} \sum_{t=0}^{\infty} \sum_{\tau \vdash t} \frac{w(\tau, \alpha)}{|\tilde\tau|!\,(\alpha^t t!)^2} \frac{C_{\tilde\tau}^{(\alpha)}(\mathcal{A})\,C_\tau^{(\alpha)}(B)}{C_{\tilde\tau}^{(\alpha)}(I_r)} \left\langle C_{\tilde\tau}^{(\alpha)}(\mathcal{Z}), C_{\tilde\tau}^{(\alpha)}(\mathcal{Z}) \right\rangle_\alpha .
$$

Using Lemma A1, (37), and equality $\prod_{j=1}^{r} a_j^{-(p-r+1)/\alpha+1} C_{\tilde\tau}^{(\alpha)}(\mathcal{A}) = C_\tau^{(\alpha)}(\mathcal{A}) = C_\tau^{(\alpha)}(A)$, we get after some cancellations

$$
RHS = \sum_{t=0}^{\infty} \sum_{\tau \vdash t} \tilde\gamma^{(\alpha)} \frac{1}{t!} \frac{C_\tau^{(\alpha)}(A)\,C_\tau^{(\alpha)}(B)}{C_\tau^{(\alpha)}(I_p)}, \tag{38}
$$

where

$$
\tilde\gamma^{(\alpha)} = \frac{\alpha^{2|\tilde\tau|}\,|\tilde\tau|!}{w(\tilde\tau, \alpha)} \frac{w(\tau, \alpha)}{\alpha^{2t} t!} \frac{C_\tau^{(\alpha)}(I_p)}{C_{\tilde\tau}^{(\alpha)}(I_r)} \prod_{j=1}^{r} \frac{\Gamma((p+1-j)/\alpha)}{\Gamma(1+(r-j)/\alpha)} \prod_{s \in \tilde\tau} \frac{r + a'(s)\alpha - l'(s)}{r + (a'(s)+1)\alpha - l'(s) - 1}.
$$

In the above expression for $\tilde\gamma^{(\alpha)}$, substitute $C_\tau^{(\alpha)}(I_p)$ and $C_{\tilde\tau}^{(\alpha)}(I_r)$ by their

explicit forms, that can be obtained from a general formula

$$C_\kappa^{(\alpha)}(I_m) = \frac{\alpha^{|\kappa|}|\kappa|!}{w(\kappa,\alpha)} \prod_{s\in\kappa}(m+\alpha a'(s)-l'(s)). \tag{39}$$

A variant of this formula, that uses the generalized Pochhammer symbol, can be found, for example, in Dumitriu et al (2007, Table 5). Then, after cancellations, we get

$$\tilde\gamma^{(\alpha)} = \frac{\alpha^{|\tilde\tau|}}{\alpha^t} \prod_{j=1}^r \frac{\Gamma((p+1-j)/\alpha)}{\Gamma(1+(r-j)/\alpha)} \frac{\prod_{s\in\tau}(p+\alpha a'(s)-l'(s))}{\prod_{s\in\tilde\tau}(r+(a'(s)+1)\alpha-l'(s)-1)}.$$

Now consider the last ratio of the products in the above expression. For the product term in the numerator that corresponds to square s in the position (i,j) in the diagram of τ, there exists exactly the same term in the denominator, which corresponds to square s in the position $(i, j + (p-r+1)/\alpha - 1)$ in the diagram of $\tilde\tau$. Therefore, we can write

$$\tilde\gamma^{(\alpha)} = \frac{\alpha^{|\tilde\tau|}}{\alpha^t} \prod_{j=1}^r \frac{\Gamma((p+1-j)/\alpha)}{\Gamma(1+(r-j)/\alpha)} \frac{1}{\prod_{s\in\hat\tau}(r+(a'(s)+1)\alpha-l'(s)-1)},$$

where $\hat\tau$ is the partition that consists of r identical parts $(p-r+1)/\alpha - 1$.

Finally, note that

$$\prod_{s\in\hat\tau}(r+(a'(s)+1)\alpha-l'(s)-1) = \alpha^{|\tilde\tau|-t} \prod_{s\in\hat\tau}((r-l'(s)-1)/\alpha + a'(s)+1)$$

$$= \alpha^{|\tilde\tau|-t} \prod_{j=1}^r \frac{\Gamma((r-j)/\alpha+(p-r+1)/\alpha)}{\Gamma((r-j)/\alpha+1)} = \alpha^{|\tilde\tau|-t} \prod_{j=1}^r \frac{\Gamma((p-j+1)/\alpha)}{\Gamma((r-j)/\alpha+1)}.$$

Therefore, $\tilde\gamma^{(\alpha)}=1$ and the statement of the lemma follows from (6) and (38).□

PROOF OF PROPOSITION 2

Proposition 1 and Corollary 1 directly imply (18) and the following formula for

$L\left(h;\mu\right)$

$$L\left(h;\mu\right) = k_1 \frac{n^{np} S^{pr-r(1+r)/2}}{\Gamma\left(np\right)} \int_0^\infty y^{p(n-r)+r(r+1)/2-1} e^{-ny} \det R \mathrm{d}y, \qquad (40)$$

where R is an $r \times r$ matrix with

$$R_{ij} = \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{K}} e^{\frac{y}{S}\frac{nh_i}{1+h_i}z} z^{j-1} \prod_{s=1}^p (z-\lambda_s)^{-1} \mathrm{d}z.$$

Let us write $\det R$ as

$$\det R = \sum_\rho \operatorname{sgn}\rho \prod_{j=1}^r \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{K}} e^{\frac{y}{S}\frac{nh_{\rho(j)}}{1+h_{\rho(j)}}z} z^{j-1} \prod_{s=1}^p (z-\lambda_s)^{-1} \mathrm{d}z,$$

or equivalently as

$$\det R = \sum_\rho \frac{\operatorname{sgn}\rho}{(2\pi\mathrm{i})^r} \oint_{\mathcal{K}} \cdots \oint_{\mathcal{K}} \prod_{j=1}^r \left\{ e^{\frac{y}{S}\frac{nh_{\rho(j)}}{1+h_{\rho(j)}}z_j} z_j^{j-1} \prod_{s=1}^p (z_j-\lambda_s)^{-1} \right\} \mathrm{d}z_r...\mathrm{d}z_1. \quad (41)$$

Using this representation, we have

$$\int_0^\infty y^{p(n-r)+r(r+1)/2-1} e^{-ny} \det R(y)\mathrm{d}y = \sum_\rho \frac{\operatorname{sgn}\rho}{(2\pi\mathrm{i})^r} \times$$

$$\int_0^\infty \oint_{\mathcal{K}} \cdots \oint_{\mathcal{K}} y^{p(n-r)+r(r+1)/2-1} \exp\left\{ -\left(n - \sum_{j=1}^r \frac{nh_{\rho(j)}}{1+h_{\rho(j)}} \frac{z_j}{S} \right) y \right\} \times$$

$$\prod_{j=1}^r \left\{ z_j^{j-1} \prod_{s=1}^p (z_j-\lambda_s)^{-1} \right\} \mathrm{d}z_r...\mathrm{d}z_1\mathrm{d}y.$$

Since the contour $\mathcal{K}$ is chosen so that for any $z \in \mathcal{K}$, $\operatorname{Re} z < \left(\sum_{j=1}^r \frac{h_j}{1+h_j} \right)^{-1} S$, the integrand in the above multiple integral is absolutely integrable on $[0,\infty) \times \mathcal{K} \times ... \times \mathcal{K}$, and Fubini's theorem justifies the interchange of the order of the integrals,

so that

$$\int_0^\infty y^{p(n-r)+r(r+1)/2-1}e^{-ny}\det R(y)\mathrm{d}y = \frac{\Gamma\left(p\left(n-r\right)+r(r+1)/2\right)}{n^{p(n-r)+r(r+1)/2}} \times$$

$$\sum_\rho \frac{\operatorname{sgn}\rho}{(2\pi\mathrm{i})^r}\oint_{\mathcal{K}}\cdots\oint_{\mathcal{K}}\left(1-\sum_{j=1}^r \frac{h_{\rho(j)}}{1+h_{\rho(j)}}\frac{z_j}{S}\right)^{-p(n-r)-r(r+1)/2}\times$$

$$\prod_{j=1}^r\left\{z_j^{j-1}\prod_{s=1}^p (z_j-\lambda_s)^{-1}\right\}\mathrm{d}z_r...\mathrm{d}z_1.$$

Combining this with (40), we get (19). $\square$

PROOF OF LEMMA 2

The lemma can be proven using arguments very similar to those in the proof of Lemmas 4 and 6 in Onatski, Moreira and Hallin (2012) (OMH in what follows), and we omit the proof to save space.$\square$

PROOF OF LEMMA 3.

To save space, we will only establish (23), relegating a conceptually similar but more technical proof of (24) to the Supplementary Appendix. Lemma 5 in OMH implies that

$$\oint_{\mathcal{K}_i} e^{-nf_i(z)}g\left(z\right)\mathrm{d}z = e^{-nf_{i0}}\left[\frac{g\left(z_{i0}\right)\pi^{1/2}}{f_{i2}^{1/2}n^{1/2}}+\frac{O_p\left(1\right)}{h_i n^{3/2}}\right], \tag{42}$$

where $g\left(z\right) = \exp\left\{-\frac{1}{2}\Delta_p\left(z\right)\right\}$ and $O_p\left(1\right)$ is uniform in $h_i \in \left(0,\bar{h}\right]$. A careful inspection of OMH's proof of their Lemma 5 reveals that a version of (42) remains valid for general functions $g\left(z\right)$ that are analytic in the open ball $B\left(z_{i0},r_i\right)$ with center at z_{i0} and radius $r_i = \min\left\{z_{i0}-\max\left\{\bar{b}_p,\lambda_1\right\},\frac{1+h_i}{h_i}S-z_{i0}\right\}$ with probability approaching 1 as $n,p \to \infty$. Precisely, for such general $g\left(z\right)$ we have

$$\oint_{\mathcal{K}_i} e^{-nf_i(z)}g\left(z\right)\mathrm{d}z = e^{-nf_{i0}}\frac{g\left(z_{i0}\right)\pi^{1/2}}{f_{i2}^{1/2}n^{1/2}}+\Psi_1+\Psi_2+\Psi_3 \tag{43}$$

with

$$|\Psi_1| < C_1 e^{-nf_{i0}} h_i^{-1} n^{-3/2} \sup_{z \in \bar{B}} |g(z)|, \tag{44}$$

$$|\Psi_2| < C_1 e^{-nf_{i0}} e^{-nC_2} h_i^{-1} \sup_{z \in \mathcal{K}_{i1} \cup \bar{\mathcal{K}}_{i1}} |g(z)|, \text{ and} \tag{45}$$

$$|\Psi_3| < C_1 \left| \oint_{\mathcal{K}_{i2} \cup \bar{\mathcal{K}}_{i2}} e^{-nf_i(z)} g(z) \, dz \right|, \tag{46}$$

where C_1 and C_2 are some positive constants, and $\bar{B}$ is a closed ball with center at z_{i0} and radius $r_i/2$.

Now, let $g(z) = g_j(z) = z^{j-1} \exp\{-\Delta_p(z)\}$. Lemma A2 in OMH implies that $\sup_{z \in \bar{B} \cup \mathcal{K}_{i1} \cup \bar{\mathcal{K}}_{i1}} |g(z)| = h_i^{1-j} O_p(1)$ uniformly in $h_i \in (0, \bar{h}]$. Therefore, by (44) and (45),

$$\Psi_1 + \Psi_2 = e^{-nf_{i0}} h_i^{-j} n^{-3/2} O_p(1). \tag{47}$$

Turning to the analysis of Ψ_3, note that by definition of $f_i(z)$ and $g(z)$,

$$e^{-nf_i(z)} g(z) = e^{n \frac{h_i}{1+h_i} z} z^{j-1} \prod_{j=1}^{p} (z - \lambda_j)^{-1}. \tag{48}$$

For $z \in \mathcal{K}_{i2} \cup \bar{\mathcal{K}}_{i2}$, we have $\left| (z - \lambda_j)^{-1} \right| < (3z_{i0})^{-1}$, and $\left| z(z - \lambda_j)^{-1} \right| < 2$, for any $j = 1, ..., p$. Therefore, using (48), we get

$$\left| \oint_{\mathcal{K}_{i2} \cup \bar{\mathcal{K}}_{i2}} e^{-nf_i(z)} g(z) \, dz \right| < 2^{j-1} (3z_{i0})^{-p+j-1} \oint_{\mathcal{K}_{i2} \cup \bar{\mathcal{K}}_{i2}} \left| e^{n \frac{h_i}{1+h_i} z} dz \right|$$

$$= 2^j (3z_{i0})^{-p+j-1} \left(n \frac{h_i}{1+h_i} \right)^{-1} e^{n \frac{h_i}{1+h_i} z_{i0}}$$

$$= 2^j (3z_{i0})^{j-1} \left(n \frac{h_i}{1+h_i} \right)^{-1} e^{-n \left(c_p \ln(3z_{i0}) - \frac{h_i}{1+h_i} z_{i0} \right)}$$

$$= 2^j (3z_{i0})^{j-1} \left(n \frac{h_i}{1+h_i} \right)^{-1} 3^{-p} e^{-n(c_p \ln(z_{i0}) - h_i - c_p)}.$$

On the other hand, for any $h_i \in [0, \bar{h}]$, $h_i < \sqrt{c_p}$ for sufficiently large n and p, and

$c_p \ln(z_{i0}) - h_i - c_p > f_{i0}$. Indeed, using the definition of z_{i0} and the fact, established in OMH's Lemma 11, that $f_{i0} = -c_p - (1 - c_p) \ln(1 + h_i) + c_p \ln \frac{c_p}{h_i}$, we have

$$c_p \ln(z_{i0}) - h_i - c_p - f_{i0} = \ln(1 + h_i) + c_p \ln(c_p + h_i) - h_i - c_p \ln c_p.$$

The right hand side of this equality equals 0 at $h_i = 0$ and has a non-negative derivative with respect to h_i for all $0 \leq h_i \leq \sqrt{c_p}$. Therefore,

$$\left| \oint_{\mathcal{K}_{i2} \cup \bar{\mathcal{K}}_{i2}} e^{-n f_i(z)} g(z) \, dz \right| < 2^j (3z_{i0})^{j-1} \left(n \frac{h_i}{1 + h_i} \right)^{-1} 3^{-p} e^{-n f_{i0}},$$

and thus, $\Psi_3 = e^{-n f_{i0}} h_i^{-j} n^{-3/2} O_p(1)$, uniformly in $h_i \in (0, \bar{h}]$. Combining this with (43) and (47), we obtain (23). $\square$

PROOF OF THEOREM 1

Proposition 2 and Lemma 3 imply that

$$L(h; \lambda) = \frac{k_1 \exp\{-n \sum_{i=1}^r f_{i0}\}}{(2i)^r (\pi n)^{r/2}} \det \left(\frac{z_{i0}^{j-1} \exp\{-\Delta_p(z_{i0})\}}{f_{i2}^{1/2}} + \frac{O_p(1)}{h_i^j n} \right)_{1 \leq i, j \leq r}$$

As is shown in OMH (see their Lemma 11 and (A8))[3], for $h_i \leq \bar{h}$,

$$f_{i0} = -c_p - (1 - c_p) \ln(1 + h_i) + c_p \ln \frac{c_p}{h_i}, \text{ and} \tag{49}$$

$$f_{i2} = -\frac{h_i^2}{2(1 + h_i)^2 (c_p - h_i^2)}. \tag{50}$$

Moreover, by OMH's Lemma A2, $\exp\{-\Delta_p(z_{i0})\} = O_p(1)$ uniformly in $h \in (0, \bar{h}]^r$. Using these facts and the definition of k_1 given in Proposition 2, we get after some algebra

---

[3] Note that the expressions given in OMH are half times the expressions given below because the equivalent of f_i in the real-valued data case considered by OMH is $f_i/2$.

$$L\left(h;\lambda\right) \;=\; n^{r^2/2}\prod_{t=1}^{r}\left[\left(p-t\right)!\left(\frac{c_p-h_t^2}{2\pi}\right)^{1/2}\right]e^{rp}p^{-rp}\times$$

$$\exp\left\{-\sum_{i=1}^{r}\Delta_p\left(z_{i0}\right)\right\}\prod_{i>j}^{r}\left(c_p-h_ih_j\right)\left(1+O_p\left(n^{-1}\right)\right)$$

Applying Stirling's formula

$$\left(p-t\right)! = e^{-p}p^{p-t+1}\left(\frac{2\pi}{p}\right)^{1/2}\left(1+O\left(p^{-1}\right)\right)$$

we get

$$L\left(h;\lambda\right) = \exp\left\{-\sum_{i=1}^{r}\Delta_p\left(z_{i0}\right)\right\}\prod_{t=1}^{r}\left(1-\frac{h_t^2}{c_p}\right)^{1/2}\prod_{i>j}^{r}\left(1-\frac{h_ih_j}{c_p}\right)\left(1+O_p\left(n^{-1}\right)\right),$$

which implies (25).

Turning to the proof of (26), Proposition 2 and Lemma 3 imply that

$$L\left(h;\mu\right) = \left(-1\right)^{r\left(r-1\right)/2}n^{-pr+r\left(r+1\right)/2}\prod_{i>j}^{r}\left(h_i-h_j\right)^{-1}\times \tag{51}$$

$$\prod_{t=1}^{r}\left[h_t^{r-p}\left(1+h_t\right)^{p-n-1}\left(p-t\right)!\right]\left(nS\right)^{pr-r\left(r+1\right)/2}\frac{\Gamma\left(p\left(n-r\right)+r\left(r+1\right)/2\right)}{\Gamma\left(np\right)}\times$$

$$\sum_{\rho}\frac{\operatorname{sgn}\rho}{\left(2\pi\mathrm{i}\right)^{r}}q_\rho\left(\mathbf{z}_0\right)\prod_{j=1}^{r}e^{-nf_{\rho\left(j\right)0}}\frac{g_j\left(z_{\rho\left(j\right)0}\right)\pi^{1/2}}{f_{\rho\left(j\right)2}^{1/2}n^{1/2}}\left(1+O_p\left(n^{-1}\right)\right)$$

Using the definition of $q_\rho\left(\mathbf{z}_0\right)$ and of z_{i0}, we get

$$q_\rho\left(\mathbf{z}_0\right) \;=\; \left(1-\sum_{i=1}^{r}\frac{h_i}{1+h_i}\frac{z_{i0}}{S}\right)^{-p\left(n-r\right)-r\left(r+1\right)/2}\exp\left\{-\sum_{i=1}^{r}\frac{nh_iz_{i0}}{1+h_i}\right\}$$

$$=\; \left(1-\sum_{i=1}^{r}\frac{h_i+c_p}{S}\right)^{-p\left(n-r\right)-r\left(r+1\right)/2}\exp\left\{-pr-\sum_{i=1}^{r}nh_i\right\}. \tag{52}$$

Further, using the definition of $g_j\left(z_{\rho(j)0}\right)$, the fact that $\sum_\rho \operatorname{sgn} \rho z_{\rho(j)0}^{j-1}$ equals the Vandermonde determinant $\prod_{i>j}^r \left(z_{i0} - z_{j0}\right)$, we get

$$\sum_\rho \operatorname{sgn} \rho \prod_{j=1}^r \frac{e^{-nf_{\rho(j)0}} g_j\left(z_{\rho(j)0}\right) \pi^{1/2}}{f_{\rho(j)2}^{1/2} n^{1/2}} = \frac{\pi^{r/2}}{n^{r/2}} \exp\left\{-\sum_{i=1}^r \left(nf_{i0} + \Delta_p\left(z_{i0}\right)\right)\right\} \times (53)$$

$$\prod_{i=1}^r f_{i2}^{-1/2} \prod_{i>j}^r \left(z_{i0} - z_{j0}\right).$$

Substituting (52) and (53) into (51), and using (49) and (50) together with the fact that the branch of the square root in $f_{i2}^{-1/2}$ is chosen so that $\sqrt{-1} = -i$, we get after some algebra

$$L\left(h;\mu\right) = \prod_{t=1}^r \left[(p-t)! \left(c_p - h_t^2\right)^{1/2}\right] S^{pr-r(r+1)/2} \times$$

$$\frac{\Gamma\left(p\left(n-r\right) + r(r+1)/2\right)}{\Gamma\left(np\right)} \left(1 - \sum_{j=1}^r \frac{h_j + c_p}{S}\right)^{-p(n-r)-r(r+1)/2} \times$$

$$\exp\left\{-\sum_{j=1}^r nh_j\right\} \frac{1}{(2\pi n)^{r/2}} c_p^{-pr} \prod_{i>j}^r \left(c_p - h_i h_j\right) \left(1 + O_p\left(n^{-1}\right)\right).$$

Now, using the fact that $S - p = O_p\left(1\right)$, we get $\ln\left(\frac{S}{p}\right) = \frac{S-p}{p} + O_p\left(p^{-2}\right)$ and

$$\ln\left(1 - \sum_{j=1}^r \frac{h_j + c_p}{S}\right) = -\frac{\sum_{j=1}^r \left(h_j + c_p\right)}{p} - \frac{1}{2} \frac{\left(\sum_{j=1}^r \left(h_j + c_p\right)\right)^2}{p^2} +$$

$$\frac{\sum_{j=1}^r \left(h_j + c_p\right)}{p^2} \left(S - p\right) + O_p\left(p^{-3}\right).$$

Further, the Stirling approximations give

$$(p-t)! = e^{-p} p^{p-t+1} \left(\frac{2\pi}{p}\right)^{1/2} \left(1 + O\left(p^{-1}\right)\right) \text{ and}$$

$$\frac{\Gamma\left(p\left(n-r\right) + r(r+1)/2\right)}{\Gamma\left(np\right)} = \left(pn\right)^{-pr+r(r+1)/2} e^{\frac{1}{2}c_p r^2} \left(1 + O\left(n^{-1}\right)\right).$$

So finally, after some cancellations,

$$
\begin{aligned}
L\left(h;\mu\right) \;=\; & e^{\frac{1}{2c_p}\left(\sum_{j=1}^{r} h_j\right)^2} e^{-\frac{\sum_{j=1}^{r} h_j}{c_p}(S-p)} e^{-\sum_{i=1}^{r}\Delta_p(z_{i0})} \\
& \prod_{t=1}^{r}\left(1-\frac{h_t^2}{c_p}\right)^{1/2} \prod_{i>j}^{r}\left(1-\frac{h_i h_j}{c_p}\right)\left(1+O_p\left(n^{-1}\right)\right),
\end{aligned}
$$

which implies (26).

To establish the rest of the statements of Theorem 1 we will need the following lemma.

Lemma A3. *Suppose that our null hypothesis holds. Denote $\sum_{j=1}^{p}\lambda_j^2$ as T. Then, for any fixed r and $\bar{h}<\sqrt{c}$, and any $(h_1,...,h_r)\in\left(0,\bar{h}\right]^r$, as $n,p\to\infty$ so that $p/n\to c$, the vector $(S-p,T-(1+c_p)\,p,\Delta_p\left(z_{10}\right),...,\Delta_p\left(z_{r0}\right))$ converges in distribution to a Gaussian vector $(\eta,\zeta,\xi_1,...,\xi_r)$ with*

$$
\begin{aligned}
\mathrm{E}\left(\eta\right) \;=\;& \mathrm{E}\left(\zeta\right)=\mathrm{E}\left(\xi_i\right)=0, \\
\mathrm{Var}\left(\eta\right) \;=\;& c,\, \mathrm{Var}\left(\zeta\right)=2c\left(2+5c+2c^2\right),\, \mathrm{Cov}\left(\eta,\zeta\right)=2c\left(1+c\right), \\
\mathrm{Cov}\left(\eta,\xi_i\right) \;=\;& -h_i,\, \mathrm{Cov}\left(\zeta,\xi_i\right)=-h_i\left(h_i+2+2c\right),\ \text{and} \\
\mathrm{Cov}\left(\xi_i,\xi_k\right) \;=\;& -\ln\left(1-h_i h_k/c\right)
\end{aligned}
$$

PROOF: The proof of the lemma is similar to that of Lemma 12 in OMH. The convergence to the Gaussian distribution follows from Theorem 1.1 of Bai and Silverstein (2004). The formulas for the means, variances and covariances of η and ξ_j are obtained using Theorem 1.1 iii) of Bai and Silverstein (2004) similarly to how the corresponding formulas in Lemma 12 of OMH are obtained using Theorem 1.1 ii). Therefore, below we only derive the formulae for the mean, variance, and covariances that involve ζ. Variable ζ does not appear in Lemma 12 of OMH because the lemma does not study the asymptotics of $T-(1+c_p)\,p$.

The fact that $\mathrm{E}\zeta=0$ follows directly from Theorem 1.1 iii) of Bai and Silver-

stein (2004). The same theorem implies that

$$\mathrm{Cov}\left(\xi_j, \zeta\right) = -\frac{1}{4\pi^2} \oint \oint \frac{z_2^2 \ln\left(\bar{z}_{j0} - z_1\right)}{\left(\underline{m}\left(z_1\right) - \underline{m}\left(z_2\right)\right)^2} \frac{\mathrm{d}\underline{m}\left(z_1\right)}{\mathrm{d}z_1} \frac{\mathrm{d}\underline{m}\left(z_2\right)}{\mathrm{d}z_2} \mathrm{d}z_1 \mathrm{d}z_2, \qquad (54)$$

where $\bar{z}_{j0} = \lim z_{j0}$ as $n, p \to \infty$,

$$\mathrm{Cov}\left(\eta, \zeta\right) = -\frac{1}{4\pi^2} \oint \oint \frac{z_2^2 z_1}{\left(\underline{m}\left(z_1\right) - \underline{m}\left(z_2\right)\right)^2} \frac{\mathrm{d}\underline{m}\left(z_1\right)}{\mathrm{d}z_1} \cdot \frac{\mathrm{d}\underline{m}\left(z_2\right)}{\mathrm{d}z_2} \mathrm{d}z_1 \mathrm{d}z_2, \qquad (55)$$

and

$$\mathrm{Var}\left(\zeta\right) = -\frac{1}{4\pi^2} \oint \oint \frac{z_1^2 z_2^2}{\left(\underline{m}\left(z_1\right) - \underline{m}\left(z_2\right)\right)^2} \frac{\mathrm{d}\underline{m}\left(z_1\right)}{\mathrm{d}z_1} \frac{\mathrm{d}\underline{m}\left(z_2\right)}{\mathrm{d}z_2} \mathrm{d}z_1 \mathrm{d}z_2, \qquad (56)$$

where

$$\underline{m}\left(z\right) = -\left(1 - c\right) z^{-1} + cm(z)$$

with $m\left(z\right)$ given by (3.6) of OMH, where c_p is replaced by c. That is,

$$\underline{m}\left(z\right) = \frac{-z + c - 1 + \sqrt{\left(z - c - 1\right)^2 - 4c}}{2z}, \qquad (57)$$

where the branch of the square root is chosen so that the real and the imaginary parts of $\sqrt{\left(z - c - 1\right)^2 - 4c}$ have the same signs as the real and the imaginary parts of $z - c - 1$, respectively. The contours of integration in (54)-(56) are closed, oriented counterclockwise, enclose zero and the support of the Marchenko-Pastur distribution with parameter c, and do not enclose $\bar{z}_{j0}$.

The above expressions can be simplified. Use formula 1.16 of Bai and Silverstein (2004), to get

$$\mathrm{Cov}\left(\xi_j, \zeta\right) = -\frac{1}{4\pi^2} \oint \oint \frac{\ln\left(\bar{z}_{j0} - z\left(m_1\right)\right)\left(z(m_2)\right)^2}{\left(m_1 - m_2\right)^2} \mathrm{d}m_1 \mathrm{d}m_2, \qquad (58)$$

$$\text{Cov}\left(\eta, \zeta\right) = -\frac{1}{4\pi^2} \oint \oint \frac{z\left(m_1\right)\left(z(m_2)\right)^2}{\left(m_1 - m_2\right)^2} dm_1 dm_2, \text{ and} \tag{59}$$

$$\text{Var}\left(\zeta\right) = -\frac{1}{4\pi^2} \oint \oint \frac{\left(z\left(m_1\right)\right)^2 \left(z(m_2)\right)^2}{\left(m_1 - m_2\right)^2} dm_1 dm_2, \tag{60}$$

where

$$z\left(m\right) = -\frac{1}{m} + \frac{c}{1+m} \tag{61}$$

and the contours of integration over m_1 and m_2 in (58-60) are obtained from the contours of integration over z_1 and z_2 in (54-56) by transformation $\underline{m}\left(z\right)$. Recall that by assumption the contours over z_1 and z_2 intersect the real line to the left of zero and in between the upper boundary of the support of the Marchenko-Pastur distribution, $\left(1 + \sqrt{c}\right)^2$, and $\bar{z}_{j0}$. Therefore, as can be shown using the definition (57) of $\underline{m}\left(z\right)$, the m_1-contour and m_2-contour are clockwise oriented and intersect the real line in between $-\left(1 + \sqrt{c}\right)^{-1}$ and $\underline{m}\left(\bar{z}_{j0}\right) = -h_j\left(h_j + c\right)^{-1}$ and to the right of zero. In particular, both contours enclose 0 and $-h_j\left(h_j + c\right)^{-1}$, but not -1 and $-\left(1 + h_j\right)^{-1}$.

Assuming without loss of generality that m_1-contour lies inside the m_2-contour, from (A64) in the Supplementary appendix of OMH, we have

$$\oint \frac{\ln\left(\bar{z}_{j0} - z\left(m_1\right)\right)}{\left(m_1 - m_2\right)^2} dm_1 = 2\pi\mathrm{i}\left(-\frac{1}{m_2} + \frac{1}{m_2 + h_j\left(h_j + c\right)^{-1}}\right). \tag{62}$$

Denoting $-h_j\left(h_j + c\right)^{-1}$ as x_j, we get from (62) and (58)

$$
\begin{aligned}
\text{Cov}\left(\xi_j, \zeta\right) &= \frac{2\pi\mathrm{i}}{4\pi^2} \oint \left(z(m_2)\right)^2 \left(\frac{1}{m_2} - \frac{1}{m_2 - x_j}\right) dm_2 \\
&= \frac{2\pi\mathrm{i}}{4\pi^2} \oint \left(-\frac{1}{m_2} + \frac{c}{1 + m_2}\right)^2 \left(\frac{1}{m_2} - \frac{1}{m_2 - x_j}\right) dm_2 \\
&= -h_j\left(h_j + 2 + 2c\right),
\end{aligned}
$$

where the last equality follows from Cauchy's residue theorem and the fact that

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the contour is oriented clock-wise.

For $\mathrm{Cov}\,(\eta, \zeta)$, we have

$$\oint \frac{z\,(m_1)}{(m_1 - m_2)^2}\mathrm{d}m_1 = \oint \frac{\left(-\frac{1}{m_1} + \frac{c}{1+m_1}\right)}{(m_1 - m_2)^2}\mathrm{d}m_1 = \frac{2\pi\mathrm{i}}{m_2^2}$$

so that

$$\begin{aligned}
\mathrm{Cov}\,(\eta, \zeta) &= -\frac{2\pi\mathrm{i}}{4\pi^2} \oint \oint \frac{(z(m_2))^2}{m_2^2}\mathrm{d}m_2 \\
&= -\frac{2\pi\mathrm{i}}{4\pi^2} \oint \left(-\frac{1}{m_2} + \frac{c}{1+m_2}\right)^2 \frac{1}{m_2^2}\mathrm{d}m_2 \\
&= 2c\,(1+c)
\end{aligned}$$

by Cauchy's theorem.

For $\mathrm{Var}\,(\zeta)$, we have

$$\begin{aligned}
\oint \frac{z\,(m_1)^2}{(m_1 - m_2)^2}\mathrm{d}m_1 &= \oint \frac{\left(-\frac{1}{m_1} + \frac{c}{1+m_1}\right)^2}{(m_1 - m_2)^2}\mathrm{d}m_1 \\
&= \frac{4\pi\mathrm{i}}{m_2^2}\left(c - \frac{1}{m_2}\right)
\end{aligned}$$

so that

$$\begin{aligned}
\mathrm{Var}\,(\zeta) &= -\frac{4\pi\mathrm{i}}{4\pi^2} \oint \oint \frac{(z(m_2))^2}{m_2^2}\left(c - \frac{1}{m_2}\right)\mathrm{d}m_2 \\
&= -\frac{4\pi\mathrm{i}}{4\pi^2} \oint \oint \frac{\left(-\frac{1}{m_2} + \frac{c}{1+m_2}\right)^2\left(c - \frac{1}{m_2}\right)}{m_2^2}\mathrm{d}m_2 \\
&= 2c\,(2 + 5c + 2c^2)
\end{aligned}$$

by Cauchy's theorem. $\square$

Lemma A3 and formulae (25) and (26) imply the convergence of finite dimensional-distributions of the random fields $\ln L\,(h; \lambda)$ and $\ln L\,(h; \mu)$ to the Gaussian distri-

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butions with means and covariance matrices characterized by (27-30).

To complete the proof of Theorem 1, we need to establish the tightness of $\ln L(h; \lambda)$ and $\ln L(h; \mu)$, viewed as random elements of the space $C\left[0, \overline{h}\right]^r$, as $n, p \to \infty$ so that $p/n \to c$. Formulae (25-26) and the facts that $S - p = O_p(1)$, and that $\Delta_p(z_{i0}) = O_p(1)$ for $i = 1, ..., r$, where $O_p(1)$ are uniform in $h \in \left(0, \overline{h}\right]^r$, imply that for an arbitrarily small positive ε, there must exist $B > 0$ such that $\Pr\left(\sup_{h \in (0, \overline{h}]^r} |\ln L(h; \lambda)| > B\right) < \varepsilon$ and $\Pr\left(\sup_{h \in (0, \overline{h}]^r} |\ln L(h; \mu)| > B\right) < \varepsilon$ for sufficiently large n and p. Since, as implied by Proposition 1, $\ln L(h; \lambda)$ and $\ln L(h; \mu)$ are continuous functions on $h \in \left[0, \overline{h}\right]^r$, $\sup_{h \in (0, \overline{h}]^r} |\ln L(h; \lambda)| = \sup_{h \in [0, \overline{h}]^r} |\ln L(h; \lambda)|$, and $\sup_{h \in (0, \overline{h}]^r} |\ln L(h; \mu)| = \sup_{h \in [0, \overline{h}]^r} |\ln L(h; \mu)|$, so that the tightness of $\ln L(h; \lambda)$ and $\ln L(h; \mu)$ follows.□

PROOF OF THEOREM 2

To save space, we only derive the asymptotic power envelope for the relatively more difficult case of real-valued data and μ-based tests. According to the Neyman-Pearson lemma, the most powerful test of the null $h = 0$ against a point alternative $h = (h_1, ..., h_r)$ is the test which rejects the null when $\ln L(h; \mu)$ is larger than a critical value C. It follows from Theorem 1 that, for such a test to have asymptotic size α, C must be

$$C = \sqrt{W(h)}\Phi^{-1}(1 - \alpha) + m(h), \tag{63}$$

where

$$m(h) = \frac{1}{2}\sum_{i,j=1}^{r}\left(\ln\left(1 - \frac{h_i h_j}{c}\right) + \frac{h_i h_j}{c}\right) \text{ and}$$

$$W(h) = -\sum_{i,j=1}^{r}\left(\ln\left(1 - \frac{h_i h_j}{c}\right) + \frac{h_i h_j}{c}\right).$$

Now, according to Le Cam's third lemma and Theorem 1, under $h = (h_1, ..., h_r)$, $\ln L(h; \mu) \overset{d}{\to} N(m(h) + W(h), W(h))$. Therefore, the asymptotic power $\beta_\mu(h)$

is (32).□

References

[1] Andreief, C. (1883). "Note sur une relation les intégrales définies des produits des fonctions", Mém.de la Soc. Sci. Bordeaux 2.

[2] Bai, Z.D. and J.W. Silverstein (2004) "CLT for Linear Spectral Statistics of Large-Dimensional Sample Covariance Matrices", *Annals of Probability* 32, 553-605.

[3] Baik, J., Ben Arous, G. and S. Péché. (2005) "Phase transition of the largest eigenvalue for non-null complex sample covariance matrices" *Annals of Probability* 33, 1643–1697.

[4] Bianchi, P., M. Debbah, M. Maïda and J. Najim (2010) "Performance of Statistical Tests for Single Source Detection using Random Matrix Theory", manuscript.

[5] Collins, B. and P. Śniady (2007), "New scaling of Itzykson-Zuber integrals", *Annales de l'IHP Prob. Stats.* 43 (2), 139–146.

[6] Dumitriu, I., Edelman, A., and G. Shuman (2007) "MOPS: Multivariate orthogonal polynomials (symbolically)", *Journal of Symbolic Computation* 42, 587–620

[7] Forrester, P.J. (2011), "Probability densities and distributions for spiked Wishart β-ensembles", arXiv:1101.2261.v1

[8] Goodman, N. R. (1963) "Statistical analysis based on a certain multivariate complex Gaussian distribution, (An introduction)." *Annals of Mathematical Statistics* 34, 152-177.

[9] Gross, K.I., and Richards, D.S.P. (1987), "Special Functions of Matrix Argument.I: Algebraic Induction, Zonal Polynomials, and Hypergeometric Functions", *Transactions of the American Mathematical Society* 301 (2), 781-811

[10] Guionnet, A. and M. Maïda (2005), "A Fourier view on the R-transform and related asymptotics of spherical integrals", *Journal of Functional Analysis* 222 (2), 435–490.

[11] Harish-Chandra (1957) "Differential Operators on Semi-simple Lie Algebra", *American journal of Mathematics* 79, 87-120.

[12] Itzykson, C., and Zuber, J.B. (1980) "The Planar Approximation. II", *Journal of Mathematical Physics* 21, 411-421.

[13] James, A. T. (1964) "Distributions of matrix variates and latent roots derived from normal samples", *Annals of Mathematical Statistics* 35, 475-501.

[14] Johnstone, I.M. (2001) "On the distribution of the largest eigenvalue in principal components analysis." *Annals of Statistics* 29, 295–327.

[15] Johnstone, I.M. (2007). "High dimensional statistical inference and random matrices", Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006. European Mathematical Society, 307-333.

[16] Johnstone, I.M., and D.M. Titterington (2009). "Statistical challenges of high-dimensional data", *Philosophical Transactions of Royal Society A* 367, 4237-4253

[17] Koev, P. and A. Edelman (2006). "The Efficient Evaluation of the Hypergeometric Function of a matrix Argument", *Mathematics of Computation* 75 (254), 833-846

[18] Krichman, S., and Nadler, B. (2009) "Non-Parametric Detection of the Number of Signals: Hypothesis Testing and Random Matrix Theory", *IEEE Transactions on Signal Processing* 57, 3930-3941.

[19] Macdonald, I. G. (1995) Symmetric functions and Hall polynomials. Second edition. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York.

[20] Mo, M.Y. (2011) "The rank 1 real Wishart spiked model", arXiv:1101.5144v1

[21] Marchenko, V.A., and L.A. Pastur (1967) "Distribution of eigenvalues for some sets of random matrices", Math. USSR-Sbornik, vol. 1, no. 4, 457-483

[22] Marinari, E., Parisi, G., and Ritort, F. (1994), "Replica field theory for deterministic models. II. A non-random spin glass with glassy behavior", *J. Phys. A* 27 (23), 7647–7668.

[23] Muller, R.R., Guo, D., and Moustakas, A.L (2008) "Vector precoding for wireless MIMO systems and its replica analysis" *IEEE Journal of Selected Areas in Communications* 26 (3), 530-540

[24] Nadakuditi, R.R. and A. Edelman (2008) "Sample Eigenvalue Based Detection of High-Dimensional Signals in White Noise Using Relatively Few Samples", *IEEE Transactions on Signal Processing* 56, 2625-2638

[25] Nadakuditi, R.R. and J.W. Silverstein (2010) "Fundamental Limit of Sample Generalized Eigenvalue Based Detection of Signals in Noise Using Relatively Few Signal-Bearing and Noise-Only Samples", *IEEE Journal of Selected Topics in Signal Processing* 4, 468-480.

[26] Onatski, A. (2009) "Testing Hypotheses About the Number of Factors in Large Factor Models", *Econometrica* 77, 1447-1479.

[27] Onatski, A., Moreira, M. J., and Hallin, M. (2012) "Asymptotic Power of Sphericity Tests for High-dimensional Data", manuscript, Economics Faculty, University of Cambridge.

[28] Patterson, N., A. L. Price, and D. Reich (2006) "Population Structure and Eigenanalysis", *PLoS Genetics* 2 (12), 2074-2093

[29] Péché, S. (2003). "Universality of local eigenvalue statistics for random sample covariance matrices," Ph.D. thesis, Ecole Polytechnique Fédérale de Lausanne.

[30] Perry, P.O. and P.J. Wolfe (2010) "Minimax Rank Estimation for Subspace Tracking", *IEEE Journal of Selected Topics in Signal Processing* 4, 504-513

[31] Ratnarajah, T. and R. Vaillancourt (2005) "Complex Singular Wishart Matrices and Applications", *Computers & Mathematics with Applications* 50, 399–411.

[32] Schreier, P.J., and L. L. Scharf (2010), Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals, Cambridge University Press.

[33] Telatar, E. (1999), "Capacity of Multi-antenna Gaussian Channels", *European Transactions on Telecommunications* 10 (6), 585-595.

[34] Tracy, C.A., and Widom, H. (2009) "The Distributions of Random Matrix Theory and their Applications", in V. Sidoravičius (ed.), New Trends in Mathematical Physics, Springer Science + Business Media B.V., 753-765

[35] Tulino, A. and Verdú, S. (2004). Random Matrix Theory and Wireless Communications. Foundations and Trends in Communications and Information Theory 1. Now Publishers, Hanover, MA.

[36] van der Vaart, A.W. (1998) Asymptotic Statistics, Cambridge University Press.

[37] Wang, D. (2010) "The largest eigenvalue of real symmetric, Hermitian and Hermitian self-dual random matrix models with rank one external source, part I." arXiv:1012.4144

[38] Zinn-Justin, P. and J.-B. Zuber (2003) "On some integrals over the U(N) unitary group and their large N limit", *Journal of Physics A.* 36 (12), 3173-3193