Firm-Specific Training

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ABSTRACT. This paper investigates the market provision of firm-specific training, and identifies the inefficiencies associated with it. Within a general stochastic learning-by-doing model, market provision of firm-specific training is potentially inefficient. In order to determine whether this inefficiency is in fact present, we analyze two special cases of the model: the accelerated productivity-enhancement model and the accelerated learning model. In both models, the inefficiency is indeed present. However, the nature of the inefficiency depends on the balance between the two key components of training, namely productivity enhancement and employee evaluation. In the accelerated productivity-enhancement model, training results in an increase in productivity enhancement but no change in employee evaluation, and training is overprovided by the market. In the accelerated learning model, training results in a proportionate increase in both productivity enhancement and employee evaluation, and training is underprovided by the market. In both cases, turnover is inefficiently low.

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1. Introduction

A widely documented feature of the US labour market is the high mobility of young workers. A typical male worker will hold seven jobs during his first ten years in the labour market. This number amounts to about two thirds of the total number of jobs he holds during his entire career.[1]

There is a positive aspect to this mobility: job-shopping early in a worker's career may help him to settle into a good match relatively quickly. The worker does not therefore spend too much time accumulating human capital specific to a bad match. There is also a negative aspect: job-shopping early in a worker's career may enable him to accumulate a small amount of firm-specific human capital in each of a large number of firms, but prevent him from accumulating a significant amount of firm-specific human capital in any one firm.[2]

In a world in which the accumulation of firm-specific human capital is passive, it can be argued that the market will achieve the optimal trade off between these two aspects of mobility. Indeed, in such a world, the principal decision is made by the worker, who must choose his employer. Moreover, firms can influence this choice via their wage offers. The mobility decisions of the worker should therefore be socially efficient.[3]

In a world in which the accumulation of firm-specific human capital is active, the situation is more complicated. There are now two decisions to be made: the worker must choose his employer; and the employer must choose whether or not to train the worker so as to enhance the accumulation of firm-specific human capital. Moreover, while firms can still influence the worker's choice of employer via their wage offers, a non-employer cannot influence the training choices of an employer. Training choices cannot therefore be expected to be socially efficient.[4]

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[1] See Topel and Ward (1992).

[2] Cf. Section 4B of Heckman (1993).

[3] See Felli and Harris (1996). See also Bergemann and Välimäki (1996) for a closely related model that looks at a different application, namely sellers competing for a buyer through dynamic pricing. See also Eeckhout and Weng (2015) for a model where human capital is "general" but learning and search frictions generate workers' mobility across jobs.

[4] Prendergast (1993) develops a model where an employer can create efficient incentives for a worker (actively) to accumulate firm-specific human capital, by committing to a pay scale that associates different remunerations to different tasks associated, in turn, with different levels of firm-specific human capital. The key difference between the analysis in Prendergast (1993) and our analysis is the ability of the employer to commit to a pay scale. In our environment employers cannot commit to a long-term contract. Therefore, the only mechanism through which different levels of firm-specific human capital can be associated to different wage rates in equilibrium is the employers' competition for the worker.

The purpose of the present paper is to investigate the market provision of firm-specific training, and to analyze the inefficiencies associated with it. To this end, we introduce a stochastic learning-by-doing model.[5]

In the model, there are two firms and one worker. Within each firm the worker can be allocated to one of two distinct activities that can be interpreted as a job (i.e. a productive activity leading to market-valued output), or a training program (i.e. an activity not necessarily associated with tangible output). In each period, the two firms compete for the worker on the basis of her expected productivity in the two competing matches. Each firm offers her a two-part contract, which specifies: (i) her wage; and (ii) whether she will be assigned to the job or to a training program. The worker then chooses between the two offers, and undertakes the assignment specified in her contract. This results in a change in the expected productivity of the worker within the match. We think of the mean and variance of this change in productivity as the *productivity-enhancement* and *employee-evaluation* components of learning-by-doing, respectively.[6]

Both firms are free to adjust the wage element of their offers. The employer therefore internalizes the preferences of the worker as to whether she should be assigned to the job or to the training program. However, the other firm has no way to express its preference as to whether the employer should assign the worker to the job or to the training program. We should therefore expect that training will be inefficiently provided in equilibrium.

The natural question is then whether this inefficiency leads to underprovision or over-

[5] We follow Jovanovic (1979) in modelling the accumulation of firm-specific human capital in a continuous-time setting. However, in our setting, firm-specific human capital is the current marginal product of the worker.

[6] Case-study evidence seems to suggest that employee evaluation is a key component of the firm-specific training programs offered by some European companies. For example, the Association of Retailers in France provides a rather extended period of training for new employees. The training is formal, and trainees who succeed in the final exam are awarded a professional diploma. Employees are initially selected by individual retailers (normally supermarkets) and then enrolled in the training program. After the training program, employees sometimes end up changing retailers. In other words, training does seem to foster worker's mobility. (Cf. The Retail Sector in France: Report for the Force Programme, CEDEFOP, Berlin 1993.) Another example is the training provided by Mercedes Benz Car Dealers in Germany. The employees are offered a whole range of training courses by the employers. New employees are offered basic introductory training, and highly technical training programs are offered to the specialized work force of the company through the Mercedez Benz training center. Training is clearly aimed at rendering the participants fully familiar with new car models, and the human capital accumulated in these courses is highly specific since it dies when the model is taken out of production. Evidence suggests that mobility following the introductory course is particularly high: on average, only one in six trainees are retained as employees. (Cf. Motor Vehicle Repair and Sales Sector: Germany Report for the Force Programme, Berlin 1993.) This can be interpreted as evidence that employee evaluation is a relevant component of the training program.

provision of training. In order to answer this question, we need only determine whether the other firm, the non-employer, assigns a positive or a negative value to training by the employer. The assumption that, while on the job, a worker exclusively accumulates firm-specific human capital implies that, from the point of view of the non-employer, what matters is whether the worker will remain forever with the current employer or whether she will eventually move. The critical step is therefore to identify how allocating the worker to the training program affects the timing and probability that the worker will eventually move. To this end, we analyze two distinct special cases of our model that differ in the impact that training has on the productivity-enhancement and employee-evaluation components of learning-by-doing.

The first special case that we consider is the *Accelerated Learning Model* (or ALM for short). In this model, when the worker is assigned to the training program: productivity enhancement and employee evaluation both increase by the same factor; but output is foregone.[7] For example, if the factor in question is 2 then, when the worker spends a day training: the change in her productivity is the same as it would have been if she had spent 2 days on the job; but she produces no output. Hence the entire future timepath of the worker's productivity with the current employer is traversed more quickly. In particular: in those states of the world in which the worker originally remained with her current employer forever, she will still remain with her current employer forever; and, in those states of the world in which she eventually moved to the other firm, she will move to the other firm sooner. In the first case, the other firm neither gains nor loses. In the second case, the other firm gains. Indeed, it can put the worker to productive use sooner. Training by the employer is therefore unambiguously good from the point of view of the non-employer and, since the other firm's preferences are not taken into account in the training decision, training is underprovided in equilibrium. Moreover, given that the benefit to the other firm of training is precisely that it causes the worker to change employer sooner, turnover is inefficiently low.

The second special case that we consider is the *Accelerated Productivity-Enhancement Model* (or APEM for short). In this model, when the worker is assigned to the training program: the productivity-enhancement component of the change in productivity increases, but the employee-evaluation component remains unchanged; and output is, of course, foregone. Hence the entire future timepath of the worker's productivity with the

[7] There is also a direct cost of training.

current employer is raised. This has two effects. First, in some states of the world in which the worker eventually switched to the other firm, she will now remain forever with the current employer. Secondly, even in states of the world in which she does eventually switch to the other firm, the switch will occur later. In the first case, the other firm loses an opportunity to put the worker to productive use altogether. In the second case, the other firm has to wait longer before it has an opportunity to put her to productive use. Training by the current employer is therefore unambiguously bad from the point of view of the other firm and, since the other firm's preferences are not taken into account in the training decision, training is overprovided in equilibrium.[8] Moreover, given that the cost to the other firm of training is precisely that it delays the time at which the worker changes employer, turnover is inefficiently low.[9]

The conclusions drawn from these two special cases of our learning-by-doing model differ with regard to the nature of the inefficiency that arises in equilibrium: the ALM predicts that training is underprovided; whereas the APEM predicts that training is overprovided. The difference between these conclusions is attributable to the different balance between the productivity-enhancement and employee-evaluation components of training. However, both cases agree on the conclusion that turnover is inefficiently low. In particular, the high mobility of young workers in the US labour market should not be interpreted as an alarming indicator of the inefficiency of this labour market.

We would like to conclude this introduction by highlighting the methodological contribution of our analysis.

In the setting considered in the existing literature, equilibrium is efficient. Building on this, it is possible to show that equilibrium is unique, and to find all the elements of the unique equilibrium. For example, one can find the joint value function of the three

[8] The inefficiency in the second model of training is similar to that in Pissarides (1994). In that model, workers underinvest in on-the-job search because they do not take into account the positive externality that this search activity exerts on the other firm. As a result, turnover is inefficiently low and on-the-job accumulation of firm-specific skills is inefficiently high.

[9] The argument that firm-specific training may inhibit turnover is already present in Becker (1993). Becker suggests two reasons why this may be the case. First, the marginal product of a worker who possesses firm-specific human capital may exceed her wage. This implies that such a worker is more likely to be retained in the face of an exogenous downward shift in productivity (or an exogenous upward shift in wages) than a worker who possesses only general human capital. Secondly, even if the marginal product of such a worker falls below her wage, the firm may still choose to retain her. This is because, if the firm lets her go during a downturn, it may be unable to rehire her during a subsequent upturn. In other words, there is an option value associated with a worker who possesses firm-specific human capital (Becker 1993, Chapter III.1). Becker's is, however, a partial equilibrium model, and he does not therefore comment on whether turnover is efficient.

market participants, and the allocation of the worker between the two firms, by solving an optimization problem.[10] In our setting, we do not have efficiency. However, we are able to identify two partial efficiency properties. Building on these, we can formulate a two-step procedure that allows us to characterize equilibrium in our model. In particular, since the output of the procedure is unique, equilibrium is unique. Moreover the procedure allows us to find all the elements of the unique equilibrium. For example: one can find the joint value function of firm k and the worker, and the equilibrium training allocation in firm k, by solving an optimization problem; and one can find the joint value function of the three market participants, and the allocation of the worker between the two firms, by solving a further optimization problem.[11]

Section 2 sets up the general stochastic learning-by-doing model. Section 3 defines and characterizes equilibrium. Section 4 derives the equilibrium wage and the identity of the employer, while Section 5 describes the worker's mobility between the two employers as well as the dynamics of wages. Section 6 discusses the efficiency properties of the equilibrium allocation. In particular, the underprovision of training in the ALM is established in Theorem 14 and the overprovision of training in the APEM is established in Theorem 16. Section 7 concludes.

2. The Model

There are three players: firm 1, firm 2 and the worker. At the outset of any given period $t \in [0, \infty)$, all three of them observe the marginal products $m_1 \in [\underline{m}_1, \overline{m}_1]$ and $m_2 \in [\underline{m}_2, \overline{m}_2]$ of the worker in firms 1 and 2 respectively. The two firms then simultaneously

[10] Cf. Felli and Harris (1996) and Bergemann and Välimäki (1996).

[11] Larsen, Pastorino and Kehoe (2016) consider an experience-good model with a variety of goods. Their model is analogous to our model, with their consumer corresponding to our worker and their variety of goods offered by a firm corresponding to our variety of tasks offered by a given firm. From a game-theoretic point of view, they allow for more than two tasks and more than two firms. They show that our two-step characterization of equilibrium generalizes to this setting: the task chosen by each firm is that which maximizes the joint payoff of the firm and the worker; and the firm chosen by the worker maximizes the total payoff of all market participants conditional on the task choices of the individual firms. (They call these properties match efficiency and conditional efficiency.) Like us, they also show (in two-firm, two-task examples) that the market may both under- and over-provide information. However, their example of overprovision complements our example in an interesting way. Finally, they show that efficiency may fail even when there is no choice of task within a firm: if there are three or more firms, then the third firm may have a preference as to whether the first or the second firm should be the employer. Since the market does not provide any means for the third firm to express this preference, inefficiency may result. See also Pastorino and Kehoe (2014) for a characterization of the optimal allocation in a setting where a decision maker faces a multi-armed dependent bandit problem

offer the worker contracts

$$(w_1, a_1), (w_2, a_2) \in \mathbb{R} \times [0, 1],$$

where w_k is the wage that the worker will receive, and a_k is the fraction of her time that she will devote to training, if she accepts the contract proposed by firm k. The worker then chooses an employer $i \in \{1, 2\}$, and works/trains for that employer for the duration dt of the current period. As a result:

1. The employer i receives a payoff of $(\pi_{i,a_i}(m_i) - w_i)\, dt$, where:
   (a) $\pi_{i,a_i}(m_i) = \gamma_i(1 - a_i)\, \pi_{i,\mathfrak{W}}(m_i) + a_i\, \pi_{i,\mathfrak{T}}(m_i)$;
   (b) $\pi_{i,\mathfrak{W}}(m_i) = m_i$;
   (c) $\pi_{i,\mathfrak{T}}(m_i) = -\kappa_i$;
   (d) γ_i is a twice continuously differentiable function such that $\gamma_i(0) = 0$, $\gamma_i(1) = 1$, $\gamma_i' > 0$ on $[0, 1]$ and $\gamma_i'' < 0$ on $[0, 1]$;
   (e) κ_i is the cost per unit time to firm i of training the worker.
2. The worker receives a payoff of $w_i\, dt$.
3. The non-employer $j = 3 - i$ receives a payoff of $0\, dt$.

In other words, if the worker spends all of her time at firm i working – i.e. $a_i = 0$ – then she generates output m_i, and the revenue of the firm is $\pi_{i,a_i}(m_i) = \pi_{i,\mathfrak{W}}(m_i) = m_i$; if she spends all of her time at firm i training – i.e. $a_i = 1$ – then she imposes a cost κ_i, and the revenue of the firm is $\pi_{i,a_i}(m_i) = \pi_{i,\mathfrak{T}}(m_i) = -\kappa_i$; and if she spends a fraction of time at firm i training then the revenue of the firm is a combination of the revenue from working and the revenue from training that is concave in the time spent working and linear in the time spent training.

Next, the change dm_i in the marginal product of the worker in firm i is distributed normally with mean $\mu_{i,a_i}(m_i)\, dt$ and variance $\Sigma_{i,a_i}(m_i)\, dt$, where

$$\begin{aligned}
\mu_{i,a_i}(m_i) &= (1 - a_i)\, \mu_{i,\mathfrak{W}}(m_i) + a_i\, \mu_{i,\mathfrak{T}}(m_i), \\
\Sigma_{i,a_i}(m_i) &= (1 - a_i)\, \Sigma_{i,\mathfrak{W}}(m_i) + a_i\, \Sigma_{i,\mathfrak{T}}(m_i);
\end{aligned}$$

and the change dm_j in the marginal product of the worker in firm j is distributed normally with mean $0\,dt$ and variance $0\,dt$. Here, for $\alpha \in \{\mathfrak{W}, \mathfrak{T}\}$, we have

($\mu1$) $\mu_{i,\alpha}(\underline{m}_i) > 0$;

($\mu2$) $\mu_{i,\alpha}(\overline{m}_i) = 0$;

($\mu3$) $\mu'_{i,\alpha} \leq 0$ for all $m_i \in [\,\underline{m}_i, \overline{m}_i\,]$;

($\mu4$) $\mu''_{i,\alpha} \geq 0$ for all $m_i \in [\,\underline{m}_i, \overline{m}_i\,]$;

($\Sigma1$) $\Sigma_{i,\alpha}(m_i) > 0$ for all $m_i \in (\,\underline{m}_i, \overline{m}_i\,)$;

($\Sigma2$) $\Sigma_{i,\alpha}(\underline{m}_i) = \Sigma_{i,\alpha}(\overline{m}_i) = 0$.

In other words, if the worker spends all her time at firm i working, then the change dm_i in her marginal product is distributed normally with mean $\mu_{i,\mathfrak{W}}(m_i)\,dt$ and variance $\Sigma_{i,\mathfrak{W}}(m_i)\,dt$; if she spends all her time at firm i training, then the change dm_i in her marginal product is distributed normally with mean $\mu_{i,\mathfrak{T}}(m_i)\,dt$ and variance $\Sigma_{i,\mathfrak{T}}(m_i)\,dt$; and therefore, if she spends a fraction $1 - a_i$ of her time working and a fraction a_i of her time training, then the change dm_i in her marginal product is distributed normally with a mean that is the corresponding convex combination of $\mu_{i,\mathfrak{W}}(m_i)\,dt$ and $\mu_{i,\mathfrak{T}}(m_i)\,dt$ and with a variance that is the corresponding convex combination of $\Sigma_{i,\mathfrak{W}}(m_i)\,dt$ and $\Sigma_{i,\mathfrak{T}}(m_i)\,dt$.

The functional form chosen for the mean and variance of dm_i can be explained as follows. At its most basic level, the idea is that working for firm i results in learning by doing. This can be captured by temporarily dropping Assumption $\Sigma1$ (which requires that $\Sigma_{i,\mathfrak{W}} > 0$ on $(\,\underline{m}_i, \overline{m}_i\,)$) and requiring instead that $\Sigma_{i,\mathfrak{W}} = 0$ on $(\,\underline{m}_i, \overline{m}_i\,)$, so that

$$dm_i = \mu_{i,\mathfrak{W}}(m_i)\,dt.$$

In other words, the simple fact of working for firm i results in a change in the marginal product of the worker in that firm. Assumptions $\mu2$ and $\mu3$ ensure that this change is positive (i.e. $\mu_{i,\mathfrak{W}} \geq 0$); but, in view of Assumption $\mu3$, increases in productivity are harder to achieve for a more productive worker (i.e. $\mu'_{i,\mathfrak{W}} \leq 0$); and, in view of Assumption $\mu4$, this latter effect becomes weaker as productivity increases (i.e. $\mu''_{i,\mathfrak{W}} \geq 0$). Finally, Assumption $\mu 1$ (which implies that $\mu_{i,\mathfrak{W}}(\underline{m}_i) \geq 0$) and Assumption $\mu 2$ (which implies that $\mu_{i,\mathfrak{W}}(\overline{m}_i) \leq 0$) together ensure that m_i does not drift out of the interval $[\,\underline{m}_i, \overline{m}_i\,]$.

At the next level up, working for firm i still results in learning by doing, but now that learning is stochastic instead of deterministic. This can be captured by reinstating Assumption $\Sigma 1$ (i.e. by requiring once again that $\Sigma_{i,\mathfrak{W}} > 0$ on $(\,\underline{m}_i, \overline{m}_i\,)$), so that

$$dm_i = \mu_{i,\mathfrak{W}}(m_i)\, dt + \sqrt{\Sigma_{i,\mathfrak{W}}(m_i)}\, dZ_i,$$

where Z_i is a standard Wiener process.[12] In other words, the change in the marginal product of the worker that results from working for firm i is now stochastic. Assumption $\Sigma 2$ (which requires that $\Sigma_{i,\mathfrak{W}}(\underline{m}_i) = \Sigma_{i,\mathfrak{W}}(\overline{m}_i) = 0$) ensures that m_i does not diffuse out of the interval $[\,\underline{m}_i, \overline{m}_i\,]$.

At the third and final level, the worker can engage in one of two activities when employed by firm i: she can work or she can train. If she works, then

$$dm_i = \mu_{i,\mathfrak{W}}(m_i)\, dt + \sqrt{\Sigma_{i,\mathfrak{W}}(m_i)}\, dZ_i$$

as before; and, if she trains, then

$$dm_i = \mu_{i,\mathfrak{T}}(m_i)\, dt + \sqrt{\Sigma_{i,\mathfrak{T}}(m_i)}\, dZ_i.$$

Furthermore she can, if she wishes, divide her time between the two activities, spending a fraction $1 - a_i$ of her time working and a fraction a_i training. In that case

$$dm_i = \big((1 - a_i)\, \mu_{i,\mathfrak{W}}(m_i) + a_i\, \mu_{i,\mathfrak{T}}(m_i)\big)\ dt + \sqrt{(1 - a_i)\, \Sigma_{i,\mathfrak{W}}(m_i) + a_i\, \Sigma_{i,\mathfrak{T}}(m_i)}\, dZ_i.$$

In other words, the means and the variances combine in the natural way.[13]

[12] This formulation of the stochastic differential equation for m_i ensures that the mean and variance of dm_i are $\mu_{i,\mathfrak{W}}(m_i)\, dt$ and $\big(dm_i - \mu_{i,\mathfrak{W}}(m_i)\, dt\big)^2 = \Big(\sqrt{\Sigma_{i,\mathfrak{W}}(m_i)}\, dZ_i\Big)^2 = \Sigma_{i,\mathfrak{W}}(m_i)\, dt$ respectively.

[13] The case is analogous to that of taking the sum of $n_{\mathfrak{W}}$ i.i.d. random variables with mean $\mu_{\mathfrak{W}}$ and variance $\Sigma_{\mathfrak{W}}$ and $n_{\mathfrak{T}}$ i.i.d. random variables with mean $\mu_{\mathfrak{T}}$ and variance $\Sigma_{\mathfrak{T}}$. In that case the sum has mean $n_{\mathfrak{W}}\, \mu_{\mathfrak{W}} + n_{\mathfrak{T}}\, \mu_{\mathfrak{T}}$, variance $n_{\mathfrak{W}}\, \Sigma_{\mathfrak{W}} + n_{\mathfrak{T}}\, \Sigma_{\mathfrak{T}}$ and standard deviation $\sqrt{n_{\mathfrak{W}}\, \Sigma_{\mathfrak{W}} + n_{\mathfrak{T}}\, \Sigma_{\mathfrak{T}}}$.

3. Equilibrium

Suppose that the current state is

$$m = (m_1, m_2) \in [\underline{m}_1, \overline{m}_1] \times [\underline{m}_2, \overline{m}_2],$$

that the continuation payoffs of firm 1, firm 2 and the worker are given by the functions

$$U_1,\, U_2 \text{ and } V : [\underline{m}_1, \overline{m}_1] \times [\underline{m}_2, \overline{m}_2] \longrightarrow \mathbb{R}$$

and that the actions chosen by the three players are (w_1, a_1), (w_2, a_2) and i. Then, if the employer i applies the discount factor $r \exp(-r\,(s-t))$ to payoffs received at time $s \in [t, t+dt]$, its expected payoff is

$$(1 - \exp(-r\,dt))\,(\pi_{i,a_i}(m_i) - w_i) + \mathrm{E}\left[\exp(-r\,dt)\,U_i(m + dm)\right].$$

In other words, it receives a flow payoff of $\pi_{i,a_i}(m_i) - w_i$ over the interval $[t, t+dt)$, followed by the continuation payoff $U_i(m+dm)$ at time $t+dt$.[14] Using Itô's Lemma, this payoff can be written in the form

$$\begin{aligned}(r\,dt)\,(\pi_{i,a_i}(m_i) - w_i) &+ (1 - r\,dt)\,(U_i(m) + (L_{i,a_i}U_i)(m)\,dt) \\ &= (1 - r\,dt)\,U_i(m) + r\,dt\left(\pi_{i,a_i}(m_i) - w_i + \tfrac{1}{r}\,(L_{i,a_i}U_i)(m)\right),\end{aligned}$$

where the operator $L_{i,a_i} : H \mapsto L_{i,a_i}H$ is defined by the formula

$$(L_{i,a_i}H)(m) = \mu_{i,a_i}(m_i)\,\tfrac{\partial H}{\partial m_i}(m) + \tfrac{1}{2}\,\Sigma_{i,a_i}(m_i)\,\tfrac{\partial^2 H}{\partial m_i^2}(m).$$

In particular, it is a positive affine transformation of the payoff

$$\pi_{i,a_i}(m_i) - w_i + \tfrac{1}{r}\,(L_{i,a_i}U_i)(m). \tag{1}$$

[14] The expected payoff of the employer can be understood in more detail as follows. If her flow payoff $\pi_{i,a_i}(m_i) - w_i$ is constant over the interval $[t, t+dt)$, then the present discounted value of her flow payoff over this interval is

$$\int_t^{t+dt} r\,\exp(-r\,(s-t))\,(\pi_{i,a_i}(m_i) - w_i)\,ds = (1 - \exp(-r\,dt))\,(\pi_{i,a_i}(m_i) - w_i).$$

If, furthermore, the change in m over this interval is dm, then the present discounted value of her continuation payoff at time $t+dt$ is $\exp(-r\,dt)\,U_i(m+dm)$. Adding these two terms together and taking expectations leads to the formula given.

Similarly, assuming that the non-employer $j = 3 - i$ uses the same discount factor as the employer, its expected payoff is

$$(1 - r\,dt)\,U_j(m) + r\,dt\,\left(\tfrac{1}{r}\,(L_{i,a_i}U_j)(m)\right).$$

This is a positive affine transformation of the payoff

$$\tfrac{1}{r}\,(L_{i,a_i}U_j)(m). \tag{2}$$

Finally, the expected payoff of the worker is

$$(1 - r\,dt)\,V(m) + r\,dt\,\left(w_i + \tfrac{1}{r}\,(L_{i,a_i}V)(m)\right).$$

This is a positive affine transformation of the payoff

$$w_i + \tfrac{1}{r}\,(L_{i,a_i}V)(m). \tag{3}$$

Since the set of subgame-perfect equilibria of a game is unaffected by positive affine transformations of the payoffs of the players, this motivates the following definition.

Definition 1 [The Constituent Game]**.** *Suppose that the state is m and that the continuation payoffs of firm 1, firm 2 and the worker are given by the functions U_1, U_2 and V. Then the constituent game is the two-stage game in which:*

1. *In the first stage, firms 1 and 2 simultaneously propose contracts (w_1, a_1) and (w_2, a_2).*
2. *In the second stage, the worker observes (w_1, a_1) and (w_2, a_2) and chooses an employer $i \in \{1, 2\}$.*
3. *The payoffs of the employer, the non-employer and the worker are given by (1), (2) and (3).*

Notice that the payoff of the employer in the constituent game has three components, namely: (i) the output $\pi_{i,a_i}(m_i)$ produced by the worker; less (ii) the wage w_i paid to the worker; plus (iii) the discounted shadow value $\frac{1}{r}\,(L_{i,a_i}U_i)(m)$ to the employer of the change in the worker's productivity that occurs as a result of the worker working/training for the employer. Similarly, the payoff of the non-employer has only one component, namely the discounted shadow value to the non-employer $\frac{1}{r}(L_{i,a_i}U_j)(m)$ of the change in the worker's productivity that occurs as a result of working for the employer. Finally, the payoff of the worker has two components, namely: (i) the wage w_i received from the employer; plus (ii) the discounted shadow value $\frac{1}{r}(L_{i,a_i}V)(m)$ to the worker of the change in the worker's productivity that occurs as a result of working for the employer.

Next, let $\text{SPEP}(m, U_1, U_2, V)$ denote the set of subgame-perfect equilibrium payoff vectors of the constituent game. Then we have the following definition:

Definition 2 [Markov-Perfect Equilibrium Value Functions]**.** *The continuation-payoff functions U_1, U_2 and V are the value functions of a Markov-perfect equilibrium of the dynamic game iff, for all states m, we have*

$$(U_1(m), U_2(m), V(m)) \in \text{SPEP}(m, U_1, U_2, V).$$

This definition departs from the usual approach to defining equilibrium in dynamic games in two ways. First, instead of defining equilibrium in terms of the strategies of the players and then characterizing it using dynamic programming, it defines equilibrium directly using dynamic programming. This approach allows us to move more quickly to the economics of the problem. Second, instead of including both strategies and value functions in the dynamic program, it includes only value functions. It can certainly be expanded to include strategies as well. However, the focus on value functions is better suited to the refinement of Markov-perfect equilibrium that we introduce below.

Unfortunately, there is a large multiplicity of Markov-perfect equilibria in the dynamic game. This corresponds to the large multiplicity of subgame-perfect equilibria in the constituent game. To understand how this latter multiplicity arises, it will be helpful to consider a simple two-stage game. In this game, there are two firms and a worker. In the first stage of the game, the firms simultaneously and independently offer wages w_1 and w_2. In the second stage, the worker chooses an employer i. The payoff of the employer i is then $m_i - w_i$; the payoff of the non-employer $j = 3 - i$ is 0; and the payoff of the worker is w_i.

Even this simple game typically has a continuum of subgame-perfect equilibria. If $m_1 < m_2$, then an action profile (w_1, w_2, i) is the outcome of a subgame-perfect equilibrium iff $m_1 \leq w_1 = w_2 \leq m_2$ and $i = 2$. Similarly, if $m_1 > m_2$, then an action profile (w_1, w_2, i) is the outcome of a subgame-perfect equilibrium iff $m_2 \leq w_1 = w_2 \leq m_1$ and $i = 1$.

However, in the former case, any outcome in which $w_1 = w_2 > m_1$ seems implausible: firm 1 is bidding more than the worker is worth to it, and it is unclear what could possibly motivate such a bid. Similarly, in the latter case, any outcome in which $w_1 = w_2 > m_2$ seems implausible.

Now, the constituent game is more complicated than this example in two respects. First, firms offer contracts (w_1, a_1) and (w_2, a_2) rather than simply wages w_1 and w_2. As a result, there is a second dimension to the continuum of equilibrium: the training allocations a_k can vary as well as the wages w_k. Second, there are non-pecuniary externalities: if the worker works/trains for firm 1, then firm 2 still receives the payoff $\frac{1}{r}(L_{1,a_1}U_2)(m)$, which depends directly on the action chosen by firm 1. We think of this as the outside option of firm 2. Similarly, if the worker works/trains for firm 2, then firm 1 still receives its outside option $\frac{1}{r}(L_{2,a_2}U_1)(m)$.

Nonetheless, the multiplicity of subgame-perfect equilibria in the constituent game can be eliminated using a simple economic argument. For example, in any subgame-perfect equilibrium in which firm 2 is the employer, firm 2 matches the offer of firm 1. The worker is therefore indifferent between working for firm 1 and working for firm 2. Hence, while the equilibrium prescribes that the worker should choose firm 2, she does not have any particular reason to do so. Hence firm 1 should bear in mind the possibility that the worker will depart from the equilibrium prescription, i.e. that she will choose firm 1 instead. In particular, firm 1 should not make any offer that makes sense only on the assumption that it will be rejected.

Now, firm 1 can always ensure that its offer (w_1, a_1) is rejected by making w_1 low enough. In this way, it ensures that it receives its outside option

$$\tfrac{1}{r}(L_{2,a_2}U_1)(m).$$

On the other hand, if – for any reason – its offer is accepted, then it receives

$$\pi_{1,a_1}(m_1) - w_1 + \tfrac{1}{r}(L_{1,a_1}U_1)(m).$$

Hence it should confine itself to offers for which the latter quantity is at least as big as the former. The same logic applies to firm 2. This motivates the following refinement of subgame-perfect equilibrium in the constituent game.

Definition 3 [Cautious Equilibrium in the Constituent Game]**.** *A cautious equilibrium* *of the constituent game is a subgame-perfect equilibrium* $((w_1, a_1), (w_2, a_2), \iota)$ *of that game*[15] *such that*

$$\pi_{1,a_1}(m_1) - w_1 + \tfrac{1}{r}\,(L_{1,a_1}U_1)(m) \geq \tfrac{1}{r}\,(L_{2,a_2}U_1)(m) \tag{4}$$

and

$$\pi_{2,a_2}(m_2) - w_2 + \tfrac{1}{r}\,(L_{2,a_2}U_2)(m) \geq \tfrac{1}{r}\,(L_{1,a_1}U_2)(m). \tag{5}$$

We shall refer to (4) and (5) as the cautious-equilibrium constraints of firm 1 and firm 2 respectively.

Remark. Cautious equilibrium as we have defined it here is a refinement of subgame-perfect equilibrium in the sense that it selects a subset of the set of all subgame-perfect equilibria. We have motivated this selection using a simple economic argument. However, there are at least two ways of providing a more formal justification for the selection: one can build a game-theoretic foundation in the spirit of trembling-hand perfection;[16] and one can build a behavioral foundation based on probabilistic choice.[17]

Remark. The cautious-equilibrium constraint of the employer is always satisfied in a subgame-perfect equilibrium. This is because the employer's offer is accepted, and the employer therefore gives explicit consideration to the alternative of making a lower offer and receiving its outside option instead. It is the cautious-equilibrium constraint of the non-employer that bites.

Our next theorem gives explicit formulae for the cautious-equilibrium payoffs of the three players in the constituent game. In order to state this theorem, it will be helpful to have some notation. For all states m and all continuation-payoff functions U_1, U_2 and V, put

$$\begin{aligned} a_1^*(m, U_1, U_2, V) &= \operatorname*{argmax}_{\widetilde{a}_1} \left\{\pi_{1,\widetilde{a}_1}(m_1) + \tfrac{1}{r}\,(L_{1,\widetilde{a}_1}(U_1 + V))(m)\right\}, \\ a_2^*(m, U_1, U_2, V) &= \operatorname*{argmax}_{\widetilde{a}_2} \left\{\pi_{2,\widetilde{a}_2}(m_2) + \tfrac{1}{r}\,(L_{2,\widetilde{a}_2}(U_2 + V))(m)\right\} \end{aligned}$$

[15] Here $(w_1, a_1), (w_2, a_2) \in \mathbb{R} \times [0, 1]$ are the strategies of the two firms, and $\iota : (\mathbb{R} \times [0, 1])^2 \to \{1, 2\}$ is the strategy of the worker.

[16] See Appendix A.1.

[17] See Appendix A.2.

and

$$S = U_1 + U_2 + V.$$

In other words: let a_1^* be the training allocation that maximizes the joint payoff of firm 1 and the worker; let a_2^* be the training allocation that maximizes the joint payoff of firm 2 and the worker; and let S be the joint continuation-payoff function of the three players. Then:

Theorem 4.

1. *The constituent game has a cautious equilibrium in which firm* 1 *is the employer iff*

$$\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S \geq \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S$$

and, in this case, there is precisely one such equilibrium. Similarly, the constituent game has a cautious equilibrium in which firm 2 *is the employer iff*

$$\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S \leq \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S$$

and, in this case, there is precisely one such equilibrium. In particular, cautious equilibrium always exists, and it is unique iff $\pi_{1,a_1^*} + \frac{1}{r}\, L_{1,a_1^*} S \neq \pi_{2,a_2^*} + \frac{1}{r}\, L_{2,a_2^*} S$.

2. *The cautious-equilibrium training allocations offered by firm* 1 *and firm* 2 *are* a_1^* *and* a_2^**; and the cautious-equilibrium payoffs of firm* 1*, firm* 2 *and the worker are*

$$\begin{aligned}
&\tfrac{1}{r}\, L_{2,a_2^*} U_1 + \max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right),\ 0\right\},\\
&\tfrac{1}{r}\, L_{1,a_1^*} U_2 + \max\left\{\left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right) - \left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right),\ 0\right\},\\
&\min\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right\} - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*} U_2.
\end{aligned}$$

In particular, the cautious-equilibrium training allocations offered by firm 1 *and firm* 2*, and the cautious-equilibrium payoffs of firm* 1*, firm* 2 *and the worker, are all unique (irrespective of whether cautious equilibrium itself is unique).*[18]

In other words, there is a cautious equilibrium in which firm 1 is the employer iff the joint payoff of all three players when firm 1 is the employer (and the training allocation is

[18] We shall see below that the cautious-equilibrium wages offered by firm 1 and firm 2 are also unique. However, we do not need this result in the present section.

a_1^*) is at least as high as the joint payoff of all three players when firm 2 is the employer (and the training allocation is a_2^*). Similarly, there is a cautious equilibrium in which firm 2 is the employer iff the joint payoff of all three players when firm 2 is the employer (and the training allocation is a_2^*) is at least as high as the joint payoff of all three players when firm 1 is the employer (and the training allocation is a_1^*). In particular, while the cautious-equilibrium choice of employer by the worker is not necessarily unique, it is always constrained efficient (in the sense that it maximizes the joint payoff of the three players taking the training allocations offered by the two firms as given). Furthermore the training allocations offered by the two firms are themselves pairwise efficient, in the sense that a_k^* maximizes the joint payoff of firm k and the worker.

Remark. This pairwise efficiency makes economic sense: given that the wage component of the contract offered to the worker by firm k effectively allows side payments to be made, one would expect that Coasian bargaining between firm k and the worker would lead to a pairwise efficient training allocation.

Turning to the formulae for the cautious-equilibrium payoffs of the three players, note first that the cautious-equilibrium payoff of firm 1 has two components. The first component is

$$\tfrac{1}{r}\, L_{2,a_2^*} U_1 .$$

This is the discounted shadow value to firm 1 of the learning that takes place about the worker's productivity in firm 2 if the worker works/trains for firm 2 and training is given by a_2^*. It can be thought of as the outside option of firm 1. The second component is

$$\max\left\{ \left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right),\ 0 \right\} .$$

This is, roughly speaking, the marginal contribution of firm 1 to the joint payoff of the three players. The cautious-equilibrium payoff of firm 2 is completely analogous.

As for the equilibrium payoff of the worker, this has three components. The first component is

$$\min\left\{ \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S \right\} .$$

This is, roughly speaking, the part of the joint payoff of the three players that the worker can capture thanks to the competition between the two firms for her services. The second and third components are

$$-\tfrac{1}{r}\, L_{2,a_2^*} U_1 \quad \text{and} \quad -\tfrac{1}{r}\, L_{1,a_1^*} U_2.$$

These are (minus) the cost to the worker of compensating firm 1 for the loss of its outside option and (minus) the cost to the worker of compensating firm 2 for the loss of its outside option.

Overall, then, the employer gets its outside option plus its marginal contribution to the joint payoff of the three players; the non-employer gets only its outside option; and the worker gets the non-employer's contribution to the joint payoff of the three players less the outside options of the two firms.

Proof of Theorem 4. See Appendix A.3. ■

Next, let us denote the unique cautious-equilibrium payoff vector of the constituent game by $\mathrm{CEP}(m, U_1, U_2, V)$. Then we can refine our preliminary definition of equilibrium in the dynamic game, namely Definition 2 above, to obtain:

Definition 5 [Markov-Cautious Equilibrium Value Functions]**.** *The continuation-payoff functions U_1, U_2 and V are the value functions of a Markov-cautious equilibrium of the dynamic game iff, for all states m, we have*

$$(U_1(m), U_2(m), V(m)) = \mathrm{CEP}(m, U_1, U_2, V).$$

Taken in conjunction with the explicit formulae for the cautious-equilibrium payoffs of the three players given in Theorem 4, this definition yields a system of three partial differential equations for the value functions of the three players. We refer to this system as the Bellman system of the dynamic game.[19]

The single most important step in our analysis is our characterization of Markov-cautious equilibrium value functions. This is the content of the next theorem.

Theorem 6. *Suppose that we construct functions*

$$F_1, F_2, G : [\,\underline{m}_1, \overline{m}_1\,] \times [\,\underline{m}_2, \overline{m}_2\,] \to \mathbb{R}$$

and

$$A_1, A_2 : [\,\underline{m}_1, \overline{m}_1\,] \times [\,\underline{m}_2, \overline{m}_2\,] \to [0, 1]$$

[19] This system is set out explicitly in equations (22-24) of Appendix A.4.

using the following procedure:

Step 1 *For $k \in \{1, 2\}$, let F_k be the unique solution of the Bellman equation*

$$F_k = \max_{\widetilde{a}_k} \left\{ \pi_{k,\widetilde{a}_k} + \tfrac{1}{r} L_{k,\widetilde{a}_k} F_k \right\} . \tag{6}$$

Step 2 *For $k \in \{1, 2\}$, put*

$$A_k = \operatorname*{argmax}_{\widetilde{a}_k} \left\{ \pi_{k,\widetilde{a}_k} + \tfrac{1}{r} L_{k,\widetilde{a}_k} F_k \right\} . \tag{7}$$

Step 3 *Let G be the unique solution of the Bellman equation*

$$G = \max_{k} \left\{ \pi_{k,A_k} + \tfrac{1}{r} L_{k,A_k} G \right\} . \tag{8}$$

Suppose furthermore that we put $U_1 = G - F_2$, $U_2 = G - F_1$ and $V = F_1 + F_2 - G$. Then U_1, U_2 and V solve the Bellman system of the dynamic game. Conversely, if U_1, U_2 and V solve the Bellman system of the dynamic game, and if we put $F_1 = U_1 + V$, $F_2 = U_2 + V$, $G = U_1 + U_2 + V$,

$$A_1 = \operatorname*{argmax}_{\widetilde{a}_1} \left\{ \pi_{1,\widetilde{a}_1} + \tfrac{1}{r} L_{1,\widetilde{a}_1} (U_1 + V) \right\}$$

and

$$A_2 = \operatorname*{argmax}_{\widetilde{a}_2} \left\{ \pi_{2,\widetilde{a}_2} + \tfrac{1}{r} L_{2,\widetilde{a}_2} (U_2 + V) \right\} ,$$

then F_1, F_2, G, A_1 and A_2 are the functions generated by the procedure above.

In the procedure described in the statement of the theorem, the pairwise value function F_k is the value function obtained when firm k and the worker operate as a team under autarky. It is necessarily unique, because it is the value function of an optimization problem. The training function A_k is the optimal policy associated with F_k. It is unique because the maximand $\pi_{k,\widetilde{a}_k} + \frac{1}{r} L_{k,\widetilde{a}_k} F_k$ is strictly concave in $\widetilde{a}_k$. The grand value function G is the value function obtained when all three players operate as a team, with the crucial proviso that they take the A_k as exogenous when they do so. Like F_k, G is necessarily unique: it is the value function of a two-armed bandit problem.

Proof of Theorem 6. See Appendix A.4.

The final step in our analysis is to establish the existence and uniqueness of equilibrium in the dynamic game. This follows easily by combining the characterization of equilibrium provided by Theorem 6 with the fact that the functions F_k, A_k and G are all unique.

Corollary 7. *The Bellman system of the dynamic game has a unique solution.*

Proof. See Appendix A.5. ■

Remark. In the models of Bergemann and Välimäki (1996) and Felli and Harris (1996), it was shown that Markov-cautious equilibrium was efficient. Furthermore, this efficiency allowed them to solve for equilibrium by solving a three-player-team optimization problem which took the form of a bandit problem. In the current paper, Markov-cautious equilibrium is no longer efficient. However, Theorem 6 and Corollary 7 show that it is still possible to solve for equilibrium. This is achieved by replacing efficiency with constrained efficiency, and by replacing the team optimization problem with a two-step procedure. The first step of this procedure involves solving two two-player-team optimization problems to find the training allocations. The second step involves solving a further three-player-team optimization problem which again takes the form of a bandit problem.[20]

4. WAGE OFFERS AND THE CHOICE OF EMPLOYER

Our analysis of Markov-cautious equilibrium so far can be summarized as follows. If the functions F_1, F_2, G, A_1 and A_2 are constructed as in Theorem 6, then the equilibrium value functions of firm 1, firm 2 and the worker are given by the formulae $U_1 = G - F_2$, $U_2 = G - F_1$ and $V = F_1 + F_2 - G$. Furthermore, the training component of the equilibrium strategies of firm 1 and firm 2 are given directly by A_1 and A_2. In this section, we fill out this picture of equilibrium in two ways. First, we give expressions for the wage components W_1 and W_2 of the equilibrium strategies of firm 1 and firm 2. Second, we will give a partial characterization of the worker's equilibrium choice of employer.[21]

Theorem 8. *We have*

$$W_1 = F_2 - \max\left\{\left(\pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2}G\right) - \left(\pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1}G\right), 0\right\}$$

[20] One can think of the model of Felli and Harris (1996) as the special case of the current model in which training is constrained to be 0: $A_1 = A_2 = 0$. (I.e. the worker must always work.)

[21] We already have one partial characterization of the worker's equilibrium choice of employer: if $\pi_{1,A_1} + \frac{1}{r}\, L_{1,A_1}G > \pi_{2,A_2} + \frac{1}{r}\, L_{2,A_2}G$, then firm 1 is the employer; and if $\pi_{1,A_1} + \frac{1}{r}\, L_{1,A_1}G < \pi_{2,A_2} + \frac{1}{r}\, L_{2,A_2}G$, then firm 2 is the employer. This characterization is certainly of conceptual interest. But it does not give much insight into concrete questions such as: in which states is firm k the employer?

and

$$W_2 = F_1 - \max\left\{\left(\pi_{1,A_1} + \tfrac{1}{r}\,L_{1,A_1}G\right) - \left(\pi_{2,A_2} + \tfrac{1}{r}\,L_{2,A_2}G\right), 0\right\}.$$

In other words W_1 has two components, namely: (i) the pairwise value F_2 of firm 2 and the worker; less (ii) the marginal contribution of firm 2 to the joint payoff of the three players. Similarly, W_2 has two components, namely: (i) the pairwise value F_1 of firm 1 and the worker; less (ii) the marginal contribution of firm 1 to the joint payoff of the three players.

More explicitly, if $\pi_{1,A_1} + \frac{1}{r}\,L_{1,A_1}G > \pi_{2,A_2} + \frac{1}{r}\,L_{2,A_2}G$, then: (i) firm 1 is the employer; (ii) the marginal contribution of firm 2 to the joint payoff of the three players is zero, and therefore $W_1 = F_2$; and (iii) the marginal contribution of firm 1 to the joint payoff the three players is strictly positive, and therefore $W_2 < F_1$. In other words: the employer pays the worker the flow equivalent of the autarky value of the match between the worker and the non-employer; but the non-employer offers a wage that is less than the flow equivalent of the autarky value of the match between the worker and the employer. If $\pi_{1,A_1} + \frac{1}{r}\,L_{1,A_1}G < \pi_{2,A_2} + \frac{1}{r}\,L_{2,A_2}G$ then analogous remarks apply. In particular, firm 2 is the employer. If $\pi_{1,A_1} + \frac{1}{r}\,L_{1,A_1}G = \pi_{2,A_2} + \frac{1}{r}\,L_{2,A_2}G$, then nothing in our analysis so far tells us which firm will be the employer. However, we do know that the marginal contribution of both firms to the joint payoff the three players is zero, and therefore that $W_1 = F_2$ and $W_2 = F_1$. Hence, unless $F_1 = F_2$ (and we shall see below that this is hardly ever the case), the wage will change discontinuously when the worker switches employer. Moreover, the jump in her wage could be either positive or negative, depending on the direction of the switch.

Proof. See Appendix A.6. ■

At one level, Theorem 8 tells us what the equilibrium wage will be: it will be F_2 when firm 1 is the employer and F_1 when firm 2 is the employer. However, we do not yet have a concrete answer to the question as to which firm will be the employer. In order to obtain one, we begin by introducing what we shall call the retirement problem of team k. In this problem, the team consisting of firm k and the worker takes the training function A_k under autarky as exogenous. However, it has the option of ceasing production at any time, and taking the lump sum $b \in \mathbb{R}$ instead. We denote the value function for the retirement problem of team k by $R_k = R_k(m_k, b)$. We then have:

Definition 9. *The Gittins index of firm k is the function $\Gamma_k : [\,\underline{m}_k, \overline{m}_k\,] \to \mathbb{R}$ given by* *the formula*

$$\Gamma_k(m_k) = \min\{b \mid b \in \mathbb{R},\ R_k(m_k, b) = b\}.$$

In other words, $\Gamma_k(m_k)$ is the smallest lump sum b that ensures that team k is willing to retire when its productivity is m_k.

The Gittins index will allow us to make precise the idea that the employer will be the firm in which the worker's productivity is – in an appropriate sense – higher. The first step towards this goal is to establish some important properties of the Gittins index in our model.

Lemma 10. *Put $\underline{F}_k = F_k(\,\underline{m}_k\,)$ and $\overline{F}_k = F_k(\,\overline{m}_k\,)$. Then Γ_k is strictly increasing and continuous, with $\Gamma_k(\,\underline{m}_k\,) = \underline{F}_k$ and $\Gamma_k(\,\overline{m}_k\,) = \overline{F}_k$.*

Proof. See Appendix A.7. ■

The second step is to note that the Gittins index does indeed allow us to identify the equilibrium employer.

Theorem 11. *If $\Gamma_1(m_1) > \Gamma_2(m_2)$ then firm 1 is the employer, and if $\Gamma_2(m_2) > \Gamma_1(m_1)$ then firm 2 is the employer.*

Proof. The proof follows standard lines. Cf. Karatzas (1984). ■

Putting Lemma 10 and Theorem 11 together, we see that the employer will be the firm in which the worker's productivity is higher, in the sense that the Gittins index of that firm is higher. Furthermore, because the Γ_k are strictly increasing and continuous, there must be a strictly increasing and continuous switching curve in the productivity rectangle $[\,\underline{m}_1, \overline{m}_1\,] \times [\,\underline{m}_2, \overline{m}_2\,]$ such that firm 2 is the employer above the switching curve and firm 1 is the employer below the switching curve. However, neither our earlier criterion in terms of the joint payoffs $\pi_{k,A_k} + \frac{1}{r}\, L_{k,A_k} G$, nor the Gittins indices $\Gamma_k(m_k)$, give a complete answer to the question as to which firm is the employer at any given point *on* the switching curve.[22]

[22] In practice, the points of the switching curve fall into one of two sets. The time spent in the first set by any solution to the dynamics is always 0, so that the choice of employer there is irrelevant. As for the second set, the choice of employer at points in that set is uniquely determined by the requirement that the dynamics be soluble. Either way, the equilibrium dynamics are unique, even though the choice of employer is not always so.

5. Equilibrium Dynamics

Armed with the formulae for the equilibrium wage offers which were given in Theorem 8 and the Gittins index, the properties of which were set out in Theorems 10 and 11, we can now provide a detailed description of the mobility of the worker and the dynamics of wages.

It follows from Theorems 10 and 11 that firm 1 will always be the employer if $\underline{F}_1 > \overline{F}_2$ and firm 2 will always be the employer if $\underline{F}_2 > \overline{F}_1$. For in those cases one firm is uniformly more productive than the other. Leaving knife-edge cases aside, this leaves us with four main scenarios:

Scenario 1 $\underline{F}_2 < \underline{F}_1 < \overline{F}_1 < \overline{F}_2$.

Scenario 2 $\underline{F}_2 < \underline{F}_1 < \overline{F}_2 < \overline{F}_1$.

Scenario 3 $\underline{F}_1 < \underline{F}_2 < \overline{F}_2 < \overline{F}_1$.

Scenario 4 $\underline{F}_1 < \underline{F}_2 < \overline{F}_1 < \overline{F}_2$.

In Scenario 1, the Gittins interval $\left[\underline{F}_1, \overline{F}_1\right]$ of firm 1 is nested within the Gittins interval $\left[\underline{F}_2, \overline{F}_2\right]$ of firm 2. Firm 2 therefore has greater upside and greater downside than firm 1. In Scenario 2, the two Gittins intervals overlap: firm 1 has greater upside than firm 2, and firm 2 has greater downside than firm 1. Scenarios 3 and 4 are just Scenarios 1 and 2 with the roles of the two firms reversed. We shall therefore focus on Scenarios 1 and 2.

Figure 1 shows the Gittins rectangle $\left[\underline{F}_1, \overline{F}_1\right] \times \left[\underline{F}_2, \overline{F}_2\right]$ for Scenario 1. This rectangle is divided into 6 regions. In the first region (light green trapezoid), the Gittins index Γ_2 of firm 2 is higher than the Gittins index Γ_1 of firm 1 but lower than its maximum possible value $\overline{F}_2$. Hence firm 2 is the employer and the state $m = (m_1, m_2)$ moves vertically and stochastically until Γ_2 either converges to its upper bound $\overline{F}_2$ or hits its lower bound Γ_1. (The state moves vertically because m_1 only changes when the worker is employed by firm 1, and it moves stochastically because $\Sigma_{2,A_2} > 0$ throughout the trapezoid.) If Γ_2 hits Γ_1, then the worker will switch from firm 2.[23] Finally, since there is a strictly positive

[23] Notice that, while it is accurate to say that "the worker will switch from firm 2", it would not be accurate to say that "the worker will switch from firm 2 to firm 1". Indeed, if τ is the first time at which $\Gamma_2 = \Gamma_1$, then – for all $\varepsilon > 0$ – the interval $(\tau, \tau + \varepsilon)$ contains an employment spell with firm 1 and an employment spell with firm 2.

probability that the worker will switch from firm 2, we have $G > F_2$ (i.e. the grand value of all three players is greater than the pairwise value of firm 2 and the worker).[24]

In the second region (dark green horizontal line at the top of the figure), Γ_2 has reached its maximum possible value $\overline{F}_2$. Hence firm 2 is the employer and m does not change. (Here m_1 does not change because the worker is not employed by firm 1, and m_2 does not change because $\mu_{2,A_2} = \Sigma_{2,A_2} = 0$.) Finally, since the worker remains with firm 2 forever, we have $G = F_2$.

The third region (light blue triangle) is similar to the first. In this region, the Gittins index Γ_1 of firm 1 is higher than the Gittins index Γ_2 of firm 2 but lower than its maximum possible value $\overline{F}_1$. Furthermore $\Gamma_2 > \underline{F}_1$. Firm 1 is the employer, and the state moves horizontally and stochastically until Γ_1 either converges to its upper bound $\overline{F}_1$ or hits its lower bound Γ_2. If Γ_1 hits Γ_2, then the worker will switch from firm 1. Finally, since there is a strictly positive probability that the worker will switch from firm 1, we have $G > F_1$.[25]

The fourth region (dark blue vertical line on the right-hand side of the figure) is similar to the second. In this region, Γ_1 has reached its maximum possible value $\overline{F}_1$. Furthermore $\underline{F}_1 < \Gamma_2 \leq \overline{F}_1$. Firm 1 is the employer and the state does not change. Finally, since the worker remains with firm 1 forever, we have $G = F_1$.

In the fifth region (dark blue rectangle at the bottom of the figure), the Gittins index Γ_2 of firm 2 is less than or equal to the minimum possible value $\underline{F}_1$ of the Gittins index of firm 1. Firm 1 is therefore uniformly more productive than firm 2. Hence firm 1 is the employer and the state moves horizontally. Indeed, the dynamics of m_1 are exactly what they would be in the team problem for firm 1 and the worker. Finally, since the worker remains with firm 1 forever, we have $G = F_1$.

In the sixth and final region (red diagonal), $\underline{F}_1 < \Gamma_1 = \Gamma_2 < \overline{F}_1$. I.e. the Gittins indices of the two firms are equal, and lie strictly in between the minimum and maximum possible values of the Gittins index of firm 1. This is the interior of the switching line.[26] The first thing to note about this line is that the time spent on it does not contribute directly to the dynamics (not even in the sense of local time): the dynamics simply cross it.

[24] The analogue for U_1 of Lemma 13 below (which is stated for U_2) shows that $G > F_2$ in the first region.

[25] Lemma 13 below shows that $G > F_1$ in the third region.

[26] The switching line is the locus of points such that $\underline{F}_1 \leq \Gamma_1 = \Gamma_2 \leq \overline{F}_1$. The interior of the switching line is the locus of points such that $\underline{F}_1 < \Gamma_1 = \Gamma_2 < \overline{F}_1$. In what follows, it should be clear from the context whether we have in mind the switching line or its interior.

Indeed, in a typical realization of the dynamics, the worker will have spells of employment with both of the two firms. Each of these spells will consist of a (relatively) open subset of the timeline $[0, \infty)$. The union of these spells will also be open, and its complement – which is the time spent on the switching line – will be closed. The union will have full measure, and its complement will have measure 0.

The second thing to note about the switching line is that it provides a convenient benchmark for the progress of the dynamics. Specifically, if the worker is currently employed by firm 1, then $\Gamma_1(m_1) > \Gamma_2(m_2)$ and the next point to be visited on the switching line (if any) will be $(\Gamma_2(m_2), \Gamma_2(m_2))$. Similarly, if the worker is currently employed by firm 2, then $\Gamma_2(m_2) > \Gamma_1(m_1)$ and the next point to be visited on the switching line (if any) will be $(\Gamma_1(m_1), \Gamma_1(m_1))$. Hence $B(m) = \min\{\Gamma_1(m_1), \Gamma_2(m_2)\}$ provides the required benchmark. (Notice that B is non-increasing, and that it decreases only when the dynamics are on the switching line. It can be thought of as a measure of the worker's current standing in the market.) More explicitly, we can conceptualize the dynamics as consisting of progress down the switching line interspersed with excursions into the employment regions of the two firms.

Building on the information contained in Figure 1, we can also describe the dynamics of the equilibrium wage W. In the employment region of firm 1 (i.e. where $\Gamma_1 > \Gamma_2$), we have $W = W_1 = F_2$. So W will remain constant until the dynamics – which are horizontal – reach the switching line.[27] Similarly, in the employment region of firm 2 (i.e. where $\Gamma_2 > \Gamma_1$), we have $W = W_2 = F_1$. So W will remain constant until the dynamics – which are vertical – reach the switching line. It can also be shown that, on the switching line, we have $F_1 > F_2$.[28] Hypothetically speaking, then: if the worker switches from firm 1 to firm 2, then her wage jumps up; and if she switches from firm 2 to firm 1, then her wage jumps down.[29]

Figure 2 shows the Gittins rectangle $\left[\underline{F}_1, \overline{F}_1\right] \times \left[\underline{F}_2, \overline{F}_2\right]$ for Scenario 2. This rectangle is again divided into 6 regions, with only minor differences in the details of each region. There is, however, one important change: on the switching line we have $F_1 > F_2$ for

[27] By contrast, the wage W_2 offered by firm 2 will not remain constant. This is because the second component of W_2, namely the marginal contribution of firm 1 to the joint payoff of the three players, does not remain constant.

[28] This is particularly apparent at the endpoints of the line (i.e. the points $\Gamma_1 = \Gamma_2 = \underline{F}_1$ and $\Gamma_1 = \Gamma_2 = \overline{F}_1$). At these points, we have: (i) $G > F_2$ (because the worker switches from firm 2 at these points); and (ii) $G = F_1$ (because the worker remains at firm 1 forever once these points are reached).

[29] These statements are hypothetical because switches from firm 1 to firm 2, and from firm 2 to firm 1, do not in fact occur on the equilibrium path. Cf. footnote 23.

low values of the Gittins index and $F_2 > F_1$ for high values of the Gittins index.[30] In particular, there is a single point on the switching line at which $F_1 = F_2$.

A clear lesson about wages on the switching line therefore emerges: at any point on the line, one firm has a higher upside and the other is safer. The safer firm is worth more (in the sense that the pairwise value of that firm and the worker is greater), but it makes sense for the firm with the higher upside to bid the worker away from the safer firm. To do so it has to pay a higher wage.

6. INEFFICIENCY

Theorem 6 shows that Markov-cautious equilibrium has two precise, but limited, efficiency properties:

Property 1 For $k \in \{1, 2\}$, the equilibrium training function A_k is pairwise efficient. That is, it would be optimal if firm k and the worker operated as a team under autarky.

Property 2 The equilibrium choice of employer is constrained efficient. That is, it would be optimal if the three players operated as a team, provided that they took the A_k as exogenous when they did so.

The very precision of these properties makes it easy to identify their limits. Property 1 tells us that the level of training offered by a firm takes the interests of that firm and the worker, and only those interests, fully into account. In particular, the interests of the other firm are completely excluded from the decision. Property 2 tells us that, while the choice of employer takes the interests of all three players fully into account, it takes the training levels within the firm as given. Overall, then, the provision of training may be inefficient but, if so, then this will be the only inefficiency.

In this section, we introduce two special cases of our stochastic learning-by-doing model. We show that, in both of these cases, the potential inefficiency identified in the previous paragraph is present. Furthermore, in one case training is underprovided, and in the other case it is overprovided.

[30] This is easy to verify at the endpoints of the line. At the point $\Gamma_1 = \Gamma_2 = \underline{F}_1$, we have: (i) $G > F_2$ (because the worker switches from firm 2 at this point); and (ii) $G = F_1$ (because the worker remains at firm 1 forever once these points are reached). By the same token, at the point $\Gamma_1 = \Gamma_2 = \overline{F}_1$, we have: (i) $G > F_1$ (because the worker switches from firm 1 at this point); and (ii) $G = F_2$ (because the worker remains at firm 2 forever once this point is reached).

The first special case is the ALM (Accelerated Learning Model). In this model, the benefit of training is quantitative: the marginal product of the worker evolves faster, but the pattern of evolution is unchanged. In effect: when the worker works, 1 hour's worth of evolution takes place in 1 hour; whereas, when she trains, $\lambda > 1$ hours' worth of evolution take place in 1 hour. More explicitly,

$$\left(\mu_{k,\mathfrak{T}}, \Sigma_{k,\mathfrak{T}}\right) = \left(\lambda\, \mu_{k,\mathfrak{W}}, \lambda\, \Sigma_{k,\mathfrak{W}}\right).$$

The cost of training is twofold: there is the direct cost of training, captured by κ_k, and there is an opportunity cost, in that the output associated with working is lost.

The second special case is the APEM (Accelerated Productivity Enhancement Model). In this model, the benefit of training is that the expected increase in the worker's productivity when she trains is $\lambda > 1$ times what it would have been if she had worked. The variance of the increase is unchanged. More explicitly,

$$\left(\mu_{k,\mathfrak{T}}, \Sigma_{k,\mathfrak{T}}\right) = \left(\lambda\, \mu_{k,\mathfrak{W}}, \Sigma_{k,\mathfrak{W}}\right).$$

The cost of training is the same as in the ALM.

We will show that, in the ALM, training is underprovided (in the sense that the shadow value of training to the non-employer is positive). By contrast, in the APEM, training is overprovided (in the sense that the shadow value of training to the non-employer is negative). In this way we establish two things. First, there really is an inefficiency in the provision of training. Second, the sign of this inefficiency could go either way, depending on how training affects the evolution of productivity.

6.1. The Accelerated Learning Model (ALM). The equilibrium training allocation within firm 1 is based on the marginal joint payoff to firm 1 and the worker of having the worker work a little less and train a little more in firm 1. In the ALM, this marginal payoff is

$$\begin{aligned} & -\gamma_1'(1-a_1)\, m_1 - \kappa_1 + \tfrac{1}{r}\, L_{1,\mathfrak{T}} F_1 - \tfrac{1}{r}\, L_{1,\mathfrak{W}} F_1 \\ = \;& -\gamma_1'(1-a_1)\, m_1 - \kappa_1 + \tfrac{1}{r}\,(\lambda - 1)\, L_{1,\mathfrak{W}} F_1. \end{aligned}$$

In other words, a small increase in the training allocation: (i) reduces output by $\gamma_1'(1-a_1)\, m_1$; (ii) increases training costs by κ_1; (iii) generates an increase in the learning from training with discounted shadow value $\frac{1}{r}\,L_{1,\mathfrak{T}}F_1$; and (iv) generates a reduction in the learning from working with discounted shadow value $\frac{1}{r}\,L_{1,\mathfrak{W}}F_1$. On the other hand, the social value of training within firm 1 is measured by the marginal joint payoff to all three players of having the worker work a little less and train a little more in firm 1. In the ALM, this is

$$
\begin{aligned}
& -\gamma_1'(1-a_1)\,m_1 - \kappa_1 + \tfrac{1}{r}\,L_{1,\mathfrak{T}}G - \tfrac{1}{r}\,L_{1,\mathfrak{W}}G \\
= \;& -\gamma_1'(1-a_1)\,m_1 - \kappa_1 + \tfrac{1}{r}\,(\lambda-1)\,L_{1,\mathfrak{W}}G.
\end{aligned}
$$

Hence, corner solutions aside, training is underprovided in equilibrium iff the latter exceeds the former, i.e. iff $\frac{1}{r}\,L_{1,\mathfrak{W}}G \geq \frac{1}{r}\,L_{1,\mathfrak{W}}F_1$. This is equivalent to

$$\tfrac{1}{r}\,L_{1,\mathfrak{W}}U_2 \geq 0. \tag{9}$$

That is, corner solutions aside, training is underprovided iff the shadow value to firm 2 of learning in firm 1 is positive. This makes intuitive sense: the payoff of firm 2 is not taken into account when the training allocation in firm 1 is chosen, and hence there will be too little training if firm 2 places a positive value on training by firm 1.

In what follows we shall show that inequality (9) does indeed hold in the ALM. Furthermore we shall show that it holds as a strict inequality whenever firm 2 is still in contention, in the sense that: either (i) firm 2 is currently the employer (so firm 2 will employ the worker for a non-trivial length of time starting immediately); or (ii) there is a positive probability that the Gittins index of firm 1 will fall below that of firm 2 (so there is a positive probability that firm 2 will employ the worker for a non-trivial length of time in the future). More precisely, we have the following definition.

Definition 12. *Firm* 2 *is* ***still in contention*** *iff: either (i)* $\Gamma_2 \in \left(\underline{F}_1, \overline{F}_1\right]$ *and* $\Gamma_1 < \overline{F}_1$*; or (ii)* $\Gamma_2 \in \left(\overline{F}_1, \overline{F}_2\right]$.

Lemma 13. $U_2 \geq 0$*, with strict inequality iff firm* 2 *is still in contention.*

It is easy to see why the weak inequality holds. When the grand team consisting of all three players solves for its value function G (taking A_1 and A_2 as given), one of the strategies available to it is to have the worker work in firm 1 at all times. In this way it will obtain the value F_1. Hence $G \geq F_1$ and $U_2 = G - F_1 \geq 0$. As for the strict inequality, firm 2 always earns its marginal contribution to the joint payoff of the three players, and this marginal contribution is strictly positive iff $\Gamma_2 > \Gamma_1$. So we certainly expect a strict inequality whenever firm 2 is in contention.

Proof. See Appendix A.8. ■

Theorem 14. *Suppose that $\Gamma_1 \geq \Gamma_2$. Then $L_{1,\mathfrak{W}} U_2 \geq 0$, with strictly inequality iff firm 2 is still in contention.*

The intuition behind this result is as follows. Suppose that the worker is currently employed by firm 1. Then there are two possible outcomes. First, the worker may change employers in due course. Second, the worker may remain with firm 1 forever. Increasing the amount of time that the worker spends training does not change the probabilities of these two outcomes, but it does bring forward the time at which the first outcome occurs. Doing so is therefore good for firm 2, which receives its continuation payoff – which is strictly positive by Lemma 13 – sooner.

As for the implications of the theorem for training as such, there are three possibilities. At one extreme, we could have

$$\tfrac{1}{r}\,(\lambda - 1)\,L_{1,\mathfrak{W}} G \leq \gamma_1'(1)\,m_1 + \kappa_1.$$

In other words, the grand shadow value of training is less than or equal to the marginal cost of training, even if the worker works full time. In this case, there will be no training whether or not we take firm 2's preferences into account. At the other extreme, we could have

$$\tfrac{1}{r}\,(\lambda - 1)\,L_{1,\mathfrak{W}} F_1 \geq \gamma_1'(0)\,m_1 + \kappa_1.$$

In other words, the pairwise shadow value of training is greater than or equal to the marginal cost of training, even if the worker does no work. In this case there will be full-time training whether or not we take firm 2's preferences into account. Finally, we could have

$$\tfrac{1}{r}\,(\lambda - 1)\,L_{1,\mathfrak{W}} F_1 < \gamma_1'(0)\,m_1 + \kappa_1 \text{ and } \tfrac{1}{r}\,(\lambda - 1)\,L_{1,\mathfrak{W}} G > \gamma_1'(1)\,m_1 + \kappa_1.$$

In other words, we are not in either of the two extreme cases. In this case, taking account of firm 2's preferences will result in a strict increase in training (since γ_1 is strictly concave).

Proof. By construction of F_1 and A_1, we have

$$F_1 = \max_{\widetilde{a}_1} \left\{ \pi_{1,\widetilde{a}_1} + \tfrac{1}{r} L_{1,\widetilde{a}_1} F_1 \right\} = \pi_{1,A_1} + \tfrac{1}{r} L_{1,A_1} F_1.$$

Also, since $\Gamma_1 \geq \Gamma_2$, we have

$$\pi_{1,A_1} + \tfrac{1}{r} L_{1,A_1} G \geq \pi_{2,A_2} + \tfrac{1}{r} L_{2,A_2} G$$

and therefore

$$G = \max_k \left\{ \pi_{k,A_k} + \tfrac{1}{r} L_{k,A_k} G \right\} = \pi_{1,A_1} + \tfrac{1}{r} L_{1,A_1} G.$$

Hence

$$\begin{aligned} U_2 &= G - F_1 = \left(\pi_{1,A_1} + \tfrac{1}{r} L_{1,A_1} G \right) - \left(\pi_{1,A_1} + \tfrac{1}{r} L_{1,A_1} F_1 \right) \\ &= \tfrac{1}{r} L_{1,A_1} G - \tfrac{1}{r} L_{1,A_1} F_1 = \tfrac{1}{r} L_{1,A_1} U_2 = \tfrac{1}{r} \left((1 - A_1) + A_1 \lambda \right) L_{1,\mathfrak{W}} U_2. \end{aligned}$$

Hence

$$L_{1,\mathfrak{W}} U_2 = \frac{U_2}{\tfrac{1}{r} \left((1 - A_1) + A_1 \lambda \right)}.$$

Hence the sign of $L_{1,\mathfrak{W}} U_2$ is the same as that of U_2. The result therefore follows from Lemma 13. ■

Notice that the reason why firm 2 likes training is that it shortens the time that the worker spends with firm 1. In other words, firm 2 would like to increase turnover. But this means that turnover is inefficiently low.

Corollary 15. *Suppose that $\Gamma_1 \geq \Gamma_2$. Then $L_{1,\mathfrak{W}} V \leq 0$, with strictly inequality iff firm 2 is still in contention.*

In other words, the shadow value to the worker of training in firm 1 is non-positive, and it is strictly negative iff firm 2 is still in contention.

The intuition behind this result parallels the intuition behind Theorem 14. Suppose that the worker is currently employed by firm 1, that her productivity in firm 2 is m_2 and that m_1 is the point to which her productivity in form 1 would have to fall to induce her to switch from firm 1. (I.e. m_1 solves the equation $\Gamma_1(m_1) = \Gamma_2(m_2)$.) As long as she remains with firm 1, her wage remains unchanged at $F_2(m_2)$. Furthermore, at the time at which she switches from firm 1, her continuation payoff will be $V(m_1, m_2)$.

Now, $V(m_1, m_2) = F_2(m_2) - U(m_1, m_2)$; and $U(m_1, m_2) > 0$ as long as firm 2 is still in contention. Hence, noting that the discount rate is 1, and expressing the worker's continuation payoff in flow terms, we see that increasing the amount of time that she spends training brings forward the time at which her flow payoff drops from $F_2(m_2)$ to $F_2(m_2) - U(m_1, m_2)$. Doing so is therefore bad from her point of view.

Proof. We have $L_{1,\mathfrak{W}}V = L_{1,\mathfrak{W}}(F_2 - U_2) = -L_{1,\mathfrak{W}}U_2$, where the last equality follows from the fact that F_2 is independent of m_1. The result therefore follows at once from Theorem 14. ■

Remark. Theorem 14 tells us that the non-employer would like to see more training by the employer, whereas Corollary 15 tells us that the worker would like to see less. This inverse relationship is an instance of a more general feature of the stochastic learning-by-doing model: as far as any activity undertaken by the employer is concerned, the interests of the non-employer and the worker are diametrically opposed.

To summarize, firm 2 likes training when it takes the form of accelerated learning, and this implies that training is socially underprovided – and turnover is inefficiently low – in this case.

6.2. The Accelerated Productivity Enhancement Model (APEM). In the APEM, the marginal joint payoff to firm 1 and the worker of having the worker work a little less and train a little more in firm 1 is

$$\begin{aligned} & -\gamma_1'(1-a_1)\, m_1 - \kappa_1 + \tfrac{1}{r}\, L_{1,\mathfrak{T}}F_1 - \tfrac{1}{r}\, L_{1,\mathfrak{W}}F_1 \\ = \quad & -\gamma_1'(1-a_1)\, m_1 - \kappa_1 + \tfrac{1}{r}(\lambda - 1)\, \mu_{1,\mathfrak{W}}\, \tfrac{\partial F_1}{\partial m_1}. \end{aligned}$$

On the other hand, the marginal joint payoff to all three players of having the worker work a little less and train a little more in firm 1 is

$$\begin{aligned} & -\gamma_1'(1-a_1)\, m_1 - \kappa_1 + \tfrac{1}{r}\, L_{1,\mathfrak{T}}G - \tfrac{1}{r}\, L_{1,\mathfrak{W}}G \\ = \quad & -\gamma_1'(1-a_1)\, m_1 - \kappa_1 + \tfrac{1}{r}(\lambda - 1)\, \mu_{1,\mathfrak{W}}\, \tfrac{\partial G}{\partial m_1}. \end{aligned}$$

Hence, corner solutions aside, training is overprovided in equilibrium iff the former exceeds the latter, i.e. iff $\frac{1}{r}\, \mu_{1,\mathfrak{W}}\, \frac{\partial F_1}{\partial m_1} \geq \frac{1}{r}\, \mu_{1,\mathfrak{W}}\, \frac{\partial G}{\partial m_1}$. This is equivalent to

$$\tfrac{1}{r}\, \mu_{1,\mathfrak{W}}\, \tfrac{\partial U_2}{\partial m_1} \leq 0. \tag{10}$$

That is, corner solutions aside, training is overprovided iff the shadow value to firm 2 of productivity enhancement in firm 1 is negative.

Our first result implies that inequality (10) does indeed hold in the APEM. Furthermore it tells us precisely when it holds as a strict inequality.

Theorem 16. $\frac{\partial U_2}{\partial m_1} \leq 0$, *with strict inequality iff firm* 2 *is still in contention.*

The intuition for this result is relatively simple in the case in which the worker is currently employed by firm 1, which also happens to be the case that we need in order to show that training is overprovided in the APEM. In this case, raising the productivity of the worker in firm 1 increases the time before she switches employer. Indeed, there may be some states of the world in which she no longer switches at all. This is bad from the point of view of firm 2, which is now less likely to reach the point at which it receives a strictly positive continuation payoff and, if it does ever reach that point, reaches it later.[31]

Proof. See Appendix A.9. ■

Notice that the reason why firm 2 dislikes training is that it lengthens the time that the worker spends with firm 1. In other words, firm 2 dislikes the reduction in turnover associated with training. But this means that turnover is once again inefficiently low.

Corollary 17. $\frac{\partial V}{\partial m_1} \geq 0$, *with strict inequality iff firm* 2 *is still in contention.*

We give the intuition behind this result in the case in which the worker is currently employed by firm 1. As long as she remains with firm 1, her flow payoff is $F_2(m_2)$, where m_2 is her productivity in firm 2. However, when she switches from firm 1, her flow payoff effectively drops from $F_2(m_2)$ to $F_2(m_2) - U(m_1, m_2)$, where m_1 is her productivity in firm 1 at the time of the switch. Now, increasing the amount of time that she spends training delays the time at which the switch occurs. Doing so is therefore good from her point of view.[32]

[31] The general case of the theorem is more subtle. However, one observation emerges clearly from the proof: the main driver of the result is the fact that raising the productivity of the worker in firm 1 increases the wage that firm 2 must pay the worker. This is obviously bad from the point of view of firm 2. The subtlety lies in the way in which this increase in the wage paid by firm 2 gets aggregated into a reduction in the payoff of firm 2.

[32] The general case of the corollary is more subtle. The best way of obtaining some insight into this case is to construct a direct proof of the corollary. Such a proof shows that the main driver of the result is the fact that raising the productivity of the worker in firm 1 increases the wage that firm 2 must pay the worker. This is obviously good from the point of view of the worker. The subtlety lies in the way in which this increase in the wage gets aggregated into an increase in the payoff of the worker.

Proof. We have $\frac{\partial V}{\partial m_1} = \frac{\partial F_2}{\partial m_1} - \frac{\partial U_2}{\partial m_1} = -\frac{\partial U_2}{\partial m_1}$, where the last equality follows from the fact that F_2 is independent of m_1. The result therefore follows at once from Theorem 16. ■

Remark. The inverse relationship between Theorem 16, which tells us that the non-employer would like to see less training by the employer, and Corollary 17, which tells us that the worker would like to see more training by the employer, is another instance of the more general feature of the stochastic learning-by-doing model mentioned in the remark following Corollary 15 above: as far as any activity undertaken by the employer is concerned, the interests of the non-employer and the worker are diametrically opposed.

To summarize, firm 2 dislikes training when it takes the form of accelerated productivity enhancement, and this implies that training is socially overprovided – and turnover is inefficiently low – in this case.

7. CONCLUSION

There are good reasons to expect the market to provide workers with the correct incentives to invest in general training.[33] The main question in the context of general training is therefore whether workers are in a position to respond to those incentives. They may not be. For example, the most efficient time for a worker to make an investment in general human capital may be at the outset of her career, when she may not have any financial resources of her own. If so, then she will need to borrow the resources required to make the investment. Unfortunately, financial lenders may not be willing to provide these resources, if the only collateral the worker can offer is her own future labour. Moreover employers may not be willing to underwrite the investment either, because employment law may prevent them from writing the long-term contract necessary to recoup their investment. There may therefore be a case for government intervention, if only partially to reinstate the market that government itself has eliminated. For example, the government could offer a training loan to the worker, and use its powers of taxation to recoup the loan once the worker returns to productive employment. There does not, however, appear to be any case for the direct regulation of employers.

On the other hand, as the analysis of the current paper has shown, there is no reason to expect the market to provide firms and workers with the correct incentives to invest

[33] See Becker (1993).

in firm-specific training. The main question in this context is therefore whether there is a case for government intervention to rectify those incentives.

There are at least two obstacles to such intervention. First, while we do not have a formal result to this effect, a comparison between the two special cases of our stochastic learning-by-doing model suggests that training tends to be overprovided when the rate of acceleration of productivity enhancement is large relative to the rate of acceleration of employee evaluation, and underprovided when the rate of acceleration of productivity enhancement is small relative to the rate of acceleration of employee evaluation. Which of these two effects of training dominates in practice is an empirical question. Therefore, in order to determine whether firm-specific training is underprovided or overprovided, it is necessary to identify the productivity-enhancement and employee-evaluation components of training. This is a difficult problem. For one thing, the standard assumption – namely that a worker's productivity can be identified with her wage – is not correct in a context in which human capital is firm specific.[34] Moreover, identifying the mean and variance of changes in a worker's productivity is even more challenging than identifying her productivity as such.

Second, even if it is possible to identify the two components of firm-specific training, any policy intervention designed to rectify the inefficiency in the provision of such training is likely to encounter opposition from workers. Indeed, as we show in Corollary 15 above, if the employee-evaluation component of training predominates, then: although training is *already underprovided*, workers would prefer to see *less* training. They will therefore resist a policy designed to increase training. Similarly, as we show in Corollary 17 above, if the productivity-enhancement component of training predominates, then: although training is *already overprovided*, workers would prefer to see *more* training. They will therefore resist a policy designed to decrease training.

[34] Postel-Vinay and Robin (2002) explicitly address the problem of identifying the worker's productivity in a world where the worker's wage differs from her marginal productivity. In particular, they analyze and estimate a model with heterogeneous workers and firms. Workers do not accumulate any human capital during their life cycle, but progressively learn about their alternative job opportunities while on the job. Whenever a worker receives a new offer, she might either accept it or renegotiate her wage with her current employer. In equilibrium, the worker's wage is then her outside option given the sequence of offers she has received up to that point in time. The wage is therefore a lower bound to the worker's productivity in her current employment. Using the identifying restrictions imposed by their theoretical model, Postel-Vinay and Robin estimate a structural model and explicitly identify the worker's productivity within a match.

References

Becker, G. S. (1993): *Human Capital.* Third edn. Chicago: The University of Chicago Press.

Bergemann, D., and J. Välimäki (1996): "Learning and Strategic Pricing," *Econometrica*, 64, 1125–1149.

Eeckhout, J., and X. Weng (2015): "Common Value Experimentation," *Journal of Economic Theory*, 160, 317–339.

Felli, L., and C. Harris (1996): "Learning, Wage Dynamics and Firm-Specific Human Capital," *Journal of Political Economy*, 104, 838–68.

Heckman, J. (1993): "Assessing Clinton's Program on Job Training, Workfare and Education in the Workplace," Working Paper No. 4428, N.B.E.R.

Jovanovic, B. (1979): "Job Matching and the Theory of Turnover," *Journal of Political Economy*, 87, 972–90.

Karatzas, I. (1984): "Gittins Indices in the Dynamic Allocation Problem for Diffusion Processes," *Annals of Probability*, 12, 173–92.

Larsen, B., E. Pastorino, and P. J. Kehoe (2016): "Dynamic Competition in the Era of Big Data," *mimeo.*

Pastorino, E., and P. J. Kehoe (2014): "A Characterization of Dependent Armed Bandit Problems with Applications," *mimeo.*

Pissarides, C. A. (1994): "Search Unemployment with On-the-job Search," *Review of Economic Studies*, 61, 457–75.

Postel-Vinay, F., and J.-M. Robin (2002): "Equilibrium Wage Dispersion with Worker and Employer Heterogeneity," *Econometrica*, 70, 2295–2350.

Prendergast, C. (1993): "The Role of Promotion in Inducing Specific Human Capital Acquisition," *Quarterly Journal of Economics*, 108, 523–34.

Topel, R. H., and M. P. Ward (1992): "Job Mobility and the Careers of Young Men," *Quarterly Journal of Economics*, 107, 439–79.

A. Appendix

A.1. Trembles in the Constituent Game. Suppose that we are given a small $\varepsilon > 0$ and a large $C < \infty$ such that:

1. If the worker decides to work for firm k then, with probability ε, she will tremble when executing her choice and pick firm $3 - k$ instead.
2. Wages are bounded below by $-C$.

Then the set of subgame-perfect equilibria of this perturbed game converges, as $\varepsilon \downarrow 0$, to the cautious equilibrium of the original game. This approach does capture the basic idea of the simple economic argument above, namely that there is a risk that the offer of the non-employer will be accepted. It is, however, somewhat clumsy: the non-employer knows that its offer will be accepted with probability ε. It therefore has an incentive to make the very low wage offer $w_2 = -C$. Indeed, that is why we need to add the lower bound $-C$.

A.2. Probabilistic Choice in the Constituent Game. An alternative foundation can be obtained by having the worker make a probabilistic choice. Suppose that we are given a small $\varepsilon > 0$ and that – when faced with the two contracts (w_1, a_1) and (w_2, a_2) – the worker chooses firm 1 with probability proportional to

$$\exp\left(\varepsilon^{-1}\left(w_1 + \tfrac{1}{r}\, L_{1,a_1} V\right)\right)$$

and firm 2 with probability proportional to

$$\exp\left(\varepsilon^{-1}\left(w_2 + \tfrac{1}{r}\, L_{2,a_2} V\right)\right).$$

In particular: the probability of choosing firm 1 depends only on

$$\Delta = \left(w_1 + \tfrac{1}{r}\, L_{1,a_1} V\right) - \left(w_2 + \tfrac{1}{r}\, L_{2,a_2} V\right);$$

it is $\frac{1}{2}$ when $\Delta = 0$; and it converges exponentially (at the very fast rate ε^{-1}) to 1 as Δ goes to $+\infty$ and to 0 as Δ goes to $-\infty$. Then, for ε sufficiently small, there is a unique Nash equilibrium of the resulting two-firm game. Moreover this equilibrium converges, as $\varepsilon \downarrow 0$, to the cautious equilibrium of the original game. This approach captures the simple economic argument rather better than the trembling-hand approach: it is only when her payoffs from the two offers are very close that there is a significant risk that the worker will choose the lower offer.

A.3. Proof of Theorem 4. Recall from the text preceding Theorem 4 that

$$\begin{aligned} a_1^* &= \operatorname*{argmax}_{\widetilde{a}_1} \left\{ \pi_{1,\widetilde{a}_1} + \tfrac{1}{r} L_{1,\widetilde{a}_1}(U_1 + V) \right\}, \\ a_2^* &= \operatorname*{argmax}_{\widetilde{a}_2} \left\{ \pi_{2,\widetilde{a}_2} + \tfrac{1}{r} L_{2,\widetilde{a}_2}(U_2 + V) \right\} \end{aligned}$$

and

$$S = U_1 + U_2 + V.$$

Then:

Lemma 18. *The constituent game has a cautious equilibrium in which firm* 1 *is the employer iff*

$$\pi_{1,a_1^*} + \tfrac{1}{r} L_{1,a_1^*} S \geq \pi_{2,a_2^*} + \tfrac{1}{r} L_{2,a_2^*} S. \tag{11}$$

In any such equilibrium, we have

$$\begin{aligned} a_1 &= a_1^*, \\ a_2 &= a_2^*, \\ w_1 &= \pi_{2,a_2^*} + \tfrac{1}{r} L_{2,a_2^*}(U_2 + V) - \tfrac{1}{r} L_{1,a_1^*}(U_2 + V), \\ w_2 &= \pi_{2,a_2^*} + \tfrac{1}{r} L_{2,a_2^*} U_2 - \tfrac{1}{r} L_{1,a_1^*} U_2. \end{aligned}$$

Here: a_1^* is the training offer that maximizes the joint payoff of firm 1 and the worker; a_2^* is the training offer that maximizes the joint payoff of firm 2 and the worker; and S is the joint continuation-value function of all three players. The first part of the lemma then tells us that there exists a cautious equilibrium of the constituent game in which firm 1 is the employer iff, taking the offers a_1^* and a_2^* as given, the joint payoff of all three players is higher when the worker works/trains for firm 1 than it would be if the worker instead worked/trained for firm 2. Furthermore, in any such equilibrium: the training offers a_1 and a_2 coincide with a_1^* and a_2^*; w_1 is equal to the joint payoff that firm 2 and the worker would obtain if the worker instead worked/trained for firm 2, less the joint discounted shadow value to firm 2 and the worker of the learning that occurs in firm 1; and w_2 is equal to the gross payoff that firm 2 would obtain if the worker instead worked/trained for firm 2, less the discounted shadow value to firm 2 of the learning that occurs in firm 1.

Proof. Consider a cautious equilibrium of the constituent game in which firm 1 is the employer. Suppose first that firm 2's offer (w_2, a_2) is given, and consider firm 1's offer (w_1, a_1). The worker will be willing to accept this offer if and only if

$$w_1 + \tfrac{1}{r} L_{1,a_1} V \geq w_2 + \tfrac{1}{r} L_{2,a_2} V, \tag{12}$$

i.e. if and only if the value that she places on firm 1's offer is at least as high as the value that she places on firm 2's offer. In that case, the payoff to firm 1 is

$$\pi_{1,a_1} - w_1 + \tfrac{1}{r} L_{1,a_1} U_1. \tag{13}$$

Now, at the very minimum, firm 1's offer must maximize her payoff among all those offers that will be accepted by the worker. Hence, holding a_1 fixed, w_1 must maximize (13) subject to the constraint (12). This yields

$$w_1 = w_2 + \tfrac{1}{r} L_{2,a_2} V - \tfrac{1}{r} L_{1,a_1} V.$$

Furthermore, with this choice of w_1, (13) becomes

$$\pi_{1,a_1} + \tfrac{1}{r} L_{1,a_1}(U_1 + V) - w_2 - \tfrac{1}{r} L_{2,a_2} V.$$

This in turn is maximized by choosing $a_1 = a_1^*$, where

$$a_1^* = \underset{a_1}{\operatorname{argmax}} \left\{ \pi_{1,a_1} + \tfrac{1}{r} L_{1,a_1}(U_1 + V) \right\}. \tag{14}$$

With this choice of a_1, our expression for w_1 becomes

$$w_1 = w_2 + \tfrac{1}{r} L_{2,a_2} V - \tfrac{1}{r} L_{1,a_1^*} V. \tag{15}$$

Finally, firm 1's best winning offer must be at least as good as making a losing offer. Hence

$$\pi_{1,a_1^*} + \tfrac{1}{r} L_{1,a_1^*}(U_1 + V) - w_2 - \tfrac{1}{r} L_{2,a_2} V \geq \tfrac{1}{r} L_{2,a_2} U_1. \tag{16}$$

Suppose second that firm 1's offer (w_1, a_1) is given, and consider firm 2's offer (w_2, a_2). As in the previous paragraph, we see that firm 2's best winning offer (w_2, a_2) is obtained by choosing $a_2 = a_2^*$, where

$$a_2^* = \operatorname*{argmax}_{a_2} \left\{ \pi_{2,a_2} + \tfrac{1}{r} \, L_{2,a_2}(U_2 + V) \right\},$$

and by putting

$$w_2 = w_1 + \tfrac{1}{r} \, L_{1,a_1} V - \tfrac{1}{r} \, L_{2,a_2^*} V.$$

This yields a payoff of

$$\pi_{2,a_2^*} + \tfrac{1}{r} \, L_{2,a_2^*}(U_2 + V) - w_1 - \tfrac{1}{r} \, L_{1,a_1} V.$$

However, this best winning offer must be no better for firm 2 than losing. Hence

$$\pi_{2,a_2^*} + \tfrac{1}{r} \, L_{2,a_2^*}(U_2 + V) - w_1 - \tfrac{1}{r} \, L_{1,a_1} V \leq \tfrac{1}{r} \, L_{1,a_1} U_2. \tag{17}$$

Suppose third that (w_1, a_1) and (w_2, a_2) are the cautious-equilibrium offers of firms 1 and 2. Since firm 2 would be willing to go through with its offer if it were accepted, we have

$$\pi_{2,a_2} - w_2 + \tfrac{1}{r} \, L_{2,a_2} U_2 \geq \tfrac{1}{r} \, L_{1,a_1^*} U_2. \tag{18}$$

Hence

$$\pi_{2,a_2} + \tfrac{1}{r} \, L_{2,a_2} U_2 - \tfrac{1}{r} \, L_{1,a_1^*} U_2 \geq w_2$$

(on rearranging (18))

$$\geq \pi_{2,a_2^*} + \tfrac{1}{r} \, L_{2,a_2^*}(U_2 + V) - \tfrac{1}{r} \, L_{1,a_1^*} U_2 - \tfrac{1}{r} \, L_{2,a_2} V$$

(using (15) and (14) to substitute for w_1 and a_1 in (17) and rearranging)

$$\geq \pi_{2,a_2} + \tfrac{1}{r} \, L_{2,a_2}(U_2 + V) - \tfrac{1}{r} \, L_{1,a_1^*} U_2 - \tfrac{1}{r} \, L_{2,a_2} V$$

(by definition of a_2^*)

$$= \pi_{2,a_2} + \tfrac{1}{r} \, L_{2,a_2} U_2 - \tfrac{1}{r} \, L_{1,a_1^*} U_2$$

(on simplifying).

In other words, we have obtained a chain of inequalities, the beginning and end of which are identical. It follows that all the inequalities must in fact hold as equalities. In particular, we have

$$w_2 = \pi_{2,a_2} + \tfrac{1}{r} L_{2,a_2} U_2 - \tfrac{1}{r} L_{1,a_1^*} U_2 \tag{19}$$

and

$$\pi_{2,a_2} + \tfrac{1}{r} L_{2,a_2}(U_2 + V) = \pi_{2,a_2^*} + \tfrac{1}{r} L_{2,a_2^*}(U_2 + V) \tag{20}$$

(from the first and third inequalities respectively). Now, equality (20) implies that

$$a_2 = a_2^*.$$

Substituting for a_2 in equality (19), we therefore obtain an expression for w_2, namely

$$w_2 = \pi_{2,a_2^*} + \tfrac{1}{r} L_{2,a_2^*} U_2 - \tfrac{1}{r} L_{1,a_1^*} U_2.$$

By the same token, substituting for a_2 and w_2 in equality (15), we obtain an expression for w_1, namely

$$w_1 = \pi_{2,a_2^*} + \tfrac{1}{r} L_{2,a_2^*}(U_2 + V) - \tfrac{1}{r} L_{1,a_1^*}(U_2 + V).$$

Finally, substituting for a_2 and w_2 in (16) and rearranging, we obtain (11).

It remains only to show that, if (11) holds, then there exists a cautious equilibrium in which firm 1 is the employer. But it is easy to construct such an equilibrium using the formulae above, so we omit the details. ■

In the light of Lemma 18, we see that the cautious-equilibrium training offers a_1 and a_2 are unique. This is because they are given by the same formula irrespective of which firm is the employer. However, it is not immediately clear whether the cautious-equilibrium wage offers w_1 and w_2 are unique. This is because the formulae for w_1 and w_2 change when the employer changes. Similarly, it is not immediately clear whether the payoffs of the three players are unique. The next Lemma settles these questions in the affirmative.

Lemma 19. *Let a_1^*, a_2^* and S be defined as in Lemma 18. Put*

$$\begin{aligned}
w_{1,1}^* &= \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*}(U_2 + V), \\
w_{1,2}^* &= \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*}(U_2 + V), \\
w_{2,1}^* &= \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S - \tfrac{1}{r}\, L_{1,a_1^*} U_2 - \tfrac{1}{r}\, L_{2,a_2^*}(U_1 + V), \\
w_{2,2}^* &= \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S - \tfrac{1}{r}\, L_{1,a_1^*} U_2 - \tfrac{1}{r}\, L_{2,a_2^*}(U_1 + V).
\end{aligned}$$

Then, in any cautious equilibrium of the constituent game, we have:

$$\begin{aligned}
w_1 &= \min\left\{w_{1,1}^*, w_{1,2}^*\right\}, \\
w_2 &= \min\left\{w_{2,1}^*, w_{2,2}^*\right\}.
\end{aligned}$$

Furthermore, the payoffs of firm 1*, firm* 2 *and the worker can be written*

$$\begin{aligned}
&\max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right),\ 0\right\} + \tfrac{1}{r}\, L_{2,a_2^*} U_1, \\
&\max\left\{\left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right) - \left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right),\ 0\right\} + \tfrac{1}{r}\, L_{1,a_1^*} U_2, \\
&\min\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right\} - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*} U_2.
\end{aligned}$$

In other words, there are two candidates for w_1, namely $w_{1,1}^*$ and $w_{1,2}^*$. The first of these, namely $w_{1,1}^*$, is the wage offered by firm 1 in a cautious equilibrium in which firm 1 is the employer. It is made up of three components: the joint payoff $\pi_{2,a_2^*} + \frac{1}{r}\, L_{2,a_2^*} S$ to the three players if firm 2 is the employer; less the discounted shadow value $\frac{1}{r}\, L_{2,a_2^*} U_1$ to firm 1 of the learning that takes place in firm 2 if firm 2 is the employer; less the discounted joint shadow value $\frac{1}{r}\, L_{1,a_1^*}(U_2 + V)$ to firm 2 and the worker of the learning that takes place in firm 1 if firm 1 is the employer. Similarly, $w_{1,2}^*$ is the wage offered by firm 1 in a cautious equilibrium in which firm 2 is the employer. Like $w_{1,1}^*$, it is made up of three components. The first of these is the joint payoff $\pi_{1,a_1^*} + \frac{1}{r}\, L_{1,a_1^*} S$ to the three players if firm 1 is the employer. The remaining two are the same as for $w_{1,1}^*$. Analogous remarks apply to w_2, $w_{2,2}^*$ and $w_{2,1}^*$.

Furthermore, the cautious-equilibrium payoff of firm 1 has two parts: (i) its outside option $\frac{1}{r}\, L_{2,a_2^*} U_1$, which is the discounted shadow value to it of the learning that will occur if the worker is employed by firm 2; and (ii) the difference between the joint payoff of all three players when firm 1 is the employer and the joint payoff of all three players when firm 2 is the employer. In other words, firm 1 gets its outside option plus its net contribution to the grand payoff. Analogous remarks apply to the cautious-equilibrium payoff of firm 2. As for the worker, there are three components to her cautious-equilibrium payoff: (i) the minimum of the joint payoff of all three players when firm 1 is the employer and the joint payoff of all three players when firm 2 is the employer; (ii) minus the discounted shadow value to firm 1 of the learning that will occur if the worker is employed by firm 2; and (iii) minus the discounted shadow value to firm 2 of the learning that will occur if the worker is employed by firm 1. In other words, she gets as much of the grand payoff as is consistent with wage competition between the two firms, less the sum of the outside options of the two firms.

Proof. Note first that

$$\begin{aligned} w^*_{1,1} &= \pi_{2,a^*_2} + \tfrac{1}{r}\, L_{2,a^*_2} S - \tfrac{1}{r}\, L_{2,a^*_2} U_1 - \tfrac{1}{r}\, L_{1,a^*_1}(U_2 + V) \\ &= \pi_{2,a^*_2} + \tfrac{1}{r}\, L_{2,a^*_2}(U_2 + V) - \tfrac{1}{r}\, L_{1,a^*_1}(U_2 + V). \end{aligned}$$

I.e. $w^*_{1,1}$ is the wage offered by firm 1 in any cautious equilibrium in which it is the employer. (Cf. the formula for w_1 in the statement of Lemma 18.) Similarly,

$$\begin{aligned} w^*_{1,2} &= \pi_{1,a^*_1} + \tfrac{1}{r}\, L_{1,a^*_1} S - \tfrac{1}{r}\, L_{1,a^*_1}(U_2 + V) - \tfrac{1}{r}\, L_{2,a^*_2} U_1 \\ &= \pi_{1,a^*_1} + \tfrac{1}{r}\, L_{1,a^*_1} U_1 - \tfrac{1}{r}\, L_{2,a^*_2} U_1. \end{aligned}$$

I.e. $w^*_{1,2}$ is the wage offered by firm 1 in any cautious equilibrium in which firm 2 is the employer. (Cf. the formula for w_2 in the statement of Lemma 18.) Furthermore it follows directly from the formulae for $w^*_{1,1}$ and $w^*_{1,2}$ that: $w^*_{1,1} \leq w^*_{1,2}$ iff

$$\pi_{1,a^*_1} + \tfrac{1}{r}\, L_{1,a^*_1} S \geq \pi_{2,a^*_2} + \tfrac{1}{r}\, L_{2,a^*_2} S,$$

i.e. (11) holds; and $w^*_{1,2} \leq w^*_{1,1}$ iff

$$\pi_{2,a^*_2} + \tfrac{1}{r}\, L_{2,a^*_2} S \geq \pi_{1,a^*_1} + \tfrac{1}{r}\, L_{1,a^*_1} S, \tag{21}$$

i.e. the analogue of (11) for firm 2 holds. Hence $w_1 = \min\left\{w^*_{1,1}, w^*_{1,2}\right\}$, as required. Analogous reasoning applies to $w^*_{2,2}$, $w^*_{2,1}$ and w_2.

Next, it follows from the proof of Lemma 18 that the cautious-equilibrium payoff of firm 1 is the maximum of the payoff

$$\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}(U_1 + V) - w_2 - \tfrac{1}{r}\, L_{2,a_2^*}V$$

from its best winning offer and the payoff

$$\tfrac{1}{r}\, L_{2,a_2^*}U_1$$

from a losing offer. Bearing in mind that

$$\begin{aligned} w_2 &= \min\left\{w_{2,1}^*, w_{2,2}^*\right\} \\ &= \min\Big\{\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S - \tfrac{1}{r}\, L_{2,a_2^*}(U_1+V) - \tfrac{1}{r}\, L_{1,a_1^*}U_2, \\ &\quad \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S - \tfrac{1}{r}\, L_{2,a_2^*}(U_1+V) - \tfrac{1}{r}\, L_{1,a_1^*}U_2\Big\} \\ &= \min\left\{\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S,\ \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right\} - \tfrac{1}{r}\, L_{2,a_2^*}(U_1+V) - \tfrac{1}{r}\, L_{1,a_1^*}U_2, \end{aligned}$$

it follows that the payoff of firm 1 is

$$\begin{aligned} &\max\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}(U_1+V) - w_2 - \tfrac{1}{r}\, L_{2,a_2^*}V,\ \tfrac{1}{r}\, L_{2,a_2^*}U_1\right\} \\ =\ &\max\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}(U_1+V) - w_2 - \tfrac{1}{r}\, L_{2,a_2^*}(U_1+V),\ 0\right\} + \tfrac{1}{r}\, L_{2,a_2^*}U_1 \\ =\ &\max\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S - \min\left\{\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S,\ \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right\},\ 0\right\} + \tfrac{1}{r}\, L_{2,a_2^*}U_1 \\ =\ &\max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right),\ 0\right\} + \tfrac{1}{r}\, L_{2,a_2^*}U_1. \end{aligned}$$

Similarly, the payoff of firm 2 is

$$\max\left\{\left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right) - \left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right),\ 0\right\} + \tfrac{1}{r}\, L_{1,a_1^*}U_2.$$

Finally, the payoff of the worker is $w_{1,1}^* + \frac{1}{r}\, L_{1,a_1^*}V$ when firm 1 is the employer and $w_{2,2}^* + \frac{1}{r}\, L_{2,a_2^*}V$ when firm 2 is the employer. But

$$\begin{aligned} w_{1,1}^* + \tfrac{1}{r}\, L_{1,a_1^*}V &= \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}(U_2+V) - \tfrac{1}{r}\, L_{1,a_1^*}U_2 \\ &= \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S - \tfrac{1}{r}\, L_{2,a_2^*}U_1 - \tfrac{1}{r}\, L_{1,a_1^*}U_2 \end{aligned}$$

and, by the same token,

$$w^*_{2,2} + \tfrac{1}{r}\, L_{2,a_2^*} V = \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*} U_2.$$

Hence, bearing in mind (11) and (21), the cautious-equilibrium payoff of the worker is

$$\min\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right\} - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*} U_2.$$

■

Theorem 4 now follows from the formulae for the payoffs of firm 1, firm 2 and the worker obtained in Lemma 19.

A.4. Proof of Theorem 6. Combining Definition 5 with Theorem 4, we see that the Bellman system of the dynamic game can be written

$$\begin{aligned}
U_1 &= \tfrac{1}{r}\, L_{2,a_2^*} U_1 + \max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right),\ 0\right\}, && (22)\\
U_2 &= \tfrac{1}{r}\, L_{1,a_1^*} U_2 + \max\left\{\left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right) - \left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right),\ 0\right\}, && (23)\\
V &= \min\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right\} - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*} U_2, && (24)
\end{aligned}$$

where

$$\begin{aligned}
a_1^* &= \underset{\widetilde{a}_1}{\operatorname{argmax}} \left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1}(U_1 + V)\right\},\\
a_2^* &= \underset{\widetilde{a}_2}{\operatorname{argmax}} \left\{\pi_{2,\widetilde{a}_2} + \tfrac{1}{r}\, L_{2,\widetilde{a}_2}(U_2 + V)\right\}
\end{aligned}$$

and $S = U_1 + U_2 + V$.

We begin by showing that, if F_1, F_2, G, A_1 and A_2 are constructed using the procedure given in the statement of the Theorem, and if $U_1 = G - F_2$, $U_2 = G - F_1$ and $V = F_1 + F_2 - G$, then U_1, U_2 and V satisfy (22-24). The first step is to show that the a_1^* constructed from U_1, U_2 and V is equal to A_1. We have

$$\begin{aligned}
a_1^* &= \underset{\widetilde{a}_1}{\operatorname{argmax}} \left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1}(U_1 + V)\right\}\\
&= \underset{\widetilde{a}_1}{\operatorname{argmax}} \left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1} F_1\right\}
\end{aligned}$$

(since $U_1 + V = (G - F_2) + (F_1 + F_2 - G) = F_1$)

$$= A_1$$

(from Step 2 of the procedure). By the same token, $a_2^* = A_2$.

The second step is to show that (22) is satisfied. We have

$$\max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right),\ 0\right\} + \tfrac{1}{r}\, L_{2,a_2^*}U_1$$

(which is the RHS of (22))

$$= \max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}G\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}G\right),\ 0\right\} + \tfrac{1}{r}\, L_{2,a_2^*}(G - F_2)$$

(since $S = U_1 + U_2 + V = (G - F_2) + (G - F_1) + (F_1 + F_2 - G) = G$ and $U_1 = G - F_2$)

$$= \max\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}G,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}G\right\} - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}F_2\right)$$

(rearranging)

$$= \max\left\{\pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1}G,\ \pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2}G\right\} - \left(\pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2}F_2\right)$$

(since $a_i^* = A_i$)

$$= G - F_2$$

(by Step 3 of the procedure in the case of the first term and by Steps 1 and 2 of the procedure in the case of the second term)

$$= U_1$$

(which is the LHS of (22)). By the same token, (23) is satisfied.

The third and final step is to show that (24) is satisfied. We have

$$\begin{aligned}
& \min\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right\} - \tfrac{1}{r}\, L_{2,a_2^*}U_1 - \tfrac{1}{r}\, L_{1,a_1^*}U_2 \\
= \ & \max\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right\} \\
& - \max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right), 0\right\} \\
& - \max\left\{\left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right) - \left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right), 0\right\} - \tfrac{1}{r}\, L_{2,a_2^*}U_1 - \tfrac{1}{r}\, L_{1,a_1^*}U_2
\end{aligned}$$

(using the formula $\min\{x,y\} = \max\{x,y\} - \max\{x-y,0\} - \max\{y-x,0\}$)

$$\begin{aligned} = \quad & \max\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right\} \\ & - \left(U_1 - \tfrac{1}{r}\, L_{2,a_2^*} U_1\right) - \left(U_2 - \tfrac{1}{r}\, L_{1,a_1^*} U_2\right) \\ & - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*} U_2 \end{aligned}$$

(using (22) and (23))

$$= S - U_1 - U_2$$

(using the fact that $\max\left\{\pi_{1,a_1^*} + \frac{1}{r}\, L_{1,a_1^*} S,\ \pi_{2,a_2^*} + \frac{1}{r}\, L_{2,a_2^*} S\right\} = S$, which can easily be derived using ideas from the second step, and rearranging)

$$= V$$

(by definition of S).

We now show that, if U_1, U_2 and V satisfy (22-24), and if we put $F_1 = U_1 + V$, $F_2 = U_2 + V$, $G = U_1 + U_2 + V$,

$$A_1 = \operatorname*{argmax}_{\widetilde{a}_1} \left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1} (U_1 + V)\right\}$$

and

$$A_2 = \operatorname*{argmax}_{\widetilde{a}_2} \left\{\pi_{2,\widetilde{a}_2} + \tfrac{1}{r}\, L_{2,\widetilde{a}_2} (U_2 + V)\right\},$$

then F_1, F_2, G, A_1 and A_2 are the functions generated by the procedure.

The first step is to show that F_1 solves the Bellman equation in Step 1 of the procedure. We have

$$F_1 = U_1 + V$$

(by construction of F_1)

$$\begin{aligned} = \quad & \max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right),\ 0\right\} \\ & + \min\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*} S\right\} \\ & - \tfrac{1}{r}\, L_{1,a_1^*} U_2 \end{aligned}$$

(by (22) and (23))

$$= \left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S\right) - \tfrac{1}{r}\, L_{1,a_1^*} U_2$$

(using the formula $\max\{x-y,0\}+\min\{x,y\}=x$)

$$= \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}(U_1+V)$$

(rearranging)

$$= \max_{\widetilde{a}_1}\left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1}(U_1+V)\right\}$$

(by definition of a_1^*)

$$= \max_{\widetilde{a}_1}\left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1}F_1\right\}$$

(by construction of F_1). By the same token, F_2 solves the Bellman equation in Step 1 of the procedure.

The second step is to show that $A_1 = a_1^*$. We have

$$A_1 = \operatorname*{argmax}_{\widetilde{a}_1}\left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1}F_1\right\}$$

(as required in Step 2 of the procedure)

$$= \operatorname*{argmax}_{\widetilde{a}_1}\left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1}(U_1+V)\right\}$$

(by construction of F_1)

$$= a_1^*$$

(by definition of a_1^*). By the same token, $A_2 = a_2^*$.

The third and final step is to show that G solves the Bellman equation in Step 3 of the procedure. We have

$$G = U_1 + U_2 + V$$

(by construction of G)

$$\begin{aligned} = \quad & \max\left\{\left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right) - \left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right),\ 0\right\} \\ & + \max\left\{\left(\pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right) - \left(\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S\right),\ 0\right\} \\ & + \min\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right\} \end{aligned}$$

(using all three of the equations of the Bellman system)

$$= \max\left\{\pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*}S,\ \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S\right\}$$

(using the formula $\max\{x-y,0\} + \max\{y-x,0\} + \min\{x,y\} = \max\{x,y\}$)

$$= \max\left\{\pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1}G,\ \pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2}G\right\}$$

(because $A_i = a_i^*$ and $S = G$).

A.5. Proof of Corollary 7. In order to establish existence, we use the procedure in the statement of the theorem to construct functions F_1, F_2, G, A_1 and A_2. We then put $U_1 = G - F_2$, $U_2 = G - F_1$ and $V = F_1 + F_2 - G$. The latter satisfy the Bellman system of the dynamic game, which gives us the required existence. In order to establish uniqueness, we begin from any solution U_1, U_2 and V of the Bellman system of the dynamic game. We then define functions F_1, F_2, G, A_1 and A_2 using the formulae $F_1 = U_1 + V$, $F_2 = U_2 + V$, $G = U_1 + U_2 + V$,

$$A_1 = \operatorname*{argmax}_{\widetilde{a}_1}\left\{\pi_{1,\widetilde{a}_1} + \tfrac{1}{r}\, L_{1,\widetilde{a}_1}(U_1 + V)\right\}$$

and

$$A_2 = \operatorname*{argmax}_{\widetilde{a}_2}\left\{\pi_{2,\widetilde{a}_2} + \tfrac{1}{r}\, L_{2,\widetilde{a}_2}(U_2 + V)\right\},$$

Since these functions are the functions produced by the procedure, they must be unique. But then U_1, U_2 and V must also be unique, since they can be expressed in terms of F_1, F_2 and G using the formulae $U_1 = G - F_2$, $U_2 = G - F_1$ and $V = F_1 + F_2 - G$.

A.6. Proof of Theorem 8. The proof of Theorem 6 shows that $a_1^* = A_1$ and $a_2^* = A_2$. As for $w_{1,1}^*$, we have

$$w_{1,1}^* = \pi_{2,a_2^*} + \tfrac{1}{r}\, L_{2,a_2^*}S - \tfrac{1}{r}\, L_{2,a_2^*}U_1 - \tfrac{1}{r}\, L_{1,a_1^*}(U_2 + V)$$

(as in the statement of Lemma 19)

$$= \pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2}G - \tfrac{1}{r}\, L_{2,A_2}(G - F_2) - \tfrac{1}{r}\, L_{1,A_1}F_2$$

(substituting for S, U_1, U_2, V, a_1^* and a_2^*)

$$= \pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2} F_2 - \tfrac{1}{r}\, L_{1,A_1} F_2$$

(rearranging)

$$= F_2 - \tfrac{1}{r}\, L_{1,A_1} F_2$$

(since F_2 satisfies (6) for $k = 2$)

$$= F_2$$

(since the pairwise value function F_2 necessarily depends only on m_2, whereas L_{1,A_1} involves only derivatives w.r.t. m_1). Similarly,

$$\begin{aligned} w_{1,2}^* &= \pi_{1,a_1^*} + \tfrac{1}{r}\, L_{1,a_1^*} S - \tfrac{1}{r}\, L_{2,a_2^*} U_1 - \tfrac{1}{r}\, L_{1,a_1^*}(U_2 + V) \\ &= \pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1} G - \tfrac{1}{r}\, L_{2,A_2}(G - F_2) - \tfrac{1}{r}\, L_{1,A_1} F_2 \\ &= \pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2} F_2 - \tfrac{1}{r}\, L_{1,A_1} F_2 + \left(\pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1} G\right) - \left(\pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2} G\right) \\ &= F_2 + \left(\pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1} G\right) - \left(\pi_{2,A_2} + \tfrac{1}{r}\, L_{2,A_2} G\right). \end{aligned}$$

The arguments for firm 2 are analogous.

A.7. Proof of Theorem 10. We begin by introducing three functions. For all $\delta \in [\,\underline{m}_1, \overline{m}_1\,)$, the function $Q_0 = Q_0(m_1; \delta)$ takes the value 0 on $[\,\underline{m}_1, \delta\,]$ and solves the differential equation

$$Q_0 = \pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1} Q_0$$

on $(\,\delta, \overline{m}_1\,]$. For all $\delta \in [\,\underline{m}_1, \overline{m}_1\,)$, the function $Q_1 = Q_1(m_1; \delta)$ takes the value 1 on $[\,\underline{m}_1, \delta\,]$ and solves the differential equation

$$Q_1 = \tfrac{1}{r}\, L_{1,A_1} Q_1$$

on $(\,\delta, \overline{m}_1\,]$. Finally, the function $Q = Q(m_1; \delta, b)$ is given by the formula

$$Q = Q_0 + b\, Q_1.$$

The main idea of the proof is then to show that there is a unique $b = B(\delta)$ such that $Q'(\delta+; \delta, b) = Q_0'(\delta+; \delta) + b\, Q_1'(\delta+; \delta) = 0$.

To see that there is such a b, note first that $Q_1 > 0$ on $[\,\delta, \overline{m}_1\,)$ (with $Q_1 = 0$ at $\overline{m}_1$), that $Q_1' < 0$ on $(\,\delta, \overline{m}_1\,)$, and that $Q_1'' > 0$ on $(\,\delta, \overline{m}_1\,)$. In particular,

$$Q_1'(\delta+;\delta) < 0.$$

Second, we have

$$Q(m_1;\delta,F_1(\delta)) = \left\{ \begin{array}{cl} F_1(\delta) & \text{for all } m_1 \in [\,\underline{m}_1, \delta\,] \\ F_1 & \text{for all } m_1 \in (\,\delta, \overline{m}_1\,] \end{array} \right\}. \tag{25}$$

In particular,

$$F_1'(\delta) = Q'(\delta+;\delta,F_1(\delta)) = Q_0'(\delta+;\delta) + F_1(\delta)\,Q_1'(\delta+;\delta);$$

and therefore

$$Q_0'(\delta+;\delta) = F_1'(\delta) - F_1(\delta)\,Q_1'(\delta+;\delta) > 0.$$

Hence: (i) $Q'(\delta+;\delta,0) > 0$; and (ii) $Q'(\delta+;\delta,b)$ is strictly decreasing in b for $b \in (0, F_1(\overline{m}_1))$. It can also be shown that $Q'(\delta+;\delta,F_1(\overline{m}_1)) < 0$. Hence there does indeed exist a unique $b \in (0, F_1(\overline{m}_1))$ such that $Q'(\delta+;\delta,b) = 0$, namely

$$b = F_1(\delta) - \frac{F_1'(\delta)}{Q_1'(\delta+;\delta)}.$$

This b depends continuously on $\delta \in (\,\underline{m}_1, \overline{m}_1\,)$, since $F_1'(\delta)$ and $Q_1'(\delta+;\delta)$ both do.

Next, in view of equation (25), we have

$$Q_0(m_1;\delta) = F_1 - F_1(\delta)\,Q_1(m_1;\delta)$$

for all $m_1 \in [\,\delta, \overline{m}_1\,]$. Hence

$$Q(m_1;\delta,b) = F_1 + (b - F_1(\delta))\,Q_1(m_1;\delta)$$

for all $m_1 \in [\,\delta, \overline{m}_1\,]$. Now, by construction of $B(\delta)$, we have $Q'(\delta+;\delta,B(\delta)) = 0$. Moreover: $B(\delta) > F_1(\delta)$ (since $Q'(\delta+;\delta,F_1(\delta)) > 0$); $Q_1'' > 0$ on $(\,\delta, \overline{m}_1\,)$; and F_1 is convex on the whole of $[\,\underline{m}_1, \overline{m}_1\,]$. Hence $Q''(m_1;\delta,B(\delta)) > 0$ for all $m_1 \in (\,\delta, \overline{m}_1\,)$. It follows that $Q(\,\cdot\,;\delta,B(\delta)) = R_1(\,\cdot\,;B(\delta))$. That is: $Q(\,\cdot\,;\delta,B(\delta))$ is the value function of the retirement problem with lump sum $B(\delta)$; and δ is the cutoff in this retirement problem.

Finally putting $\Delta_1 = B^{-1}$, we see that $R_1(m_1, b) = b$ iff $m_1 \leq \Delta_1(b)$. Hence

$$\begin{aligned} \Gamma_1(m_1) &= \min\{b \mid b \in \mathbb{R},\ m_1 \leq \Delta_1(b)\} \\ &= \min\{b \mid b \in \mathbb{R},\ B(m_1) \leq b\} \\ &= B(m_1). \end{aligned}$$

This completes the proof.

A.8. Proof of Lemma 13. By construction of G, we have

$$G = \max_k \left\{\pi_{k,A_k} + \tfrac{1}{r}\, L_{k,A_k} G\right\} \geq \pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1} G,$$

with strict inequality iff $\Gamma_2 > \Gamma_1$. Moreover, by construction of F_1,

$$F_1 = \pi_{1,A_1} + \tfrac{1}{r}\, L_{1,A_1} F_1.$$

Hence, subtracting the second relation from the first, we obtain

$$G - F_1 \geq \tfrac{1}{r}\, L_{1,A_1}(G - F_1)$$

or

$$U_2 \geq \tfrac{1}{r}\, L_{1,A_1} U_2,$$

with strict inequality iff $\Gamma_2 > \Gamma_1$. More explicitly, putting

$$g = U_2 - \tfrac{1}{r}\, L_{1,A_1} U_2,$$

we have

$$\tfrac{1}{r}\, L_{1,A_1} U_2 - U_2 + g = 0,$$

where $g > 0$ iff $\Gamma_2 > \Gamma_1$.

Now, the operator L_{1,A_1} describes the horizontal evolution of m_1 when firm 1 and the worker operate as a team. Furthermore, holding m_2 fixed, the locus of points m_1 for which $\Gamma_2 > \Gamma_1$ is an interval of the form $[\,\underline{m}_1, \overline{m}_1\,] \cap (-\infty, c)$. Hence, continuing to hold m_2 fixed, $U_2 > 0$ iff: either (i) $c > \overline{m}_1$ (so that $g > 0$ on the whole of $[\,\underline{m}_1, \overline{m}_1\,]$); or (ii) $c \in (\,\underline{m}_1, \overline{m}_1\,)$ (so that $g > 0$ on a non-trivial subset of $[\,\underline{m}_1, \overline{m}_1\,]$) and $m_1 < \overline{m}_1$ (so that there is a strictly positive probability that the dynamics of the team problem will spend a strictly positive amount of time in the interval $[\,\underline{m}_1, c\,)$ where $g > 0$). In a word, $U_2 > 0$ iff firm 2 is still in contention.

A.9. Proof of Theorem 16. Let $\overline{\pi}$, $\overline{\mu}$, $\overline{\Sigma}$ be defined by the formulae $\overline{\pi}(k,m) = \pi_{k,A_k(m_k)}(m_k)$, $\overline{\mu}(k,m) = \mu_{k,A_k(m_k)}(m_k)$ and $\overline{\Sigma}(k,m) = \Sigma_{k,A_k(m_k)}(m_k)$, and let the operator $\overline{L}_{k,m}$ be given by the formula

$$\overline{L}_{k,m}H(m) = \overline{\mu}(k,m)\,\tfrac{\partial H}{\partial m_k}(m) + \tfrac{1}{2}\,\overline{\Sigma}(k,m)\,\tfrac{\partial^2 H}{\partial m_k^2}(m).$$

Then we have

$$U_2 = \max\Big\{\big(\overline{\pi}(2,m) + \tfrac{1}{r}\,\overline{L}_{2,m}G\big) - \big(\overline{\pi}(1,m) + \tfrac{1}{r}\,\overline{L}_{1,m}G\big)\,, 0\Big\} + \tfrac{1}{r}\,\overline{L}_{1,m}U_2$$

(by the second equation of the Bellman system of the dynamic game, namely (23))

$$= \max\Big\{\big(\overline{\pi}(2,m) + \tfrac{1}{r}\,\overline{L}_{2,m}(U_2 + F_1)\big) - \big(\overline{\pi}(1,m) + \tfrac{1}{r}\,\overline{L}_{1,m}F_1\big)\,, \tfrac{1}{r}\,\overline{L}_{1,m}U_2\Big\}$$

(taking the term $\frac{1}{r}\,\overline{L}_{1,m}U_2$ inside the maximum and noting that $G = U_2 + F_1$)

$$= \max\Big\{\big(\overline{\pi}(2,m) + \tfrac{1}{r}\,\overline{L}_{2,m}U_2\big) - F_1,\, \tfrac{1}{r}\,\overline{L}_{1,m}U_2\Big\}$$

(since $\overline{L}_{2,m}F_1 = 0$ and $\overline{\pi}(1,m) + \frac{1}{r}\,\overline{L}_{1,m}F_1 = F_1$). Hence, differentiating with respect to m_1 and putting $\widehat{U}_2 = \frac{\partial U_2}{\partial m_1}$, we obtain

$$\begin{aligned}\widehat{U}_2 &= \tfrac{1}{r}\,\overline{L}_{2,m}\widehat{U}_2 + \tfrac{\partial\overline{\pi}}{\partial m_1}(2,m) + \tfrac{1}{r}\Big(\tfrac{\partial\overline{\mu}}{\partial m_1}(2,m)\,\tfrac{\partial U_2}{\partial m_2} + \tfrac{1}{2}\,\tfrac{\partial\overline{\Sigma}}{\partial m_1}(2,m)\,\tfrac{\partial^2 U_2}{\partial m_2^2}\Big) - \tfrac{\partial F_1}{\partial m_1}\\ &= \tfrac{1}{r}\,\overline{L}_{2,m}\widehat{U}_2 - \tfrac{\partial F_1}{\partial m_1}\end{aligned}$$

if $\overline{\pi}(2,m) + \frac{1}{r}\,\overline{L}_{2,m}G > \overline{\pi}(1,m) + \frac{1}{r}\,\overline{L}_{1,m}G$ (because $\frac{\partial\overline{\pi}}{\partial m_1}(2,m) = \frac{\partial\overline{\mu}}{\partial m_1}(2,m) = \frac{\partial\overline{\Sigma}}{\partial m_1}(2,m) = 0$); and we obtain

$$\begin{aligned}\widehat{U}_2 &= \tfrac{1}{r}\,\overline{L}_{1,m}\widehat{U}_2 + \tfrac{1}{r}\Big(\tfrac{\partial\overline{\mu}}{\partial m_1}(1,m)\,\tfrac{\partial U_2}{\partial m_1} + \tfrac{1}{2}\,\tfrac{\partial\overline{\Sigma}}{\partial m_1}(1,m)\,\tfrac{\partial^2 U_2}{\partial m_1^2}\Big)\\ &= \tfrac{1}{r}\tfrac{\partial\overline{\mu}}{\partial m_1}(1,m)\,\widehat{U}_2 + \tfrac{1}{r}\Big(\overline{\mu}(1,m) + \tfrac{1}{2}\,\tfrac{\partial\overline{\Sigma}}{\partial m_1}(1,m)\Big)\,\tfrac{\partial\widehat{U}_2}{\partial m_1} + \tfrac{1}{2\,r}\,\overline{\Sigma}(1,m)\,\tfrac{\partial^2\widehat{U}_2}{\partial m_1^2}\end{aligned}$$

or

$$\Big(1 - \tfrac{1}{r}\tfrac{\partial\overline{\mu}}{\partial m_1}(1,m)\Big)\,\widehat{U}_2 = \tfrac{1}{r}\Big(\overline{\mu}(1,m) + \tfrac{1}{2}\,\tfrac{\partial\overline{\Sigma}}{\partial m_1}(1,m)\Big)\,\tfrac{\partial\widehat{U}_2}{\partial m_1} + \tfrac{1}{2\,r}\,\overline{\Sigma}(1,m)\,\tfrac{\partial^2\widehat{U}_2}{\partial m_1^2}$$

if $\overline{\pi}(1,m) + \frac{1}{r}\,\overline{L}_{1,m}G > \overline{\pi}(2,m) + \frac{1}{r}\,\overline{L}_{2,m}G$.[35]

In other words, for all $x \in [\,\underline{m}_1, \overline{m}_1\,] \times [\,\underline{m}_2, \overline{m}_2\,]$, we have the representation

$$\widehat{U}_2(x) = \mathrm{E}\left[\int_0^\infty \exp\left(-\int_0^t \alpha(m(s))\,ds\right)\beta(m(t))\,dt\right],$$

where: $m(0) = x$; $dm(t)$ is distributed normally with mean $\zeta(m(t))\,dt$ and variance $\eta(m(t))\,dt$;

$$\alpha = 1 - \tfrac{1}{r}\,\tfrac{\partial\overline{\mu}}{\partial m_1}(1,m), \quad \beta = 0, \quad \zeta = \tfrac{1}{r}\left(\overline{\mu}(1,m) + \tfrac{1}{2}\,\tfrac{\partial\overline{\Sigma}}{\partial m_1}(1,m)\right), \quad \eta = \tfrac{1}{r}\,\overline{\Sigma}(1,m)$$

if $\overline{\pi}(1,m) + \frac{1}{r}\,\overline{L}_{1,m}G > \overline{\pi}(2,m) + \frac{1}{r}\,\overline{L}_{2,m}G$; and

$$\alpha = 1, \quad \beta = -\tfrac{\partial F_1}{\partial m_1}, \quad \zeta = \tfrac{1}{r}\,\overline{\mu}(2,m), \quad \eta = \tfrac{1}{r}\,\overline{\Sigma}(2,m)$$

if $\overline{\pi}(2,m) + \frac{1}{r}\,\overline{L}_{2,m}G > \overline{\pi}(1,m) + \frac{1}{r}\,\overline{L}_{1,m}G$.

Now, α is locally bounded. (It may in principle be negative.) Hence the cumulative discount rate

$$\int_0^t \alpha(m(s))\,ds$$

is well defined. (It too may in principle be negative.) Hence the discount factor

$$\exp\left(-\int_0^t \alpha(m(s))\,ds\right)$$

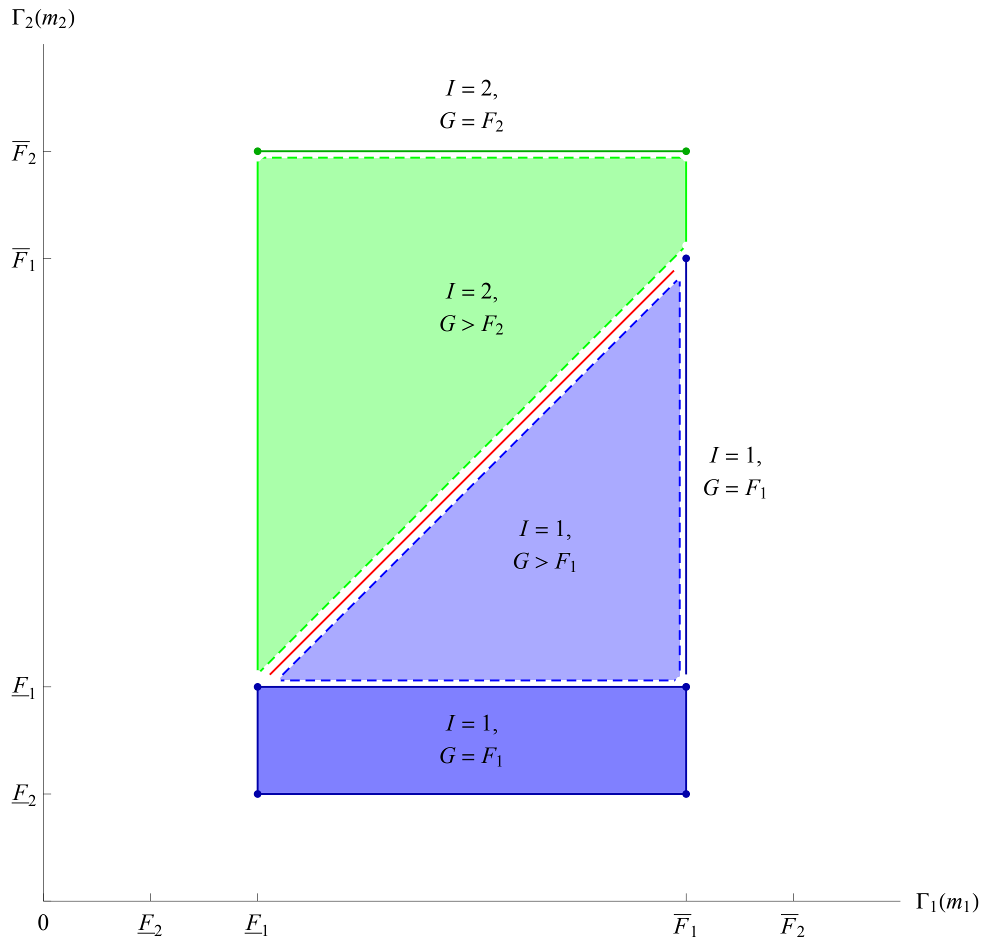
is well defined and strictly positive. Moreover $\beta \leq 0$, with strict inequality if $\overline{\pi}(2,m) + \frac{1}{r}\,\overline{L}_{2,m}G > \overline{\pi}(1,m) + \frac{1}{r}\,\overline{L}_{1,m}G$. Hence the integrand

$$\exp\left(-\int_0^t \alpha(m(s))\,ds\right)\beta(m(t))$$

is non-positive. Moreover it is strictly negative with positive probability iff firm 2 is still in contention. Hence the representation is well defined in the extended sense. (It could in principle be $-\infty$.) Hence $\widehat{U}_2(x) \leq 0$, with strict inequality iff firm 2 is still in contention.

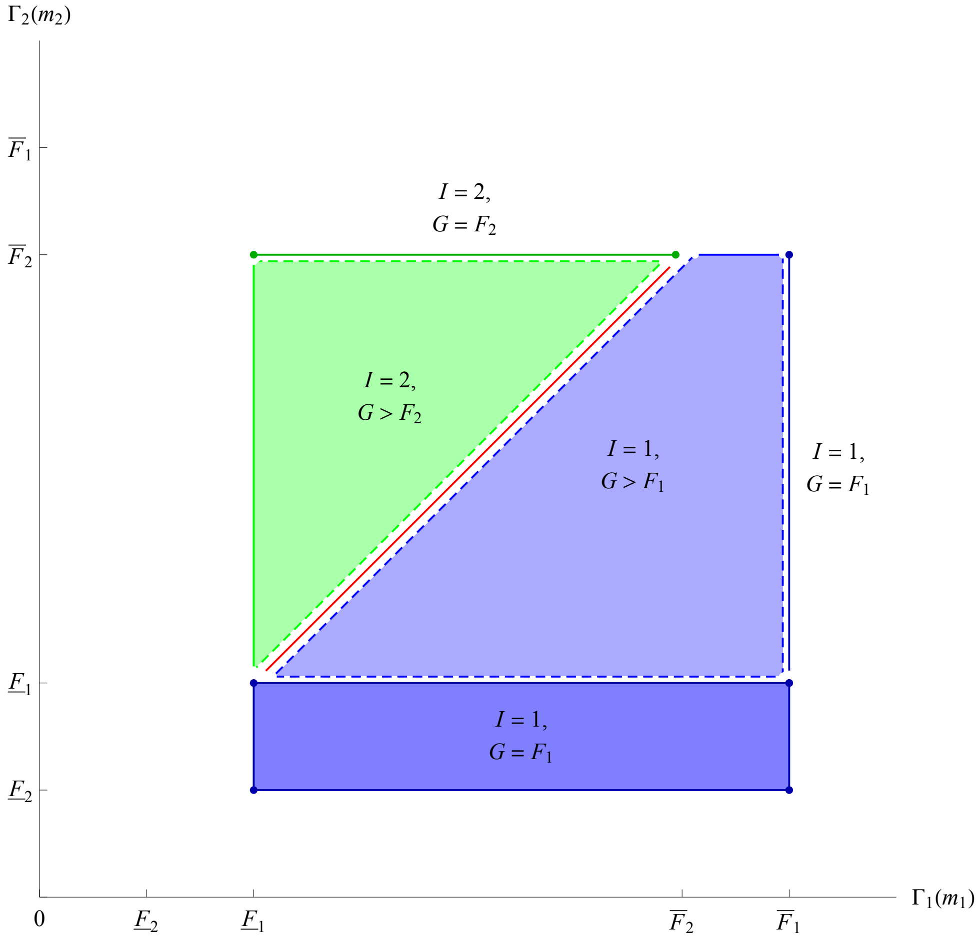
[35] If $\overline{\pi}(2,m) + \frac{1}{r}\,\overline{L}_{2,m}G = \overline{\pi}(1,m) + \frac{1}{r}\,\overline{L}_{1,m}G$, then we will get one of these two equalities, but there is no simple criterion to tell us which. In particular, the result may depend on the direction in which we take the derivative w.r.t. m_1.

Figure 1: The Worker's Choice of Employer in Scenario 1 $\left(\underline{F}_2 < \underline{F}_1 < \overline{F}_1 < \overline{F}_2\right)$



The worker's choice of employer I depicted in (Γ_1, Γ_2)−space, where Γ_k is the Gittins index of firm k. (1) If $\Gamma_1 > \Gamma_2$ (blue region) then $I = 1$, and if $\Gamma_2 > \Gamma_1$ (green region) then $I = 2$. (2) If $\Gamma_1 = \Gamma_2 = \underline{F}_1$ or $\Gamma_1 = \Gamma_2 = \overline{F}_1$ (two of the five blue points) then I is determined by the need to ensure that a solution to the dynamics exists. (3) If $\underline{F}_1 < \Gamma_1 = \Gamma_2 < \overline{F}_1$ (red line) then I is indeterminate, but the solution of the dynamics does not depend upon I. (4) If the worker is currently employed by firm 2, and if there is a strictly positive probability that she will later switch to firm 1, then $G > F_2$. (This is the case $\Gamma_1 < \Gamma_2 < \overline{F}_2$.) (5) If the worker is currently employed by firm 2, but the probability that she will later switch to firm 1 is zero, then $G = F_2$. (This is the case $\Gamma_2 = \overline{F}_2$.) (6) If the worker is currently employed by firm 1, and if there is a strictly positive probability that she will later switch to firm 2, then $G > F_1$. (This is the case $\Gamma_2 < \Gamma_1 < \overline{F}_1$ and $\underline{F}_1 < \Gamma_2$.) (7) If the worker is currently employed by firm 1, but the probability that she will later switch to firm 2 is zero, then $G = F_1$. (This happens when either $\Gamma_2 \le \Gamma_1 = \overline{F}_1$ or $\Gamma_2 \le \underline{F}_1$.)

Figure 2: The Worker's Choice of Employer in Scenario 2 $\left(\underline{F}_2 < \underline{F}_1 < \overline{F}_2 < \overline{F}_1\right)$



The worker's choice of employer I in Scenario 2 depicted in (Γ_1, Γ_2)−space, where Γ_k is the Gittins index of firm k. (1) If $\Gamma_1 > \Gamma_2$ (blue region) then $I = 1$, and if $\Gamma_2 > \Gamma_1$ (green region) then $I = 2$. (2) If $\Gamma_1 = \Gamma_2 = \underline{F}_1$ (one of the five blue points) or $\Gamma_1 = \Gamma_2 = \overline{F}_2$ (one of the two green points) then I is determined by the need to ensure that a solution to the dynamics exists. (3) If $\underline{F}_1 < \Gamma_1 = \Gamma_2 < \overline{F}_2$ (red line) then I is indeterminate, but the solution of the dynamics does not depend upon I. (4) If the worker is currently employed by firm 2, and if there is a strictly positive probability that she will later switch to firm 1, then $G > F_2$. (This is the case $\Gamma_1 < \Gamma_2 < \overline{F}_2$.) (5) If the worker is currently employed by firm 2, but the probability that she will later switch to firm 1 is zero, then $G = F_2$. (This is the case $\Gamma_1 \leq \Gamma_2 = \overline{F}_2$.) (6) If the worker is currently employed by firm 1, and if there is a strictly positive probability that she will later switch to firm 2, then $G > F_1$. (This is the case $\Gamma_2 < \Gamma_1 < \overline{F}_1$ and $\underline{F}_1 < \Gamma_2$.) (7) If the worker is currently employed by firm 1, but the probability that she will later switch to firm 2 is zero, then $G = F_1$. (This happens when either $\Gamma_1 = \overline{F}_1$ or $\Gamma_2 \leq \underline{F}_1$.)