Supplementary Material

("Multi-Stage Voting, Sequential Elimination and Condorcet Consistency" by Bag, Sabourian and Winter)

Proof of Theorem 2. (Existence of a Markov equilibrium for the weakest link voting when $n \geq 2k-1$.)

To prove existence we need to show that there exists a Markov strategy profile s^* such that at each stage it is Nash and undominated assuming that all players play according to s^* in any later stages. This is done by defining s^* inductively in subgames with a given number of candidates as the inductive variable, as follows.

First, let $k(h)$ denote the number of candidates at h. Then at any h with $k(h) = 2$, assume that voter i chooses *sincerely*. Clearly such a strategy profile is an undominated Nash equilibrium in this last stage and is independent of h.

Induction hypothesis. *Now suppose for all h' such that $k(h') \leq J-1$, $s^*(h')$ is defined, and is undominated Nash and Markov*[1] *from here onwards.*

We need to define a profile of choices for all voters $s^*(h)$, for all h such that $k(h) = J$, such that $s^*(h)$ is an undominated Nash equilibrium and Markov from here onwards, assuming that all follow $s^*(h')$ at all later stages h' s.t. $k(h') \leq J-1$.

Fix any h s.t. $k(h) = J$. Let $C = \{c^1, ..., c^J\}$ be the set of candidates at h. Without any loss of generality assume that $c^{j'}$ is higher in the tie-breaking rule than c^j (i.e., if at all, c^j is eliminated before eliminating $c^{j'}$) if and only if $j' < j$.

Also let $\sigma(c)$ be the winner if c is eliminated at the start of play of the subgame $\Gamma(h)$. Notice that $\sigma(c)$ is unique because by the induction hypothesis $s^*(h)$, when there are $J-1$ candidates, is independent of the past history.

Next define, for any $i \in \mathcal{N}$,

$$M_i = \begin{cases} \Theta_i & \text{if } \exists c \text{ and } c' \in C \text{ s.t. } \sigma(c) \neq \sigma(c'); \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\Theta_i = \{c \in C \mid \nexists c' \in C \text{ s.t. } \sigma(c') \succ_i \sigma(c)\}$ consists of voter i's best elimination candidate(s) in this round of play. Note that M_i is empty-valued when the subgame is degenerate (the identity of the eventual winner is independent of who is eliminated at this round). Finally, let $M_i^c = C \backslash M_i$. Clearly, $M_i^c \neq \emptyset$.

[1]That is, the strategies depend only on the candidates around and not on the precise history leading up to it.

Lemma 1. *In the subgame* $\widehat{\Gamma}(h)$, *any* $c \in M_i^c$ *is not weakly dominated for voter* i.

Proof of Lemma 1. Suppose $M_i \neq \emptyset$ (if $M_i = \emptyset$, Lemma 1 holds trivially). Fix $c \in M_i^c$ and any $c' \in C$, $c' \neq c$. We want to argue that switching his vote from c to c' would be worse for voter i for at least one profile of other voters' votes.

If the tie-breaker places *some* $\hat{c} \in M_i$ ahead of c and $\hat{c} \neq c'$, let the distribution of votes of all the voters other than i be as follows:

$$\vartheta(\hat{c}) = \vartheta(c) = 0 \text{ and } \vartheta(\tilde{c}) > 0, \ \forall \tilde{c} \neq \hat{c}, c,$$

where $\vartheta(\cdot)$ denotes the number of votes in favor of a candidate by all voters other than i. Now if i votes for c then the distribution of votes as above leads to the elimination of $\hat{c}$. However, if i switches to c' while the rest stay with their votes as above, candidates $\hat{c}$ and c will be tied with minimal votes and by the tie-breaker c will be eliminated, which is worse for voter i. If $\hat{c} = c'$, the argument holds with even greater force as c would be eliminated (as i switches to c') without having to invoke the tie-breaker.

If the tie-breaker is such that c is placed ahead of *all* $\hat{c} \in M_i$, let the distribution of votes of all the voters other than i be as follows:

$$\vartheta(c) = 0, \ \vartheta(\hat{c}) = 1 \ \ \forall \ \hat{c} \in M_i, \ \text{ and } \vartheta(\tilde{c}) \geq 2 \ \ \forall \tilde{c} \neq c, \ \tilde{c} \in M_i^c.$$

Now if i votes for c then this leads to the elimination of some $\hat{c}$. However, if i switches to c' while the rest stay with their votes as above, c will be unique with minimal votes and therefore be eliminated, which is worse for voter i. This completes the proof of Lemma 1. ||

Next for any $r = 1, ..., J$ we define the following property.

Definition 1. *Any* $r \in \{2, ..., J\}$ *satisfies property* α *if there exists a set of voters* $\Omega = (u_1, v_1, u_2, v_2, ..., u_{r-1}, v_{r-1})$ *consisting of* $2(r-1)$ *different voters such that*

$$c^j \in M_i^c \ \textit{ for } \ i = u_j, v_j \textit{ for all } \ j < r. \tag{1}$$

Lemma 2. *Suppose that the following two conditions hold for some* $1 \leq r < J$: *(i) either* $r = 1$ *or* r *satisfies property* α; *and (ii)* $r + 1$ *does not satisfy property* α. *Then there exist a choice profile* $s^*(h)$ *that is Nash, is not weakly dominated, and is Markov.*

Proof of Lemma 2. Given that r satisfies (i) and (ii) above, there exists a set of voters Ω (that is empty if $r = 1$) consisting of $2(r-1)$ different voters $(u_1, v_1, u_2, v_2, ..., u_{r-1}, v_{r-1}) \subset \mathcal{N}$ such that

$$c^j \in M_i^c \text{ for } i = u_j, v_j \text{ for all } j < r \tag{2}$$

and there exists a set of voters $V \subset \mathcal{N} \backslash \Omega$ such that

$$| V |= n - 2(r-1) - 1 \tag{3}$$

and

$$c^r \in M_v \text{ for any } v \in V. \tag{4}$$

Let

$$C^r = \{c \in C \,|\sigma(c) = \sigma(c^r)\} \quad \text{and} \quad \overline{C}^r = \{C \backslash C^r\} \cap \{c^{r+1}, ..., c^J\}.$$

Since the preferences of each voter is strict, it follows that

$$\overline{C}^r \subset M_v^c \text{ for any } v \in V. \tag{5}$$

Also since $| V |= n - 2(r-1) - 1$, $| \overline{C}^r |\leq J - r$ and by assumption $n \geq 2k - 1 \geq 2J - 1$ and $r < J$, it follows that the number of voters in V is at least twice the number of candidates in $\overline{C}^r$. But this implies that there exists a choice profile $\{s_v^*(h)\}_{v \in V}$ such that

$$s_v^*(h) \in \overline{C}^r \text{ for each } v \in V, \tag{6}$$

$$|\{v \in V \,|s_v^*(h) = c\}| \geq 2 \text{ for each } c \in \overline{C}^r. \tag{7}$$

(The second condition says that each candidate $c \in \overline{C}^r$ receive at least two votes). Next set the choice $s_i^*(h)$ of each $i \in \Omega$ to be such that

$$s_i^*(h) = c^j \text{ for } i = u_j, v_j. \tag{8}$$

Finally, denote the remaining voter $\mathcal{N} \backslash \{V \cup \Omega\}$ by x and set the choice of voter x to be such that

$$\begin{array}{ll} s_x^*(h) \in M_x^c \backslash c^r & \text{if } M_x^c \backslash c^r \text{ is not empty;} \\ s_x^*(h) = c^r & \text{otherwise.} \end{array} \tag{9}$$

Now by Lemma 1 and conditions (2), (5), (6), (8) and (9), the choice $s_\ell^*(h)$ is undominated in this round for any voter $\ell \in \mathcal{N}$ and is Markov.[2] Next we show that $s^*(h) = \{s_\ell^*(h)\}_{\ell \in \mathcal{N}}$ is Nash. There are two possible cases.

[2] If $M_x^c \backslash c^r$ is an empty set then c^r must be an element of M_x^c because $M_x^c \neq \emptyset$. Thus, $s_x^*(h)$ is undominated.

Case A. $M_x^c \neq c^r$**.** First, note that by (7) and (8), in this round each candidate $c \in \overline{C}^r \cup \{c^1, ..., c^{r-1}\}$ receives at least two votes, c^r receives zero vote (follows from (9), given the fact that $M_x^c \neq \emptyset$ and $M_x^c \neq c^r$), and any other $c' \in C^r \cap \{c^{r+1}, \dots, c^J\}$ receives at most one vote. This means that some candidate $c \in C^r$ is eliminated and $\sigma(c^r)$ will be the final winner. Moreover, since c^r receives zero vote, it must be that the eliminated candidate $c^e \in C^r$ receives zero vote and $e \geq r$.

Since, by (4), $\sigma(c^r)$ is a best outcome for each $v \in V$, it follows that $s_v^*(h)$ is a best choice for any $v \in V$. Moreover, no voter $i \in \Omega$ can change the final outcome $\sigma(c^r)$ by changing its action because the choice $s_i^*(h) \in \{c^1, ..., c^{r-1}\}$ receives at least two votes, the eliminated candidate c^e has zero vote and $e \geq r$. Finally, voter x cannot change the final outcome $\sigma(c^r)$ by changing its action because either the choice $s_x^*(h) \in \{c^1, ..., c^{r-1}\} \cup \overline{C}^r$, in which case $s_x^*(h)$ receives at least three votes and as before c^e has zero vote, or $s_x^*(h) \in C^r \cap \{c^{r+1}, \dots, c^J\}$ in which case $s_x^*(h)$ receives one vote and any deviation results in some candidate in the set C^r to be eliminated.

Case B. $M_x^c = c^r$**.** Then for each $c' \neq c^r$, $c' \in M_x$. Therefore

$$\forall c', c'' \neq c^r, \;\; \sigma(c') = \sigma(c''). \tag{10}$$

This implies that $\overline{C}^r = \{c^{r+1}, ..., c^J\}$. But then, by (7) and (8), in this round each candidate $c \neq c^r$ receives at least two votes, c^r receives one vote (the vote of x), c^r is eliminated and $\sigma(c^r)$ will be the final winner. As in the previous case, since this is a best outcome for each $v \in V$ it follows that $s_v^*(h)$ is a best choice for any $v \in V$. Next note that for each voter $i = u_j, v_j$ for $j < r$ we have $s_i^*(h) = c^j \in M_i^c$ and thus $c^r \in M_i$. Therefore, eliminating c^r is also the best outcome for any $i \in \Omega$. Finally, note that voter x cannot change the final outcome $\sigma(c^r)$ by changing its action because every $c \neq c^r$ receives two votes. $||$

Lemma 3. *Suppose that J satisfies property α. Then there exist a choice profile $s^*(h)$ that is Nash, is not weakly dominated, and is Markov.*

Proof of Lemma 3. Given that J satisfies property α, there exists a set of voters $\Omega = (u_1, v_1, u_2, v_2, ..., u_{J-1}, v_{J-1})$ consisting of $2(J-1)$ different voters such that

$$c^j \in M_i^c \;\text{ for }\; i = u_j, v_j \text{ for all } \; j < J. \tag{11}$$

Set the choice profile $\{s_i^*(h)\}_{i \in \Omega}$ to be such that

$$s_i^*(h) = c^j \text{ if } i = u_j, v_j. \tag{12}$$

Also partition the remaining voters as follows:

$$\begin{aligned}\Gamma^J &= \left\{v \in \mathcal{N}\backslash\Omega | M_v^c = c^J\right\} \\ \overline{\Gamma}^J &= \left\{v \in \mathcal{N}\backslash\Omega | M_v^c \neq c^J\right\}.\end{aligned}$$

Let the choice profile $\{s_v^*(h)\}_{v\in\mathcal{N}\backslash\Omega}$ be such that

(i)
$$s_v^*(h) \in \begin{cases} c^J & \text{if } v \in \Gamma^J \\ M_v^c\backslash c^J & \text{if } v \in \overline{\Gamma}^J; \end{cases}$$

(ii) if Γ^J is non-empty,

$$|n(c) - n(c')| \leq 1 \;\; \forall c, c' \neq c^J \;\text{ s.t. }\; \sigma(c) = \sigma(c'), \tag{13}$$

where

$$n(c) = |\{v \in \mathcal{N}\,|s_v^*(h) = c\}| \quad \text{for any } \; c.$$

(Note that $M_v^c\backslash c^J \neq \emptyset$ for $v \in \overline{\Gamma}^J$.) Notice that if Γ^J is non-empty, (13) is possible for the following reasons. First, since Γ^j is non-empty,

$$\forall c', c'' \neq c^J, \;\; \sigma(c') = \sigma(c'') \neq \sigma(c^J). \tag{14}$$

Next note that each c^j, $j < J$ receives two votes from the set of voters Ω. The only other voters that vote for the candidates c^j, $j < J$ are from the set $\overline{\Gamma}^J$. Because Γ^J is non-empty it follows from (14) that for each $v \in \Gamma^J$, $M_v = \{c^1, \ldots, c^{J-1}\}$; therefore votes by the members of $\overline{\Gamma}^J$ can be arranged so that (13) is satisfied: the first member of $\overline{\Gamma}^J$ votes for c^1, the second for c^2 etc. until the $(J-1)$st member votes for c^{J-1}, the J-th member for c^1, $(J+1)$st for c^2 etc.

By Lemma 1, $s^*(h)$ is not weakly dominated. Also by definition, $s^*(h)$ is Markov. Next we show that it is a Nash equilibrium.

Case A. Γ^J is empty. Then every $c \neq c^J$ receives at least two votes, c^J receives no votes and is eliminated. This together with c^J having the lowest rank in the tie-breaking rule imply that no player can change the final outcome by changing their choices and thus $s^*(h)$ constitutes a Nash equilibrium.

Case B. Γ^J is non-empty. By (14), since for each $v \notin \Gamma^J$ there exists a $c \neq c^J$ such that $c \in M_v^c$, it follows that

$$\forall v \notin \Gamma^J, \;\; c^J \in M_v. \tag{15}$$

Now there are two possibilities.

Subcase 1: Candidate c^J is eliminated. Then, by (15), this is the best outcome for any $v \notin \Gamma^J$ and therefore, each such v is choosing his optimal action. Moreover, each $v \in \Gamma^J$ cannot change the outcome by deviating from $s_v^*(h)$ because $s_v^*(h) = c^J$ and c^J is the candidate that is eliminated.

Subcase 2: Some $c \neq c^J$ is eliminated. Then, by the tie-breaking rule

$$n(c) < n(c^J). \tag{16}$$

Next note that by (14) and the definition of Γ^J, this is the best outcome for any $v \in \Gamma^J$ and therefore, each such v is choosing his optimal action. Next we show that no voter $v \notin \Gamma^J$ can change the outcome by deviating. Suppose not; then some voter $v \notin \Gamma^J$ can deviate from $s_v^*(h) = c^j (\neq c)$ for some $j < J$ and change the final outcome $\sigma(c)$ by voting for another candidate. Since the outcome is changed, by (14), it must be that c^J is eliminated. This implies that

$$n(c^j) - 1 \geq n(c^J).$$

But this together with (16) imply that

$$n(c^j) > n(c) + 1$$

But this contradicts condition (13). Therefore no $v \notin \Gamma^J$ can change the final outcome by deviating. ||

The last two lemmas together establish that there exists a choice profile $s^*(h)$ that is Nash, not weakly dominated, and is Markov. **Q.E.D.**

Justifying the use of Markov strategies

Consider any voting game described in section 2, where at each round at least one candidate is eliminated. Recall, S_i is the strategy set of voter i with $s_i : \mathcal{H} \to \cup_{C,j} A_i(C,j)$ s.t. $s_i(h) \in A_i(C,j) \quad \forall h \in \mathcal{H}_C^j$, where $\mathcal{H}_C^j = \mathcal{H}_C \cap \mathcal{H}^j$. Also, let $S = \Pi_i S_i$.

Definition 2. *A strategy $s_i \in S_i$ is <u>more complex</u> than another strategy $s_i' \in S_i$ if $\exists C$ and j s.t.*
(i) $s_i(h) = s_i'(h) \quad \forall h \notin \mathcal{H}_C^j$;
(ii) $s_i'(h) = s_i'(h') \quad \forall h, h' \in \mathcal{H}_C^j$;
(iii) $s_i(h) \neq s_i(h')$ *for some* $h, h' \in \mathcal{H}_C^j$.

The above ordering of complexity is only a partial ordering. Nevertheless, it will prove a powerful one for our purpose. Based on this ordering, let us refine our original definition of *equilibrium* as follows.

Definition 3. *A equilibrium strategy profile $s^* \in S$ will be called a simple equilibrium if for any $i \in \mathcal{N}$*

$$\nexists s_i \in S_i \text{ s.t. } w(s_i, s_{-i}^*) = w(s_i^*, s_{-i}^*) \text{ and } s_i^* \text{ is more complex than } s_i, \tag{17}$$

where $w(s)$ is the winner if profile s is adopted.

Note that while the definition of *simple* equilibrium allows history-dependent (i.e., non-Markov) strategies, the condition in (17) reflects the implicit assumption that the voters are averse to complexity *unless* it helps to change the final outcome. Thus, simplicity of the simple equilibrium is a very weak, and in our view plausible, requirement for any descriptive analysis. We can therefore use the simplicity criterion for equilibrium selection.

Theorem. *Any simple equilibrium is also a Markov equilibrium.*

Proof. Suppose $s^* \in S$ is a simple equilibrium but not a Markov equilibrium. Then there exists some i, C, j and $h, h' \in \mathcal{H}_C^j$ s.t. $s_i^*(h) \neq s_i^*(h')$. Clearly, if $\mathcal{H}_C^j \cap E \neq \emptyset$ where E is the equilibrium path corresponding to the simple equilibrium s^*, then $\mathcal{H}_C^j \cap E$ is *unique*; that is, C happens on the equilibrium path at stage j at most once. Now consider another strategy $s_i \in S_i$ s.t.

$$\begin{aligned} s_i(h) &= s_i^*(h) \quad \forall h \notin \mathcal{H}_C^j; \\ \forall h \in \mathcal{H}_C, \quad s_i(h) &= \begin{cases} s_i^*(\mathcal{H}_C^j \cap E) & \text{if } \mathcal{H}_C^j \cap E \neq \emptyset, \\ a_i \in C & \text{if } \mathcal{H}_C^j \cap E = \emptyset, \end{cases} \end{aligned}$$

where a_i denotes any arbitrary element of C (note that s_i is well-defined because $\mathcal{H}_C^j \cap E$ is unique when defined).

It is easy to see that s_i is simpler than s_i^*. Moreover, because s_i differs from s_i^* only possibly for histories in $\mathcal{H}_C^j$ that are off-the-equilibrium path, (s_i, s_{-i}^*) will result in the same winner as the equilibrium s^*, so that $w_i(s_i, s_{-i}^*) = w_i(s_i^*, s_{-i}^*)$. Hence, s^* cannot be a simple equilibrium – a contradiction. **Q.E.D.**

Proof of Proposition 3. We prove that the voting rules considered satisfy the two conditions in Definition 6 separately.

Condition 1

To show this, fix any x and y and any two strategies $R_i = (X_1, ..., X_J)$ and $R'_i = (X'_1, ..., X'_J)$ such that $x \in X_\tau$, $y \in X_{\tau'}$ and $\tau < \tau'$, so that $x \, P_i \, y$. Suppose also $x \in X'_\nu$, $y \in X'_{\nu'}$ and $\nu' < \nu$, so that $y \, P'_i \, x$. Also, let $m = (n-1)/2$ and consider the set of voters other than i. Enumerate this set (we are assuming an odd number of voters) and denote the enumeration by $\{\alpha_1, ..., \alpha_{2m}\}$, with a typical voter denoted as α_ℓ. Also enumerate the candidates other than x, y as $\{c^1, c^2, \ldots, c^{k-2}\}$.

Next, for any voter α_ℓ consider any strategy $R_{\alpha_\ell} = (\hat{X}_1, ..., \hat{X}_J)$ satisfying[3]

$$\begin{array}{rcll} x & \in & \begin{cases} \hat{X}_1 & \text{if either } \ \ell \leq m \ \text{ or } \ M(1) > 1 \\ \hat{X}_2 & \text{if } \ \ell > m \ \text{ and } \ M(1) = 1 \end{cases} \\ y & \in & \begin{cases} \hat{X}_1 & \text{if either } \ \ell > m \ \text{ or } \ M(1) > 1 \\ \hat{X}_2 & \text{if } \ \ell \leq m \ \text{ and } \ M(1) = 1 \end{cases} \\ \text{and} \quad c^r & \in & \hat{X}_J & \text{for voter } \ \alpha_r, \quad 1 \leq r \leq k-2. \end{array}$$

Thus, each of the candidates other than x and y is placed in at least one voter's lowest-ranked cell. This is possible because there are $k-2$ such candidates and $k - 2 \leq 2m$ (by assumption $k - 1 \leq n$).

Next, consider the different voting rules under consideration.

Scoring rules and Approval Voting: Denote the score attached to the j-th cell in either of the two voting rules by ς_{J-j+1}. Also, denote respectively the total score that any candidate c receives for strategy profile (R_i, R_{-i}) and (R'_i, R_{-i}) by $TS(c, R_i, R_{-i})$ and $TS(c, R'_i, R_{-i})$. Then it follows from the definition of R_{-i} above that:

$$\left.\begin{array}{r} TS(x, R_i, R_{-i}) = m\varsigma_J + m\gamma + \varsigma_{J-\tau+1} \\ TS(y, R_i, R_{-i}) = m\varsigma_J + m\gamma + \varsigma_{J-\tau'+1} \\ TS(c^r, R_i, R_{-i}) \leq \varsigma_J + (2m-1)\gamma + \varsigma_1, \qquad 1 \leq r \leq k-2, \end{array}\right\} \tag{18}$$

$$\text{where} \qquad \gamma = \begin{cases} \varsigma_J & \text{if } \ M(1) > 1 \\ \varsigma_{J-1} & \text{if } \ M(1) = 1. \end{cases}$$

Therefore, it follows from (18) and $\varsigma_{J-\tau+1} > \varsigma_{J-\tau'+1} \geq \varsigma_1$ that $TS(x, R_i, R_{-i}) - TS(y, R_i, R_{-i}) > 0$, and $TS(x, R_i, R_{-i}) - TS(c^r, R_i, R_{-i}) > 0$. Therefore, (R_i, R_{-i}) results in x being elected.

[3] One should index the cells to reflect individualistic voting, but we keep to minimal notations.

Also, it follows from the definition of R_{-i} above that:

$$\begin{aligned} TS(x, R_i', R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\nu+1} \\ TS(y, R_i', R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\nu'+1} \\ TS(c^r, R_i', R_{-i}) &\leq \varsigma_J + (2m-1)\gamma + \varsigma_1, \qquad 1 \leq r \leq k-2. \end{aligned}$$

But this together with $\varsigma_{J-\nu'+1} > \varsigma_{J-\nu+1} \geq \varsigma_1$ imply that $TS(y, R_i', R_{-i}) - TS(x, R_i', R_{-i}) \geq \varsigma_{J-\nu'+1} - \varsigma_{J-\nu+1} > 0$, and $TS(y, R_i', R_{-i}) - TS(c^r, R_i', R_{-i}) \geq \varsigma_{J-\nu'+1} - \varsigma_1 > 0$. Thus, (R_i', R_{-i}) results in y being elected.

Instant runoff voting (with and without the majority top-rank trigger). For both variants of instant runoff voting, the strategy profile (R_i, R_{-i}) described above results in candidate x having the highest number of votes at each round and therefore in x being elected, and the strategy profile (R_i', R_{-i}) described above results in candidate y having the highest number of votes at each round and therefore in y being elected (this follows from the same reasoning as in the previous case with scoring rules and approval voting).

Copeland rule. To calculate Copeland scores for (R_i, R_{-i}) submissions, let us do binary comparisons: comparing x against any other candidate yields x each time a score of $+1$, thus the Copeland score of x is $k-1$. Since $k-1$ is the maximum possible Copeland score, it follows that (R_i, R_{-i}) results in x being the winner. By the same reasoning, the Copeland score of y when (R_i', R_{-i}) is chosen is $k-1$; therefore in this case the Copeland winner is y. Thus, condition 1 is satisfied.

Simpson rule. The strategy profile (R_i, R_{-i}) described above results in candidate x having the highest Simpson score and therefore being elected. This is because the Simpson score of x in this case is $m+1$ ($N(x,a) = 2m$ for all $a \neq x, y$ and $N(x,y) = m+1$), whereas the Simpson score of y is m and that of any other candidate $a \neq x, y$ is no greater than 1. By the same reasoning, it follows that the strategy profile (R_i', R_{-i}) described above results in candidate y having the highest Simpson score and therefore being elected. ||

Condition 2

Scoring rules, approval voting and instant runoff voting (with and without the majority top-rank trigger). For these voting rules condition 2 holds vacuously because these voting rules are not CC with respect to sincere voting. To see this, consider each of the voting rules under consideration.

For scoring rules, the assertion follows from a three candidates, seventeen voters example due to Fishburn (1973) with a CW that fails to be elected under sincere voting (see also Theorem 9.1 in Moulin, 1988). To show that the same holds for arbitrary number of candidates k, consider Fishburn's example and add $k-3$ more candidates below the three candidates for all voters.

For approval voting and instant runoff voting (with and without the majority top-rank trigger), consider the following example of five voters and k candidates with strict preferences over candidates in a descending order:

$$\begin{aligned} 1,4 &: \; y,\; x,\; z,\; w, X_5, \ldots, X_k \\ 2,5 &: \; z,\; x,\; y,\; w, X_5, \ldots, X_k \\ 3 &: \; w,\; x,\; z,\; y, X_5, \ldots, X_k. \end{aligned}$$

While x is the CW, sincere voting (for approval voting, 'sincere' in the sense defined in our paper) will eliminate x under these voting rules and thus will not be CC.

Copeland rule and Simpson rule. Consider any three candidates $X = \{x, y, z\}$. Suppose that the tie-breaker places z above x and y. Fix any two strategies $R_i = (x, y, X_3, X_4, \ldots, X_k)$ and $R_i' = (y, x, X_3, X_4, \ldots, X_k)$, for any $(X_3, X_4, \ldots, X_k)$ with $X_\tau = z$ for some specific $\tau > 2$. Next, specify R_{-i} as follows: $(n-1)/2$ voters submit $(x, z, \underbrace{\ldots, X_{\tau-1}, y, X_{\tau+1}, \ldots})$ and $(n-1)/2$ other voters submit $(z, y, \underbrace{\ldots,\; X_{\tau-1}, x, X_{\tau+1} \ldots})$, where the two underbraced $(\underbrace{\ldots})$ rankings by the two groups of $(n-1)/2$ voters are otherwise the same as the ranking $(X_3, X_4, \ldots, X_k)$ except that X_τ's slot is filled in respectively by y and x.

Let us now calculate Copeland scores first. For R_i submission by i, comparing x against any other candidate yields x each time a score of $+1$, so candidate x's Copeland score $CSc(x) = k-1$, and thus x is the Copeland winner. On the other hand if i submits R_i' instead, the Copeland scores are calculated as follows. Candidate x: comparison x, y yields x the score -1 and comparison of x against any other candidate yields each time x the score $+1$, so $CSc(x) = k-3$. Candidate y: comparison y, z yields y the score -1 and comparison of y against any other candidate yields each time y the score $+1$, so $CSc(y) = k-3$. Candidate z: comparison z, x yields z the score -1 and comparison of z against any other candidate yields each time z the score $+1$, so $CSc(z) = k-3$. Since z is ahead of x and y in the tie-breaker, it follows that if R_i' is chosen z will be the Copeland winner (for any other candidate w, $CSc(w) \leq k-7$).

Next, consider the Simpson rule. For R_i submission by i, the Simpson scores are $SSc(x) = (n-1)/2 + 1,\ SSc(y) = 1,\ SSc(z) = (n-1)/2$ and $SSc(w) = 0$ for any other w; thus the Simpson winner is x. On the other hand, for R_i' submission the Simpson scores are $SSc(x) = (n-1)/2,\ SSc(y) = 1,\ SSc(z) = (n-1)/2$ and $SSc(w) \leq 1$ for any other w. With a tie-breaker placing z ahead of x and y, the Simpson winner is x. $||$

Our required verifications for the specific one-shot voting rules are now complete. Thus, by Theorem 4, none of the voting rules considered in Proposition 3 are CC under strategic voting. **Q.E.D.**

Proof of Proposition 4.

[Plurality runoff] Suppose there are three voters and four candidates, w, x, y, z. Fix a tie-breaking rule y, z, w, x. The voters' ranking over candidates are as follows:

$$\begin{aligned} 1:&\quad x, w, y, z \\ 2:&\quad x, y, z, w \\ 3:&\quad z, w, y, x. \end{aligned}$$

The CW is x. Also, $y\,T\,z\,T\,w\,T\,y$.

Under plurality-runoff rule, an equilibrium strategy profile is

$$1:\ y\ ;\qquad 2:\ z\ ;\qquad 3:\ \ w$$

in stage 1, followed by sincere voting in stage 2. In stage 1, x and w are eliminated, so that y is picked as the ultimate winner.

Given that sincere voting in stage 2 constitute a Nash equilibrium that is also weakly undominated, we only need to check that the proposed stage 1 strategies will be Nash equilibrium and weakly undominated. Clearly, the strategies are best responses to each other and therefore Nash in stage 1. So we will only verify that for any voter no other strategy weakly dominates his proposed equilibrium strategy.

The votes by voters 1 and 2 are the unique best responses, thus also weakly undominated. So let us consider voter 3's strategy. Let voters 1 and 2 choose in stage 1 respectively x and z. If voter 3 chooses w the outcome is z; on the other hand, if voter 3 chooses x or z the outcome is x, and if he chooses y the outcome is y, and both are worse compared to z.

Thus, plurality-runoff rule is not CC.[4]

[4]Note that given a counter example for a particular tie-breaking rule, one can construct similar

[Exhaustive ballot] There are three types of voters, with three voters of each type. There are three alternatives with the following preferences:

type 1 : x, y, z

type 2 : y, z, x

type 3 : x, z, y.

The CW is x.

Consider the following strategy profile: In round 1, type 1 voters vote for x and types 2 and 3 vote for z. In round 2, if reached, each type vote for the alternative (from the remaining two) which he prefers most.

The above strategy induces z as the winner. We claim that this strategy will be an equilibrium. That round 2 voting satisfies the equilibrium conditions is trivial. So consider round 1 voting. First, note that no player can gain by deviating unilaterally in this round (this is because each type has three voters). It thus remains to argue that no weakly dominated strategies are used in round 1. As type 1 voters vote for their top-ranked candidate, clearly the strategy is undominated. So we need to argue that voter types 2 and 3 are not using weakly dominated strategies in round 1. First consider type 3 voters. Let the strategy combination in round 1 be as follows: all type 1 vote for y, all type 2 vote for z, two type 3 vote for z and one type 3 votes for x. This leads to z being elected. If on the other hand one of the two type 3 voters who voted for z now switches to either y or x, then x will be eliminated and y is the ultimate outcome, which is worse for a type 3 voter. Consider now type 2 voters. Let the strategy combination in round 1 be as follows: all type 1 vote for x, all type 3 vote for z, one of type 2 votes for x and the other two vote for z. For this profile the outcome is z. If, however, one of the voters who earlier voted for z now switches to either y or x, then y will be eliminated and x is the ultimate outcome, which is worse for a type 2 voter.

[One-shot version of the weakest link voting] Consider the one-shot version of the weakest link game. Suppose there are four candidates, w, x, y, z. Fix a tie-breaking rule y, z, w, x. Below we specify voter preferences for which CW will not be elected. For any other tie-breaking rule, a counter example can be constructed by permuting voter preferences appropriately (see footnote 4).

counter examples for other tie-breakers by simply permuting voter preferences in the same way the alternatives are permuted to obtain the new tie-breaking rule.

Consider three types of voters with two voters of each type with the following preferences:

type 1 : y, x, z, w

type 2 : z, x, y, w

type 3 : w, x, z, y.

The CW is x.

For only two candidates sincere submission (i.e., voting for one's favorite candidate) is clearly the only undominated strategy for any voter. We now specify the strategies for each type of voter for three and four candidates:

$$s_1(w,y,z) = y, \quad \mathbf{s_1(x,y,z) = x}, \quad s_1(w,x,z) = x, \quad s_1(w,x,y) = y, \quad s_1(w,x,y,z) = y$$
$$s_2(w,y,z) = z, \quad s_2(x,y,z) = z, \quad s_2(w,x,z) = z, \quad s_2(w,x,y) = x, \quad s_2(w,x,y,z) = z$$
$$s_3(w,y,z) = w, \quad s_3(x,y,z) = x, \quad \mathbf{s_3(w,x,z) = x}, \quad \mathbf{s_3(w,x,y) = x}, \quad s_3(w,x,y,z) = w.$$

We verify that the strategies constitute a Nash equilibrium. For candidates $\{w, y, z\}$, the proposed strategies lead to z as the ultimate winner (given that for any two candidates under consideration the voters would vote sincerely). For a type 1 voter deviation to w or z, and for a type 3 voter deviation to y or z, still result in z as the winner; for a type 2 voter clearly deviation cannot be optimal.

For candidates $\{x, y, z\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to y or z, for a type 2 voter deviation to y or x, and for a type 3 voter deviation to y or z – all leave the voting outcome unchanged (i.e., x is the winner).

For candidates $\{w, x, z\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to w or z, and for a type 3 voter deviation to y or w, still result in x as the winner; for a type 2 voter deviation to w or x also result in x as the winner (in the first deviation, w gets eliminated using the tie-breaker).

For candidates $\{w, x, y\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to w or x, and for a type 3 voter deviation to w or y, leave the winner x unchanged; and for a type 2 voter clearly deviation cannot be optimal.

For candidates $\{w, x, y, z\}$, the proposed strategies lead to x's elimination in the first round and ultimately lead to z being the ultimate winner. Given that x is placed below other alternatives in the tie-breaker and there are two voters of each type with the same types choosing the same strategy in the proposed equilibrium, no individual voter can prevent x's elimination by altering his vote. Thus, deviation by any of the voters is never optimal.

We next argue why the proposed strategies are undominated as well. We will make our assertions for only the three vote choices indicated in bold; for the remainder, to verify that the choices are undominated is easy because in each case the voter votes for his top-ranked candidate.

First, consider a type 1 voter's decision $s_1(x, y, z) = x$. To show that for this voter voting for y cannot (weakly) dominate, suppose all other five voters vote for z, and for only two candidates remaining all vote sincerely. Then the type 1 voter by voting for y will induce z as the winner, whereas if he votes for x the winner is x; and he prefers x over y. To show that voting for z cannot dominate either, suppose the other type 1 voter votes for x and the remaining four voters all vote for y. Then if the type 1 voter votes for z the winner is z, whereas if he votes for x the winner is x; and he prefers x over z.

Next consider a type 3 voter's decision $s_3(w, x, z) = x$. To show that voting for w cannot dominate, suppose the other type 3 voter votes for x and the remaining four voters all vote for z. Then if the type 3 voter votes for w, the tie-breaker would eliminate x and sincere voting in the next round of elimination would elect z; on the other hand, if the type 3 voter voted for x then in the next round of elimination (with x and z as the candidates) sincere voting would elect x, which the type 3 voter prefers over z. To show that voting for z cannot dominate either, suppose the other type 3 voter votes for x, one from the remaining four votes for w and three others vote for z. Then if the type 3 voter votes for z, the tie-breaker eliminates x and sincere voting (with two candidates remaining) elects z as the winner. On the other hand, if the type 3 voter votes for x, alternative w will be eliminated and sincere voting with two candidates remaining would elect x, which the type 3 voter prefers.

Finally, by an argument similar to the one just given, one can show that a type 3 voter's decision $s_3(w, x, y) = x$ is undominated.

Thus, we have constructed a Nash equilibrium with undominated strategies in which the *CW*, x, is eliminated and z gets elected. **Q.E.D.**