Repeated Games with One – Memory*

Mehmet Barlo
Sabancı University

Guilherme Carmona
Universidade Nova de Lisboa

Hamid Sabourian
University of Cambridge

April 2, 2008

Abstract

We study the extent to which equilibrium payoffs of discounted repeated games can be obtained by 1 – memory strategies. We establish the following in games with perfect (rich) action spaces: First, when the players are sufficiently patient, the subgame perfect Folk Theorem holds with 1 – memory. Second, for arbitrary level of discounting, all strictly enforceable subgame perfect equilibrium payoffs can be approximately supported with 1 – memory if the number of players exceeds two. Furthermore, in this case all subgame perfect equilibrium payoffs can be approximately supported by an ε – equilibrium with 1 – memory. In two-player games, the same set of results hold if an additional restriction is assumed: players must have common punishments.

Finally, to illustrate the role of our assumptions, we present robust examples of games in which there is a subgame perfect equilibrium payoff profile that cannot be obtained with 1 –

*This is a revised version of chapter 2 in Barlo (2003), and of Sabourian (1989). We thank Beth Allen, Pedro Amaral, Kemal Badur, Kevin Hasker, Ehud Kalai, Narayana Kocherlakota, Andy McLennan, Han Özsöylev, Aldo Rustichini, Jan Werner, an associate editor, an anonymous referee and participants at the Applied Microeconomics Workshop and Mathematical Economics Workshop at the University of Minnesota, the Games 2002 conference, the Mid-West Mathematical Economics 2002 meeting, the SAET 2003 conference, and the ASSET 2003 and 2006 conferences, and the SED 2006 conference, and the Winter 2007 North American Meeting of the Econometric Society for helpful comments and suggestions. Any remaining errors are ours.

memory. Thus, our results are the best that can be hoped for.

Journal of Economic Literature Classification Numbers: C72; C73; C79

Keywords: Repeated Games; Memory; Bounded Rationality; Folk Theorem

1 Introduction

The (subgame perfect) Folk Theorem of complete information repeated games states that any individually rational payoffs can be sustained as a subgame perfect equilibrium (SPE) if the players are sufficiently patient — see [13] and [3]. Such a multiplicity of equilibria arises because in repeated games at any stage each player can condition his behavior on the past behavior of all the players. Such long memories are clearly unreasonable.

Many papers have modeled memory in repeated games by considering strategies that recall only a finite number past periods (see Section 2). Following this approach, in this paper, we restrict the strategies to those that depend only on what has happened in the previous period. We refer to such behavior by *1 – memory strategies*. (In this paper, memory refers to the number of past periods the players can recall; in the literature such memory limitations are also known as bounded recall.)

As suggested by Aumann [2], it is reasonable to expect that the extensive multiplicity of equilibria described by the Folk Theorem may be reduced by restricting players to limited memory strategies. In contrast, we show that if the set of actions in the stage game is sufficiently "rich" (i.e., it has a large number of actions), the Folk Theorem continues to hold with *1 – memory strategies*.

Notice that except for stationary strategies (that take the same action following every history), 1 – memory strategies form, in terms of period recall, the simplest class of repeated game strategies. Despite being simple and highly restrictive in their dependence on the past, they are, in games with rich action spaces, able to generate a large set of equilibria. This is in sharp contrast to the case of stationary strategies, which can only implement outcomes that consist of repetitions of Nash equilibria of the stage game. Thus, it is surprising that, by increasing players' memory from zero to one period, the equilibrium set expands so significantly.

The richness of the set of actions assumption is critical in establishing the Folk Theorem (and the other results in this paper) with one period memory. Before discussing the fundamental role of this assumption, note that the Folk Theorem provides a characterization of the equilibrium set when the players discount the future by arbitrarily small amounts. But even when players are impatient, equilibrium strategies often require them to remember distant pasts. Therefore, we also ask whether or not, for any *arbitrary* level of discounting, we can obtain all SPE payoffs with 1 – memory strategies, if the action sets are sufficiently rich.

For an arbitrary level of discounting, Abreu [1] showed that any SPE payoff profile can be supported by a *simple* strategy profile. We define a strictly enforceable SPE payoff profile as one that can be sustained by a simple strategy profile with the property that the players *strictly* prefer to follow the associated simple strategy at every subgame. Then our main result here establishes that if the action spaces are rich (formally, we assume that they are perfect sets[1]) and the number of players exceeds two, then any strictly enforceable SPE can be approximated by a 1 – memory SPE. We also show that the same result holds with two players if, in addition, the equilibrium considered is such that the two players have a common punishment path, i.e., the punishment path induced when one player deviates is the same as that induced when the other player deviates.

The proof of our Folk Theorem with 1 – memory follows from the above results on the implementability of strictly enforceable SPEs. This is because when players are sufficiently patient, any strictly individually rational payoff profile can be sustained by a strictly enforceable SPE, and in the case of 2-players, with a common punishment path (see [13] and [14]).[2]

Implementing equilibrium payoffs with 1 – memory is not easy and the richness of the action spaces is critical to establishing our results. The first difficulty comes from implementing an outcome path that involves playing the same action profile s a finite number of times T. Clearly, such a path cannot be implemented with 1 – memory if $T > 1$. However, if the action space is rich, then players can instead implement with 1 – memory a payoff profile close to the one of the original path by playing T *distinct* action profiles $s^1, \ldots, s^T$, each generating payoffs close to s.

But to implement (even approximately) an equilibrium payoff vector with 1 – memory, we need a great deal more than simply being able to implement a specific path. We basically require to know the state of play by observing the outcome in the previous period. If the action spaces were not rich then it would be difficult to know the state of play, and thus implement even approximately a SPE with 1 – memory strategies. To illustrate this point, in Subsection 4, we provide a repeated game example in which no efficient outcome can be (approximately) sustained by a 1 – memory SPE, even when the players are almost fully patient. In this example, the difficulty arises because each player has two pure actions and, as a result, it is difficult to identify, by recalling the previous period outcome, if a player has singly deviated. Introducing rich action spaces by allowing mixed

[1] A subset of a topological space X is a *perfect* set if X is equal to the set of all limit points of X, or equivalently, X is a closed set with no isolated points. Therefore, perfect subsets of locally compact spaces are uncountable.

[2] In establishing our Folk theorem, we show that any strictly individually rational payoff profile can be indeed induced *exactly* by a 1 – memory SPE (approximation is not necessary), when the players are sufficiently patient.

strategies in this example, on the other hand, overcomes the problem because it allows one to construct appropriate punishment paths such that the state of play is identifiable.

More generally, to sustain a simple strategy profile with 1 – memory, not only the equilibrium paths and all the n punishment paths, where n denotes the number of the players, need to be implementable with 1 – memory, but also it should be the case that (i) the action profiles used in the punishment phase for any player occur neither on the equilibrium path nor be used in the punishment phase for other players, and (ii) any single deviation can be detected by observing the previous period.[3] Otherwise, it may not be possible for players to know the state of play with 1 – memory. To rule out such ambiguities, we consider simple strategies that are such that at any stage players can, by observing only what has happened in the previous period, find out in which phase of the $n+1$ paths the play is in. We call such simple strategies *confusion-proof* and show that simple strategies have 1 – memory if and only if they are *confusion-proof.*

The assumption of rich (perfect) action spaces ensures that there are uncountably many actions close to each action of each player. Thus, any simple strategy profile can be perturbed so that each player chooses different actions at different phases and periods, and as result, in the case of more than two players, it is confusion-proof. This observation allows us to establish, for any arbitrary discount factor, our main approximation result for the case with more than two players: any strictly enforceable SPE can be perturbed to make it implementable with 1 – memory. The idea here is that in any strictly enforceable simple strategy profile, players strictly prefer playing the simple strategies at every subgame and hence there are slacks in the incentive conditions associated with such SPE; these slacks, together with the richness assumption, then allow one to perturb the equilibrium to make it confusion-proof, and hence 1 – memory, without destroying the players' incentives.

It is very important to point out that, even with rich action spaces, the above perturbation argument that replaces memory with a complex set of actions is not sufficient to implement all SPE with 1 – memory. First, if a SPE is not strictly enforceable, the incentive conditions are not all strict and as a result it might not be possible to perturb the equilibrium to make it confusion-proof without violating some incentive condition, i.e., without some slack in the incentive conditions, information about the past cannot be coded into agents' behavior. Second, with two players, in

[3]For instance, it must be the case that a player being punished cannot, by deviating from the action that the punishments prescribe, give rise to an action profile on the equilibrium outcome.

contrast to the case with more than two, a simple SPE that involve choosing different actions at different phases and periods is not necessarily confusion-proof. For instance, with two players, if at any period player 1 takes an action that belongs to the equilibrium path of a simple strategy profile and player 2 does an action that belongs to a punishment path, then with $1-$ memory it cannot be concluded whether or not it was player 2 who deviated from the equilibrium path, or if it was player 1 who deviated from the punishment path. This is clearly a problem unless both players have a common punishment path, in which case they do not need to know who has deviated.

These difficulties that arise when implementing equilibria with $1-$ memory with impatient players are illustrated in Section 7. (As our Folk Theorem demonstrates, these difficulties disappear if the players are sufficiently patient). There, we provide some robust repeated game examples with rich action spaces and discounting showing that some (efficient) SPE payoffs cannot be obtained with $1-$ memory strategies. In both examples, the action space for each player at any stage consists of the set of mixed strategies over two pure strategies and is therefore convex (and rich). In the first of the two examples an efficient SPE cannot be approximately implemented with $1-$ memory because the game has two players. In the second example the problem is due to the lack of *strict* enforceability. Moreover, these examples are generic since they remain valid for any small perturbations of the payoffs and the discount factor. Thus, the two examples show that our main results are the best that can be hoped for in terms of implementing $1-$ memory SPEs with an arbitrary level of discount factor.

In Section 7, we also briefly consider the question of $1-$ memory implementation when an approximate equilibrium concept is used. Here, we show that every SPE payoff profile, and not just those that are strictly enforceable, can be approximately supported with a $1-$ memory $\varepsilon-$ equilibrium, for all $\varepsilon > 0$ if either the number of players exceeds two or in the 2-player games a common punishment can be used.[4]

Despite the examples in Section 7, our results establish, at least for games with more than two players, that if the action spaces are rich the restriction to $1-$ memory strategies will not place severe limitations on equilibrium payoffs. Rich action spaces endow agents with the capacity to "code" information about who-deviated-when into their play; thereby, allowing us to establish our results. However, we emphasize that this ability to encode histories into play is not the same as

[4]Here $\varepsilon-$ equilibrium refers to contemporaneous perfect $\varepsilon-$ equilibrium (see [22]).

the situation in which players have an arbitrary rich set of messages: Since in our framework the kind of messages players can send are restricted to be their actions, it follows that they are costly to send.

Note, however, that the rich (perfect) action space assumption is consistent with most standard games with infinite action spaces (e.g., Bertrand and Cournot competition) because it is often assumed that the action space is a convex (and hence perfect) subset of some finite dimensional Euclidean space. Also, since the set of mixed strategies are also convex, our richness assumption is satisfied in any repeated game (with finite or infinite pure action space) in which at each stage the players choose mixed strategies and past mixed actions are observable.

In Section 2, we discuss some related literature. In Section 3, we provide the notation and the definitions. Section 4 presents an example illustrating the difficulties associated with non-rich action spaces. In Section 5, we discuss when a simple strategy profile can be implemented with $1-$ memory, by characterizing the concept of confusion-proof simple strategies. By appealing to these characterizations, we then present our main results in Section 6. In Section 7, we discuss whether or not our main results on $1-$ memory implementation for an arbitrary discount factor can be weakened. All the proofs are in the Appendix and in the supplementary material to this paper.

2 Related Literature on Bounded Memory

This paper deals with $1-$ memory recall in repeated games with large action spaces. There is also a significant literature on equilibria of repeated games with finite memory (in terms of recall). In particular, Sabourian [26] obtained a perfect Folk Theorem with bounded memory strategies for the case of repeated games with no discounting and finite number of pure actions. This paper concludes that "in order to obtain a Folk Theorem like result with finite actions, one faces a trade-off between the number of elements of the action space of the stage game and the length of the memory". Barlo, Carmona, and Sabourian [7] presents a perfect Folk Theorem for discounted 2-player repeated games with bounded memory and finite action spaces, showing that, in order for the Folk Theorem to hold, the lack of a rich action space can be compensated by expanding the size of players' memory.

Bounded memory restriction has also been studied in the context of repeated games with imperfect monitoring. Cole and Kocherlakota [11] consider the repeated prisoners' dilemma with imperfect public monitoring and finite memory. They show that for some set of parameters defection every period is the only strongly symmetric public perfect equilibrium with bounded memory (regardless of the discount factor), whereas the set strongly symmetric public perfect strategies with unbounded recall is strictly larger. The particular example considered by Cole and Kocherlakota [11] does not satisfy the identifiability condition used in Fudenberg, Levine, and Maskin [12] to establish their Folk Theorems for repeated games with imperfect monitoring. By strenghtening those identifiability conditions and by allowing asymmetric strategies, Hörner and Olszewski [15] obtained a perfect Folk Theorem with bounded memory strategies for games with (public or private but almost public) imperfect monitoring and finite action and outcome spaces. Their result, however, requires a rich set of public signal and displays a trade-off between the discount factor and the length of the memory. However, all these papers are about large but finite memory. In contrast, our present paper shows that when the action space is rich and monitoring is perfect, then $1-$ memory strategies are enough to establish the perfect Folk Theorem.[5]

Memory in terms of recall used in this paper captures one aspect of complexity of a strategy. There are clearly other aspects of complexity of a strategy which we do not address here.[6] In particular, we obtain our approximation results with $1-$ memory/recall by using paths that involve different action profiles at each date. Such paths may be complex if we use an alternative definition of complexity to the notion of memory we use in this paper. The objective here is not to tackle this general issue of complexity but simply to characterize the implications of recall restriction, and

[5]Other works on repeated games with bounded (recall) memory include Kalai and Stanford [17], Lehrer [18], Aumann and Sorin [4], Lehrer [19], Neyman and Okada [23], Bhaskar and Vega-Redondo [9], Mailath and Morris [20] and Mailath and Morris [21].

[6]A particular important class of repeated strategies that has received considerable attention are those represented by finite automata (see Kalai [16] for an early survey). Similar results to the ones obtained here appeared in Kalai and Stanford [17] and Ben-Porath and Peleg [8]. The former shows that in repeated games all SPE payoffs can be approximately supported by finite automata as an $\varepsilon-$ equilibrium for sufficiently large automata. The latter shows that if the players are sufficiently patient, the approximation with finite automata can be done in terms of full SPE, as opposed to $\varepsilon-$ equilibrium, and thereby establishes a Folk Theorem with finite automata. Neither papers assume that the action spaces are rich, but they allow for any finite size automata. Furthermore, in the case of Ben-Porath and Peleg [8], the bound on the size of the automata depends on the level of approximation and hence is not uniform (with a different measure of complexity a uniform bound on complexity may be sufficient; see Cho [10] for a neural network approach). Our result are different because we only consider one period recall.

in particular, to explain how, with some qualifications, in repeated games with rich action spaces players do not need to use much memory: remembering yesterday is almost enough to support all SPE payoffs.

3 Notation and Definitions

The stage game:

A *normal form game* G is defined by $G = \left(N, (S_i)_{i\in N}, (u_i)_{i\in N}\right)$, where $N = \{1, \dots, n\}$ is a finite set of players, S_i is the set of player i's actions and $u_i : \prod_{i\in N} S_i \to \mathbb{R}$ is player i's payoff function.

We assume that S_i is a perfect and compact metric space and that u_i is continuous for all $i \in N$. Note that if S_i is convex, then S_i is perfect. Therefore, the mixed extension of any finite normal form game satisfies the above assumptions.[7]

Let $S = \prod_{i\in N} S_i$ and $S_{-i} = \prod_{j\neq i} S_i$. Also, for any $i \in N$ denote respectively the *minmax payoff* and a *minmax profile* for player i by $v_i = \min_{s_{-i}\in S_{-i}} \max_{s_i\in S_i} u_i(s_i, s_{-i})$ and $m^i \in \arg\min_{s_{-i}\in S_{-i}} \max_{s_i\in S_i} u_i(s_i, s_{-i})$. If G is a $2-$player game, a *mutual minmax profile* is $\bar{m} = (m_1^2, m_2^1)$.

Let $D = \mathrm{co}\,(u(S)), \mathcal{U} = \{y \in D : y_i \geq v_i \text{ for all } i \in N\}$ and $\mathcal{U}^0 = \{y \in D : y_i > v_i \text{ for all } i \in N\}$. The set $\mathcal{U}$ (resp. $\mathcal{U}^0$) is *the set of (resp. strictly) individually rational payoffs.*

The repeated game:

For all $\delta \in [0, 1)$, the *supergame* $G^\infty(\delta)$ *of* G consists of an infinite sequence of repetitions of G with a common discount factor δ. We denote the action of any player i in $G^\infty(\delta)$ at any date $t = 1, 2, 3, \dots$ by $s_i^t \in S_i$. Also, let $s^t = (s_1^t, \dots, s_n^t)$ be the profile of choices at t.

For $t \geq 1$, a $t-$stage history is a sequence $h_t = (s^1, \dots, s^t)$. The set of all $t-$stage histories is denoted by $H_t = S^t$ (the $t-$fold Cartesian product of S). We use H_0 to represent the initial ($0-$stage) history. The set of all histories is defined by $H = \bigcup_{t=0}^{\infty} H_t$.

For all $i \in N$, player i's strategy is a function $f_i : H \to S_i$.[8] The set of player i's strategies is denoted by F_i, and $F = \prod_{i\in N} F_i$ is the joint strategy space with a typical element $f \in F$.

[7]More generally, the mixed extension of any normal form game with compact metric strategy spaces and continuous payoff functions also satisfies the above assumptions.

[8]Notice that when G refers to the mixed extension of a normal form game, then the strategy in the repeated game at any period may depend on past randomization choices which in such cases must be publicly observable.

Given a strategy $f_i \in F_i$ and a history $h \in H$, denote the *strategy induced at* h by $f_i|h$; thus $(f_i|h)(\bar{h}) = f_i(h, \bar{h})$, for every $\bar{h} \in H$. Also, let $f|h = (f_1|h, \ldots, f_n|h)$ for every $f \in F$ and $h \in H$.

Any strategy profile $f \in F$ induces an outcome at any date as follows: $\pi^1(f) = f(H_0)$ and $\pi^t(f) = f(\pi^1(f), \ldots, \pi^{t-1}(f))$ for any $t > 1$. Denote the set of outcome paths by $\Pi = S \times S \times \cdots$ and define the outcome path induced by any strategy profile $f \in F$ by $\pi(f) = \{\pi^1(f), \pi^2(f), \ldots\} \in \Pi$.

We consider the following memory restriction on the set of strategies in this paper. For any history $h \in H \setminus H_0$, the *1 – period tail* of h is $T(h) = s^t$ if $h = (s^1, \ldots, s^t)$ for some $t \geq 1$.

Definition 1 *A strategy $f_i \in F_i$ has* one period memory *(henceforth called 1 – memory) if $f_i(h) = f_i(\bar{h})$ for any two histories $h, \bar{h} \in H \setminus H_0$ such that $T(h) = T(\bar{h})$.*

Notice that in the above definition the choice of action at any date t depends only on the last stage of the supergame and not t; thus, 1 – memory strategies are independent of the calendar time.

Since the players discount the future by a common discount factor $\delta < 1$, the payoff in the supergame $G^\infty(\delta)$ of G is then given by

$$U_i(f, \delta) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(\pi^t(f)).$$

Also, for any $\pi \in \Pi$, $t \in \mathbb{N}$, and $i \in N$, let $V_i^t(\pi, \delta) = (1 - \delta) \sum_{r=t}^{\infty} \delta^{r-t} u^i(\pi^r)$ be the continuation payoff of player i at date t if the outcome path π is played. For simplicity, when the meaning is clear, we write $U_i(f)$, $V_i^t(\pi)$ and $V_i(\pi)$ instead of $U_i(f, \delta)$, $V_i^t(\pi, \delta)$ and $V_i^1(\pi, \delta)$, respectively.

A strategy profile $f \in F$ is a *Nash equilibrium* of $G^\infty(\delta)$ if $U_i(f) \geq U_i(\hat{f}_i, f_{-i})$ for all $i \in N$ and all $\hat{f}_i \in F_i$. A strategy profile $f \in F$ is a *SPE* of $G^\infty(\delta)$ if $f|h$ is a Nash equilibrium for all $h \in H$. An outcome path π is a *subgame perfect outcome path* if there exists a SPE f such that $\pi = \pi(f)$.

We also define a 1 – memory SPE as a SPE with the additional property that it has 1 – memory.[9]

4 An Example with Non-Rich Action Spaces

In this section, we show, by means of an example, that when the action spaces are not rich, it is possible that no efficient payoff vector can be supported by a 1 – memory SPE even if the discount

[9]Note that with this definition the equilibrium strategy of each player has 1 – memory but is best amongst *all* strategies, including those with memory longer than one. Alternatively, we could have just required optimality amongst the set of 1 – memory strategies. For the purpose of the results in this paper the two possible definitions are equivalent.

factor is near one.

Consider the following normal form game described by Table 1.

$1\backslash 2$	c	d
c	4,4	0,10
d	10,0	1,1

Table 1: Payoff function for the example with non-rich action spaces

This game is a prisoners' dilemma in which the outcome (c, c) is not efficient. In fact, the set of efficient individually rational payoffs are those that are a convex combination of $(10, 0)$ and $(0, 10)$ and give at least a payoff of 1 to each player. If there is no limit on the memory then any efficient individually rational payoffs can be sustained by a SPE if δ is sufficiently large. In particular, the path that alternates between playing (d, c) and (c, d), and thereby induces the payoff of $\frac{10}{1+\delta}$ for one player and $\frac{10\delta}{1+\delta}$ for the other player, can be sustained by a SPE if and only if $\delta \geq 1/9$. The necessity part of this claim holds because if $\delta < 1/9$ the payoff that one of the player obtains will be less than the minmax payoff of 1. The sufficiency part holds because if $\delta \geq 1/9$ then the following grim type strategy profile is a SPE: (i) on the equilibrium path play the path of alternating between (d, c) and (c, d) and (ii) if in the past there has been a deviation from the equilibrium path then play the Nash equilibrium of the one-shot game, (d, d), forever. This strategy profile cannot, however, be implemented with 1 – memory because then if (d, c) or (c, d) is observed it cannot be inferred if in the previous period the game was following the equilibrium path or if the game was in the punishment phase and one of the players had deviated from (d, d).

The problem of inferring the state of play with 1 – memory may not matter if appropriate incentives were not needed to ensure that the players follow the equilibrium strategies. For example, when $\delta = 1/9$ the path consisting of playing (d, d) forever is no worse for a player than deviating from (d, d) for one period followed by the (equilibrium) path of alternating between playing (d, c) and (c, d) — both induce an average payoff of 1. Therefore, when $\delta = 1/9$, the path that alternates between playing (d, c) and (c, d) can be supported with a 1 – memory SPE f that is otherwise identical to the grim profile except that if there is a single player deviation from (d, d) then the game follows the equilibrium path of alternating between (d, c) and (c, d).[10] But such a profile is not a SPE when $\delta > 1/9$ because then it pays to deviate from the punishment path consisting of (d, d) forever.

More generally, with 1 – memory, no efficient payoffs can be sustained by a SPE for any $\delta \neq 1/9$. To show this, suppose otherwise and assume that there is a 1 – memory SPE f such that $U(f)$ is efficient. Then, it must be that $f(H_0)$ is either (d, c) or (c, d). For concreteness, we consider the case $f(H_0) = (d, c)$ (the case $f(H_0) = (c, d)$ is analogous).

Efficiency of $U(f)$ also implies that $f(d, c)$ is either (d, c) or (c, d). However, if $f(d, c) = (d, c)$, then (d, c) is played forever implying that $U_2(f)$ is not individually rational and f is not SPE. Thus, $f(d, c) = (c, d)$. Similarly, it follows that $f(c, d) = (d, c)$. Therefore, the equilibrium path must consist of alternating between (d, c) and (c, d). Since such a path can be sustained as a SPE only if $\delta \geq 1/9$ and, by assumption, $\delta \neq 1/9$, it follows that δ must exceed 1/9.

The above shows that implementing efficient payoffs by 1 – memory SPE strategies imposes severe restrictions on the equilibrium path. We now show that 1 – memory also restricts the off-the-equilibium paths so that f cannot be SPE, and hence we obtain a contradiction to the supposition that a 1 – memory efficient SPE exists.

First, note that $f(d, d) \neq (d, d)$. This is because, otherwise player 1 can profitably deviate at the history $h = (d, d)$: instead of playing $f_1(d, d) = d$, that would yield him a discounted payoff of 1, he can play c and then follow f_1. By doing so, and despite losing a payoff of 1 in the current period, he is able to bring the play back to the equilibrium path, obtaining a discounted payoff of $10\delta/(1 + \delta) > 1 = U_1(f|h)$.

Second, it must be that $f(d, d) = (c, c)$. To establish this, note first that if $f(d, d) = (c, d)$, then player 1 has a profitable deviation at the history $h = (d, c)$: by playing according to f_1 at h, he receives $10\delta/(10 + \delta)$; by playing d at h and then following f_1, he gets $1 - \delta + 10\delta/(1 + \delta) > 10\delta/(1 + \delta) = U_1(f|h)$. A symmetric argument shows that if $f(d, d) = (c, d)$, then player 2 can profitably deviate. Hence, it follows that $f(d, d) = (c, c)$.

Third, it must be that $f(c, c) \neq (c, c)$. The argument is similar to the one used to show that $f(d, d) \neq (d, d)$. As in there, if $f(c, c) = (c, c)$, then player 1 can profitably deviate at $h = (c, c)$ by playing d and bringing the play back to the equilibrium path: his payoff would equal

[10] Formally, f is such that $f(H_0) = (d, c)$, $f(d, c) = (c, d)$, $f(c, d) = (d, c)$ and $f(c, c) = f(d, d) = (d, d)$.

$10/(1+\delta) > 4 = U_1(f|h)$.

Fourth, we also cannot have that $f(c,c) = (d,c)$ nor $f(c,c) = (c,d)$. If $f(c,c) = (d,c)$, then player 1 can profitably deviate at $h = (d,d)$: he receives $4(1-\delta)+10\delta/(1+\delta)$ by playing according with f at h, while, by playing d instead of c and then following f_1, he gets $10/(1+\delta) > 4(1-\delta)+10\delta/(1+\delta)$. Similarly, if $f(c,c) = (c,d)$, then player 2 can deviate at $h = (d,d)$.

Finally, there must be a profitable deviation by player 1 at $h = (d,d)$, and hence f cannot be a SPE. This is because, by the previous points, $f(c,c)$ must equal (d,d), and thus $f|h$ induces the repetition of $((c,c),(d,d))$ and an average payoff of $(4+\delta)/(1+\delta)$ for each player; while, player 1 by playing d instead of c at h and then following f_1 obtains $10/(1+\delta) > (4+\delta)/(1+\delta)$.

In the above example the lack of efficient 1 – memory SPEs arises because the action space for each player consists of two elements. In order to illustrate this point, assume that players could choose a third action, consisting of choosing c with probability 0.01 and d with probability 0.99. We assume that the randomization devices are observable. Denoting this strategy by r, the resulting normal form game can be described by Table 2.

$1\backslash 2$	c	d	r
c	4,4	0,10	4/100, 994/100
d	10,0	1,1	109/100,99/100
r	994/100,4/100	99/100,109/100	2159/2000,2159/2000

Table 2: Payoff function when players can randomize

In this case, we claim that the efficient outcome path $\pi = ((d,c),(c,d),\ldots)$ can be supported by a 1 – memory strategy whenever $\delta > 1/4$. In order to prove this claim, we shall define a strategy profile f as follows: $f(H_0) = (c,d)$ and, for all $h \in H\backslash H_0$, $f(h) = g(T(h))$ where $g : \{c,d,r\}^2 \to \{c,d,r\}^2$ is given by

$$g(a,b) = \begin{cases} (b,a) & \text{if } a,b \in \{c,d\} \text{ and } a \neq b \\ (c,d) & \text{if } (a,b) = (r,r) \\ (r,r) & \text{otherwise.} \end{cases}$$

Thus, f is a 1 – memory and simple, and plays the path $\pi = ((d,c),(c,d),(d,c),(c,d),\ldots)$ on the equilibrium path and has a common punishment path $((r,r),\pi)$. Furthermore, a single deviation from π or from (r,r) will be punished by playing (r,r) once, and then returning afterwards to π.

To show that f is SPE, one only needs to establish the follows inequalities:

$$\begin{aligned}\frac{10\delta}{1+\delta} &\geq 1-\delta+\delta(1-\delta)\frac{2159}{2000}+\delta^2\frac{10}{1+\delta}\\ (1-\delta)\frac{2159}{2000}+\delta y &\geq (1-\delta)\frac{109}{100}+\delta(1-\delta)\frac{2159}{2000}+\delta^2 y,\end{aligned}$$

for all $y \in \{10/(1+\delta), 10\delta/(1+\delta)\}$. Since $\delta > 1/4$, it is easy to verify that the above inequalities are indeed satisfied.

The above establishes that if the (random) action r, defined above, is available then the efficient outcome path $\pi = ((d,c),(c,d),\ldots)$ can be supported by a $1-$memory SPE whenever $\delta > 1/4$. More generally, if at each stage the players could choose any mixed action (randomize between c and d), so that they had a rich action space, and their randomized strategies were to be publicly observable, then the efficient outcome path $\pi = ((d,c),(c,d),\ldots)$ can be supported by a $1-$ memory strategy profile whenever $\delta > 1/9$. Furthermore, in this case, it can be shown (see Theorem 2 below) that all individually rational payoffs (efficient or not) can be sustained by a $1-$ memory SPE provided that δ is sufficiently close to one.

5 Confusion-Proof Paths and $1-$ Memory

Abreu [1] used the concept of simple strategies to characterize the set of subgame perfect equilibria. In this section, we consider simple strategy profiles that can be implemented with $1-$ memory. For this purpose, we introduce the notion of a confusion-proof profile of outcome paths and show in Proposition 1 below that a profile of outcome paths can be supported by a $1-$ memory simple strategy profile if and only if it is confusion-proof. The concept of confusion-proof outcome paths is our main tool and is used throughout the paper to establish our main results for $1-$ memory SPEs.

Following Abreu [1], $f \in F$ is *a simple strategy profile represented by* $n+1$ *paths* $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ if f specifies: (i) play $\pi^{(0)}$ until some player deviates singly from $\pi^{(0)}$; (ii) for any $j \in N$, play $\pi^{(j)}$ if the jth player deviates singly from $\pi^{(i)}$, $i = 0,1,\ldots,n$, where $\pi^{(i)}$ is the ongoing previously specified path; (iii) if two or more players deviate simultaneously from $\pi^{(i)}$, $i = 0,1,\ldots,n$, then play $\pi^{(j)}$ for some $j \in \{0,1,\ldots,n\}$; (iv) continue with the ongoing specified path $\pi^{(i)}$, $i = 0,1,\ldots,n$, if no deviations occur. These strategies are simple because the play of the game is always in only $n+1$ *states*, namely, in state $j \in \{0,\ldots,n\}$ where $\pi^{(j),t}$ is played, for some $t \in \mathbb{N}$. In this case, we say that the play is in *phase t of state j*. A profile $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ of $n+1$ outcome paths is *subgame perfect* if the simple strategy profile represented by it is a SPE.

Henceforth, when the meaning is clear, we shall use the term $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ to refer to both an $n+1$ outcome paths as well as to the simple strategy profile represented by these paths. Also, when referring to a profile of $n+1$ outcome paths, we shall not always explicitly mention $n+1$ and simply refer to it by a profile of outcome paths.

Now fix a simple strategy profile given by $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$. The notion of a confusion-proof profile of outcome paths (that is used to characterize 1 – memory simple strategies) is motivated by the following observations. For the simple strategy profile to be supported by a 1 – memory simple strategy profile, players need to infer the correct phase of the correct state of play by only observing the action profile played in the previous period. This is not always possible because three kinds of complications can arise if the strategies have 1 – memory.

The first kind of complication happens when $\pi^{(i),t} = \pi^{(j),r}$ for some $i, j \in \{0, 1, \ldots, n\}$ and $t, r \in \mathbb{N}$. That is, the action profile in phase t of state i is the same as that in phase r of state j. Since players condition their behavior only on the last period's action profile, the players cannot distinguish between phase t of state i and phase r of state j, and therefore the simple strategy profile cannot be implemented, unless $\pi^{(i),t+1} = \pi^{(j),r+1}$.

The second kind of complication arises when $\pi_k^{(i),t} \neq \pi_k^{(j),r}$ and $\pi_{-k}^{(i),t} = \pi_{-k}^{(j),r}$ for some $i, j \in \{0, 1, \ldots, n\}$, $k \in N$ and $t, r \in \mathbb{N}$; i.e., every player other than $k \in N$ takes the same action in phase t of state i and in phase r of state j. Then if, for example, the last period's action profile is $\pi^{(j),r}$, the players would not be able to deduce whether the play in the previous period was in phase t of state i and player k deviated to $\pi_k^{(j),r}$ or whether it was in phase r of state j and no deviation has occured. Since a deviation by player k from $\pi_k^{(i),t}$ to $\pi_k^{(j),r}$ in phase t of state i is impossible to detect by observing only the action in the last period, the simple strategy profile cannot be implemented, unless $\pi^{(i),t+1} = \pi^{(j),r+1} = \pi^{(k),1}$.

The third kind of complication appears when $\pi_l^{(i),t} \neq \pi_l^{(j),r}$, $\pi_m^{(i),t} \neq \pi_m^{(j),r}$ and $\pi_{-l,m}^{(i),t} = \pi_{-l,m}^{(j),r}$ for some $i, j \in \{0, 1, \ldots, n\}$, $l, m \in N$ and $t, r \in \mathbb{N}$. In words, all players other than l and $m \in N$ take the same action both in phase t of state i, and in phase r of state j. Then, if the last period's action profile is given by $(\pi_l^{(i),t}, \pi_m^{(j),r}, (\pi_k^{(i),t})_{k \neq l,m}) = (\pi_l^{(i),t}, \pi_m^{(j),r}, (\pi_k^{(j),r})_{k \neq l,m})$, players, looking back one period, can conclude that either player l or player m has deviated. But, they cannot be certain of the identity of the deviator. Consequently, both of them must be punished. This requires $\pi^{(l)} = \pi^{(m)}$.

These observations are formalized below as follows. For any profile of outcome paths $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)}) \subseteq \Pi^{n+1}$, let

$$\Omega\left(\{i,t\},\{j,r\}\right) = \{k \in N : \pi_k^{(i),t} \neq \pi_k^{(j),r}\}$$

be the set of players whose actions in phase t of stage i and in phase r stage j are different.

Definition 2 *A profile* $(\pi^{(0)}, \ldots, \pi^{(n)}) \in \Pi^{n+1}$ *of outcome paths is* confusion-proof *if for any* $i, j \in \{0, 1, \ldots, n\}$ *and* $t, r \in \mathbb{N}$ *the following holds:*

1. *If* $\Omega(\{i,t\},\{j,r\}) = \emptyset$, *then* $\pi^{(i),t+1} = \pi^{(j),r+1}$.
2. *If* $\Omega(\{i,t\},\{j,r\}) = \{k\}$ *for some* $k \in N$, *then* $\pi^{(i),t+1} = \pi^{(j),r+1} = \pi^{(k),1}$.
3. *If* $\Omega(\{i,t\},\{j,r\}) = \{k,l\}$ *for some* k *and* $l \in N$, *then* $\pi^{(k)} = \pi^{(l)}$.

The above observations, which motivated the definition of confusion-proof outcome paths, suggest that confusion-proofness is necessary to support a profile of outcome paths with an $1-$memory simple strategy profile. The next Proposition asserts that confusion-proofness is, in fact, not only a necessary but also a sufficient condition to support a profile of outcome paths with $1-$memory.

Proposition 1 *A profile of outcome paths is confusion-proof if and only if there exists a* $1-$*memory simple strategy profile represented by it.*

The $1-$memory strategy profile f supporting the confusion-proof profile of outcome paths $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)}) \subseteq \Pi^{n+1}$ is as follows: If the last period of a given history equals $\pi^{(j),t}$, for some $j = 0, 1, \ldots, n$ and $t \in \mathbb{N}$, then player i chooses $\pi_i^{(j),t+1}$. If only player $k \in N$ deviated from the outcome $\pi^{(j)}$ in the last period of the history, then player i chooses $\pi_i^{(k),1}$. Finally, if more then one player deviated from the outcome $\pi^{(j)}$ in the last period of the history, then player i chooses $\pi_i^{(m),k}$ for some $m \in \{0, 1, \ldots, n\}$ and $k \in \mathbb{N}$. Since f has $1-$memory and has the structure of a simple strategy profile, we say that f is a *$1-$memory simple strategy.* As before, the profile $(\pi^{(0)}, \ldots, \pi^{(n)})$ represents f. The main task of the sufficiency part of the proof of Proposition 1 is to show that f is well defined, which we show follows from $(\pi^{(0)}, \ldots, \pi^{(n)})$ being confusion-proof.

Before turning to the equilibrium characterization with 1 – memory, we shall next provide a set of easily tractable sufficient conditions for a profile of outcome paths to be confusion-proof. These are different for the case of 2 – player games from that with three or more players because, as in the implementation literature, when there are only two players, it may not be possible to detect which of the two players have deviated and as a result both must be punished with the same punishment path whenever a deviation is detected.

Lemma 1 *A profile of outcome paths $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ is confusion-proof if one of the following conditions holds:*

1. *The number of players exceeds two and for all $i, j \in \{0, 1, \ldots, n\}$ and $t, r \in \mathbb{N}$ satisfying $(i, t) \neq (j, r)$ the number of players whose actions in phase t of stage i and in phase r stage j are different is at least three: $|\Omega(\{i, t\}, \{j, r\})| \geq 3$.*
2. *The number of players equals two and the following two conditions hold:*
   - *(a) players have the same punishment path: $\pi^{(1)} = \pi^{(2)}$;*
   - *(b) for all $i, j \in \{0, 1, 2\}$ and $t, r \in \mathbb{N}$ satisfying $(i, t) \neq (j, r)$ and $\{i, j\} \neq \{1, 2\}$ the actions of each agent is distinct: $\pi_l^{(i),t} \neq \pi_l^{(j),r}$ for any $l = 1, 2$.*

The sufficient conditions specified in Lemma 1 are not necessary for confusion-proofness. However, it is worth pointing out that when $n = 2$ the common punishment condition specified in 2a is almost necessary: confusion-proof in this case implies that the punishment paths for the two players are identical beyond the first period of punishments (See [6] and the supplementary material for details).

6 Main Results

6.1 1 – Memory SPE with an Arbitrary Discount Factor

To establish our results for SPE payoffs which can be implemented with 1 – memory, we start with Abreu's [1] characterization of SPE outcome paths in terms of simple strategies.[11] A profile of

[11] One can ask whether all 1 – memory SPE payoffs can be obtained using 1 – memory simple strategies. Although interesting, we do not address this question.

outcome paths $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)}) \in \Pi^{n+1}$ is *weakly enforceable* if

$$V_i^t(\hat{\pi}^{(j)}) \geq (1-\delta) \sup_{s_i \neq \hat{\pi}_i^{(j),t}} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) + \delta V_i(\hat{\pi}^{(i)}) \tag{1}$$

for all $i \in N$, $j \in \{0, 1, \ldots, n\}$ and $t \in \mathbb{N}$. Abreu [1] showed that subgame perfection is equivalent to weak enforceability. More precisely, an outcome path $\hat{\pi}^{(0)}$ is a SPE outcome path if and only if there exists a weakly enforceable profile of outcome paths $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$.

It then follows from Proposition 1 that any weakly enforceable, confusion-proof profile of outcome paths can be supported by a 1 – memory simple SPE strategy profile. Thus, in terms of payoffs, we have the following corollary to Proposition 1.

Corollary 1 *Let u be SPE payoff vector that can be supported by a weakly enforceable, confusion-proof profile of outcome paths. Then, there is a 1 – memory SPE strategy profile f such that $U(f) = u$.*[12]

In general, for an arbitrarily discount factor, we cannot support all subgame perfect payoff vectors by 1 – memory strategies. In fact, the best that can be hoped for is to obtain them approximately. Our approach will thus be to approximate any SPE with an arbitrarily close 1 – memory strategy profile. The difficulty is to ensure that the latter is also a SPE. This can be achieved if the original equilibrium is robust to perturbations.

More formally, since any SPE has to be weakly enforceable, it turns out that a slack in the incentive conditions (1) is needed in order to perform the required approximations. This leads us to introduce the notion of strictly enforceable SPE.

Definition 3 *A simple strategy profile defined by $n+1$ outcome paths $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$ is a* strictly enforceable SPE *if*

$$\inf_{t \in \mathbb{N}} \left(V_i^t(\hat{\pi}^{(j)}) - \left((1-\delta) \sup_{s_i \neq \hat{\pi}_i^{(j),t}} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) + \delta V_i(\hat{\pi}^{(i)}) \right) \right) > 0 \tag{2}$$

for all $i \in N$ and all $j \in \{0, 1, \ldots, n\}$. Furthermore, a payoff vector $u \in \mathbb{R}^n$ is a strictly enforceable SPE payoff profile *if there exists a strictly enforceable SPE strategy profile f such that $u = U(f)$.*

[12] Note that the rich action spaces assumption (S_i is perfect for all $i \in N$) is not needed to establish this corollary.

The next theorem demonstrates that under certain conditions, any strictly enforceable SPE payoff can be approximated by a 1 – memory strictly enforceable SPE payoff profile.

Theorem 1 *Let u be a strictly enforceable SPE payoff vector induced by the simple strategy profile $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$. Then, for any approximation level $\eta > 0$, there is a 1 – memory strictly enforceable SPE payoff profile u' such that $\|u' - u\| < \eta$, provided that either $n \geq 3$, or $n = 2$ and $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$.*

The proof of Theorem 1 uses the perfectness of S_k and the continuity of u_k for all $k \in N$ to construct a confusion-proof profile of outcome paths $(\bar{\pi}^{(0)}, \ldots, \bar{\pi}^{(n)})$ that is arbitrarily close to $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$ in terms of the distance in payoffs. In fact, since S_k is perfect, there are uncountably many actions close to $\hat{\pi}_k^{(i),t}$, for all $k \in N$, $i \in \{0, 1, \ldots, n\}$ and $t \in \mathbb{N}$. Hence, fixing any i and t, it is possible to choose an action profile $\bar{\pi}^{(i),t}$ arbitrarily close to $\hat{\pi}^{(i),t}$ such that, for all $k \in N$, $\bar{\pi}_k^{(i),t} \neq \bar{\pi}_k^{(j),r}$ for all $j \in \{0, 1, \ldots, n\}$ and $r \in \mathbb{N}$ such that either $r < t$ or $r = t$ and $j < i$. But this means that when $n > 2$, by induction, we can construct a profile of outcome paths $(\bar{\pi}^{(0)}, \ldots, \bar{\pi}^{(n)})$ satisfying (α) $\bar{\pi}_k^{(i),t} \neq \bar{\pi}_k^{(j),r}$ holds for all $k \in N$, $i, j \in \{0, 1, \ldots, n\}$ and $t, r \in \mathbb{N}$ such that $(i, t) \neq (j, r)$ and (β) $u_k(\bar{\pi}^{(i),t})$ is close to $u_k(\hat{\pi}^{(i),t})$ for all $k \in N$, $i \in \{0, 1, \ldots, n\}$ and $t \in \mathbb{N}$. Property (α) together with Lemma 1 then imply that $(\bar{\pi}^{(0)}, \ldots, \bar{\pi}^{(n)})$ is confusion-proof when $n \geq 3$ and property (β) implies that it is also a strictly enforceable SPE. Furthermore, when $n = 2$, the assumption that $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$ implies that $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \bar{\pi}^{(2)})$ can be chosen so that $\bar{\pi}^{(1)} = \bar{\pi}^{(2)}$ and still satisfies properties (α) (modified by replacing $i, j \in \{0, 1, 2\}$ by $i, j \in \{0, 1\}$) and (β).

6.2 Folk Theorem with 1 – Memory

In Theorem 1, the approximate implementation with 1 – memory is obtained for the set of SPE payoffs that are strictly enforceable and, for the case with two players, have common punishment paths. For an arbitrary discount factor, these restrictions to strictly enforceable SPE payoffs and common punishment paths when $n = 2$, cannot be weakened as the examples in the next section demonstrate. On the other hand, it turns out that in the limit, as the discount factor tends to 1, these constraints on the set of SPEs restrict nothing and, furthermore, there is no need for approximating the set of SPE payoffs. Thus, the Folk Theorem remains valid with 1 – memory.

Indeed, when the stage game is full-dimensional ($\dim(\mathcal{U}) = n$), an *approximate* Folk Theorem with 1 – memory follows immediately from Theorem 1 and Proposition 1 in Fudenberg and Maskin

[14] and its proof. First, by the latter, if $\dim(\mathcal{U}) = n$, every strictly individually rational payoff profile can be supported by a strictly enforceable SPE strategy profile whenever the discount factor is sufficiently close to one. Second, by the former, every strictly enforceable SPE can be approximately implemented by a $1 -$ memory SPE if $n > 2$. Therefore, we have the following approximate Folk Theorem as a corollary to Theorem 1.

Corollary 2 *Suppose that $n > 2$ and $\dim(\mathcal{U}) = n$. Then, for every individually rational payoff $u \in \mathcal{U}$ and any level of approximation $\eta > 0$, there exists $\delta^* \in (0, 1)$ such that for all $\delta \geq \delta^*$, there is a $1 -$ memory SPE $f \in F$ with $\|U(f, \delta) - u\| < \eta$.*

When $n = 2$, the same approximate Folk Theorem with $1 -$ memory also holds (in this case the full-dimensionality assumption is not needed). But to demonstrate this there is an additional difficulty because, with two players, Theorem 1 requires a common punishment path. Therefore, to overcome the difficulty one needs to extend the results in Fudenberg and Maskin [14] to show that, with $n = 2$, every strictly individually rational payoff can be supported by a strictly enforceable SPE simple strategy profile with a common punishment, whenever the discount factor is sufficiently close to one. This can be done using the punishment used in Fudenberg and Maskin's [13] Folk Theorem with two players that consists of playing the mutual minmax action profile for a finite number of periods (see the proof of Theorem 2 below).

The approximate Folk Theorem described in Corollary 2 (and its extension to 2-player games) can in fact be made stronger by showing that any payoff profile belonging to the relative interior of the set of individually rational payoff profiles $\mathcal{U}$ can be sustained *exactly* by a $1 -$ memory SPE. Formally, let $\mathrm{ri}(\mathcal{U})$ be the relative interior of the set $\mathcal{U}$ with respect to the topology induced by the affine hull of $\mathcal{U}$ (if $\dim(\mathcal{U}) = n$, then $\mathrm{ri}(\mathcal{U})$ equals the interior of $\mathcal{U}$); then we can establish the following exact Folk Theorem with $1 -$ memory.

Theorem 2 *Suppose that either $\dim(\mathcal{U}) = n$ or $n = 2$ and $\mathcal{U}^0 \neq \emptyset$. Then, for all $u \in \mathrm{ri}(\mathcal{U})$, there exists $\delta^* \in (0, 1)$ such that for all $\delta \geq \delta^*$, there is a $1 -$ memory SPE $f \in F$ with $U(f, \delta) = u$.*[13]

[13]As in the full memory case, it can be shown that our $1 -$ memory Folk Theorem can be extended to the case when the players do not discount the future, without assuming the full-dimensionality condition (see the supplementary material of this paper for the details).

There are two parts to the proof of Theorem 2. First, we find conditions that ensure that a strictly enforceable SPE payoff profile can be implemented exactly with 1 – memory. Second, by extending Proposition 1 in Fudenberg and Maskin [14], we show that any $u \in \mathrm{ri}(\mathcal{U})$ is strictly enforceable and satisfies these conditions, if the players are sufficiently patient.

Turning to the first part of the proof, recall that by Theorem 1 any strictly enforceable payoff u, induced by a simple strategy profile $(\hat{\pi}^{(0)}, \dots, \hat{\pi}^{(n)})$, can approximately implemented by a 1 – memory SPE if either $n \geq 3$, or $n = 2$ and $\pi^{(1)} = \pi^{(2)}$. To implement any such u exactly by a 1 – memory SPE, it is sufficient to assume that, in addition, the equilibrium path $\hat{\pi}^{(0)}$ is implementable with 1 – memory (i.e., $\hat{\pi}^{(0)}$ does not involve any confusing instances). Formally, a single outcome path π is confusion-proof (and so can be implemented with 1 – memory) if for any two periods $t, r \in \mathbb{N}$ the following two conditions (similar to those made in Definition 2) hold:

$$\text{If } \pi^t = \pi^r, \text{ then } \pi^{t+1} = \pi^{r+1}.$$
$$\text{If } \pi^t \neq \pi^r, \text{ then } \left|\{k \in N : \pi_k^t \neq \pi_k^r\}\right| \geq \min\{n, 3\}.$$

Also, let Π^{cp} to be the set of all single confusion-proof paths. Then, we have the following exact 1 – memory implementation analogue of Theorem 1.[14]

Lemma 2 *Suppose that the payoff vector u can be induced by a strictly enforceable simple strategy profile $(\hat{\pi}^{(0)}, \dots, \hat{\pi}^{(n)})$ and $\hat{\pi}^{(0)} \in \Pi^{cp}$. Then, there is a 1 – memory SPE strategy profile $f \in F$ such that $U(f) = u$, provided that either $n \geq 3$, or $n = 2$ and $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$.*

The above lemma can be established as follows. Since $(\hat{\pi}^{(0)}, \hat{\pi}^{(1)}, \dots, \hat{\pi}^{(n)})$ is strictly enforceable, the profile $(\hat{\pi}^{(0)}, \bar{\pi}^{(1)}, \dots, \bar{\pi}^{(n)})$ is also a strictly enforceable SPE for any punishment profile $(\bar{\pi}^{(1)}, \dots, \bar{\pi}^{(n)})$ that is sufficiently close to the punishment profile $(\hat{\pi}^{(1)}, \dots, \hat{\pi}^{(n)})$; furthermore, since S_i is perfect and u_i is continuous for all $i \in N$, it follows, as in the proof of Theorem 1, that $(\bar{\pi}^{(1)}, \dots, \bar{\pi}^{(n)})$ can be appropriately chosen so that $(\hat{\pi}^{(0)}, \bar{\pi}^{(1)}, \dots, \bar{\pi}^{(n)})$ is confusion-proof and hence 1 – memory implementable.[15]

[14] In this paper, we have assumed that 1 – memory strategy cannot depend on the calendar time. When actions can be conditioned on the calender time, then all outcome paths, and not just all the single confusion-proof ones, can be implemented with 1 – memory strategies. This allows one to show that, with time-dependence, any strictly enforceable SPE can be implemented exactly as a 1 – memory SPE (see the supplementary material for the details).

[15] To save space, we shall omit the details of the proof of this lemma.

To complete the proof of Theorem 2, it then follows from Lemma 2 that it is sufficient to show that, when δ is close to 1, every $u \in \text{ri}(\mathcal{U})$ can be exactly induced by a strictly enforceable path profile $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$ such that $\hat{\pi}^{(0)} \in \Pi^{cp}$ and $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$ when $n = 2$. Thus, we need to extend Proposition 1 in Fudenberg and Maskin [14] (and its extension to $n = 2$) to ensure that the equilibrium path $\hat{\pi}^{(0)}$ corresponding to any such u belongs to the set Π^{cp} and, as in [14], is such that, for every player, his continuation payoff at any date is bounded away from his minmax payoff. This we establish with the help of the following Lemma.[16]

Lemma 3 *Suppose that either* $\dim(\mathcal{U}) = n$ *or* $n = 2$ *and* $\mathcal{U}^0 \neq \emptyset$. *Then, for all* $u \in \text{ri}(\mathcal{U})$, *there exist* $\delta' \in (0, 1)$ *and* $\zeta > 0$ *with the following property: for all* $\delta \geq \delta'$ *there exists* $\pi \in \Pi^{cp}$ *such that* $u = V(\pi, \delta)$ *and* $V_i^t(\pi, \delta) > v_i + \zeta$ *for all* $i \in N$.

7 Counter-Examples and Approximate Equilibrium

Theorem 2 above demonstrates that 1 – memory does not restrict the set of SPE payoffs when the discount factor is allowed to be arbitrarily close to 1. On the other hand, when the players are impatient, in Theorem 1 we could only show that every SPE payoff profile is approximately implementable with 1 – memory if (i) it is strictly enforceable and (ii) either the number of players exceeds two or punishment paths were common. Below, using two examples, we show that both of these conditions are essential in the analysis with arbitrary discount factors, and the lack of either could result in 1 – memory SPE being a strict subset of SPE. The first example considers a 2-player game in which an efficient payoff profile can be supported (even approximately) as a strictly enforceable SPE only if the players punishment paths are different. Imposing 1 – memory, however, restricts the punishment paths that can be implemented and, as a result, rules out such an efficient payoff profile as a SPE. The second example deals with a 3-player game in which the lack of strict enforceability delivers the same conclusion.

Finally, we end this section by briefly considering the notion of 1 – memory ϵ-equilibrium implementation, for any small $\epsilon > 0$. As we mentioned before in footnote 6, Kalai and Stanford [17] showed that, for any arbitrary discount factor, every SPE payoff profile can be implemented by a finite complexity ε – equilibrium. We show that the same conclusion holds for 1 – memory

[16]We are thankful to an anonymous referee for suggesting to us this lemma and the statement of Theorem 2.

implementation if either the number of players exceeds two or the SPE has a common punishment path. Thus, even though the assumption of a common punishment for 2-players games cannot be weakened, the strict enforceability assumption is not needed to implement the set of SPEs with 1 – memory as ε – equilibria.

7.1 A Two Player Example

Consider the following normal form game described by Table 3. Assume that players can randomize. Let $S_i = [0,1]$ for all $i = 1,2$, where $s_i \in S_i$ is to be interpreted as the probability assigned by player i to action a. Note that the minmax payoff is 2 for each player. In the case of player 1, it can only be obtained by $m^1 = (a,b)$, while $m^2 = (b,a)$ is the only action profile leading to the minmax payoff for player 2. Moreover, both m^1 and m^2 are Nash equilibria of the stage game.

1\2	a	b
a	4,4	2,5
b	5,2	0,0

Table 3: Payoff function for the 2-player example with rich action spaces

Now suppose that the above game is played infinitely often and each player's randomization device is observable to the other. If there are no bounds on the memory and $\delta \geq 1/3$, then the payoff vector $(4,4)$ can be sustained as a SPE by a simple strategy profile that plays (a,a) at each date on the equilibrium path and punishes deviations from player $i = 1,2$ by playing m^i forever. Furthermore, this simple strategy profile is strictly enforceable if $\delta > 1/3$.

Since the punishment paths for the two players in the above simple profile are different, this profile cannot be implemented with 1 – memory because it may not be possible to know who has deviated and therefore whom to punish (for example, with 1 – memory, if $m^2 = (b,a)$ is observed it cannot be inferred if in the previous period player 1 has deviated from (a,a) or if player 2 was being punished). The impact of the 1 – memory restriction is, in fact, more profound than not being able to implement the above strategy profile. Below we show that that there exists $\delta' > 1/3$ such that, for all $\delta \in [1/3, \delta')$, the SPE payoff of $(4,4)$ cannot be even approximately implemented by any 1 –

memory SPE strategies.[17]

To show this, fix any $\delta \geq 1/3$, any $\eta > 0$ and consider any 1 – memory SPE f such that $\|U(f) - (4,4)\| < \eta$. For each player $i = 1, 2$, denote the equilibrium action taken by player i at date 1 by $p_i = f_i(H_0)$ and let $g_i : [0,1]^2 \to [0,1]$ be such that $f_i(h) = g_i(T(h))$ for all $h \in H \setminus H_0$. Then the following claims hold for any such strategy profile f.

Claim 1: $p_i \geq 1 - \frac{2\eta}{7(1-\delta)}$ **for each** i**.**

Let $V^t = V_1^t(\pi(f)) + V_2^t(\pi(f))$ for ant $t \in \mathbb{N}$. Clearly, $V^t \leq 8$. Moreover,

$$V^1 = (1-\delta)(8p_1p_2 + 7p_1(1-p_2) + 7p_2(1-p_1)) + \delta V^2 \leq (1-\delta)(7p_1 + 7p_2 - 6p_1p_2) + 8\delta. \quad (3)$$

Since $\|U(f) - (4,4)\| < \eta$, it follows that $8 - 2\eta < V^1$. But then, by (3), $8 - 2\eta < (1-\delta)(p_i + 7) + 8\delta$ for each $i = 1, 2$. Therefore, $p_i \geq 1 - \frac{2\eta}{7(1-\delta)}$.

Claim 2: For any $q \in S_2$**,** $g_1(p_1, q) < \alpha_1 + \beta_1 q$ **where** $\alpha_1 = \frac{4+\eta-2\delta-5(1-\delta)p_1}{2\delta(1-\delta)}$ **and** $\beta_1 = \frac{3p_1-2}{2\delta}$**.**

Consider a deviation by player 2 to a strategy $\bar{f}_2$ defined by $\bar{f}_2(H_0) = q$ and $\bar{f}_2(h) = 1$ for all $h \in H \setminus H_0$. Then, since player 2 can guarantee himself a payoff of 2 in every period, this deviation would him a payoff of

$$U_2(f_1, \bar{f}_2) \geq (1-\delta)[4p_1q + 5p_1(1-q) + 2(1-p_1)q] + (1-\delta)\delta[4g_1(p_1,q) + 2(1 - g_1(p_1,q))] + 2\delta^2. \quad (4)$$

Then since f is a SPE, the claim follows from (4) and $4 + \eta > U_2(f) \geq U_2(f_1, \bar{f}_2)$.

Claim 3: For any $q \in S_1$**,** $g_2(q, p_2) < \alpha_1 + \beta_1 q$ **where** $\alpha_2 = \frac{4+\eta-2\delta-5(1-\delta)p_2}{2\delta(1-\delta)}$ **and** $\beta_2 = \frac{3p_2-2}{2\delta}$**.**

This follows by an analogous reasoning as in Claim 2.

Claim 4: For all $q \in S_2$**,** $g_2(p_1, q) \geq \alpha_3 - \beta_3 g_1(p_1, q)$ **where** $\alpha_3 = \frac{2-5\delta}{3(1-\delta)}$ **and** $\beta_3 = \frac{2}{3}$**.**

To see this, consider for player 1 the strategy $\bar{f}_1$ of playing a at every history: $\bar{f}_1(h) = 1$ for all $h \in H$. Note that for all $q \in S_2$

$$U_1(\bar{f}_1, f_2 | (p_1, q)) \geq (1-\delta)[4g_2(p_1,q) + 2(1 - g_2(p_1,q))] + 2\delta = 2 + (1-\delta)2g_2(p_1,q). \quad (5)$$

We also have that

$$\begin{aligned} U_1(f|(p_1,q)) &\leq (1-\delta)[4g_1(p_1,q)g_2(p_1,q) + 2g_1(p_1,q)(1-g_2(p_1,q)) + 5(1-g_1(p_1,q))g_2(p_1,q)] + 5\delta \\ &= (1-\delta)[2g_1(p_1,q) + 5g_2(p_1,q) - 3g_1(p_1,q)g_2(p_1,q)] + 5\delta \leq (1-\delta)[2g_1(p_1,q) + 5g_2(p_1,q)] + 5\delta. \end{aligned} \quad (6)$$

[17] In the supplementary material to this paper, we also prove that this conclusion is robust to any small perturbations in the stage game payoffs.

Again, since f is SPE, we have $U_1(f|(1,q)) \geq U_1(\bar{f}_1, f_2|(1,q))$. This, together with (5) and (6), implies the claim.

Next, consider player 1 deviating from the equilibrium path by choosing strategy $\tilde{f}_1$ defined by $\tilde{f}_1(H_0) = 0$, $\tilde{f}_1(0,p_2) = p_1$ and $\tilde{f}_1(h) = 1$ for all $h \in H \setminus \{H_0, (0,p_2)\}$. Then,

$$\begin{aligned} U_1(\tilde{f}_1, f_2) &\geq 5p_2(1-\delta) + \delta(1-\delta)u_1(p_1, g_2(0,p_2)) + \delta^2(1-\delta)u_1(1, g_2(p_1, g_2(0,p_2))) + 2\delta^3 \quad (7) \\ &\geq 5p_2(1-\delta) + 2\delta(1-\delta)p_1 + \delta^2(1-\delta)2g_2(p_1, g_2(0,p_2)) + 2\delta^3. \end{aligned}$$

Now note that, by Claim 3, we have

$$g_2(0,p_2) \leq \frac{4+\eta-2\delta-5(1-\delta)p_2}{2\delta(1-\delta)}. \tag{8}$$

This, together with Claim 2, implies that

$$g_1(p_1, g_2(0,p_2)) \leq \frac{4+\eta-2\delta-5(1-\delta)p_1}{2\delta(1-\delta)} + \frac{(3p_1-2)}{2\delta}g_2(0,p_2). \tag{9}$$

But then, by Claim 4, we have a lower bound to $g_2(p_1, g_2(0,1))$ given by:

$$\bar{g}_2(p_1, g_2(0,1)) = \frac{2-5\delta}{3(1-\delta)} - \frac{2}{3}g_1(p_1, g_2(0,p_2)). \tag{10}$$

But as $\delta \to 1/3$ and $\eta \to 0$, we have by Claim 1, (8), (9) and (10) respectively that $p_i \to 1$ for each i, $g_2(0,p_2) \to 0$, $g_1(p_1, g_2(0,p_2)) \to 0$ and $\bar{g}_2(p_1, g_2(0,1)) \to 1/6$. Hence, as $\delta \to 1/3$ and $\eta \to 0$, by (7), any limit point of $U_1(\tilde{f}_1, f_2)$ is no less than $4 + \frac{2}{81}$. Since by assumption $U_1(f) \to 4$ as $\delta \to 1/3$, then $U_1(\tilde{f}_1, f_2) > U_1(f)$ for all δ and η sufficiently close to 1/3 and 0, respectively, yielding the desired contradiction.

The basic idea in the above proofs is that players cannot be as severely punished in the 1 – memory case as in the full memory case. This can be illustrated by considering the values of $g_2(0,p_2)$ and $g_2(p_1, g_2(0,1))$ for δ and η sufficiently close to 1/3 and 0, respectively. In the full memory case, a deviation by player 1 from a can lead player 2 to choose b forever, while in the 1 – memory case, although it leads player 2 to choose b almost for sure in the first period after the deviation ($g_2(0,p_2)$ is close to 0), in the second period after the deviation player 2 would have to play a with a probability bounded away from zero ($g_2(p_1, g_2(0,1))$ is close to 1/6). Consequently, the punishment with 1 – memory is less severe and for such values of δ and η, a profitable deviation for player 1 exists.

7.2 A Three Player Example with Lack of Strict Enforceability

Let G be the mixed extension of the following normal form game with three players: all players have pure action spaces given by $A_i = \{a, b\}$,

$$u_3(a_1, a_2, a_3) = \begin{cases} 4 & \text{if } a_3 = a \text{ and} \\ 2 & \text{if } a_3 = b. \end{cases}$$

for all $a_1 \in A_1$ and $a_2 \in A_2$, u_1 and u_2 are defined by Table 3 above if $a_3 = a$ and arbitrarily if $a_3 = b$.

Clearly, $u_3(a_1, a_2, a) > u_3(a_1', a_2', b)$ for all $a_1, a_1' \in A_1$ and $a_2, a_2' \in A_2$. Therefore, if f is a SPE, then $f_3(h) = a$ for all $h \in H$. Thus, we are effectively in the same situation as in the 2-player example in the previous subsection. Therefore, the payoff vector $(4, 4, 4)$ is a SPE if there are no bounds on the memory and $\delta \geq 1/3$. However, arguing as in the 2-player example, one can show that there exists a discount factor $\delta' > 1/3$ such that such that the SPE payoff of $(4, 4, 4)$ cannot be supported by a $1-$memory SPE for any $\delta \in [1/3, \delta')$.[18]

7.3 Implementation in Approximate Equilibrium

The concept of approximate equilibrium we employ is the notion of contemporaneous perfect $\varepsilon-$equilibrium (ε–CPE) that is formally defined as follows (see Mailath, Postlewaite, and Samuelson [22]): For all $\varepsilon \geq 0$, a strategy profile $f \in F$ is a *contemporaneous ε – Nash equilibrium* of the supergame of G if for all $i \in N$, $V_i^t(\pi(f)) \geq V_i^t(\pi(\hat{f}_i, f_{-i})) - \varepsilon$ for all $t \in \mathbb{N}$ and $\hat{f}_i \in F_i$. A strategy profile $f \in F$ is a *contemporaneous perfect ε – equilibrium* of the supergame of G if $f|h$ is a contemporaneous ε – Nash equilibrium for every $h \in H$. Using an argument analogous to that of Theorem 1, we can infer analogous conclusion as that in Kalai and Stanford [17] with $1-$memory strategies. This is because (i) any SPE payoff profile can supported by a weakly enforceable simple strategy $(\hat{\pi}^{(0)}, \hat{\pi}^{(1)}, \ldots, \hat{\pi}^{(n)})$, and (ii) for any $\varepsilon > 0$, we can construct a confusion-proof profile of outcome paths $(\bar{\pi}^{(0)}, \ldots, \bar{\pi}^{(n)})$ that is sufficiently close to $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$ so that it is an ε – CPE (simply ensure that $|u_k(\bar{\pi}^{(i),t}) - u_k(\hat{\pi}^{(i),t})| < \varepsilon/2$ and $|\max_{s_k \in S_k} u_k(s_k, \bar{\pi}_{-k}^{(i),t}) - \max_{s_k \in S_k} u_k(s_k, \hat{\pi}_{-k}^{(i),t})| < \varepsilon/2$ for all $k \in N$, $i \in \{0, 1, \ldots, n\}$ and $t \in N$). Using these two simple observations, we can state an analogous result to Theorem 4.1 of [17] as follows.

[18] Note that, by the same arguments as in the supplementary material, this conclusion is robust to any small perturbations in the stage game payoffs.

Corollary 3 *For every SPE payoff vector* $u \in \mathbb{R}^n$*, every* $\eta > 0$ *and every* $\varepsilon > 0$*, there is a 1 – memory* ε *– CPE,* $f \in F$*, such that* $\|U(f) - u\| < \eta$*, whenever either* $n \geq 3$*, or* $n = 2$ *and there exist a SPE simple strategy profile described by* $(\hat{\pi}^{(0)}, \hat{\pi}^{(1)}, \hat{\pi}^{(2)})$ *such that* $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$ *and* $V(\hat{\pi}^{(0)}) = u$*.*

Note that in Corollary 3, it is assumed that the punishment paths are common when $n = 2$. This assumption again cannot be weakened even for the case of ε – CPE. In fact, we can extend the analysis of the 2 – player example of Subsection 7.1 to show that the SPE payoff vector $(4, 4)$ cannot even be approximated by a 1 – memory ε – CPE, for small $\varepsilon > 0$ for δ near 1/3.[19] And again the reason for this is that to enforce the payoff vector $(4, 4)$ (or a payoff close to it) as a ε – CPE requires different punishment paths, which are not feasible with 1 – memory.

On the other hand, it follows from Corollary 3 that for $\delta \geq 1/3$, the SPE payoff vector $(4, 4, 4)$ in the 3 – player example of Subsection 7.2 can be approximated arbitrarily closely with a 1 – memory ε – CPE, for all $\varepsilon > 0$.[20] Since this payoff vector cannot be approximately supported by any 1 – memory SPE for δ near 1/3, it follows that there is a *discontinuity* in the sense that $(4, 4, 4)$ can be obtained with a 1 – memory ε – CPE for all $\varepsilon > 0$, whereas it cannot when $\varepsilon = 0$ (the same discontinuity holds if we perturb the stage game payoffs). To see the nature of this discontinuity, note that the payoff profile $(4, 4, 4)$ cannot be obtained with a strictly enforceable simple strategy profile because player 3 has a dominant strategy that induces a payoff of 4 at every stage game (therefore, the continuation payoff of player 3 is the same at every history).[21] This implies that the hypothesis of Theorem 1 on strict enforceability does not hold for $(4, 4, 4)$ when $\varepsilon = 0$. On the other hand, when ε exceeds zero the payoffs at different histories do not have to be the same, and as a result, $(4, 4, 4)$ can be obtained as an ε – CPE.

8 Concluding Remarks

In this paper, we have tried to characterize the set of subgame perfect equilibrium strategies with 1 – memory recall when the stage game action spaces are sufficiently rich. The availability of a large number of actions may suggest that the restriction to 1 – period memory is of little consequence,

[19] We demonstrate this in the supplementary material to this paper.

[20] Indeed, since this payoff can be supported by a confusion-proof single path (repeating (a, a, a) forever), it can be obtained *exactly* as an ε – CPE with 1 – memory.

[21] Notice that, in the 2 player example, there is no discontinuity of the ε – CPE correspondence at $\varepsilon = 0$.

as action sets are so rich that they allow players to encode all past information at almost no cost. In this paper, we tried to see what is required to formalize this intuition. This has turned out to be rather intricate. Our characterization of 1 – memory SPE for arbitrary discount factors then involved first characterizing simple equilibrium paths that can be implemented with 1 – memory strategies (confusion-proof simple paths) and second, finding appropriate slackness in the incentive constraints to ensure that an equilibrium can be perturbed to make it 1 – memory implementable. To illustrate the intricacies of the issues, we also provide robust examples of SPE payoffs with patient players that cannot be sustained even approximately with 1 – memory strategies. Our characterization results enable us to show that 1 – memory strategies are enough to obtain all SPE payoffs if the players are sufficiently patient.

A Appendix: Proofs

Proof of Proposition 1.

(Sufficiency) Let $(\pi^{(0)}, \ldots, \pi^{(n)})$ be a confusion-proof profile of outcome paths. Let $i \in N$ and define f_i as follows: $f_i(H_0) = \pi_i^{(0),1}$ and for any $h \in H \setminus H_0$, $j \in \{0, \ldots, n\}$, $l \in N$ and $t \in \mathbb{N}$

$$f_i(h) = \begin{cases} \pi_i^{(j),t+1} & \text{if } T(h) = \pi^{(j),t}, \\ \pi_i^{(l),1} & \text{if } T(h) = (s_l, \pi_{-l}^{(j),t}) \text{ and } s_l \neq \pi_l^{(j),t}, \\ \pi_i^{(0),1} & \text{otherwise.} \end{cases}$$

Now we show that f is a well defined function. First, suppose that $\pi^{(j),t} = \pi^{(k),r}$ for some $k, j \in \{0, 1, \ldots, n\}$ and $r, t \in \mathbb{N}$. Then, f is well defined if $\pi^{(k),r+1} = \pi^{(j),t+1}$. Since $(\pi^{(0)}, \ldots, \pi^{(n)})$ is confusion-proof, it follows from part 1 of Definition 2 that this is indeed the case.

Second, suppose that $\pi^{(k),r} = (s_l, \pi_{-l}^{(j),t})$ and $s_l \neq \pi_l^{(j),t}$ for $k, j \in \{0, 1, \ldots, n\}$, $l \in N$ and $r, t \in \mathbb{N}$. Then, f is well defined only if $\pi^{(k),r+1} = \pi^{(l),1}$. Since $(\pi^{(0)}, \ldots, \pi^{(n)})$ is confusion-proof and $\Omega(\{k, r\}, \{j, t\}) = \{l\}$, it follows from part 2 of Definition 2 that this is indeed the case.

Finally, suppose that $(s_l, \pi_{-l}^{(j),t}) = (s_k, \pi_{-k}^{(m),r})$, $s_k \neq \pi_k^{(m),r}$ and $s_l \neq \pi^{(j),t}$ for some $j, m \in \{0, 1, \ldots, n\}$, $k, l \in N$ and $r, t \in \mathbb{N}$. Then f is well defined only if $\pi^{(l),1} = \pi^{(k),1}$. Note that it must be that $s_l = \pi_l^{(m),r}$ and $s_k = \pi_k^{(j),t}$. Hence, $\pi_l^{(m),r} \neq \pi_l^{(j),t}$ and $\pi_k^{(m),r} \neq \pi_k^{(j),t}$, implying that $\Omega(\{m, r\}, \{j, t\}) = \{k, l\}$. Since $(\pi^{(0)}, \ldots, \pi^{(n)})$ is confusion-proof, it follows from part 3 of Definition 2 that $\pi^{(l),1} = \pi^{(k),1}$.

It is clear that the strategy profile $f = (f_1 \ldots, f_n)$ has 1 – memory, since, by definition, f_i depends only on $T(h)$ for all $i \in N$.

Note, also that f has the following property: $\pi(f) = \pi^{(0)}$ and if player $i \in N$ deviates unilaterally in phase t in any state j, then $\pi^{(i)}$ will be played starting from period $t+1$. Therefore, f defined by $(\pi^{(0)}, \ldots, \pi^{(n)})$ is a 1 – memory simple strategy profile.

(Necessity) Let f be a 1 – memory simple strategy profile represented by $(\pi^{(0)}, \ldots, \pi^{(n)})$. Let $i, j \in \{0, \ldots, n\}$ and $t, r \in \mathbb{N}$.

Suppose that $\Omega(\{i,t\},\{j,r\}) = \emptyset$. Then, $\pi^{(i),t} = \pi^{(j),r}$. Let $h_1 = (\pi^{(i),t})$ and $h_2 = (\pi^{(j),r})$. Since $T(h_1) = h_1 = h_2 = T(h_2)$ and f has 1 – memory, we have $f(h_1) = f(h_2)$. But then part 1 of Definition 2 is satisfied because $f(h_1) = \pi^{(i),t+1}$ and $f(h_2) = \pi^{(j),r+1}$.

Suppose next that $\Omega(\{i,t\},\{j,r\}) = \{k\}$ for some $k \in N$. Then, $\pi_l^{(i),t} = \pi_l^{(j),r}$ for all $l \neq k$, while $\pi_k^{(i),t} \neq \pi_k^{(j),r}$. Consider $s_k = \pi_k^{(i),t}$ and $\bar{s}_k = \pi_k^{(j),r}$. Then, $(s_k, \pi_{-k}^{(j),r}) = \pi^{(i),t}$ and since f is a 1 – memory simple strategy profile, it follows that $\pi^{(k),1} = f((s_k, \pi_{-k}^{(j),r})) = f(\pi^{(i),t}) = \pi^{(i),t+1}$. Similarly, $(\bar{s}_k, \pi_{-k}^{(i),t}) = \pi^{(j),r}$ and so, $\pi^{(k),1} = f((\bar{s}_k, \pi_{-k}^{(i),t})) = f(\pi^{(j),r}) = \pi^{(j),r+1}$. Hence, $\pi^{(k),1} = \pi^{(j),r+1} = \pi^{(i),t+1}$ and part 2 of Definition 2 is satisfied.

Finally, suppose that $\Omega(\{i,t\},\{j,r\}) = \{k,l\}$ for some $k,l \in N$. Then, $\pi_m^{(i),t} = \pi_m^{(j),r}$ for all $m \notin \{k,l\}$, while $\pi_k^{(i),t} \neq \pi_k^{(j),r}$ and $\pi_l^{(i),t} \neq \pi_l^{(j),r}$. Consider $s_k = \pi_k^{(j),r}$ and $s_l = \pi_l^{(i),t}$. Then, $(s_l, \pi_{-l}^{(j),r}) = (s_k, \pi_{-k}^{(i),t})$ and since f is a 1 – memory simple strategy profile, it follows that $\pi^{(l),1} = f((s_l, \pi_{-l}^{(j),r})) = f((s_k, \pi_{-k}^{(i),t})) = \pi^{(k),1}$. Hence, by induction, $\pi^{(l)} = \pi^{(k)}$ and part 3 of Definition 2 is satisfied. ■

Proof of Lemma 1. Condition 1 is clearly sufficient for confusion-proofness when the number of players exceeds two. Similarly, in a game with two players, if $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ satisfies conditions 2a and 2b, then for all $i, j \in \{0,1,2\}$ and $t, r \in \mathbb{N}$ such that $(i,t) \neq (j,r)$, it follows that $|\Omega(\{i,t\},\{j,r\})| = 2$, except when $\{i,j\} = \{1,2\}$ and $t = r$. Indeed, if $\{i,j\} \neq \{1,2\}$, then $|\Omega(\{i,t\},\{j,r\})| = 2$ by 2b, whereas if $\{i,j\} = \{1,2\}$ and $t \neq r$, then $\pi^{(j),r} = \pi^{(i),r}$ by 2a, and so $|\Omega(\{i,t\},\{j,r\})| = |\Omega(\{i,t\},\{i,r\})| = 2$ by 2b. Finally, if $\{i,j\} = \{1,2\}$ and $t = r$, arguing as above, we obtain that $|\Omega(\{i,t\},\{j,r\})| = |\Omega(\{i,t\},\{i,r\})| = 0$.

The above claim, together with condition 2a, imply that $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ is confusion-proof when $n = 2$. ■

Proof of Theorem 1. Consider any $\eta > 0$ and any strictly enforceable SPE payoff vector u

described by $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$. For all $j \in \{0, 1, \ldots, n\}$ and $i \in N$, define $\zeta_i^{(j)}$ by

$$\zeta_i^{(j)} = \inf_{t \in \mathbb{N}} \left(V_i^t(\hat{\pi}^{(j)}) - \left((1-\delta) \sup_{s_i \neq \hat{\pi}_i^{(j),t}} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) + \delta V_i(\hat{\pi}^{(i)}) \right) \right).$$

Let γ be defined by

$$\gamma = \min \left\{ \eta, \frac{1}{3} \left(\min_{j \in \{0,1,\ldots,n\}, i \in N} \{\zeta_i^{(j)}\} \right) \right\}. \tag{11}$$

Since $\eta > 0$ and u is a strictly enforceable SPE payoff vector, it follows that $\gamma > 0$.

Let $\psi > 0$ be such that $d(x, y) < \psi$ implies $|u_i(x) - u_i(y)| < \gamma$ and $|\max_{z_i} u_i(z_i, x_{-i}) - \max_{z_i} u_i(z_i, y_{-i})| < \gamma$, for all $i \in N$. Since S_i is perfect, it follows that for every $i \in N$, $j = 0, 1, \ldots n$ and $t \in \mathbb{N}$, $B_\psi(\hat{\pi}_i^{(j),t}) \cap S_i$ is uncountable. Thus, we can construct a simple outcome paths $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$ satisfying the conditions described in Lemma 1. Thus, $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$ is confusion-proof. Therefore, by Proposition 1, there exists a 1 – memory strategy profile f that is represented by it. Moreover, $\gamma \leq \eta$ implies $|U_i(f) - u_i| = |V_i(\bar{\pi}^{(0)}) - V_i(\hat{\pi}^{(0)})| < \eta$ for all i.

To complete the proof we need to show f is a strictly enforceable SPE. Fix any $t \in \mathbb{N}$, $i \in N$ and $j \in \{0, 1, \ldots, n\}$. Since $V_i^t(\hat{\pi}^{(j)}) - \gamma < V_i^t(\bar{\pi}^{(j)})$, $V_i(\bar{\pi}^{(i)}) < V_i(\hat{\pi}^{(i)}) + \gamma$ and $\max_{s_i} u_i(s_i, \bar{\pi}_{-i}^{(j),t}) < \max_{s_i} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) + \gamma$, it follows from (11) that

$$\begin{aligned} &V_i^t(\bar{\pi}^{(j)}) - (1-\delta) \sup_{s_i \neq \bar{\pi}_i^{(j),t}} u_i(s_i, \bar{\pi}_{-i}^{(j),t}) - \delta V_i(\bar{\pi}^{(i)}) > \\ &\quad V_i^t(\hat{\pi}^{(j)}) - (1-\delta) \sup_{s_i \neq \hat{\pi}_i^{(j),t}} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) - \delta V_i(\hat{\pi}^{(i)}) - 2\gamma \geq \zeta_i^{(j)} - 2\gamma \geq 3\gamma - 2\gamma = \gamma. \end{aligned}$$

Hence, any deviation by player i from the path induced by state (j) makes i worse off by a positive amount γ. Thus, f is a 1 – memory strictly enforceable SPE. ■

Proof of Lemma 3. Let $u \in \mathrm{ri}(\mathcal{U})$.

Claim 1 *There exists* $\{s^1, \ldots, s^K\} \subseteq S$ *and* $\{\lambda^k\}_{k=1}^K$ *such that* $\mathrm{aff}(\mathcal{U}) = \mathrm{aff}\{u(s^1), \ldots, u(s^K)\}$, $u = \sum_{k=1}^K \lambda^k u(s^k)$, $\lambda^k > 0$ *for all* k *and* $\sum_{k=1}^K \lambda^k = 1$.

Proof of Claim 1. Let m denote $\dim(\mathcal{U})$. First, we show that there exists a set of vectors $\{u^1, \ldots, u^{2m}\} \subseteq \mathcal{U}$ such that (i) $u = \sum_{j=1}^{2m} \alpha^j u^j$ with $\alpha^j > 0$ for all $j = 1, \ldots, 2m$ and $\sum_{j=1}^{2m} \alpha^j = 1$, and (ii) $\{u^1 - u^{2m}, \ldots, u^{2m-1} - u^{2m}\}$ spans L, where L is the unique subspace of $\mathbb{R}^n$ parallel to $\mathrm{aff}(\mathcal{U})$ (i.e., $L = \mathrm{aff}(\mathcal{U}) + \bar{u}$ for all $\bar{u} \in \mathrm{aff}(\mathcal{U})$).

To show this consider first the case where $\dim(\mathcal{U}) = n$. Since $u \in \mathrm{ri}(\mathcal{U}) = \mathrm{int}(\mathcal{U})$, there exists $\eta > 0$ such that $B_{2\eta}(u) \subseteq \mathcal{U}$. Let $\{e^1, \ldots, e^n\}$ denote the standard basis of $\mathbb{R}^n$, and define $u^j = u + \eta e^j$ for all $j = 1, \ldots, n$ and $u^j = u - \eta e^j$ for all $j = n+1, \ldots, 2n$. Then, $\{u^1 - u^{2n}, \ldots, u^{2n-1} - u^{2n}\}$ spans $\mathbb{R}^n$ and $u = \sum_{j=1}^{2n} \alpha^j u^j$ with $\alpha^j > 0$ for all $j = 1, \ldots, 2n$ and $\sum_{j=1}^{2n} \alpha^j = 1$ (set $\alpha^j = 1/2n$ for all j). Next, consider the case where $n = 2$, $\mathcal{U}^0 \neq \emptyset$ and $\dim(\mathcal{U}) < 2$. In this case $\dim(\mathcal{U}) = 1$, and so $\mathrm{aff}(\mathcal{U})$ is a line. Thus, let u^1 and u^2 be points in $\mathcal{U}$ lying on opposite sides of u such that $\{u^1 - u^2\}$ spans L and $u = \sum_{j=1}^{2} \alpha^j u^j$ with $\alpha^j > 0$ for all $j = 1, 2$ and $\alpha^1 + \alpha^2 = 1$.

Since, for all $1 \leq j \leq 2m$, $u^j \in \mathcal{U} \subseteq \mathrm{co}(u(S))$, then there exist $\{\tilde{s}_k^j\}_{k=1}^{K_j}$ and $\{\tilde{\lambda}_k^j\}_{k=1}^{K_j}$ such that $u^j = \sum_{k=1}^{K_j} \tilde{\lambda}_k^j u(\tilde{s}_k^j)$, $\tilde{s}_k^j \in S$, $\tilde{\lambda}_k^j > 0$ and $\sum_{k=1}^{K_j} \tilde{\lambda}_k^j = 1$. We then define $K = \sum_{j=1}^{2m} K_j$, $\{s^1, \ldots, s^K\} = \cup_{j=1}^{2m} \{\tilde{s}_k^j\}_{k=1}^{K_j}$ and $\{\lambda^1, \ldots, \lambda^K\} = \cup_{j=1}^{2m} \{\alpha^j \tilde{\lambda}_k^j\}_{k=1}^{K_j}$.

We finally show that $\{s^1, \ldots, s^K\}$ and $\{\lambda^1, \ldots, \lambda^K\}$ satisfy the desired properties. First note that $u = \sum_{j=1}^{2m} \left(\sum_{k=1}^{K_j} \alpha^j \tilde{\lambda}_j^k u(\tilde{s}_k^j) \right) = \sum_{k=1}^{K} \lambda^k u(s^k)$, that $\lambda^k > 0$ for all k and $\sum_{k=1}^{K} \lambda^k = 1$.

Furthermore, $\mathrm{aff}(\mathcal{U}) = \mathrm{aff}(\{u(s^1), \ldots, u(s^K)\})$. To see this, let $u' \in \mathrm{aff}(\mathcal{U})$. Then, since $\{u^1 - u^{2m}, \ldots, u^{2m-1} - u^{2m}\}$ spans L and $L = \mathrm{aff}(\mathcal{U}) + u^{2m}$, then $u' = \sum_{j=1}^{2m-1} \theta^j (u^j - u^{2m}) + u^{2m}$ for some $\{\theta^j\}_{j=1}^{2m-1}$. Then, letting $\theta^{2m} = 1 - \sum_{j=1}^{2m-1} \theta^j$, we obtain $u' = \sum_{j=1}^{2m} \theta^j u^j = \sum_{j=1}^{2m} \theta^j \left(\sum_{k=1}^{K_j} \tilde{\lambda}_j^k u(\tilde{s}_k^j) \right)$, $\sum_{j=1}^{2m} \theta^j = 1$ and $\sum_{j=1}^{2m} \theta^j \left(\sum_{k=1}^{K_j} \tilde{\lambda}_j^k \right) = 1$. Conversely, note that $u(s^k) \in \mathrm{co}(u(S)) \subseteq \mathrm{aff}(\mathcal{U})$ for all k and so $\mathrm{aff}(\{u(s^1), \ldots, u(s^K)\}) \subseteq \mathrm{aff}(\mathcal{U})$. $\square$

For all $\gamma \in \mathbb{R}_+$, define $X_\gamma = \{w \in \mathbb{R}^n : w = \sum_{k=1}^{K} \mu_k u(s_k), \sum_{k=1}^{K} \mu_k = 1 \text{ and } \mu_k > \lambda_k - \gamma \text{ for all } k\}$.

Claim 2 *There exist $0 < \varepsilon < \min_k \lambda_k$ such that $w \in X_\varepsilon$ implies $w_i > v_i + \varepsilon$ for all $i \in N$.*

Proof of Claim 2. Recall that $u \in \mathrm{ri}(\mathcal{U})$. We first show that $u \in \mathcal{U}^0$. This is clear when $\dim(\mathcal{U}) = n$ since in this case $u - (\varepsilon, \ldots, \varepsilon) \in \mathcal{U}$ for some $\varepsilon > 0$. Consider next the case where $n = 2$ and $\mathcal{U}^0 \neq \emptyset$. Let $u' \in \mathcal{U}^0$. Then, since $u \in \mathrm{ri}(\mathcal{U})$, it follows by Rockafellar [24, Theorem 6.4] that there exists $\mu > 1$ such that $\mu u + (1-\mu) u' \in \mathcal{U}$ and let $\varepsilon = \mu - 1 > 0$. If $u_i = v_i$ for some i, then $\mu u_i + (1-\mu) u_i' = u_i + \varepsilon(u_i - u_i') < v_i$, a contradiction. Hence, $u_i > v_i$ for all i and so $u \in \mathcal{U}^0$.

Since $u \in \mathcal{U}^0$, there exists $\zeta > 0$ such that $u > v + 2\zeta$ where $v = (v_1, \ldots, v_n)$. Furthermore, $X_0 = \{u\}$ and so it follows from the definition of X_γ that there exists $\gamma' > 0$ such that $\|w - u\| < \zeta$ for all $w \in X_{\gamma'}$. Since $X_\gamma \subseteq X_{\gamma'}$ for all $\gamma \leq \gamma'$, it then follows that for any $\varepsilon < \min\{\zeta, \gamma'\}$, we have $w > v + \varepsilon$ for all $w \in X_\varepsilon$. $\square$

Let $\delta' \in (0,1)$ be such that $\lambda_k > (1-\delta) + \delta(\lambda_k - \varepsilon)$ for all $1 \leq k \leq K$ and $\delta \geq \delta'$.

Claim 3 *For all $\delta \geq \delta'$, there exist sequences $\{w_t\}_{t=1}^{\infty}$ and $\pi = \{\pi^t\}_{t=1}^{\infty}$ such that $\pi \in \Pi^{cp}$, $w_1 = u$, $w_t \in X_\varepsilon$ and $w_t = (1-\delta)u(\pi^t) + \delta w_{t+1}$ for all $t \in \mathbb{N}$.*

Proof of Claim 3. Fix $\delta \geq \delta'$. We next define, by induction, a sequence $\{w_t\}_{t=1}^{\infty}$ and $\{\pi^t\}_{t=1}^{\infty}$. Let $w_1 = u$ and suppose that we are given $w_1, \ldots, w_t$ and $\pi^1, \ldots, \pi^{t-1}$, with $w_l \in X_\varepsilon$ for all $1 \leq l \leq t$. Since $w_t \in X_\varepsilon$, then $w_t = \sum_{k=1}^{K} \mu_k^t u(s_k)$ for some $\{\mu_k^t\}_k$ such that $\mu_k^t \geq \lambda_k - \varepsilon$ for all k and $\sum_k \mu_k^t = 1$. Pick any $1 \leq \bar{k} \leq K$ such that $\mu_{\bar{k}}^t \geq \lambda_{\bar{k}}$.

Define a continuous function $g : S \to \mathrm{aff}(\mathcal{U})$ by $g(s) = (w_t - (1-\delta)u(s))/\delta$. Since $u(S) \subseteq \mathrm{aff}(\mathcal{U})$ and $w^t \in \mathrm{aff}(\mathcal{U})$, it follows that $g(s)$ does indeed belong to $\mathrm{aff}(\mathcal{U})$. In particular, $g(s_{\bar{k}}) = \sum_{k=1}^{K}(\mu_k^t/\delta)u(s_k) - [(1-\delta)/\delta]u(s_{\bar{k}})$. Using the fact that $\mu_{\bar{k}}^t \geq \lambda_{\bar{k}}$, $\lambda_{\bar{k}} > (1-\delta) + \delta(\lambda_{\bar{k}} - \varepsilon)$ and $\mu_k^t \geq \lambda_k - \varepsilon$ for all $1 \leq k \leq K$, it is easy to check that $\mu_k^t/\delta > \lambda_k - \varepsilon$ for all $k \neq \bar{k}$ and $\mu_{\bar{k}}^t/\delta - (1-\delta)/\delta > \lambda_{\bar{k}} - \varepsilon$. Thus, $g(s_{\bar{k}}) \in \mathrm{ri}(X_\varepsilon) = \{w \in \mathbb{R}^n : w = \sum_{k=1}^{K} \mu_k u(s_k), \sum_{k=1}^{K} \mu_k = 1$ and $\mu_k > \lambda_k - \varepsilon$ for all $k\}$.

Since $\mathrm{ri}(X_\varepsilon)$ is open in $\mathrm{aff}(\mathcal{U})$ and g is continuous, there exists an open neighborhood V_i of $s_i^{\bar{k}}$ for all $i \in N$, such that if $\pi = (\pi_1, \ldots, \pi_n) \in V_1 \times \cdots \times V_n$, then $g(\pi) \in \mathrm{ri}(X_\varepsilon)$. Since S is perfect, then V_i is uncountable for all i. Hence, choose $\pi^t \in V_1 \times \cdots \times V_n$ such that $\pi_i^t \neq \pi_i^r$ for all $r < t$ (if $t = 1$, we can let $\pi^1 = s_{\bar{k}}$). Finally, define $w_{t+1} = g(\pi^t) \in X_\varepsilon$.

The above construction of the two sequence $\{w_t\}_{t=1}^{\infty}$ and $\{\pi^t\}_{t=1}^{\infty}$ satisfy the conditions of the claim because $w_{t+1} = g(\pi^t)$ imply that $w_t = (1-\delta)u(\pi^t) + \delta w_{t+1}$ and $\pi_i^t \neq \pi_i^r$ for all $i \in N$ and $t, r \in \mathbb{N}$ with $t \neq r$ imply that $\pi \in \Pi^{cp}$. □

Finally, it follows easily by induction that $u = (1-\delta)\sum_{t=1}^{T} \delta^{t-1}u(\pi^t) + \delta^T w_{T+1}$. Since $w_t \in X_\varepsilon$ for all t, it follows that $\{w_t\}_{t=1}^{\infty}$ is bounded. Therefore, $\delta^T w_{T+1} \to 0$ and so $u = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}u(\pi^t)$. Similarly, $w_t = (1-\delta)\sum_{k=t}^{\infty} u(\pi^k) = V^t(\pi, \delta)$ and so $V^t(\pi, \delta) \in X_\varepsilon$ for all $t \in \mathbb{N}$. Since $w \in X_\varepsilon$ implies that $w_i > v_i + \varepsilon$ for all i, we obtain $V_i^t(\pi, \delta) > v_i + \varepsilon$ for all $i \in N$ and $t \in \mathbb{N}$. Since $\pi \in \Pi^{cp}$, this completes the proof of Lemma 3. ■

Proof of Theorem 2. We consider the two cases of $n \geq 3$ (**Case A**) and the $n = 2$ (**Case B**) separately.

Case A: $n \geq 3$ and $\dim(\mathcal{U}) = n$.

Let $u \in \mathrm{ri}(\mathcal{U}) = \mathrm{int}(\mathcal{U})$. It then follows by Lemma 3 that there exist $\delta' \in (0,1)$ and $\zeta > 0$ such that for all $\delta \geq \delta'$ there exists $\pi(\delta) \in \Pi^{cp}$ such that $u = V(\pi(\delta), \delta)$ and $V_i^t(\pi(\delta), \delta) > v_i + \zeta$. It follows from Fudenberg and Maskin [14, Proposition 1] and its proof that there exists $\delta^* \in (0,1)$

such that for all $\delta \geq \delta^*$, there exists a strictly enforceable SPE simple strategy profile $(\pi^{(0)}, \ldots, \pi^{(n)})$ such that $u = V(\pi^{(0)}, \delta)$ and $\pi^{(0)} = \pi(\delta)$. Thus, by Proposition 2, there exists a 1 – memory simple strategy profile f such that $U(f, \delta) = u$.

Case B: $n = 2$ and $\mathcal{U}^0 \neq \emptyset$.

Let $u \in \mathrm{ri}(\mathcal{U})$. It then follows by Lemma 3 that there exist $\delta' \in (0, 1)$ and $\zeta > 0$ such that for all $\delta \geq \delta'$ there exists $\pi_\delta \in \Pi^{cp}$ such that $u = V(\pi_\delta, \delta)$ and $V_i^t(\pi_\delta, \delta) > v_i + \zeta$.

Finally, as in Fudenberg and Maskin [13, Theorem 1], there exists $\delta^* \in (\delta', 1)$ and $N^* \in \mathbb{N}$ such that

$$(1 - \delta^*)M + (\delta^*)^{N^*+1} u_i < u_i - \zeta \tag{12}$$

and

$$(\delta^*)^{N^*+1} u_i > v_i \tag{13}$$

for all i. Furthermore, for all $\delta \geq \delta^*$, let $N(\delta) \in \mathbb{N}$ be such that (12) and (13) hold for $(\delta, N(\delta))$.

The proof is now standard. Define a simple strategy profile by $\pi^{(0)} = \{\pi_\delta\}_{t=1}^{\infty}$ and $\pi^{(1)} = \pi^{(2)}$ defined by

$$\pi^{(1),t} = \begin{cases} \bar{m} & \text{if } 1 \leq t \leq N(\delta), \\ \pi_\delta^{t-N(\delta)} & \text{if } t > N(\delta). \end{cases}$$

That this simple strategy profile is a strictly enforceable SPE follows from (12) and (13) (corresponding to $(\delta, N(\delta))$) as in Fudenberg and Maskin [13, Theorem 1]).

In conclusion, for all $\delta \geq \delta^*$, there exists a strictly enforceable SPE simple strategy profile that supports u. Thus, by by Proposition 2, there exists a 1 – memory simple strategy profile f such that $U(f, \delta) = u$. ■

References

[1] Abreu, D. (1988): "On the Theory of Infinitely Repeated Games with Discounting," *Econometrica*, 56, 383–396.

[2] Aumann, R. (1981): "Survey of Repeated Games," in *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern*. Bibliographisches Institut, Mannheim.

[3] Aumann, R., and L. Shapley (1994): "Long-Term Competition – A Game-Theoretic Analysis," in *Essays in Game Theory in Honor of Michael Maschler*, ed. by N. Megiddo. Springer-Verlag, New York.

[4] Aumann, R., and S. Sorin (1989): "Cooperation and Bounded Recall," *Games and Economic Behavior*, 1, 5–39.

[5] Barlo, M. (2003): "Essays in Game Theory," Ph.D. thesis, University of Minnesota.

[6] Barlo, M., G. Carmona, and H. Sabourian (2006): "One - Memory in Repeated Games," working paper 500, Universidade Nova de Lisboa.

[7] ——— (2007): "Bounded Memory with Finite Action Spaces," Sabancı University, Universidade Nova de Lisboa and University of Cambridge.

[8] Ben-Porath, E., and B. Peleg (1987): "On the Folk Theorem and Finite Automata," The Hebrew University of Jerusalem.

[9] Bhaskar, V., and F. Vega-Redondo (2002): "Asynchronous Choice and Markov Equilibria," *Journal of Economic Theory*, 103, 334–350.

[10] Cho, I. K. (1994): "Bounded Rationality, Neural Network and Folk Theorem in Repeated Games with Discounting," *Economic Theory*, 4, 935–957.

[11] Cole, H., and N. Kocherlakota (2005): "Finite Memory and Imperfect Monitoring," *Games and Economic Behavior*, 53, 59–72.

[12] Fudenberg, D., D. Levine, and E. Maskin (1994): "The Folk Theorem with Imperfect Public Information," *Econometrica*, 62(5), 997–1039.

[13] Fudenberg, D., and E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54, 533–554.

[14] ——— (1991): "On the Dispensability of Public Randomization in Discounted Repeated Games," *Journal of Economic Theory*, 53, 428–438.

[15] Hörner, J., and W. Olszewski (2007): "How Robust is the Folk Theorem with Imperfect Public Monitoring?," Northwestern University.

[16] Kalai, E. (1990): "Bounded Rationality and Strategic Complexity in Repeated Games," in *Game Theory and Applications*, ed. by T. Ichiishi, A. Neyman, and Y. Tauman. Academic Press, New York.

[17] Kalai, E., and W. Stanford (1988): "Finite Rationality and Interpersonal Complexity in Repeated Games," *Econometrica*, 56, 397–410.

[18] Lehrer, E. (1988): "Repeated Games with Stationary Bounded Recall Strategies," *Journal of Economic Theory*, 46, 130–144.

[19] ——— (1994): "Finitely Many Players with Bounded Recall in Infinitely Repeated Games," *Games and Economic Behavior*, 6, 97–113.

[20] Mailath, G., and S. Morris (2002): "Repeated Games with Almost-Public Monitoring," *Journal of Economic Theory*, 102, 189–228.

[21] ——— (2006): "Coordination Failure in Repeated Games with Almost-Public Monitoring," *Theoretical Economics*, 1, 311–340.

[22] Mailath, G., A. Postlewaite, and L. Samuelson (2005): "Contemporaneous Perfect Epsilon-Equilibrium," *Games and Economic Behavior*, 53, 126–140.

[23] Neyman, A., and D. Okada (1999): "Strategic Entropy and Complexity in Repeated Games," *Games and Economic Behavior*, 29, 191–223.

[24] Rockafellar, R. T. (1970): *Convex Analysis.* Princeton University Press, Princeton.

[25] Sabourian, H. (1989): "The Folk Theorem of Repeated Games with Bounded Recall (One-Period) Memory," *Economic Theory Discussion Paper 143, University of Cambridge.*

[26] ——— (1998): "Repeated Games with M-period Bounded Memory (Pure Strategies)," *Journal of Mathematical Economics*, 30, 1–35.