Supplementary Materials for Bounded Memory Folk Theorem

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This document presents supplementary materials for our paper "Bounded Memory Folk Theorem". While the first section concerns those related to the bounded memory Folk Theorem with pure strategies, the second involves those associated with the one employing mixed (behavioral) strategies.

A Theorem 1 (pure Folk Theorem)

A.1 Auxiliary results

Claims A.1 to A.15 establish results needed to show that the strategy given in (28) is well-defined and subgame perfect.

The first two claims establish some properties of the paths $\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)}$.

Claim A.1 *For all $i \in \{0, \ldots, n\}$ and $d, d' \in N$ with $d \neq d'$:*

$$-B(1-\delta^{(n+5)(T+1)}) + \delta^{(n+5)(T+1)} V_d(\hat{\pi}^{(d')}) > B(1-\delta^{n+6}) + \delta^{n+6+T} V_d(\hat{\pi}^{(d)}), \quad \text{(A.1)}$$

$$-B(1-\delta^{K}) + \delta^{K} V_d(\hat{\pi}^{(i)}) > (1-\delta^{n+6})B + \delta^{n+6+T} V_d(\hat{\pi}^{(d)}), \quad \text{(A.2)}$$

$$-(1-\delta^{n+5})B + \delta^{n+5+(n+4)T} V_d(\hat{\pi}^{(d)}) > (1-\delta^{n+5})B + \delta^{(n+5)(T+1)} V_d(\hat{\pi}^{(d)}). \quad \text{(A.3)}$$

Proof. First, by (25) and the definitions of $\{x_j\}_{j=0}^{n}$, $\{y_j\}_{j=0}^{n}$, $\{\hat{\pi}^{(j)}\}_{j=0}^{n}$, ξ, ζ, and ζ', we have the following two conditions:

$$V_d(\hat{\pi}^{(d)}) > x_d^d - \xi > y_d^d - 2\xi > \zeta, \tag{A.4}$$

$$V_d(\hat{\pi}^{(d')}) > x_d^{d'} - \xi > y_d^{d'} - 2\xi > y_d^d + \zeta' - 2\xi > V_d(\hat{\pi}^{(d)}) + \zeta' - 4\xi = V_d(\hat{\pi}^{(d)}) + \zeta \tag{A.5}$$

By (24), we have $\delta^{(n+5)(T+1)}\zeta > B(2 - 2\delta^{(n+5)(T+1)} - \delta^{n+6}(1-\delta^T))$. Then (A.1) follows immediately from (A.5).

Consider inequality (A.2). Since $K < T$, (A.1) implies (A.2) when $d, i \in N$ and $d \neq i$. Therefore, to demonstrate (A.2), it suffices to consider two cases: $i = 0$ and $d = i \in N$.

By (A.5) and (24), we have $(V_d(\hat{\pi}^{(0)}) - V_d(\hat{\pi}^{(d)}))\delta^K > B(2 - \delta^K - \delta^{n+6})$. This together with $K < T$ and (A.4) implies that (A.2) holds when $i = 0$.

By (A.4) and (24), $V_d(\hat{\pi}^{(d)})(\delta^K - \delta^{n+6+T}) > \zeta(\delta^K - \delta^{n+6+T}) > B(2 - \delta^K - \delta^{n+6})$. This implies that (A.2) holds when $d = i \in N$.

Finally, consider (A.3). By (A.4) and (24), $V_d(\hat{\pi}^{(d)})\delta^{n+5+(n+4)T}(1-\delta^T) > \zeta\delta^{n+5+(n+4)T}(1-\delta^T) > 2B(1-\delta^{n+5})$. This implies that (A.3) holds. ■

Claim A.2 *For all $i \in \{0, \ldots, n\}$, the following hold:*

1. *All actions $a \neq s'$ are played for $t \geq n+i+2$ consecutive periods in $\hat{\pi}^{(i)}$.*
2. *Suppose that, for some $t \in \mathbb{N}$, $(\hat{\pi}^{(i),t}, \ldots, \hat{\pi}^{(i),t+l+1}) = ((s';2),(s;l))$ and $l > 0$. Then either $\hat{\pi}^{(i),t+l+2} = s$ or $\hat{\pi}^{(i),t+l+2} = s'$ and $l = n+i+2$.*
3. *Suppose that, for some $t \in \mathbb{N}$, $\hat{\pi}^{(i),t} \in D(s')$ and $\hat{\pi}^{(i),t+1} \in D(s)$. Then $\hat{\pi}^{(i),t} = s'$ and $\hat{\pi}^{(i),t+1} = s$. Furthermore, either (i) $\hat{\pi}^{(i),t-1} = s'$ and $t = 2 + \beta K$ or (ii) $\hat{\pi}^{(i),t-1} = s$ and $t = n+i+5+\beta K$ for some $\beta \in \mathbb{N}_0$.*
4. *Suppose that, for some $t \in \mathbb{N}$, $\hat{\pi}^{(i),t} \in D(s)$ and $\hat{\pi}^{(i),t+1} \in D(s')$. Then $\hat{\pi}^{(i),t+1} = s'$.*

Proof. This follows immediately from the ordering of A and the definition of $\hat{\pi}^{(i)}$. ■

Claim A.3 *If $h \in H_{1,a}^{(i),k}$ for some $i \in \{0, \ldots, n\}$ and $k \in \{n+i+5, \ldots, M\}$, then $T^{k'}(h) \notin \Sigma^{d,k'}$ for all $d \in N$ and $k' \in \mathbb{N}$ such that $d+4 \leq k' \leq k$.*

Proof. Suppose otherwise; then there exists $h \in H_{1,a}^{(i),k}$ such that $T^{k'}(h) \in \Sigma^{d,k'}$ for some $d \in N$ and $k' \in \mathbb{N}$ with $d+4 \leq k' \leq k$. Let $T^k(h) = (a^1, \ldots, a^k)$. Since $T^{k'}(h) \in \Sigma^{d,k'}$, we have $a^{k-k'+2} \in D(s')$, $a^{k-k'+3} \in D(s)$ and $a^{k-k'+d+4} \in D(s')$. Therefore, there is an action, namely $a^{k-k'+3}$, which is different from s' and is played at most $d+1$ consecutive periods in $T^k(h)$. Since $h \in H_{1,a}^{(i),k}$, this contradicts Claim A.2.1. ■

Next, for all $\tau \in \mathbb{N}$ and $i \in \{0, \ldots, n\}$, define

$$\Lambda^{i,\tau} = \left\{ h \in H : h = (a^t)_{t=1}^{\tau} \text{ such that } a^t = \hat{\pi}^{(i),t} \text{ if } t \leq n+i+5 \right\}.$$

Claim A.4 *If $h \in H_{1,a}^{(i),k}$ for some $i \in \{0, \ldots, n\}$ and $k \in \{n+i+5, \ldots, M\}$ and $T^{k'}(h) \in \Lambda^{i',k'}$ for some $i' \in \{0, \ldots, n\}$ and $n+i'+5 \leq k' \leq k$, then $i = i'$ and $k = k' + \beta K$ for some $\beta \in \mathbb{N}_0$.*

Proof. Since $h \in H_{1,a}^{(i),k}$ and $T^{k'}(h) \in \Lambda^{i',k'}$, $(\hat{\pi}^{(i),t}, \ldots, \hat{\pi}^{(i),t+n+i'+4}) = ((s';2),(s;n+i'+2),s')$ where $t = k+1-k'$. It then follows from Claim A.2.2 that $n+i'+2 = n+i+2$ and so $i = i'$. Hence, by Claim A.2.3, $t+1 = 2+\beta K$ for some $\beta \in \mathbb{N}_0$ and so $k = k' + \beta K$. ∎

Claim A.5 *Suppose $h \in H_3^{k,d} \cup H_4^{k,d,0}$ for some $k \in \{1, \ldots, M\}$ and $d \in N$. Then $T^{k'}(h) \notin \Lambda^{i,k'} \cup \Sigma^{d',k'}$ for all $i \in \{0, \ldots, n\}$, $d' \in N$ and $k' \in \mathbb{N}$ such that $3 \leq k' < k$.*

Proof. Suppose otherwise; then $T^{k'}(h) \in \Lambda^{i,k'} \cup \Sigma^{d',k'}$ for some $i \in \{0, \ldots, n\}$, $d' \in N$ and $k' \in \mathbb{N}$ such that $3 \leq k' < k$. Let $T^k(h) = (a^1, \ldots, a^k)$. Since $T^{k'}(h) \in \Lambda^{i,k'} \cup \Sigma^{d',k'}$ and $h \in H_3^{k,d} \cup H_4^{k,d,0}$ it follows, respectively, that $a^{k-k'+1}, a^{k-k'+2} \in D_{d'}(s')$ and $a^3, \ldots, a^{d+3} \in D(s)$. Consequently, $k - k' + 1 \geq d + 4$. This means that $a_{-d}^{k-k'+2} = a_{-d}^{k-k'+3} = m_{-d}^d$. Also, by appealing again to $T^{k'}(h) \in \Lambda^{i,k'} \cup \Sigma^{d',k'}$, one has $a^{k-k'+3} \in D_{d'}(s)$. Therefore, by $a^{k-k'+2} \in D_{d'}(s')$, we have that $s_{-d,d'} = s'_{-d,d'}$; a contradiction. ∎

Claim A.6 *If $h \in H_4^{k,d,r}$ for some $k \in \{1, \ldots, M\}$ and $d \in N$ and $r \in \{1, \ldots, n+d+4\}$ and $T^{k'}(h) \in \Lambda^{i,k'}$ for some $i \in \{0, \ldots, n\}$ and $3 \leq k' < k$, then $k' = r$ and $k' < n+i+5$.*

Proof. Let $T^k(h) = (a^1, \ldots, a^k)$. By $T^{k'}(h) \in \Lambda^{i,k'}$, $a^{k-k'+1} = a^{k-k'+2} = s'$ and, by $h \in H_4^{k,d,r}$, $a^t \in D(s)$ for all $t \in \{3, \ldots, d+3\} \cup \{k-r+3, \ldots, k\}$. It then follows that $k - k' + 1 \in \{2, d+4, \ldots, k-r+2\}$.

Note, however, that it cannot be that $k - k' + 1 = k - r + 2$; otherwise, $a^{k-r+3} = s'$. Also, $k - k' + 1 = k - r - 1$ is not possible because otherwise, by $T^{k'}(h) \in \Lambda^{i,k'}$, $a^{k-k'+3} = s$ and, by $h \in H_4^{k,d,r}$, $a^{k-k'+3} = a^{k-r+1} = s'$. Furthermore, $k - k' + 1 = k - r$ is not possible because otherwise, by $T^{k'}(h) \in \Lambda^{i,k'}$, $a^{k-k'+3} = s$ and, by $h \in H_4^{k,d,r}$, $a^{k-r+2} = s'$.

By Claim A.5, it also cannot be that $k - k' + 1 \leq k - r - 2$. The reasoning for this is as follows. Since $h \in H_4^{k,d,r}$ and $T^{k'}(h) \in \Lambda^{i,k'}$, it follows, respectively, that $B^r(h) \in H_4^{k-r,d,0}$ and $T^{k'-r}(B^r(h)) \in \Lambda^{i,k'-r}$. But then by Claim A.5 it must be that $k' - r < 3$.

Hence, it follows from all the above that $k - k' + 1 = k - r + 1$, i.e. $k' = r$.

Finally, $k' < n+i+5$ because otherwise, by $T^{k'}(h) \in \Lambda^{i,k'}$, $a^{k-k'+n+i+5} = s'$ and, by $h \in H_4^{k,d,r}$, $a^{k-r+n+i+5} = s$. ∎

Claim A.7 *If $h \in H_4^{k,d,r}$ for some $k \in \{1,\ldots,M\}$ and $d \in N$ and $r \in \{1,\ldots,n+d+4\}$, then $T^{k'}(h) \notin \Sigma^{d',k'}$ for all $d' \in N$ and $k' \in \mathbb{N}$ such that $d'+4 \le k' < k$.*

Proof. Suppose otherwise. Let $T^k(h) = (a^1,\ldots,a^k)$. Since $h \in H_4^{k,d,r}$ and $T^{k'}(h) \in \Sigma^{d',k'}$, it follows, respectively, that $B^r(h) \in H_4^{k-r,d,0}$ and $T^{k'-r}(B^r(h)) \in \Sigma^{d',k'-r}$ if $k' > r$. But then by Claim A.5 it must be that $k'-r < 3$. Therefore, $k-k'+d'+4 > k-r+2$. Hence, $k-k'+d'+4 = k-r+t$ for some $t > 2$. But this is a contradiction since, by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{k-k'+d'+4} \in D(s')$ and, by $h \in H_4^{k,d,r}$, $a^{k-r+t} = s$. ■

Claim A.8 *If $h \in H_5^{k,d,\tau}$ for some $k \in \{1,\ldots,M\}$ and $d \in N$ and $\tau \in \{0,\ldots,d+3\}$, then $T^{k'}(h) \notin \Sigma^{d',k'}$ for all $d' \in N$ and $k' \in \mathbb{N}$ such that $d'+4 \le k' < k$.*

Proof. Suppose not. Then there exist $h = (a^1,\ldots,a^t) \in H_5^{k,d,\tau}$ such that $T^{k'}(h) \in \Sigma^{d',k'}$ for the parameters given in the statement of the claim. We now derive a contradiction by considering six different possibilities. Before doing so, we let $\bar{h} = B^{\tau+1}(h)$ and point out that, by $h \in H_5^{k,d,\tau}$, $T^k(h) \in \Sigma^{i,k}$ for some $i \in N$.

Case 1: $k' - (\tau+1) \ge d'+4$. Then, $T^{k'-(\tau+1)}(\bar{h}) \in \Sigma^{d',k'-(\tau+1)}$ and $\bar{h} \in H_3^{k-(\tau+1),i} \cup \left(\cup_{r=0}^{n+i+4} H_4^{k-(\tau+1),i,r}\right)$. But this contradicts either Claim A.5 or Claim A.7.

Case 2: $k' - (\tau+1) < d'+1$. Then, by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{t-k'+d'+4} \in D(s')$ and, by $h \in H_5^{k,d,\tau}$ and $t-k'+d'+4 > t-\tau+2$, $a^{t-k'+d'+4} \in D(s)$; a contradiction.

Case 3: $k' - (\tau+1) = d'+1$. Then, by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{t-k'+d'+3} \in D(s)$ and, by $h \in H_5^{k,d,\tau}$ and $t-k'+d'+3 = t-\tau+1$, $a^{t-k'+d'+3} \in D(s')$; a contradiction.

Case 4: $k' - (\tau+1)$ *equals either* $d'+2$ *or* $d'+3$ *and* $\bar{h} \in H_3^{k-(\tau+1),i} \cup H_4^{k-(\tau+1),i,0}$. Then $3 \le k'-(\tau+1) < k-(\tau+1)$ and $T^{k'-(\tau+1)}(\bar{h}) \in \Sigma^{d',k'-(\tau+1)}$. This contradicts Claim A.5.

Case 5: $k' - (\tau+1) = d'+3$ *and* $\bar{h} \in H_4^{k-(\tau+1),i,r}$ *with* $r > 0$. By $T^{k'}(h) \in \Sigma^{d',k'}$, (i) $a^{t-k'+2} \in D(s')$, (ii) $a^{t-k'+l} \in D(s)$ for all $3 \le l \le d'+3$ and (iii) $a^{t-k'+d'+4} \in D(s')$. By $h \in H_5^{k,d,\tau}$ and $\bar{h} \in H_4^{k-(\tau+1),i,r}$, (iv) $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(i),r+1})$ and (v) $a^{t-\tau-r'} = \hat{\pi}^{(i),r-r'+1}$ for all $1 \le r' \le r$.

By $r > 0$ and $t-k'+d'+4 = t-\tau$, (iii) and (iv) imply that either $r = 1$ or $r = n+i+4$. If $r = 1$, by (v), $a^{t-\tau-1} = \hat{\pi}^{(i),1} = s'$. This together with $t-\tau-1 = t-k'+d'+3$ contradicts (ii). If $r = n+i+4$, then, by (v) and $n+i-d'+3 \ge 3$, $a^{t-\tau-d'-2} = \hat{\pi}^{(i),n+i-d'+3} = s$. This together with $t-\tau-d'-2 = t-k'+2$ contradicts (i).

Case 6: $k' - (\tau + 1) = d' + 2$ and $\bar{h} \in H_4^{k-(\tau+1),i,r}$ with $r > 0$. Then, by (ii) and $t - \tau = t - k' + d' + 3$, $a^{t-\tau} \in D_{d'}(s)$. Therefore, by (iv), $a^{t-\tau} \in \bar{D}_d(s)$ and $d = d'$.

Next, we show that $r = d+2$. If $r > d+2$, then, by (v), $a^{t-\tau-d-1} = \hat{\pi}^{(i),r-d} = s$. But this is a contradiction because $t-k'+2 = t-\tau-d-1$ and, by (i), $a^{t-k'+2} \in D_d(s')$. If $r < d+2$, then, by (ii), $a^{t-k'+d-r+4} \in D(s)$. But this is a contradiction because $t-\tau-r+1 = t-k'+d-r+4$ and, by (v), $a^{t-\tau-r+1} = \hat{\pi}^{(i),2} = s'$.

By $\bar{h} \in H_4^{k-(\tau+1),i,r}$ and $r = d+2$, we have $T^{d+2}(\bar{h}) = ((s';2),(s;d))$. Since $a^{t-\tau} \in \bar{D}_d(s)$, it then follows from the definition of $H_5^{k,d,\tau}$ that $\tau = 0$. But this contradicts $k'-(\tau+1) = d'+2$ and $k' \geq d' + 4$. ■

Claim A.9 *If $T^k(h) \in \Sigma^{d,k} \cap \Sigma^{d',k}$ for some $d, d' \in N$ and $k \in \{\min\{d,d'\} + 4, \ldots, M\}$, then $d = d'$.*

Proof. Suppose not; assume that $d > d'$. Let $T^k(h) = (a^1, \ldots, a^k)$. Since $T^k(h) \in \Sigma^{d,k}$, $a^{d'+4} \in D_d(s)$ and, by $T^k(h) \in \Sigma^{d',k}$, $a^{d'+4} \in D_{d'}(s')$. But this is a contradiction. ■

Claim A.10 *Let $h \in H \setminus \cup_{l=1}^5 H_l$ and $a \in A$. Then one of the following conditions hold: (a) $h \cdot a \notin \cup_{l=1}^5 H_l$, (b) $h \cdot a \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0, \ldots, n\}$ and (c) $T^{d+4}(h \cdot a) \in \Sigma^{d,d+4}$ for some $d \in N$. Furthermore, if $a \in D(s)$, then $h \cdot a \notin \cup_{l=1}^5 H_l$.*

Proof. Suppose that $h \cdot a$ does not satisfy (a)–(c). Then there are six cases to consider. (i) *$h \cdot a \in H_{1,a}^{(i),k}$ for some i and $k > n + i + 5$*: then $h \in H_{1,a}$; a contradiction. (ii) $h \cdot a \in H_{1,b}$: then $h \in H_{1,b}$; a contradiction. (iii) $h \cdot a \in H_2$: then $h \in H_1 \cup H_2$; a contradiction. (iv) *$h \cdot a \in H_3$ and $T^{d+4}(h \cdot a) \notin \Sigma^{d,d+4}$ for all $d \in N$*: then $h \in H_3$; a contradiction. (v) $h \cdot a \in H_4$: then $h \in H_3 \cup H_4$; a contradiction. (vi) $h \cdot a \in H_5$: then $h \in H_3 \cup H_4 \cup H_5$; a contradiction.

Furthermore, if $a \in D(s)$, by the definition of $H_{1,a}^{(i),n+i+5}$ and $\Sigma^{d,d+4}$, (b) and (c) cannot hold. Therefore, (a) must hold, i.e. $h \cdot a \notin \cup_{l=1}^5 H_l$. ■

Claim A.11 *If $h \in H_{1,a}^{(i),M}$ for some $i \in \{0, \ldots, n\}$, then $h \in H_{1,a}^{(i),k}$ for some $k < M$.*

Proof. Since $h \in H_{1,a}^{(i),M}$, $T^M(h) = (\hat{\pi}^{(i),1}, \ldots, \hat{\pi}^{(i),M})$. Also, by $T > K$ and by the choice of M, $M > K + n + i + 5$. Therefore, $h \in H_{1,a}^{(i),M-K}$. ■

Claim A.12 *If $h \in H_2^{M,d,\tau}$ for some $d \in N$ and $\tau \in \{0, \ldots, d+3\}$, then $h \in H_2^{k,d,\tau}$ for some $k < M$.*

Proof. Since $h \in H_2^{M,d,\tau}$, $T^M(h) = \bar{h} \cdot a \cdot \tilde{h}$ where $\bar{h}$, a, and $\tilde{h}$ are as in the definition of $H_2^{M,d,\tau}$. In particular, $\bar{h} \in H_{1,a}^{(i),k'} \cup H_{1,b}^{(0),k'}$ for some $i \in \{0, \ldots, n\}$ and $k' \leq M - (\tau + 1)$. Therefore, $h \in H_2^{k'+\tau+1,d,\tau}$.

If $k' + \tau + 1 < M$, the claim follows. If $k' + \tau + 1 = M$, then $\bar{h} = (\hat{\pi}^{(i),1}, \ldots, \hat{\pi}^{(i),M-(\tau+1)})$ for some $i \in \{0, \ldots, n\}$. Also, by $T > K$ and by the choice of M, $M \geq (n+4)+(2n+5)+K$. Therefore, $\bar{h} \in H_{1,a}^{(i),M-(\tau+1)-K}$ and so $h \in H_2^{M-K,d,\tau}$. ■

Claim A.13 *If $h \in H_3^{k,d}$ for some $k \in \{1, \ldots, M\}$ and $d \in N$, then $k < M$.*

Proof. Let $\bar{h}$ and $\tilde{h}$ be such that $T^k(h) = \bar{h} \cdot \tilde{h}$ and satisfy the conditions in the definition of $H_3^{k,d}$. Then $k = \ell(\bar{h}) + \ell(\tilde{h}) < d + 4 + (\theta(\bar{h}) + 1)T \leq n + 4 + (n + 5)T < M$. ■

Claim A.14 *If $h \in H_4^{k,d,r}$ for some $k \in \{1, \ldots, M\}$ and $d \in N$ and $r \in \{0, \ldots, n + d + 4\}$, then $k < M$.*

Proof. Let $\bar{h}$, $\hat{h}$ and $\tilde{h}$ be such that $T^k(h) = \bar{h} \cdot \hat{h} \cdot \tilde{h}$ and satisfy the conditions in the definition of $H_4^{k,d,r}$. Then $k \leq \ell(\bar{h}) + \ell(\hat{h}) + \ell(\tilde{h}) = d + 4 + (\theta(\bar{h}) + 1)T + n + d + 4 \leq n + 4 + (n + 5)T + 2n + 4 < M$. ■

Claim A.15 *If $h \in H_5^{k,d,\tau}$ for some $k \in \{1, \ldots, M\}$ and $d \in N$ and $\tau \in \{0, \ldots, d+3\}$, then $k < M$.*

Proof. Let $\bar{h}$, a and $\tilde{h}$ be such that $T^k(h) = \bar{h} \cdot a \cdot \tilde{h}$ and satisfy the conditions in the definition of $H_5^{k,d,\tau}$. By the proof of Claim A.13 and Claim A.14, $\ell(\bar{h}) \leq n + 4 + (n + 5)T + 2n + 4$. Therefore, $k = \ell(\bar{h}) + \ell(a \cdot \tilde{h}) \leq [n + 4 + (n + 5)T + 2n + 4] + 1 + d + 3 < M$. ■

A.2 f is well-defined

Now we show that $f \in F^p$, described in (28), is well-defined.

Claim A.16 *If $h \in H_{1,a}^{(i),k} \cap H_{1,a}^{(i'),k'}$ for some $i, i' \in \{0, \ldots, n\}$ and $k \in \{n + i + 5, \ldots, M\}$ and $k' \in \{n + i' + 5, \ldots, M\}$, then $i = i'$ and $k = k' + \beta K$ for some $\beta \in \mathbb{Z}$.*

Proof. It follows immediately by Claim A.4. ■

Claim A.17 *For all $i \in \{0, \ldots, n\}$ and $k \in \{n + i + 5, \ldots, M\}$ and $k' \in \{0, \ldots, n + 4\}$, $H_{1,a}^{(i),k} \cap H_{1,b}^{(0),k'} = \emptyset$.*

Proof. If $h \in H_{1,b}^{(0),k'}$, then $\ell(h) < n + 5$, whereas if $h \in H_{1,a}^{(i),k}$, then $\ell(h) \geq n + i + 5$ for some $i \in \{0, \ldots, n\}$. Hence, $H_{1,a}^{(i),k} \cap H_{1,b}^{(0),k'} = \emptyset$. ■

Claim A.18 *For all $i \in \{0,\dots,n\}$ and $k \in \{n+i+5,\dots,M\}$ and $k' \in \{1,\dots,M\}$ and $d \in N$ and $\tau \in \{0,\dots,d+3\}$, $H_{1,a}^{(i),k} \cap H_2^{k',d,\tau} = \emptyset$.*

Proof. Suppose not; then there exist $h = (a^1,\dots,a^t) \in H_{1,a}^{(i),k} \cap H_2^{k',d,\tau}$ for some i, k, k', d and τ as described in the claim. Since $h \in H_{1,a}^{(i),k}$, then $\hat{\pi}^{(i),r} = a^{t-k+r}$ for all $1 \le r \le k$. Also since $h \in H_2^{k',d,\tau}$, then $T^{k'}(h) = \bar{h} \cdot a^{t-\tau} \cdot \tilde{h}$ where $\tilde{h} \in \Sigma^{d,\tau}$, $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(i'),k'-\tau})$ and $\bar{h} \in H_1^{(i'),k'-(\tau+1)}$ for some $i' \in \{0,\dots,n\}$ satisfying either $k'-(\tau+1) \ge n+i'+5$ or $i'=0$ and $k'=t$ and $k'-(\tau+1) < n+5$. Next, consider each of these possibilities separately.

Case 1: $\bar{h} \in H_{1,b}^{(0),k'-(\tau+1)}$, $k'=t$ and $k'-(\tau+1) < n+5$. In this case $a^{k'-k+1} = a^{k'-k+2} = s'$, $\bar{h} = ((s';2), s,\dots,s)$ and $\ell(\bar{h}) = k'-(\tau+1)$. Thus, since $\tau \le d+3$ and $k \ge n+i+5$, it must be that $k'-k+1 < k'-\tau$ and so $k'=k$. But then $a^{t-\tau} \in \bar{D}_d(s) \cup \bar{D}_d(s')$ and $a^{t-\tau} = \hat{\pi}^{(i),t-\tau} \in \{s,s'\}$, a contradiction.

Case 2: $\bar{h} \in H_{1,a}^{(i'),k'-(\tau+1)}$ and $k'-(\tau+1) \ge n+i'+5$. We consider four subcases.

Subcase 1: $k \ge n+i+5+\tau+1$. Let $\hat{h} = B^{\tau+1}(h)$. Since $k-(\tau+1) \ge n+i+5$, it follows that $\hat{h} \in H_{1,a}^{(i),k-(\tau+1)}$. Also, since $T^{k'-(\tau+1)}(\hat{h}) = \bar{h}$, $\hat{h} \in H_{1,a}^{(i'),k'-(\tau+1)}$. Then, by Claim A.16, $i=i'$ and $k-(\tau+1) = k'-(\tau+1)+\beta K$ for some $\beta \in \mathbb{Z}$. This together with $a = \hat{\pi}^{(i),k-\tau}$ and $a \in \bar{D}_d(\hat{\pi}^{(i'),k'-\tau})$ implies that $\hat{\pi}^{(i),k'-\tau} = \hat{\pi}^{(i),k-\tau} \in \bar{D}_d(\hat{\pi}^{(i),k'-\tau})$, a contradiction.

Subcase 2: $k = n+i+5+\tau$. By $h \in H_{1,a}^{(i),k}$, $a^{t-\tau-1} = s$ and $a^{t-\tau} = s'$. Also, by $h \in H_2^{k',d,\tau}$, $a^{t-\tau-1} = \hat{\pi}^{(i'),k'-\tau-1}$ and $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(i'),k'-\tau})$. Hence, it follows from $a^{t-\tau-1} = s$ and $a^{t-\tau} = s'$, respectively, that $\hat{\pi}^{(i'),k'-\tau-1} = s$ and $\hat{\pi}^{(i'),k'-\tau} \in \bar{D}_d(s')$. But this contradicts Claim A.2.4.

Subcase 3: $k = n+i+4+\tau$. First, we show that $k' \ge k$. Suppose otherwise. Since $\bar{h} \in H_{1,a}^{(i'),k'-(\tau+1)}$, $T^{k'}(h) \in \Lambda^{i',k'}$. By Claim A.4, this together with $h \in H_{1,a}^{(i),k}$ implies that $k = k'+\beta K$ for some $\beta \in \mathbb{N}$. Also, by the supposition that $k < n+i+5+\tau+1$ and $k'-(\tau+1) \ge n+i'+5$, we have $k-k' \le n$. Since $n < K$, we have a contradiction.

By $h \in H_{1,a}^{(i),k}$, $(a^{t-\tau-(n+i+3)},\dots,a^{t-\tau}) = ((s';2),(s;n+i+2))$. Also, by $h \in H_2^{k',d,\tau}$, $(a^{t-\tau-(n+i+3)},\dots,a^{t-\tau}) = (\hat{\pi}^{(i'),k'-\tau-(n+i+3)},\dots,\hat{\pi}^{(i'),k'-\tau-1},a^{t-\tau})$. Hence, it follows from $a^{t-\tau} = s$ that $\hat{\pi}^{(i'),k'-\tau} \in \bar{D}_d(s)$. But this contradicts Claim A.2.2.

Subcase 4: $k < n+i+4+\tau$. By $k \ge n+i+5$ and $\tau \le d+3$, it follows that $\tau > 0$ and $k-\tau \ge 2$. Hence, by $h \in H_2^{k',d,\tau}$, $a^{t-\tau+1} \in D(s')$ and, by $h \in H_{1,a}^{(i),k}$ and $k-\tau < n+i+4$, $a^{t-\tau+1} = s$; but this is a contradiction. ■

Claim A.19 *For all* $k \in \{0, \ldots, n+4\}$ *and* $k' \in \{1, \ldots, M\}$ *and* $d \in N$ *and* $\tau \in \{0, \ldots, d+3\}$, $H_{1,b}^{(0),k} \cap H_2^{k',d,\tau} = \emptyset$.

Proof. Suppose otherwise; then there exists $h \in H_{1,b}^{(0),k} \cap H_2^{k',\tau,d}$. This means that $h = (\hat{\pi}^{(0),1}, \ldots, \hat{\pi}^{(0),k})$ and $T^{k'}(h) = \bar{h} \cdot a \cdot \tilde{h}$ satisfying the remaining conditions in the definition of $H_2^{k',d,\tau}$. Therefore, for some $\hat{k} < n+5$, $\bar{h} \cdot a \in H_{1,b}^{(0),\hat{k}+1}$. But this is a contradiction as, by $h \in H_2^{k',d,\tau}$, $a \in \bar{D}(\hat{\pi}^{(0),\hat{k}+1})$. ■

Claim A.20 *If* $h \in H_2^{k,d,\tau} \cap H_2^{k',d',\tau'}$ *for some* $k, k' \in \{1, \ldots, M\}$ *and* $d, d' \in N$ *and* $\tau \in \{0, \ldots, d+3\}$ *and* $\tau' \in \{0, \ldots, d'+3\}$, *then* $\tau = \tau'$ *and* $d = d'$ *and* $k = k' + \beta K$ *for some* $\beta \in \mathbb{Z}$.

Proof. Let $h = (a^1, \ldots, a^t)$. First, we establish that $\tau = \tau'$. Suppose, without loss of generality, that $\tau < \tau'$. Define $\hat{h} = B^{\tau+1}(h) = (a^1, \ldots, a^{t-(\tau+1)})$ and note that $\hat{h} \in \left(\cup_{i=0}^{n} H_1^{(i),k-(\tau+1)}\right) \cap H_2^{k'-(\tau+1),d',\tau'-(\tau+1)}$. But this contradicts Claim A.18 or Claim A.19.

By $\tau = \tau'$ and the definition of $H_2^{k,d,\tau}$ and $H_2^{k',d',\tau'}$, we have that $\hat{h} \in H_1^{(i),k-(\tau+1)}$ and $\hat{h} \in H_1^{(i'),k'-(\tau'+1)}$ for some $i, i' \in \{0, \ldots, n\}$. It then follows from Claim A.16 and Claim A.17 that $k = k' + \beta K$ for some $\beta \in \mathbb{Z}$ and $i = i'$. Also, by $h \in H_2^{k,d,\tau}$, $a^{t-\tau} = (a_d, \hat{\pi}_{-d}^{(i),k-\tau})$ and, by $h \in H_2^{k',d',\tau'}$, $a^{t-\tau'} = (a_{d'}, \hat{\pi}_{-d'}^{(i'),k'-\tau'})$. Since $i = i'$, $k = k' + \beta K$ and $\tau = \tau'$, then $(a_d, \hat{\pi}_{-d}^{(i),k-\tau}) = (a_{d'}, \hat{\pi}_{-d'}^{(i),k-\tau})$ and so $d = d'$. ■

Claim A.21 *For all* $i \in \{0, \ldots, n\}$ *and* $k \in \{n+i+5, \ldots, M\}$ *and* $k' \in \{1, \ldots, M\}$ *and* $d \in N$, $H_{1,a}^{(i),k} \cap H_3^{k',d} = \emptyset$.

Proof. Suppose that $h \in H_{1,a}^{(i),k} \cap H_3^{k',d}$. Since $H_3^{k',d} \subseteq \Sigma^{d,k'}$ and, by $h \in H_3^{k',d}$, $k' \geq d+4$, we have a contradiction to Claim A.3 when $k \geq k'$. Also, since $H_{1,a}^{(i),k} \subseteq \Lambda^{i,k}$ and $k \geq n+i+5$, we have a contradiction to Claim A.5 when $k < k'$. ■

Claim A.22 *For all* $k \in \{0, \ldots, n+4\}$ *and* $k' \in \{1, \ldots, M\}$ *and* $d \in N$, $H_{1,b}^{(0),k} \cap H_3^{k',d} = \emptyset$.

Proof. Suppose there exists $h = (a^1, \ldots, a^t) \in H_{1,b}^{(0),k} \cap H_3^{k',d}$. By $h \in H_3^{k',d}$, $a^{t-k'+d+4} \in D(s')$ and, by $h \in H_{1,b}^{(0),k}$, $a^r = s$ for all $r > 2$. But this is a contradiction as it implies $d+4 \leq t-k'+d+4 \leq 2$. ■

Claim A.23 *For all* $k, k' \in \{1, \ldots, M\}$ *and* $d, d' \in N$ *and* $\tau \in \{0, \ldots, d+3\}$, $H_2^{k,d,\tau} \cap H_3^{k',d'} = \emptyset$.

Proof. Suppose that $h = (a^1, \dots, a^t) \in H_2^{k,d,\tau} \cap H_3^{k',d'}$. There are two possibilities.

Case 1: $k' > k$. First, note that, for some $i \in \{0, \dots, n\}$, $B^{\tau+1}(h) \in H_{1,a}^{(i),k-(\tau+1)}$ with $k-(\tau+1) \geq n+i+5$; otherwise, $B^{\tau+1}(h) \in H_{1,b}^{(0),k-(\tau+1)}$ and $\ell(h) = k < k'$; a contradiction.

It follows from $B^{\tau+1}(h) \in H_{1,a}^{(i),k-(\tau+1)}$, for some $i \in \{0, \dots, n\}$, and $k-(\tau+1) \geq n+i+5$, that $T^k(h) \in \Lambda^{i,k}$. But this contradicts Claim A.5 because $h \in H_3^{k',d'}$ and $k' > k \geq n+i+5$.

Case 2: $k \geq k'$. There are two possibilities.

Subcase 1: $k' - \tau > d' + 4$. In this case, $B^{\tau+1}(h)$ belongs to $H_1^{(i),k-(\tau+1)}$, for some $i \in \{0, \dots, n\}$, and to $H_3^{k'-(\tau+1),d'}$. But this contradicts Claim A.21 or A.22.

Subcase 2: $k' - \tau \leq d' + 4$. Since $h \in H_3^{k',d'}$, we have (i) $a^{t-k'+r} \in D_{d'}(s')$ for $r = 1, 2$, (ii) $a^{t-k'+r} \in D_{d'}(s)$ for $r = 3, \dots, d'+3$ and (iii) $a^{t-k'+d'+4} \in D_{d'}(s')$. When $k' - \tau = d' + 4$, by (i), (ii) and $B^{\tau+1}(h) \in H_1^{(i),k-(\tau+1)}$ for some $i \in \{0, \dots, n\}$, Claim A.2.3 implies that $(a^{t-\tau-d'-3}, \dots, a^{t-\tau-1}) = ((s'; 2), (s; d'+1))$ and $a^{t-\tau} \in \bar{D}_d(s)$. But the latter contradicts (iii). Therefore, it must be that $k' - \tau < d' + 4$.

By $h \in H_2^{k,d,\tau}$, it must be that (iv) $a^{t-\tau+r} \in D_d(s')$ for all $r = 1, 2$ and (v) $a^{t-\tau+r} \in D_d(s)$ for all $r \geq 3$. But then by $t - k' + d' + 4 > t - \tau$, (iii) and (v), it must be that either $t - k' + d' + 4 = t - \tau + 1$ or $t - k' + d' + 4 = t - \tau + 2$. The latter, however, cannot hold because, by (ii), $a^{t-k'+d'+3} \in D(s)$ and, by (iv), $a^{t-\tau+1} \in D(s')$. Therefore, assume the former. When $k' - \tau = d' + 3$, by (i), (ii) and $B^{\tau+1}(h) \in H_1^{(i),k-(\tau+1)}$ for some $i \in \{0, \dots, n\}$, Claim A.2.3 implies that $(a^{t-k'+1}, \dots, a^{t-k'+d'+2}) = ((s'; 2), (s; d'))$ and $a^{t-k'+d'+3} \in \bar{D}_d(s)$. But the latter together with (ii) implies $d = d'$. Hence, by part (4) of the definition of $H_2^{k,d,\tau}$, $\tau = 0$. Thus, $k' < d' + 4$; but this contradicts $h \in H_3^{k',d'}$. ■

Claim A.24 *If $h \in H_3^{k,d} \cap H_3^{k',d'}$ for some $k, k' \in \{1, \dots, M\}$ and $d, d' \in N$, then $k = k'$ and $d = d'$.*

Proof. First we show that $k = k'$. Suppose otherwise and assume, without loss of generality, that $k > k'$. By $h \in H_3^{k',d'}$, we have $T^{k'}(h) \in \Sigma^{d',k'}$ and $k > k' \geq d' + 4$. But this contradicts Claim A.5 because $h \in H_3^{k,d}$.

To show that $d = d'$, by $h \in H_3^{k,d} \cap H_3^{k,d'}$, we have $T^k(h) \in \Sigma^{d,k} \cap \Sigma^{d',k}$, $k \geq d + 4$ and $k \geq d' + 4$. Hence, by Claim A.9, $d = d'$. ■

Claim A.25 *For all $i \in \{0, \dots, n\}$ and $k \in \{n+i+5, \dots, M\}$ and $k' \in \{1, \dots, M\}$ and $d \in N$ and $r \in \{0, \dots, n+d+4\}$, $H_{1,a}^{(i),k} \cap H_4^{k',d,r} = \emptyset$.*

Proof. Suppose that $h \in H_{1,a}^{(i),k} \cap H_4^{k',d,r}$. Then $T^k(h) \in \Lambda^{i,k}$, $k \geq n+i+5$, $T^{k'}(h) \in \Sigma^{d,k'}$ and $k' \geq d+4$. But these contradict Claim A.3 if $k \geq k'$ and Claim A.5 and Claim A.6 if $k < k'$. ■

Claim A.26 *For all $k \in \{0, \ldots, n+4\}$ and $k' \in \{1, \ldots, M\}$ and $d \in N$ and $r \in \{0, \ldots, n+d+4\}$, $H_{1,b}^{(0),k} \cap H_4^{k',d,r} = \emptyset$.*

Proof. If $h \in H_4^{k',d,r}$, then $\ell(h) > T$. If $h \in H_{1,b}^{(0),k}$, then $\ell(h) < n+5 < T$ by (23). ■

Claim A.27 *For all $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$ and $r \in \{0, \ldots, n+d'+4\}$ and $\tau \in \{0, \ldots, d+3\}$, $H_2^{k,d,\tau} \cap H_4^{k',d',r} = \emptyset$.*

Proof. Suppose that $h \in H_2^{k,d,\tau} \cap H_4^{k',d',r}$. By $h \in H_2^{k,d,\tau}$, $B^{\tau+1}(h) \in H_1^{(i),k-(\tau+1)}$. Also, since $h \in H_4^{k',d',r}$ and $T > \tau+1$, then $B^{\tau+1}(h)$ belongs to $H_4^{k'-(\tau+1),d',r-(\tau+1)}$ if $r-(\tau+1) \geq 0$ or to $H_3^{k'-(\tau+1),d'}$ otherwise. But by Claim A.21, Claim A.22, Claim A.25 or Claim A.26, this is a contradiction. ■

Claim A.28 *For all $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$ and $r \in \{0, \ldots, n+d'+4\}$, $H_3^{k,d} \cap H_4^{k',d',r} = \emptyset$.*

Proof. Suppose that $h \in H_3^{k,d} \cap H_4^{k',d',r}$. Assume first that $k = k'$. By Claim A.9, this implies $d = d'$. Since $h \in H_3^{k,d}$, it follows that $k < (\theta(T^k(h))+1)T + d + 4$ and, since $h \in H_4^{k',d',r}$, $k = k'$ and $d = d'$, it follows that $k \geq (\theta(T^k(h))+1)T + d + 4$. But this is a contradiction.

Suppose next that $k > k'$. Then, by $h \in H_4^{k',d',r}$, $T^{k'}(h) \in \Sigma^{d',k'}$ with $d'+4 \leq k' < k$. But this together with $h \in H_3^{k,d}$ contradicts Claim A.5.

Finally, suppose that $k' > k$. Then, by $h \in H_3^{k,d}$, $T^k(h) \in \Sigma^{d,k}$ with $d+4 \leq k < k'$. But this together with $h \in H_4^{k',d',r}$ contradicts Claim A.5 and Claim A.7. ■

Claim A.29 *If $h \in H_4^{k,d,r} \cap H_4^{k',d',r'}$ for some $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$ and $r, r' \in \mathbb{N}_0$, then $k = k'$ and $d = d'$ and $r = r'$.*

Proof. To show that $k = k'$, suppose, without loss of generality, that $k > k'$. Then, by $h \in H_4^{k',d',r'}$, $T^{k'}(h) \in \Sigma^{d',k'}$ with $d'+4 \leq k' < k$. But this together with $h \in H_4^{k,d,r}$ contradict Claim A.5 and Claim A.7. Hence, $k = k'$ and, by Claim A.9, $d = d'$. Letting $\theta = \theta(T^k(h)) = \theta(T^{k'}(h))$, we have that $k = d+4+(\theta+1)T+r$ and $k' = d'+4+(\theta+1)T+r'$. Since $d = d'$ and $k = k'$, it follows that $r = r'$. ■

Claim A.30 *For all $i \in \{0,\dots,n\}$ and $k \in \{n+i+5,\dots,M\}$ and $k' \in \{1,\dots,M\}$ and $d \in N$ and $\tau \in \{0,\dots,d+3\}$, $H_{1,a}^{(i),k} \cap H_5^{k',d,\tau} = \emptyset$.*

Proof. Suppose that $h = (a^1,\dots,a^t) \in H_{1,a}^{(i),k} \cap H_5^{k',d,\tau}$. Then $h \in H_{1,a}^{(i),k}$, $T^{k'}(h) \in \Sigma^{d,k'}$ and $k' \geq d+4$. Therefore, $k' > k$; otherwise we would contradict Claim A.3.

Consider next the case $k' > k$ and let $\hat{h} = B^{\tau+1}(h)$. Then $\hat{h} \in H_3^{k'-(\tau+1),d'} \cup H_4^{k'-(\tau+1),d',r}$ for some $d' \in N$ and $0 \leq r \leq n+d'+4$. Also, by $h \in H_{1,a}^{(i),k}$, $T^{k-(\tau+1)}(\hat{h}) \in \Lambda^{i,k-(\tau+1)}$. Furthermore, since $0 \leq \tau \leq d+3 \leq n+3$, we have that $k-(\tau+1) \geq 1$. Therefore, by Claim A.5 and Claim A.6, one of the following must hold: (1) $k-(\tau+1) = 1$, (2) $k-(\tau+1) = 2$ and (3) $B^{\tau+1}(h) \in H_4^{k'-(\tau+1),d',r}$, $r > 0$, $k-(\tau+1) = r$ and $3 \leq k-(\tau+1) < n+i+5$.

Case (1) implies that $t-\tau+1 = t-k+3$ and case (2) implies that $t-\tau+1 = t-k+4$. Since $T^k(h) \in H_{1,a}^{(i),k}$, we have $a^{t-k+3} = a^{t-k+4} = s$, and so, $a^{t-\tau+1} = s$ in both cases. Since $h \in H_5^{k',d,\tau}$, we also have $T^\tau(h) \in \Sigma^{d,\tau}$. But this implies $a^{t-\tau+1} \in D_d(s')$; a contradiction.

In case (3), we have that $t-\tau \leq t-k+n+i+5$. This together with $h \in H_{1,a}^{(i),k}$ implies that $a^{t-\tau} \in \{s,s'\}$. Since $h \in H_5^{k',d,\tau}$, $B^{\tau+1}(h) \in H_4^{k'-(\tau+1),d',r}$ and $\hat{\pi}^{(d'),r+1} \in \{s,s'\}$, it must be that $a^{t-\tau} \in \bar{D}_d(s) \cup \bar{D}_d(s')$; a contradiction. ■

Claim A.31 *For all $k \in \{0,\dots,n+4\}$ and $k' \in \{1,\dots,M\}$ and $d \in N$ and $\tau \in \{0,\dots,d+3\}$, $H_{1,b}^{(0),k} \cap H_5^{k',d,\tau} = \emptyset$.*

Proof. Suppose $h = (a^1,\dots,a^k) \in H_{1,b}^{(0),k} \cap H_5^{k',d,\tau}$. By $h \in H_5^{k',d,\tau}$, $a^{k-k'+1}, a^{k-k'+i+4} \in D(s')$ for some $i \in N$ and, by $h \in H_{1,b}^{(0),k}$, $h = ((s';2),(s;k-2))$; a contradiction. ■

Claim A.32 *For all $k,k' \in \{1,\dots,M\}$ and $d,d' \in N$ and $\tau \in \{0,\dots,d+3\}$ and $\tau' \in \{0,\dots,d'+3\}$, $H_2^{k,d,\tau} \cap H_5^{k',d',\tau'} = \emptyset$.*

Proof. Suppose that $h \in H_2^{k,d,\tau} \cap H_5^{k',d',\tau'}$. Assume first that $\tau \leq \tau'$. Then $B^{\tau+1}(h) \in H_1$ and $B^{\tau+1}(h) \in H_3 \cup H_4 \cup H_5$. But this contradicts Claims A.21, A.22, A.25, A.26, A.30 or A.31. If $\tau > \tau'$, then $B^{\tau'+1}(h) \in H_2$ and $B^{\tau'+1}(h) \in H_3 \cup H_4$. But this contradicts Claim A.23 or Claim A.27. ■

Claim A.33 *For all $k,k' \in \{1,\dots,M\}$ and $d,d' \in N$ and $\tau \in \{0,\dots,d'+3\}$, $H_3^{k,d} \cap H_5^{k',d',\tau} = \emptyset$.*

Proof. Suppose that $h = (a^1,\dots,a^t) \in H_3^{k,d} \cap H_5^{k',d',\tau}$. By $h \in H_3^{k,d}$, $T^k(h) \in \Sigma^{d,k}$ with $k \geq d+4$; by $h \in H_5^{k',d',\tau}$, $T^{k'}(h) \in \Sigma^{i',k'}$ with $k' \geq i'+4$ for some $i' \in N$. By appealing to Claims A.5, A.8 and A.9, it then follows that $k = k'$ and $d = i'$.

By $h \in H_5^{k',d',\tau}$ and the last two equalities, $k-(\tau+1) \geq d+4$. But then, by $h \in H_3^{k,d}$, we have $B^{\tau+1}(h) \in H_3^{k-(\tau+1),d}$. This together with $h \in H_5^{k',d',\tau}$ implies $a^{t-\tau} \in \bar{D}_{d'}(m^d)$ and $d' \neq d$. But since $h \in H_3^{k,d}$ and $k-(\tau+1) \geq d+4$, $a^{t-\tau} \in D_d(m^d)$; a contradiction. ■

Claim A.34 *For all $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$ and $r \in \{0, \ldots, n+d+4\}$ and $\tau \in \{0, \ldots, d'+3\}$, $H_4^{k,d,r} \cap H_5^{k',d',\tau} = \emptyset$.*

Proof. Suppose that $h = (a^1, \ldots, a^t) \in H_4^{k,d,r} \cap H_5^{k',d',\tau}$. By $h \in H_4^{k,d,r}$, $T^k(h) \in \Sigma^{d,k}$ with $k \geq d+4$; by $h \in H_5^{k',d',\tau}$, $T^{k'}(h) \in \Sigma^{i',k'}$ with $k' \geq i'+4$ for some $i' \in N$. By appealing to Claim A.7 and Claim A.8, it then follows that $k = k'$.

Suppose $r \geq \tau+1$. By $h \in H_4^{k,d,r}$, $B^{\tau+1}(h) \in H_4^{k-(\tau+1),d,r-(\tau+1)}$. This implies that $a^{t-\tau} = \hat{\pi}^{(d),r-\tau}$ and, by $h \in H_5^{k',d',\tau}$ and Claim A.28 and Claim A.29, $a^{t-\tau} \in \bar{D}_{d'}(\hat{\pi}^{(d),r-\tau})$; a contradiction.

Finally, suppose $r < \tau+1$. By $h \in H_4^{k,d,r}$, $B^{\tau+1}(h) \in H_3^{k-(\tau+1),d}$. This implies that $a^{t-\tau} \in D_d(m^d)$ and, by $h \in H_5^{k',d',\tau}$ and Claim A.24 and ClaimA.28, $a^{t-\tau} \in \bar{D}_{d'}(m^d)$ and $d' \neq d$; a contradiction. ■

Claim A.35 *If $h \in H_5^{k,d,\tau} \cap H_5^{k',d',\tau'}$ for some $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$ and $\tau \in \{0, \ldots, d+3\}$ and $\tau' \in \{0, \ldots, d'+3\}$, then $k = k'$, $\tau = \tau'$ and $d = d'$.*

Proof. Suppose $h \in H_5^{k,d,\tau} \cap H_5^{k',d',\tau'}$. First, note that $\tau = \tau'$. Otherwise, say $\tau < \tau'$; then, by $h \in H_5^{k',d',\tau'}$, $B^{\tau+1}(h) \in H_5$ and, by $h \in H_5^{k,d,\tau}$, $B^{\tau+1}(h) \in H_3 \cup H_4$. This contradicts Claim A.33 or Claim A.34. Second, note that $B^{\tau+1}(h) \in \cup_{i \in N} \left(H_3^{k-(\tau+1),i} \cup \left(\cup_r H_4^{k-(\tau+1),i,r} \right) \right)$ and $B^{\tau'+1}(h) \in \cup_{i \in N} \left(H_3^{k'-(\tau'+1),i} \cup \left(\cup_r H_4^{k'-(\tau'+1),i,r} \right) \right)$. Since $\tau = \tau'$, by Claim A.24, Claim A.28 and A.29, $k = k'$. Finally, it follows immediately from the definitions of $H_5^{k,d,\tau}$ and $H_5^{k',d',\tau'}$, $k = k'$ and $\tau = \tau'$ that $d = d'$. ■

Claim A.36 *If $h \in H_6^{d,\tau} \cap H_6^{d',\tau'}$ for some $d, d' \in N$ and $\tau \in \{0, \ldots, d+3\}$ and $\tau' \in \{0, \ldots, d'+3\}$, then $d = d'$.*

Proof. Let $h = (a^1, \ldots, a^t) \in H_6^{d,\tau} \cap H_6^{d',\tau'}$. We may assume, without loss of generality, that $\tau \geq \tau'$. Then, by $h \in H_6^{d',\tau'}$, $a^{t-\tau'} \in \bar{D}_{d'}(s) \cup \bar{D}_{d'}(s')$ and, by $\tau \geq \tau'$ and $h \in H_6^{d,\tau}$, $a^{t-\tau'} \in D_d(s) \cup D_d(s')$. Thus, $d = d'$. ■

A.3 Proofs of Claims 9 – 19

Proof of Claim 9. By Claim A.11, we may assume that $k < M$. Hence, $f(h) = \hat{\pi}^{(i),k+1}$ and $h \cdot f(h) \in H_{1,a}^{(i),k+1}$. Thus, by induction, $\pi(f|h) = (\hat{\pi}^{(i),k+1}, \hat{\pi}^{(i),k+2}, \ldots)$. ■

Proof of Claim 10. If $k < n+4$, then $h \cdot f(h) \in H_{1,b}^{(0),k+1}$. If $k = n+4$, then $h \cdot f(h) \in H_{1,a}^{(0),n+5}$. Thus, by Claim 9, $\pi(f|h) = (\pi^{(0),k+1}, \pi^{(0),k+2}, \ldots)$. ■

Proof of Claim 11. If $r = n+d+4$, then $h \cdot f(h) \in H_{1,a}^{(d),n+d+5}$, thus, $\pi(f|h) = (\hat{\pi}^{(d),r+1}, \hat{\pi}^{(d),r+2}, \ldots)$ by Claim 9. If $r < n+d+4$, then $f(h) = \hat{\pi}^{(d),r+1}$. Therefore, $h \cdot f(h) \in H_4^{k+1,d,r+1}$. Furthermore, by Claim A.14, $k < M$. Hence, by induction, $\pi(f|h) = (\hat{\pi}^{(d),r+1}, \hat{\pi}^{(d),r+2}, \ldots)$. ■

Proof of Claim 12. Since $f(h) = m^d$, we have $h \cdot f(h) \in H_4^{k+1,d,0}$ if $k-(d+4) = (\theta+1)T-1$ and $h \cdot f(h) \in H_3^{k+1,d}$ if $k-(d+4) < (\theta+1)T-1$. Also, by Claim A.13, $k < M$. Therefore, by Claim 11, the result follows by induction. ■

Proof of Claim 13. We establish this claim by considering the different possible cases.

Case 1: One of the following conditions hold: (a) $\tau = d+3$, (b) $\tau = 0$, $T^{d+3}(h) = ((s';2),(s;d),a)$ and $a \in \bar{D}_d(s)$, and (c) $h \in H_6^{d,0}$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$. In this case, it follows that $f(h) = s'$ and $\bar{a} \in D_d(s')$ and $h \cdot \bar{a} \in H_3^{d+4,d}$. Thus, by Claim 12, $\pi(f|h \cdot \bar{a}) = \pi^{(d)}(\bar{\theta}(\bar{a}), t(\bar{a}))$ where $t(\bar{a}) = d+5$ and $\bar{\theta}(\bar{a}) = \theta(T^{d+3}(h) \cdot \bar{a})$. Furthermore, if $\bar{a} \in \bar{D}_d(f(h))$, then $\theta(T^{d+3}(h) \cdot \bar{a}) = \theta(T^{d+3}(h) \cdot f(h)) + 1$. Hence, $\bar{\theta}(f(h)) < \bar{\theta}(\bar{a})$.

Case 2: $h \in H_2^{k,d,\tau} \cup H_5^{k,d,\tau}$ and none of the conditions (a)–(c) hold. In this case, $f(h) = \pi^{(d),\tau+1}$ and $h \cdot \bar{a} \in H_2^{k+1,d,\tau+1} \cup H_5^{k+1,d,\tau+1}$. Thus, by Claims A.12 and A.15, $f(h \cdot \bar{a}) = \pi^{(d),\tau+2}$. Then, by induction, it follows that $f(h \cdot \bar{a} \cdot \pi^{(d),\tau+2}) = \pi^{(d),\tau+3}, \ldots, f(h \cdot \bar{a} \cdot \pi^{(d),\tau+2} \cdot \cdots \cdot \pi^{(d),d+2}) = \pi^{(d),d+3}$. Therefore, by appealing to Case 1, we have $\pi(f|h \cdot \bar{a}) = \pi^{(d)}(\bar{\theta}(\bar{a}), t(\bar{a}))$ where $t(\bar{a}) = \tau+2$ and $\bar{\theta}(\bar{a}) = \theta(T^{\tau}(h) \cdot (\bar{a}, \pi^{(d),\tau+2}, \ldots, \pi^{(d),d+4}))$. It also follows from the latter that $\bar{\theta}(\bar{a}) = \bar{\theta}(f(h)) + 1$ if $\bar{a} \in \bar{D}_d(f(h))$.

Case 3: $h \in H_6^{d,\tau}$ and none of the conditions (a)–(c) hold. For any history $h' \in H$, define $v(h') = \max\{t \in \{0, \ldots, d+3\} : h' \in H_6^{d,t}\}$ and let $\tau' = v(h)$. Then (i) $h \in H_6^{d,\tau'}$, (ii) $T^1(h) \in \bar{D}_d(s) \cup \bar{D}_d(s')$ if $\tau' = 0$, (iii) $T^1(h) \in D_d(s')$ if $\tau' = 1$ and (iv) $f(h) = \pi^{(d),\tau'+1}$. To complete the proof in this case, we first establish two subclaims.

Subclaim 1: $h \cdot \bar{a} \in H_6^{d,\tau'+1}$. Suppose not. By (iv), $h \cdot \bar{a} \in \tilde{\Sigma}^{d,\tau'+1}$, hence, $h \cdot \bar{a} \in \cup_{l=1}^{5} H_l$.

Also, if $\tau' \geq 2$, then $\bar{a} \in D(s)$. Therefore, by Claim A.10, $\tau' = 0$ or 1 and either $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0, \dots, n\}$ or $T^{\hat{d}+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+4}$ for some $\hat{d} \in N$.

If $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$, then $T^1(h) = s$, contradicting (ii) and (iii). If $T^{\hat{d}+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+4}$, then $T^1(h) \in D_{\hat{d}}(s)$. Therefore, by (iii), $\tau' = 0$. Then, by (ii), it follows that $T^1(h) \in \bar{D}_{\hat{d}}(s)$ and $d = \hat{d}$. This implies that $T^{d+3}(h) \in \Sigma^{d,d+3}$. Thus, by (i) and $\tau' = 0$, condition (c) is satisfied. But this contradicts our supposition.

Subclaim 2: $v(h \cdot \bar{a}) = \tau' + 1$. Since $h \cdot \bar{a} \in H_6^{d,\tau'+1}$, $v(h \cdot \bar{a}) \geq \tau' + 1 > 0$. Therefore, $h \in \tilde{\Sigma}^{d,v(h \cdot \bar{a})-1}$. Hence, $\tau' \geq v(h \cdot \bar{a}) - 1$ and thus $v(h \cdot \bar{a}) = \tau' + 1$.

It follows from the above two subclaims that $f(h \cdot \bar{a}) = \pi^{(d),\tau'+2}$. Then, by induction, it follows that $f(h \cdot \bar{a} \cdot \pi^{(d),\tau'+2}) = \pi^{(d),\tau'+3}, \dots, f(h \cdot \bar{a} \cdot \pi^{(d),\tau'+2} \cdot \dots \cdot \pi^{(d),d+2}) = \pi^{(d),d+3}$. Therefore, by appealing to Case 1, we have $\pi(f|h \cdot \bar{a}) = \pi^{(d)}(\bar{\theta}(\bar{a}), t(\bar{a}))$, where $t(\bar{a}) = \tau' + 2$ and $\bar{\theta}(\bar{a}) = \theta(T^{\tau'}(h) \cdot (\bar{a}, \pi^{(d),\tau'+2}, \dots, \pi^{(d),d+4}))$. Furthermore, from the latter, $\bar{\theta}(\bar{a}) = \bar{\theta}(f(h)) + 1$ for all $\bar{a} \in \bar{D}_{d'}(f(h))$. ■

Proof of Claim 14. If $h \notin \cup_{k=2}^{n+3} H_7^k$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$ for some $d \in N$, then $f(h) = s'$ and $T^{d+4}(h \cdot f(h)) \in \Sigma^{d,d+4}$. Hence, $h \cdot f(h) \in H_3^{d+4,d}$ and the conclusion follows from Claim 12.

If $h \notin \cup_{k=2}^{n+3} H_7^k$ and $T^{n+i+4}(h) = ((s';2),(s;n+i+2))$ for some $i \in \{0, \dots, n\}$, then $f(h) = s'$ and $T^{n+i+5}(h \cdot f(h)) = ((s';2),(s;n+i+2),s')$. Hence, $h \cdot f(h) \in H_{1,a}^{(i),n+i+5}$ and the conclusion follows from Claim 9. For the remainder of the proof, therefore assume that the following holds:

$$\begin{aligned}&\text{if } h \notin \cup_{k=2}^{n+3} H_7^k, \text{ then } T^{n+i+4}(h) \neq ((s';2),(s;n+i+2)),\\ &\text{for all } i \in \{0, \dots, n\}, \text{ and } T^{d+3}(h) \notin \Sigma^{d,d+3}, \text{ for all } d \in N.\end{aligned} \tag{A.6}$$

Next, for any history $h' \in H$, define $v(h') = \max\{t \in \{0, \dots, n+4\} : T^t(h') \in H_{1,b}^{(0),t}\}$ and let $k' = v(h)$. If $k' = n+4$, then $f(h) = s'$ and $h \cdot f(h) \in H_{1,a}^{(0),n+5}$, and the conclusion follows from Claim 9.

If $k' \in \{0, \dots, n+3\}$, then the claim follows by induction if it is the case that $h \cdot f(h) \in H_7^{k'+1}$ and $v(h \cdot f(h)) = k' + 1$. We show that the latter is indeed the case in several steps.

Step 1: $h \cdot f(h) \notin H_6$. Otherwise, $h \cdot f(h) \in H_6^{d,\tau}$ for some $d \in N$ and $\tau \in \{0, \dots, d+3\}$. Then if $\tau > 0$, $T^{\tau-1}(h) \in \tilde{\Sigma}^{d,\tau-1}$; but this is a contradiction as this implies that $h \in H_6$.

If $\tau = 0$, then by the definition of H_6^0, $f(h) = T^1(h \cdot f(h)) \in \bar{D}(s) \cup \bar{D}(s')$. But this is a contradiction because, by $h \in H_7^{k'}$, $f(h) \in \{s, s'\}$.

Step 2: $h \cdot f(h) \notin \cup_{l=1}^5 H_l$. When $k' \geq 2$, then $f(h) = s$. Therefore, the claim in this step follows immediately from Claim A.10. Next suppose that $k' \in \{0, 1\}$ and that $h \cdot f(h) \in \cup_{l=1}^5 H_l$. Then Claim A.10 implies that either $h \in H_{1,a}^{(i),n+i+4}$ for some $i \in \{0, \ldots, n\}$ or $T^{d+3}(h) \in \Sigma^{d,d+3}$ for some $d \in N$. But this contradicts our supposition in (A.6).

Step 3: $T^{k'+1}(h \cdot f(h)) \in H_{1,b}^{(0),k'+1}$ and $v(h \cdot f(h)) = k' + 1$. By $h \in H_7^{k'}$, $f(h) = \pi^{(0),k'+1}$ and so $T^{k'+1}(h \cdot f(h)) \in H_{1,b}^{(0),k'+1}$. Hence, $v(h \cdot f(h)) \geq k' + 1$. If it were the case that $v(h \cdot f(h)) > k' + 1$, then $T^{v(h \cdot f(h))-1}(h) \in H_{1,b}^{(0),v(h \cdot f(h))-1}$, implying $v(h) \geq v(h \cdot f(h)) - 1$. Since $k' = v(h)$ and $v(h \cdot f(h)) > k' + 1$, this is a contradiction. ■

Proof of Claim 15. By assumption, $h \in H_{1,a}^{(i),k} \cup H_{1,b}^{(0),l}$ for some i, k and l, and $\bar{a} \in \bar{D}_d(f(h))$. Thus, $h \cdot \bar{a} \in H_2^{k+1,d,0}$. Also, by Claim 13, $\pi(f | h \cdot \bar{a} \cdot f(h \cdot \bar{a})) = \pi^{(d)}(\theta, t)$ for some θ and t. ■

Proof of Claim 16. We first argue that it is sufficient to show that either $h \cdot \bar{a} \in H_3^{d+4,d}$ or $h \cdot \bar{a} \notin \cup_{l=1}^5 H_l$. If the former holds, then $\pi(f | h \cdot \bar{a}) = \pi^{(d)}(\theta, d + 5)$ and the claim follows from Claim 12. If the latter holds, then this together with $f(h) \in \{s, s'\}$ and $\bar{a} \in \bar{D}_d(f(h))$ implies that $h \cdot \bar{a} \in H_6^{d,0}$. Then the claim follows from Claim 13.

We next establish that either $h \cdot \bar{a} \in H_3^{d+4,d}$ or $h \cdot \bar{a} \notin \cup_{l=1}^5 H_l$. Suppose not; then we derive a contradiction for each of the different possible cases as follows.

<u>Case 1</u>: $T^{\hat{d}+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+4}$ for some $\hat{d} \in N$. Then $\bar{a} \in D_{\hat{d}}(s')$, which together with $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$ implies that $\hat{d} = d$. Hence, $h \cdot \bar{a} \in H_3^{d+4,d}$; a contradiction.

<u>Case 2</u>: $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0, \ldots, n\}$. Then $\bar{a} = s'$ and this contradicts $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$.

<u>Case 3</u>: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ and $h \cdot \bar{a} \in H_{1,a}^{(i),k'} \cup H_{1,b} \cup H_2^{\hat{k},\hat{d},0} \cup H_5^{\hat{k},\hat{d},0}$ for some $i \in \{0, \ldots, n\}$ and $k' > n + i + 5$ and $\hat{k} \in \{1, \ldots, M\}$ and $\hat{d} \in N$. Then, by the latter, $h \in H_1 \cup H_3 \cup H_4$ and, by Claims A.18, A.19, A.23, A.27, A.30, A.31, A.33 and A.34, this contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

<u>Case 4</u>: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ and $h \cdot \bar{a} \in H_2^{\hat{k},\hat{d},\hat{\tau}} \cup H_5^{\hat{k},\hat{d},\hat{\tau}}$ for some $\hat{k} \in \{1, \ldots, M\}$ and $\hat{d} \in N$ and $\hat{\tau} \in \{1, \ldots, \hat{d} + 3\}$. By the latter, $\bar{a} \in D_{\hat{d}}(s) \cup D_{\hat{d}}(s')$. Since we also have $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$, it follows that $\hat{d} = d$. This together with $h \cdot \bar{a} \in H_2^{\hat{k},\hat{d},\hat{\tau}} \cup H_5^{\hat{k},\hat{d},\hat{\tau}}$ implies that $h \in H_2^{\hat{k}-1,d,\hat{\tau}-1} \cup H_5^{\hat{k}-1,d,\hat{\tau}-1}$. Since $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$, Claims A.20, A.32 and A.35

imply that $\tau = \hat{\tau} - 1$ and $k = \hat{k} - 1 + \beta K$ for some $\beta \in \mathbb{Z}$. Thus, $\bar{a} = f(h)$, contradicting the supposition that $\bar{a} \in \bar{D}_d(f(h))$.

<u>Case 5</u>: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ *and* $h \cdot \bar{a} \in H_3$ *and* $T^{\hat{d}+4}(h \cdot \bar{a}) \notin \Sigma^{\hat{d},\hat{d}+4}$ *for all* $\hat{d} \in N$. Then $h \in H_3$ and, by Claims A.23 and A.33, this contradicts $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

<u>Case 6</u>: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ *and* $h \cdot \bar{a} \in H_4$. Then $h \in H_3 \cup H_4$ and, by Claims A.23, A.27, A.33 and A.34, this contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

<u>Case 7</u>: $h \in H_6^{d',\tau}$ *and* $h \cdot \bar{a} \notin H_{1,a}^{(i),n+i+5}$ *for all* $i \in \{0, \ldots, n\}$ *and* $T^{\hat{d}+4}(h \cdot \bar{a}) \notin \Sigma^{\hat{d},\hat{d}+4}$ *for all* $\hat{d} \in N$. Then $h \notin \cup_{l=1}^{5} H_l$ and, by Claim A.10, we have $h \cdot \bar{a} \notin \cup_{l=1}^{5} H_l$; a contradiction. ∎

Proof of Claim 17. If $d = d'$, then $h \cdot \bar{a} \in H_4^{k+1,d,0}$ if $k + 1 - (d + 4) = (\theta + 1)T$ and $h \cdot \bar{a} \in H_3^{k+1,d}$, otherwise. Thus, in this case, the result follows from Claim 11 and Claim 12, respectively.

If $d \neq d'$, then $h \cdot \bar{a} \in H_5^{k+1,d,0}$. Thus, by Claim 13, $\pi(f|h \cdot \bar{a} \cdot f(h \cdot \bar{a})) = \pi^{(d)}(\theta, t)$ for some $\theta \in \{0, \ldots, d + 4\}$ and $t \in \{1, \ldots, d + 5\}$. ∎

Proof of Claim 18. We have that $h \cdot \bar{a} \in H_5^{k+1,0,d}$. Thus, by Claim 13, $\pi(f|h \cdot \bar{a} \cdot f(h \cdot \bar{a})) = \pi^{(d)}(\theta, t)$ for some $\theta \in \{0, \ldots, d + 4\}$ and $t \in \{1, \ldots, d + 5\}$. ∎

Proof of Claim 19. If $h \notin \cup_{k=2}^{n+3} H_7^k$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$, then $f(h) = s'$ and $\bar{a} \in \bar{D}_d(s')$ and $T^{d+4}(h \cdot \bar{a}) \in \Sigma^{d,d+4}$. Hence, $h \cdot \bar{a} \in H_3^{d+4,d}$ and the conclusion follows from Claim 12.

For the remainder of the proof, therefore, assume that the following holds:

$$\text{If } h \notin \cup_{k=2}^{n+3} H_7^k, \text{ then } T^{d+3}(h) \notin \Sigma^{d,d+3}. \tag{A.7}$$

Then we will show that $h \cdot \bar{a} \in H_6^{d,0}$ which, by Claim 13, establishes the conclusion of the claim. We prove the former in two steps.

Step 1: $h \cdot \bar{a} \in \tilde{\Sigma}^{d,0}$. Since $h \in H_7^k$, $f(h) \in \{s, s'\}$. Therefore, $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$.

Step 2: $h \cdot \bar{a} \notin \cup_{l=1}^{5} H_l$. Suppose otherwise. Then, by Claim A.10, $\bar{a} \notin D(s)$. Hence, by $h \in H_7^k$, (i) $\bar{a} \in \bar{D}_d(s')$ and (ii) $h \notin \cup_{k=2}^{n+3} H_7^k$. Furthermore, Claim A.10 implies that either $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0, \ldots, n\}$ or $T^{d'+4}(h \cdot \bar{a}) \in \Sigma^{d',d'+4}$ for some $d' \in N$. If $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0, \ldots, n\}$, then $\bar{a} = s'$, a contradiction to (i). If $T^{d'+4}(h \cdot \bar{a}) \in \Sigma^{d',d'+4}$ for some $d' \in N$, then $\bar{a} \in D_{d'}(s')$ which, by (i), implies that $d = d'$. Thus, $T^{d+3}(h) \in \Sigma^{d,d+3}$. But this together with (ii) contradicts our supposition in (A.7). ∎

B Theorem 2 (mixed Folk Theorem)

In this section, we provide the proofs of Lemmas 1 and 3.

B.1 Proof of Lemma 1

Proof of Lemma 1. In order to economize on notation, we shall omit the dependence of $\hat{\Sigma}^d(.), S(.), \lambda(.), \lambda_i(.), \Phi_i^d(.), \alpha_i^d(.)$ on $\hat{h}, Q, \delta$ or η whenever the meaning is clear.

Fix any $\zeta > 0$ and $0 < \varepsilon_1 < 1$. Let $Q \in \mathbb{N}$ be such that $Q > \gamma$ and

$$\frac{1}{\zeta^{Q/(Q+1)}(Q+1)} < \frac{\varepsilon_1}{2B}.$$

Moreover, let $\eta > 0$ be such that $(n-1)B|A|^2\eta < \varepsilon_1/2$.

We next fix player $d \in N$ and consider any $\delta \in (0,1)$, $t \in \mathbb{N}$ and $\hat{h} = (\hat{a}^1, \ldots, \hat{a}^t) \in H_t$ such that $\delta^t \geq \zeta$ and $\alpha_i^d(\hat{h}) = 1$ for all $i \neq d$. For ease of exposition we reorder the players so that $d = n$. Then for every $a \in A$ and every $i \neq n$:

$$|\lambda(a) - \lambda_i(a_{-i})\mu_i^n(a_i)| < \eta\frac{1-\delta^t}{1-\delta}$$

Fix $a = (a_1, a_2, \ldots, a_n) \in A$. In particular, we have that

$$\left|\lambda(a) - \sum_{b_1 \in A_1} \lambda(b_1, a_2, \ldots, a_n)\mu_1^n(a_1)\right| < \eta\frac{1-\delta^t}{1-\delta}$$

Since, for every $b_1 \in A_1$,

$$\left|\lambda(b_1, a_{-1}) - \sum_{b_2 \in A_2} \lambda(b_1, b_2, a_3, \ldots, a_n)\mu_2^n(a_2)\right| < \eta\frac{1-\delta^t}{1-\delta},$$

we obtain

$$\left|\sum_{b_1 \in A_1} \lambda(b_1, a_{-1})\mu_1^n(a_1) - \sum_{(b_1,b_2) \in A_1 \times A_2} \lambda(b_1, b_2, a_3, \ldots, a_n)\mu_1^n(a_1)\mu_2^n(a_2)\right| \leq \mu_1^n(a_1)|A_1|\eta\frac{1-\delta^t}{1-\delta} < |A|\eta\frac{1-\delta^t}{1-\delta}.$$

Hence,

$$\left|\lambda(a) - \sum_{(b_1,b_2) \in A_1 \times A_2} \lambda(b_1, b_2, a_3, \ldots, a_n)\mu_1^n(a_1)\mu_2^n(a_2)\right| < (1+|A|)\eta\frac{1-\delta^t}{1-\delta} < 2|A|\eta\frac{1-\delta^t}{1-\delta}.$$

Repeating the same procedure $n-1$ times implies that

$$\left|\lambda(a) - \sum_{(b_1,\ldots,b_{n-1})\in A_{-n}} \lambda(b_1, b_2, \ldots, b_{n-1}, a_n) \prod_{j=1}^{n-1} \mu_j^n(a_j)\right| < (n-1)|A|\eta\frac{1-\delta^t}{1-\delta}.$$

Hence,

$$\frac{|\lambda(a) - \sum_{(b_1,\ldots,b_{n-1})\in A_{-n}} \lambda(b_1, b_2, \ldots, b_{n-1}, a_n) \prod_{j=1}^{n-1} \mu_j^n(a_j)|}{\sum_{k\notin S} \delta^{k-1}} < \frac{\eta(1-\delta^t)|A|}{(1-\delta)\sum_{k\notin S}\delta^{k-1}}. \quad \text{(B.1)}$$

Next, define for each $a_n \in A_n$,

$$r_n(a_n) = \frac{\sum_{b_{-n}\in A_{-n}} \lambda(b_{-n}, a_n)}{\sum_{k\notin S}\delta^{k-1}}.$$

It follows from the definition of r_n and (B.1) that $r_n \in \Delta(A_n)$ and, for all $a \in A$,

$$\frac{\lambda(a)}{\sum_{k\notin S}\delta^{k-1}} < r_n(a_n)\prod_{j=1}^{n-1}\mu_j^n(a_j) + \frac{\eta(1-\delta^t)|A|}{(1-\delta)\sum_{k\notin S}\delta^{k-1}}.$$

Hence, because μ_{-n}^n is a minmax profile,

$$\frac{\sum_{a\in A}\lambda(a)u_n(a)}{\sum_{k\notin S}\delta^{k-1}} < u_n(r_n, \mu_{-n}^n) + \frac{\eta(1-\delta^t)|A|^2 B}{(1-\delta)\sum_{k\notin S}\delta^{k-1}} < \frac{\varepsilon_1}{2}\frac{1-\delta^t}{1-\delta}\frac{1}{\sum_{k\notin S}\delta^{k-1}}. \quad \text{(B.2)}$$

Then it follows from (B.2) that

$$\begin{aligned}
\frac{1-\delta}{1-\delta^t}\sum_{k=1}^{t}\delta^{k-1}u_n(\hat{a}^k) &= \frac{1-\delta}{1-\delta^t}\left\{\sum_{k\in S}\delta^{k-1}u_n(\hat{a}^k) + \sum_{a\in A}\lambda(a)u_n(a)\right\} \\
&\le \frac{1-\delta}{1-\delta^t}\left\{B\sum_{k\in S}\delta^{k-1} + \frac{\varepsilon_1}{2}\frac{1-\delta^t}{1-\delta}\right\} \\
&\le \frac{1-\delta}{1-\delta^t}\left\{B\frac{1-\delta^{|S|}}{1-\delta} + \frac{\varepsilon_1}{2}\frac{1-\delta^t}{1-\delta}\right\} \\
&= B\frac{1-\delta^{|S|}}{1-\delta^t} + \frac{\varepsilon_1}{2}.
\end{aligned}$$

Hence, to complete the proof, it remains to be shown that $B\frac{1-\delta^{|S|}}{1-\delta^t} < \frac{\varepsilon_1}{2}$. Note that $\frac{|S|}{t}$ is bounded by $\frac{1}{Q+1}$. Letting $f(x) = x^{1/(Q+1)}$ and $g(x) = x$, and using the generalized mean value theorem, we have that, for some $b \in (\delta^t, 1)$,

$$\frac{1-\delta^{|S|}}{1-\delta^t} \le \frac{1-\delta^{\frac{t}{Q+1}}}{1-\delta^t} = \frac{f(1)-f(\delta^t)}{g(1)-g(\delta^t)} = \frac{f'(b)}{g'(b)} = \frac{1}{b^{\frac{Q}{Q+1}}(Q+1)}.$$

Since $\delta^t \ge \zeta$, it then follows that $\frac{1-\delta^{|S|}}{1-\delta^t} \le \frac{1}{\zeta^{\frac{Q}{Q+1}}(Q+1)} < \frac{\varepsilon_1}{2B}$. This concludes the proof. ■

B.2 Proof of Lemma 3

In what follows, we present the proof of Lemma 3.

B.2.1 Auxiliary results

We first prove two lemmas.

Lemma B.1 *For all $i, d \in N$, with $i \neq d$,*

$$P_{f^d,T}\left(\{\hat{h} \in H_T : \Phi_i(d, \hat{h}) \geq \eta\}\right) < 2\varepsilon_2.$$

Proof. Suppose not. Then for some $i, d \in N$, with $i \neq d$,

$$P_{f^d,T}\left(\{\hat{h} \in H_T : \Phi_i(d, \hat{h}) \geq \eta\}\right) \geq 2\varepsilon_2.$$

Suppose that in the calculation described by (a)-(d) in section B.2.2 of the main text, player i plays strategy μ_i^d at every history $h \in \cup_{t=0}^{T-1} H_t$ and all other players play f_{-i}^d. Then the expected payoff of i is given by

$$V_i^* = (1-\delta) \sum_{\hat{h}=(\hat{a}^1,\ldots,\hat{a}^T)\in H_T} P_{(\tilde{\mu}_i^d, f_{-i}^d),T}(\hat{h}) \left(\sum_{k=1}^{T} \delta^{k-1} u_i(\hat{a}^k) + \delta^T w_i(d, \hat{h})\right).$$

Given the definition of f^d, we have that

$$V_i^d(H_0) \geq V_i^*. \tag{B.3}$$

Furthermore, by Lemma 2 and (47) and (48),

$$V_i^* \geq -B(1-\delta^T) + \delta^T(u_i' + \rho(1-\varepsilon_2) - 2\xi).$$

By (47) and (48),

$$V_i^d(H_0) \leq B(1-\delta^T) + \delta^T(u_i' + \rho(1-2\varepsilon_2) + 2\xi).$$

Hence, by (51),

$$V_i^d(H_0) - V_i^* \leq 2B(1-\delta^T) + 4\xi\delta^T - \rho\varepsilon_2\delta^T < 0.$$

But this contradicts (B.3). ■

Lemma B.2 *For all $d \in N$,*

$$(1-\delta) \sum_{\hat{h}=(\hat{a}^1,\ldots,\hat{a}^T)\in H_T} P_{f^d,T}(\hat{h}) \sum_{k=1}^{T} \delta^{k-1} u_d(\hat{a}^k) \geq -4\xi\delta^T.$$

Proof. Suppose not. Then for some $d \in N$,

$$(1-\delta) \sum_{\hat{h}=(\hat{a}^1,\ldots,\hat{a}^T)\in H_T} P_{f^d,T}(\hat{h}) \sum_{k=1}^{T} \delta^{k-1} u_d(\hat{a}^k) < -4\xi\delta^T.$$

Consider strategy f'_d for player d defined by setting, for all $h \in \cup_{t=0}^{T-1} H_t$, $f'_d(h)$ to be such that it solves $\max_{a_d \in A_d} u_d(a_d, f^d_{-d}(h))$. Since $\max_{a_d \in A_d} u_d(a_d, f^d_{-d}(h)) \geq \tilde{v}_i = 0$, it follows that

$$(1-\delta) \sum_{\hat{h}=(\hat{a}^1,\ldots,\hat{a}^T)\in H_T} P_{(f'_d, f^d_{-d}),T}(\hat{h}) \sum_{k=1}^{T} \delta^{k-1} u_d(\hat{a}^k) \geq 0.$$

Letting V_d be player d's expected discounted payoff from (f'_d, f^d_{-d}), by (47) and (48),

$$\begin{aligned} & V_d - V_d^d(H_0) = \\ & (1-\delta) \sum_{\hat{h}\in H_T} P_{(f'_d, f^d_{-d}),T}(\hat{h}) \sum_{k=1}^{T} \delta^{k-1} u_d(\hat{a}^k) - (1-\delta) \sum_{\hat{h}\in H_T} P_{f^d,T}(\hat{h}) \sum_{k=1}^{T} \delta^{k-1} u_d(\hat{a}^k) \\ & +\delta^T \left\{ \sum_{\hat{h}\in H_T} P_{(f'_d, f^d_{-d}),T}(\hat{h}) w_d(d,\hat{h}) - \sum_{\hat{h}\in H_T} P_{f^d,T}(\hat{h}) w_d(d,\hat{h}) \right\} \\ > \ & 4\xi\delta^T - 4\xi\delta^T = 0. \end{aligned}$$

But this contradicts $V_d \leq V_d^d(H_0)$. ■

Next, we provide some results that are similar to those for the pure strategy case in Appendix A of the main text and Section A of the supplementary materials; for the sake of completeness we will describe them fully.

Claim B.1 *For all $\omega \in \{0, \ldots, \bar{\omega}\}$, the following hold:*

1. *All actions $a \neq s'$ are played for $t \geq n + \overline{\omega} + Q + 2$ consecutive periods in $\hat{\pi}^{(\omega)}$.*
2. *Suppose that, for some $t \in \mathbb{N}$, $(\hat{\pi}^{(\omega),t}, \ldots, \hat{\pi}^{(\omega),t+l+1}) = ((s';2),(s;l))$ and $l > 0$. Then either $\hat{\pi}^{(\omega),t+l+2} = s$ or $\hat{\pi}^{(\omega),t+l+2} = s'$ and $l = n + \omega + Q + 2$.*
3. *Suppose that, for some $t \in \mathbb{N}$, $\hat{\pi}^{(\omega),t} \in D(s')$ and $\hat{\pi}^{(\omega),t+1} \in D(s)$. Then $\hat{\pi}^{(\omega),t} = s'$ and $\hat{\pi}^{(\omega),t+1} = s$. Furthermore, either (i) $\hat{\pi}^{(\omega),t-1} = s'$ and $t = 2 + \beta K$ or (ii) $\hat{\pi}^{(\omega),t-1} = s$ and $t = n + \omega + Q + 5 + \beta K$ for some $\beta \in \mathbb{N}_0$.*

4. Suppose that, for some $t \in \mathbb{N}$, $\hat{\pi}^{(\omega),t} \in D(s)$ and $\hat{\pi}^{(\omega),t+1} \in D(s')$. Then $\hat{\pi}^{(\omega),t+1} = s'$.

Proof. This follows immediately from the ordering of A and the definition of $\hat{\pi}^{(\omega)}$. ∎

Claim B.2 *If $h \in H_{1,a}^{(\omega),k}$ for some $\omega \in \{0,\ldots,\bar{\omega}\}$ and $k \in \{n+\omega+Q+5,\ldots,M\}$, then $T^{k'}(h) \notin \Sigma^{d,k'}$ for all $d \in N$ and $k' \in \mathbb{N}$ such that $d+Q+4 \le k' \le k$.*

Proof. Suppose otherwise; then there exists $h \in H_{1,a}^{(\omega),k}$ such that $T^{k'}(h) \in \Sigma^{d,k'}$ for some $d \in N$ and $k' \in \mathbb{N}$ with $d+Q+4 \le k' \le k$. Let $T^k(h) = (a^1,\ldots,a^k)$. Since $T^{k'}(h) \in \Sigma^{d,k'}$, we have $a^{k-k'+2} \in D(s')$, $a^{k-k'+3} \in D(s)$ and $a^{k-k'+d+Q+4} \in D(s')$. Therefore, there is an action, namely $a^{k-k'+3}$, which is different from s' and is played at most $d+Q+1$ consecutive periods in $T^k(h)$. Since $h \in H_{1,a}^{(i),k}$, this contradicts Claim B.1.1. ∎

Next, for all $\tau \in \mathbb{N}$ and $\omega \in \{0,\ldots,\bar{\omega}\}$, define

$$\Lambda^{\omega,\tau} = \left\{h \in H : h = (a^t)_{t=1}^{\tau} \text{ such that } a^t = \hat{\pi}^{(\omega),t} \text{ if } t \le n+\omega+Q+5\right\}.$$

Claim B.3 *If $h \in H_{1,a}^{(\omega),k}$ for some $\omega \in \{0,\ldots,\bar{\omega}\}$ and $k \in \{n+\omega+Q+5,\ldots,M\}$and $T^{k'}(h) \in \Lambda^{\omega',k'}$ for some $\omega' \in \{0,\ldots,\bar{\omega}\}$ and $n+\omega'+Q+5 \le k' \le k$, then $\omega = \omega'$ and $k = k' + \beta K$ for some $\beta \in \mathbb{N}_0$.*

Proof. Since $h \in H_{1,a}^{(\omega),k}$ and $T^{k'}(h) \in \Lambda^{\omega',k'}$, for some $t = k+1-k'$ it must be that $(\hat{\pi}^{(\omega),t},\ldots,\hat{\pi}^{(\omega),t+n+\omega'+Q+4}) = ((s';2),(s;n+\omega'+Q+2),s')$. By Claim B.1.2, $n+\omega'+Q+2 = n+\omega+Q+2$ and so $\omega = \omega'$. Hence, by Claim B.1.3, $t+1 = 2+\beta K$ for some $\beta \in \mathbb{N}_0$ and so $k = k'+\beta K$. ∎

Claim B.4 *Suppose that, for some $k \in \{1,\ldots,M\}$ and $d \in N$, $\omega \in W_d$ and $h \in H_{3,a}^{k,d} \cup H_{3,b}^{k,d} \cup H_4^{k,\omega,0}$. If $T^{k'}(h) \in \Lambda^{\omega',k'} \cup \Sigma^{d',k'}$ for some $\omega' \in \{0,\ldots,\bar{\omega}\}$ and $d' \in N$ and $k' \in \mathbb{N}$ such that $Q+2 \le k' < k$, then $k' = Q+2$ and $h \in H_{3,b}^{k,d} \cup H_4^{k,\omega,0}$. Hence, $T^{k'}(h) \notin \Lambda^{\omega',k'} \cup \Sigma^{d',k'}$ for all $\omega' \in \{0,\ldots,\bar{\omega}\}$ and $d' \in N$ and $k' \in \mathbb{N}$ such that $Q+3 \le k' < k$.*

Proof. Suppose that $T^{k'}(h) \in \Lambda^{\omega',k'} \cup \Sigma^{d',k'}$ for some $\omega' \in \{0,\ldots,\bar{\omega}\}$ and $d' \in N$ and $k' \in \mathbb{N}$ such that $Q+2 \le k' < k$. Let $T^k(h) = (a^1,\ldots,a^k)$. First, note that by $T^{k'}(h) \in \Lambda^{\omega',k'} \cup \Sigma^{d',k'}$,

$$a^{k-k'+\tau} \in \begin{cases} D_{d'}(s') & \text{if } \tau = 1,2 \\ D_{d'}(s) & \text{if } \tau = 3,\ldots,Q+4. \end{cases} \tag{B.4}$$

Since, by $h \in H_{3,a}^{k,d} \cup H_{3,b}^{k,d} \cup H_4^{k,\omega,0}$, it must be that $a^3,\ldots,a^{d+Q+3} \in D(s)$, it follows that $k-k'+1 \ge d+Q+4$. We next consider three cases regarding $k-k'+1$.

Case 1: $k - k' + 1 = d + Q + 4 + (\gamma + 1)l + \tau$ for some $l \in \{0, \ldots, d + Q + 3\}$ and some $\tau = 0, \ldots, \gamma - 2$. Since $h \in H_{3,a}^{k,d} \cup H_{3,b}^{k,d} \cup H_4^{k,\omega,0}$ and $\gamma \geq 3$, in this case we must have $a_{-d}^{k-k'+2} = a_{-d}^{k-k'+3} \in \{m_{-d}^d, \bar{a}_{-d}^{(d)}\}$. Also, by (B.4), $a^{k-k'+2} \in D_{d'}(s')$ and $a^{k-k'+3} \in D_{d'}(s)$. But this a contradiction.

Case 2: $k - k' + 1 = d + Q + 4 + (\gamma + 1)l + \gamma - 1$ for some $l \in \{0, \ldots, d + Q + 3\}$. In this case, $a_{-d}^{k-k'+3} = s'_{-d}$ by $h \in H_{3,a}^{k,d} \cup H_{3,b}^{k,d} \cup H_4^{k,\omega,0}$. But this contradicts (B.4).

Case 3: $k - k' + 1 = d + Q + 4 + (\gamma + 1)l + \gamma$ for some $l \in \{0, \ldots, d + Q + 2\}$. Since $k' \geq Q + 2 > \gamma + 2$, $a^{k-k'+\gamma+3} \in D_d(s')$ by $h \in H_{3,a}^{k,d} \cup H_{3,b}^{k,d} \cup H_4^{k,\omega,0}$. Also, by (B.4), it must be that $a^{k-k'+\tau} \in D_{d'}(s)$ for all $\tau = 3, \ldots, Q + 3$. Since $Q > \gamma$ we then have a contradiction.

It follows from the above three cases that $k - k' + 1 \geq (d + Q + 4)(\gamma + 2) - 1$. Since $k' \geq Q + 2 > \gamma + 2$ we have $h \notin H_{3,a}^{k,d}$. We next complete the proof by showing that $k' = Q + 2$.

Suppose otherwise; hence, $k' \geq Q + 3$. If $k - k' + 1 = (d + Q + 3)(\gamma + 2) - 1$, then $a^{k-k'+3} \in D_d(s')$ by $h \in H_{3,b}^{k,d} \cup H_4^{k,\omega,0}$; but this contradicts (B.4). So suppose that $k - k' + 1 > (d + Q + 3)(\gamma + 2) - 1$. Then, since we have $a^{k-k'+\tau} \in D_{d'}(s)$ for all $Q + 2 \geq \tau \geq 3$ by (B.4), we obtain a contradiction to (B.4) because $a^{k-k'+Q+3} \in D_d(s')$ due to $h \in H_{3,b}^{k,d} \cup H_4^{k,\omega,0}$. ■

Claim B.5 *If $h \in H_4^{k,\omega,r}$ for some $k \in \{1, \ldots, M\}$ and $\omega \in \{0, \ldots, \bar{\omega}\}$ and $r \in \{1, \ldots, n + \omega + Q + 4\}$ and $T^{k'}(h) \in \Lambda^{\omega',k'}$ for some $\omega' \in \{0, \ldots, \bar{\omega}\}$ and $Q + 3 \leq k' < k$, then $k' = r$ and $k' < n + \omega' + Q + 5$.*

Proof. Let $T^k(h) = (a^1, \ldots, a^k)$ and $d \in N$ be such that $\omega \in W_d$. By $T^{k'}(h) \in \Lambda^{\omega',k'}$, $a^{k-k'+1} = a^{k-k'+2} = s'$; by $h \in H_4^{k,\omega,r}$, $a^t \in D(s)$ for all $t \in \{3, \ldots, d + Q + 3\} \cup \{k - r + 3, \ldots, k\}$. It then follows that $k - k' + 1 \in \{2, d + Q + 4, \ldots, k - r + 2\}$.

By Claim B.4, it must be that $k - k' + 1 > k - r - Q - 2$. The reasoning for this is as follows. Since $h \in H_4^{k,\omega,r}$ and $T^{k'}(h) \in \Lambda^{\omega',k'}$, it follows, respectively, that $B^r(h) \in H_4^{k-r,\omega,0}$ and $T^{k'-r}(B^r(h)) \in \Lambda^{\omega',k'-r}$. But then by Claim B.4 it must be that $k' - r < Q + 3$.

Note, however, that it cannot be that $k - k' + 1 \in \{k - r - Q - 1, \ldots, k - r\}$; otherwise, $a^{k-r+2} = s$ by $T^{k'}(h) \in \Lambda^{\omega',k'}$ and $k' \geq Q + 3$, and $a^{k-r+2} = s'$ by $h \in H_4^{k,\omega,r}$. Furthermore, it cannot be that $k - k' + 1 = k - r + 2$; otherwise, $a^{k-r+3} = s'$ by $T^{k'}(h) \in \Lambda^{\omega',k'}$ and $a^{k-r+3} = s$ by $h \in H_4^{k,\omega,r}$. Hence, $k - k' + 1 = k - r + 1$, i.e. $k' = r$.

Finally, $k' < n + \omega' + Q + 5$ because otherwise, by $T^{k'}(h) \in \Lambda^{\omega',k'}$, $a^{k-k'+n+\omega'+Q+5} = s'$ and, by $h \in H_4^{k,\omega,r}$ and $k' = r$, $a^{k-r+n+\omega+Q+5} = s$. ■

Claim B.6 *If $h \in H_4^{k,\omega,r}$ for some $k \in \{1, \dots, M\}$ and $\omega \in \{0, \dots, \bar{\omega}\}$ and $r \in \{1, \dots, n + \omega + Q + 4\}$, then $T^{k'}(h) \notin \Sigma^{d',k'}$ for all $d' \in N$ and $k' \in \mathbb{N}$ such that $d' + Q + 4 \leq k' < k$.*

Proof. Suppose otherwise. Let $T^k(h) = (a^1, \dots, a^k)$ and $d \in N$ be such that $\omega \in W_d$. Since $h \in H_4^{k,\omega,r}$ and $T^{k'}(h) \in \Sigma^{d',k'}$, it follows, respectively, that $B^r(h) \in H_4^{k-r,\omega,0}$ and $T^{k'-r}(B^r(h)) \in \Sigma^{d',k'-r}$ if $k' > r$. Then, by Claim B.4, $k' - r < Q + 3$. Therefore, $k - k' + d' + Q + 4 > k - r + 2$. Hence $k - k' + d' + Q + 4 = k - r + t$ for some $t > 2$. But this is a contradiction: by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{k-k'+d'+Q+4} \in D(s')$ and, by $h \in H_4^{k,\omega,r}$, $a^{k-r+t} = s$. ∎

Claim B.7 *If $h \in H_5^{k,d,\tau}$ for some $k \in \{1, \dots, M\}$ and $d \in N$ and $\tau \in \{0, \dots, d + Q + 3\}$, then $T^{k'}(h) \notin \Sigma^{d',k'}$ for all $d' \in N$ and $k' \in \mathbb{N}$ such that $d' + Q + 4 \leq k' < k$.*

Proof. Suppose not. Then there exist $h = (a^1, \dots, a^t) \in H_5^{k,d,\tau}$ such that $T^{k'}(h) \in \Sigma^{d',k'}$ for the parameters given in the statement of the claim. Note that, by $h \in H_5^{k,d,\tau}$, $T^k(h) \in \Sigma^{i,k}$ for some $i \in N$. We now derive a contradiction by considering five different possibilities. Before doing so, let $\bar{h} = B^{\tau+1}(h)$.

<u>Case 1</u>: $k' - (\tau + 1) \geq d' + Q + 4$. Then $T^{k'-(\tau+1)}(\bar{h}) \in \Sigma^{d',k'-(\tau+1)}$ and $\bar{h} \in H_{3,a}^{k-(\tau+1),i} \cup H_{3,b}^{k-(\tau+1),i} \cup H_4^{k-(\tau+1),\omega,r}$ for some $\omega \in \{0, \dots, \bar{\omega}\}$ and $r \in \{0, \dots, n + \omega + Q + 4\}$. But this contradicts either Claim B.4 or Claim B.6.

<u>Case 2</u>: $k' - (\tau + 1) < d' + Q + 1$. Then, by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{t-k'+d'+Q+4} \in D(s')$ and, by $h \in H_5^{k,d,\tau}$ and $t - k' + d' + Q + 4 > t - \tau + 2$, $a^{t-k'+d'+Q+4} \in D(s)$; a contradiction.

<u>Case 3</u>: $k' - (\tau + 1) = d' + Q + 1$. Then, by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{t-k'+d'+Q+3} \in D(s)$ and, by $h \in H_5^{k,d,\tau}$ and $t - k' + d' + Q + 3 = t - \tau + 1$, $a^{t-k'+d'+Q+3} \in D(s')$; a contradiction.

<u>Case 4</u>: *$\bar{h} \in H_{3,a}^{k-(\tau+1),i} \cup H_{3,b}^{k-(\tau+1),i} \cup H_4^{k-(\tau+1),\omega,0}$ for some $\omega \in \{0, \dots, \bar{\omega}\}$ and either $k' - (\tau + 1) = d' + Q + 2$ or $d' + Q + 3$.* Then $Q + 3 \leq k' - (\tau + 1) < k - (\tau + 1)$ and $T^{k'-(\tau+1)}(\bar{h}) \in \Sigma^{d',k'-(\tau+1)}$. This contradicts Claim B.4.

<u>Case 5</u>: *$\bar{h} \in H_4^{k-(\tau+1),\omega,r}$ with $\omega \in \{0, \dots, \bar{\omega}\}$ and $r > 0$ and either $k' - (\tau+1) = d' + Q + 3$ or $k' - (\tau + 1) = d' + Q + 2$.* By $T^{k'}(h) \in \Sigma^{d',k'}$, (i) $a^{t-k'+2} \in D(s')$, (ii) $a^{t-k'+l} \in D(s)$ for all $3 \leq l \leq d' + Q + 3$ and (iii) $a^{t-k'+d'+Q+4} \in D(s')$. By $h \in H_5^{k,d,\tau}$ and $\bar{h} \in H_4^{k-(\tau+1),\omega,r}$, (iv) $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(\omega),r+1})$ and (v) $a^{t-\tau-r'} = \hat{\pi}^{(\omega),r-r'+1}$ for all $1 \leq r' \leq r$. Next consider two subcases.

<u>Subcase 5A</u>: $k' - (\tau + 1) = d' + Q + 3$. By $r > 0$ and $t - k' + d' + Q + 4 = t - \tau$, (iii) and (iv) imply that either $r = 1$ or $r = n + \omega + Q + 4$. If $r = 1$, by (v), $a^{t-\tau-1} = \hat{\pi}^{(\omega),1} = s'$.

This together with $t-\tau-1 = t-k'+d'+Q+3$ contradicts (ii). If $r = n+\omega+Q+4$, then, by (v) and $n+\omega-d'+3 \geq 3$, $a^{t-\tau-d'-Q-2} = \hat{\pi}^{(\omega),n+\omega-d'+3} = s$. This together with $t-\tau-d'-Q-2 = t-k'+2$ contradicts (i).

Subcase 5B: $k'-(\tau+1) = d'+Q+2$. Then by (ii) and $t-\tau = t-k'+d'+Q+3$, $a^{t-\tau} \in D_{d'}(s)$. Therefore, by (iv), $a^{t-\tau} \in \bar{D}_d(s)$ and $d = d'$.

Next, we show that $r = d+Q+2$. If $r > d+Q+2$, then, by (v), $a^{t-\tau-d-Q-1} = \hat{\pi}^{(\omega),r-d-Q} = s$. But this is a contradiction because $t-k'+2 = t-\tau-d-Q-1$ and, by (i), $a^{t-k'+2} \in D_d(s')$. If $r < d+Q+2$, then, by (ii), $a^{t-k'+d-r+Q+4} \in D(s)$. But this is a contradiction because $t-\tau-r+1 = t-k'+d-r+Q+4$ and, by (v), $a^{t-\tau-r+1} = \hat{\pi}^{(\omega),2} = s'$.

By $\bar{h} \in H_4^{k-(\tau+1),\omega,r}$ and $r = d+Q+2$, we have $T^{d+Q+2}(\bar{h}) = ((s';2),(s;d+Q))$. Since $a^{t-\tau} \in \bar{D}_d(s)$, it then follows from the definition of $H_5^{k,d,\tau}$ that $\tau = 0$. But this contradicts $k'-(\tau+1) = d'+Q+2$ and $k' \geq d'+Q+4$. ■

Claim B.8 *If $T^k(h) \in \Sigma^{d,k} \cap \Sigma^{d',k}$ for some $d, d' \in N$ and $k \in \{\min\{d,d'\}+Q+4,\ldots,M\}$, then $d = d'$.*

Proof. Suppose not; assume that $d > d'$. Let $T^k(h) = (a^1,\ldots,a^k)$. Since $T^k(h) \in \Sigma^{d,k}$, $a^{d'+Q+4} \in D_d(s)$ and, since $T^k(h) \in \Sigma^{d',k}$, $a^{d'+Q+4} \in D_{d'}(s')$. But this is a contradiction. ■

Claim B.9 *Let $h \in H \setminus \cup_{l=1}^5 H_l$ and $a \in A$. Then one of the following conditions hold: (a) $h \cdot a \notin \cup_{l=1}^5 H_l$, (b) $h \cdot a \in H_{1,a}^{(\omega),n+\omega+Q+5}$ for some $\omega \in \{0,\ldots,\bar{\omega}\}$ and (c) $T^{d+Q+4}(h \cdot a) \in \Sigma^{d,d+Q+4}$ for some $d \in N$. Furthermore, if $a \in D(s)$, then $h \cdot a \notin \cup_{l=1}^5 H_l$.*

Proof. Suppose that $h \cdot a$ does not satisfy (a)–(c). Then there are six cases to consider. (i) $h \cdot a \in H_{1,a}^{(\omega),k}$ *for some ω and* $k > n+\omega+Q+5$: then $h \in H_{1,a}$; a contradiction. (ii) $h \cdot a \in H_{1,b}$: then $h \in H_{1,b}$; a contradiction. (iii) $h \cdot a \in H_2$: then $h \in H_1 \cup H_2$; a contradiction. (iv) $h \cdot a \in H_3$ *and* $T^{d+Q+4}(h \cdot a) \notin \Sigma^{d,d+Q+4}$ *for all* $d \in N$: then $h \in H_3$; a contradiction. (v) $h \cdot a \in H_4$: then $h \in H_3 \cup H_4$; a contradiction. (vi) $h \cdot a \in H_5$: then $h \in H_3 \cup H_4 \cup H_5$; a contradiction.

Furthermore, if $a \in D(s)$, by the definition of $H_{1,a}^{(\omega),n+\omega+Q+5}$ and $\Sigma^{d,d+Q+4}$, (b) and (c) cannot hold. Therefore, (a) must hold, i.e. $h \cdot a \notin \cup_{l=1}^5 H_l$. ■

Claim B.10 *If $h \in H_{1,a}^{(\omega),M}$ for some $\omega \in \{0,\ldots,\bar{\omega}\}$, then $h \in H_{1,a}^{(\omega),k}$ for some $k < M$.*

Proof. Since $h \in H_{1,a}^{(\omega),M}$, $T^M(h) = (\hat{\pi}^{(\omega),1},\ldots,\hat{\pi}^{(\omega),M})$. Also, by $T > K$ and by the choice of M, $M > K+n+\omega+Q+5$. Therefore, $h \in H_{1,a}^{(\omega),M-K}$. ■

Claim B.11 *If $h \in H_2^{M,d,\tau}$ for some $d \in N$ and $\tau \in \{0, \ldots, d+Q+3\}$, then $h \in H_2^{k,d,\tau}$ for some $k < M$.*

Proof. Since $h \in H_2^{M,d,\tau}$, $T^M(h) = \bar{h} \cdot a \cdot \tilde{h}$, where $\bar{h}$, a, and $\tilde{h}$ are as in the definition of $H_2^{M,d,\tau}$. In particular, $\bar{h} \in H_{1,a}^{(\omega),k'} \cup H_{1,b}^{(0),k'}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ and $k' \leq M - (\tau+1)$. Therefore, $h \in H_2^{k'+\tau+1,d,\tau}$.

If $k' + \tau + 1 < M$, the claim follows. If $k' + \tau + 1 = M$, then $\bar{h} = (\hat{\pi}^{(\omega),1}, \ldots, \hat{\pi}^{(\omega),M-(\tau+1)})$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$. Also, by $T > K$ and by the choice of M, $M \geq (n+Q+4) + (n + \bar{\omega} + Q + 5) + K$. Therefore, $\bar{h} \in H_{1,a}^{(\omega),M-(\tau+1)-K}$, hence, $h \in H_2^{M-K,d,\tau}$. ■

Claim B.12 *If $h \in H_{3,a}^{k,d} \cup H_{3,b}^{k,d}$ for some $k \in \{1, \ldots, M\}$ and $d \in N$, then $k < M$.*

Proof. We have that $k \leq (\gamma+2)(d+Q+4) + T < M$. ■

Claim B.13 *If $h \in H_4^{k,\omega,r}$ for some $k \in \{1, \ldots, M\}$ and $\omega \in \{0, \ldots, \bar{\omega}\}$ and $r \in \{0, \ldots, n + \omega + Q + 4\}$, then $k < M$.*

Proof. We have that $k \leq (\gamma+2)(n+Q+4) + T + n + \bar{\omega} + Q + 4 < M$. ■

Claim B.14 *If $h \in H_5^{k,d,\tau}$ for some $k \in \{1, \ldots, M\}$ and $d \in N$ and $\tau \in \{0, \ldots, d+Q+3\}$, then $k < M$.*

Proof. Let $\bar{h}$, a and $\tilde{h}$ be such that $T^k(h) = \bar{h} \cdot a \cdot \tilde{h}$ and satisfy the conditions in the definition of $H_5^{k,d,\tau}$. By the proof of Claims B.12 and B.13, $\ell(\bar{h}) < (\gamma+2)(n+Q+4)+T+n+\bar{\omega}+Q+4$. Therefore, $k = \ell(\bar{h}) + \ell(a \cdot \tilde{h}) \leq [(\gamma+2)(n+Q+4) + T + n + \bar{\omega} + Q + 4] + 1 + n + Q + 3 = (\gamma+3)(n+Q+4) + T + n + \bar{\omega} + Q + 4 \leq M$. ■

B.2.2 f is well-defined

Now we show that f is well-defined.

Claim B.15 *If $h \in H_{1,a}^{(\omega),k} \cap H_{1,a}^{(\omega'),k'}$ for some $\omega, \omega' \in \{0, \ldots, \bar{\omega}\}$ and $k \in \{n + \omega + Q + 5, \ldots, M\}$ and $k' \in \{n + \omega' + Q + 5, \ldots, M\}$, then $\omega = \omega'$ and $k = k' + \beta K$ for some $\beta \in \mathbb{Z}$.*

Proof. It follows immediately by Claim B.3. ■

Claim B.16 *For all $\omega \in \{0, \ldots, \bar{\omega}\}$ and $k \in \{n + \omega + Q + 5, \ldots, M\}$ and $k' \in \{0, \ldots, n + Q + 4\}$, $H_{1,a}^{(\omega),k} \cap H_{1,b}^{(0),k'} = \emptyset$.*

Proof. If $h \in H_{1,b}^{(0),k'}$, then $\ell(h) < n+Q+5$; whereas if $h \in H_{1,a}^{(\omega),k}$, then $\ell(h) \geq n+\omega+Q+5$. Hence, $H_{1,a}^{(\omega),k} \cap H_{1,b}^{(0),k'} = \emptyset$. ■

Claim B.17 *For all $\omega \in \{0, \ldots, \bar{\omega}\}$ and $k \in \{n + \omega + Q + 5, \ldots, M\}$ and $k' \in \{1, \ldots, M\}$ and $d \in N$ and $\tau \in \{0, \ldots, d + Q + 3\}$, $H_{1,a}^{(\omega),k} \cap H_2^{k',d,\tau} = \emptyset$.*

Proof. Suppose not; then there exist $h = (a^1, \ldots, a^t) \in H_{1,a}^{(\omega),k} \cap H_2^{k',d,\tau}$ for some ω, k, k', d and τ as described in the claim. Since $h \in H_{1,a}^{(\omega),k}$, then $\hat{\pi}^{(\omega),r} = a^{t-k+r}$ for all $1 \leq r \leq k$. Also since $h \in H_2^{k',d,\tau}$, then $T^{k'}(h) = \bar{h} \cdot a^{t-\tau} \cdot \tilde{h}$ where $\tilde{h} \in \Sigma^{d,\tau}$, $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(\omega'),k'-\tau})$ and $\bar{h} \in H_1^{(\omega'),k'-(\tau+1)}$ for some $\omega' \in \{0, \ldots, \bar{\omega}\}$ satisfying either $k' - (\tau + 1) \geq n + \omega' + Q + 5$ or $\omega' = 0$, $k' = t$ and $k' - (\tau + 1) < n + Q + 5$. We consider each of these separately.

<u>*Case 1*</u>*: $\bar{h} \in H_{1,b}^{(0),k'-(\tau+1)}$, $k' = t$ and $k' - (\tau + 1) < n + Q + 5$.* In this case $a^{k'-k+1} = a^{k'-k+2} = s'$, $\bar{h} = ((s'; 2), s, \ldots, s)$ and $\ell(\bar{h}) = k' - (\tau + 1)$. Thus, since $\tau \leq d + Q + 3$ and $k \geq n + \omega + Q + 5$, it must be that $k' - k + 1 < k' - \tau$ and so $k' = k$. But then $a^{t-\tau} \in \bar{D}_d(s) \cup \bar{D}_d(s')$ and $a^{t-\tau} = \hat{\pi}^{(\omega),t-\tau} \in \{s, s'\}$, a contradiction.

<u>*Case 2*</u>*: $\bar{h} \in H_{1,a}^{(\omega'),k'-(\tau+1)}$ and $k' - (\tau + 1) \geq n + \omega' + Q + 5$.* We consider four subcases.

<u>*Subcase 1*</u>*: $k \geq n+\omega+Q+5+\tau+1$.* Let $\hat{h} = B^{\tau+1}(h)$. Since $k-(\tau+1) \geq n+\omega+Q+5$, it follows that $\hat{h} \in H_{1,a}^{(\omega),k-(\tau+1)}$. Also, since $T^{k'-(\tau+1)}(\hat{h}) = \bar{h}$, $\hat{h} \in H_{1,a}^{(\omega'),k'-(\tau+1)}$. Then, by Claim B.15, $\omega = \omega'$ and $k - (\tau + 1) = k' - (\tau + 1) + \beta K$ for some $\beta \in \mathbb{Z}$. This together with $a = \hat{\pi}^{(\omega),k-\tau}$ and $a \in \bar{D}_d(\hat{\pi}^{(\omega'),k'-\tau})$ implies that $\hat{\pi}^{(\omega),k'-\tau} = \hat{\pi}^{(\omega),k-\tau} \in \bar{D}_d(\hat{\pi}^{(\omega),k'-\tau})$, a contradiction.

<u>*Subcase 2*</u>*: $k = n + \omega + Q + 5 + \tau$.* By $h \in H_{1,a}^{(\omega),k}$, $a^{t-\tau-1} = s$ and $a^{t-\tau} = s'$. Also, by $h \in H_2^{k',d,\tau}$, $a^{t-\tau-1} = \hat{\pi}^{(\omega'),k'-\tau-1}$ and $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(\omega'),k'-\tau})$. Hence, it follows from $a^{t-\tau-1} = s$ and $a^{t-\tau} = s'$, respectively, that $\hat{\pi}^{(\omega'),k'-\tau-1} = s$ and $\hat{\pi}^{(\omega'),k'-\tau} \in \bar{D}_d(s')$. But this contradicts Claim B.1.4.

<u>*Subcase 3*</u>*: $k = n + \omega + Q + 4 + \tau$.* First, we show that $k' \geq k$. Suppose otherwise. Since $\bar{h} \in H_{1,a}^{(\omega'),k'-(\tau+1)}$, then $T^{k'}(h) \in \Lambda^{\omega',k'}$. By Claim B.3, this together with $h \in H_{1,a}^{(\omega),k}$ implies that $k = k' + \beta K$ for some $\beta \in \mathbb{N}$. Also, by the supposition that $k < n + \omega + Q + 5 + \tau + 1$ and $k' - (\tau + 1) \geq n + \omega' + Q + 5$, we have $k - k' \leq n$. Since $n < K$, we have a contradiction.

By $h \in H_{1,a}^{(\omega),k}$, $(a^{t-\tau-(n+Q+3)}, \ldots, a^{t-\tau}) = ((s'; 2), (s; n + \omega + Q + 2))$; by $h \in H_2^{k',d,\tau}$, $(a^{t-\tau-(n+\omega+Q+3)}, \ldots, a^{t-\tau}) = (\hat{\pi}^{(\omega'),k'-\tau-(n+\omega+Q+3)}, \ldots, \hat{\pi}^{(\omega'),k'-\tau-1}, a^{t-\tau})$. Hence, it follows from $a^{t-\tau} = s$ that $\hat{\pi}^{(\omega'),k'-\tau} \in \bar{D}_d(s)$. But this contradicts Claim B.1.2.

<u>*Subcase 4*</u>*: $k < n + \omega + Q + 4 + \tau$.* By $k \geq n + \omega + Q + 5$ and $\tau \leq d + Q + 3$, it follows that $\tau > 0$ and $k - \tau \geq 2$. Hence, by $h \in H_2^{k',d,\tau}$, $a^{t-\tau+1} \in D(s')$ and, by $h \in H_{1,a}^{(\omega),k}$ and

$k-\tau < n+\omega+Q+4$, $a^{t-\tau+1} = s$; a contradiction. ■

Claim B.18 *For all $k \in \{0,\ldots,n+Q+4\}$ and $k' \in \{1,\ldots,M\}$ and $d \in N$ and $\tau \in \{0,\ldots,d+Q+3\}$, $H_{1,b}^{(0),k} \cap H_2^{k',d,\tau} = \emptyset$.*

Proof. Suppose otherwise; then there exists $h \in H_{1,b}^{(0),k} \cap H_2^{k',\tau,d}$. This means that $h = (\pi^{(0),1},\ldots,\pi^{(0),k})$ and $T^{k'}(h) = \bar{h}\cdot a\cdot\tilde{h}$ satisfying the remaining conditions in the definition of $H_2^{k',d,\tau}$. Therefore, for some $\hat{k} < n+Q+5$, $\bar{h}\cdot a \in H_{1,b}^{(0),\hat{k}+1}$. But this is a contradiction as $a \in \bar{D}(\hat{\pi}^{(0),\hat{k}+1})$ due to $h \in H_2^{k',d,\tau}$. ■

Claim B.19 *If $h \in H_2^{k,d,\tau} \cap H_2^{k',d',\tau'}$ for some $k,k' \in \{1,\ldots,M\}$ and $d,d' \in N$ and $\tau \in \{0,\ldots,d+Q+3\}$ and $\tau' \in \{0,\ldots,d'+Q+3\}$, then $\tau=\tau'$ and $d=d'$ and $k=k'+\beta K$ for some $\beta \in \mathbb{Z}$.*

Proof. Let $h = (a^1,\ldots,a^t)$. First, we establish that $\tau=\tau'$. Suppose, without loss of generality, that $\tau<\tau'$. Define $\hat{h} = B^{\tau+1}(h) = (a^1,\ldots,a^{t-(\tau+1)})$ and note that $\hat{h} \in \left(\cup_{i=0}^{\bar{\omega}} H_1^{(\omega),k-(\tau+1)}\right) \cap H_2^{k'-(\tau+1),d',\tau'-(\tau+1)}$. But this contradicts Claims B.17 or B.18.

By $\tau=\tau'$ and the definition of $H_2^{k,d,\tau}$ and $H_2^{k',d',\tau'}$, we have that $\hat{h} \in H_1^{(\omega),k-(\tau+1)}$ and $\hat{h} \in H_1^{(\omega'),k'-(\tau'+1)}$ for some $\omega,\omega' \in \{0,\ldots,\bar{\omega}\}$. It then follows from Claims B.15 and B.16 that $k=k'+\beta K$ for some $\beta \in \mathbb{Z}$ and $\omega=\omega'$. Also, by $h \in H_2^{k,d,\tau}$, $a^{t-\tau} = (a_d, \hat{\pi}_{-d}^{(\omega),k-\tau})$ and, by $h \in H_2^{k',d',\tau'}$, $a^{t-\tau'} = (a_{d'}, \hat{\pi}_{-d'}^{(\omega'),k'-\tau'})$. Since $\omega=\omega'$, $k=k'+\beta K$ and $\tau=\tau'$, then $(a_d, \hat{\pi}_{-d}^{(\omega),k-\tau}) = (a_{d'}, \hat{\pi}_{-d'}^{(\omega),k-\tau})$ and so $d=d'$. ■

Claim B.20 *For all $\omega \in \{0,\ldots,\bar{\omega}\}$ and $k \in \{n+\omega+Q+5,\ldots,M\}$ and $k' \in \{1,\ldots,M\}$ and $d \in N$, $H_{1,a}^{(\omega),k} \cap H_3^{k',d} = \emptyset$ where $H_3^{k',d} = (H_{3,a}^{k',d} \cup H_{3,b}^{k',d})$.*

Proof. Suppose that $h \in H_{1,a}^{(\omega),k} \cap H_3^{k',d}$. Since $H_3^{k',d} \subseteq \Sigma^{d,k'}$ and $k' \geq d+Q+4$ (due to $h \in H_3^{k',d}$), we have a contradiction to Claim B.2 when $k \geq k'$. Also, since $H_{1,a}^{(\omega),k} \subseteq \Lambda^{\omega,k}$ and $k \geq n+\omega+Q+5$, we have a contradiction to Claim B.4 when $k<k'$. ■

Claim B.21 *For all $k \in \{0,\ldots,n+Q+4\}$ and $k' \in \{1,\ldots,M\}$ and $d \in N$, $H_{1,b}^{(0),k} \cap H_3^{k',d} = \emptyset$.*

Proof. Suppose there exists $h = (a^1,\ldots,a^t) \in H_{1,b}^{(0),k} \cap H_3^{k',d}$. By $h \in H_3^{k',d}$, $a^{t-k'+d+Q+4} \in D(s')$ and, by $h \in H_{1,b}^{(0),k}$, $a^r = s$ for all $r>2$. But this is a contradiction as it implies $d+Q+4 \leq t-k'+d+Q+4 \leq 2$. ■

Claim B.22 *For all $k,k' \in \{1,\ldots,M\}$ and $d,d' \in N$ and $\tau \in \{0,\ldots,d+Q+3\}$, $H_2^{k,d,\tau} \cap H_3^{k',d'} = \emptyset$.*

Proof. Suppose that $h = (a^1, \ldots, a^t) \in H_2^{k,d,\tau} \cap H_3^{k',d'}$. There are two possibilities.

Case 1: $k' > k$. First, note that, for some $\omega \in \{0, \ldots, \bar{\omega}\}$, $B^{\tau+1}(h) \in H_{1,a}^{(\omega),k-(\tau+1)}$ with $k - (\tau + 1) \geq n + \omega + Q + 5$; otherwise, $B^{\tau+1}(h) \in H_{1,b}^{(0),k-(\tau+1)}$ and $\ell(h) = k < k'$; a contradiction.

It follows from $B^{\tau+1}(h) \in H_{1,a}^{(\omega),k-(\tau+1)}$ that, for some $\omega \in \{0, \ldots, \bar{\omega}\}$ and $k-(\tau+1) \geq n+\omega+Q+5$, $T^k(h) \in \Lambda^{\omega,k}$ contradicting Claim B.4 because $h \in H_3^{k',d'}$ and $k' > k \geq n+\omega+Q+5$.

Case 2: $k \geq k'$. There are two possibilities.

Subcase 1: $k' - \tau > d' + Q + 4$. In this case, $B^{\tau+1}(h)$ belongs to $H_1^{(\omega),k-(\tau+1)}$, for some $\omega \in \{0, \ldots, \bar{\omega}\}$, and to $H_3^{k'-(\tau+1),d'}$. But this contradicts Claim B.20 or B.21.

Subcase 2: $k' - \tau \leq d' + Q + 4$. Since $h \in H_3^{k',d'}$, we have (i) $a^{t-k'+r} \in D_{d'}(s')$ for $r = 1, 2$, (ii) $a^{t-k'+r} \in D_{d'}(s)$ for $r = 3, \ldots, d'+Q+3$, and (iii) $a^{t-k'+d'+Q+4} \in D_{d'}(s')$. When $k' - \tau = d' + Q + 4$, by (i), (ii) and $B^{\tau+1}(h) \in H_1^{(\omega),k-(\tau+1)}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$, Claim B.1.3 implies that $(a^{t-\tau-d'-Q-3}, \ldots, a^{t-\tau-1}) = ((s'; 2), (s; d'+Q+1))$ and $a^{t-\tau} \in \bar{D}_d(s)$. But the latter contradicts (iii). Therefore, it must be that $k' - \tau < d' + Q + 4$.

By $h \in H_2^{k,d,\tau}$, it must be that (iv) $a^{t-\tau+r} \in D_d(s')$ for all $r = 1, 2$ and (v) $a^{t-\tau+r} \in D_d(s)$ for all $r \geq 3$. But then by $t - k' + d' + Q + 4 > t - \tau$, (iii) and (v), it must be that either $t - k' + d' + Q + 4 = t - \tau + 1$ or $t - k' + d' + Q + 4 = t - \tau + 2$. The latter, however, cannot hold because, by (ii), $a^{t-k'+d'+Q+3} \in D(s)$ and, by (iv), $a^{t-\tau+1} \in D(s')$. Therefore, assume the former. This together with (i) and (ii) implies $B^{\tau+1}(h) \in H_1^{(\omega),k-(\tau+1)}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ and Claim B.1.3 imply that $(a^{t-k'+1}, \ldots, a^{t-k'+d'+Q+2}) = ((s'; 2), (s; d'+Q))$ and $a^{t-k'+d'+Q+3} \in \bar{D}_d(s)$. But the latter together with (ii) implies $d = d'$. Hence, by part (4) of the definition of $H_2^{k,d,\tau}$, $\tau = 0$. Thus, $k' < d' + Q + 4$; but this contradicts $h \in H_3^{k',d'}$.

■

Claim B.23 *If $h \in H_3^{k,d} \cap H_3^{k',d'}$ for some $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$, then $k = k'$ and $d = d'$.*

Proof. First we show that $k = k'$. Suppose otherwise and assume, without loss of generality, that $k > k'$. By $h \in H_3^{k',d'}$, we have $T^{k'}(h) \in \Sigma^{d',k'}$ and $k > k' \geq d' + Q + 4$. But this contradicts Claim B.4 because $h \in H_3^{k,d}$.

To show that $d = d'$, by $h \in H_3^{k,d} \cap H_3^{k,d'}$, we have $T^k(h) \in \Sigma^{d,k} \cap \Sigma^{d',k}$ where $k \geq d+Q+4$ and $k \geq d' + Q + 4$. Hence, by Claim B.8, $d = d'$. ■

Claim B.24 *For all* $\omega, \omega' \in \{0, \dots, \bar{\omega}\}$ *and* $k \in \{n+\omega+Q+5, \dots, M\}$ *and* $k' \in \{1, \dots, M\}$ *and* $r \in \{0, \dots, n+\omega'+Q+4\}$*,* $H_{1,a}^{(\omega),k} \cap H_4^{k',\omega',r} = \emptyset$*.*

Proof. Suppose that $h \in H_{1,a}^{(\omega),k} \cap H_4^{k',\omega',r}$ and let $d \in N$ be such that $\omega' \in W_d$. Then $T^k(h) \in \Lambda^{\omega,k}$ where $k \geq n+\omega+Q+5$ and $T^{k'}(h) \in \Sigma^{d,k'}$ and $k' \geq d+Q+4$; a contradiction to Claim B.2 if $k \geq k'$ and to Claims B.4 and B.5 if $k < k'$. ■

Claim B.25 *For all* $k \in \{0, \dots, n+Q+4\}$ *and* $k' \in \{1, \dots, M\}$ *and* $\omega \in \{0, \dots, \bar{\omega}\}$ *and* $r \in \{0, \dots, n+\omega+Q+4\}$*,* $H_{1,b}^{(0),k} \cap H_4^{k',\omega,r} = \emptyset$*.*

Proof. Suppose there exists $h = (a^1, \dots, a^t) \in H_{1,b}^{(0),k} \cap H_4^{k',\omega,r}$. By $h \in H_4^{k',\omega,r}$, $a^{t-k'+d+Q+4} \in D(s')$ and, by $h \in H_{1,b}^{(0),k}$, $a^\tau = s$ for all $\tau > 2$. But this is a contradiction as it implies $d+Q+4 \leq t-k'+d+Q+4 \leq 2$. ■

Claim B.26 *For all* $k, k' \in \{1, \dots, M\}$ *and* $d \in N$ *and* $\omega \in \{0, \dots, \bar{\omega}\}$ *and* $r \in \{0, \dots, n+\omega+Q+4\}$ *and* $\tau \in \{0, \dots, d+Q+3\}$*,* $H_2^{k,d,\tau} \cap H_4^{k',\omega,r} = \emptyset$*.*

Proof. Suppose that $h \in H_2^{k,d,\tau} \cap H_4^{k',\omega,r}$ and let $d' \in N$ be such that $\omega \in W_{d'}$. By $h \in H_2^{k,d,\tau}$, $B^{\tau+1}(h) \in H_1^{(\omega'),k-(\tau+1)}$ for some $\omega' \in \{0, \dots, \bar{\omega}\}$. And since $h \in H_4^{k',\omega,r}$ and $T > \tau+1$ (by (50)) $B^{\tau+1}(h)$ belongs to $H_4^{k'-(\tau+1),\omega,r-(\tau+1)}$ if $r-(\tau+1) \geq 0$ or to $H_3^{k'-(\tau+1),d'}$ otherwise. But, by Claims B.20, B.21, B.24 or B.25, this is a contradiction. ■

Claim B.27 *For all* $k, k' \in \{1, \dots, M\}$ *and* $d \in N$ *and* $\omega \in \{0, \dots, \bar{\omega}\}$ *and* $r \in \{0, \dots, n+\omega+Q+4\}$*,* $H_3^{k,d} \cap H_4^{k',\omega,r} = \emptyset$*.*

Proof. Suppose that $h \in H_3^{k,d} \cap H_4^{k',\omega,r}$ and let $d' \in N$ be such that $\omega \in W_{d'}$. Assume first that $k = k'$. By Claim B.8, this implies $d = d'$. Since $h \in H_3^{k,d}$, it follows that $k < (\gamma+2)(d+Q+4)+T$ and, since $h \in H_4^{k',\omega,r}$ with $k = k'$ and $d = d'$, it follows that $k \geq (\gamma+2)(d+Q+4)+T$. But this is a contradiction.

Suppose next that $k > k'$. Then, by $h \in H_4^{k',\omega,r}$, $T^{k'}(h) \in \Sigma^{d',k'}$ with $d'+Q+4 \leq k' < k$. But this together with $h \in H_3^{k,d}$ contradicts Claim B.4.

Finally, suppose that $k' > k$. Then, by $h \in H_3^{k,d}$, $T^k(h) \in \Sigma^{d,k}$ with $d+Q+4 \leq k < k'$. But this together with $h \in H_4^{k',\omega,r}$ contradicts Claims B.4 and B.6. ■

Claim B.28 *If* $h \in H_4^{k,\omega,r} \cap H_4^{k',\omega',r'}$ *for some* $k, k' \in \{1, \dots, M\}$ *and* $\omega, \omega' \in \{0, \dots, \bar{\omega}\}$ *and* $r, r' \in \mathbb{N}_0$*, then* $k = k'$*,* $\omega = \omega'$ *and* $r = r'$*.*

Proof. Let $d \in N$ be such that $\omega \in W_d$ and $d' \in N$ be such that $\omega' \in W_{d'}$. To show that $k = k'$, without loss of generality, suppose that $k > k'$. Then, by $h \in H_4^{k',\omega',r'}$, $T^{k'}(h) \in \Sigma^{d',k'}$ with $d' + Q + 4 \leq k' < k$. But this together with $h \in H_4^{k,\omega,r}$ contradicts Claims B.4 and B.6. Hence, $k = k'$ and, by Claim B.8, $d = d'$. Thus, $r = k - ((\gamma + 2)(d + Q + 4) + T) = k' - ((\gamma + 2)(d' + Q + 4) + T) = r'$. Furthermore, letting $\bar{h} = B^r(T^k(h)) = B^{r'}(T^{k'}(h))$, it follows that $\omega = \omega(d, T^T(\bar{h})) = \omega(d', T^T(\bar{h})) = \omega'$. ■

Claim B.29 *For all* $\omega \in \{0, \ldots, \bar{\omega}\}$ *and* $k \in \{n + \omega + Q + 5, \ldots, M\}$ *and* $k' \in \{1, \ldots, M\}$ *and* $d \in N$ *and* $\tau \in \{0, \ldots, d + Q + 3\}$, $H_{1,a}^{(\omega),k} \cap H_5^{k',d,\tau} = \emptyset$.

Proof. Suppose that $h = (a^1, \ldots, a^t) \in H_{1,a}^{(\omega),k} \cap H_5^{k',d,\tau}$. Then $h \in H_{1,a}^{(\omega),k}$, $T^{k'}(h) \in \Sigma^{d,k'}$ and $k' \geq d + Q + 4$. Therefore, $k' > k$; otherwise we would contradict Claim B.2.

Consider next the case $k' > k$ and let $\hat{h} = B^{\tau+1}(h)$. Then $\hat{h} \in H_3^{k'-(\tau+1),d'} \cup H_4^{k'-(\tau+1),\omega',r}$ for some $d' \in N$ and $\omega' \in \{0, \ldots, \bar{\omega}\}$ and $0 \leq r \leq n + \omega + Q + 4$. Also, by $h \in H_{1,a}^{(\omega),k}$, $T^{k-(\tau+1)}(\hat{h}) \in \Lambda^{\omega,k-(\tau+1)}$. Furthermore, since $0 \leq \tau \leq d + Q + 3 \leq n + Q + 3$, we have that $k - (\tau + 1) \geq 1$. Therefore, by Claims B.4 and B.5, one of the following must hold: (1) $1 \leq k - (\tau + 1) \leq Q + 2$ or (2) $\hat{h} \in H_4^{k'-(\tau+1),\omega',r}$ and $Q + 3 \leq k - (\tau + 1) = r < n + \omega + Q + 5$.

Case (1) implies that $t - \tau + 1 \in \{t - k + 3, \ldots, t - k + Q + 4\}$. By $T^k(h) \in H_{1,a}^{(\omega),k}$, $a^{t-\tau+1} = s$; and, by $h \in H_5^{k',d,\tau}$, $T^\tau(h) \in \Sigma^{d,\tau}$. These imply $a^{t-\tau+1} \in D_d(s')$; a contradiction.

In case (2), $t - \tau \leq t - k + n + \omega + Q + 5$. This together with $h \in H_{1,a}^{(\omega),k}$ implies that $a^{t-\tau} \in \{s, s'\}$. Since $h \in H_5^{k',d,\tau}$ and $\hat{h} \in H_4^{k'-(\tau+1),\omega',r}$ and $\hat{\pi}^{(\omega'),r+1} \in \{s, s'\}$, it must be that $a^{t-\tau} \in \bar{D}_d(s) \cup \bar{D}_d(s')$; a contradiction. ■

Claim B.30 *For all* $k \in \{0, \ldots, n + Q + 4\}$ *and* $k' \in \{1, \ldots, M\}$ *and* $d \in N$ *and* $\tau \in \{0, \ldots, d + Q + 3\}$, $H_{1,b}^{(0),k} \cap H_5^{k',d,\tau} = \emptyset$.

Proof. Suppose $h = (a^1, \ldots, a^k) \in H_{1,b}^{(0),k} \cap H_5^{k',d,\tau}$. By $h \in H_5^{k',d,\tau}$, we have $a^{k-k'+1}, a^{t-k'+i+Q+4} \in D(s')$ for some $i \in N$; and, by $h \in H_{1,b}^{(0),k}$, $h = ((s'; 2), (s; k - 2))$; a contradiction. ■

Claim B.31 *For all* $k, k' \in \{1, \ldots, M\}$, $d, d' \in N$, $\tau \in \{0, \ldots, d + Q + 3\}$ *and* $\tau' \in \{0, \ldots, d' + Q + 3\}$, $H_2^{k,d,\tau} \cap H_5^{k',d',\tau'} = \emptyset$.

Proof. Suppose that $h \in H_2^{k,d,\tau} \cap H_5^{k',d',\tau'}$. Assume first that $\tau \leq \tau'$. Then $B^{\tau+1}(h) \in H_1$ and $B^{\tau+1}(h) \in H_3 \cup H_4 \cup H_5$. But this contradicts Claims B.20, B.21, B.24, B.25, B.29 or B.30. If $\tau > \tau'$, then $B^{\tau'+1}(h) \in H_2$ and $B^{\tau'+1} \in H_3 \cup H_4$ contradicting Claim B.22 or Claim B.26. ■

Claim B.32 *For all $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$ and $\tau \in \{0, \ldots, d' + Q + 3\}$, $H_3^{k,d} \cap H_5^{k',d',\tau} = \emptyset$.*

Proof. Suppose that $h = (a^1, \ldots, a^t) \in H_3^{k,d} \cap H_5^{k',d',\tau}$. By $h \in H_3^{k,d}$, $T^k(h) \in \Sigma^{d,k}$ with $k \geq d + Q + 4$; by $h \in H_5^{k',d',\tau}$, $T^{k'}(h) \in \Sigma^{i',k'}$ with $k' \geq i' + Q + 4$ for some $i' \in N$. By appealing to Claims B.4, B.7 and B.8, it then follows that $k = k'$ and $d = i'$.

By $h \in H_5^{k',d',\tau}$ and the last two equalities, $k - (\tau + 1) \geq d + Q + 4$. But then, by $h \in H_3^{k,d}$, $B^{\tau+1}(h) \in H_3^{k-(\tau+1),d}$. This and $h \in H_5^{k',d',\tau}$ imply $a^{t-\tau} \in \bar{D}_{d'}(f(B^{\tau+1}(h)))$ and $d' \neq d$. Since $h \in H_3^{k,d}$ and $k - (\tau + 1) \geq d + Q + 4$, $a^{t-\tau} \in D_d(f(B^{\tau+1}(h)))$; a contradiction. ■

Claim B.33 *For all $k, k' \in \{1, \ldots, M\}$ and $d' \in N$ and $\omega \in \{0, \ldots, \bar{\omega}\}$ and $r \in \{0, \ldots, n + \omega + Q + 4\}$ and $\tau \in \{0, \ldots, d' + Q + 3\}$, $H_4^{k,\omega,r} \cap H_5^{k',d',\tau} = \emptyset$.*

Proof. Suppose that $h = (a^1, \ldots, a^t) \in H_4^{k,\omega,r} \cap H_5^{k',d',\tau}$. By $h \in H_4^{k,\omega,r}$, there is some $d \in N$ such that $T^k(h) \in \Sigma^{d,k}$ with $k \geq d + Q + 4$; by $h \in H_5^{k',d',\tau}$, $T^{k'}(h) \in \Sigma^{i',k'}$ with $k' \geq i' + Q + 4$ for some $i' \in N$. So by Claims B.6 and B.7, $k = k'$.

Suppose $r \geq \tau + 1$. By $h \in H_4^{k,\omega,r}$, $B^{\tau+1}(h) \in H_4^{k-(\tau+1),\omega,r-(\tau+1)}$ implying $a^{t-\tau} = \hat{\pi}^{(\omega),r-\tau}$. By $h \in H_5^{k',d',\tau}$ and Claims B.27 and B.28, $a^{t-\tau} \in \bar{D}_{d'}(\hat{\pi}^{(\omega),r-\tau})$. But this is a contradiction.

Finally, suppose $r < \tau + 1$. By $h \in H_4^{k,\omega,r}$, $B^{\tau+1}(h) \in H_3^{k-(\tau+1),d}$. This implies that $a^{t-\tau} \in D_d(f(B^{\tau+1}(h)))$. Also, by $h \in H_5^{k',d',\tau}$ and Claims B.23 and B.27, $a^{t-\tau} \in \bar{D}_{d'}(f(B^{\tau+1}(h)))$ and $d' \neq d$. But this is a contradiction. ■

Claim B.34 *If $h \in H_5^{k,d,\tau} \cap H_5^{k',d',\tau'}$ for some $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$ and $\tau \in \{0, \ldots, d + Q + 3\}$ and $\tau' \in \{0, \ldots, d' + Q + 2\}$, then $k = k'$ and $\tau = \tau'$ and $d = d'$.*

Proof. Suppose $h \in H_5^{k,d,\tau} \cap H_5^{k',d',\tau'}$. First, note that $\tau = \tau'$. Otherwise, say $\tau < \tau'$; then, by $h \in H_5^{k,d,\tau}$, $B^{\tau+1}(h) \in H_5$ and, by $h \in H_5^{k',d',\tau'}$, $B^{\tau+1}(h) \in H_3 \cup H_4$. This contradicts Claim B.32 or Claim B.33. Second, note that $B^{\tau+1}(h) \in \left(\cup_{i \in N} H_3^{k-(\tau+1),i}\right) \cup \left(\cup_{\omega,r} H_4^{k-(\tau+1),\omega,r}\right)$ and $B^{\tau'+1}(h) \in \left(\cup_{i \in N} H_3^{k'-(\tau+1),i'}\right) \cup \left(\cup_{\omega,r} H_4^{k'-(\tau'+1),\omega,r}\right)$. Since $\tau = \tau'$, by Claims B.23, B.27 and B.28, $k = k'$. Finally, it follows immediately from the definitions of $H_5^{k,d,\tau}$ and $H_5^{k',d',\tau'}$, $k = k'$ and $\tau = \tau'$ that $d = d'$. ■

Claim B.35 *If $h \in H_6^{d,\tau} \cap H_6^{d',\tau'}$ for some $d, d' \in N$ and $\tau \in \{0, \ldots, d + Q + 3\}$ and $\tau' \in \{0, \ldots, d' + Q + 3\}$, then $d = d'$.*

Proof. Let $h = (a^1, \ldots, a^t) \in H_6^{d,\tau} \cap H_6^{d',\tau'}$. We may assume, without loss of generality, that $\tau \geq \tau'$. Then, by $h \in H_6^{d',\tau'}$, $a^{t-\tau'} \in \bar{D}_{d'}(s) \cup \bar{D}_{d'}(s')$; and, by $\tau \geq \tau'$ and $h \in H_6^{d,\tau}$, $a^{t-\tau'} \in D_d(s) \cup D_d(s')$. Thus, $d = d'$. ■

B.2.3 Outcome paths induced by f and by one-shot deviations from f

Claim B.36 *If $h \in H_{1,a}^{(\omega),k}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ and $k \in \{n + \omega + Q + 5, \ldots, M\}$, then $\pi(f|h) = (\hat{\pi}^{(\omega),k+1}, \hat{\pi}^{(\omega),k+2}, \ldots)$.*

Proof. By Claim B.10, we may assume that $k < M$. Hence, $f(h) = \hat{\pi}^{(\omega),k+1}$ and $h \cdot f(h) \in H_{1,a}^{(\omega),k+1}$. Thus, by induction, $\pi(f|h) = (\hat{\pi}^{(\omega),k+1}, \hat{\pi}^{(\omega),k+2}, \ldots)$. ■

Claim B.37 *If $h \in H_{1,b}^{(0),k}$ for some $k \in \{0, \ldots, n+Q+4\}$, then $\pi(f|h) = (\pi^{(0),k+1}, \pi^{(0),k+2}, \ldots)$.*

Proof. If $k < n+Q+4$, then $h \cdot f(h) \in H_{1,b}^{(0),k+1}$. If $k = n+Q+4$, then $h \cdot f(h) \in H_{1,a}^{(0),n+Q+5}$. Thus, by induction and Claim B.36, $\pi(f|h) = (\pi^{(0),k+1}, \pi^{(0),k+2}, \ldots)$. ■

Claim B.38 *If $h \in H_4^{k,\omega,r}$ for some $k \in \{1, \ldots, M\}$, $\omega \in \{1, \ldots, \bar{\omega}\}$ and $r \in \{0, \ldots, n+\omega+Q+4\}$, then $\pi(f|h) = (\hat{\pi}^{(\omega),r+1}, \hat{\pi}^{(\omega),r+2}, \ldots)$.*

Proof. If $r = n+\omega+Q+4$, then $h \cdot f(h) \in H_{1,a}^{(\omega),n+\omega+Q+5}$. Thus, $\pi(f|h) = (\hat{\pi}^{(\omega),r+1}, \hat{\pi}^{(\omega),r+2}, \ldots)$ by Claim B.36. If $r < n + \omega + Q + 4$, then $f(h) = \hat{\pi}^{(\omega),r+1}$. Therefore, $h \cdot f(h) \in H_4^{k+1,\omega,r+1}$. Furthermore, by Claim B.13, $k < M$. Hence, by induction, $\pi(f|h) = (\hat{\pi}^{(\omega),r+1}, \hat{\pi}^{(\omega),r+2}, \ldots)$. ■

Claim B.39 *If $h \in H_{3,b}^{k,d}$ for some $d \in N$ and $k \in \{1, \ldots, M\}$ and $T^{k-(\gamma+2)(d+Q+4)}(h) = \hat{h}$, then for any $h' \in H_{T-k+(\gamma+2)(d+Q+4)}$ the strategy $f|h$ induces the infinite path*

$$h' \cdot (\hat{\pi}^{(\omega(d,\widehat{h} \cdot h')),1}, \hat{\pi}^{(\omega(d,\widehat{h} \cdot h')),2}, \ldots)$$

with probability

$$P_{f^d|\hat{h},T-k+(\gamma+2)(d+Q+4)}(h').$$

Furthermore, if $k = (\gamma+2)(d+Q+4)$, then $P_{f^d|\hat{h},T}(\{h' \in H_T : \widehat{u}_i^{\omega(d,h')} = u_i' + \rho\}) > 1 - 2\varepsilon_2$ for all $i \neq d$.

Proof. If $T^{Q+1}(h) \notin \cup_{l \in N} \hat{\Sigma}^{l,Q+1}$ for all $l \in N$ then, for any $a \in A$, we have $h \cdot a \in H_4^{k+1,\omega(d,T^T(h \cdot a)),0}$ if $k - (\gamma+2)(d+Q+4) = T - 1$ and $h \cdot a \in H_{3,b}^{k+1,d}$ if $k - (\gamma+2)(d+Q+4) < T - 1$. Similarly, if $T^{Q+1}(h) \in \hat{\Sigma}^{l,Q+1}$ for some $l \in N$, then $h \cdot f(h) \in H_4^{k+1,\omega(d,T^T(h \cdot f(h))),0}$ if $k - (\gamma+2)(d+Q+4) = T - 1$ and $h \cdot f(h) \in H_{3,b}^{k+1,d}$ if $k - (\gamma+2)(d+Q+4) < T - 1$. Also, by Claim B.12, $k < M$. Therefore, by Claim B.38, the first part of the claim follows by induction.

Furthermore, since $\{h' \in H_T : \widehat{u}_i^{\omega(d,h')} = u_i' + \rho\} = \{h' \in H_T : \Phi_i(d, h') < \eta\}$, Lemma B.1 implies that if $k = (\gamma+2)(d+Q+4)$, then $P_{f^d|\hat{h},T}(\{h' \in H_T : \widehat{u}_i^{\omega(d,h')} = u_i' + \rho\}) > 1 - 2\varepsilon_2$ for all $i \neq d$. ■

Claim B.40 *Suppose that* $h \in H_{3,a}^{k,d}$ *for some* $d \in N$ *and* $k \in \{1, \ldots, M\}$ *and let*

$$h' = (a(1); \gamma), (s_d^*, s'_{-d}), \ldots, (a(d+Q+4); \gamma), (s_d^*, s'_{-d})),$$

where, for all $l = 1, \ldots, d+Q+4$,

$$a(l) = \begin{cases} \bar{a}^{(d)} & \text{if } a^l \in \{s', s\}, \\ m^d & \text{if } a^l \notin \{s', s\}. \end{cases}$$

Then strategy $f|h$ *induces* $T^{(\gamma+2)(d+Q+4)-k}(h')$ *in the first* $(\gamma+2)(d+Q+4)-k$ *periods. Furthermore,* $h \cdot T^{(\gamma+2)(d+Q+4)-k}(h') \in H_{3,b}^{d,(\gamma+2)(d+Q+4)}$.

Proof. If $k = (\gamma+2)(d+Q+4)-1$, then $f(h) = (s_d^*, s'_{-d})$ and $h \cdot f(h) \in H_{3,b}^{k+1,d}$. Hence, the conclusion follows. If $k < (\gamma+2)(d+Q+4)-1$, then $h \cdot f(h) \in H_{3,a}^{k+1,d}$ and the result follows by induction and the definition of f. ■

Next, for any $d \in N$, let

$$(\widehat{a}^{(d),1}, \ldots, \widehat{a}^{(d),d+Q+4}) = (s', s', (s; d+Q+1), s').$$

Claim B.41 *Let* $h \in H_2^{k,d,\tau} \cup H_5^{k,d,\tau} \cup H_6^{d,\tau}$ *for some* $k \in \{1, \ldots, M\}$ *and* $d \in N$ *and* $\tau \in \{0, \ldots, d+Q+3\}$. *Then for all* $\bar{a} \in D_d(f(h))$, *we have either* $h \cdot \bar{a} \in H_{3,a}^{d+Q+4,d}$ *or* $h \cdot \bar{a} \cdot (\pi^1(f|h \cdot \bar{a}), \ldots, \pi^t(f|h \cdot \bar{a})) \in H_{3,a}^{d+Q+4,d}$ *for some* $t \in \{1, \ldots, d+Q+3\}$.

Proof. We establish this claim by considering the different possible cases.

<u>Case 1</u>: *One of the following conditions hold: (a)* $\tau = d+Q+3$, *(b)* $\tau = 0$, $T^{d+Q+3}(h) = ((s'; 2), (s; d+Q), a)$ *and* $a \in \bar{D}_d(s)$, *and (c)* $h \in H_6^{d,0}$ *and* $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$. Since $f(h) = s'$ it must be that $\bar{a} \in D_d(s')$ and $h \cdot \bar{a} \in H_{3,a}^{d+Q+4,d}$.

<u>Case 2</u>: $h \in H_2^{k,d,\tau} \cup H_5^{k,d,\tau}$ *and none of the conditions (a)–(c) in case 1 hold.* In this case, let $t = d+Q+3-\tau$. Indeed, we have that $h \cdot \bar{a} \in H_2^{k+1,d,\tau+1} \cup H_5^{k+1,d,\tau+1}$. Thus, by Claims B.11 and B.14, $f(h \cdot \bar{a}) = \widehat{a}^{(d),\tau+2}$. Then, by induction, it follows that $f(h \cdot \bar{a} \cdot \widehat{a}^{(d),\tau+2}) = \widehat{a}^{(d),\tau+3}, \ldots, f(h \cdot \bar{a} \cdot (\widehat{a}^{(d),\tau+2}, \ldots, \widehat{a}^{(d),d+Q+3})) = \widehat{a}^{(d),d+Q+4}$.

<u>Case 3</u>: $h \in H_6^{d,\tau}$ *and none of the conditions (a)–(c) in case 1 hold.* For any history $h' \in H$, define $v(h') = \max\{t \in \{0, \ldots, d+Q+3\} : T^t(h') \in H_6^{d,t}\}$ and let $\tau' = v(h)$. In this case, let $t = d+Q+3-\tau'$. Since (a) in case 1 does not hold, $\tau' < d+Q+3$. Moreover, (i) $h \in H_6^{d,\tau'}$, (ii) $T^1(h) \in \bar{D}_d(s) \cup \bar{D}_d(s')$ if $\tau' = 0$, (iii) $T^1(h) \in D_d(s')$ if $\tau' = 1$ and (iv) $h \cdot \bar{a} \in \tilde{\Sigma}^{d,\tau'+1}$. To complete the proof in this case, we first establish two subclaims.

Subclaim 1: $h \cdot \bar{a} \in H_6^{d,\tau'+1}$. Suppose not. By (iv), $h \cdot \bar{a} \in \cup_{l=1}^{5} H_l$. Also, if $\tau' \geq 2$, then $\bar{a} \in D(s)$ (recall that $\tau' < d + Q + 3$). Therefore, by Claim B.9, $\tau' = 0$ or 1 and either $h \cdot \bar{a} \in H_{1,a}^{(\omega),n+\omega+Q+5}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ or $T^{\hat{d}+Q+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+Q+4}$ for some $\hat{d} \in N$.

If $h \cdot \bar{a} \in H_{1,a}^{(\omega),n+\omega+Q+5}$, then $T^1(h) = s$, contradicting (ii) and (iii). If $T^{\hat{d}+Q+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+Q+4}$, then $T^1(h) \in D_{\hat{d}}(s)$. Therefore, by (iii), $\tau' = 0$. Then, by (ii), it follows that $T^1(h) \in \bar{D}_{\hat{d}}(s)$ and $d = \hat{d}$. This implies that $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$. Thus, by (i) and $\tau' = 0$, condition (c) is satisfied. But this contradicts our supposition.

Subclaim 2: $v(h \cdot \bar{a}) = \tau' + 1$. Since $h \cdot \bar{a} \in H_6^{d,\tau'+1}$, $v(h \cdot \bar{a}) \geq \tau' + 1 > 0$. Therefore, $h \in \tilde{\Sigma}^{d,v(h \cdot \bar{a})-1}$. Hence, $\tau' \geq v(h \cdot \bar{a}) - 1$ and thus $v(h \cdot \bar{a}) = \tau' + 1$.

It follows from the above two subclaims that $f(h \cdot \bar{a}) = \widehat{a}^{(d),\tau'+2}$. Then, by induction, it follows that $f(h \cdot \bar{a} \cdot \widehat{a}^{(d),\tau'+2}) = \widehat{a}^{(d),\tau'+3}, \ldots, f(h \cdot \bar{a} \cdot (\widehat{a}^{(d),\tau'+2}, \cdots, \widehat{a}^{(d),d+Q+3}) = \widehat{a}^{(d),d+Q+4}$. ■

Claim B.42 *Let $h \in H_7^k$ for some $k \in \{0, \ldots, n+Q+4\}$. If $h \notin \cup_{k=2}^{n+Q+3} H_7^k$ and $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$ for some $d \in N$, then $f(h) = s'$ and $h \cdot f(h) \in H_{3,a}^{d+Q+4,d}$. Otherwise, $\pi(f|h) = \left(\hat{\pi}^{(\omega),k'+1}, \hat{\pi}^{(\omega),k'+2}, \ldots\right)$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ and $k' \in \{0, \ldots, n + \omega + Q + 4\}$.*

Proof. First note that if $h \notin \cup_{k=2}^{n+Q+3} H_7^k$ then $f(h) = s'$. Thus, if $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$ for some $d \in N$ we have $T^{d+Q+4}(h \cdot f(h)) \in \Sigma^{d,d+Q+4}$; hence, $h \cdot f(h) \in H_{3,a}^{d+Q+4,d}$. Also, if $T^{n+\omega+Q+4}(h) = ((s'; 2), (s; n + \omega + Q + 2))$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ then, by $f(h) = s'$, $T^{n+\omega+Q+5}(h \cdot f(h)) = ((s'; 2), (s; n + \omega + Q + 2), s')$; thus, $h \cdot f(h) \in H_{1,a}^{(\omega),n+\omega+Q+5}$; but then, by Claim B.36, $\pi(f|h) = \left(\hat{\pi}^{(\omega),k'+1}, \hat{\pi}^{(\omega),k'+2}, \ldots\right)$. Therefore, for the remainder of the proof, assume that the following holds:

$$\begin{aligned} &\text{if } h \notin \cup_{k=2}^{n+Q+3} H_7^k, \text{ then } T^{n+\omega+Q+4}(h) \neq ((s'; 2), (s; n + \omega + Q + 2)), \\ &\text{for all } \omega \in \{0, \ldots, \bar{\omega}\}, \text{ and } T^{d+Q+3}(h) \notin \Sigma^{d,d+Q+3}, \text{ for all } d \in N. \end{aligned} \tag{B.5}$$

Next, for any history $h' \in H$, define $v(h') = \max\{t \in \{0, \ldots, n+Q+4\} : T^t(h') \in H_{1,b}^{(0),t}\}$ and let $k' = v(h)$. If $k' = n + Q + 4$, then $f(h) = s'$ and $h \cdot f(h) \in H_{1,a}^{(0),n+Q+5}$, and the conclusion follows from Claim B.36.

If $k' \in \{0, \ldots, n + Q + 3\}$, then the claim follows by induction if it is the case that $h \cdot f(h) \in H_7^{k'+1}$ and $v(h \cdot f(h)) = k' + 1$. Next, we complete the proof by showing in several steps that these two conditions indeed hold.

Step 1: $h \cdot f(h) \notin H_6$. Otherwise, $h \cdot f(h) \in H_6^{d,\tau}$ for some $d \in N$ and $\tau \in \{0, \ldots, d+Q+3\}$. Then if $\tau > 0$, $T^{\tau-1}(h) \in \tilde{\Sigma}^{d,\tau-1}$; but this is a contradiction as this implies that $h \in H_6$.

If $\tau = 0$, then by the definition of H_6^0, $f(h) = T^1(h \cdot f(h)) \in \bar{D}(s) \cup \bar{D}(s')$. But this is a contradiction because, by $h \in H_7^{k'}$, $f(h) \in \{s, s'\}$.

Step 2: $h \cdot f(h) \notin \cup_{l=1}^5 H_l$. When $k' \geq 2$, then $f(h) = s$. Therefore, the claim in this step follows immediately from Claim B.9. Next suppose that $k' \in \{0, 1\}$ and that $h \cdot f(h) \in \cup_{l=1}^5 H_l$. Then Claim B.9 implies that either $h \in H_{1,a}^{(\omega),n+\omega+Q+4}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ or $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$ for some $d \in N$. But this contradicts our supposition in (B.5).

Step 3: $T^{k'+1}(h \cdot f(h)) \in H_{1,b}^{(0),k'+1}$ and $v(h \cdot f(h)) = k' + 1$. By $h \in H_7^{k'}$, $f(h) = \pi^{(0),k'+1}$ and so $T^{k'+1}(h \cdot f(h)) \in H_{1,b}^{(0),k'+1}$. Hence, $v(h \cdot f(h)) \geq k' + 1$. If it were the case that $v(h \cdot f(h)) > k' + 1$, then $T^{v(h \cdot f(h))-1}(h) \in H_{1,b}^{(0),v(h \cdot f(h))-1}$, implying $v(h) \geq v(h \cdot f(h)) - 1$. Since $k' = v(h)$ and $v(h \cdot f(h)) > k' + 1$, this is a contradiction. ■

Claim B.43 *Let $h \in H_1$ and $\bar{a} \in \bar{D}_d(f(h))$ for some $d \in N$. Then there exists $t \in \{1, \ldots, d+Q+4\}$ such that $h \cdot \bar{a} \cdot (\pi^1(f|h \cdot \bar{a}), \ldots, \pi^t(f|h \cdot \bar{a})) \in H_{3,a}^{d+Q+4,d}$.*

Proof. By assumption, $h \in H_{1,a}^{(\omega),k'} \cup H_{1,b}^{(0),l}$ for some ω and k' and l with $l < M$ and (by Claim B.10) $k' < M$. Thus, $h \cdot \bar{a} \in H_2^{k,d,0}$ with either $k = k' + 1$ or $k = l + 1$. The result then follows by Claim B.41. ■

Claim B.44 *Let $h \in H_2^{k,d',\tau'} \cup H_5^{k,d',\tau'} \cup H_6^{d',\tau'}$ for some $k \in \{1, \ldots, M\}$ and $d' \in N$ and $\tau' \in \{0, \ldots, d' + 3\}$. Let $d \neq d'$ and $\bar{a} \in \bar{D}_d(f(h))$. Then either $h \cdot \bar{a} \in H_{3,a}^{d+Q+4,d}$ or there exists $t \in \{1, \ldots, d + Q + 4\}$ such that $h \cdot \bar{a} \cdot (\pi^1(f|h \cdot \bar{a}), \ldots, \pi^t(f|h \cdot \bar{a})) \in H_{3,a}^{d+Q+4,d}$.*

Proof. We first argue that it is sufficient to show that either $h \cdot \bar{a} \in H_3^{d+Q+4,d}$ or $h \cdot \bar{a} \notin \cup_{l=1}^5 H_l$. Indeed, if the latter holds, then this together with $f(h) \in \{s, s'\}$ and $\bar{a} \in \bar{D}_d(f(h))$ implies that $h \cdot \bar{a} \in H_6^{d,0}$. Then the claim follows from Claim B.41.

We next establish that either $h \cdot \bar{a} \in H_3^{d+Q+4,d}$ or $h \cdot \bar{a} \notin \cup_{l=1}^5 H_l$. Suppose not; then we derive a contradiction for the different possible cases as follows.

<u>Case 1</u>: *$T^{\hat{d}+Q+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+Q+4}$ for some $\hat{d} \in N$.* Then $\bar{a} \in D_{\hat{d}}(s')$ which together with $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$ implies that $\hat{d} = d$. Hence, $h \cdot \bar{a} \in H_3^{d+Q+3,d}$; a contradiction.

<u>Case 2</u>: *$h \cdot \bar{a} \in H_{1,a}^{(\omega),n+\omega+Q+5}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$.* Then $\bar{a} = s'$ and this contradicts $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$.

<u>Case 3</u>: *$h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ and $h \cdot \bar{a} \in H_{1,a}^{(\omega),k'} \cup H_{1,b} \cup H_2^{\hat{k},\hat{d},0} \cup H_5^{\hat{k},\hat{d},0}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ and $k' > n + \omega + Q + 5$ and $\hat{k} \in \{1, \ldots, M\}$ and $\hat{d} \in N$.* Then, by the latter,

$h \in H_1 \cup H_3 \cup H_4$. By Claims B.17, B.18, B.22, B.26, B.29, B.30, B.32 and B.33, this contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

<u>Case 4</u>: *$h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ and $h \cdot \bar{a} \in H_2^{\hat{k},\hat{d},\hat{\tau}} \cup H_5^{\hat{k},\hat{d},\hat{\tau}}$ for some $\hat{k} \in \{1,\ldots,M\}$ and $\hat{d} \in N$ and $\hat{\tau} \in \{1,\ldots,\hat{d}+Q+3\}$.* By the latter, $\bar{a} \in D_{\hat{d}}(s) \cup D_{\hat{d}}(s')$. Since we also have $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$ it follows that $\hat{d} = d$. This together with $h \cdot \bar{a} \in H_2^{\hat{k},\hat{d},\hat{\tau}} \cup H_5^{\hat{k},\hat{d},\hat{\tau}}$ implies that $h \in H_2^{\hat{k}-1,d,\hat{\tau}-1} \cup H_5^{\hat{k}-1,d,\hat{\tau}-1}$ which, by Claims B.19, B.31 and B.34, contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

<u>Case 5</u>: *$h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ and $h \cdot \bar{a} \in H_3$ and $T^{\hat{d}+Q+4}(h \cdot \bar{a}) \notin \Sigma^{\hat{d},\hat{d}+Q+4}$ for all $\hat{d} \in N$.* Then $h \in H_3$ and, by Claims B.22 and B.32, this contradicts $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

<u>Case 6</u>: *$h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ and $h \cdot \bar{a} \in H_4$.* Then $h \in H_3 \cup H_4$ and, by Claims B.22, B.26, B.32 and B.33, this contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

<u>Case 7</u>: *$h \in H_6^{d',\tau}$ and $h \cdot \bar{a} \notin H_{1,a}^{(\omega),n+\omega+Q+5}$ for all $i \in \{0,\ldots,n\}$ and $T^{\hat{d}+Q+4}(h \cdot \bar{a}) \notin \Sigma^{\hat{d},\hat{d}+Q+4}$ for all $\hat{d} \in N$.* Then $h \notin \cup_{l=1}^{5} H_l$ and, by Claim B.9, we have $h \cdot \bar{a} \notin \cup_{l=1}^{5} H_l$; a contradiction. ■

Claim B.45 *Let $h \in H_{3,a}^{k,d'}$ for some $k \in \{1,\ldots,M\}$ and $d' \in N$, and $\bar{a} \in \bar{D}_d(f(h))$ with $d \neq d'$. Then there exists $t \in \{1,\ldots,d+Q+4\}$ such that $h \cdot \bar{a} \cdot (\pi^1(f|h \cdot \bar{a}),\ldots,\pi^t(f|h \cdot \bar{a})) \in H_{3,a}^{d+Q+4,d}$.*

Proof. The claim follows by Claim B.41 since $h \cdot \bar{a} \in H_5^{k+1,d,0}$. ■

Claim B.46 *Let $h \in H_{3,b}^{k,d'}$ for some $k \in \{1,\ldots,M\}$ and $d' \in N$, $T^{Q+1}(h) \in \cup_{l \in N} \hat{\Sigma}^{l,Q+1}$, and $d \neq d'$ and $\bar{a} \in \bar{D}_d(f(h))$. Then there exists $t \in \{1,\ldots,d+Q+4\}$ such that $h \cdot \bar{a} \cdot (\pi^1(f|h \cdot \bar{a}),\ldots,\pi^t(f|h \cdot \bar{a})) \in H_{3,a}^{d+Q+4,d}$.*

Proof. The claim follows by Claim B.41 since $h \cdot \bar{a} \in H_5^{k+1,d,0}$. ■

Claim B.47 *Let $h \in H_4^{k,\omega,r}$ for some $k \in \{1,\ldots,M\}$ and $\omega \in \{1,\ldots,\bar{\omega}\}$ and $r \in \{0,\ldots,n+\omega+Q+4\}$, and let $\bar{a} \in \bar{D}_d(f(h))$ for some $d \in N$. Then there exists $t \in \{1,\ldots,d+Q+4\}$ such that $h \cdot \bar{a} \cdot (\pi^1(f|h \cdot \bar{a}),\ldots,\pi^t(f|h \cdot \bar{a})) \in H_{3,a}^{d+Q+4,d}$.*

Proof. The claim follows by Claim B.41 since $h \cdot \bar{a} \in H_5^{k+1,d,0}$. ■

Claim B.48 *Let $h \in H_7^k$ for some $k \in \{0,\ldots,n+Q+4\}$ and $\bar{a} \in \bar{D}_d(f(h))$ for some $d \in N$. If $h \notin \cup_{k=2}^{n+Q+3} H_7^k$ and $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$, then $h \cdot \bar{a} \in H_{3,a}^{d+Q+4,d}$. Otherwise, there exists $t \in \{1,\ldots,d+Q+4\}$ such that $h \cdot \bar{a} \cdot (\pi^1(f|h \cdot \bar{a}),\ldots,\pi^t(f|h \cdot \bar{a})) \in H_{3,a}^{d+Q+4,d}$.*

Proof. If $h \notin \cup_{k=2}^{n+Q+3} H_7^k$ and $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$, then $f(h) = s'$ and $\bar{a} \in \bar{D}_d(s')$ and $T^{d+Q+4}(h \cdot \bar{a}) \in \Sigma^{d,d+Q+4}$. Hence, $h \cdot \bar{a} \in H_{3,a}^{d+Q+4,d}$.

For the remainder of the proof, therefore assume that the following holds:

$$\text{If } h \notin \cup_{k=2}^{n+Q+3} H_7^k, \text{ then } T^{d+Q+3}(h) \notin \Sigma^{d,d+Q+3}. \tag{B.6}$$

Then we will show that $h \cdot \bar{a} \in H_6^{d,0}$ which, by Claim B.41, establishes the conclusion of the claim. We prove the former in two steps.

Step 1: $h \cdot \bar{a} \in \tilde{\Sigma}^{d,0}$. Since $h \in H_7^k$, $f(h) \in \{s, s'\}$. Therefore, $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$.

Step 2: $h \cdot \bar{a} \notin \cup_{l=1}^{5} H_l$. Suppose otherwise. Then, by Claim B.9, $\bar{a} \notin D(s)$. Hence, by $h \in H_7^k$, (i) $\bar{a} \in \bar{D}_d(s')$ and (ii) $h \notin \cup_{k=2}^{n+Q+3} H_7^k$. Furthermore, Claim B.9 implies that either $h \cdot \bar{a} \in H_{1,a}^{(\omega),n+\omega+Q+5}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ or $T^{d'+Q+4}(h \cdot \bar{a}) \in \Sigma^{d',d'+Q+4}$ for some $d' \in N$. If $h \cdot \bar{a} \in H_{1,a}^{(\omega),n+\omega+Q+5}$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$, then $\bar{a} = s'$; a contradiction to (i). If $T^{d'+Q+4}(h \cdot \bar{a}) \in \Sigma^{d',d'+Q+4}$ for some $d' \in N$, then $\bar{a} \in D_{d'}(s')$ which, by (i), implies that $d = d'$. Thus, $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$. But this together with (ii) contradicts our supposition in (B.6). ■

B.2.4 f is subgame perfect and $\pi(f) = \hat{\pi}^{(0)}$

By Claim B.36 and Claim B.37, $\pi(f) = \hat{\pi}^{(0)}$. To show that f is subgame perfect, we next establish the following for all $h \in H$:

$$U_d(f|h) \geq (1-\delta)u_d(\bar{a}) + \delta U_d(f|h \cdot \bar{a}) \text{ for all } d \in N \text{ and } \bar{a} \in \bar{D}_d(f(h)). \tag{B.7}$$

By construction, (B.7) holds in the following cases:

1. $h \in H_{3,a}^{k,d}$ for some $k \in \{1, \ldots, M\}$,
2. $h \in H_{3,b}^{k,d}$ for some $k \in \{1, \ldots, M\}$ and $T^{Q+1}(h) \in \cup_{l \in N} \hat{\Sigma}^{l,Q+1}$, and
3. $h \in H_{3,b}^{k,d'}$ for some $k \in \{1, \ldots, M\}$ and $d' \in N$, and $T^{Q+1}(h) \notin \cup_{l \in N} \hat{\Sigma}^{l,Q+1}$ for all $l \in N$.

In the first case, the future play is independent of player d's current action and such current action is a static best-reply against the action of the other players. In the second and third cases, player d's current action is optimal since by the construction of f^d.

In what follows, we fix h and $\bar{a} \in \bar{D}_d(f(h))$ and consider the remaining cases.

<u>*Case 1*</u>*: Either $h \in H_1 \cup H_4$ or the following holds: $h \in H_7^k$ for some $k \in \{0, \ldots, n+Q+4\}$ and either $h \in \cup_{k=2}^{n+Q+3} H_7^k$ or $T^{d'+Q+3}(h) \notin \Sigma^{d',d'+Q+3}$ for all $d' \in N$.* In this case, by Claims B.36, B.37, B.38 and B.42, $\pi(f|h) = (\hat{\pi}^{(\omega),k}, \ldots)$ for some $\omega \in \{0, \ldots, \bar{\omega}\}$ and $k \leq M$. Therefore, the left-hand side of (B.7) is greater or equal to

$$-B(1-\delta^{K-k+1}) + \delta^{K-k+1} V_d(\hat{\pi}^{(\omega)}) \geq -B(1-\delta^K) + \delta^K V_d(\hat{\pi}^{(\omega)}) > -B(1-\delta^K) + \delta^K(u_d' - 2\xi).$$

By Claims B.43, B.47 and B.48, $h \cdot \bar{a} \cdot (\pi^1(f|h \cdot \bar{a}), \ldots, \pi^t(f|h \cdot \bar{a})) \in H_{3,a}^{d+Q+4,d}$ for some $t \in \{0, \ldots, d+Q+3\}$. Therefore, by Claims B.40 and B.39 and Lemma 1 and Lemma B.1, the right-hand side of (B.7) is less than or equal to

$$\begin{aligned}&(1-\delta^{t+1+(\gamma+1)(d+Q+4)})B + \delta^{t+1+(\gamma+1)(d+Q+4)}(1-\delta^T)\bar{\varepsilon} + \delta^{t+1+(\gamma+1)(d+Q+4)+T}(u_d' + 2\xi) \leq \\ &(1-\delta^{(\gamma+2)(n+Q+4)+1})B + \delta^{(\gamma+2)(n+Q+4)+1}(1-\delta^T)\bar{\varepsilon} + \delta^{(\gamma+2)(n+Q+4)+1+T}(u_d' + 2\xi).\end{aligned}$$

Thus, by (53), (B.7) holds.

<u>*Case 2*</u>*: One of the following conditions holds for some $d' \in N$ with $d \neq d'$: (i) $h \in H_{3,a}^{k,d'}$ for some $k \in \{1, \ldots, M\}$, (ii) $h \in H_{3,b}^{k,d'}$ for some $k \in \{1, \ldots, M\}$ and $T^{Q+1}(h) \in \cup_{l \in N} \hat{\Sigma}^{l,Q+1}$, (iii) $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau} \cup H_6^{d',\tau}$ for some $k \in \{1, \ldots, M\}$ and $\tau \in \{0, \ldots, d'+Q+3\}$, and (iv) $h \in H_7^k$ for some $k \in \{0, \ldots, n+Q+4\}$ and $h \notin \cup_{k=2}^{n+Q+3} H_7^k$ and $T^{d'+Q+3}(h) \in \Sigma^{d',d'+Q+3}$.* Claims B.40, B.39, B.41 and B.42 can be applied to cases (i), (ii), (iii) and (iv), respectively; and we obtain that the left-hand side of (B.7) is greater or equal to

$$-(1-\delta^{(\gamma+2)(n+Q+4)+T})B + \delta^{(\gamma+2)(n+Q+4)+T}(u_d' + (1-2\varepsilon_2)\rho - 2\xi).$$

The right-hand side of (B.7) is, as in Case 1, less than or equal to

$$(1-\delta^{(\gamma+2)(n+Q+4)+1})B + \delta^{(\gamma+2)(n+Q+4)+1}(1-\delta^T)\bar{\varepsilon} + \delta^{(\gamma+2)(n+Q+4)+1+T}(u_d' + 2\xi).$$

Thus, by (54), (B.7) holds.

<u>*Case 3*</u>*: Either $h \in H_2^{k,d,\tau} \cup H_5^{k,d,\tau} \cup H_6^{d,\tau}$ for some $k \in \{1, \ldots, M\}$ and $\tau \in \{0, \ldots, d+Q+3\}$ or $h \in H_7^k$ for some $k \in \{0, \ldots, n+Q+4\}$, $h \notin \cup_{k=2}^{n+Q+3} H_7^k$ and $T^{d+Q+3}(h) \in \Sigma^{d,d+Q+3}$.* By Claim B.41 (when $h \in H_2^{k,d,\tau} \cup H_5^{k,d,\tau} \cup H_6^{d,\tau}$ for some $k \in \{1, \ldots, M\}$ and

$\tau \in \{0, \ldots, d+Q+3\}$) and Claims B.42 and B.48 (in the remaining case), we have that the difference between the left and the right-hand sides of (B.7) is greater or equal to

$$-(1-\delta)2B + \delta^{(\gamma+2)(n+Q+4)}(1-\delta^{\gamma})(u_d(\bar{a}^{(d)}) - u_d(m^d)).$$

Indeed, if the deviation takes place in the tth action of player d's signalling phase, then the tth sequence of $(\gamma+1)$ actions in the phase between the signalling phase and the minmax phase is $((m^d;\gamma),(s_d^*, s'_{-d}))$, whereas it is $((a^{(d)};\gamma),(s_d^*, s'_{-d}))$ if no deviation occurs. Furthermore, these are the only differences that occur in future play. Furthermore, this different sequence of actions can start no later than $(\gamma+2)(n+Q+4)$ periods after the deviation. Thus, (B.7) holds if

$$\delta^{(\gamma+2)(n+Q+3)} \frac{(1-\delta^{\gamma})(u_d(\bar{a}^{(d)}) - u_d(m^d))}{(1-\delta)2B} > 1;$$

hence, by (52), (B.7) holds.

This concludes the proof of Lemma 3.