Optimal Treatment of an SIS Disease with Two Strains

Selma Telalagic

May 2012

CWPE 1229

Optimal Treatment of an SIS Disease with Two Strains

Selma Telalagic*

May 29, 2012

Abstract

This paper explores optimal treatment of an SIS (Susceptible-Infected-Susceptible) disease that has two strains with different infectivities. When we assume that neither eradication nor full infection are possible, it is shown that there are two categories of equilibria. First, there are two continua of interior equilibria characterised by a fixed, positive total level of infection, where both strands of the disease prevail. It is hypothesised that a Skiba curve of indifference lies between them. Second, there are two sets of equilibria where one strand of the disease is eradicated asymptotically. The feasibility of equilibria depends on parameter assumptions; a combination of low natural rate of recovery and large difference between infectivities leaves only a small proportion of equilibria as feasible. Simulations exploring the relationship between cost and optimal policy are carried out. There exists a parameter range such that, counter-intuitively, it is optimal to allow the high-infectivity strain of the disease to prevail, while asymptotically eradicating the low-infectivity strain. Within this parameter range, there is added benefit from policy flexibility. At higher costs, simulations of the interior equilibria demonstrate the existence of a Skiba curve. The curve delineates two regions, each of which has a clear optimal policy.

Keywords: Epidemiological modelling, Optimal control, Simulations

JEL Classification: I18, I19, C61, C63

*Faculty of Economics, University of Cambridge, CB3 9DD England. Email address: st390@cam.ac.uk.
The author is thankful to Prof Robert Rowthorn and Flavio Toxvaerd for their valuable guidance. Financial support from the ESRC is gratefully acknowledged.

1 Introduction

Epidemiology, as it is studied today, originated in the early 20th century and has since developed into a multi-faceted field that combines the skills of mathematicians, biologists and, most recently, economists. One predominant area of epidemiology focuses on transmission system models. These models are built on differential equations that describe the evolution of disease prevalence over time as a function of parameters. Often a point of criticism, these models assume homogeneous mixing within populations, identical agents and no behavioural adaptation. Although this produces the benefit of parsimony, allowing significant predictive power and the ability to work with data, there is a strand of the literature that argues this simplicity comes at the price of applicability (Epstein 2009). Nevertheless, few advances have been made in other approaches to epidemiology that have received the backing that these types of models have.

In the standard Susceptible-Infected-Susceptible (SIS) model, individuals move between two states, susceptible and infected, based on exogenous probabilities. The probability of an individual catching a disease when he encounters an infected person depends on an infectivity or transmission parameter. This parameter is predominantly assumed to be homogeneous, a simplification that does not allow for policy differentiation if there exist several strands of infection. Infections in reality are frequently present in more than one form. To motivate an infection stratified by transmission parameter, consider the case of HIV as an example. HIV has two main strains: HIV-1 and HIV-2. Studies show that the less common HIV-2 strain is also less infectious than its counterpart for most of its infectious period.[1] This motivates us to ask how a policymaker deals with the presence of several variants of an infection in the population that vary by transmission risk. There is a trade-off between treating individuals, which is costly but offers a welfare benefit, and saving money. There is a further trade-off between treating individuals infected with the more infectious versus less infectious strand. Supposing the policymaker can differentiate policy by infection type, does she treat the more infectious or less infectious first? What is the prevalence of the different infection strands in equilibrium?

Variations on standard epidemiological models are common in the mathematical literature, where researchers detail dynamics and equilibria but do not look at optimality and intervention. In direct relevance to this paper, Castillo-Chavez, Huang and Li (1999) develop an SIS model with a two-strand disease where individuals are genetically predisposed to a specific strand. They derive stability conditions on the various equilibria of the model, which include boundary (one or both strands eradicated) and coexistence (both strands prevail) equilibria. These are equilibria the system tends towards when there is no intervention. Hyman and Li (1997) analyse an SIS STD model with multiple groups where interaction between

[1] More information on this can be found at the Centers for Disease Control and Prevention (http://www.cdc.gov/hiv/topics/basic) as well as the charity AVERT (http://www.avert.org/hiv-types.htm).

groups is behaviourally variable and depends on prevalence levels in the different groups. The development of the infection is complex and depends on how these interactions take place; individuals may reduce their contacts with individuals in high prevalence groups, which may reduce overall prevalence. Biological epidemiology brings models to the data. Truscott et al (2011) show that accurate modelling of influenza should not neglect the presence of several strains; they construct a model with two strains and show that its predictions are close to data on influenza. This paper develops a model with multiple strains akin to those studied by mathematical and biological epidemiologists while simultaneously introducing the economic consideration of optimal intervention.

Economic research into epidemiology is fast-growing. Research has focused on two main areas relevant to the present paper: optimal intervention and empirical work. Optimal intervention has been studied in several extensions to the standard model, including the consideration of spatial factors, budget constraints and the availability of several policy instruments. Rowthorn, Laxminarayan and Gilligan (2009) focus on the spatial dynamics of disease. They answer the question of optimal control of infections via treatment in the case of metapopulations, defined as subpopulations within a population that mix at a lower rate than individuals within each subpopulation. Although intuition may suggest that equalising infection rates across subpopulations leads to the highest level of welfare, this turns out to be the worst possible solution. Another policy-relevant aspect is the role of budget constraints. Rowthorn (2004) examines this in the context of optimal control of a disease using treatment and shows that funds should never be retained as long as there are individuals that can be treated. Rowthorn and Toxvaerd (2011) examine analytically the trade-off between vaccination and treatment as two instruments available to the policy-maker. Gersovitz and Hammer (2004) provide a fruitful discussion of this issue of targeting when these two policy instruments are available. In particular, they argue that while naturally one would assume only infected people are treated and susceptible people are vaccinated, there may be other targeting functions, especially if the policymaker is unable to perfectly observe infection levels in the population.

Significant empirical work has been carried out on infectious disease. Several studies have been carried out aiming to verify the responsiveness of risky behaviour to changes in perceived risk of contracting infections such as HIV/AIDS. St. Lawrence et al (1991) look at differences in risky behaviour across two cities with different prevalence rates. They find startling differences with risky behaviour being as much as three times more common in the low-prevalence city as compared to the high-prevalence city. Similarly, Dupas (2005) looks at whether a public health information program that teaches teenagers about relative risks of contracting HIV/AIDS depending on partner age group has an effect on their behaviour. Dupas finds that the information campaign reduces childbearing by 1.7% in the treatment group, representing a 31% decrease in childbearing. In terms of age group, there is a reduction in cross-generational pregnancies of 65%.

Oster (2005) provides a detailed simulation-based analysis on the effects of changes in

transmission rates and partner choice on national HIV/AIDS prevalence levels. Using actual transmission rates and sexual behaviour parameters, the paper predicts an HIV/AIDS infection prevalence of 0.23% in the United States and 12.7% in Africa, close to actual prevalence rates of 0.15% and 11.9%. Estimates are then carried out using US sexual behaviour parameters but Sub-Saharan African transmission rates. This results in an estimated prevalence rate of over 11% for the United States, suggesting that it is the transmission rate that is driving the higher HIV/AIDS prevalence rates observed in Africa when compared to the United States. This shows that the transmission rate is an important determinant of prevalence levels.

The present paper is a natural next step for the literature. While the case of several policy instruments has been considered in the theoretical literature, it has not been considered in conjunction with more than one infection type, a scenario that brings this type of modelling closer to the realities faced by policymakers. Further, the empirical literature highlights the importance of modelling transmission parameters correctly. This paper explores an SIS model with two infection strains and provides answers on optimal policy in various situations. We show that the model has two categories of steady state. First, there are two continua of steady states where both infection strains prevail. Second, there are asymptotic equilibria where one of the strains is eliminated asymptotically, while the other is endemic. Under certain parameter assumptions, it is optimal to asymptotically eliminate the less infectious strand while allowing the high infectivity strand to prevail. This interesting case is explored by way of simulations, where optimality under fixed policy and variable policy is explored. The role of cost of treatment in governing optimal policy is explored in detail. We also examine the interior equilibria where both infection strains prevail and demonstrate the existence of a Skiba curve of indifference between them.

The rest of the paper is structured as follows. Section 2 introduces the basic model and Section 3 develops the model to encompass two infection types. Section 4 provides examples of simulations. Section 5 concludes.

2 The basic SIS model

2.1 Overview

The results of the basic SIS model with treatment are discussed in this section, following Rowthorn (2004) and Goldman and Lightwood (2002). It has become standard to assume random mixing in models of this type. Typically, these models also assume a homogeneous transmission parameter. This assumption does not provide an accurate representation of the way disease spreads when it exists in different forms. Indeed, awareness of the improved predictions resulting from accurate transmission parameters has been raised by Oster (2005).

These observations provide the impetus for an extension to the basic SIS model, which is presented in Section 4.

The standard SIS model is in continuous time. There are two possible states: individuals are susceptible (proportion S of the total population) or infected (proportion I). They can move between the two states an unlimited number of times. Agents are homogeneous and the population is closed. Perfect or homogeneous mixing is assumed between agents, with a uniform transmission probability (β). A proportion f of infected individuals is treated, with the success rate of treatment (which can be interpreted as a rate of recovery) given by the parameter α. There is also the possibility of spontaneous or natural recovery at rate τ. The evolution of the two subpopulations, susceptible and infected, is described by the following differential equations:

$$\dot{I}(t) = I(t)S(t)\beta - I(t)(f(t)\alpha + \tau), \tag{1}$$

$$\dot{S}(t) = I(t)(f(t)\alpha + \tau) - I(t)S(t)\beta. \tag{2}$$

Optimal policy is derived via the introduction of an objective function. Typically intervention takes the form of either vaccination or treatment. The latter is studied in the present paper. Intervention affects welfare consequences directly by influencing the rate of movement between the susceptible and infected subpopulations.

Objective functions can take many forms, from a social planner's welfare maximisation function, to an individual's utility maximisation function. There is also the possibility of cost minimisation, prevalence minimisation, and so on and so forth. One natural objective function to add to this model is social welfare, determined by the proportion of infected and susceptible individuals and the expenditure on treatment:

$$W(I^0) = \int_0^\infty e^{-\delta t}(pN(1 - I(t)) - cf(t)I(t))dt. \tag{3}$$

In this simple case, infected individuals have a value of zero while susceptible individuals have a value of p; treatment has a constant marginal cost of c per instant per individual. The problem is solved as a Hamiltonian optimal control problem, normalising population to 1: $S(t) + I(t) = N = 1$ for all t. This allows (1) and (2) to collapse to one constraint.

It is assumed that

$$\begin{aligned} f &\in [0, 1] \\ I(0) &= I^0 > 0 \text{ given} \end{aligned}$$

The current value Hamiltonian function is

$$H = p(1-I) - cfI + \gamma I((1-I)\beta - f\alpha - \tau) \tag{4}$$

where γ is the shadow price of infection. Differentiating the Hamiltonian with respect to the control variable gives us the solution, which is of "bang-bang" form:

$$f^* \begin{Bmatrix} = 0 \\ \in (0,1) \\ = 1 \end{Bmatrix} if\ \gamma^* \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} - \frac{c}{\alpha}. \tag{5}$$

Policy can either be at an interior level $f^* \in (0,1)$, or at a boundary level, $f^* = 0$ or 1. The interpretation is as follows. The multiplier is the shadow price of another infected individual. It will always be negative. The higher is this shadow price in absolute terms, the more costly it is to social welfare to have an additional infected person. On the other hand, $\frac{c}{\alpha}$ is the relative price of treating an infected individual - it is the ratio of cost to treatment effectiveness. The negative of this price of treatment is the welfare cost from treating an additional individual. These concepts clarify the intuition behind the solution; it is clear that if the cost of infection exceeds the cost of treatment, everyone is treated. Similarly, when the cost of treatment is higher than the cost of infection, no one is treated. When they are equal, any interior level of treatment is optimal subject to parameters.

Note that the equation of motion for the multiplier is

$$\dot{\gamma} = \delta\gamma - \frac{\partial H}{\partial I} = p + cf - \gamma((1-2I)\beta - f\alpha - \tau - \delta). \tag{6}$$

Let us examine the cases of interior and boundary policies more closely.

2.2 Policy is interior

For an interior policy to be optimal, the Hamiltionian conditions require that $\gamma = -\frac{c}{\alpha}$. Differentiating this gives us $\dot{\gamma}(t) = 0$. Further, it must be that $\dot{I}(t) = 0$ if we are in steady state. These three conditions give us steady state solutions $I = I^*, \gamma = \gamma^*$ and $f = f^*$:

$$I^* = \frac{\alpha p + c(\beta - \delta - \tau)}{2c\beta}, \tag{7}$$

$$f^* = \frac{c(\beta + \delta - \tau) - \alpha p}{2c\alpha}, \tag{8}$$

$$\gamma^* = -\frac{c}{\alpha}. \tag{9}$$

Thus, a path with interior policy has $I = I^*, \gamma = \gamma^*$ and $f = f^*$. Note that f^* may lie outside the range $(0, 1)$, in which case no feasible interior policy exists.

2.3 Policy is at a boundary

There are two feasible boundary policies that can be optimal in steady state: $f = 0$ or $f = 1$. Consider the case where $\gamma > -\frac{c}{\alpha}$. Under this policy it must be that $f = f^{**} = 0$. Solving $\dot{I}(t) = 0$ yields

$$I^{**} = 1 - \frac{\tau}{\beta}. \tag{10}$$

The disease is endemic as long as $\tau < \beta$. It is eradicated if $\tau \geqslant \beta$. Setting $\dot{\gamma}(t) = 0$ yields

$$\gamma^{**} = \frac{p}{\tau - \beta - \delta}. \tag{11}$$

Another possibility is that $\gamma > -\frac{c}{\alpha}$. In this case, $f = f^{***} = 1$. Solving $\dot{I}(t) = 0$ and $\dot{\gamma}(t) = 0$ yields

$$I^{***} = 1 - \frac{\alpha + \tau}{\beta},$$

$$\gamma^{***} = \frac{p + c}{\alpha + \tau - \beta - \delta}.$$

The disease is endemic as long as $\alpha + \tau < \beta$. It is eradicated if $\alpha + \tau \geqslant \beta$.

2.4 Optimal policy

Policy can be either at one of the boundaries or at an interior level, depending on the value of the shadow price. Rowthorn (2004) and Goldman and Lightwood (2002) show that optimal policy will take on one of the two boundary values. It is never optimal to treat partially. This is because the shadow price is a single-valued function of the state variable, so optimal policy can have at most one switch point. The interior steady state can only be reached by a path that zig-zags back on itself. In contrast, each of the boundary steady states can be reached by a path with at most one switch point, with the precise path depending on the initial infection level. Which policy of the two boundaries is optimal will depend on the value of parameters.

3 The SIS model with two strains of infection

3.1 Overview

In the model of the previous section, the transmission rate is uniform and there is one policy instrument. In this section we relax both of these assumptions. Suppose there are two variants of infection, one more infectious than the other. The more infectious variant H has transmission rate β_H while the less infectious variant L is characterised by transmission rate β_L. The policymaker has two policy instruments at her disposal (f_H and f_L), each targeting one of the infection strands. There is an implicit assumption that the policymaker can distinguish the two strains and therefore target therapy perfectly. This we can relate to the discussion of Gersovitz and Hammer (2004). We assume that the policymaker has perfect information about each strain of the disease and its prevalence in the population. Other versions are possible of course; for example, a policymaker may know an individual is ill but not which strain of the disease he has. In this case the policymaker would effectively have only one policy instrument at hand.

A further assumption of the model is that individuals can catch either infection strand at the outset. When infected they transmit the strand that they themselves are infected with. Super-infection is not possible: individuals cannot become infected with both strains of the infection at the same time. Similar to the previous section, there is a possibility of exogenous recovery. If individuals recover, they are again susceptible to either infection strain. The proportion of the total population infected with H is I_H. The proportion infected with L is I_L. The total population is normalised to size 1: $I_H + I_L + S = 1$. The policymaker maximises the social welfare function

$$V(I_H^0, I_L^0) = \int_0^\infty e^{-\delta t}(p(1 - I_H(t) - I_L(t)) - c(f_H(t)I_H(t) + f_L(t)I_L(t)))dt \qquad (12)$$

subject to the equations of motion for the two infection types:

$$\begin{aligned} \dot{I}_H &= \beta_H I_H(t)(1 - I_H(t) - I_L(t)) - I_H(t)(\tau + \alpha f_H(t)), \quad (13) \\ \dot{I}_L &= \beta_L I_L(t)(1 - I_H(t) - I_L(t)) - I_L(t)(\tau + \alpha f_L(t)). \quad (14) \end{aligned}$$

All parameters are strictly positive. Further,

$$\begin{aligned} f_H, f_L &\in [0,1] \\ I_H(0) &= I_H^0 > 0 \text{ given} \\ I_L(0) &= I_L^0 > 0 \text{ given} \\ I_H^0 + I_L^0 &< 1 \end{aligned}$$

In addition,

$$\beta_H > \beta_L > \tau + \alpha. \quad (15)$$

The inequalities in (15) ensure that neither variant of the disease can be eliminated even asymptotically by treating all infected people. Thus, at any steady state, $I_H, I_L > 0$. They also ensure that $I_H(t) + I_L(t) < 1$ for all t.

The current value Hamiltonian is

$$\begin{aligned} H &= p(1 - I_H - I_L) - c(f_H I_H + f_L I_L) \\ &\quad + \lambda_H(\beta_H I_H(1 - I_H - I_L) - I_H(\tau + f_H \alpha) \\ &\quad + \lambda_L(\beta_L I_L(1 - I_H - I_L) - I_L(\tau + f_L \alpha) \quad (16) \end{aligned}$$

The first order-conditions yield the following solution:

$$f_H^* \begin{Bmatrix} = 0 \\ \in (0,1) \\ = 1 \end{Bmatrix} if\ \lambda_H^* \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} - \frac{c}{\alpha}, \quad (17)$$

$$f_L^* \begin{Bmatrix} = 0 \\ \in (0,1) \\ = 1 \end{Bmatrix} if\ \lambda_L^* \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} - \frac{c}{\alpha}. \quad (18)$$

The equations of motion for the two costate variables are

$$\begin{aligned}\dot{\lambda}_H &= \delta\lambda_H - \frac{\partial H}{\partial I_H} \\ &= p + cf_H - \lambda_H\left(-\delta + \beta_H(1 - I_H - I_L) - \tau - \alpha f_H\right) \\ &\quad +(\lambda_H\beta_H I_H + \lambda_L\beta_L I_L), \end{aligned} \tag{19}$$

$$\begin{aligned}\dot{\lambda}_L &= \delta\lambda_L - \frac{\partial H}{\partial I_L} \\ &= p + cf_L - \lambda_L\left(-\delta + \beta_L(1 - I_H - I_L) - \tau - \alpha f_L\right) \\ &\quad +(\lambda_H\beta_H I_H + \lambda_L\beta_L I_L). \end{aligned} \tag{20}$$

In the next section we derive the equilibria of this model.

3.2 Fixed points

3.2.1 The set of feasible fixed points

We are interested in the fixed points or steady states of the system. The reason we are interested in fixed points is because they are the states the system converges to. This convergence takes place only in the limit in the special case of the Asymptotic Fixed Points (AFPs), which will be defined shortly.

We assume that there are no cycles: the system always converges to one of our potential fixed points or AFPs. A proof of the non-existence of cycles is beyond the scope of this paper. Let us examine the fixed points in more detail.

Definition 1 *A fixed point (FP) is a solution* $(f_H^*, f_L^*, I_H^*, I_L^*, \lambda_H^*, \lambda_L^*)$ *satisfying equations (13), (14), (17), (18), (19) and (20) as well as* $\dot{I}_H = \dot{I}_L = \dot{\lambda}_H = \dot{\lambda}_L = \dot{f}_H = \dot{f}_L = 0$.

There are nine potential fixed points, listed below. The notation A_{ab} denotes the fixed point with policy $f_H^* = a, f_L^* = b$ for $a, b = 0$ or 1. The notation $a, b = 2$ denotes an interior policy.

$$
\begin{aligned}
A_{00} &: \quad f_H = 0, f_L = 0 \\
A_{01} &: \quad f_H = 0, f_L = 1 \\
A_{02} &: \quad f_H = 0, f_L \in (0,1) \\
A_{10} &: \quad f_H = 1, f_L = 0 \\
A_{11} &: \quad f_H = 1, f_L = 1 \\
A_{12} &: \quad f_H = 1, f_L \in (0,1) \\
A_{20} &: \quad f_H \in (0,1), f_L = 0 \\
A_{21} &: \quad f_H \in (0,1), f_L = 1 \\
A_{22} &: \quad f_H \in (0,1), f_L \in (0,1)
\end{aligned}
$$

Definition 2 *An asymptotic fixed point (AFP) is a solution* $(f_H^*, f_L^*, I_H^*, I_L^*, \lambda_H^*, \lambda_L^*)$ *where at least one component in each of the pairs* $(I_H, I_L), (\lambda_H, \lambda_L)$ *comes arbitrarily close to its solution but only converges to it in the limit. At least one equality in each of the following pairs does not hold:* $\{\dot{I}_H = 0, \dot{I}_L = 0\}, \{\dot{\lambda}_H = 0, \dot{\lambda}_L = 0\}$. *The condition* $\dot{f}_H = \dot{f}_L = 0$ *holds.*

We refer to AFPs and FPs jointly as 'equilibria'. The key difference between AFPs and FPs is that some variables at an AFP are not constant. They move towards a constant but only reach it in the limit. There are two potential types of AFPs, each encompassing several potentially optimal policies. They are listed below:

$$
\begin{aligned}
A_{13} &: \quad I_H \to 0, I_L = I_L^*, f_H^* = 1, f_L = f_L^* \in [0,1] \\
A_{31} &: \quad I_L \to 0, I_H = I_H^*, f_L^* = 1, f_H = f_H^* \in [0,1]
\end{aligned}
$$

Overall this gives us a large set of potential equilibria. Let us reduce this set by showing the infeasibility of certain fixed points.

Lemma 3 *All fixed points are of type* $A_{10}, A_{12},$ *or* A_{20}.

Proof. Consider A_{22}. Suppose $f_H \in (0,1)$ and $f_L \in (0,1)$ during a finite interval of time. Then $\lambda_H = \lambda_L = -\frac{c}{\alpha}$ and thus $\dot{\lambda}_H = \dot{\lambda}_L = 0$ within this interval. Subtracting (20) from (19) yields:

$$\frac{c}{\alpha}(\beta_H - \beta_L)(1 - I_L - I_H) = 0$$

This is not possible since the left hand side is strictly positive. This demonstrates that A_{22} does not satisfy the Hamiltonian conditions and is not feasible. Thus, the Hamiltonian conditions imply that at least one of the control variables at a steady state is on the boundary.

Since $I_H, I_L > 0$, we can rewrite the equations of motion as follows:

$$\frac{\dot{I}_H}{I_H} = \beta_H(1 - I_H - I_L) - \tau - f_H\alpha \tag{21}$$

$$\frac{\dot{I}_L}{I_L} = \beta_L(1 - I_H - I_L) - \tau - f_L\alpha \tag{22}$$

At a fixed point the right hand sides of the above equations must be zero. This implies that

$$f_H = \frac{\beta_H(1 - I_H - I_L) - \tau}{\alpha}, \tag{23}$$

$$f_L = \frac{\beta_L(1 - I_H - I_L) - \tau}{\alpha}. \tag{24}$$

Subtracting (24) from (23) yields

$$f_H - f_L = \frac{(\beta_H - \beta_L)(1 - I_H - I_L)}{\alpha} > 0. \tag{25}$$

This is not satisfied by fixed points A_{00}, A_{01}, A_{11}, A_{02} and A_{21}. This reduces the set of feasible FPs to $F = \{A_{10}, A_{12}, A_{20}\}$. ■

3.2.2 Characteristics of fixed points A_{10} and A_{12}

Fixed points A_{10} and A_{12} are the case when $f_H^* = 1$ and $f_L^* = 1$ in the former while $f_L^* \in (0, 1)$ in the latter. By setting $\dot{I}_H = 0$, $\dot{I}_L = 0$ and $f_H^* = 1$ we obtain the treatment levels that characterise these fixed points. Except in the special case of strict equality of $1 \geq \frac{(\beta_H - \beta_L)}{\beta_H}\frac{\tau+\alpha}{\alpha}$, the fixed point is type A_{12}. The fixed point A_{10} is a boundary fixed point and will be addressed in Section 4.2.5. It can be shown that there is a line of fixed points in (I_H, I_L) space of type A_{12} that satisfies the Hamiltonian conditions with the following properties (derivations can be found in the Appendix):

$$
\begin{aligned}
I_H^* + I_L^* &= 1 - \frac{\tau + \alpha}{\beta_H} \\
\lambda_H^* &= -\frac{c}{\alpha}\left[1 + \frac{(\beta_H - \beta_L)}{\beta_H}\frac{\tau + \alpha}{\delta}\right] < -\frac{c}{\alpha} \\
\lambda_L^* &= -\frac{c}{\alpha} \\
f_H^* &= 1 \\
f_L^* &= 1 - \frac{(\beta_H - \beta_L)}{\beta_H}\frac{\tau + \alpha}{\alpha}
\end{aligned}
$$

3.2.3 Characteristics of fixed point A_{20}

Fixed point A_{20} is the case when $f_H^* \in (0,1)$ and $f_L^* = 0$. At fixed point A_{20}, setting $f_L^* = 0$, $\dot{I}_H = 0$ and $\dot{I}_L = 0$ yields the optimal treatment levels. Except in the special case of strict equality of $1 \geq \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\alpha}$, the fixed point is of type A_{20}. In the case of strict equality, this fixed point becomes A_{10}. It can be shown that there is a line of fixed points of type A_{20} that satisfy the Hamiltonian conditions with the following properties:

$$
\begin{aligned}
I_H^{**} + I_L^{**} &= 1 - \frac{\tau}{\beta_L} \\
\lambda_H^{**} &= -\frac{c}{\alpha} \\
\lambda_L^{**} &= -\frac{c}{\alpha}\left[1 - \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\delta}\right] > -\frac{c}{\alpha} \\
f_H^{**} &= \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\alpha} \\
f_L^{**} &= 0
\end{aligned}
$$

Derivations of this can be found in the Appendix. Let us label fixed points A_{12} and A_{20} as Interior Fixed Points (IFPs) for ease of exposition, as they are fixed points that induce one policy instrument to be at an interior level. It is interesting to notice the similarity between the fixed points when policies are at boundary levels in the two-strain and one-strain cases. Setting policy to $(1,1)$ or $(0,0)$ in A_{12} or A_{20} gives us prevalence levels that look similar to what we observed when we had one strain. They differ in the constants due to the fact that the transmission parameter in the two-type case is not homogeneous; rather, it is a weighted average depending on the prevalence of each strain in the population.

3.2.4 Characteristics of asymptotic fixed points

For the analysis of asymptotic fixed points we need to define the concept of a Most Rapid Approach Path (MRAP).

Definition 4 *An MRAP is a path with a policy that ensures convergence to the fixed point in less time than any other policy.*

First, consider A_{13}. I_H tends asymptotically towards zero and I_L converges to some equilibrium level:

$$A_{13} : I_H \longrightarrow 0, I_L = I_L^*$$

For I_H to asymptotically tend to zero, we require $\frac{\dot{I}_H}{I_H} < 0$ at all points in time, for which the MRAP is $f_H^* = 1$. Combining these features gives $\frac{\dot{I}_H}{I_H} = \beta_H(1 - I_H - I_L^*) - \alpha - \tau \approx \beta_H(1 - I_L^*) - \alpha - \tau$, for I_H sufficiently close to zero. This needs to be negative, so the condition required for this to be a feasible AFP is

$$1 - \frac{\alpha + \tau}{\beta_H} < I_L^*. \tag{26}$$

Similarly, I_L converges to I_L^*, which requires $\frac{\dot{I}_L}{I_L} = 0$. Using this we can solve for I_L^*:

$$I_L^* = 1 - \frac{\tau + \alpha f_L^*}{\beta_L}. \tag{27}$$

Thus, (26) simplifies to

$$\frac{\tau + \alpha f_L^*}{\beta_L} < \frac{\alpha + \tau}{\beta_H}. \tag{28}$$

Any f_L^* chosen to satisfy this will cause I_H to converge asymptotically to zero and I_L to I_L^* as defined above. Thus there exists a fixed point of type A_{13}, which involves asymptotic convergence of I_H to zero:

$$\begin{aligned} I_H &\longrightarrow 0 \\ I_L^* &= 1 - \frac{\tau + \alpha f_L^*}{\beta_L} \\ f_H^* &= 1 \end{aligned}$$

Similarly, consider fixed point A_{31}, where I_L tends towards zero asymptotically and I_H converges to I_H^*:

$$A_{31} : I_H = I_H^*, I_L \longrightarrow 0$$

For I_L to tend asymptotically to zero, we require $\frac{\dot{I}_L}{I_L} < 0$ at all points in time, for which the MRAP is $f_L^* = 1$. Combining these features gives $\frac{\dot{I}_L}{I_L} = \beta_L(1 - I_H^* - I_L) - \alpha - \tau \approx \beta_L(1 - I_H^*) - \alpha - \tau$, for I_L sufficiently close to zero. This needs to be negative, which requires

$$1 - \frac{\alpha + \tau}{\beta_L} < I_H^*. \tag{29}$$

Similarly, I_H converges to I_H^*, which requires $\frac{\dot{I}_H}{I_H} = 0$. Using this we can solve for I_H^*:

$$I_H^* = 1 - \frac{\tau + \alpha f_H^*}{\beta_H}. \tag{30}$$

Thus, (29) simplifies to

$$\frac{\tau + \alpha f_H^*}{\beta_H} < \frac{\alpha + \tau}{\beta_L}. \tag{31}$$

Any f_H^* chosen to satisfy this will cause I_L to converge asymptotically to zero and I_H to I_H^* as defined above. Thus, there exists a fixed point of type A_{31}, which involves asymptotic convergence of I_L to zero:

$$\begin{aligned} I_L &\longrightarrow 0 \\ I_H^* &= 1 - \frac{\tau + \alpha f_H^*}{\beta_H} \\ f_L^* &= 1 \end{aligned}$$

Note that both asymptotic fixed points can be feasible at the same time. Rearranging (28) and (31) gives:

$$\begin{aligned} \tau(\frac{1}{\beta_L} - \frac{1}{\beta_H}) &< \frac{\alpha}{\beta_H} - \frac{\alpha f_L^*}{\beta_L}, \\ \tau(\frac{1}{\beta_L} - \frac{1}{\beta_H}) &> \frac{\alpha f_H^*}{\beta_H} - \frac{\alpha}{\beta_L}. \end{aligned}$$

Both conditions can be satisfied as long as $f_H^*, f_L^* < 1$. The case of $f_H = f_L = 1$ deserves further attention and is examined more fully in Section 4.2.6. Although both asymptotic fixed points can be feasible at the same time, asymptotic eradication of both strains of the disease is never possible:

Proposition 5 *Both variants of the disease cannot be simultaneously eradicated even asymptotically in equilibrium, i.e. we cannot have both $I_H^* \to 0$ and $I_L^* \to 0$, if we assume that*

$$\begin{aligned} \frac{\tau + \alpha}{\beta_H} &< 1, \\ \frac{\tau}{\beta_L} &< 1, \\ \frac{\tau + \alpha}{\beta_L} &< 1. \end{aligned}$$

Proof. In the Appendix. ■

This Proposition shows that in the case of the AFPs, one strain of the disease always prevails. This becomes important when it is shown in a later section that under some parameter constellations, the only feasible fixed points are of the asymptotic type and thus in these cases asymptotic eradication of both strains of the disease is not possible.

3.2.5 Regimes of feasibility: fixed points

Following Wagener (2003), the parameter space can be split into different regimes which mandate which fixed points are feasible under every possible parameter constellation. Define the constant K as

$$K = \frac{\beta_H - \beta_L}{\beta_L} \frac{\tau}{\alpha}$$

The parameter space can now be divided into three regimes of feasibility.

Proposition 6 *If $K < 1$, there exist a line of fixed points of type A_{12} and a line of fixed points of type A_{20}. If $K = 1$, there exists a fixed point of type A_{10}. If $K > 1$, there are no ordinary fixed points.*

Proof. In the Appendix. ■

The proof of this Proposition shows that if $K < 1$, there are two lines of fixed points with total infection levels:

$$\begin{aligned} I_H^* + I_L^* &= 1 - \frac{\tau + \alpha}{\beta_H}, \\ I_H^{**} + I_L^{**} &= 1 - \frac{\tau}{\beta_L}. \end{aligned}$$

Subtracting,

$$(I_H^{**} + I_L^{**}) - (I_H^* + I_L^*) = \frac{\alpha}{\beta_H}\left(1 - \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\alpha}\right) > 0$$

This shows that A_{20} always has higher total infection than A_{12}. This is obvious as in the latter, both treatment levels are higher.

3.2.6 Regimes of feasibility: asymptotic fixed points

We examine further the role of K in the feasibility of the AFPs. Let us denote A_{13}^0 as the AFP A_{13} when $f_H^* = 1$ and $f_L^* = 0$. Further denote the AFP A_{13} when $f_H^* = 1$ and $f_L^* = 1$ as A_{13}^1. Last, A_{13}^i is the AFP A_{13} when $f_H^* = 1$ and $f_L^* \in (0, 1)$. Symmetrically, we can define A_{31}^0, A_{31}^i and A_{31}^1 as the AFP A_{31} when $f_H^* = 0, f_H^* \in (0, 1)$ and $f_H^* = 1$ respectively and $f_L^* = 1$ in all cases.

Proposition 7 *If $K < 1$, there exist AFPs of type A_{31}^0, A_{31}^i, A_{31}^1, A_{13}^0 and A_{13}^i. If $K \geq 1$, there exist AFPs of type A_{31}^0, A_{31}^i, and A_{31}^1.*

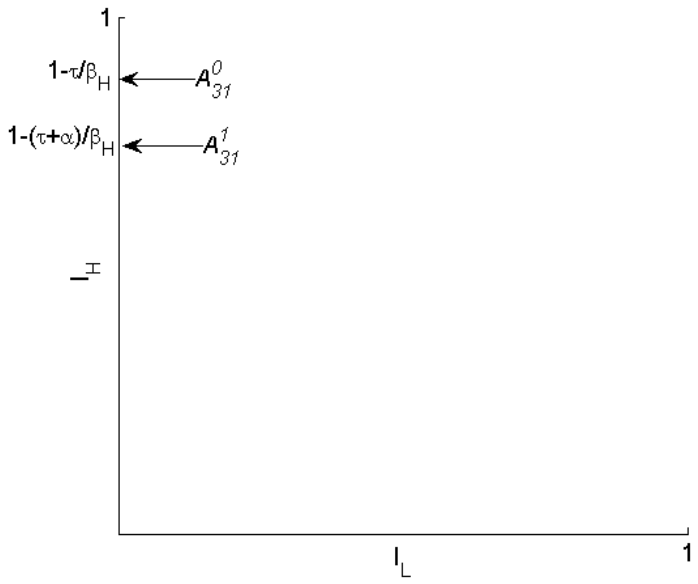

Figure 1: The set of feasible equilibria when $K > 1$.

Proof. In the Appendix. ■

The set of feasible equilibria when $K > 1$ is depicted in Figure 1. The feasible set when $K = 1$ is shown in Figure 2. Figure 3. shows the set of feasible equilibria when $K < 1$. The AFPs with interior policies are not depicted in these graphs as they are only pinned down once the treatment levels are known.

3.2.7 Feasible policy along the path

It is necessary to consider the path towards each of the steady states, and in particular which policies are feasible under which conditions. Policies along the path will always be boundary policies, as these are Most Rapid Approach Paths (MRAPs). We take each of the boundary policies in turn and examine the feasibility conditions required for I_H and I_L to converge to their steady state values. Details of this are provided in the Appendix. Letting P_{ab} denote the policy $f_H = a$, $f_L = b$, the conditions for feasibility are summarised in the table below:

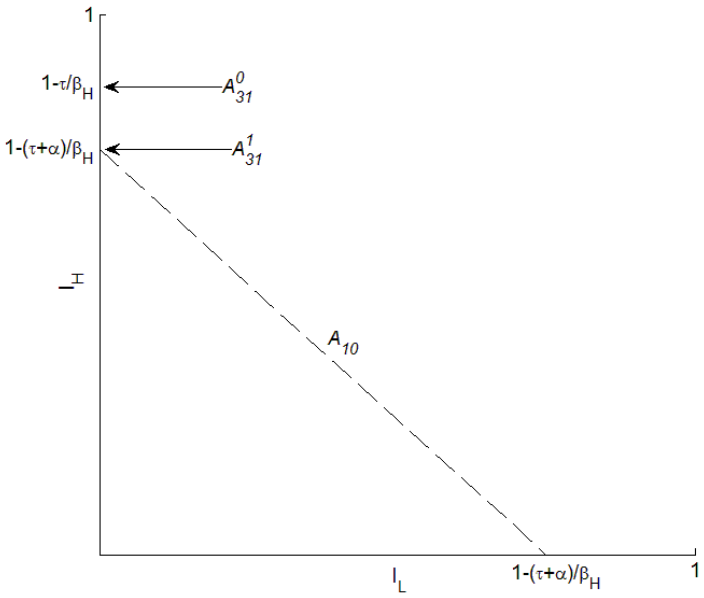

Figure 2: The set of feasible equilibria when $K = 1$.

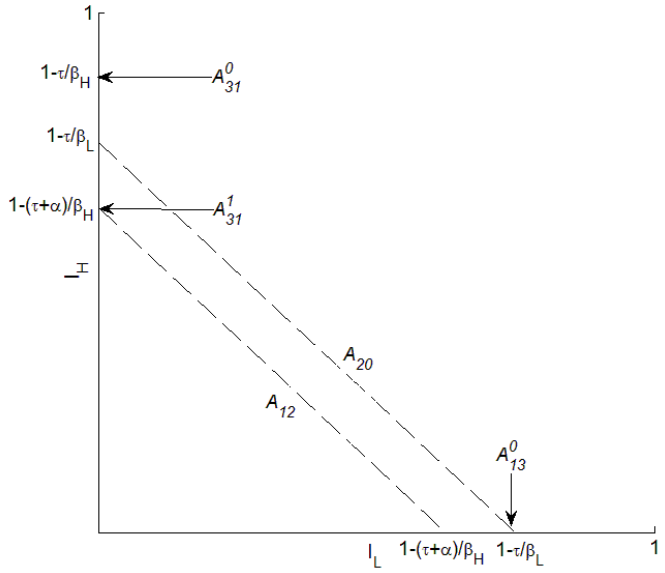

Figure 3: The set of feasible equilibria when $K < 1$.

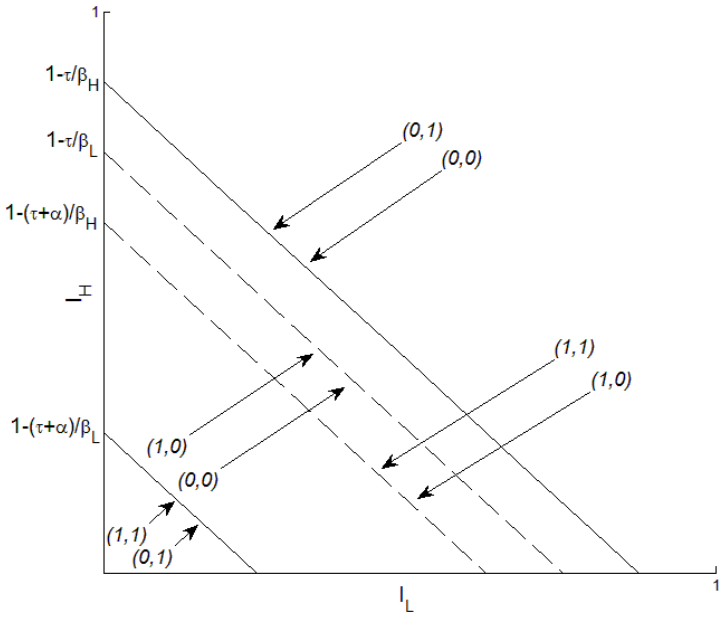

Figure 4: Feasible policies depicted for the case when $K < 1$.

Table 1 (Feasible policies along the path)

	$\dot{I}_H, \dot{I}_L > 0$	$\dot{I}_H, \dot{I}_L < 0$
P_{00}	$1 - \frac{\tau}{\beta_L} > I_H + I_L$	$1 - \frac{\tau}{\beta_H} < I_H + I_L$
P_{10}	$1 - \frac{\tau}{\beta_L} > I_H + I_L$	$1 - \frac{\tau+\alpha}{\beta_H} < I_H + I_L$
P_{11}	$1 - \frac{\tau+\alpha}{\beta_L} > I_H + I_L$	$1 - \frac{\tau+\alpha}{\beta_H} < I_H + I_L$
P_{01}	$1 - \frac{\tau+\alpha}{\beta_L} > I_H + I_L$	$1 - \frac{\tau}{\beta_H} < I_H + I_L$

Figure 4 shows which policies are feasible in different regions of initial infection levels.

3.3 Optimal policy

3.3.1 Optimal policy in the neighbourhood of the IFPs

Having derived feasibility conditions for the various policies, the obvious question is which policies are optimal. We explore the behaviour of the path in approaching each of the interior fixed points. Specifically, what is the policy along the path near to the fixed point? We know that policies along the path will be at a boundary as these are MRAPs. Therefore, we examine those policies that are at an interior level at the steady state, as they are likely to have a switch point along the path. The approach is to perturb the fixed point slightly and derive the policy in the neighbourhood of the fixed point.

Proposition 8 *The optimal policy when approaching A_{12} from above is $f_L^* = 1$. The optimal policy when approaching this fixed point from below is $f_L^* = 0$. The optimal value of f_H^* is equal to its steady state value, $f_H^* = 1$, throughout. For fixed point A_{20}, the optimal policy when approaching from above is $f_H^* = 1$ and $f_H^* = 0$ when approaching from below. The optimal value of f_L^* is at its steady state value, $f_L^* = 0$.*

Proof. In the Appendix. ■

Comparing these optimal policies to the feasibility conditions of the previous section, we find that all of the optimal policies are feasible. From the conditions derived for policy along the path, it is clear that, for example, the upper line $I_H^{**} + I_L^{**} = 1 - \frac{\tau}{\beta_L}$ is attainable from the top using both P_{11} and P_{10}. It is also clear that P_{11} is the MRAP. However, we find that P_{10} is the optimal policy. The intuition for this is as follows. This is because if $f_L^* = 0$ at the fixed point, then $\lambda_L^* > -\frac{c}{\alpha}$. Since λ_L is continuous it must be that $\lambda_L^* > -\frac{c}{\alpha}$ in the vicinity of the fixed point. Hence $f_L^* = 0$ in the vicinity of the fixed point and it cannot be optimal to reach this fixed point with P_{11}. Similar intuition applies for the optimal policy for A_{12}.

3.3.2 Skiba Hypothesis

Definition 9 *A Skiba point is a point of indifference where two separate solutions for the optimal control problem exist (Wagener 2003).*

We hypothesise the existence of a Skiba curve along which conditions prescribe indifference between selecting the path towards A_{20} versus A_{12}. Note that these paths are the optimal paths derived in the previous section. Following on from these results, the hypothesis is that there is a Skiba curve lying between the two lines of fixed points. If the initial point $\left(I_H^0, I_L^0\right)$ lies between the origin and the Skiba curve, then optimal policy is

$$\begin{aligned} f_H &= 1, f_L = 0 \text{ for } I_H^0 + I_L^0 < 1 - \frac{\tau + \alpha}{\beta_H} \\ f_H &= 1, f_L = 1 \text{ for } I_H^0 + I_L^0 > 1 - \frac{\tau + \alpha}{\beta_H} \\ f_H &= 1, f_L = 1 - \frac{(\beta_H - \beta_L)}{\beta_H} \frac{\tau + \alpha}{\alpha} \text{ for } I_H^0 + I_L^0 = 1 - \frac{\tau + \alpha}{\beta_H} \end{aligned}$$

If the initial point $\left(I_H^0, I_L^0\right)$ lies on the opposite side of the Skiba curve from the origin, then optimal policy is

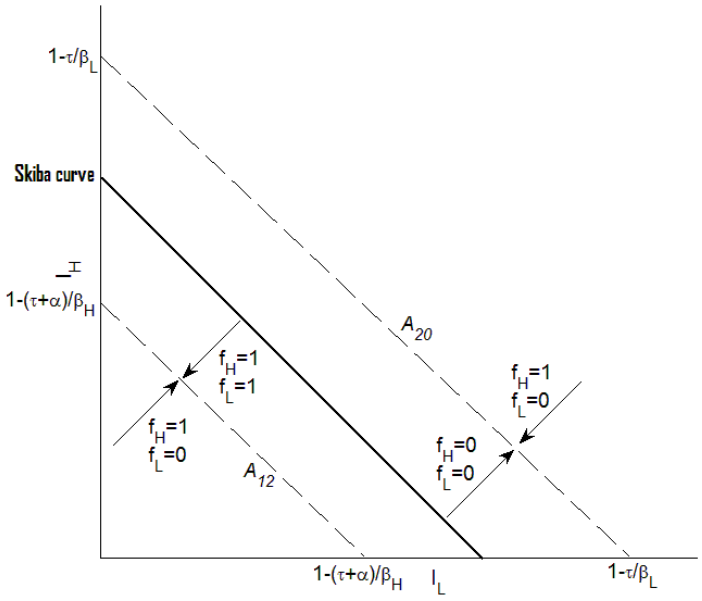

Figure 5: Optimal policies and the Skiba curve.

$$
\begin{aligned}
f_H &= 0, f_L = 0 \text{ for } I_H^0 + I_L^0 < 1 - \frac{\tau}{\beta_L} \\
f_H &= 1, f_L = 0 \text{ for } I_H^0 + I_L^0 > 1 - \frac{\tau}{\beta_L} \\
f_H &= \frac{(\beta_H - \beta_L)}{\beta_L} \frac{\tau}{\alpha}, f_L = 0 \text{ for } I_H^0 + I_L^0 = 1 - \frac{\tau}{\beta_L}
\end{aligned}
$$

Note that all these optimal policies satisfy the feasibility conditions set out in Table 1. The sets of optimal policies are depicted in Figure 5. Skiba curves cannot be derived analytically. Their presence can only be detected by means of simulations.

3.3.3 Optimal policy in the neighbourhood of the AFPs

We have already shown that optimal policy for A_{31} will involve $f_L^* = 1$ as this is the MRAP. Similarly, optimal policy for A_{13} will involve $f_H^* = 1$. The question is which policy is optimal of the range available to f_H in A_{31} and f_L in A_{13}. In order to draw conclusions on this we observe that the asymptotic fixed points always involve one strand of the infection that is asymptotically eradicated. As a result, the behaviour of the system in the neighbourhood of the fixed point can be approximated by the behaviour of a one-infection system. This is because the behaviour of the system for small I_L is very similar to the behaviour when $I_L = 0$. Naturally this holds for I_H close to zero as well.

The behaviour of a one-infection system has been analysed in Section 3. As was discussed, Rowthorn (2004) shows that only extreme values for policy are optimal. This is because a one-infection system has a costate variable that is single-valued in the infection level along the optimal path, which implies that the optimum path cannot be a spiral. In our case the costate variable λ_H is defined as

$$\lambda_H = \frac{\partial V(I_H, I_L)}{\partial I_H},$$

which for small I_L is single-valued along the optimal path. Similarly,

$$\lambda_L = \frac{\partial V(I_H, I_L)}{\partial I_L},$$

which for small I_H is single-valued along the optimal path. Interior policies involve spirals. This implies that optimal policy for the AFPs will only ever involve boundary values, which allows us to eliminate A^i_{31} and A^i_{13} as steady states that are never optimal. Therefore, when $K > 1$, the set of feasible equilibria is $F = \{A^0_{31}, A^1_{31}\}$, one of which will be optimal. When $K < 1$, the set of feasible equilibria is $F = \{A_{12}, A_{20}, A^0_{31}, A^1_{31}, A^0_{13}\}$, one of which will be optimal. Similarly, when $K = 1$, the feasible set is $F = \{A_{10}, A^0_{31}, A^1_{31}\}$. We cannot make any further conclusions on the optimality of these remaining feasible fixed points. Optimality will depend on parameter values. This will be explored by way of simulations in the next section.

4 Simulations

The purpose of simuations is to enable the identification of optimal policy under different parameters. Examples are provided of optimal policy in the case of various parameter assumptions. Simulations are carried out using the fourth-order Runge-Kutta method. Recalling the constant $K = \frac{\beta_H - \beta_L}{\beta_L} \frac{\tau}{\alpha}$, there are three cases that can be evaluated: $K < 1$, $K = 1$ and $K > 1$. We separate these into two cases: the case when $K > 1$ and there are only two feasible fixed points: A^0_{31} and A^1_{31}, and the case when $K \leq 1$ and the interior fixed points are feasible.

4.1 Paths with fixed policy

The case when $K > 1$ is interesting because it suggests that with an appropriate set of parameters, it may be optimal to only eradicate the low infectivity strain, while allowing the high infectivity strain to be endemic, with full or maybe even no treatment. Further, this case will allow the clearest policy recommendations as the number of possible optimal policies is small.

The following parameter assumptions ensure that $K > 1$:

Table 2 (Parameter values)

Parameter	Value
β_H	0.95
β_L	0.4
τ	0.15
α	0.2

In addition, we assume that $p = 1$ and $\delta = 0.111$. The goal is to evaluate whether, under different scenarios, it is better to move towards A^0_{31} or A^1_{31}. In Section 4.2.4 it was shown that in the neighbourhood of A^0_{31}, optimal policy is $(f^*_H, f^*_L) = (0, 1)$. In the neighbourhood of A^1_{31}, optimal policy is $(f^*_H, f^*_L) = (1, 1)$. These policies may not be optimal along the entire path towards these fixed points. However, we begin with a simple thought experiment where we assume that the policymaker can only choose one policy and cannot change it. This may happen in reality, for example, if the policymaker commits to a certain treatment level and purchases the requisite amount of material. Organising additional treatment may take time. Further, there may be political factors as agencies responsible for treatment may not be able to secure additional funds from governements in the short run. We carry out simulations where we assume that this is the case. In the next section, we allow for flexibility of treatment across time. The simulations in this section are carried out with $t = 90$. We can interpret each t as being one day, which implies that the results simulate an infection evolving over approximately 90 days.

Optimal policy is evaluated based on the value of the integral, V, under each policy. We fix policy at the beginning and allow the system to converge to steady state. In order to analyse policy under various scenarios, we focus on the cost paremeter c, which we vary. The initial value for the L infection is constant across all simulations and is set at a value close to zero: $I^0_L = 0.1$. We find that there are three regions of values for c, each of which involve a different optimal policy. These are shown in the table below:

Table 3 (Regions of optimal policy as c varies when $I_L^0 = 0.1$)

Region	c	f_H^*	f_L^*
I (low costs)	$c < 0.2875$	1	1
II (intermediate costs)	$0.2875 \leq c \leq 0.3006$	0 or 1, depending on I_H^0	1
III (high costs)	$c > 0.3006$	0	1

Let us look at examples from each region and compare the value of V when starting at different initial points I_H^0 and setting $f_H = 0$ or $f_H = 1$. Note that when we are at A_{31}^1, $I_H^* = 1 - \frac{\tau+\alpha}{\beta_H} = 0.6316$. When we are at A_{31}^0, $I_H^* = 1 - \frac{\tau}{\beta_H} = 0.8421$. We take five initial infection levels for the H type, distributed evenly across the interval $I_H^0 \in [0.6316, 0.8421]$.

First, consider Region I. Let $c = 0.1$. The table below gives the prevalence of each infection type when steady state is reached and the value of the integral of moving to that steady state. The policy with the higher value of V - the optimal policy - is emphasised in bold.

Table 4 ($c = 0.1$)

	$f_H = 1$ (path towards A_{31}^1)			$f_H = 0$ (path towards A_{31}^0)		
I_H^0	I_L^*	I_H^*	V	I_L^*	I_H^*	V
0.6667	0.0000110	0.6316	**2.3134**	0.00000027	0.8421	1.3883
0.7018	0.0000109	0.6316	**2.2641**	0.00000026	0.8421	1.3425
0.7369	0.0000108	0.6316	**2.2170**	0.00000026	0.8421	1.2989
0.7719	0.0000107	0.6316	**2.1720**	0.00000026	0.8421	1.2572
0.8070	0.0000106	0.6316	**2.1287**	0.00000026	0.8421	1.2172

In this scenario, policy is independent of the initial value. It is always optimal to set $f_H^* = 1$ and treat everyone. As costs rise, we enter Region II. As an example of policy evaluation for costs in this region, we set $c = 0.295$. The table below shows details of the value of the integral and the infection levels for this parameter combination:

Table 5 ($c = 0.295$)

	$f_H = 1$ (path towards A_{31}^1)			$f_H = 0$ (path towards A_{31}^0)		
I_H^0	I_L^*	I_H^*	V	I_L^*	I_H^*	V
0.6667	0.0000110	0.6316	**1.4806**	0.00000027	0.8421	1.4636
0.7018	0.0000109	0.6316	**1.4223**	0.00000026	0.8421	1.4184
0.7369	0.0000108	0.6316	1.3668	0.00000026	0.8421	**1.3753**
0.7719	0.0000107	0.6316	1.3138	0.00000026	0.8421	**1.3342**
0.8070	0.0000106	0.6316	1.2628	0.00000026	0.8421	**1.2947**

From the simulations it is clear that for $I_H^0 \leq 0.7018$, the policy $f_H^* = 1$ is optimal. For $I_H^0 \geq 0.7369$, the policy $f_H^* = 0$ is optimal. However, we can be more specific than this. In the region $I_H^0 \in (0.7018, 0.7369)$, there is a Skiba point as hypothesised, where the initial value is such that policy is indifferent between setting $f_H^* = 0$ and $f_H^* = 1$. Simulations show that this value is $\tilde{I}_H^0 = 0.7125$, where $V = 1.4051$ for both policies. Optimal policy when $c = 0.295$ is summarised in the table below:

Table 6 (Optimal policy when $c = 0.295$)

I_H^0	f_H^*	f_L^*
$I_H^0 < 0.7125$	1	1
$I_H^0 = 0.7125$ (Skiba point)	0 or 1	1
$I_H^0 > 0.7125$	0	1

The remaining region to be considered is Region III, where $c > 0.3006$ and optimal policy is $f_H^* = 0$. Let us take $c = 0.5$ as an example. The table below details the values of the relevant variables from the simulations:

Table 7 ($c = 0.5$)

	$f_H = 1$ (path towards A_{31}^1)			$f_H = 0$ (path towards A_{31}^0)		
I_H^0	I_L^*	I_H^*	V	I_L^*	I_H^*	V
0.6667	0.0000110	0.6316	0.2900	0.00000027	0.8421	**1.3806**
0.7018	0.0000109	0.6316	0.2228	0.00000026	0.8421	**1.3361**
0.7369	0.0000108	0.6316	0.1585	0.00000026	0.8421	**1.2936**
0.7719	0.0000107	0.6316	0.0971	0.00000026	0.8421	**1.2530**
0.8070	0.0000106	0.6316	0.0381	0.00000026	0.8421	**1.2140**

When $c = 0.5$, the optimal policy is $f_H^* = 0$. This is the optimal policy for any c in Region III. Note that all of the above simulations show the same qualitative results for smaller values of I_L^0, namely $I_L^0 = 0.01$, $I_L^0 = 0.001$ and $I_L^0 = 0.0001$.

These simulations show a fairly intuitive result, namely that as costs rise, we move from the optimal treatment of everyone to the optimal treatment of only the L strain. They also demonstrate an interesting finding, whereby there is a small range of costs for which optimal policy is dependent on initial prevalence of infection.

4.2 Hamiltonian paths with variable policy

In this section we allow policy to vary in the case where $K > 1$. We also look at strictly Hamiltonian paths i.e. those that satisfy the Hamiltonian conditions for optimality. In order

to check that paths are Hamiltonian paths, costate variables are required. We examine each initial point studied above and solve for values of the costate variables at these points using the facts that

$$\begin{aligned} \lambda_H &= \frac{\partial V(I_H, I_L)}{\partial I_H}, \\ \lambda_L &= \frac{\partial V(I_H, I_L)}{\partial I_L}. \end{aligned}$$

These partial derivatives can be approximated by perturbing the infection levels slightly. Thus, for initial infection levels I_H^0 and I_L^0,

$$\lambda_H^0 \approx \frac{V(I_H^0 + \Delta, I_L^0) - V(I_H^0, I_L^0)}{\Delta}, \tag{32}$$

$$\lambda_L^0 \approx \frac{V(I_H^0, I_L^0 + \Delta) - V(I_H^0, I_L^0)}{\Delta}, \tag{33}$$

for small Δ. In these simulations we set $\Delta = 0.001$. Table 8 depicts the Hamiltonian conditions required for our two potential policies to be optimal:

Table 8 (Hamiltonian optimality conditions)

Policy	Condition
$f_H^* = 1, f_L^* = 1$	$\lambda_H^* < -\frac{c}{\alpha}, \lambda_L^* < -\frac{c}{\alpha}$
$f_H^* = 0, f_L^* = 1$	$\lambda_H^* > -\frac{c}{\alpha}, \lambda_L^* < -\frac{c}{\alpha}$

In order to test whether our paths are Hamiltonian paths, for each initial infection level we find the initial costate variables, λ_H^0 and λ_L^0, using (32) and (33). We then test whether either of the two candidate policies satisfies the Hamiltonian conditions. If one does, we simulate the path from this initial point, using values for our costate variables to test for optimal policy at each time increment. This allows policy to vary optimally. We then plot graphs of the evolution of the policy variables over time along with the state variables I_H and I_L. This will allow us to see whether there are any switch points (i.e. changes) in policy, and at what levels of I_H and I_L they occur. We also simulate the paths 'backwards', from the initial point in the direction away from steady state. This gives us an indication of the path and any possible switch points before the system reaches $I_H = I_H^0, I_L = I_L^0$. The backwards paths are necessarily Hamiltonian paths as long as the Hamiltonian conditions for the forward paths are satisfied. This is because the backwards paths are merely a continuation, albeit in the opposite direction, of the Hamiltonian paths.

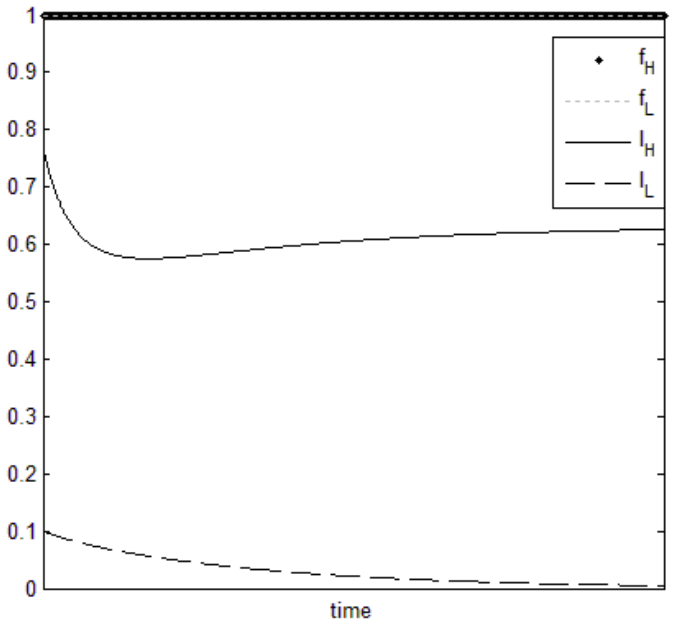

Figure 6: Evolution of system towards fixed point. $I_H^0 = 0.7719, c = 0.1$.

Let us begin with the lowest costs, $c = 0.1$. Solving for the costate variables and checking the Hamiltonian conditions shows that there is a Hamiltonian path from each initial point, with optimal policy $(1, 1)$ along the entire path. There are no switch points. One example of such a path is depicted in Figure 6. The simulations in this section are carried out with $t = 30$. Therefore, the graphs show the infection evolving over approximately 30 days. This path is simple. It is optimal to treat everyone. The control variables do not change over time and the system gradually moves towards the fixed point, with I_H dipping slightly before converging to the steady state level. Fixing policy is optimal. This is in line with our findings from the previous section, where moving to A_{31}^1 was optimal when $c = 0.1$.

When the paths are run backwards, four of the five initial points exhibit a switch point to the policy $(0, 1)$. One example of such a path is given in Figure 7. Careful interpretation of this graph is required. It depicts movement away from steady state, so it in effect needs to be read 'backwards', from right to left. The switch point implies that if we begin at an initial infection level further away from our (I_H^0, I_L^0) - in particular, before the switch point - then optimal policy will begin at $(0, 1)$ and switch to $(1, 1)$, remaining at $(1, 1)$ until convergence.

Further simulations are carried out on paths when $c = 0.295$. Each of our five initial points has a Hamiltonian path. All paths have one switch point. Simulations show that all paths converge to fixed point A_{31}^1, somewhat unexpectedly as, for all, initial optimal policy is $(0, 1)$. These details are shown in Table 9. It is interesting to compare this to the results of the previous section. We find that only treating the L strain is initially optimal, but early on there is an optimal switch to the policy of treating everyone The system never optimally

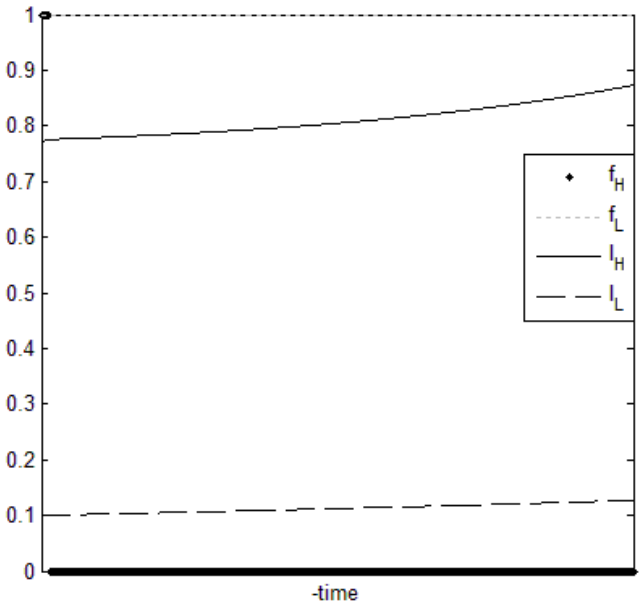

Figure 7: Evolution of system away from fixed point. $I_H^0 = 0.7719, c = 0.1$.

converges to A_{31}^0 in contrast to the case of fixed policy. Policy is dependent on initial values, but not in the same way that we observed in the previous section.

Table 9 (Switch points for Hamiltonian paths when $c = 0.295$)

I_H^0	Initial (f_H^*, f_L^*)	Δf_H^*	Δf_L^*	I_H at switch	I_L at switch	I_H^*
0.6667	$(0, 1)$	$0 \to 1$	None	0.7401	0.0734	0.6316
0.7018	$(0, 1)$	$0 \to 1$	None	0.7482	0.0728	0.6316
0.7369	$(0, 1)$	$0 \to 1$	None	0.7558	0.0722	0.6316
0.7719	$(0, 1)$	$0 \to 1$	None	0.7626	0.0717	0.6316
0.8070	$(0, 1)$	$0 \to 1$	None	0.7692	0.0714	0.6316

An example of one of these paths is depicted in Figure 8. The behaviour is different to what we observed in Figure 6. There is a switch point early on, after which optimal policy is to treat everyone. Prior to the switch point, prevalence of the H strand rises. It then undershoots, growing slightly to converge to the low prevalence steady state. The intuition behind the switch point is that initially, prevalence of the H strand is not high enough to justify full treatment - the marginal cost of an additional infected person is lower than the relative cost of treatment. As I_H rises, there comes a point when this marginal cost exceeds the relative cost of treatment. At this point, policy switches to treating everyone.

When running the path backwards, there is again one switch point. This is shown in Figure 9. Reading the graph from right to left, it is clear that optimal policy begins with

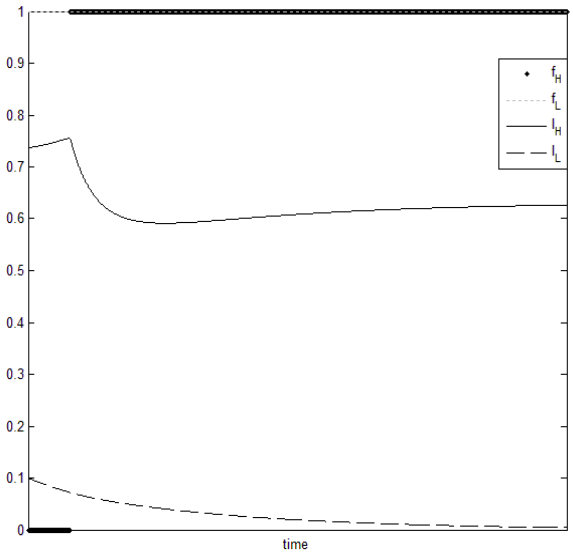

Figure 8: Evolution of system towards fixed point. $I_H^0 = 0.7369, c = 0.295$.

treating no one, with f_L^* switching to full treatment as we approach $I_H = 0.7369$. Thereafter, the path follows what is depicted in Figure 8.

Next, we turn to the example of high costs, when $c = 0.5$. Only three of our five initial points have a Hamiltonian path. Despite each path beginning with only treating the L strain, similar to our results in the previous section with non-variable policy, all three paths switch to the policy of full treatment after a short period and converge to the fixed point with lower infection level. Thus, the paths we derived in the previous section when $c = 0.5$ were not Hamiltonian along their entirety. The details of the switch points when $c = 0.5$ are given in the table below:

Table 10 (Switch points for Hamiltonian paths when $c = 0.5$)

I_H^0	Initial (f_H^*, f_L^*)	Δf_H^*	Δf_L^*	I_H at switch	I_L at switch	I_H^*
0.6667	$(0, 1)$	$0 \to 1$	None	0.7608	0.0619	0.6316
0.7018	$(0, 1)$	$0 \to 1$	None	0.7641	0.0617	0.6316
0.7369	$(0, 1)$	$0 \to 1$	None	0.7672	0.0615	0.6316

Figure 10 depicts an example of such a path. The behaviour of the system is similar to the case when $c = 0.295$. There is a sharper rise in I_H than in the previous example, but the system still converges to A_{31}^1 after the policy switch. The path approaching I_H^0 is shown in Figure 11. When approaching the fixed point, optimal policy does not treat anyone for a large segment of the path. There is a policy switch very close to I_H^0. This is similar to the behaviour observed at low costs, $c = 0.1$, and in contrast to the behaviour observed when $c = 0.295$. This is likely to be because $c = 0.295$ is within the range of costs where optimal

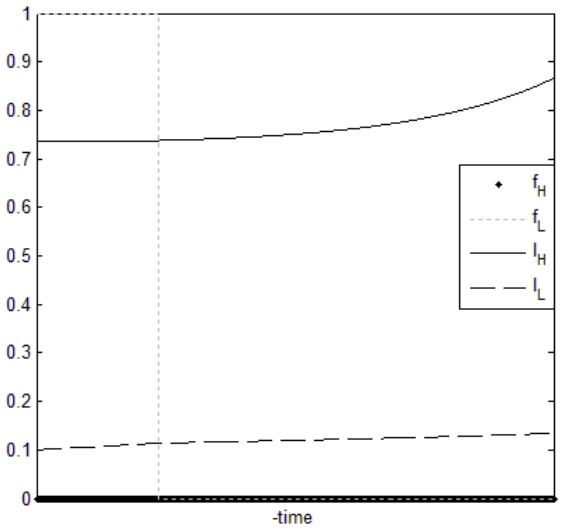

Figure 9: Evolution of system away from fixed point. $I_H^0 = 0.7369, c = 0.295$.

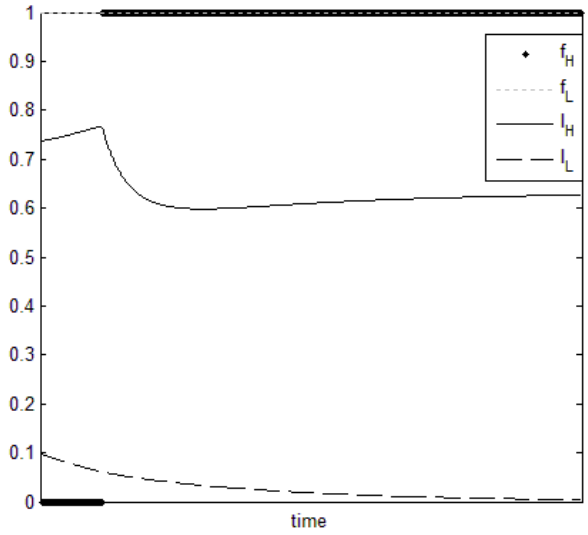

Figure 10: Evolution of system towards fixed point. $I_H^0 = 0.7369, c = 0.5$.

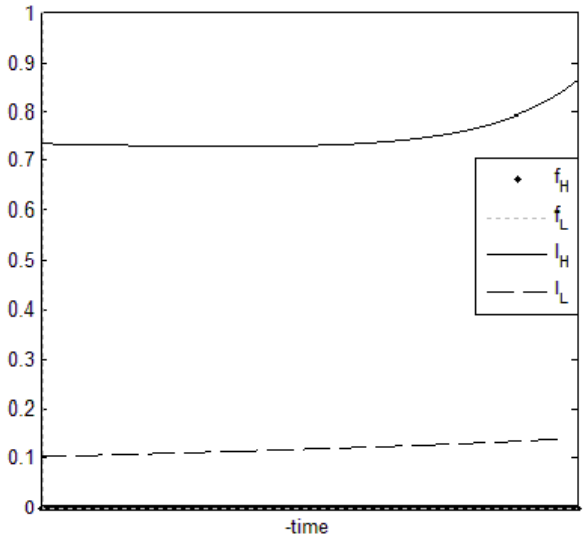

Figure 11: Evolution of system away from fixed point. $I_H^0 = 0.7369, c = 0.5$.

policy is not clear and instead depends on prevalence levels. Thus, we expect to observe more policy switches in this intermediate case than we observe at the extreme cases.

There are several points to take away from these simulations. First, when policy is variable, all simulations converge to the fixed point A_{31}^1. This is interesting because, despite the policy of only treating the L strain being optimal on segments of some of the paths, it is still optimal to converge to the low infection state. Thus, the simulations suggest that it is always better to attempt to lower prevalence of the H strand as much as possible. Another observation to note is that when costs are sufficiently low, our paths are simple, with no switch points. They retain the same optimal policy that they began with, namely treating everyone. Fixed policy is optimal at low cost levels. As costs rise, the paths become more complex. We observe switch points and the policymaker is better off if she has flexibility in her actions. This is relevant in particular for the paths modelled moving away from the fixed point. Here, the extreme examples of $c = 0.1$ and $c = 0.5$ exhibit simple paths with only one policy switch. In contrast, the paths when $c = 0.295$ are complex. This provides evidence for the intuitive idea that when costs are extremely low or extremely high, optimal policy is likely to be simple to predict. When costs are somewhere in the intermediate range, optimal policy is likely to be complex and change frequently over the course of the epidemic.

The examples in this section pointed towards A_{31}^1 as the sole optimal fixed point. We ask ourselves whether there are any parameters such that it is optimal to move to or remain at fixed point A_{31}^0, where I_H^* is higher than at A_{31}^1. This question is addressed in the next section.

4.3 The Skiba curve

In this section we address the case when $K \leq 1$. In this scenario, it is possible to have steady states where both strains of the disease prevail. Section 3.3.2 hypothesised the existence of a Skiba curve between the two lines of interior fixed point. This hypothesis is explored in this section. We set $\beta_H = 0.6$ in order to ensure that $K < 1$. Note that $K = 1$ is a simplified case of $K < 1$ with no Skiba curve, so $K < 1$ is the more interesting case to simulate. All other parameters are the same as in the previous section, apart from costs which are set to $c = 0.6$. We run paths between the two lines of fixed points and look for initial values (I_H^0, I_L^0) where the values of the integral of taking the path towards A_{12} versus A_{20} are equal. We take several paths starting at A_{12} and looking for the Skiba point along each of those paths. We then use these Skiba points to infer the Skiba curve.

The value of total infection and optimal policy at each line of fixed points is given in the table below:

Table 11 (Characteristics of interior fixed points)

Fixed point	$I = I_H + I_L$	f_H^*	f_L^*
A_{12}	$1 - \frac{\tau+\alpha}{\beta_H} = 0.4167$	1	$1 - \frac{(\beta_H - \beta_L)}{\beta_H}\frac{\tau+\alpha}{\alpha} = 0.416$
A_{20}	$1 - \frac{\tau}{\beta_L} = 0.6250$	$\frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\alpha} = 0.375$	0

We split the total value of infection at A_{12}, which we will refer to as I_{12}, into twenty equal segments. This gives us twenty initial points $(I_H^0, I_L^0)_i, i = 1, ..., 20$. We run the path from these points towards A_{20}. In each of these cases it is better to stay at A_{12} than to move towards A_{20}. However, it is also better to stay at A_{20} than to move towards A_{12}. This suggests that there is a point of indifference along each path where moving towards A_{12} versus A_{20} leaves the policymaker indifferent. We take several points along each path and examine the value of the integral of going forward towards A_{20} or back towards A_{12}. We record the (I_H^0, I_L^0) where $V(\text{go to } A_{20}) = V(\text{go to } A_{12})$. In addition, we explore in further detail initial values where the point of indifference is likely to lie *on* one of the fixed point line. Put simply, these are the two points where the Skiba line intersects each fixed point line. Exploring these points with greater precision allows a more accurate Skiba line to be plotted. This is shown in Figure (12).

It is clear that the Skiba curve exists, which confirms our hypothesis. However, it is not a straight line and it intersects each of the fixed point lines. It is also of a slightly steeper slope than the two lines of fixed points. This is because of the higher infectivity of the H strain. When I_H^0 is high and I_L^0 is low, we reach A_{20} faster than when I_H^0 is low and I_L^0 is high. Similarly, when I_H^0 is low and I_L^0 is high, we reach A_{12} faster than when I_H^0 is high and I_L^0 is low. We are more likely to observe indifference closer to A_{20} when I_H^0 is high and closer to A_{12} when I_H^0 is low.

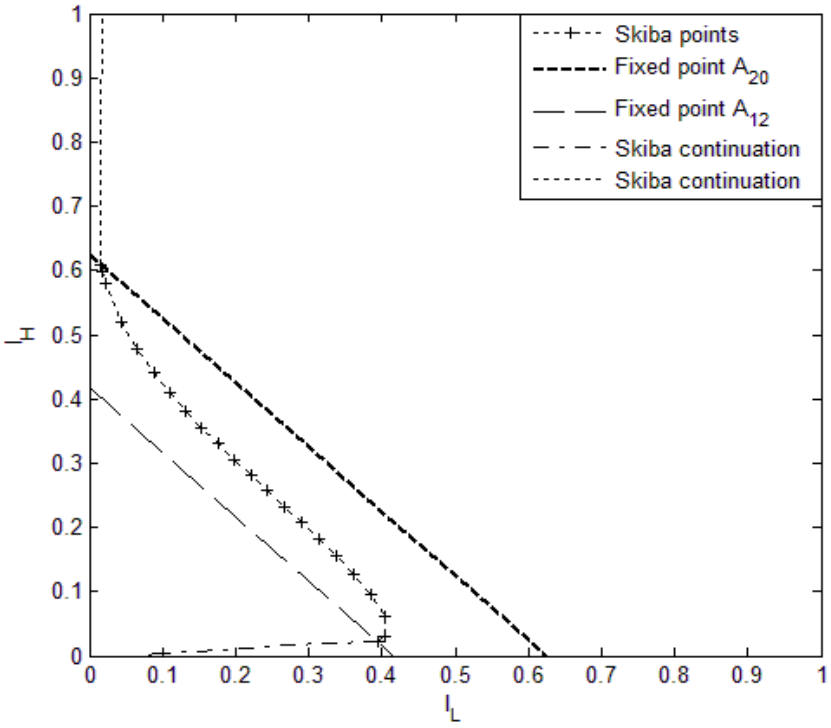

Figure 12: The estimated Skiba curve.

We also hypothesise that the Skiba line continues beyond the points where it intersects the fixed point lines. Specifically, the two paths that begin at the points of intersection with the fixed point lines and move away from each of these lines are assumed to represent the continuation of the Skiba line. We call this continuation, combined with the estimated Skiba points, the 'extended' Skiba line. The extended Skiba line in fact delineates two areas. Above the extended Skiba line, it is always optimal to go to A_{20}, even if this involves crossing fixed point line A_{12}. Below the line, it is always optimal to go to A_{12}, even if this involves crossing A_{20}. It is optimal to go to the fixed point that is further away.

These results show that when costs are high but $K < 1$, we can reach the interior fixed points. There exists a curve of indifference between the two lines where moving towards the fixed point with higher total prevalence is just as valuable as moving towards the fixed point with lower total prevalence. Further, there are areas where although it is faster to reach one fixed point, it is optimal to take the longer path towards the other fixed point. These areas are shown by the Skiba continuation lines. The Skiba curve provides a delineation of two areas, each of which have a clear optimal policy. This is useful: a policymaker tackling any initial infection level that lies in one of these two areas immediately knows the optimal policy for the situation at hand.

4.4 Optimal policy lessons from the simulations

Three sets of simulations were carried out. Each taught us different lessons about optimal policy. First, simulations were carried out with fixed policy, where both infection strains were initially prevalent at a positive level. In this case, treating everyone and converging to a lower prevalence of the H strain was optimal with low costs. As costs rose, optimal policy was ambiguous and began to depend on the initial prevalence level. The further away we started from A^1_{31}, the more likely it was that converging to A^0_{31} would be optimal. As costs rose further, optimal policy unambiguously pointed towards not treating anyone and allowing the system to converge to the fixed point with higher prevalence of H. From this set of simulations we learn that when policy needs to be fixed, optimal policy is definitely dependent on costs and may also be dependent on initial prevalence levels.

The second set of simulations allowed policy to vary. Here we observed that regardless of the costs and the initial prevalence level, the system always optimally instructed us to treat everyone when close to steady state. It was never optimal to allow the H strain to prevail at the higher level. However, the policy almost always exhibited switch points, the exception being the case when $c = 0.1$. Let us consider the intuition behind switch points. In most cases optimal policy began with treating only the L strain, switching to full treatment when the system was closer to equilibrium. This suggests that as long as policy can vary, we can reach A^1_{31} without needing to set $f_H = 1$ throughout. In fact, as long as we are allowed some periods where the I_H types are not treated, it is always optimal to, eventually, bring them down to a low prevalence level. Thus: as long as we have flexibility in the policy instrument, we can always reach A^1_{31}. This is true for intermediate and high costs. The reason why this is not the case with high costs and fixed policy is because treating everyone for the whole duration of the epidemic proves to be too expensive to be worth it. Some interesting lessons were observed from the backwards simulations as well, which simply reinforce the idea that flexibility often induces switch points which allow us to reach 'better' equilibria.

The third set of simulations focused on the Skiba line. It was shown that the Skiba hypothesis is valid: there exists a line of indifference that lies between the two lines of fixed points. There is a further continuation of the Skiba line beyond the two lines of fixed points, which helps delineate two areas that determine which fixed point is optimal. In general, the higher the total infection level $I^0_H + I^0_L$, the more likely it is that movement to A_{20} is optimal. This is intuitive: if the infection is very prevalent, it is more costly to move to the fixed point with lower total infection. For a given *total* initial infection level, the higher the proportion of I^0_H in the total, the more likely it is that fixed point A_{20} is optimal. This is because the H strain is more difficult to treat due to its higher infectivity. Similarly the higher the proportion of I^0_L in the total, the more likely it is that fixed point A_{12} is optimal. Thus, we learn that when we are in the interior region of initial infection levels, two factors determine which policy is optimal: the total initial infection level and its composition of H and L types.

5 Conclusion

This paper has explored an SIS model with two variants of infection differentiated by transmission risk. It has been shown that there are two types of steady states. First, there is a set of fixed points with one treatment level at the boundary and one at an interior level. These fixed points form two lines in (I_H, I_L) space and are only feasible under certain parameter combinations. Only the total level of infection is pinned down here; the distribution of this total infection between the two strains will depend on initial levels. Optimal policy for these steady states is derived; along the path, optimal policy is always at a boundary, after which it may switch to an interior level when steady state is reached. There are also asymptotic fixed points that involve asymptotic eradication of one strand, while the other strand remains endemic. Under the same parameter combinations that eliminate the interior fixed points, we are left only with those asymptotic fixed points that asymptotically eradicate the L strand, leaving the H strand to prevail. This is interesting as it suggests that sometimes it may be optimal for the policymaker to focus treatment on the less infective strand, which may seem counterintuitive. It is also shown that simultaneous asymptotic eradication of both strands is not possible.

Simulations focus on two cases: when only asymptotic eradication of the L strand is feasible, and when the interior fixed points are feasible. In the first case, we consider two situations: when policy is fixed throughout the epidemic and when policy is flexible. We vary costs and compare policy across different parameter combinations. The results are insightful. When policy is fixed, there is a clear relationship between costs and optimal policy. There is a small intermediate range of costs where optimal policy is dependent on initial value. This range contains a Skiba point. When policy is allowed to vary, all paths converge to the steady state with full treatment, even at high costs. At low costs, optimal policy is fixed and there is no added benefit from being able to vary policy. As costs rise, policy exhibits switch points and there is additional benefit from variable policy. In the second case the interior fixed points are feasible and simulations show the Skiba line. The total initial infection level and the proportion of H versus L types determine whether the policymaker should move towards A_{20} or A_{12} or whether he is indifferent between the two.

There are several points to take away from these results. There are many possible steady states, and feasibility will depend on paremeters. Simulations show that in the case of a reduced set of feasible steady states, optimal fixed policy is clearly related to cost of treatment. The non-equality of optimal fixed policy and optimal variable policy in most cases suggests that there is an added benefit from policy flexibility. Therefore, our policy recommendation is that treatment agencies should negotiate flexible terms with their suppliers and their governments so that they have the option to change policy over time.

Further research should consider extending the simulations to look at other parameter combinations. It would also be interesting to extend this model to include protection via

vaccination as another instrument available to the policymaker.

A Analysis of interior fixed points

A.1 Fixed point A_{12}

At fixed point A_{12}, we can characterise the total level of infection:

$$I_H^* + I_L^* = 1 - \frac{\tau + \alpha}{\beta_H}.$$

Further, the equation of motion for λ_H is given by

$$\begin{aligned}
\dot{\lambda}_H &= 0 \\
&= p + cf_H^* + \delta\lambda_H^* - \lambda_H^* \left(\frac{\dot{I}_H}{I_H^*} \right) + (\lambda_H^* \beta_H I_H^* + \lambda_L^* \beta_L I_L^*) \\
&= p + cf_H^* + \delta\lambda_H^* + (\lambda_H^* \beta_H I_H^* + \lambda_L^* \beta_L I_L^*).
\end{aligned}$$

where the second equality follows from the fact that $\dot{I}_H = 0$ at a steady state. Similarly,

$$\begin{aligned}
\dot{\lambda}_L &= 0 \\
&= p + cf_L^* + \delta\lambda_L^* - \lambda_L^* \left(\frac{\dot{I}_L}{I_L^*} \right) + (\lambda_H^* \beta_H I_H^* + \lambda_L^* \beta_L I_L^*) \\
&= p + cf_L^* + \delta\lambda_L^* + (\lambda_H^* \beta_H I_H^* + \lambda_L^* \beta_L I_L^*).
\end{aligned}$$

By subtraction,

$$c(f_H^* - f_L^*) + \delta(\lambda_H^* - \lambda_L^*) = 0.$$

Since f_L^* is interior, it must be that $\lambda_L^* = -\frac{c}{\alpha}$. Thus,

$$\lambda_H^* = -\frac{c}{\alpha} \left[1 + \frac{(\beta_H - \beta_L)}{\beta_H} \frac{\tau + \alpha}{\delta} \right].$$

Since $\beta_H - \beta_L > 0$, it follows that $\lambda_H^* < -\frac{c}{\alpha}$, as required by the Hamiltonian conditions. Thus, there is a line of fixed points in (I_H, I_L) space of type A_{12} that satisfies the Hamiltonian conditions with the following properties:

$$
\begin{aligned}
I_H^* + I_L^* &= 1 - \frac{\tau + \alpha}{\beta_H} \\
\lambda_H^* &= -\frac{c}{\alpha}\left[1 + \frac{(\beta_H - \beta_L)}{\beta_H}\frac{\tau + \alpha}{\delta}\right] < -\frac{c}{\alpha} \\
\lambda_L^* &= -\frac{c}{\alpha} \\
f_H^* &= 1 \\
f_L^* &= 1 - \frac{(\beta_H - \beta_L)}{\beta_H}\frac{\tau + \alpha}{\alpha}
\end{aligned}
$$

A.2 Fixed point A_{20}

Rearranging $\dot{I}_H = 0$ and $\dot{I}_L = 0$ gives us the total level of infection,

$$I_H^{**} + I_L^{**} = 1 - \frac{\tau}{\beta_L}.$$

The equations of motion for the costate variables are,

$$
\begin{aligned}
\dot{\lambda}_H &= 0 \\
&= p + cf_H^{**} + \delta\lambda_H^{**} - \lambda_H^{**}\left(\frac{\dot{I}_H}{I_H^{**}}\right) + (\lambda_H^{**}\beta_H I_H^{**} + \lambda_L^{**}\beta_L I_L^{**}) \\
&= p + cf_H^{**} + \delta\lambda_H^{**} + (\lambda_H^{**}\beta_H I_H^{**} + \lambda_L^{**}\beta_L I_L^{**}),
\end{aligned}
$$

$$
\begin{aligned}
\dot{\lambda}_L &= 0 \\
&= p + cf_L^{**} + \delta\lambda_L^{**} - \lambda_L^{**}\left(\frac{\dot{I}_L}{I_L^{**}}\right) + (\lambda_H^{**}\beta_H I_H^{**} + \lambda_L^{**}\beta_L I_L^{**}) \\
&= p + cf_L^{**} + \delta\lambda_L^{**} + (\lambda_H^{**}\beta_H I_H^{**} + \lambda_L^{**}\beta_L I_L^{**}).
\end{aligned}
$$

Employing the same method as in the previous section, we subtract to yield

$$c(f_H^{**} - f_L^{**}) + \delta(\lambda_H^{**} - \lambda_L^{**}) = 0.$$

Since f_H^{**} is interior it must be that $\lambda_H^{**} = -\frac{c}{\alpha}$. Thus,

$$\lambda_L^{**} = -\frac{c}{\alpha}\left[1 - \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\delta}\right].$$

Since $\beta_H - \beta_L > 0$ it follows that $\lambda_L^* > -\frac{c}{\alpha}$ always holds, as required by the Hamiltonian conditions. Thus, there is a line of fixed points of type A_{20} that satisfy the Hamiltonian conditions with the following properties:

$$\begin{aligned}
I_H^{**} + I_L^{**} &= 1 - \frac{\tau}{\beta_L} \\
\lambda_H^{**} &= -\frac{c}{\alpha} \\
\lambda_L^{**} &= -\frac{c}{\alpha}\left[1 - \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\delta}\right] > -\frac{c}{\alpha} \\
f_H^{**} &= \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\alpha} \\
f_L^{**} &= 0
\end{aligned}$$

B Asymptotic eradication

Proposition 10 *Both variants of the disease cannot be simultaneously eradicated even asymptotically in equilibrium, i.e. we cannot have both $I_H^* \to 0$ and $I_L^* \to 0$, if we assume that*

$$\begin{aligned}
\frac{\tau + \alpha}{\beta_H} &< 1, \\
\frac{\tau}{\beta_L} &< 1, \\
\frac{\tau + \alpha}{\beta_L} &< 1.
\end{aligned}$$

Proof. To see this, first consider the case of the interior fixed points. Here, it is trivial. In the case of A_{12}, $I_H^* + I_L^* = 1 - \frac{\alpha+\tau}{\beta_H}$. We cannot have $I_H^* + I_L^* = 0$. Similarly for A_{20} where $I_H^* + I_L^* = 1 - \frac{\tau}{\beta_L}$, it is not possible to have $I_H^* + I_L^* = 0$. Next, consider the asymptotic fixed points. In the case of A_{31}, we know that $I_L^* \to 0$ so the question is what happens to I_H^*. For I_L to tend towards zero asymptotically, the necessary condition is $I_H > 1 - \frac{\alpha+\tau}{\beta_L}$. Clearly $I_H \neq 0$ is necessary for this to be satisfied. Similarly, fixed point A_{13} implies that $I_H^* \to 0$. The necessary condition for this is $I_L > 1 - \frac{\alpha+\tau}{\beta_H}$, which can only be satisfied if $I_L \neq 0$. Thus, both variants of the disease cannot be eradicated in equilibrium, even asymptotically. ■

C Regimes of feasibility

Proposition 11 *If $K < 1$, there exist a line of fixed points of type A_{12} and a line of fixed points of type A_{20}. If $K = 1$, there exists a fixed point of type A_{10}. If $K > 1$, there are no ordinary fixed points.*

Proof. The conditions for the two kinds of interior fixed points to exist are as follows:

$$\begin{aligned} A_{12} &: \quad f_H = 1, f_L \in (0,1) \text{ needs } 1 > \frac{(\beta_H - \beta_L)}{\beta_H}\frac{\tau+\alpha}{\alpha} \\ A_{20} &: \quad f_H \in (0,1), f_L = 0 \text{ needs } 1 > \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\alpha} \end{aligned}$$

These two conditions are, in fact, identical. To see this, consider the following rearrangement of the condition for A_{12}:

$$\begin{aligned} 1 \quad &> \quad \frac{(\beta_H - \beta_L)}{\beta_H}\frac{\tau+\alpha}{\alpha} \\ &\Leftrightarrow \quad \beta_H\alpha > (\beta_H - \beta_L)(\tau+\alpha) \\ &\Leftrightarrow \quad \beta_L(\tau+\alpha) > \beta_H\tau \\ &\Leftrightarrow \quad \beta_L\alpha > (\beta_H - \beta_L)\tau \\ &\Leftrightarrow \quad 1 > \frac{(\beta_H - \beta_L)}{\beta_L}\frac{\tau}{\alpha} \end{aligned}$$

This demonstrates that both conditions are equivalent to $\beta_L(\tau+\alpha) > \beta_H\tau$. The condition $1 > \frac{(\beta_H-\beta_L)}{\beta_L}\frac{\tau}{\alpha}$ is identical to $K < 1$.

If $K = 1$, this implies that $\frac{(\beta_H-\beta_L)}{\beta_L}\frac{\tau}{\alpha} = \frac{(\beta_H-\beta_L)}{\beta_H}\frac{\tau+\alpha}{\alpha} = 1$, and both A_{12} and A_{20} become the fixed point of type A_{10}. Total infection is characterised by the equation:

$$I_H + I_L = 1 - \frac{\tau+\alpha}{\beta_H} = 1 - \frac{\tau}{\beta_L}.$$

If $K > 1$, none of the conditions for A_{12}, A_{20} nor A_{10} are satisfied. Therefore there are no ordinary fixed points. ■

Proposition 12 *If $K < 1$, there exist AFPs of type A^0_{31}, A^i_{31}, A^1_{31}, A^0_{13} and A^i_{13}. If $K \geq 1$, there exist AFPs of type A^0_{31}, A^i_{31}, and A^1_{31}.*

Proof. First, note that the necessary condition for the feasibility of A_{31} ($\frac{\tau+\alpha f^*_H}{\beta_H} < \frac{\alpha+\tau}{\beta_L}$) is satisfied independently of the value of K.

Next, consider $K < 1$ and A_{13}. The necessary condition for this AFP to be feasible is $\frac{\tau+\alpha f_L^*}{\beta_L} < \frac{\alpha+\tau}{\beta_H}$. This is never satisfied for $f_L^* = 1$. However, it may be satisfied for small enough f_L^*. In particular, when $K < 1$, it is satisfied when $f_L^* = 0$. Thus, the AFPs that are feasible when $K < 1$ are $A_{31}^0, A_{31}^i, A_{31}^1, A_{13}^0$ and A_{13}^i. Here, A_{13}^i is defined such that f_L^* is small enough to satisfy the feasibility condition for this AFP.

Next, consider $K \geq 1$. For A_{13} to be a feasible equilibrium, we require $\frac{\tau+\alpha}{\beta_H} > \frac{\tau+\alpha f_L^*}{\beta_L}$, which is violated for all values of f_L^* when $K \geq 1$. Thus, the set of AFPs that are feasible when $K \geq 1$ is A_{31}^0, A_{31}^i and A_{31}^1. ■

D Feasible policy along the path

In order to derive which policies are feasible along the path towards steady state, it is necessary to examine each MRAP policy in turn and derive the conditions that are required for I_H and I_L to converge.

First, consider P_{00}. This policy implies

$$\begin{aligned} \frac{\dot{I}_H}{I_H} &= \beta_H(1 - I_H - I_L) - \tau, \\ \frac{\dot{I}_L}{I_L} &= \beta_L(1 - I_H - I_L) - \tau. \end{aligned}$$

There are two ways of approaching a fixed point with this policy. First, we can have $\dot{I}_H, \dot{I}_L > 0$ (i.e. I_H and I_L are increasing towards I_H^* and I_L^*). The required conditions for this are

$$\begin{aligned} 1 - \frac{\tau}{\beta_L} &> I_H + I_L \\ 1 - \frac{\tau}{\beta_H} &> I_H + I_L \end{aligned}$$

which collapse to

$$1 - \frac{\tau}{\beta_L} > I_H + I_L.$$

Similarly, we can have $\dot{I}_H, \dot{I}_L < 0$ (i.e. I_H and I_L are decreasing towards I_H^* and I_L^*). The required conditions for this are

$$\begin{aligned} 1 - \frac{\tau}{\beta_L} &< I_H + I_L \\ 1 - \frac{\tau}{\beta_H} &< I_H + I_L \end{aligned}$$

which collapse to

$$1 - \frac{\tau}{\beta_H} < I_H + I_L.$$

Next, consider P_{10}. For $\dot{I}_H, \dot{I}_L > 0$, the required conditions are

$$\begin{aligned} 1 - \frac{\tau}{\beta_L} &> I_H + I_L \\ 1 - \frac{\tau + \alpha}{\beta_H} &> I_H + I_L \end{aligned}$$

where the overriding condition is

$$1 - \frac{\tau}{\beta_L} > I_H + I_L.$$

For $\dot{I}_H, \dot{I}_L < 0$, we require

$$\begin{aligned} 1 - \frac{\tau}{\beta_L} &< I_H + I_L \\ 1 - \frac{\tau + \alpha}{\beta_H} &< I_H + I_L \end{aligned}$$

both of which are satisfied when

$$1 - \frac{\tau + \alpha}{\beta_H} < I_H + I_L.$$

Third, take P_{11}. For $\dot{I}_H, \dot{I}_L > 0$, we need to satisfy

$$\begin{aligned} 1 - \frac{\tau + \alpha}{\beta_H} &> I_H + I_L \\ 1 - \frac{\tau + \alpha}{\beta_L} &> I_H + I_L \end{aligned}$$

where the overriding condition is

$$1 - \frac{\tau + \alpha}{\beta_L} > I_H + I_L.$$

For $\dot{I}_H, \dot{I}_L < 0$, we require

$$\begin{aligned} 1 - \frac{\tau + \alpha}{\beta_H} &< I_H + I_L \\ 1 - \frac{\tau + \alpha}{\beta_L} &< I_H + I_L \end{aligned}$$

both of which are satisfied when

$$1 - \frac{\tau + \alpha}{\beta_H} < I_H + I_L.$$

Last, consider P_{01}. For $\dot{I}_H, \dot{I}_L > 0$, we need to satisfy

$$\begin{aligned} 1 - \frac{\tau}{\beta_H} &> I_H + I_L \\ 1 - \frac{\tau + \alpha}{\beta_L} &> I_H + I_L \end{aligned}$$

where the overriding condition is

$$1 - \frac{\tau + \alpha}{\beta_L} > I_H + I_L.$$

For $\dot{I}_H, \dot{I}_L < 0$, we require

$$\begin{aligned} 1-\frac{\tau}{\beta_H} &< I_H+I_L \\ 1-\frac{\tau+\alpha}{\beta_L} &< I_H+I_L \end{aligned}$$

both of which are satisfied when

$$1-\frac{\tau}{\beta_H} < I_H+I_L.$$

E Optimal policy in the region of the IFPs

Proposition 13 *The optimal policy when approaching A_{12} from above is $f_L^*=1$. The optimal policy when approaching this fixed point from below is $f_L^*=0$. The optimal value of f_H^* is equal to its steady state value, $f_H^*=1$, throughout. For fixed point A_{20}, the optimal policy when approaching from above is $f_H^*=1$ and $f_H^*=0$ when approaching from below. The optimal value of f_L^* is at its steady state value, $f_L^*=0$.*

Proof. First, take A_{12}. Let us perturb the solution by changing f_L from f_L^* to $f_L^*+\Delta f_L$ whilst leaving f_H unchanged at its steady state value. Immediately following this change, $\dot{I}_H=0, \dot{I}_L=-\alpha I_L^*\Delta f_L\neq 0, \dot{\lambda}_H=0$ and $\dot{\lambda}_L=0$. Differentiating (20) yields

$$\begin{aligned} \ddot{\lambda}_L &= (c+\alpha\lambda_L^*)\dot{f}_L-\dot{\lambda}_L\left(-\delta+\beta_L(1-I_L^*-I_H^*)-\tau-\alpha f_L^*\right)+\lambda_L^*\beta_L(\dot{I}_H+\dot{I}_L) \\ &\quad +(\dot{\lambda}_H\beta_H I_H^*+\dot{\lambda}_L\beta_L I_L^*)+(\lambda_H^*\beta_H\dot{I}_H+\lambda_L^*\beta_L\dot{I}_L) \\ &= 2\lambda_L^*\beta_L\dot{I}_L \\ &= -2\frac{c}{\alpha}\beta_L(-\alpha I_L^*)\Delta f_L \\ &= 2c\beta_L I_L^*\Delta f_L\neq 0 \end{aligned}$$

Thus, there is a policy switch. To see this, consider the following. If $\Delta f_L>0$ then $\ddot{\lambda}_L>0$ and $\dot{I}_L<0$. Since we require $\lambda_L=-\frac{c}{\alpha}$ at the fixed point, this implies that $\lambda_L<-\frac{c}{\alpha}$ when approaching the fixed point from above. The Hamiltonian conditions imply that $f_L=1$ along this segment of the path. Likewise, if $\Delta f_L<0$ then $\ddot{\lambda}_L<0$ and $\dot{I}_L>0$. This implies that $\lambda_L>-\frac{c}{\alpha}$ when approaching the fixed point from below, and hence from the Hamiltonian conditions it must be that $f_L=0$. Since $\dot{\lambda}_H=0$ and $\lambda_H<-\frac{c}{\alpha}$ at the fixed point, it must be that $\lambda_H<-\frac{c}{\alpha}$ holds on either side of the fixed point, by continuity. This demonstrates that there is a Hamiltonian path to A_{12} which involves boundary values of f_H and f_L until it reaches the fixed point, when it switches to an interior value of f_L. There is no change in f_H.

Next, take A_{20}. Perturb the solution by altering f_H from f_H^{**} to $f_H^{**} + \Delta f_H$, leaving $f_L^* = 0$. Immediately following this change, $\dot{I}_H = -\alpha I_H^{**} \Delta f_H \neq 0$, $\dot{I}_L = 0$, $\dot{\lambda}_H = 0$ and $\dot{\lambda}_L = 0$. Differentiating $\dot{\lambda}_H$ we see that in the proximity to this fixed point,

$$\begin{aligned}
\ddot{\lambda}_H &= (c + \alpha\lambda_H^{**})\dot{f}_H - \dot{\lambda}_H\left(-\delta + \beta_H(1 - I_H^{**} - I_L^{**}) - \tau - \alpha f_H^{**}\right) + \lambda_H^{**}\beta_H(\dot{I}_H + \dot{I}_L) \\
&\quad + (\dot{\lambda}_H\beta_H I_H^{**} + \dot{\lambda}_L\beta_L I_L^{**}) + (\lambda_H^{**}\beta_H\dot{I}_H + \lambda_L^{**}\beta_L\dot{I}_L) \\
&= 2\lambda_H^{**}\beta_H\dot{I}_H \\
&= -2\frac{c}{\alpha}\beta_H(-\alpha I_H^{**})\Delta f_H \\
&= 2c\beta_H I_H^{**}\Delta f_H \neq 0
\end{aligned}$$

Again, there will be a policy switch. If $\Delta f_H > 0$ then $\ddot{\lambda}_H > 0$ and $\dot{I}_H < 0$. This implies that $\lambda_H < -\frac{c}{\alpha}$ when approaching the fixed point from above, and hence from the Hamiltonian conditions it must be that $f_H = 1$. Likewise, if $\Delta f_H < 0$ then $\ddot{\lambda}_H < 0$ and $\dot{I}_H > 0$. This implies that $\lambda_H > -\frac{c}{\alpha}$ when approaching the fixed point from below, and hence it must be the case that $f_H = 0$. By continuity, $\lambda_L > -\frac{c}{\alpha}$ and $f_L = 0$ on both sides of the fixed point. Thus, there is a Hamiltonian path to this fixed point which involves boundary values of f_H and f_L and a switch to an interior value of f_H on reaching the fixed point, while retaining the boundary value for f_L. ■

References

[1] Ahituv, A., V. J. Holz and T. Philipson. "The Responsiveness of the Demand for Condoms to the Local Prevalence of AIDS." *Journal of Human Resources* 31.4 (1996): 869-897.

[2] Anderson, R. M. and R. M. May. *Infectious Diseases of Humans: Dynamics and Control.* Oxford: Oxford University Press, 1991.

[3] Auld, M. C. "Estimating Behavioural Response to the AIDS Epidemic." *Contributions to Economic Analysis and Policy* 5.1 (2006).

[4] Barrett, S. "Global Disease Eradication." *Journal of the European Economic Association* 1.2-3 (2003): 591-600.

[5] Boulier, B. L., T. S. Datta and R. S. Goldfarb. "Vaccination Externalities". *B. E. Journal of Economic Analysis and Policy* 7.1 (2007)

[6] Brito, D.L., E. Sheshinski and M. D. Intriligator. "Externalities and Compulsory Vaccinations." *Journal of Public Economics* 45.1 (1991): 69-90.

[7] Castillo-Chavez, C., W. Huang and J. Li. "Competitive Exclusion and Coexistence of Multiple Strains in an SIS STD Model." *SIAM Journal of Applied Mathematics* 59.5 (1999): 1790-1811.

[8] Dupas, P. "Relative Risks and the Market for Sex: Teenagers, Sugar Daddies and HIV in Kenya." (2005). Mimeo.

[9] Epstein, J. M. "Modelling to Contain Pandemics." *Nature* 460.7256 (2009): 687.

[10] Geoffard, P.-Y. and T. Philipson. "Rational Epidemics and Their Public Control." *International Economic Review* 37.3 (1996): 603-624.

[11] Geoffard, P.-Y. and T. Philipson. "Disease Eradication: Private versus Public Vaccination." *American Economic Review* 87.1 (1997): 222-230.

[12] Gersovitz, M. and J. S. Hammer. "Infectious Diseases, Public Policy and the Marriage of Economics and Epidemiology." *World Bank Research Observer* 18.2 (2003): 129-157.

[13] Gersovitz, M. and J. S. Hammer. "The Economical Control of Infectious Diseases." *Economic Journal* 114.492 (2004): 1-27.

[14] Goldman, S.M. and J. Lightwood. "Cost Optimisation in the SIS Model of Infectious Disease with Treatment." *Topics in Economic Analysis and Policy* 2.1 (2002): 1-22.

[15] Greenwood, J., P. Kircher and M. Tertilt. "An Equilibrium Model of the African HIV/AIDS Epidemic." (2010). Mimeo.

[16] Hethcote, H. W. and J. W. Van Ark. "Epidemiological Models for Heterogeneous Populations: Proportionate Mixing, Parameter Estimation, and Immunization Programs." *Mathematical Biosciences* 84.1 (1987): 85-118.

[17] Hyman, J. M. and J. Li. "Behaviour Changes in SIS STD Models with Selective Mixing." *SIAM Journal of Applied Mathematics* 57.4 (1997): 1082-1094.

[18] Keeling, M. J. and P. Rohani. *Modeling Infectious Diseases in Humans and Animals.* Princeton, New Jersey: Princeton University Press, 2008.

[19] Kremer, M. "Integrating Behavioural Choice into Epidemiological Models of AIDS." *Quarterly Journal of Economics* 111.2 (1996): 549-573.

[20] Oster, E. "Sexually Transmitted Infections, Sexual Behaviour and the HIV/AIDS Epidemic." *Quarterly Journal of Economics* 120.2 (2005): 467-515.

[21] Oster, E. "HIV and Sexual Behaviour Change: Why not Africa?" (2009). Mimeo.

[22] Philipson, T. and R. A. Posner. *Private Choices and Public Health: An Economic Interpretation of the AIDS Epidemic.* Cambridge, MA: Harvard University Press, 1993.

[23] Reluga, T. C. "An SIS Epidemiology Game with Two Subpopulations." *Journal of Biological Dynamics* 3.5 (2009): 515-531.

[24] Rowthorn, R. E., R. Laxminarayan and C. A. Gilligan. "Optimal Control of Epidemics in Metapopulations." *Journal of the Royal Society Interface* 6 (2009): 1135-1144.

[25] Rowthorn, R. E. "Optimal Treatment of Disease Under a Budget Constraint." *Explorations in Environmental and Natural Resource Economics: Essays in Honor of Gardner M. Brown, Jr.* Ed. R. Halvorsen and D. F. Layton. Cheltenham, UK: Edward Elgar Pub., 2006. 20-35.

[26] Rowthorn, R.E. and F. Toxvaerd, "The Optimal Control of Infectious Diseases via Prevention and Treatment." (2011). Mimeo.

[27] St Lawrence, J. S., Kelly, J. A., Diaz, Y.E. et al. "HIV Risk Behaviour Reduction following Intervention with Key Opinion Leaders of Population: An Experimental Analysis." *American Journal of Public Health* 81.2 (1991): 168-171.

[28] Toxvaerd, F. "Recurrent Infection and Externalities in Protection." (2009). Mimeo.

[29] Toxvaerd, F. "Recurrent Infection and Externalities in Treatment" (2009). Mimeo.

[30] Truscott, J., C. Fraser, S. Cauchemez, A. Meeyai, W. Hinsley, C. A. Donnelly, A. Ghani and N. Ferguson. "Essential Epidemiological Mechanisms Underpinning the Transmission Dynamics of Seasonal Influenza. " *Journal of the Royal Society Interface* (2011). Web.

[31] Wagener, F.O.O. "Skiba Points and Heteroclinic Bifurcations, with Applications to the Shallow Lake System." *Journal of Economic Dynamics and Control* 27.9 (2003): 1533-1561.