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Asset price manipulation with several traders

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Abstract

In financial markets with asymmetric information, traders may have an incentive to forgo profitable deals today in order to preserve their informational advantage for future deals. This sort of manipulative behaviour has been studied in markets with one informed trader (Kyle 1985, Chakraborty and Yilmaz 2004). The effect is slower social learning. Using an extension of Glosten and Milgrom's (1985) trading model, we study this effect in markets with N informed traders. As N grows large, each trader's price impact subsides, and so does manipulation in equilibrium. However, the impact of manipulation on social learning can be increasing in N. As N increases, each trader individually manipulates less. But nonetheless, the increased number of manipulative actions introduces enough noise to exacerbate the impact of manipulation on learning.

Keywords: Price manipulation, asset pricing, asymmetric information, Glosten-Milgrom model.

JEL classifications: D80, D82, G10, G14.

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1 Introduction

When a trader receives a favourable tip about a stock, she usually wants to buy it. Buying a lot straight away would give a strong signal of her tip to the market and erode her informational advantage. Thus she may forgo profitable deals today to preserve more of her advantage for future deals. This could be described as price manipulation. It is important for economists and policy-makers to understand price manipulation incentives, because they slow down learning and information revelation in financial markets and make financial cycles more persistent. The existing theoretical literature on the issue concentrates on the stylised case with a single informed trader. The contribution of this paper is to examine whether manipulation incentives prevail when there are several informed traders in the market.

We study a new extension of the Glosten-Milgrom (1985) model, where N traders each get two chances to place orders with competitive but uninformed dealers. Each trader is either informed, and receive good or bad news, or a noise trader. The dealers do not know what type they are facing. Our main results are as follows:

1. In equilibrium, traders often engage in *manipulative behaviour* by choosing not to trade in the first round. They pretend to be noise traders and, in doing so, avoid unfavourable price movements by slowing down the learning process of dealers. They forgo a profitable first-round deal, but the price effect justifies this. The degree of price manipulation is decreasing in N and eventually drops to zero, since one trader's potential price impact is diminished when others are in the market.

2. The adverse *impact of manipulation on social learning,* measured relative to a non-manipulative benchmark, may be increasing in N. In our baseline numerical example, the impact is strongest in markets with about six traders. When manipulation falls a little but does not drop

to zero, its impact on learning is exacerbated because there are now more signals which it can cloud.

3. Manipulation is less pronounced in markets with *a lot of noise or high quality information signals*, because both factors reduce the potential price impact while increasing the profitability of first-round deals, thus discouraging manipulation.

We fully characterise the equilibrium of our model, showing which kinds of manipulative strategies traders may adopt. In particular, traders will engage in 'weak manipulation', which involves not trading at all when they receive news. There is no 'strong manipulation', which involves loss-making trades. We also analytically derive limiting results showing that price manipulation disappears as N grows large and as the amount of information in the markets gets close to zero. We then use numerical solutions to investigate exact comparative statics, the impact of the quality of information on price manipulation, and the impact of price manipulation on social learning.

The most closely related study to ours is Chakraborty and Yilmaz (2004). They show that in an extended Glosten-Milgrom model, a single informed trader with good news will manipulate by selling today if the time horizon is long enough, giving him enough opportunities to profit in the future. They consider an agent to manipulate prices when he engages in loss-making deals, which is called 'strong manipulation' in our paper. Their main result rules out non-manipulative behaviour under certain conditions. Strong manipulation does not occur in our model because we restrict the time horizon to two periods. We expect that similar results to ours would prevail with the stronger definition if the time horizon were longer.

Price manipulation we study can slow down learning in financial markets. News gets absorbed into prices slowly when traders follow manipulative strategies. This has interesting implications for real world markets. For example, consider a situation where rational traders discover a stock price bubble. Abreu and Brunnermeier (2003) show that they may fail to coordinate, thus allowing the bubble to last for a while. However, once the rational traders do attack in their model, the bubble bursts instantly. Historical experience (Kindleberger and Aliber 2011) suggests that the deflation of bubbles is a drawn-out process that starts slowly and then accelerates, rather than an instant crash. Slow learning, as generated by our model, could play a role in explaining this discrepancy.

Of course, slow learning has been explained through many other channels. We refer the reader to Chamley (2004) and Brunnermeier (2001) for more comprehensive surveys. Most prominently, rational traders may 'follow the herd', ignoring their own signals and slowing down the revelation of information. The most famous illustration is Bikhchandani, Hirshleifer and Welch (1992). It was long believed that the herding effect is wiped out in a financial market with efficient pricing, such as the Glosten-Milgrom model, due to the negative results of Avery and Zemsky (1998). However, Park and Sabourian (2011) shows that herding may prevail in a Glosten-Milgrom model with a general signal structure under special circumstances, which are fully characterised in their paper. In our model, herding effects are switched off due to a binary signal structure. Our model should be seen as complementary to herding models in explaining slow learning in financial markets.

The trade-off between first-round and second-round trades in our model is reminiscent of games of delayed investment and endogenous timing. For instance, in Chamley and Gale (1994), Agents trade off the opportunity cost of waiting to invest against the benefit of learning from others' actions. Similar reasoning has been applied to financial market settings by Smith (2000) and Malinova and Park (2012), and this is another important explanation of slow social learning. Note, however, that the trade-off is different in our model - traders have two chances to invest, and trade off the opportunity of placing two profitable orders against the cost of giving away information through the first one.

The definition of price manipulation considered here is weak compared to other definitions in the literature. Importantly, the manipulative strategies used by players in our model would not be considered unlawful (see Kyle (2008) for a discussion of the legal dimension). Also, they do not involve insider trading or the release of false information (see Allen and Gale (1992) for a review of papers on such strategies).

A related definition to ours is considered by Allen and Gale (1992). They show that uninformed traders may manipulate prices by emulating the behaviour of informed traders. The effect we examine is the opposite, and complementary for understanding information in markets. Whereas in their model, bogus information is 'learned' by the market, real information fails to be learned in ours.

Section 2 sets up the model and defines the equilibrium concept. Section 3 fully characterises equilibrium play analytically. Section 4 contains analytical and numerical results on comparative statics, including the effect of the number of traders on price manipulation in equilibrium. Section 5 analyses the impact of price manipulation on social learning. Section 6 concludes.

2 The environment

2.1 The model

We study a market which is open for two trading rounds $t \in \{1, 2\}$. There is one asset, which yields a random cash flow of $V \in \{0, 1\}$ per unit after the end of the second round. Both realisations are equally likely ex ante: $P[V=0] = P[V=1] = \frac{1}{2}$.[1]

[1] Abandoning this assumption would complicate the analysis, since the equilibrium would no longer involve fully symmetric action rules. However, we do not expect it to change our qualitative conclusions.

Trade timing. There are N traders and $M > N$ dealers, and everybody is risk-neutral. The timing in round t is as follows.

1. Each trader is assigned to one dealer by some (potentially random) matching mechanism.

2. Each trader i calls 'his' dealer, who quotes him an ask price A_t^i and a bid price B_t^i.

3. Each trader submits an order X_t^i to his dealer. The allowed orders are: buy one unit at the ask price (written $X_t^i = \mathsf{b}$), hold ($X_t^i = \mathsf{h}$) or sell at the bid price ($X_t^i = \mathsf{s}$).

4. The trades $\left(X_t^1, .., X_t^N\right)$ become public information.

Information. Each trader is *informed* with probability μ, and a *noise trader* with probability $(1-\mu)$. Each trader learns his information status (i.e. whether he is informed or a noise trader) at $t = 1$, and it does not change thereafter. His information status is independent of the information status of others, and of all other random variables considered here.

If i is informed, he receives a signal $S^i \in \{0, 1\}$ at $t = 1$, and no further signals at $t = 2$. The precision of each informed trader's signal is $q = P\left[S^i = v | V = v\right] > \frac{1}{2}$ for $v \in \{0, 1\}$. Thus, $S^i = 1$ is good news and $S^i = 0$ is bad news. Conditional on V, traders' signals are independent of each other.

To summarise the information of trader i, we define the random variable Ω^i as follows: $\Omega^i = g$ if i has good news, i.e. in the event $\{i \text{ is informed} \cap S^i = 1\}$, $I^i = b$ in the event $\{i \text{ is informed} \cap S^i = 0\}$, and $\Omega^i = n$ in the event that i is a noise trader.

Histories. The history of orders before round 1 is $H_1 = \emptyset$. The history of orders before round 2 is $H_2 = \left(X_1^1, ..., X_1^N\right) \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}^N$. The final history of orders before the end of the game is $H_3 = \left(X_t^1, ..., X_t^N\right)_{t \in \{1,2\}} \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}^{2N}$.

For any trader i, the history of actions of other traders before round t is $H_t^{-i} \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}^{N-1}$. All dealers and traders know the history of orders before every round.

Pricing. The dealers' prices are set at a competitive level, for reasons exogenous to the model. By competitiveness, we mean that they make zero expected profits from both buy and sell orders.[2] Hence, prices satisfy

$$\begin{aligned} A_t^i &= E\left[V|H_t, X_t^i = \mathsf{b}\right] \\ B_t^i &= E\left[V|H_t, X_t^i = \mathsf{s}\right] \end{aligned} \tag{1}$$

Note that dealers know the identity of the traders to whom they are quoting prices. Hence, not all traders will be quoted the same prices. A trader's past actions influence dealers' belief about his signals, and therefore the prices he will be quoted today.

One final piece of notation will be useful. In the first round, no trader knows exactly what prices he will be quoted in the second, because he does not know the actions of others, which affect market learning. The second-period prices trader i expects to be quoted, having placed the order x in the first period and received information Ω^i, are defined as

$$\begin{aligned} \bar{A}_2^i(\omega, \mathsf{x}) &= E\left[A_2^i|\Omega^i = \omega, X_1^i = \mathsf{x}\right] \\ \bar{B}_2^i(\omega, \mathsf{x}) &= E\left[B_2^i|\Omega^i = \omega, X_1^i = \mathsf{x}\right] \end{aligned} \tag{2}$$

[2]Assuming exogenous competitive price setting follows Glosten and Milgrom (1985). By making dealers' actions exogenous, we eliminate the need to specify responses to non-competitive price setting as part of traders' strategies, which simplifies the analysis. An easy way to make these price setting rules arise endogenously would be to assume that each trader is assigned to two dealers, both of whom quote him bid and ask prices. Bertrand competition between each trader's two dealers implies zero profits.

2.2 Strategies and equilibrium

The behaviour of noise traders is exogenously given. In each round, a noise trader buys, holds or sells with equal probabilities. A noise trader's actions are independent across periods, and independent of the actions of other traders. We now describe the strategic behaviour of informed traders.

Action rules. An *action rule* for trader i contingent on an event is a probability distribution over the action set $\{\mathsf{b}, \mathsf{h}, \mathsf{s}\}$. An action rule specifies whether, upon observing the event, trader i buys, hold, sells or randomises.

Strategies. A *strategy* for trader i prescribes what informed traders will do in every possible state of information. Formally, a strategy for i consists of

1. First-period action rules contingent on the events $\Omega^i = g$ and $\Omega^i{=}b$ (i.e. two action rules).

2. Second-period action rules contingent on the events $\{\Omega^i = g \cap H_2 = h_2\}$ and $\{\Omega^i = b \cap H_2 = h_2\}$ for all $h_2 \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}^N$ (i.e. $2 \times \left(3^N\right)$ action rules).

A strategy profile contains a strategy for each trader. We restrict attention to *symmetric strategy profiles.* The properties of a symmetric strategy profile are that every trader has the same action rules, and that their action rules contingent on bad news are the mirror image of their action rules contingent on good news. The action rules specified by a symmetric strategy profile satisfy the following for $t = 1, 2$ and all i

$$\begin{aligned} P\left[X_t^i = 0|\Omega^i = g, H_t\right] &= \alpha_t = P\left[X_t^i = 0|\Omega^i = b, H_t\right] \\ P\left[X_t^i = -1|\Omega^i = g, H_t\right] &= \beta_t = P\left[X_t^i = 1|\Omega^i = b, H_2\right] \\ P\left[X_t^i = 1|\Omega^i = g, H_t\right] &= 1 - \alpha_t - \beta_t = P\left[X_t^i = -1|\Omega^i = b, H_t\right] \end{aligned} \tag{3}$$

Equilibrium. A *Symmetric Perfect Bayesian Equilibrium* (henceforth simply called *'equilibrium'*) is a symmetric strategy profile such that

1. No informed trader can increase their expected payoff by deviating from the action rules specified by his strategy, taking competitive price setting behaviour of dealers as given.

2. The beliefs of dealers used to set competitive prices are formed by Bayesian updating, consistent with the action rules specified by traders' strategies.

3 Equilibrium behaviour

Informed traders in the standard Glosten-Milgrom model 'follow their signal'. That is, traders with good news buy with probability 1 and traders with bad news sell with probability 1. Things are different in our model. Consider a trader with good news deciding what order to place in the first round. By Lemma 1, his valuation exceeds the ask price, so buying would be profitable for him.

However, given then chance to trade again, the trader with good news may prefer to hold or sell. Doing so means forgoing a profitable trade or even making a loss, so why would he do so? Because these orders, typical of traders with bad news and noise traders, may be perceived as negative signals by dealers and depress future prices. Because he knows he will buy in the second round, he may be able to secure himself a better deal by not buying now. This is the *price manipulation incentive* this paper focuses on.

Recall from the definition of symmetric strategy profiles in Equation (3) that β_t is the probability that a trader places an order 'against his signal' in round t. We call this *strong manipulation,* which involves placing a loss-making order to manipulate prices. α_t is the probability that a trader does not trade despite having received a signal. We call this *weak manipulation*, which merely inolves forgoing a profitable deal to manipulate prices.

The following result states that market prices quoted to i will always lie below i's current asset valuation if i has good news ($\Omega^i = g$), and above it if i has bad news ($\Omega^i = b$).

Lemma 1. *For $t = 1, 2$ and all i, we have*

$$\begin{aligned} E\left[V|H_t, \Omega^i = b\right] &< A_t^i < E\left[V|H_t, \Omega^i = g\right] \\ E\left[V|H_t, \Omega^i = b\right] &< B_t^i < E\left[V|H_t, \Omega^i = g\right] \end{aligned}$$

Proof. See Appendix A □

The following result will be useful for analysing trading strategies. It states that at $t = 1$ each trader has a limited impact on the prices he expects to be quoted at $t = 2$.

Lemma 2. *For all orders $\mathsf{x}, \mathsf{x}' \in \{-1, 0, 1\}$ and all $\omega \in \{g, b\}$, we have*

$$\begin{aligned} \bar{A}_2^i(\omega, \mathsf{x}) - \bar{A}_2^i(\omega, \mathsf{x}') &< 2q - 1 \\ \bar{B}_2^i(\omega, \mathsf{x}') - \bar{B}_2^i(\omega, \mathsf{x}') &< 2q - 1 \end{aligned}$$

Proof. See Appendix A. □

Now, a backward induction argument helps us to establish that in the second period, there is no manipulation. Because the order at $t = 2$ is each traders' last action, he has no strategic considerations except for comparing the prices he is quoted to what he thinks the asset is worth. By Lemma 1, he will always consider the asset more valuable than the ask price, and hence buy it, if he has good news. Conversely, he will always sell if he has bad news.

Proposition 1. *In any equilibrium, $\alpha_2 = \beta_2 = 0$.*

Proof. See Appendix A. □

Chakraborty and Yilmaz (2004) show that when a single informed trader gets to act repeatedly, he will engage in strong manipulation if the time horizon is long enough. It turns out that two periods are never long enough, and the most we are going to find in equilibrium is weak manipulation.

Proposition 2 formalises this idea. The logic of the proof is instructive. The costs of strong manipulation are considerable, since one has to place a loss-making order in the first period. Thus strong manipulation can only occur in equilibrium if the benefits of manipulation, embodied in the price impact of the manipulative action, are very high. But we demonstrated in Lemma 2 that the price impact is inherently limited. This limit is sufficient to rule out strong manipulation in equilibrium.

Proposition 2. *In any equilibrium,* $\beta_1 = 0$.

Proof. Assume that $\beta_1 > 0$. Suppose a trader with good news deviates from her strategy at $t = 1$ by buying with probability $1 - \alpha_1$ and holding with probability α_1. Since by Proposition 1, she will buy with probability 1 in the second round, this deviation gives her an expected profit of

$$\alpha_1 \left[q - \bar{A}_2^i \left(g, \mathsf{h}\right)\right] + \left(1 - \alpha_1\right) \left[2q - A_1^i - \bar{A}_2^i \left(g, \mathsf{b}\right)\right]$$

Following her strategy yields a profit of

$$\alpha_1 \left[q - \bar{A}_2^i \left(g, \mathsf{h}\right)\right] + \beta_1 \left[B_1^i - \bar{A}_2^i \left(g, \mathsf{s}\right)\right] + \left(1 - \alpha_1 - \beta_1\right) \left[2q - A_1^i - \bar{A}_2^i \left(g, \mathsf{b}\right)\right]$$

Hence the gain from deviating is

$$\beta_1 \left[2q - \left(A_1^i + B_1^i\right) - \left(\bar{A}_2^i \left(g, \mathsf{b}\right) - \bar{A}_2^i \left(g, \mathsf{s}\right)\right)\right]$$

Symmetric strategies imply that $A_1^i = 1 - B_1^i$, so that by Lemma 1, the gain is strictly positive. This profitable deviation contradicts equilibrium. □

Now we know that in the first period, informed traders either follow their signal or hold (i.e. weakly manipulate) in equilibrium. An equilibrium can then be described by the single parameter $\alpha_1 \in [0,1]$ specifying the probability of weak manipulation at $t = 1$. The equilibrium value of α_1 is a convenient measure of the *degree of price manipulation*. The higher it is, the less likely informed traders are to follow their signals in the first round.

Take an equilibrium described by α_1, and consider the incentives of a trader with good news in the first period. He believes the asset to be worth q. Also, he will definitely buy in the second round, and expects to execute this trade at the ask price $\bar{A}_2^i(g, \mathsf{b})$ if he places order $\mathsf{x} \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}$ in the first round. Hence holding gives him an expected payoff of $q - \bar{A}_2^i(g, \mathsf{h})$, whereas buying gives him

$$2q - A_1^i - \bar{A}_2^i(g, \mathsf{b})$$

. The two payoffs depend on the strategy α_1 through the prices, which are formed by Bayesian updating given strategies. Define $G(\alpha_1)$ as the difference between the two:

$$\begin{aligned} G(\alpha_1) &\equiv \left[\bar{A}_2^i(g, \mathsf{b}) - \bar{A}_2^i(g, \mathsf{h})\right] - \left[q - A_1^i\right] \qquad (4) \\ &= \text{price impact} \;-\; \text{profit lost} \end{aligned}$$

G neatly summarises the incentives to manipulate for a trader with good news. The cost of manipulation is the profit lost by not buying today. The benefit is the negative impact manipulation has on the expected ask price tomorrow. The trader strictly prefers to buy if $G < 0$, is indifferent if $G = 0$ and strictly prefers to manipulate, i.e. hold, if $G > 0$.

Now consider the incentives of a trader with bad news. Holding gives him $\bar{B}_2^i(b, \mathsf{h}) - (1 - q)$, whereas selling gives him $B_1^i + \bar{B}_2^i(b, \mathsf{s}) - 2(1 - q)$. Let $\hat{G}(\alpha_1) \equiv \left[\bar{B}_2^i(b, \mathsf{h}) - \bar{B}_2^i(b, \mathsf{s})\right] - [B - (1 - q)]$. He strictly prefers selling if $\hat{G} < 0$, is indifferent when $\hat{G} = 0$ and strictly prefers holding if $\hat{G} > 0$. The symmetry of the model and the strategies allow us to derive the following result:

Lemma 3. *In a symmetric equilibrium, the manipulation incentives given good and bad news are exactly the same, or $G = \hat{G}$.*

Proof. See Appendix A. □

Lemma 3 is the reason why we can find symmetric equilibria - if traders with good and bad news didn't have the same incentives, it would not be optimal for them to play the same strategy. We can now completely characterise symmetric equilibria and prove their existence.

Proposition 3. *There exists an equilibrium. Furthermore, α_1 describes an equilibrium if and only if one of the following holds:*

1. $G(\alpha_1) = 0$ and $\alpha_1 \in \left(0, \frac{1}{2}\right)$.
2. $G(\alpha_1) \leq 0$ and $\alpha_1 = 0$.

Proof. We start with the second part of the proposition.

(i) "only if" direction. Let α_1 describe a symmetric equilibrium. Firstly, suppose $\alpha_1 \geq \frac{1}{2}$. Then equilibrium requires that $G(\alpha_1) \geq 0$. But in this case, a hold followed by a buy is 'better news' from the perspective of the market maker than two consecutive buys. It follows that $\bar{A}_2^i(g, \mathsf{h}) > \bar{A}_2^i(g, \mathsf{b})$ and hence $G(\alpha_1) < 0$, contradicting equilibrium. Secondly, suppose $0 < \alpha_1 < \frac{1}{2}$. Then a trader with good news must be indifferent between holding and buying, so that $G = 0$ and the first condition holds. Finally, suppose $\alpha_1 = 0$. Then we must have $G(\alpha_1) \geq 0$ and the second condition holds.

(ii) "if" direction. If one of the statements holds, then by construction, a trader with good news has no profitable deviation from α. For a trader with bad news, optimality follows from the fact that $G = \hat{G}$.

To show existence, suppose the second statement holds. If $G(0) \leq 0$, then $\alpha_1 = 0$ is an equilbrium by the second condition. If $G(0) > 0$, note that G is continuous in α_1 and that $G\left(\frac{1}{2}\right) < 0$. Hence, there exists an α_1 satisfying the first condition. □

4 Comparative statics

One of the key questions of the paper is how the number of traders (N) affects the equilibrium degree of price manipulation (α_1). Another interesting issue is how the information structure affects α_1. The parameter μ captures the quantity of information in the market because it measures what proportion of traders is informed on average. The parameter q measures the precision of their signals, which captures the quality of information. We now consider each of these parameters in turn.

4.1 The number of traders N

As the number of traders grows large, we would expect incentives to manipulate prices eventually disappear. As Equation (4) illustrates, the incentives to manipulate are strong when the lost profits from not trading in the first round are small, or when a trader's individual price impact is large. The price impact vanishes as N becomes large, whereas the lost profits are always strictly positive.

To prove this formally, we first use the law of large numbers to show that i's expected ask price $\bar{A}_2^i(\omega, \mathsf{x})$ converges to a fixed quantity as $N \to \infty$, regardless of the order x that he placed in the first round. When dealers observe a large number of first-period orders other than i's, a single order from i cannot move their beliefs by much.

Lemma 4. *For all $\omega \in \{g, b, n\}$ and all $\mathsf{x} \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}$,*

$$\lim_{N \to \infty} \bar{A}_2^i (\omega, \mathsf{x}) = P \left[V = 1 | \Omega^i = \omega \right]$$

Proof. See Appendix A. □

Next, we confirm that price manipulation in equilibrium does indeed disappear as N gets large.

Proposition 4. *For given parameters μ and q, there exists an integer $\bar{N}$ such that when $N > \bar{N}$, the only equilibrium is $\alpha_1 = 0$.*

Proof. Suppose not. Let α_1^{max} be the largest value of α_1 that describes an equilibrium in a market with N traders. Then for all N_0 there exists an $N > N_0$ with $\alpha_1^{max} > 0$. By Proposition 3, $\alpha_1^{max} > 0$ implies $G(\alpha_1^{max}) = 0$. We find a contradiction by showing that (taking a convergent subsequence if necessary), $\lim_{N \to \infty} G(\alpha_1^{max}) < 0$.

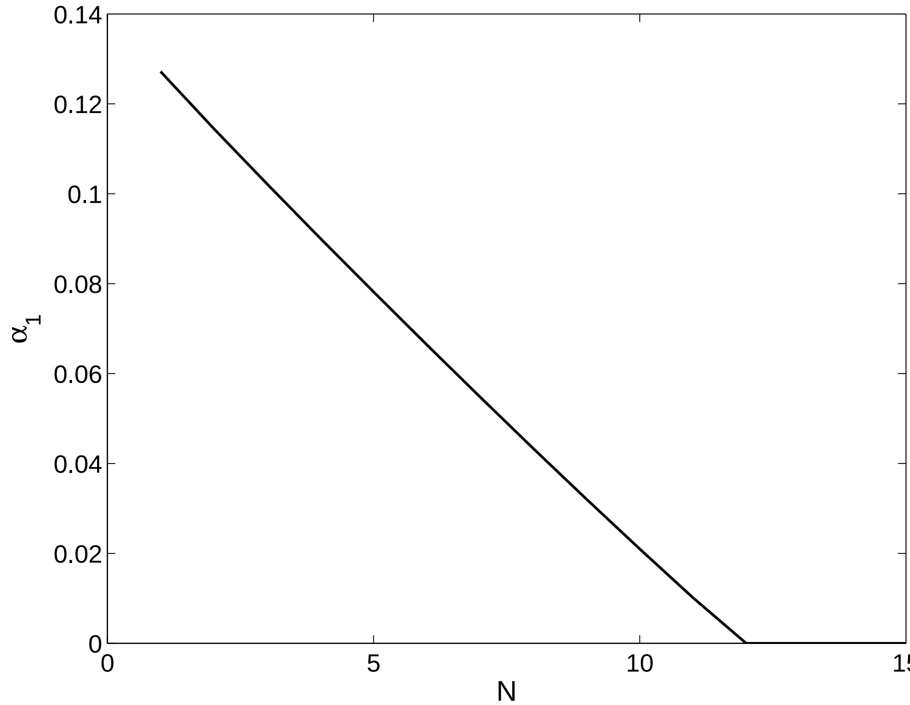
By Lemma 4, the 'price impact' term in $G(\alpha_1^{max})$ converges to zero. But then $G(\alpha_1^{max})$ converges to the same limit as $-[q - A_1^i]$, which is strictly negative as long as $\mu > 0$. □

In general, the equilibrium conditions in Proposition 3 are highly non-linear in the strategy α_1, and the equilibrium cannot be solved for analytically. Hence, use numerical solutions to evaluate how exactly the degree of price manipulation depends on N.[3] Appendix C explains the computations involved.

Figure 1 shows that the degree of price manipulation is *decreasing in N*. This can again be explained intuitively by examining the characterisation of

[3]This is relatively uncontroversial in our context, because our model only has three parameters and the robustness of the numerical results can be verified quite generally by considering different parameterisations. Also, although we have not proved uniqueness of equilibrium, all our numerical solutions yielded a unique equilibrium, which is the one we report here.

Figure 1: The degree of price manipulation and the number of traders[1]



[1] Based on $\mu = 0.5$ and $q = 0.75$.

manipulation incentives in Equation (4). An individual trader's price impact declines as N grows. However, lost profits are not affected by the number of other traders: When first-round prices are quoted, dealers have not observed any actions of other traders.

However, the figure also illustrates that the rate of decline of the degree of price manipulation is quite slow. α_1 appears to decline approximately linearly as N grows. For instance, in order to halve the degree of price manipulation, one would have to increase N from one to seven. In order to eliminate it altogether, one would have to set $N \geq 12$.

4.2 The quantity (μ) and quality (q) of information

As the quantity of information diminishes, we would again expect the incentives to manipulate prices to disappear. Intuitively, if a dealer thinks it unlikely that traders have information, then he will not pay much attention to trades when setting prices. Hence, the price impact term in Equation (4) vanishes, whereas the lost profits remain strictly negative. We have the following result:

Proposition 5. *For given parameters N and q, there exists a $\bar{\mu} \in (0,1)$ such that when $\mu < \bar{\mu}$, the only equilibrium is $\alpha_1 = 0$.*

Proof. Suppose not. Let α_1^{max} be the largest value of α_1 that describes an equilibrium in a market with N traders. Then for μ_0, there exists a $\mu < \mu_0$ with $\alpha_1^{max} > 0$. By Proposition 3, $\alpha_1^{max} > 0$ implies $G\left(\alpha_1^{max}\right) = 0$. We find a contradiction by showing that (taking a convergent subsequence if necessary), $\lim_{\mu \to 0} G\left(\alpha_1^{max}\right) < 0$.

Applying Bayes' rule to the competitive prices in Equation (1), we find that ask prices quoted to trader i satisfy

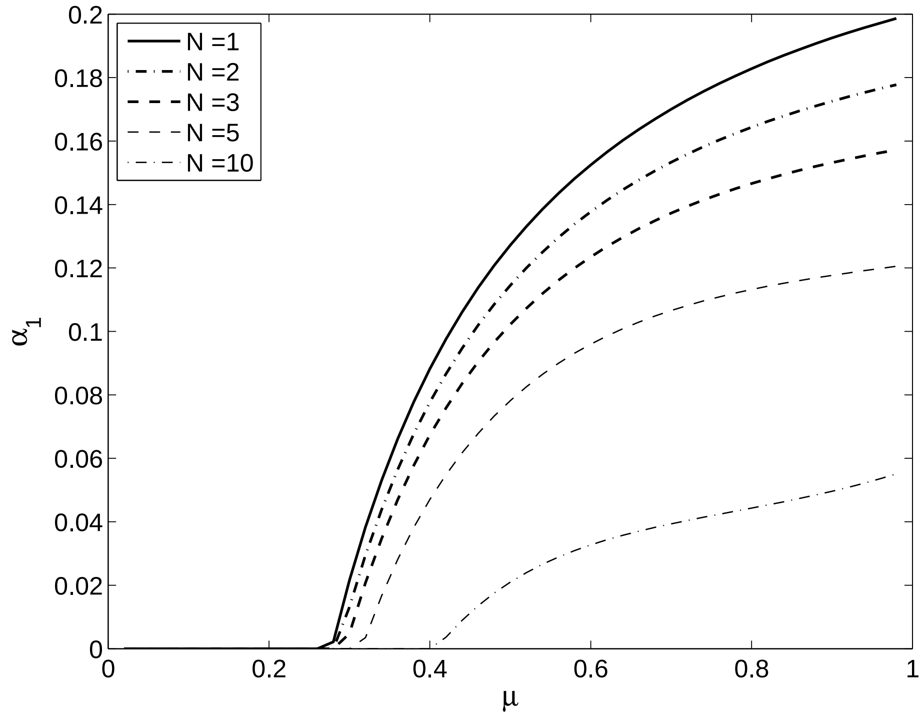
$$A_t^i = \frac{1}{1 + \frac{P\left[H_t, X_t^i = 1 | V = 0\right]}{P\left[H_t, X_t^i = 1 | V = 1\right]}}$$

Let χ denote the event that everybody is a noise trader. We have $P\left[\chi\right] = (1-\mu)^N$, which converges to 1 as $\mu \to 0$. So for all v, $P\left[H_t, X_t^i = 1 | V = v\right]$ converges to

$$P\left[H_t, X_t^i = 1 | V = v, \chi\right] = \frac{1}{3^{(t-1)N+1}}$$

Hence it follows that $A_t^i \to \frac{1}{2}$ with probability 1. Hence it follows from Equation (4) that $G\left(\alpha_1^{max}\right) \to -\left[q - \frac{1}{2}\right] < 0$ as required. □

Figure 2: The degree of price manipulation and the amount of information[1]

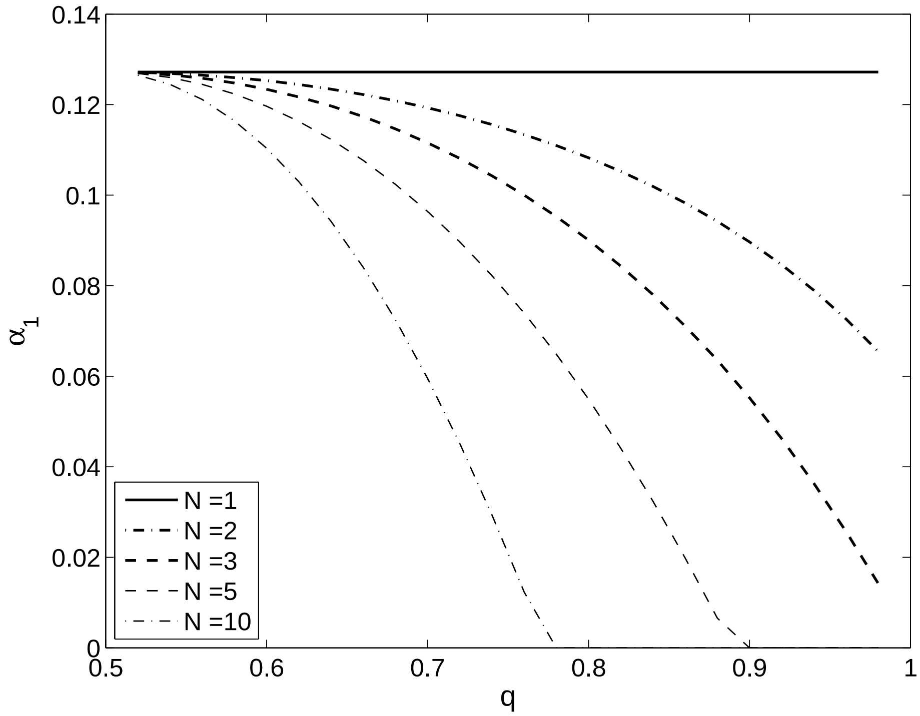


[1] Based on $q = 0.75$.

Again, we show how exactly the degree of price manipulation α_1 depends on μ using numerical solutions. Figure 2 shows that it is *increasing in* μ - there is less manipulation in noisy markets. When there is very little noise (high μ), the lost profits are small because the informational advantage of an informed trader is small, and the price impact is large because actions contain a lot of information.

We also use numerical solutions to show how the degree of price manpulation α_1 depends on the quality of information q. Figure 3 shows that it is *decreasing in* q, except in the case with one trader, where it does not depend on q. There is less manipulation when information is good.

Figure 3: The degree of price manipulation and the quality of information[1]



[1] Based on $\mu = 0.5$.

The intuition is slightly less obvious. Note that lost profits are small when information is bad (low q), again because an informed trader has a smaller informational advantage over dealers. But we would also expect bad information to reduce a trader's price impact because actions contain little information. Which effect dominates? It turns out that they exactly cancel each other out for the case with one trader. But the second effect becomes weaker with several traders: The price impact is diminished by the presence of others anyway, so that the quality of information becomes less important.

5 Social learning

Another central question of this paper is how the price manpulation incentives we have studied affect social learning in financial markets. Clearly, manipulation has an adverse impact on social learning because it means that actions will be less correlated with signals and therefore not contain as much information about the true value.

Let us start by finding a measure of social learning more precise. The *public belief* after the round t is

$$\begin{aligned} P_t &= E\left[V | \left(X_s^1, ..., X_s^N\right)_{s \leq t}\right] \\ &= E\left[V | H_{t+1}\right] \end{aligned} \quad (5)$$

This is an uninformed agent's valuation of the asset, having observed traders' orders up to t. It is a random variable, since it depends on trades, and can be interpreted as a statistical estimator for V.

Our measure of social learning is the *mean-square error* of the post-trade belief. It captures the precision of the uninformed agent's estimate of V after observing trades.[4] We define it as

$$\text{MSE}\left(P_t\right) = E\left[\left(P_t - V\right)^2\right] \quad (6)$$

The formulae necessary to calculate this are explained in Appendix B.

We now set up non-manipulative benchmark, to which we will compare equilibrium behaviour. We calculate what social learning would be if traders had non-manipulative strategies in equilibrium ($\alpha_1 = 0$). To see why this is sensible, suppose we replaced our traders by two identical generations of

[4]Note that the post-trade belief, being the conditional expectation of V, is already the minimum mean-square error estimator of V. Hence its mean-square error measures how well the public can possibly do in estimating V after observing trades.

short-lived traders who only get to place one order, thus wiping out any manipulative incentives. Then, as in the standard Glosten-Milgrom setting, every trader would follow their signal, which corresponds to the strategy $\alpha_1 = 0$. Changing α_1 to zero will affect the way an uninformed agent learns from market data, and hence the post-trade will not be equal to P_t. We denote the post-trade belief in the benchmark setting by P_t^0.

Our measure of the impact price manipulation incentives have on social learning is the percentage impact of manipulation on social learning, which was previously defined as the MSE of post-trade beliefs. It is defined as

$$\Pi_t = 100 \left(\frac{\text{MSE}\,(P_t)}{\text{MSE}\,(P_t^0)} - 1 \right) \tag{7}$$

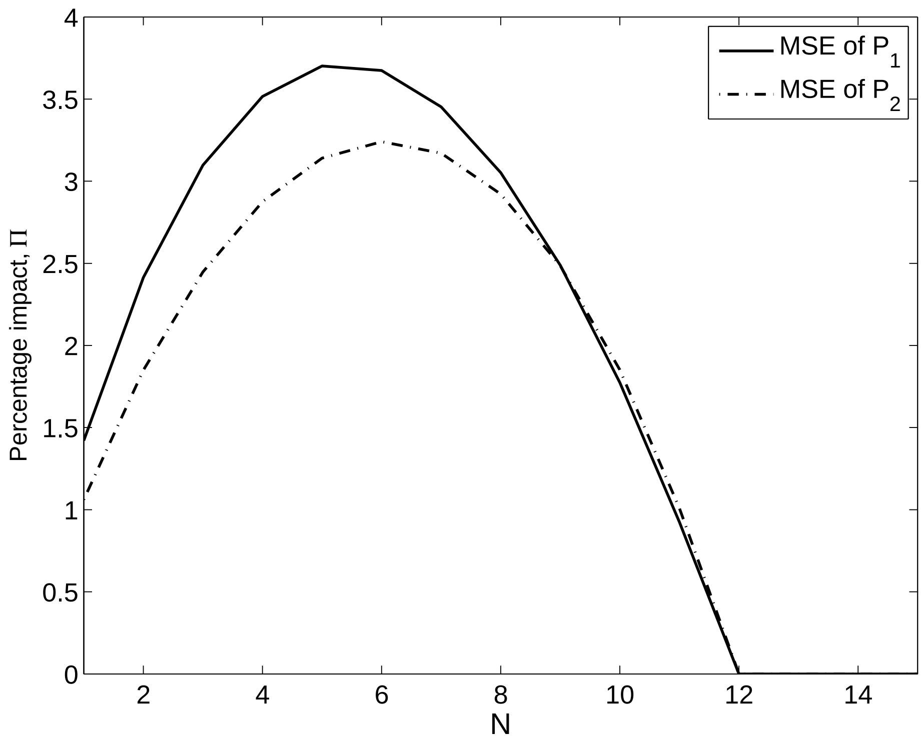
Figure 4 shows the impact of price manipulation on social learning Π_t as a function of N. For both periods, the impact is increasing in N over a significant range and peaks at around $N = 5$. This is in contrast to our results in Section 4, which showed that the *degree* of price manipulation α was monotonically decreasing in N. In other words, while the presence of more traders decreases manipulation incentives, it may increase their impact on social learning.

This result is counterintuitive at first glance. Let us disentangle the different effects behind it. When we add an extra trader, we add an extra signal from which the public can learn by observing trades. Also, in our model, manipulation incentives decrease, improving the informational content of traders. It is clear that both effects improve social learning.

But when we think about social learning *relative to the benchmark*, there are two counteracting forces.

1. The benefit of the extra signal is stronger in the benchmark without manipulation, because the information content of the new trader's orders is higher. Hence learning relative to the benchmark worsens, and

Figure 4: The impact of price manipulation on social learning[1]



[1] Based on $\mu = 0.5$ and $q = 0.75$.

the ratio $\frac{MSE(P_t)}{MSE(P_t^0)}$ rises, driving up the impact measure Π_t. This force is strongest for low values of N because this is when manipulation is most prevalent.

2. The benefit of reduced manipulation incentives is non-existent in the benchmark, because the degree of manipulation is already zero. Hence learning relative to the benchmark improves, driving down Π_t. This force is strongest for high values of N because manipulation incentives are close to zero.

These two forces help us rationalise the inverse-U shape observed in Figure 4. For low values of N, the first force dominates, driving up the impact of manipulation on learning. The opposite is true for high N.

6 Conclusion

Our analysis shows that price manipulation incentives can be important, even when there are several informed traders. When we consider markets with large numbers of traders, these incentives disappear. But in markets with intermediate numbers of traders, they remain important. Importantly, their impact on social learning is often larger than in it would be in a market with one trader.

The main conclusion of this paper is therefore that manipulation incentives should be taken seriously when thinking about learning in financial markets in general, and financial boom and bust cycles in particular. The argument that they are only relevant in stylised models with monopolistic traders should be applied with caution.

We have also shown some interesting links between manipulation incentives and other elements of the market structure. One needs to be less wary of manipulation incentives in markets that are considered very noisy, and in markets where the information is considered to be very precise.

This paper is only a first step towards understanding manipulation incentives and their impact on learning with several traders. It provides a motivation for further research on the issue. Firstly, one could extend the analysis to a model with more than two trading rounds. This might strengthen manipulation incentives as there would be more future prices to manipulate, like in Chakraborty and Yilmaz (2004). However, it is a challenging project since the characterisation of equilibrium strategies becomes very complex. Secondly, one could examine whether similar effects prevail in different models of market microstructure, such as limit order markets.

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Appendix

A Proofs

Lemma 1

Lemma. *For $t = 1, 2$ and all i, we have*

$$E\left[V|H_t, \Omega^i = b\right] < A_t^i < E\left[V|H_t, \Omega^i = g\right]$$
$$E\left[V|H_t, \Omega^i = b\right] < B_t^i < E\left[V|H_t, \Omega^i = g\right]$$

Proof. From the pricing Equations in (1), we have $A_t^i = E\left[V|H_t, X_t^i = \mathsf{b}\right]$ and $B_t^i = E\left[V|H_t, X_t^i = \mathsf{s}\right]$. It is sufficient to show that

$$E\left[V|H_t, \Omega^i = b\right] < E\left[V|H_t, X_t^i\right] < E\left[V|H_t, \Omega^i = g\right]$$

. Firstly, by the law of iterated expectations, and using Lemma B.5,

$$\begin{aligned} E\left[V|H_t, X_t^i\right] &= E\left[E\left[V|H_t, X_t^i, \Omega^i\right]|H_t, X_t^i\right] \\ &= E\left[E\left[V|H_t, \Omega^i\right]|H_t, X_t^i\right] \\ &= \sum_{\omega\in\{g,b,n\}} P\left[\Omega^i = \omega|H_t, X_t^i\right] E\left[V|H_t, \Omega^i = \omega\right] \\ &< E\left[V|H_t, \Omega^i = g\right] \end{aligned}$$

An identical argument shows that $E\left[V|H_t, X_t^i\right] > E\left[V|H_t, \Omega^i = b\right]$, which completes the proof. □

Lemma 2

Lemma. *For all orders* $\mathsf{x}, \mathsf{x}' \in \{-1, 0, 1\}$ *and all* $\omega \in \{g, b\}$*, we have*

$$\begin{aligned} \bar{A}_2^i(\omega, \mathsf{x}) - \bar{A}_2^i(\omega, \mathsf{x}') &< 2q - 1 \\ \bar{B}_2^i(\omega, \mathsf{x}') - \bar{B}_2^i(\omega, \mathsf{x}') &< 2q - 1 \end{aligned}$$

Proof. Let $\mathcal{A}_2^i(\mathsf{x}) = E\left[V|H_2^{-i}, X_1^i = \mathsf{x}, X_2^i = 1\right]$, which is the second-period ask price i will be quoted having placed order x in the first round. By Lemmas 1 and B.5, we know that for all x,

$$E\left[V|H_2^{-i}, \Omega^i = b\right] < \mathcal{A}_2^i(\mathsf{x}) < E\left[V|H_2^{-i}, \Omega^i = g\right]$$

Hence it follows from (3.) in Lemma B.5 that for all x and x'

$$\mathcal{A}_2^i(\mathsf{x}) - \mathcal{A}_2^i(\mathsf{x}') < 2q - 1$$

with probability 1. Finally, observe that

$$\begin{aligned}
\bar{A}_2^i(\omega, \mathsf{x}) - \bar{A}_2^i(\omega, \mathsf{x}') &= E\left[A_2^i | \Omega^i = \omega, X_1^i = \mathsf{x}\right] - E\left[A_2^i | \Omega^i = \omega, X_1^i = \mathsf{x}'\right] \\
&= E\left[\mathcal{A}_2^i(\mathsf{x}) - \mathcal{A}_2^i(\mathsf{x}') | \Omega^i = \omega\right] \\
&< 2q - 1
\end{aligned}$$

Next, let $\mathcal{B}_2^i(\mathsf{x}) = E\left[V | H_2^{-i}, X_1^i = \mathsf{x}, X_2^i = -1\right]$. Repeating the above argument shows that for all x and x', $\mathcal{B}_2^i(\mathsf{x}) - \mathcal{B}_2^i(\mathsf{x}) < 2q - 1$ with probability 1, and that

$$\begin{aligned}
\bar{B}_2^i(\omega, \mathsf{x}) - \bar{B}_2^i(\omega, \mathsf{x}') &= E\left[\mathcal{B}_2^i(\mathsf{x}) - \mathcal{B}_2^i(\mathsf{x}') | \Omega^i = \omega\right] \\
&< 2q - 1
\end{aligned}$$

□

Proposition 1

Proposition. *In any equilibrium,* $\alpha_2 = \beta_2 = 0$.

Proof. Assume that $\alpha_2 > 0$ or $\beta_2 > 0$ for some history $h_2 \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}^N$. Suppose a trader with good news deviates from her strategy at h_2 by buying with probability 1. This deviation gives her an expected continuation profit of

$$E\left[V - A_2^i | H_2 = h_2, \Omega^i = g\right]$$

Following her strategy yields a profit of

$$\beta_2 \cdot E\left[B_2^i - V|H_2 = h_2, \Omega^i = g\right] + (1 - \alpha_2 - \beta_2) \cdot E\left[V - A_2^i|H_2 = h_2, \Omega^i = g\right]$$

Hence the gain from deviating is

$$(\alpha_2 + \beta_2)\, E\left[V - A_2^i|H_2 = h_2, \Omega^i = g\right] + \beta_2 E\left[V - B_2^i|H_2 = h_2, \Omega^i = g\right]$$

which is strictly positive by Lemma 1. This profitable deviation contradicts equilibrium □

Lemma 3

Lemma. *In a symmetric equilibrium, the manipulation incentives given good and bad news are exactly the same, or* $G = \hat{G}$*.*

Proof. Firstly, note that in a symmetric equilibrium, $A_1^i + B_1^i = 1$ and hence $B_1^i - (1-q) = q - A_1^i$. It is then sufficient to show that

$$\bar{A}_2^i\left(g, \mathsf{b}\right) - \bar{A}_2^i\left(g, \mathsf{h}\right) = \bar{B}_2^i\left(b, \mathsf{h}\right) - \bar{B}_2^i\left(b, \mathsf{s}\right)$$

Let $\mathcal{A}_2^i\left(h, \mathsf{x}\right) = E\left[V|H_2^{-i} = h, X_1^i = \mathsf{x}, X_2^i = 1\right]$, which is the second-period ask price i will be quoted having placed order x in the first round if other traders place orders $h \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}^{N-1}$. Hence we need to show that

$$\sum_{h \in \{\mathsf{b},\mathsf{h},\mathsf{s}\}^{N-1}} P\left[H_2^{-i} = h|\Omega^i = g\right]\left[\mathcal{A}_2^i\left(h, \mathsf{b}\right) - \mathcal{A}_2^i\left(h, \mathsf{h}\right)\right]$$
$$= \sum_{h \in \{\mathsf{b},\mathsf{h},\mathsf{s}\}^{N-1}} P\left[H_2^{-i} = h|\Omega^i = b\right]\left[\mathcal{B}_2^i\left(h, \mathsf{h}\right) - \mathcal{B}_2^i\left(h, \mathsf{s}\right)\right]$$

For any history $h = \left(X_1^j\right)_{j \neq i}$ of first-round orders from traders other than i, define the 'inverse history $\hat{h} = \left(\hat{X}_1^j\right)_{j \neq i}$ as follows:

$$\hat{X}_1^j = \begin{cases} \mathsf{b} & \text{if } X_1^j = \mathsf{s} \\ \mathsf{h} & \text{if } X_1^j = \mathsf{h} \\ \mathsf{s} & \text{if } X_1^j = \mathsf{b} \end{cases}$$

Note that $h \to \hat{h}$ is a one-to-one mapping from $\{\mathsf{b}, \mathsf{h}, \mathsf{s}\}^{N-1}$ into itself, so that for any function $g : \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}^{N-1} \to \mathbb{R}$, we have

$$\sum_{h \in \{\mathsf{b},\mathsf{h},\mathsf{s}\}^{N-1}} g(h) = \sum_{h \in \{\mathsf{b},\mathsf{h},\mathsf{s}\}^{N-1}} g\left(\hat{h}\right) \tag{8}$$

Moreover, it can be shown that due to the symmetry of strategies,

$$\mathcal{A}_2^i(h, \mathsf{b}) - \mathcal{A}_2^i(h, \mathsf{h}) = \mathcal{B}_2^i(h, \mathsf{h}) - \mathcal{B}_2^i(h, \mathsf{s})$$

and

$$P\left[H_2^{-i} = h | \Omega^i = g\right] = P\left[H_2^{-i} = \hat{h} | \Omega^i = b\right]$$

Multiply the last two equalities by each other and sum up over all histories to obtain

$$\begin{aligned} & \sum_{h \in \{-1,0,1\}^{N-1}} P\left[H_2^{-i} = h | \Omega^i = g\right] \left[\mathcal{A}_2^i(h, \mathsf{b}) - \mathcal{A}_2^i(h, \mathsf{h})\right] \\ = & \sum_{h \in \{-1,0,1\}^{N-1}} P\left[H_2^{-i} = \hat{h} | \Omega^i = b\right] \left[\mathcal{B}_2^i\left(\hat{h}, \mathsf{h}\right) - \mathcal{B}_2^i\left(\hat{h}, \mathsf{s}\right)\right] \end{aligned}$$

Now the result follows from (8). □

Lemma 4

Lemma. *For all* $\omega \in \{g, b, n\}$ *and all* $\mathsf{x} \in \{\mathsf{b}, \mathsf{h}, \mathsf{s}\}$,

$$\lim_{N\to\infty} \bar{A}_2^i(\omega, \mathsf{x}) = P\left[V = 1 | \Omega^i = \omega\right]$$

Proof. Let $\mathcal{A}_2^i(\mathsf{x}) = E\left[V | H_2^{-i}, X_1^i = \mathsf{x}, X_2^i = 1\right]$, which is the second-period ask price i will be quoted having placed order x in the first round. The *expected* second-period ask price is given by

$$\begin{aligned}
\bar{A}_2^i(\omega, \mathsf{x}) &= E\left[\mathcal{A}_2^i(\mathsf{x}) | \Omega^i = \omega\right] \\
&= \sum_{v\in\{0,1\}} P\left[V = v | \Omega^i = \omega\right] E\left[\mathcal{A}_2^i(\mathsf{x}) | \Omega^i = \omega, V = v\right]
\end{aligned}$$

Furthermore, the characterisation of ask prices in Appendix B implies that

$$\mathcal{A}_2^i(\mathsf{x}) = \frac{1}{1 + \lambda^{Q_{\mathsf{b}}^{-i} - Q_{\mathsf{s}}^{-i}} \Lambda(\mathsf{x}, 1)}$$

where Q_{x}^{-i} number of traders other than i that place order x in the first round. λ and Λ are suitably defined likelihood ratios, with $\lambda \in (0, 1)$ and $\Lambda > 0$. Conditional on $V = v$ and $\Omega^i = \omega$, traders' orders are independently and identically distributed. Hence by the weak law of large numbers, $\frac{Q_{\mathsf{b}}^{-i} - Q_{\mathsf{s}}^{-i}}{N-1}$ converges to ϑ_v in probability[5] conditional on $V = v$ and $\Omega^i = \omega$, where

$$\vartheta_v = P\left[X_1^j = \mathsf{b} | V = v, \Omega^i = \omega\right] - P\left[X_1^j = \mathsf{s} | V = v, \Omega^i = \omega\right]$$

Given equilibrium strategies, it is easy to show that $\vartheta_0 < 0$ and $\vartheta_1 > 0$.

[5] For a sequence of random variable $Y^{(N)}$, we say that $Y^{(N)}$ converges to Y in probability conditional on C if for all $\varepsilon > 0$,

$$\lim_{N\to\infty} P\left[\left|Y^{(N)} - Y\right| \geq \varepsilon | C\right] = 0$$

Since $\lambda \in (0,1)$, we can then show that $\mathcal{A}_2^i(\mathsf{x})$ converges to v in probability conditional on $V = v$ and $\Omega^i = \omega$ for $v \in \{0,1\}$. Hence we know that for all ω and all x, $\bar{A}_2^i(\omega, \mathsf{x})$ converges to $P[V = 1|\Omega^i = \omega]$ as required. □

B Beliefs and prices

This appendix derives characterisations of players' beliefs and equilibrium prices. Some of these are used for the proofs of results in Section 3 and 4. Others are used for the numerical solutions presented in Sections 4 and 5.

As a guide to the proofs in this appendix, note that for any event C, the fact that $P[V = 1] = \frac{1}{2}$ and Bayes' rule imply

$$E[V|C] = \frac{1}{1 + \frac{P[C|V=0]}{P[C|V=1]}} \tag{9}$$

Applying this rule to the competitive prices in Equation (1), we have

$$\begin{aligned} A_t^i &= \frac{1}{1 + \frac{P\left[H_t, X_t^i = \mathsf{b} | V=0\right]}{P\left[H_t, X_t^i = \mathsf{b} | V=1\right]}} \\ B_t^i &= \frac{1}{1 + \frac{P\left[H_t, X_t^i = \mathsf{s} | V=0\right]}{P\left[H_t, X_t^i = \mathsf{s} | V=1\right]}} \end{aligned} \tag{10}$$

Furthermore, from the definition of public beliefs in (5), we find that public belief after round t satisfies

$$P_t = \frac{1}{1 + \frac{P[H_{t+1}|V=0]}{P[H_{t+1}|V=1]}} \tag{11}$$

First, we prove four lemmas characterising equilibrium prices in both trading rounds as well as public beliefs after both trading rounds.

Lemma B.1. *Ask and bid prices at satisfy*

$$A_1^i = \frac{1}{1+\lambda}$$
$$B_1^i = \frac{1}{1+\lambda^{-1}}$$

and therefore $A_1^i = 1 - B_1^i$, *where*

$$\lambda = \frac{\frac{1-\mu}{3} + \mu\left[q\beta_1 + (1-q)(1-\alpha_1-\beta_1)\right]}{\frac{1-\mu}{3} + \mu\left[q(1-\alpha_1-\beta_1) + (1-q)\beta_1\right]}$$

Proof. Using the characterisation of equilibrium strategies in Section 3, it follows that

$$\frac{P\left[X_t^i = \mathsf{b}|V=0\right]}{P\left[X_t^i = \mathsf{b}V=1\right]} = \lambda$$
$$\frac{P\left[X_t^i = \mathsf{h}|V=0\right]}{P\left[X_t^i = \mathsf{h}V=1\right]} = 1$$
$$\frac{P\left[X_t^i = \mathsf{s}|V=0\right]}{P\left[X_t^i = \mathsf{s}|V=1\right]} = \lambda^{-1}$$

and the result follows from Equation (10). □

Lemma B.2. *Let* $Q_x = \sum_{i=1}^{N} 1\left(X_1^i = \mathsf{x}\right)$ *be the number of traders placing order* x *in the first round. Then the public belief after* $t=1$ *satisfies*

$$P_1 = \frac{1}{1+\lambda^{Q_\mathsf{b}-Q_\mathsf{s}}}$$

Proof. Because traders' orders are independent of each other conditional on

V, we find that

$$\frac{P\left[H_2|V=0\right]}{P\left[H_2|V=1\right]} = \lambda^{Q_{\mathsf{b}}-Q_{\mathsf{s}}}$$

and the result follows from Equation (11). □

Lemma B.3. *Let* $Q_{\mathsf{x}}^{-i} = \sum_{j\neq i} 1\left(X_1^j = \mathsf{x}\right)$ *be the number of traders other than* i *placing order* x *in the first round. Then ask and bid prices at* $t=2$ *satisfy*

$$A_2^i = \frac{1}{1+\lambda^{Q_{\mathsf{b}}^{-i}-Q_{\mathsf{s}}^{-i}}\Lambda\left(X_1^i,\mathsf{b}\right)}$$
$$B_2^i = \frac{1}{1+\lambda^{Q_{\mathsf{b}}^{-i}-Q_{\mathsf{s}}^{-i}}\Lambda\left(X_1^i,\mathsf{s}\right)}$$

where λ *is as defined in Lemma B.1, and*

$$\Lambda\left(\mathsf{x}_1,\mathsf{x}_2\right) = \frac{P\left[X_1^i=\mathsf{x}_1, X_2^i=\mathsf{x}_2|V=0\right]}{P\left[X_1^i=\mathsf{x}_1, X_2^i=\mathsf{x}_2|V=1\right]}$$

Proof. Because traders' orders are independent of each other conditional on V, we find that

$$\frac{P\left[H_2, X_2^i|V=1\right]}{P\left[H_2, X_2^i|V=0\right]} = \lambda^{Q_{\mathsf{b}}^{-i}-Q_{\mathsf{s}}^{-i}}\Lambda\left(X_1^i, X_2^i\right)$$

and the result follows from Equation (10). □

Lemma B.4. *Let* $Q_{\mathsf{x}_1\mathsf{x}_2} = \sum_{i=1}^N 1\left(X_1^i=\mathsf{x}_1, X_2^i=\mathsf{x}_2\right)$ *be the number of traders placing order* x_1 *in the first round and* x_2 *in the second. Then the public belief after* $t=2$ *satisfies*

$$P_2 = \frac{1}{1+\bar{\Lambda}^{Q_{\mathsf{bb}}-Q_{\mathsf{ss}}}\underline{\Lambda}^{Q_{\mathsf{hb}}-Q_{\mathsf{hs}}}}$$

where

$$\begin{aligned}\bar{\Lambda} &= \frac{\frac{1-\mu}{9}+\mu(1-q)(1-\alpha_1)}{\frac{1-\mu}{9}+\mu q(1-\alpha_1)} \\ \textit{and } \underline{\Lambda} &= \frac{\frac{1-\mu}{9}+\mu(1-q)\alpha_1}{\frac{1-\mu}{9}+\mu q\alpha_1}\end{aligned}$$

Proof. Because traders' orders are independent of each other conditional on V, we find that

$$\frac{P[H_3|V=0]}{P[H_3|V=1]} = \prod_{i=1}^{N} \Lambda\left(X_1^i, X_2^i\right)$$

where the function Λ is as defined in Lemma B.3. Using the characterisation of equilibrium strategies in Section 3, we have

$$\begin{aligned}\Lambda(\mathsf{b},\mathsf{b}) &= \frac{\frac{1-\mu}{9}+\mu(1-q)(1-\alpha_1)}{\frac{1-\mu}{9}+\mu q(1-\alpha_1)} \equiv \bar{\Lambda} \\ \Lambda(\mathsf{s},\mathsf{s}) &= \bar{\Lambda}^{-1} \\ \Lambda(\mathsf{h},\mathsf{b}) &= \frac{\frac{1-\mu}{9}+\mu(1-q)\alpha_1}{\frac{1-\mu}{9}+\mu q\alpha_1} \equiv \underline{\Lambda} \\ \Lambda(\mathsf{h},\mathsf{s}) &= \underline{\Lambda}^{-1}\end{aligned}$$

Moreover, for all other pairs of x_1 and x_2, we have $\Lambda(\mathsf{x}_1,\mathsf{x}_2)=1$, since those sequences of orders would only ever be placed by a noise trader in equilibrium. Hence, we have

$$\frac{P[H_3|V=0]}{P[H_3|V=1]} = \bar{\Lambda}^{Q_{\mathsf{bb}}-Q_{\mathsf{ss}}}\underline{\Lambda}^{Q_{\mathsf{hb}}-Q_{\mathsf{hs}}}$$

and the result follows from Equation (11). □

Next, we prove a result stating some properties of the posterior beliefs of traders and dealers.

Lemma B.5. *For $t=1,2$, the following are true with probability 1:*

1. Traders do not learn from their own actions:

$$E\left[V|H_t, X_t^i, \Omega^i\right] = E\left[V|H_t, \Omega^i\right] = E\left[V|H_t^{-i}, \Omega^i\right]$$

2. Ordering of posterior beliefs:

$$E\left[V|H_t, \Omega^i = g\right] > E\left[V|H_t, \Omega^i = n\right] > E\left[V|H_t, \Omega^i = b\right]$$

3. The difference between posteriors is bounded by the difference between priors:

$$E\left[V|H_t, \Omega^i = g\right] - E\left[V|H_t, \Omega^i = b\right] < 2q - 1$$

4. Dealers can never rule our a noise trader:

$$P\left[\Omega^i = 0|H_t, X_t^i\right] > 0$$

Proof. 1. Note that

$$\begin{aligned} P\left[H_t, X_t^i, \Omega^i|V\right] &= P\left[X_t^i|H_t, \Omega^i, V\right] P\left[H_t, \Omega^i|V\right] \\ &= P\left[X_t^i|H_t, \Omega^i\right] P\left[H_t, \Omega^i|V\right] \end{aligned}$$

. The second line holds because traders' actions are independent of V conditional on the history and i's information. Hence,

$$\frac{P\left[H_t, X_t^i, \Omega^i|V = 0\right]}{P\left[H_t, X_t^i, \Omega^i|V = 1\right]} = \frac{P\left[H_t, \Omega^i|V = 0\right]}{P\left[H_t, \Omega^i|V = 1\right]}$$

which implies the first equality due to Equation 9. The argument for the second equality is identical.

2. By (1.), it is sufficient to show that

$$E\left[V|H_t^{-i}, \Omega^i = g\right] > E\left[V|H_t^{-i}, \Omega^i = n\right] > E\left[V|H_t^{-i}, \Omega^i = b\right]$$

Let $\Gamma^{-i} = \frac{P\left[H_t^{-i}|V=0\right]}{P\left[H_t^{-i}|V=1\right]}$. Using the fact that traders's signals are independent conditional on V, we have the following three characterisations of likelihood ratios:

$$\begin{aligned}
\frac{P\left[H_t^{-i},\Omega^i=g|V=0\right]}{P\left[H_t^{-i},\Omega^i=g|V=1\right]} &= \Gamma^{-i}\frac{1-q}{q} \\
\frac{P\left[H_t^{-i},\Omega^i=n|V=0\right]}{P\left[H_t^{-i},\Omega^i=n|V=1\right]} &= \Gamma^{-i} \\
\frac{P\left[H_t^{-i},\Omega^i=b|V=0\right]}{P\left[H_t^{-i},\Omega^i=b|V=1\right]} &= \Gamma^{-i}\frac{q}{1-q}
\end{aligned}$$

The result follows directly from the fact that $q > \frac{1}{2}$ and hence $\frac{1-q}{q} < 1 < \frac{q}{1-q}$.

3. Define the function F as follows:

$$F(y) = \frac{1}{1+y\frac{1-q}{q}} - \frac{1}{1+y\frac{q}{1-q}}$$

Note that F is twice continuously differentiable and that $F(1) = 2q-1$. Also note that

$$E\left[V|H_t,\Omega^i=g\right] - E\left[V|H_t,\Omega^i=b\right] = F\left(\Gamma^{-i}\right)$$

where Γ^{-i} is defined as in (2.) above. Since $\Gamma^{-i} > 0$, it is sufficient to show that for all $y > 0$ such that $y \neq 1$,

$$F(y) < F(1)$$

Taking the first derivative of F shows that the only $y > 0$ such that $F'(y) = 0$ is $y = 1$. Also, it can be shown that $F''(1) < 0$. Hence for all $y \in (0,1)$,

$F'(y) > 0$ and

$$F(1) - F(y) = \int_b^1 F'(z)\, dz > 0$$

Also, for all $y > 1$, $F'(y) < 0$ and

$$F(y) - F(1) = \int_1^b F'(z)\, dz < 0$$

which establishes the result.

4. By Bayes' rule,

$$P\left[\Omega^i = n | H_t, X_t^i\right] = \frac{P\left[H_t, X_t^i | \Omega^i = n\right] P\left[\Omega^i = n\right]}{\sum_{\omega \in \{g,b,n\}} P\left[H_t, X_t^i | \Omega^i = \omega\right] P\left[\Omega^i = \omega\right]}$$

Note that $P\left[\Omega^i = n\right] = \mu > 0$ and

$$\begin{aligned} P\left[H_t, X_t^i | \Omega^i = n\right] &\geq P\left[H_t, X_t^i | \cap_{i=1}^N \left\{\Omega^i = n\right\}\right] \\ &> 0 \end{aligned}$$

with probability 1, since all sequences of orders are possible if everybody is a noise trader. Hence, the expression above is strictly positive as required. □

C Numerical procedures

C.1 Solving for equilibrium strategies

This subsection describes the numerical procedure required to solve for equilibrium strategies given the parameters of the model, N, μ and q. This procedure is used to produces Figures 1, 2 and 3 in Section 4 of the paper.

In the *Matlab* package available from the author, it is executed by the script `alpha1graphs.m`, which can be used to replicate the figures.

To find all possible equilibria given parameters, it is sufficient to find all $\alpha_1 \in \left[0, \frac{1}{2}\right)$ that satisfy one of the conditions in Proposition 3. The algorithm used for every parameterisation is as follows:

1. Calculate $G(0)$. If $G(0) \leq 0$ then $\alpha_1 = 0$ is an equilibrium by the first condition of Proposition 3.

2. For starting values $\alpha_1^{ini} \in \{0, 0.05, ..., 0.45, 0.5\}$, solve the Equation $G(\alpha_1) = 0$ numerically using α_1^{ini} as the starting value. If $\alpha_1 > 0$ at the solution, then α_1 is an equilibrium by the second condition of Proposition 3.

This algorithm requires us to compute $G(\alpha_1)$ for any $\alpha_1 \in \left[0, \frac{1}{2}\right)$. In the *Matlab* package, this is done by the function `Gfun.m`. Recall the definition of G:

$$G(\alpha_1) = \left[\bar{A}_2^i(g, \mathsf{b}) - \bar{A}_2^i(g, \mathsf{h})\right] - \left[q - A_1^i\right]$$

The second term is easily calculated using the characterisation of A_1^i in Lemma B.1. For the first term, we still need to compute the expected ask price $\bar{A}_2^i(\omega, \mathsf{x})$ for $\mathsf{x} \in \{\mathsf{b}, \mathsf{h}\}$. Using the characterisation of A_2^i in Lemma B.3, this is given by

$$\begin{aligned}
\bar{A}_2^i(\omega, \mathsf{x}) &= E\left[A_2^i | \Omega^i = \omega, X_1^i = \mathsf{x}\right] \\
&= \sum_{v \in \{0,1\}} P\left[V = v | \Omega^i = \omega\right] E\left[\frac{1}{1 + \lambda^{Q_{\mathsf{b}}^{-i} - Q_{\mathsf{s}}^{-i}} \Lambda(\mathsf{x}, \mathsf{b})} | V = v\right]
\end{aligned}$$

The conditional expectation inside the sum can be calculated by using the fact that the joint distribution of Q_{b}^{-i} and Q_{s}^{-i} conditional on V is trinomial, with joint probabilities as follows:

$$P\left[Q_{\mathsf{b}}^{-i}=q_{\mathsf{b}},Q_{\mathsf{s}}^{-i}=q_{\mathsf{s}}|V=v\right] = \frac{1}{2}\frac{(N-1)!}{q_{\mathsf{b}}!q_{\mathsf{s}}!(N-1-q_{\mathsf{b}}-q_{\mathsf{s}})!} \times \left(p_{\mathsf{b}}^{v}\right)^{q_{\mathsf{b}}}\left(p_{\mathsf{s}}^{v}\right)^{q_{\mathsf{s}}}\left(p_{\mathsf{h}}^{v}\right)^{N-1-q_{\mathsf{b}}-q_{\mathsf{s}}}$$

where $p_{\mathsf{x}}^{v}=P\left[X_{1}^{i}=\mathsf{x}|V=v\right]$. The probabilities p_{x}^{v} are given by

$$\begin{aligned}
p_{\mathsf{b}}^{1} &= \frac{1-\mu}{3}+\mu q\left(1-\alpha_{1}\right)=p_{\mathsf{s}}^{0},\\
p_{\mathsf{h}}^{1} &= \frac{1-\mu}{3}+\mu\alpha_{1}=p_{\mathsf{h}}^{0}\\
p_{\mathsf{s}}^{1} &= \frac{1-\mu}{3}+\mu\left(1-q\right)\left(1-\alpha_{1}\right)=p_{\mathsf{b}}^{0}
\end{aligned}$$

C.2 Public beliefs

This subsection describes the numerical procedure required to calculate the impact of equilibrium price manipulation on social learning, as measured by the mean-square error (MSE) of the public belief after trading rounds $t\in\{1,2\}$. This procedure is used to produce Figure 4 in Section 5 of the paper. In the *Matlab* package available from the author, it is executed by the script `MSEgraphs.m`, which can be used to replicate the figures.

The algorithm used for every parameterisation is as follows:

1. Solve for the equilibrium α_1 as described in Subsection C.1 above.
2. Calculate the MSE of the public beliefs given the equilibrium α_1, i.e. $\mathrm{MSE}\left(P_t\right)$.
3. Calculate the MSE of the public beliefs given $\alpha_1=0$, i.e. $\mathrm{MSE}\left(P_t^0\right)$.
4. Calculate the percentage impact of manipulation on the MSE, i.e. Π_t as defined in Equation (7).

Steps 2 requires us to compute $\mathrm{MSE}(P_t)$. We describe the necessary calculations below. Step 3 is executed using exactly the same calculations, but replacing α_1 with 0 everywhere. In the *Matlab* package, these calculations are done by the function `MSE.m`.

First, consider P_1, the public belief after the first trading round. Using the definition of the mean-square error of P_1 in Equation (6) and the characterisation of public beliefs in Lemma B.2, we have

$$\begin{aligned}\mathrm{MSE}(P_1) &= E\left[(P_1 - V)^2\right] \\ &= \sum_{v\in\{0,1\}} P[V=v]\, E\left[\left(\frac{1}{1+\lambda^{Q_{\mathsf{b}}-Q_{\mathsf{s}}}} - v\right)^2 |V=v\right]\end{aligned}$$

The conditional expectation inside the sum can be calculated by using the fact that the joint distribution of Q_{b} and Q_{s} conditional on V is trinomial, with joint probabilities as follows:

$$\begin{aligned}P[Q_{\mathsf{b}}=q_{\mathsf{b}}, Q_{\mathsf{s}}=q_{\mathsf{s}}|V=v] &= \frac{1}{2}\frac{N!}{q_{\mathsf{b}}!q_{\mathsf{s}}!(N-q_{\mathsf{b}}-q_{\mathsf{s}})!} \\ &\times \; (p^v_{\mathsf{b}})^{q_{\mathsf{b}}}(p^v_{\mathsf{s}})^{q_{\mathsf{s}}}(p^v_{\mathsf{h}})^{N-q_{\mathsf{b}}-q_{\mathsf{s}}}\end{aligned}$$

where $p^v_{\mathsf{x}} = P[X^i_1 = \mathsf{x}|V=v]$. The probabilities p^v_{x} are given by

$$\begin{aligned}p^1_{\mathsf{b}} &= \frac{1-\mu}{3} + \mu q(1-\alpha_1) = p^0_{\mathsf{s}}, \\ p^1_{\mathsf{h}} &= \frac{1-\mu}{3} + \mu\alpha_1 = p^0_{\mathsf{h}} \\ p^1_{\mathsf{s}} &= \frac{1-\mu}{3} + \mu(1-q)(1-\alpha_1) = p^0_{\mathsf{b}}\end{aligned}$$

Second, consider P_2, the public belief after the second trading round. Using the definition of the mean-square error of P_2 in Equation (6) and the characterisation of public beliefs in Lemma B.4, we have

$$\begin{aligned}\mathrm{MSE}\,(P_2) &= E\left[(P_2 - V)^2\right] \\ &= \sum_{v\in\{0,1\}} P\,[V=v]\,E\left[\left(\frac{1}{1+\bar{\Lambda}^{Q_{\mathsf{bb}}-Q_{\mathsf{ss}}}\underline{\Lambda}^{Q_{\mathsf{hb}}-Q_{\mathsf{hs}}}} - v\right)^2 |V=v\right]\end{aligned}$$

The conditional expectation inside the sum can be calculated by using the fact that the joint distribution of the $Q_{\mathsf{x}_1\mathsf{x}_2}$ variables conditional on V is multinomial, with joint probabilities as follows:

$$\begin{aligned}P\left[(Q_{\mathsf{bb}}, Q_{\mathsf{ss}}, Q_{\mathsf{hb}}, Q_{\mathsf{hs}}) = (q_{\mathsf{bb}}, q_{\mathsf{ss}}, q_{\mathsf{hb}}, q_{\mathsf{hs}})\,|V=v\right] &= \frac{1}{2}\frac{N!}{q_{\mathsf{bb}}!q_{\mathsf{ss}}!q_{\mathsf{hb}}!q_{\mathsf{hs}}!q_{\mathsf{r}}!} \\ &\times \left(p^v_{\mathsf{bb}}\right)^{q_{\mathsf{bb}}}\left(p^v_{\mathsf{ss}}\right)^{q_{\mathsf{ss}}}\left(p^v_{\mathsf{hb}}\right)^{q_{\mathsf{hb}}}\left(p^v_{\mathsf{hs}}\right)^{q_{\mathsf{hs}}}\left(p^v_{\mathsf{r}}\right)^{q_{\mathsf{r}}}\end{aligned}$$

where

$$\begin{aligned}p^v_{\mathsf{x}_1\mathsf{x}_2} &= P\left[X^i_1 = \mathsf{x}_1, X^i_2 = \mathsf{x}_2 | V=v\right] \\ q_{\mathsf{r}} &= N - q_{\mathsf{bb}} - q_{\mathsf{ss}} - q_{\mathsf{hb}} - q_{\mathsf{hs}} \\ p^v_{\mathsf{r}} &= 1 - p^v_{\mathsf{bb}} - p^v_{\mathsf{ss}} - p^v_{\mathsf{hb}} - p^v_{\mathsf{hs}}\end{aligned}$$

The required probabilities $p^v_{\mathsf{x}_1\mathsf{x}_2}$ are given by

$$\begin{aligned}p^1_{\mathsf{bb}} &= \frac{1-\mu}{9} + \mu q\,(1-\alpha_1) = p^0_{\mathsf{ss}} \\ p^1_{\mathsf{hb}} &= \frac{1-\mu}{9} + \mu q\alpha_1 = p^0_{\mathsf{hs}}\end{aligned}$$