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Leonie Baumann
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# A Model of Weighted Network Formation*

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†Cambridge-INET, Faculty of Economics, University of Cambridge, lb714@cam.ac.uk.

**Abstract**

The paper proposes a game of weighted network formation in which each agent has a limited resource to form links of possibly different intensities with other agents and to use for private purposes. We show that every equilibrium is either "reciprocal" or "non-reciprocal". In a reciprocal equilibrium, any two agents invest equally in the link between them. In a non-reciprocal equilibrium, agents are partitioned into "concentrated" and "diversified" agents and a concentrated agent is only linked to diversified agents and vice versa. For every link, the concentrated agent invests more in the link than the diversified agent. The unweighted relationship graph of an equilibrium, in which two agents are linked if they both invest positively in each other, uniquely predicts the equilibrium values of each agent's network investment and utility level, as well as the ratio of any two agents' investments in each other. We show that equilibria are not pairwise stable and not efficient due to the positive externalities of investing in a link.

*Keywords* weighted networks, network formation, link-specific investment

*JEL Classification Codes* D85, L14, Z13, C72

# 1 Introduction

A network is a graph which describes the relations between the network's members. A link between two members of a network can represent, for example, friendship, co-authorship, trade or communication between them. Most of the literature on network formation, following the seminal papers by Jackson and Wolinsky (1996) and Bala and Goyal (2000), assumes that an agent decides whether or not to form a link, but does not determine its intensity. However, in many situations agents must choose not only with whom to interact but also the intensity of that interaction.

We analyze a symmetric game in which each agent has a limited resource that she can keep for herself (*self-investment*) and invest in forming links with other agents. A strategy of an agent specifies an allocation of her resource across all agents (including herself). We say that *two agents are linked* if they both invest positively in each other. An agent's utility is the sum of her benefits from self-investment and from each of her relationships. The benefit from self-investment is represented by an increasing and strictly concave function. The benefit from her relationship with another agent is increasing and strictly concave in the two agents' investments in each other and is represented by a function which exhibits strategic complementarity and is homogenous of degree one.

In the main analysis, we investigate the game's Nash equilibria. Special attention is devoted to the (unweighted and undirected) relationship graphs which are induced by equilibria and which include a link between two agents if they both invest positively in each other.

We show that every equilibrium is of one of two types: *reciprocal* or *non-reciprocal*. In a reciprocal equilibrium, any two linked agents invest the same amount in the link between them, and all agents choose the same self-investment and derive the same utility. Using a result from graph theory, we characterize the full set of relationship graphs associated with reciprocal equilibria. This set includes, for example, graphs in which every agent is linked to more than half of the other agents, or in which every agent has the same number of links. The set excludes, for example, graphs in which there is an agent with only one link. It is possible that two agents have a different number of links and that an agent has links of varying intensities.

In a non-reciprocal equilibrium, agents are partitioned into two sets: the set of *concentrated* agents and the set of *diversified* agents. Links only exist across the sets and never within a set. For all links, the concentrated agent invests more in the link than the diversified agent. The ratio between the investment of a concentrated agent and that of a diversified agent in their link is the same across all links (and denoted by $q^t$). All concentrated agents choose the same level of self-investment which is higher than the level of self-investment chosen by all diversified agents. Diversified agents derive greater utility than concentrated agents. The ratio of the number of concentrated agents to the number of diversified agents is positively correlated with $q^t$.

We show that the relationship graphs of reciprocal and non-reciprocal equilibria are entirely distinct. Thus, only knowing the equilibrium relationship graph is sufficient to determine whether the equilibrium is reciprocal or non-reciprocal. Furthermore, the relationship graph of a non-reciprocal equilibrium uniquely determines the partition into concentrated and diversified agents, the value of $q^t$, and each agent's level of self-investment and utility. However, the relationship graph does not always pin down the equilibrium investments in a link. We demonstrate that many equilibria with different levels of investment in links can induce the same relationship graph.

We examine the comparative statics of equilibria when relationships become more valuable relative to self-investment and when each agent's resource endowment increases. In view of the multiplicity of equilibria, we restrict ourselves to investigate how the equilibrium values which are uniquely determined by the relationship graph and the model parameters change when the corresponding model parameter is varied and the relationship graph is held fixed.

Additionally, we show that equilibria are not stable against pairwise deviations and are not efficient, in the sense that they do not maximize the sum of agents' utilities. This is due to the positive externality of an agent's investment in a link that benefits the other agent in the link. We characterize efficient networks and find that in an efficient network, any two agents invest the same amount in each other, but choose a lower self-investment than in a reciprocal equilibrium. We show that nevertheless the set of relationship graphs of efficient networks coincides with the set of relationship graphs of reciprocal equilibria.

**Related literature.** This paper adds to the literature on network formation with weighted links.

The most closely related articles are Salonen (2015), Griffith (2017), and Brueckner (2006) which analyze the formation of weighted social networks, and Goyal et al. (2008) which analyze a two-stage game in which firms first form weighted links in R&D networks and then compete in a market. These authors focus on symmetric equilibria. Restricting the analysis in this way limits the possibility of differences in link intensities in equilibrium. We extend to beyond symmetric equilibria and identify asymmetric equilibrium structures.

Bloch and Dutta (2009) and Deroïan (2009) analyze the formation of communication networks, in which agents also derive utility from indirect links, with budget constraints and without self-investment. Thus, the amount invested in the network is determined exogenously and is the same for all agents. The possibility of self-investment in our model gives rise to equilibria in which agents choose different levels of network investment. Another difference is our assumption that two agents' investments in their link are strategic (imperfect) complements. For the main part of their analysis, Bloch and Dutta (2009) assume that link quality is an additively separable function of two agents' investments in their link. Deroïan (2009) assumes that an agent's link investment benefits herself but not her link partner.

Rogers (2006) suggests a different type of network formation game in which agents invest in links in order to pursue a higher status. An agent's status is increasing in the status of agents she is linked to and in the intensity of those links.

Finally, Golub and Livne (2010), Cabrales et al. (2011), Durieu et al. (2011) and Galeotti and Merlino (2014) assume that agents can choose one parameter (quality, effort or investment level) which then affects the intensities of all their links equally. Such a constraint limits the set of weighted networks that can form in equilibrium.

**Roadmap.** Section 2 introduces the model. Section 3 presents the equilibrium analysis and is divided into the following subsections: Section 3.1 characterizes the equilibrium investment strategy profiles and utility levels; Section 3.2 analyzes the relationship graphs of reciprocal and non-reciprocal equilibria; Section 3.3 discusses the multiplicity of equilibria; and Section 3.4 presents comparative statics results. Section 4 discusses the pairwise stability of equilibria and characterizes the efficient networks.

# 2 The Model

There is a set of agents $N = \{1, ..., n\}$. Each agent $i$ possesses resource $T > 0$ which she can invest in relations with other agents and in private activity. Her investment in a relation with agent $j \neq i$ is denoted by $t_{ij}$ and her investment in private activity (*self-investment*) by $t_{ii}$. An investment strategy of agent $i$ is $t_i = (t_{i1}, ..., t_{in})$ such that $t_{ij} \geq 0$ for all $j$ and $\sum_j t_{ij} \leq T$. The analysis is restricted to pure strategies. A strategy profile is represented by a matrix $t = [t_{ij}]_{i,j}$ and can be interpreted as a weighted directed graph, with $t_{ij}$ being the weight on the link from $i$ to $j$. We will also refer to strategy profile $t$ as *network* t.

Agent $i$'s utility given network $t$ is the sum of her utilities from relations with others and from self-investment:

$$u_i(t) = \sum_{j \neq i} av(t_{ij}, t_{ji}) + f(t_{ii})$$

where $av(t_{ij}, t_{ji})$, $a > 0$ is $i$'s utility from her relation with $j$ and $f(t_{ii})$ is her utility from self-investment. The parameter $a$ determines the value of relationships relative to the value of self-investment.

The relationship utility $v$ is differentiable. The partial derivative of $v$ with respect to argument $k = 1, 2$ is denoted by $v_k$, and the second-order partial derivative of $v$ with respect to arguments $k = 1, 2$ and $l = 1, 2$ is denoted by $v_{kl}$. Apart from differentiability, $v$ satisfies the following properties:

**P1** $v(x, 0) = v(0, y) = 0$ for all $x, y \geq 0$.

A relationship yields zero benefit if one agent does not invest in the relationship.

**P2** For all $x, y > 0$, $v(x, y)$ is increasing and strictly concave, and $\lim_{x \to 0} v_1(x, y) = \infty$ for all $y > 0$.

Agent $i$'s utility from her relationship with $j$ is increasing and strictly concave in $i$'s and $j$'s investments. Marginal utility is infinite if $i$'s investment goes to zero and $j$ invests positively.

**P3** $v_{12}(x, y) > 0$, $v_{21}(x, y) > 0$ for all $x, y > 0$.

Two agents' investments in their relationship are strategic complements.

**P4** $v(\gamma x, \gamma y) = \gamma v(x, y)$ for all $\gamma > 0$.

The relationship utility $v$ is homogenous of degree 1 and exhibits constant returns to scale. P4 implies that $v_k$ is homogenous of degree 0.

For example, a Cobb-Douglas function $v(x, y) = x^{\beta}\, y^{1-\beta}$ with $\beta \in (0, 1)$ satisfies P1-P4.

The utility function from self-investment, $f$, is increasing, strictly concave and differentiable, with $\lim_{x \to 0} f'(x) = \infty$ and $\lim_{x \to T} f'(x) = 0$.

A network $t$ induces an unweighted and undirected (relationship) graph $g(t)$ on $N$ which describes the relationships with mutual positive investments in $t$. That is, agents $i$ and $j$ are linked in $g(t)$ (link $ij \in g(t)$) if $t_{ij} > 0$ and $t_{ji} > 0$.

We introduce some graph-related definitions that are necessary for the analysis of the game. In what follows, *graph* always means an unweighted and undirected graph. Consider a graph $g$ on $N$. Agent $i$'s set of *neighbors* is $N_i := \{j|ij \in g\}$. A *walk* between agents $i$ and $j$ is a sequence of links $i_1i_2, i_2i_3, ..., i_{K-1}i_K$ such that $i_{k-1}i_k \in g$ for all $k = 2, ..., K$ and $i_1 = i$ and $i_K = j$. Two *agents are connected* if there exists a walk between them, and *$g$ is connected* if all agents in $N$ are connected. A *component of $g$* is a maximal connected subgraph of $g$. This means that all agents in one component are connected to each other and not linked to any agent outside the component. An agent without any links (component of size 1) is called an *isolated* agent. To avoid unnecessary complications, we will often refer to the links, components, etc. of a network $t$, when we mean the links, components, etc. of its graph $g(t)$.

# 3 Equilibrium Networks

The analysis focusses on the Nash equilibria of the network formation game in which all agents simultaneously choose their investment strategies. A strategy profile $t$ is a Nash equilibrium if no agent $i$ can strictly increase her utility by deviating to another strategy, given all other agents' strategies.

In Section 3.1, we show that every equilibrium is either *reciprocal* or *non-reciprocal*. In a reciprocal equilibrium, any two agents invest the same amount in each other, and all agents have the same self-investment and utility level. In a non-reciprocal equilibrium, agents can be partitioned into two sets $C$ (concentrated agents) and $D$ (diversified agents). Links only exist between the sets, and never within them. For every link, the concentrated agent invests more in the link than the diversified agent. The ratio between the concentrated agent's investment in the link and the diversified agent's is the same across all links. All agents within same set have the same self-investment and utility level.

In Section 3.2, we characterize the relationship graphs of equilibria. We show that simply by observing an equilibrium relationship graph we can uniquely determine each agent's equilibrium self-investment and utility level as well as the ratio of any two agents' equilibrium investments in each other. In particular, the graph can be used to determine whether the equilibrium that induced it is reciprocal or non-reciprocal.

In Section 3.3, we discuss the multiplicity of the equilibria. A given relationship graph can induced by many equilibria, which feature different link investments. We propose a simple mechanism by which we can construct multiple equilibria from a given equilibrium.

In Section 3.4, we investigate the comparative statics of equilibria for the case that relationships become relatively more valuable (i.e. increase in $a$) and for the case that the total resource endowment increases (i.e. increase in $T$). Given the multiplicity of equilibria, we restrict ourselves to analyze the change in the equilibrium values which are uniquely determined by the relationship graph and the model parameters when the corresponding model parameter is varied and the relationship graph remains the same.

## 3.1 Investment Strategy Profiles and Utility Levels

Note that $t_{ij} = 0$ is the unique optimal choice of agent $i$ if agent $j$ chooses $t_{ji} = 0$ because self-investment is always utility-enhancing and $v(t_{ij}, 0) = 0$ for all $t_{ij}$. Thus, a trivial equilibrium is the *empty* network where all agents only invest in themselves. More generally, a network is an equilibrium if and only if the investment choices of the agents in each component of the network are an equilibrium of the network formation game reduced to the agents in that component. Therefore, in order to characterize the full set of equilibrium networks, we restrict the analysis from now on to connected equilibrium networks with $n > 1$.

The next proposition requires the following definitions: Let $\sigma : \mathbb{R}^{>0} \to (0, T)$ be the function defined by the equation $f'(\sigma(x)) = av_1(x, 1)$. Note that the properties of $f$ guarantee that the function $\sigma$ is well defined. Let $\mu : \mathbb{R}^{>0} \to \mathbb{R}$ be the function defined by $\mu(x) = (T - \sigma(x))av(1, \frac{1}{x}) + f(\sigma(x))$.

**Lemma 1.** *The function $\sigma$ is strictly increasing and the function $\mu$ is strictly decreasing.*

The proof of Lemma 1 is relegated to the appendix.

**Proposition 1.** *For every equilibrium $t$, there exists $q^t \geq 1$ such that for every $i \in N$, there exists $q_i$ where $q_i \in \left\{q^t, \frac{1}{q^t}\right\}$ and $\frac{t_{ij}}{t_{ji}} = q_i$ for all $j \in N_i$, $t_{ii} = \sigma(q_i)$ and $u_i(t) = \mu(q_i)$. Thus, every equilibrium $t$ is either*

*(i)* ***reciprocal*** *($q^t = 1$) where $q_i = 1$ for all $i \in N$.*

*or*

*(ii)* ***non-reciprocal*** *($q^t > 1$) where there is a bipartition $(C, D)$ of $N$ such that if $i$ is linked to $j$, then $i$ and $j$ are in different sets. For all $i \in C$ and $j \in D$, $q_i = q^t$ and $q_j = \frac{1}{q^t}$.*

*Proof.* We start with a lemma that establishes necessary and sufficient conditions on $t$ for it to be a Nash equilibrium.

**Lemma 2.** *A network $t$ is a Nash equilibrium if and only if, for all $i \in N$ and all $j \neq i$,*

*a)* $\sum_k t_{ik} = T$,

*b)* *if $t_{ji} = 0$, then $t_{ij} = 0$,*

*c)* *if $t_{ji} > 0$, then $t_{ij} > 0$ and $av_1(t_{ij}, t_{ji}) = f'(t_{ii})$.*

The proof of Lemma 2 is immediate from the standard conditions on each agent's utility maximization problem given all other agents' strategies and is omitted. In any equilibrium, each agent $i$ invests her entire resource and invests positively in $j$ if and only if $j$ invests positively in $i$. An agent's positive investment levels are such that her marginal utility from investing in any of her links is equal to her marginal utility from self-investment.

Now consider an equilibrium $t$ and $i \in N$. Note first that $v_1(t_{ij}, t_{ji}) = v_1\left(\frac{t_{ij}}{t_{ji}}, 1\right)$ for all $j \in N_i$ by P4. By Lemma 2c, $v_1\left(\frac{t_{ij}}{t_{ji}}, 1\right) = v_1\left(\frac{t_{ik}}{t_{ki}}, 1\right)$ for all $j, k \in N_i$. Thus, $\frac{t_{ij}}{t_{ji}} = \frac{t_{ik}}{t_{ki}}$ for all $j, k \in N_i$ because $v$ is strictly concave. Hence, there is $q_i > 0$ such that $\frac{t_{ij}}{t_{ji}} = q_i$ for all $j \in N_i$. Then, $q_j = \frac{1}{q_i}$ for all $j \in N_i$. Let $q^t = \max\left\{q_i, \frac{1}{q_i}\right\}$. Since all agents are connected, $q_k \in \left\{q^t, \frac{1}{q^t}\right\}$ for all $k \in N$.

By Lemma 2c, it then follows with regard to agent $i$'s self-investment that $f'(t_{ii}) = av_1(q_i, 1)$ and hence $t_{ii} = \sigma(q_i)$. Regarding agent $i$'s utility, observe that $u_i(t) = \sum_{j \neq i} av(t_{ij}, t_{ji}) + f(t_{ii}) = \sum_{j \neq i} t_{ij} av(1, \frac{t_{ji}}{t_{ij}}) + f(t_{ii}) = (T - t_{ii}) av(1, \frac{1}{q_i}) + f(t_{ii}) = \mu(q_i)$ because $v$ is homogenous of degree 1 and $t_{ii} = \sigma(q_i)$.

In the case of a reciprocal equilibrium in which $q^t = 1$, obviously $q_k = 1$ for all $k \in N$. In the case of a non-reciprocal equilibrium with $q^t > 1$, there exists an agent $i \in N$ for whom $q_i = q^t$. For all $j \in N_i$, $q_j = \frac{1}{q_i} = \frac{1}{q^t}$, and so on. Thus, because $q_i = \frac{1}{q_j}$ for all $i$ and all $j \in N_i$, there exists a partition $(C, D)$ of $N$ in which all $i$ with $q_i = q^t$ are in $C$ and all $j$ with $q_j = \frac{1}{q^t}$ are in $D$, and there are only links across the sets.

□

It is worthwhile summarizing the observations about equilibria which follow from Proposition 1. Every equilibrium $t$ is associated with a number $q^t$ which we call the *investment ratio* of $t$. In equilibrium $t$, for any link which agent $i$ has, the ratio of $i$'s investment to her neighbor's investment in the link is equal to $q_i$. This ratio $q_i$ is either $q^t$ or $\frac{1}{q^t}$. Agent $i$'s equilibrium self-investment level is a strictly increasing function of $q_i$ while her equilibrium utility level is a strictly decreasing function of $q_i$.

In any reciprocal equilibrium, every agent's $q_i$ is equal to one, and every agent chooses the same level of self-investment and derives the same level of utility. We call the agents in a reciprocal equilibrium *balanced* and denote their self-investment and utility by $t_{bb} := \sigma(1)$ and $u_b := \mu(1)$, respectively.

In any non-reciprocal equilibrium $t$, there exists a partition of $N$ into two sets $C$ and $D$ such that links only exist between agents in different sets. We call the agents in $C$ *concentrated* and the agents in $D$ *diversified*. For every concentrated agent $i$, $q_i = q^t$ and for every diversified agent $i$, $q_i = \frac{1}{q^t}$. This means that, for any link, the concentrated agent invests more in the link than the diversified agent. Moreover, all agents in the same set choose the same level of self-investment and derive the same level of utility. We denote the self-investment of concentrated agents and diversified agents by $t_{cc} := \sigma(q^t)$ and $t_{dd} := \sigma(\frac{1}{q^t})$ and their utility by $u_c := \mu(q^t)$and $u_d := \mu(\frac{1}{q^t})$, respectively.

Since $\sigma$ is strictly increasing and $\mu$ is strictly decreasing, the equilibrium levels of self-investment and utility are unambiguously ordered for different values of $q_i$. For any reciprocal equilibrium $t$ and any non-reciprocal equilibrium $t'$, $t'_{dd} < t_{bb} < t'_{cc}$, and $u'_d > u_b > u'_c$. In other words, diversified agents have the lowest self-investment and highest utility, concentrated agents have the highest self-investment and lowest utility, and balanced agents have both a self-investment and utility somewhere in between. Note that the ordering of self-investment levels trivially imposes an ordering on agents' total equilibrium network investment. A diversified agent chooses the highest total network investment, a concentrated agent the lowest and a balanced agent chooses somewhere in between.

The divergence of $t'_{dd}$ and $t'_{cc}$ from $t_{bb}$ is strictly increasing in $q^{t'}$, as is the divergence of $u'_d$ and $u'_c$ from $u_b$. Thus, for any equilibrium $t$, the investment ratio $q^t$ is an indication of the overall degree of inequality between agents in $t$. The differences in two agents' investments in the link between them, in agents' self-investment levels and in agents' utility levels are all strictly increasing in $q^t$.

Example 1 illustrates a reciprocal equilibrium and two non-reciprocal equilibria with different investment ratios for a specific configuration of the model.

**Example 1.** *Let $n = 5$, $T = 2$, and $u_i(t) = \sum_{j \neq i} t_{ij}^{\beta} t_{ji}^{1-\beta} + t_{ii}^{\beta}$ with $\beta \in (0, 1)$. By Lemma 2, in equilibrium, agent $i$'s marginal utilities from investing in link $ij$ and from self-investment are equal:*

$$av_1(t_{ij}, t_{ji}) = f'(t_{ii}) \quad \Leftrightarrow \quad \beta t_{ij}^{\beta-1} t_{ji}^{1-\beta} = \beta t_{ii}^{\beta-1} \quad \Leftrightarrow \quad t_{ii} = \frac{t_{ij}}{t_{ji}}.$$

*Thus, in every reciprocal equilibrium $t$, $t_{bb} = 1$ and $u_b = 2$. Figure 1 shows an example of a reciprocal equilibrium $t$.*

*In every non-reciprocal equilibrium $t$, $t_{cc} = q^t$, $t_{dd} = \frac{1}{q^t}$, $u_c = (T - q^t)\left(\frac{1}{q^t}\right)^{1-\beta} + q^{t\beta}$ and $u_d = (T - \frac{1}{q^t}) q^{t1-\beta} + \left(\frac{1}{q^t}\right)^{\beta}$. An example of a non-reciprocal equilibrium $t$ where $q^t = \frac{3}{2}$, $C = \{1, 2, 3, 4\}$ and $D = \{5\}$ is shown*

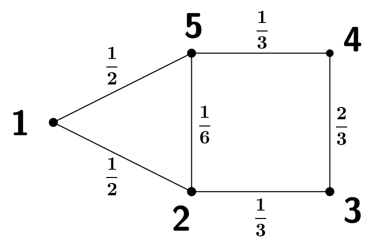
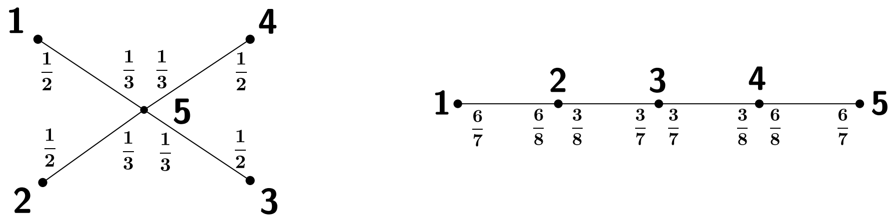



Figure 1: A reciprocal equilibrium $t$. Bold numbers indicate the identity of each agent (node), and the number at link $ij$ is $t_{ij} = t_{ji}$.

*in Figure 2a. An example of a non-reciprocal equilibrium $t'$ with a lower degree of inequality where $q^{t'} = \frac{8}{7}$, $C' = \{1, 3, 5\}$ and $D' = \{2, 4\}$ is shown in Figure 2b.*




(a) $q^t = \frac{3}{2}$ (b) $q^{t'} = \frac{8}{7}$

Figure 2: Two non-reciprocal equilibria $t$ and $t'$ with different investment ratios. Bold numbers indicate the identity of each agent (node), and the number next to agent $i$ at link $ij$ is $t_{ij}$.

## 3.2 Relationship Graphs

In this section, we investigate the graphs of equilibrium networks where a link between two agents means that both invest positively in each other. We will show that simply by observing the graph of an equilibrium we can uniquely determine $q_i$, $t_{ii}$ and $u_i(t)$ for each agent $i$, without any other information about the investment profile.

Let $G^R = \{g \mid g = g(t)$ for some reciprocal equilibrium $t\}$, that is, $G^R$ is the set of all graphs that are induced by some reciprocal equilibrium, and let $G^{NR} = \{g \mid g = g(t)$ for some non-reciprocal equilibrium $t\}$, that is, $G^{NR}$ is the set of all graphs that are induced by some non-reciprocal equilibrium.

We first provide a full characterization of $G^R$.[1] Let $g\,[N\backslash U]$ with $U \subseteq N$ be the subgraph induced in $g$ by $N\backslash U$. Denote by $W(U)$ the set of isolated agents in $g\,[N\backslash U]$ and by $|X|$ the cardinality of a set $X$.

**Proposition 2.** *A connected graph $g$ on $N$ is in $G^R$ if and only if for every $U \subseteq N$,*

*1) $|U| > |W(U)|$, or*

*2) $|U| = |W(U)|$ and for every link $ij \in g$, if $i \in U$, then $j \in W(U)$.*

Proposition 2 states that a connected graph $g$ is induced by some reciprocal equilibrium if and only if for every $U \subseteq N$ either 1) the number of agents in $U$ is strictly larger than the number of isolated agents in $g\,[N\backslash U]$, or 2) the number of agents in $U$ and of isolated agents in $g\,[N\backslash U]$ are the same, and in $g$, agents in $U$ are only linked to agents in $W(U)$. We will refer to Condition 1 and 2 of Proposition 2 as Condition 2.1 and 2.2.

*Proof of Proposition 2: Necessity.* Let a connected graph $g$ be in $G^R$ and let $t$ be a reciprocal equilibria $t$ such that $g(t) = g$.

The total network investment by agents in $U$ is $|U|(T-t_{bb})$ and by agents in $W(U)$ is $|W(U)|(T-t_{bb})$. In $g$, every $i \in W(U)$ is only linked to agents in $U$, otherwise $i \in W(U)$ would not be isolated in $g\,[N\backslash U]$. Thus, the total network investment by agents in $W(U)$ must be fully reciprocated by agents in $U$. Then either $|U|(T-t_{bb}) > |W(U)|(T-t_{bb})$ which means $|U| > |W(U)|$, or $|U|(T-t_{bb}) = |W(U)|(T-t_{bb})$ which means $|U| = |W(U)|$ and agents in $U$ must be linked only to agents in $W(U)$ in $g$. Otherwise, the network investment by agents in $U$ would not be sufficient to fully reciprocate that by agents in $W(U)$. □

The sufficiency proof of Proposition 2 relies on an existence result for a particular type of matching in a graph in Schrijver (2004, p. 584). Because it is largely technical, the proof is relegated to the appendix and only a short outline is provided here. We show first that a reciprocal equilibrium $t$ with $g(t) = g$ exists if a *perfect b-matching* for the connected graph $g$ exists and

[1] Proposition 2 was established with the help of Henning Bruhn-Fujimoto.

second that a perfect $b$-matching for $g$ exists if $g$ is connected and is such that, for every $U \subseteq N$, Condition 2.1 or 2.2 is satisfied.

Proposition 2 provides a tool to determine whether a connected graph $g$ is induced by some reciprocal equilibrium. For this, it is sufficient to determine whether Condition 2.1 or 2.2 is satisfied when $|U| < \frac{n}{2} + 1$, since for $|U| \geq \frac{n}{2} + 1$, Condition 2.1 is trivially satisfied. Some straightforward graph properties simplify this task, as shown below in Corollary 1.

An agent is *a leaf* in graph $g$ if she has only one link. A graph $g$ *is bipartite* or *has a bipartition* if there exists a bipartition $(A, B)$ of $N$ such that if $ij \in g$, then $i$ and $j$ are in different sets of the bipartition.

**Corollary 1.**

*Let $g$ be a connected graph on $N$.*

*a) If $|N_i| > \frac{n}{2}$ for all $i \in N$, then $g \in G^R$.*

*b) If $|N_i| = d > 0$ for all $i \in N$, then $g \in G^R$.*

*c) If $n > 2$ and $g$ contains a leaf, then $g \notin G^R$.*

*d) If $g$ is bipartite with $|A| \neq |B|$, then $g \notin G^R$.*

*Proof.*

*a*) Given $g$, $|U| > \frac{n}{2}$ is necessary to have at least one isolated agent in $g[N \backslash U]$. Hence, $|U| > |W(U)|$ for all $U$ and $g \in G^R$ by Proposition 2.

*b*) Given $g$, every agent in $W(U)$ is linked to $d$ agents in $U$ and thus there exist $d|W(U)|$ links between $U$ and $W(U)$. If every $i \in U$ is only linked to agents in $W(U)$, then $d|U| = d|W(U)|$ and Condition 2.2 is satisfied. If not every $i \in U$ is only linked to agents in $W(U)$, then $d|U| > d|W(U)|$ and Condition 2.1 is satisfied. Thus, $g \in G^R$ by Proposition 2.

*c*) Given $g$, let $i$ be a leaf. Take $U = N_i$. Then, $|W(U)| \geq |U|$ and the only neighbor of agent $i$ is not only linked to $i$ but also to other agents because $n > 2$. Thus, $g \notin G^R$ by Proposition 2.

*d)* Given $g$ where w.l.o.g $|A| > |B|$, take $U = B$. Then, $W(U) = A$ and $|W(U)| > |U|$. Thus, $g \notin G^R$ by Proposition 2.

□

Hence, by Corollary 1a and 1b, any connected graph that is "dense" or "regular" is induced by some reciprocal equilibrium. By Corollary 1c and 1d, graphs which contain leaves (for example, trees), or graphs that are bipartite with two unequally sized sets are never induced by a reciprocal equilibrium.

We next turn to analyze $G^{NR}$. In Proposition 3, we present necessary conditions for a graph to be in $G^{NR}$. Let $\rho : \mathbb{R}^{>1} \to \mathbb{R}^{>1}$ be the function defined by $\rho(x) = x\frac{T-\sigma(\frac{1}{x})}{T-\sigma(x)}$. Given that $\sigma$ is strictly increasing, it is straightforward to show that $\rho(x) > 1$ for all $x$ and that $\rho$ is strictly increasing.

**Proposition 3.** *If $g \in G^{NR}$, then $g$ has a unique bipartition $(A, B)$. W.l.o.g. let $|A| \geq |B|$. For any non-reciprocal equilibrium $t$ with $g(t) = g$, $\frac{|A|}{|B|} = \rho(q^t) > 1$, for every $i \in A$, $q_i = q^t$, and every $j \in B$, $q_j = \frac{1}{q^t}$ and $|N_j| > 1$.*

Proposition 3 makes several statements. Consider any $g \in G^{NR}$. The graph $g$ has exactly one bipartition and the two sets in that bipartition are of unequal size. For any non-reciprocal equilibrium that induces $g$, all concentrated agents are in the larger set of the bipartition and all diversified agents are in the smaller one. Any leaf is in the larger set and thus is a concentrated agent. There is a strictly increasing correspondence between the investment ratio $q^t$ and the ratio of concentrated to diversified agents.

*Proof of Proposition 3.* Let $g$ be in $G^{NR}$ and let $t$ be an equilibrium with $g(t) = g$. By Proposition 1, each link is between a member of $C$ (concentrated agents) and a member of $D$ (diversified agents). Thus, $(C, D)$ is a bipartition of $g$. Since $g$ is connected, a standard result from graph theory implies that the bipartition of $g$ is unique.

For all $i \in C$ and $j \in D$, $\frac{t_{ij}}{t_{ji}} = q^t$ if $ij \in g$ by Proposition 1. Thus $\sum_{i\in C, j\in D} t_{ij} = q^t \sum_{i\in C, j\in D} t_{ji}$ which is equivalent to $\sum_{i\in C}(T - t_{ii}) = q^t \sum_{j\in D}(T - t_{jj})$. Hence $|C|(T - t_{cc}) = q^t|D|(T - t_{dd})$ and $\frac{|C|}{|D|} = q^t\frac{T-t_{dd}}{T-t_{cc}} = q^t\frac{T-\sigma(\frac{1}{q^t})}{T-\sigma(q^t)} = \rho(q^t)$.

Suppose to the contrary that agent $i$ is a leaf and is in $D$. Then $i$'s only link is with some $j \in C$. This implies, using Lemma 2 and Proposition 1, that $t_{ii} + t_{ij} = T < t_{jj} + t_{ji}$ and $j$'s resource constraint is violated. □

The fact that each $g \in G^{NR}$ is bipartite implies that there exists no graph in $G^{NR}$ which includes an odd cycle.[2] Proposition 3 also provides further insight regarding the investment strategies of concentrated and diversified agents in a non-reciprocal equilibrium: A diversified agent has on average more links than a concentrated agent, since the network is connected and $|C| > |D|$.

The next result shows that a reciprocal equilibrium and a non-reciprocal equilibrium never induce the same graph. Moreover, some graphs cannot be induced by any equilibrium.

**Proposition 4.** *For every $n \geq 2$, $G^R \cap G^{NR} = \emptyset$, and for every $n \geq 4$, there exists a connected graph $g$ on $N$ such that $g \notin G^R \cup G^{NR}$.*

*Proof.* By Corollary 1d, there exists no $g \in G^R$ with a bipartition where the two sets of the bipartition are of unequal size. By Proposition 3, every $g \in G^{NR}$ has a bipartition with the two sets of unequal size. Thus, $G^R \cap G^{NR} = \emptyset$.

Let $n \geq 4$ and consider the following graph. Agents 1, 2, and 3 form a triangle. Every other agent is only linked to agent 1. Thus, $g$ includes an odd cycle and hence $g \notin G^{NR}$. Moreover, $g$ contains a leaf and hence $g \notin G^R$. □

Another family of graphs (in addition to the one described in the proof above) that cannot be induced by any equilibrium is one in which two leaves are connected via an odd number of links: If such a graph were in $G^R$, it would not include a leaf, and if it were in $G^{NR}$, both leaves would be in $C$ and thus would have to be connected via an even number of links.

Proposition 3 and 4 imply that the information about the graph $g$ induced by an equilibrium $t$ is sufficient to determine $q_i$, $t_{ii}$, and $u_i(t)$ for all $i$ in equilibrium $t$. If $g$ has a bipartition $(A, B)$ where $A$ and $B$ are of unequal

[2] An *odd cycle* is a sequence of links $i_1 i_2, ..., i_{K-1} i_K$ where $i_j \neq i_k$ for $k \notin \{1, K\}$, $i_1 = i_K$ and $K > 2$ is even.

size and w.l.o.g. $|A| > |B|$, then any equilibrium $t$ that induces $g$ is non-reciprocal where $q^t = \rho^{-1}\left(\frac{|A|}{|B|}\right)$ and for all $i \in A$, $q_i = q^t$, and for all $i \in B$, $q_i = \frac{1}{q^t}$. Otherwise, any equilibrium that induces $g$ is reciprocal, and $q_i = 1$ for all $i$. Self-investment and utility levels follow from Proposition 1.

## 3.3 Equilibrium Multiplicity

We now turn to equilibrium multiplicity. Based on the previous section, equilibria that induce the same graph must feature the same values of $q_i$, $t_{ii}$ and $u_i(t)$ for all $i$ because the equilibrium graph uniquely determines those values. However, equilibria that induce the same graph may feature different levels of investments in links and those are therefore not uniquely determined by the graph. We capture the multiplicity with a simple mechanism that derives an equilibrium $t'$ from an equilibrium $t$. The mechanism relies on appropriately shifting link investment levels in $t$ on an even-lengthed cycle.

**Proposition 5.** *Let $n \geq 4$ and let $t$ be an equilibrium with $g(t) = g$. Then, the following strategy profile $t'$ is also an equilibrium.*

*First, let $S$ be a sequence of distinct agents $i_1, i_2, ..., i_{K-1}$ and $i_K = i_1$ such that $K > 4$ is an odd integer, $i_k i_{k+1} \in g$ for odd $k$, and if $q^t > 1$, then $i_k \in C$ and $i_{k+1} \in D$ for odd $k$. Second, let $t'$ be equal to $t$, except for $t'_{i_k i_{k+1}} = t_{i_k i_{k+1}} + x > 0$ and $t'_{i_k i_{k-1}} = t_{i_k i_{k-1}} - x > 0$ for all $k \in \{1, ..., K-1\}$ where $x = -h$ if $k$ is odd, $x = l$ if $k$ is even, and $\frac{h}{l} = q^t$.*

*Proof.* Consider agent $i_k$. The only change in agent $i_k$'s strategy from $t$ to $t'$ is a shift of her investment by an amount $x$ between agents $i_{k-1}$ and $i_{k+1}$. Thus, agent $i_k$'s budget constraint remains binding in $t'$. The only investments by other agents in agent $i_k$ that have changed from $t$ to $t'$ are those of agents $i_{k-1}$ and $i_{k+1}$. Thus, $t'_{i_k j} = t'_{j i_k} = 0$ if $i_k j \notin g(t)$ and $\frac{t'_{i_k j}}{t'_{j i_k}} = q_{i_k}$ if $i_k j \in g(t)$ for all $j \neq i_{k-1}, i_{k+1}$. We next show that $\frac{t'_{i_k j}}{t'_{j i_k}} = q_{i_k}$ also for $j = i_{k-1}, i_{k+1}$. If $k$ is odd and $j = i_{k+1}$, then $q_{i_k} = q^t$ and

$$\frac{t'_{i_k,i_{k+1}}}{t'_{i_{k+1},i_k}} = \frac{t_{i_k,i_{k+1}} - h}{t_{i_{k+1},i_k} - l} = \frac{q^t\left(t_{i_k,i_{k+1}} - h\right)}{q^t\left(t_{i_{k+1},i_k} - l\right)} = \frac{q^t\left(t_{i_k,i_{k+1}} - h\right)}{\frac{t_{i_k,i_{k+1}}}{t_{i_{k+1},i_k}} t_{i_{k+1},i_k} - \frac{h}{l} l} = q^t \frac{t_{i_k,i_{k+1}} - h}{t_{i_k,i_{k+1}} - h} = q^t.$$

Similarly, for $j = i_{k-1}$ and then also for even $k$.

For any agent $i$ not in the sequence $S$, $t'_i = t_i$ and the investments in $i$ are the same in both $t'$ and $t$. Thus, $t'$ is an equilibrium by Lemma 2.

□

In Example 2, we apply the mechanism provided in Proposition 5.

**Example 2.** *Consider the environment of Example 1. In Figure 3, $t$ is a reciprocal equilibrium, and in Figure 4, $t$ is a non-reciprocal equilibrium with $q^t = \frac{8}{7}$. In both, $t'$ is an equilibrium obtained from $t$ by applying the mechanism described in Proposition 5.*

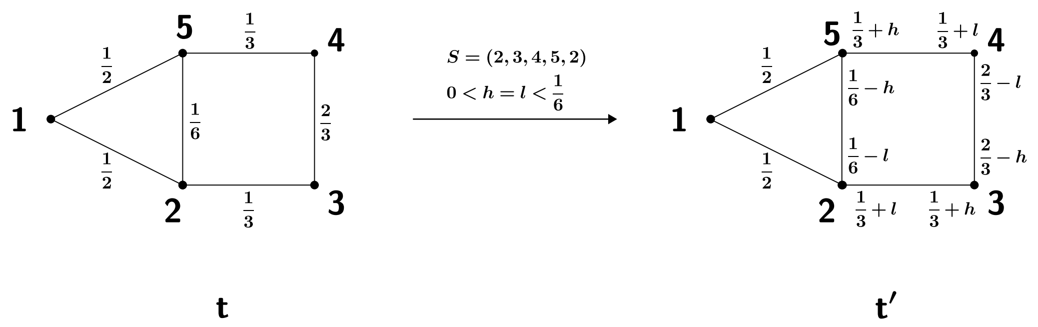



Figure 3: Deriving a reciprocal equilibrium $t'$ from the reciprocal equilibrium $t$.

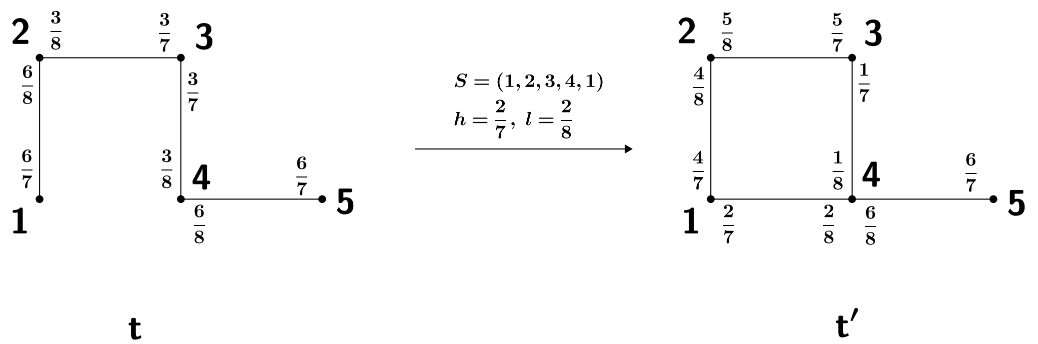



Figure 4: Deriving a non-reciprocal equilibrium $t'$ from the non-reciprocal equilibrium $t$.

## 3.4 Comparative Statics

In this section, we present the comparative statics of the equilibria for the following changes: (i) when the investment in a relationship becomes more valuable relative to self-investment (i.e. an increase in $a$) and (ii) the total resources available for investment increase (i.e. an increase in $T$). Given the multiplicity of equilibria, we limit ourselves to the following comparative statics exercise: Consider a particular graph induced by an equilibrium. Assuming that the graph remains unchanged, what is the effect of a change in the model parameters on those equilibrium values of the strategy profile that are uniquely determined by the graph and the parameters?

Note that previously we assumed for simplicity that $\lim_{x\to T} f'(x) = 0$ in order to guarantee that every agent chooses a self-investment less than $T$ in any equilibrium. Keeping this assumption and altering $T$ would change the function $f$ and render the comparative statics for $T$ impossible. In this section, we assume that $f$ is fixed and that all values of $T$ satisfy $f'(T) < v_1(n-1,1)$. That is, $f'(T)$ is low enough to guarantee equilibrium self-investments less than $T$ for all $T$.

**Proposition 6.**

a) *Consider $a < \hat{a}$. Let $t$ be an equilibrium given $a$, and let $\hat{t}$ be an equilibrium given $\hat{a}$ where $t$ and $\hat{t}$ induce the same graph ($g(t) = g(\hat{t})$).*

   *If $q^t = 1$, then $\hat{q}^t = 1$ and $t_{bb} > \hat{t}_{bb}$.*

   *If $q^t > 1$, then $\hat{q}^t > 1$, $\frac{|C|}{|D|} = \frac{|\hat{C}|}{|\hat{D}|}$, $t_{cc} > \hat{t}_{cc}$, and $t_{dd} > \hat{t}_{dd}$.*

b) *Consider $T < \hat{T}$ and suppose that $f = \hat{f}$ with $f'(T) < v_1(n-1,1)$. Let $t$ be an equilibrium given $T$, and let $\hat{t}$ be an equilibrium given $\hat{T}$ where $g(t) = g(\hat{t})$.*

   *If $q^t = 1$, then $\hat{q}^t = 1$ and $t_{bb} = \hat{t}_{bb}$.*

   *If $q^t > 1$, then $q^t < \hat{q}^t$, $\frac{|C|}{|D|} = \frac{|\hat{C}|}{|\hat{D}|}$, $t_{cc} < \hat{t}_{cc}$, $T - t_{cc} < \hat{T} - \hat{t}_{cc}$ and $t_{dd} > \hat{t}_{dd}$.*

Before presenting the proof, it is worthwhile restating Proposition 6. Consider part $a$). Unsurprisingly, since investment in a relationship is more valuable under $\hat{a}$ than under $a$, the self-investment of all types of agents is lower under $\hat{a}$ than under $a$. If $t$ is a non-reciprocal equilibrium, then the ratio of concentrated to diversified agents remains the same, and the effect on the investment ratio is not uniquely determined.

Consider part $b$). As the resource endowment increases, the self-investment level of balanced agents remains the same and therefore their network investment increases. If $t$ is a non-reciprocal equilibrium, then the concentrated-to-diversified-agents ratio remains unchanged. Both a concentrated agent's self-investment and her network investment are increasing in the resource endowment, whereas a diversified agent's self-investment decreases and thus her network investment increases by a larger amount than the increase in the resource endowment. An increase in the resource endowment also increases the ratio of a concentrated agent's investment to that of a diversified agent in the link between them.

*Proof of Proposition 6, part a*). (The proof for part $b$) proceeds similarly and is relegated to the appendix.)

Consider $q^t = 1$. Then, $g(t) \in G^R$. The set $G^R$ is independent of $a$ by Proposition 2. Therefore, $g(\hat{t}) \in \hat{G}^R$ and $\hat{q}^t = 1$. By Proposition 1, $f'(t_{bb}) = av_1(1,1)$. Then, by the implicit function theorem, $\frac{\partial t_{bb}}{\partial a} = \frac{v_1(1,1)}{f''(t_{bb})}$ which is strictly negative because $f$ is strictly concave and $v$ is strictly increasing. Thus, $t_{bb} > \hat{t}_{bb}$.

Next consider $q^t > 1$. Then $g(t) = g(\hat{t}) \notin G^R = \hat{G}^R$ and thus $\hat{q}^t > 1$. By Proposition 3, the bipartition of $g(t)$ is unique, and thus $\frac{|C|}{|D|} = \frac{|\hat{C}|}{|\hat{D}|}$. By Proposition 1 and 3, $\hat{t}$ is such that i) $f'(\hat{t}_{cc}) = \hat{a}v_1(\hat{q}^t, 1)$, ii) $f'(\hat{t}_{dd}) = \hat{a}v_1(\frac{1}{\hat{q}^t}, 1)$, and iii) $\frac{|C|}{|D|} = \hat{q}^t \frac{T-\hat{t}_{dd}}{T-\hat{t}_{cc}}$, and the analogous conditions hold for $t$. Suppose by contradiction that $\hat{t}_{cc} \geq t_{cc}$. Then $\hat{q}^t > q^t$ by i). This implies that $\hat{t}_{dd} < t_{dd}$ by ii). Then $\frac{|C|}{|D|} < \hat{q}^t \frac{T-\hat{t}_{dd}}{T-\hat{t}_{cc}}$, contradicting iii). The proof that $\hat{t}_{dd} < t_{dd}$ proceeds analogously.

□

# 4 Stability and Efficiency

In this section, we will show that Nash equilibria are not "stable" if pairwise deviations are allowed, and are not efficient, in the sense that they do not maximize the sum of agents' utilities. This is due to the positive externalities of an agent's network investment on her neighbors, which are not incorporated into an agent's individual utility maximization.

## 4.1 Equilibrium Stability

Following Bloch and Dutta (2009), we say that a strategy profile $t$ is strongly pairwise stable if it is a Nash equilibrium and if there are no two agents $(i, j)$ who would both be strictly better off by a joint deviation from $(t_i, t_j)$ to $(t'_i, t'_j)$, given all other agents' strategies.

**Proposition 7.** *No strategy profile $t$ is strongly pairwise stable.*

*Proof.* We will show that for any equilibrium, there exist two agents who gain from reducing their self-investment and establishing or intensifying a reciprocal relationship among them. Suppose $t$ is an equilibrium. Consider any $i \in N$ and any $j \neq i$. If $i$ reduces her self-investment by $c > 0$, and $i$ and $j$ each invest $c$ in order to intensify or establish a reciprocal link between them, then $i$'s change in utility is $\Delta u_i(c) = f(t_{ii} - c) - f(t_{ii}) + cav(1, 1)$. If $i$ is a balanced or concentrated agent, then there is $c > 0$ such that $\Delta u_i(c)$ is positive because $\frac{\partial \Delta u_i}{\partial c}(0) = -f'(t_{ii}) + av(1, 1)$ and $f'(t_{ii}) < av(1, 1) = a(v_1(1, 1)+v_2(1, 1))$ for $t_{ii} \in \{t_{bb}, t_{cc}\}$. If $q^t = 1$, then there exist two balanced agents, and if $q^t > 1$, then there exist two concentrated agents, and therefore, in each case there is a pair with a strict incentive to jointly deviate. □

## 4.2 Efficient Networks

Following Jackson and Wolinsky (1996), we say that a strategy profile $t$ is efficient if $t$ maximizes $\sum_{i \in N} u_i(t)$ such that $\sum_j t_{ij} = T$ for all $i$.

We will see that the set of efficient networks and the set of equilibrium networks do not intersect. However, there is no distinction between the set of all graphs that are induced some efficient network and the set of all graphs that are induced some reciprocal equilibrium.

**Proposition 8.** *A network $t$ is efficient if and only if $t_{ij} = t_{ji}$, $\sum_k t_{ik} = T$ and $f'(t_{ii}) = av(1,1)$ for all $i$ and all $j \neq i$. A graph is induced by an efficient network if and only if each of its components is induced by some reciprocal equilibrium of the network formation game reduced to the agents in that component.*

Proposition 8 states that in every efficient network any two agents invest the same amount in each other. Moreover, any agent's self-investment is such that her marginal utility from self-investment equals the marginal increase in the sum of her own and her neighbor's utility from her investment in their reciprocal link. Thus, the efficient level of self-investment accounts for the positive externalities from network investment and is lower than the level of self-investment in a reciprocal equilibrium. Since every agent's self-investment is less than $T$ (as implied by the assumptions on $f$), there is no isolated agent in an efficient network. In particular, the set of all graphs of efficient networks is identical to the set of all graphs of equilibrium networks which only consist of reciprocal equilibrium components.

*Proof of Proposition 8.* Let $t$ be efficient. Then every agent's resource constraint is binding, since self-investment is always beneficial. Moreover, for all $i$ and $j \neq i$, $t_{ii} > 0$, and $t_{ij} = 0$ if and only if $t_{ji} = 0$. By the first-order conditions on $t$ to maximize the sum of utilities, any positive link investments $t_{ii}$, $t_{ij}$ and $t_{ji}$ must satisfy $f'(t_{ii}) = av_1(t_{ij}, t_{ji}) + av_2(t_{ji}, t_{ij}) = av_1\left(\frac{t_{ij}}{t_{ji}}, 1\right) + av_2\left(1, \frac{t_{ij}}{t_{ji}}\right)$ for all $j \in N_i$ and all $i$. In other words, agent $i$'s investment in her link with agent $j$ is such that its marginal impact on the sum of utilities equals agent $i$'s marginal utility from self-investment.

We next show that any link is reciprocal, that is $t_{ij} = t_{ji}$ for all $t_{ij}, t_{ji} > 0$. Suppose to the contrary that link $ij$ is non-reciprocal and w.l.o.g $\frac{t_{ij}}{t_{ji}} > 1$. Hence, $f'(t_{ii}) = av_1\left(\frac{t_{ij}}{t_{ji}}, 1\right) + av_2\left(1, \frac{t_{ij}}{t_{ji}}\right) < f'(t_{jj}) = av_1\left(\frac{t_{ji}}{t_{ij}}, 1\right) + av_2\left(1, \frac{t_{ji}}{t_{ij}}\right)$ by the concavity of $v$ and $t_{ii} > t_{jj}$ by the concavity of $f$. Since $t_{ij} + t_{ii} > t_{ji} + t_{jj}$ and $i$'s resource constraint must bind, $j$ must have another link to some agent $k \neq i$. By efficiency and the strict concavity of $v$, $\frac{t_{ji}}{t_{ij}} = \frac{t_{jk}}{t_{kj}} < 1$, and $\frac{t_{kl}}{t_{lk}} = \frac{t_{im}}{t_{mi}} > 1$ for all $l \in N_k$ and $m \in N_i$ which implies that $i$ and $k$ are not linked and $t_{ik} = t_{ki} = 0$. Now consider strategy profile $t' \neq t$ where the self-investment of both $i$ and $k$ is reduced by $c$ and a reciprocal link between them is established with an investment of $c$ by each. As in the proof for Proposition 7, we can show that there is $c > 0$ such that $u_i(t') - u_i(t) > 0$ and $u_k(t') - u_k(t) > 0$. Moreover, $u_l(t') = u_l(t)$ for all $l \neq i, k$ and no agent's resource constraint has been affected by moving from $t$ to $t'$. Hence, a non-reciprocal link cannot exist in an efficient network.

Thus, $t_{ij} = t_{ji}$ for all $i$ and $j$, and $av_1(1,1) + av_2(1,1) = av(1,1) = f'(t_{ii})$ for all $i$ who have a link. It remains to show that every agent has a link.

Assume there exist at least two isolated agents $i$ and $j$. Then, $t_{ii} = t_{jj} = T$ by efficiency. However, by the same argument as in the proof of Proposition 7, the sum of utilities can be increased if their self-investment is decreased and a reciprocal link between them is established.

Assume there exists only one isolated agent $i$. Let agents $j$ and $k$ be linked to each other. In this case, the sum of utilities can be increased as follows: Decrease $i$'s self-investment by $2\epsilon > 0$ and the investments by $j$ and $k$ in their link $jk$ by $\epsilon$ each and establish the reciprocal links $ij$ and $ik$, with $t'_{ij} = t'_{ji} = t'_{ik} = t'_{ki} = \epsilon$. For $\epsilon$ small enough, $i$'s utility strictly increases and the utility of no other agent changes, by $v$'s homogeneity of degree 1.

Let $t$ be such that $t_{ij} = t_{ji}$, $\sum_k t_{ik} = T$ and $f'(t_{ii}) = av(1,1)$ for all $i$ and all $j \neq i$. Then, agent $i$'s utility is $u_i(t) = \sum_{j \neq i} av(t_{ij}, t_{ji}) + f(t_{ii}) = \sum_{j \in N_i} av(t_{ij}, t_{ji}) + f(t_{ii}) = \sum_{j \in N_i} at_{ij}v(1, \frac{t_{ji}}{t_{ij}}) + f(t_{ii}) = \sum_{j \in N_i} a(T - t_{ii})v(1,1) + f(t_{ii})$. Thus, the sum of utilities for any such $t$ is the same and hence any such $t$ is efficient. This concludes the first part of the proof of Proposition 8.

To prove the second part, first observe that we know from the first part of Proposition 8 that in an efficient network, every agent belongs to a component of at least two agents who are connected via reciprocal links.

The result then follows from the observation that a connected graph $g$ is induced by an efficient network if and only if $g$ is induced by a reciprocal equilibrium: Let $t$ be an efficient network that induces a connected graph $g$. Thus, $t_{ij} = t_{ji} > 0$ for all $ij \in g$, $t_{ij} = t_{ji} = 0$ for all $ij \notin g$, $\sum_{j \in N_i} t_{ij} = T - t_{ii}$ for all $i$ and $f'(t_{ii}) = av(1,1)$. Then, the following $t'$ is a reciprocal equilibrium that induces $g$. Let $t'$ be such that $t'_{ij} = t'_{ji} = t_{ij} \frac{T - t'_{ii}}{T - t_{ii}}$ for all $ij \in g$, $t'_{ij} = t'_{ji} = 0$ for all $ij \notin g$, and $f'(t'_{ii}) = av_1(1,1)$.

Let $t'$ be a reciprocal equilibrium that induces a connected graph $g$. Thus, $t'_{ij} = t'_{ji} > 0$ for all $ij \in g$, $t'_{ij} = t'_{ji} = 0$ for all $ij \notin g$, $\sum_{j \in N_i} t'_{ij} = T - t'_{ii}$ for all $i$ and $f'(t'_{ii}) = av_1(1,1)$. Then, the following $t$ is a connected efficient network that induces $g$. Let $t$ be such that $t_{ij} = t_{ji} = t'_{ij} \frac{T - t_{ii}}{T - t'_{ii}}$ for all $ij \in g$, $t_{ij} = t_{ji} = 0$ for all $ij \notin g$, and $f'(t_{ii}) = av(1,1)$. □

# 5 Concluding Comments

We analyzed a game of weighted network formation in which agents simultaneously decide how to allocate a limited budget between building links of possibly different intensities with other agents and self-investment. Expanding the discussion of network formation from unweighted to weighted networks enlarges the strategy space of agents. Nevertheless, we obtained results about the structure of the game's equilibria. In particular, we showed that an equilibrium must have one of two structures, i.e. either reciprocal or non-reciprocal, and we characterized their properties.

Some of the results are consistent with empirical findings. First, note that in both reciprocal and non-reciprocal equilibria two agents' investments in the link between them are predicted to be positively correlated. Griffith (2017) finds support for this property in his analyis of a weighted social network among school girls. He shows that the weights assigned by two girls to their relation are positively (though not perfectly) correlated.

The presence of reciprocal and non-reciprocal relations is investigated in Wang et al. (2013). They find that in a mobile phone communication network, 72% of all links are such that the two linked agents call each other with significantly different probabilities. They further suggest that the presence of reciprocal relations is more likely when the total network investment (number of calls made) by an agent is positively correlated across linked agents.

This is in line with our theoretical findings: In a reciprocal equilibrium, each agent chooses the same total network investment which is thus predicted to be perfectly and positively correlated across agents, while in a non-reciprocal equilibrium, the level of total network investments by a concentrated agent is negatively correlated with that by a diversified agent.

We also characterized the properties of the graphs for reciprocal and non-reciprocal equilibria. We showed, for example, that any sufficiently "dense" graph, where *density* is measured by the number of links in the graph, or a "regular" graph, in which every agent has the same number of neighbors, is only induced by reciprocal equilibria (Corollary 1a and 1b). Some empirical studies provide evidence for a positive correlation between reciprocity and network density and/or regularity (for example, Kovanen et al. (2010) and Wang et al. (2013) for mobile phone communication networks). We also found that in graphs of non-reciprocal equilibria, diversified agents have on average more links than concentrated agents. This again resonates with Wang et al. (2013) who suggest that "networked systems that induce anti-correlation in the number of neighbors of each vertex [agent] in a dyad [link] should  all else being equal be characterized by high levels of non-reciprocity".

On an anecdotal level, the three types of agents that arise in our model's equilibria can perhaps be observed in real life. *Diversified* agents are more popular and outgoing, they more actively network and free-ride on the efforts of other agents. *Concentrated* agents rely more on themselves, are more introverted, provide greater effort in relationships, and are exploited. *Balanced* agents are in give-and-take relationships and share responsibilities equally.

An avenue for further research would be to introduce heterogeneity between agents and to investigate how this affects the existence and properties of reciprocal and non-reciprocal equilibria. A first step could be to differentiate between two types of agents, where linking to one of the types is more profitable than linking to the other.

# Appendix

**Proof of Lemma 1.** Since $f$ and $v$ are both increasing and strictly concave, it immediately follows that $\sigma$ is strictly increasing.

We next show that $\frac{\partial \mu}{\partial x}$ is strictly negative.

$$\begin{aligned}
\frac{\partial \mu}{\partial x} &= -\sigma'(x) a v\left(1, \frac{1}{x}\right) - (T - \sigma(x)) a v_2\left(1, \frac{1}{x}\right)\frac{1}{x^2} + f'(\sigma(x))\sigma'(x) &(1)\\
&= -\sigma'(x) a v\left(1, \frac{1}{x}\right) - (T - \sigma(x)) a v_2\left(1, \frac{1}{x}\right)\frac{1}{x^2} + a v_1(x,1)\sigma'(x) &(2)\\
&= -(T - \sigma(x)) a v_2\left(1, \frac{1}{x}\right)\frac{1}{x^2} + \sigma'(x)\left[a v_1(x,1) - a v\left(1, \frac{1}{x}\right)\right] &(3)\\
&= -(T - \sigma(x)) a v_2\left(1, \frac{1}{x}\right)\frac{1}{x^2}\\
&\quad + \sigma'(x)\left\{a v_1\left(1, \frac{1}{x}\right) - a\left[v_1\left(1, \frac{1}{x}\right) + \frac{1}{x} v_2\left(1, \frac{1}{x}\right)\right]\right\} &(4)\\
&= -(T - \sigma(x)) a v_2\left(1, \frac{1}{x}\right)\frac{1}{x^2} - \sigma'(x) a \frac{1}{x} v_2\left(1, \frac{1}{x}\right) &(5)\\
&< 0 &(6)
\end{aligned}$$

To get from (1) to (2), we use $f'(\sigma(x)) = v_1(x, 1)$; from (3) to (4), we use Euler's Theorem and that $v_1$ is homogeneous of degree 0; and from (5) to (6), we use $\sigma'(x) > 0$.

□

**Sufficiency proof of Proposition 2.** In the following we prove that there exists a reciprocal equilibrium $t$ with $g(t) = g$ and hence $g \in G^R$ if $g$ is connected and is such that for every $U \subseteq N$ Condition 2.1 or 2.2 is satisfied.

For the proof, we draw on Theorem 35.1 in Schrijver (2004, p. 584), which states necessary and sufficient conditions for a *perfect b-matching* to exist for a graph $g$. A perfect $b$-matching for $g$ is a function which assigns a value to each link such that the sum of the values of links incident at one node is equal to the *b-value* of that node.

We will first show that if a perfect $b$-matching for a connected graph $g$

exists, then a reciprocal equilibrium $t$ with $g(t) = g$ exists, and second, that if $g$ is connected and is such that for all $U \subseteq N$ either condition 2.1 or 2.2 is true, then a perfect b-matching for $g$ exists (for the second part, we use the theorem in Schrijver (2004)). This will prove sufficiency for Proposition 2.

Consider a graph $g$ on $N$. Let $E^g$ be the set of all links in $g$ and let $E^g[X, Y]$ be the set of links $xy \in g$ with $x \in X \subseteq N$, $y \in Y \subseteq N$ and $X \cap Y = \emptyset$. Let $E^g[Y]$ be the set of links $ij \in g$ with $i, j \in Y \subseteq N$. Denote by $\delta(i)$ the set of links incident at node $i \in N$. Let $g[Y]$ be the subgraph induced in $g$ by $Y \subseteq N$. For every vector $w \in \mathbb{R}^Y$ with vector components $w_y$, let $w(U) := \sum_{y \in U} w_y$ for any $U \subseteq Y$. The set of integers is denoted by $\mathbb{Z}$.

Considering just a special case, Theorem 35.1 in Schrijver (2004, p. 584) can be reduced to the following statement.

**Special case of Theorem 35.1 in Schrijver (2004, p. 584).** *Let $g$ be a graph on $N$ and let $b \in \mathbb{Z}^N$ and $c \in \mathbb{Z}^{E^g}$ with every $c_{ij} > 1$. Then, there exists an $x \in \mathbb{Z}^{E^g}$ such that (i) $1 \leq x_{ij} \leq c_{ij}$ for all $ij \in E^g$ and (ii) $x(\delta(i)) = b_i$ for all $i \in N$ if and only if for each partition $\{T, V, Y\}$ of $N$, the number of components $K$ of $g[T]$ with*

*(35.2)* $b(K) + c(E^g[K, Y]) + |E^g[K, V]|$

*odd is at most*

*(35.3)* $b(V) - 2|E^g[V]| - |E^g[T, V]| - b(Y) + 2c(E^g[Y]) + c(E^g[T, Y])\,.$

Let every $c_{ij} = \gamma$ with $\gamma$ extremely large and every $b_i = \beta$ with $\beta$ sufficiently large. If $g$ is connected and $x$ given $g$ exists, then a reciprocal equilibrium $t$ with $g(t) = g$ is such that $t_{ij} = t_{ji} = \frac{x_{ij}}{\beta}(T - t_{bb})$ for all $ij \in E^g$ and $t_{ij} = t_{ji} = 0$ for all $ij \notin E^g$.

Let $g$ be connected and such that for all $U \subseteq N$, with $W(U)$ being the set of isolates in $g\,[N \backslash U]$, either

1. $|U| > |W(U)|$, or

2. $|U| = |W(U)|$ and for every link $ij \in E^g$, if $i \in U$, then $j \in W(U)$.

We will next show by contradiction that $x$ exists given $g$.

Suppose $x$ does not exist. Then, by the theorem in Schrijver (2004, p. 584), there must be a partition $\{T, V, Y\}$ of $N$ such that the number of components $K$ of $g[T]$ with (35.2) odd is greater than (35.3); otherwise $x$ would exist.

For any partition with $E^g[Y] \neq \emptyset$ and/or $E^g[T, Y] \neq \emptyset$, the number of components $K$ with (35.2) odd is always smaller than (35.3) because $\gamma$ is extremely large and the number of components $K$ is finite. Then, there must be a partition with $E^g[Y] = E^g[T, Y] = \emptyset$ with a number of components $K$ with (35.2) odd greater than (35.3).

For every partition $\{T, V, Y\}$ with $E^g[Y] = E^g[T, Y] = \emptyset$, it must be true that every $i \in Y$ has links to nodes in $V$ only and that every $i \in Y$ has at least one link to nodes in $V$ because $g$ is connected. Then, $Y$ is a subset of the set of isolates in $g[N \backslash V]$. Hence, $Y \subseteq W(U)$ for $U = V$. We know that in $g$ for all $U \subseteq N$ either 1. $|U| > |W(U)|$, or 2. $|U| = |W(U)|$ and for every link $ij \in E^g$, if $i \in U$, then $j \in W(U)$. This implies that, for any $V$, either 1. $|V| > |Y|$, or 2. $|V| = |Y|$ and for every link $ij \in E^g$, if $i \in V$, then $j \in Y$. Thus, there does not exist a partition $\{T, V, Y\}$ of $N$ for which $E^g[Y] = E^g[T, Y] = \emptyset$ and $|V| < |Y|$.

Then, there must be a partition $\{T, V, Y\}$ of $N$ for which $E^g[Y] = E^g[T, Y] = \emptyset$ and $|V| \geq |Y|$ such that the number of components $K$ with (35.2) odd is greater than (35.3).

For any partition with $E^g[Y] = E^g[T, Y] = \emptyset$ and $|V| > |Y|$, the number of components K with (35.2) odd is always smaller than (35.3) because $\beta$ is chosen sufficiently large.

For any partition with $E^g[Y] = E^g[T, Y] = \emptyset$ and $|V| = |Y|$, we know that for every link $ij \in E^g$, if $i \in V$, then $j \in Y$. (The reason is that if $|U| = |W(U)|$, then for every link $ij \in E^g$ with $i \in U$ it is true that $j \in W(U)$, and for $U = V$ in this case $W(U) = Y$.) Then, $E^g[V] = E^g[T, V] = E^g[T, Y] = \emptyset$. This implies that $T = \emptyset$. If $T$ were not empty, nodes in $T$ would not be connected to either $V$ or $Y$, and $g$ would not be connected, a contradiction. From $T = \emptyset$, it follows that the number of components $K$ is zero. (35.3) is also zero. Hence, the number of components $K$ is not greater than (35.3).

Thus, there does not exist any partition $\{T, V, Y\}$ of $N$ such that the number of components $K$ with (35.2) odd is greater than (35.3). This is a contradiction and therefore $x$ must exist. Thus, there also exists a reciprocal equilibrium $t$ with $g(t) = g$. □

**Proof of Proposition 6.** Proof of part $b$).

First, suppose that $q^t = 1$. Then $g(t) \in G^R$. The set $G^R$ is independent of $T$ by Proposition 2. Thus $g(\hat{t}) \in \hat{G}^R$ and $\hat{q}^t = 1$. By Proposition 1, $f'(t_{bb}) = av_1(1, 1)$. Thus, $t_{bb}$ is independent of $T$ and therefore, $t_{bb} = \hat{t}_{bb}$.

Second, suppose that $q^t > 1$. Then, $g(t) = g(\hat{t}) \notin G^R = \hat{G}^R$ and thus $\hat{q}^t > 1$. By Proposition 3, the bipartition of $g(t)$ is unique, and therefore $\frac{|C|}{|D|} = \frac{|\hat{C}|}{|\hat{D}|}$. By Proposition 1 and 3: i) $f'(\hat{t}_{cc}) = av_1(\hat{q}^t, 1)$, ii) $f'(\hat{t}_{dd}) = av_1(\frac{1}{\hat{q}^t}, 1)$, and iii) $\frac{|C|}{|D|} = \hat{q}^t \frac{\hat{T} - \hat{t}_{dd}}{\hat{T} - \hat{t}_{cc}}$. The analogous conditions hold for $t$. Applying the implicit function theorem to the equation $\frac{|C|}{|D|} = q^t \frac{T - t_{dd}}{T - t_{cc}}$, we get:

$$\frac{\partial q^t}{\partial T} = -\frac{\frac{(T - \sigma(q^t)) - (T - \sigma(\frac{1}{q^t}))}{(T - \sigma(q^t))^2}}{\frac{T - \sigma(\frac{1}{q^t})}{T - \sigma(q^t)} + q^t \frac{\sigma'(\frac{1}{q^t}) \frac{1}{q^{t2}} (T - \sigma(q^t)) + (T - \sigma(\frac{1}{q^t})) \sigma'(q^t)}{(T - \sigma(q^t))^2}}.$$

It is straightforward to show – keeping in mind that $\sigma$ is strictly increasing – that the numerator is negative and the denominator positive and therefore $\frac{\partial q^t}{\partial T} > 0$.

Thus, $\hat{q}^t > q^t$, $\hat{t}_{cc} > t_{cc}$ by (i) and $\hat{t}_{dd} < t_{dd}$ by (ii). Hence $\hat{T} - \hat{t}_{dd} > T - t_{dd}$. From $\hat{q}^t > q^t$, it follows that

$$\frac{\hat{T} - \hat{t}_{cc}}{\hat{T} - \hat{t}_{dd}} > \frac{T - t_{cc}}{T - t_{dd}}$$

and therefore $\hat{T} - \hat{t}_{cc} > T - t_{cc}$. □

# References

Bala, V. and Goyal, S. (2000), 'A noncooperative model of network formation', *Econometrica* **68**(5), 1181–1229.

Bloch, F. and Dutta, B. (2009), 'Communication networks with endogenous link strength', *Games and Economic Behavior* **66**, 39–56.

Brueckner, J. K. (2006), 'Friendship networks', *Journal of Regional Science* **46**(5), 847–865.

Cabrales, A., Calvó-Armengol, A. and Zenou, Y. (2011), 'Social interactions and spillovers', *Games and Economic Behavior* **72**(2), 339–360.

Deroïan, F. (2009), 'Endogenous link strength in directed communication networks', *Mathematical Social Sciences* **57**(1), 110–116.

Durieu, J., Haller, H. and Solal, P. (2011), 'Nonspecific networking', *Games* **2**(1), 87–113.

Galeotti, A. and Merlino, L. P. (2014), 'Endogenous job contact networks', *International Economic Review* **55**(4), 1201–1226.

Golub, B. and Livne, Y. (2010), 'Strategic random networks and tipping points in network formation'. *Unpublished working paper.*

Goyal, S., Moraga-González, J. L. and Konovalov, A. (2008), 'Hybrid R&D', *Journal of the European Economic Association* **6**(6), 1309–1338.

Griffith, A. (2017), 'Random assignment with non-random peers: a structural approach to counterfactual treatment assessment'. *Unpublished working paper.*

Jackson, M. O. and Wolinsky, A. (1996), 'A strategic model of social and economic networks', *Journal of Economic Theory* **71**, 44–74.

Kovanen, L., Saramäki, J. and Kaski, K. (2010), 'Reciprocity of mobile phone calls', *arXiv preprint, arXiv: 1002.0763* .

Rogers, B. (2006), 'A strategic theory of network status'. *Unpublished working paper.*

Salonen, H. (2015), 'Reciprocal equilibria in link formation games', *Czech Economic Review* **9**(3), 169–183.

Schrijver, A. (2004), *Combinatorial optimization: polyhedra and efficiency*, Springer.

Wang, C., Lizardo, O., Hachen, D., Strathman, A., Toroczkai, Z. and Chawla, N. V. (2013), 'A dyadic reciprocity index for repeated interaction networks', *Network Science* **1**(01), 31–48.