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Reference Details

2175	Cambridge Working Papers in Economics
2109	Janeway Institute Working Paper Series
Published	25 October 2021
Revised	27 January 2023
Websites	www.econ.cam.ac.uk/cwpe
	www.janeway.econ.cam.ac.uk/working-papers

Clubs and Networks

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Abstract

A recurring theme in the study of society is the concentration of influence and power that is driven through unequal membership of groups and associations. In some instances, these bodies constitute a small world while in others they are fragmented into distinct cliques. This paper presents a new model of clubs and networks to understand the sources of individual marginalization and the origins of different club networks.

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1 Introduction

Economists study group formation using the theory of coalitions/clubs and the theory of network formation. In the coalitions approach individual payoffs are defined on the partition of players into mutually exclusive groups and in the networks literature individuals can join any number of groups but each of the groups is of size 2. However, in some important instances – examples include interlocking directorates and editorial boards of journals – groups have sizes larger than 2 and individuals typically join multiple groups. Importantly, the productivity of a group depends on both its size and how it is connected to other groups through overlapping memberships. In these contexts, a major concern is that a few individuals take up most memberships while everyone else is left out thereby giving rise to a very unequal distribution of payoffs.[1] A second and related concern is that groups may be fragmented into cliques, which may undermine openness and the performance of the system as a whole. Our paper proposes a new model of clubs and networks to examine these concerns.[2]

In our model, individuals seek to become members of clubs while clubs wish to have members. Clubs have capacity constraints (due to congestion effects) and individuals can only join up to a certain number of clubs (due to time limitations). Links between two clubs arise when an individual joins both clubs. The productivity of a club is increasing in the number of its members (until the capacity is reached) and it may be increasing or decreasing in the strength of its ties with other clubs. Individual utility is increasing in the sum of the productivity of the clubs they join. We define a notion of stable memberships that takes into account the incentives of individuals and clubs. Our interest is in understanding patterns of individual memberships and on the network of connections across clubs.

The main body of the analysis focuses on a setting where club productivity is increasing in link strength: in this case, a club prefers individuals who are members of more clubs and an individual prefers a club that links with more clubs. We show that stable outcomes exhibit a strong marginalization property: when club capacity is the binding constraint, a few individuals exhaust their membership capacity, while all others join no clubs; when individual availability is the binding constraint, a few clubs are fully occupied while all others go empty.[3]

[1] Durlauf and Young (2004) present an influential account of the groups based perspective on inequality and poverty. In Section 6 we present case studies on a number of empirical contexts.

[2] See Section 2.1 for a detailed account of the relation between our approach and the coalitions and networks approaches.

[3] For concreteness suppose that the number of individuals is 8, the number of clubs is 4, every individual can join up to 4 clubs and every club has capacity 4. The total club capacity is 16, so in principle every individual could belong to 2 clubs each. We will say that a membership profile exhibits marginalization

We next show that this marginalization is not always in line with efficiency: when individual utility is strongly concave, this marginalization is inefficient. Similarly, when club productivity is a concave function of membership size, the marginalization of clubs is inefficient. Thus, the incentives of individuals and clubs and the collective interest are generally not aligned.

We then study the network of connections among the clubs. When the returns to link strength are linear, the distribution of link strength across clubs is not important for the productivity of clubs: as a result, a variety of club networks are stable. In applications, however, the marginal returns from link strength are likely to be non-linear. For instance, in case club links are used for information sharing, we would expect marginal returns to decline with link strength. On the other hand, if links help clubs coordinate activities then the marginal returns may be increasing in link strength. We show that if the marginal returns from link strength are increasing, i.e., they are convex, then incentives of clubs and individuals push towards disconnected cliques of clubs with full-strength links. If, on the other hand, the marginal returns from link strength are decreasing, i.e., they are concave, then the club network entails larger components that are held together by weak links.[4]

We also consider a setting where club productivity is decreasing in link strength with other clubs: a club prefers individuals who are not members of other clubs. In this setting, when club capacity is the binding constraint, stable outcomes entail isolated clubs. On the other hand, if individual availability is the binding constraint then clubs may be obliged to accept individuals who are also members of other clubs, which connects them.

We check the robustness of our results with four extensions. First, we explore what happens when the productivity of a club depends not only on the number and weight of links it has with other clubs but also the properties of the clubs it links to. The club properties we consider include club size, weight and degree. Note that by including the weight and degree of neighboring clubs into the production function, we allow clubs to benefit from indirect connections. We find that in all these extensions, stable membership profiles exhibit strong marginalization, indicating the robustness of our marginalization result. Nonetheless, adding the three elements leads to different predictions regarding efficient structures and stable club

when 4 individuals become members of 4 clubs each while the other four individuals are completely left out.

[4] For concreteness suppose that the number of individuals is 16, the number of clubs is 6, every individual can join up to 2 clubs and every club has capacity 5. If returns are convex in link strength then the unique clubs-efficient and stable outcome is three cliques of two clubs each, and the links have maximal strength with 5 common members. If returns are concave in link strength then the unique clubs-efficient and stable membership profile is a connected network where every club has a link with one common member with every other club. These networks of clubs are illustrated in Figure 7 in Section 4 below).

networks. Compared to the benchmark model, when clubs benefit from linking to clubs with larger sizes or weights, marginalization is more likely to be efficient. When clubs benefit from linking to clubs with higher degrees, the club networks induced by both stable and efficient membership profiles have larger components and weaker links. The other extensions we consider include adopting a stronger solution concept of strong stability, allowing heterogeneity across individuals in terms of their contributions to club productivity, and removing the capacity constraints of clubs. We show that in all these extensions, stable membership profiles exhibit marginalization.

The theoretical analysis is complemented by case studies on interlocking directorates and editorial boards of journals. There is a large and distinguished body of work on inter-locking directorates, see e.g., Brandeis (1915), Brandeis (2009), Mizruchi (1996), Levine (1977), Useem (1984), and Davis, Yoo and Baker (2003); for a recent networks perspective on this literature see Kogut (2012). This literature argues that a major function of boards is to encourage best practices and that this is facilitated when a board member also has ties with other firms' boards. If information sharing is important then it is reasonable to suppose that the marginal returns from the strength of links are declining. In this setting, the theory predicts that the stable (and efficient) club network will contain weak ties and exhibit high connectivity. This is in line with the empirical evidence: Baker, Davis and Yoo (2001) and Kogut (2012) show that interlocking directorates exhibit a small-world property – weak ties form the basis for a large connected network.[5]

Our second case study pertains to editorial boards of journals. We draw on the work of Ductor and Visser (2021) to study the membership of authors in these boards and the connections between boards defined by common editors. There exists a very significant inequality in editorial memberships: a very small fraction of authors become editors. Moreover, most editors serve only on one or two boards, but there exists a core group of editors who serve on 4 or more journals. The network of the editorial boards is held together with (mostly) weak

[5] The work on interlocking directorates is also related to a more general study of elites and power structures in sociology. In the nineteenth century, the Italian school of sociology proposed a theory of elites defined in terms of the membership of the top echelons of different – government and non-government – organizations (Pakulski (2018)). Building on this tradition, in his well-known study of mid-twentieth-century American society, Wright Mills (1956) argued that the power to make major decisions was highly concentrated: a very small group of individuals moved between the top levels of the Federal government, a few hundred largest corporations, and the military. He referred to these individuals as the *power Elite*. Similar claims have been made about the concentration of power and influence in other societies. For an overview of the theory of elites, see Bottomore (1993), and for a critique of theories of elite power and control, see Dahl (1958). Our model and case studies draw attention to economic forces that push toward concentration of power in modern society.

links. These patterns are consistent with our theoretical predictions on marginalization and club networks (in the presence of concave returns from link strength).

There is a voluminous literature on coalitions and networks; for surveys of this work see e.g., Demange and Wooders (2005), Bloch and Dutta (2012), Bramoullé, Galeotti and Rogers (2016) and Goyal (2022). Our model draws on the theory of clubs and the theory of networks to explain phenomena such as marginalization, the small world of interlocking directorates, and power elites. Specifically, we combine the ideas of congestion and capacity constraints from club theory (Buchanan, 1965; Cornes, 1996; Demange and Wooders, 2005) with the ideas of multiple memberships and returns from links from the theory of networks (Bala and Goyal, 2000; Jackson and Wolinsky, 1996; Bloch and Dutta, 2012). We now discuss two earlier papers that seek in different ways to combine networks and clubs.

In an early paper, Page and Wooders (2010) study the problem of club joining and member admission with a bipartite network formation model where an individual and a club form a link if the individual joins the club. They allow individual utility to depend on own choices as well as the choices of others and they focus on the conditions under which the game has a potential function, which guarantees existence of a Nash equilibrium. In our study, we consider how clubs form links with each other by admitting common members. We establish existence by constructions and our focus is on marginalization as a property of stable and efficient membership profiles. Additionally, we derive a mapping between marginal return to link strength and club networks. While we consider more specific functional forms and pay-off structures, these results go beyond the Page and Wooders (2010) paper.

A recent paper by Fershtman and Persitz (2021) also studies a model of clubs and networks. Motivated by the fact that individuals often interact within social clubs, they analyze the incentives for individuals to join clubs. They assume that by joining a same club, two individuals form a link, whose strength is decreasing in the size of the club; this is captured by the club congestion function. Fershtman and Persitz (2021) show that the club congestion function shapes the structure of stable networks: depending on which is more cost efficient, individuals might establish weak connections with each other by joining a big club or by forming strong links with a central player who is in many small clubs. They use their model to explain why social networks feature homophily and clustering. By contrast, our study is motivated by real-world applications such as interlocking directorates and editorial boards of directors. We build a model where individuals can choose what clubs to join and clubs can also choose what members to admit, with a goal to maximize connection with other clubs. We assume that the strength of a link between two clubs is increasing in the number of their common members.

The properties we are interested in are the degree distribution of individuals, strength of links in club networks, and connectedness of club networks. We give a more detailed discussion on the relation between Fershtman and Persitz (2021) and this paper in Section 2.2.

We close the introduction with a few words on the relation with the matching literature. The model we study can be viewed as a two-sided many-to-many matching model: each individual can be matched with several clubs and each club can be matched with several individuals. While our problem can be described using the language of matching, our analysis is nevertheless very different from that of matching studies. In matching studies, researchers are interested in uncovering the most general conditions for the existence of a stable outcome, deriving an algorithm to find a stable outcome, and the welfare properties of stable outcomes.[6] In the case of our study, although we establish existence of stable outcomes using constructive arguments, our main focus are particular properties of stable outcomes (marginalizations), the tension between efficiency and stability, as well as the structure of resulting networks of clubs. Addinitionally, since we assume the productivity of a club to depend on the affliations of its members, the choices of individuals and clubs have externality. There is a strand of matching literature that studies matching with externality; see Bando et al. (2016) for a survey. Nonetheless, we intesgite a particular kind of externality, driven by network considerations, that has not been dicssued in the matching literature.

The rest of the paper is organlized as follows. Section 2 presents the model, Section 3 presents an analysis of the marginalization and Section 4 presents our results on club networks. Section 5 checks the robustness of our results with four extensions. Section 6 presents case studies on interlocking directorates and boards of editors of journals. Section 7 concludes. All the proofs are presented in the Online Appendix.

2 The Model

There is a set of individuals $I = \{i_1, \ldots, i_n\}$ and a set of clubs $C = \{c_1, \ldots, c_m\}$. We use i to denote a typical individual and c to denote a typical club. Individuals join clubs to become members. A *membership profile* is represented by a matrix $\boldsymbol{a} = (a_{ic})_{i\in I, c\in C}$ where $a_{ic} \in \{0, 1\}$ indicates whether individual i is a member of club c.

We define a few notions based on a membership profile $\boldsymbol{a}$. The *degree* of individual i, given

[6] See Roth and Sotomayor (1992) for an overview of matching studies and Hatfield and Kominers (2015), Rostek and Yoder (2020) and Bando et al. (2021) for advances in many-to-many matching problems.

a membership profile $\boldsymbol{a}$, is the number of clubs joined by i:

$$d_i(\boldsymbol{a}) = \sum_{c \in C} a_{ic}.$$

The *membership size* of club c, given a membership profile $\boldsymbol{a}$, is the number of individuals who join c:

$$s_c(\boldsymbol{a}) = \sum_{i \in I} a_{ic}.$$

There is a *link* between two clubs if they share common members. The *link strength* between clubs c and c', given a membership profile $\boldsymbol{a}$, is the number of common members they share:

$$g_{cc'}(\boldsymbol{a}) = \sum_{i \in I} a_{ic} a_{ic'}.$$

Following the large literature in club theory, we shall assume that there are strong congestion effects that set limits to club capacity (see Buchanan (1965) and Page and Wooders (2010)).[7] Similarly, we assume that individuals can only join a certain number of clubs; this is because they have a fixed amount of time and participating in a club has a minimum time commitment. Formally, we assume that $d_i(\boldsymbol{a}) \leq D$ for all $i \in I$, and $s_c(\boldsymbol{a}) \leq S$ for all $c \in C$, where D and S are two positive integers. The set of feasible membership profiles is $A = \{\boldsymbol{a} \in \{0,1\}^{n \times m} : d_i(\boldsymbol{a}) \leq D, s_c(\boldsymbol{a}) \leq S\}$. We also assume that $2 \leq S \leq n$ and $2 \leq D \leq m$: this ensures that at least one club can be fully occupied and at least one person can join the maximum number of clubs.

A club provides goods and services to its members. The productivity of a club depends on its size and on the links it has with other clubs. We assume that until the capacity is reached, club productivity increases in the number of its members. And we assume that the productivity of a club is increasing in the strength of the ties it maintains with other clubs.[8] In different contexts, we can interpret clubs as different institutions. For example, a club can be a board of a firm: links between boards, created by overlapping directors, may help the

[7] Section 5.4 studies a variant of the model where clubs can choose how many members to admit.

[8] In some contexts, club productivity may be falling in links with other clubs. This happens for instance if the clubs are in a competitive setting and when individuals belong to many clubs, they allocate limited time to each of their clubs and that lowers their productivity. The analysis of clubs and networks with negative spillovers can be carried out using the same methods we develop for the case of positive spillovers across clubs. We comment on the implications of negative spillovers after presenting the results for positive spillovers.

transmission of best practices and the coordination of corporate strategies. A club can also be an editorial board: links between boards, generated by shared editors, can facilitate knowledge spillovers. Depending on the roles links serve, the marginal returns from link strength vary. If the link helps to convey factual information then the marginal returns from link strength may be declining. On the other hand, if the information concerns complex issues such as new technologies or standards then marginal returns to link strength may be increasing. Similarly, if we are in a context of developing common standards (technological or social) then there may be value in significant overlap of membership.

With these ideas in mind, let us define the productivity of club $c \in C$ in profile $\boldsymbol{a}$ as

$$\pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \sum_{c' \neq c} h\left(g_{cc'}(\boldsymbol{a})\right), \tag{1}$$

where returns from membership size, f, are strictly increasing with $f(0) = 0$, and the externality from links, h, is increasing with $h(0) = 0$. The next section studies the benchmark case of linear increasing returns case: $h(x) = \alpha x$, with $\alpha \geq 0$. We take up the case of convex and concave returns in Sections 3.1 and 4.

Turning to individual utility, we assume that an individual enjoys benefits from the productivity of the clubs she joins. Given a profile $\boldsymbol{a}$, the utility of individual $i \in I$ is

$$u_i(\boldsymbol{a}) = v\left(\sum_{c \in C} a_{ic}\pi_c(\boldsymbol{a})\right), \tag{2}$$

where v is strictly increasing with $v(0) = 0$. In situations where individuals are directors of boards, it is natural to assume that their utility increases at a decreasing rate with the aggregate productivity of clubs they are in, so $v''(\cdot) \leq 0$.

We study efficient and stable memberships. We consider two standards for a membership profile to be efficient: maximizing the utilitarian welfare of individuals and maximizing the aggregate productivity of clubs.

Definition 1. A membership profile $\boldsymbol{a} \in A$ is the *utilitarian optimum* if for all $\boldsymbol{a'} \in A$,

$$\sum_{i \in I} u_i(\boldsymbol{a}) \geq \sum_{i \in I} u_i(\boldsymbol{a'}).$$

A membership profile $\boldsymbol{a} \in A$ is *clubs-efficient* if for all $\boldsymbol{a'} \in A$,

$$\sum_{c\in C} \pi_c(\boldsymbol{a}) \geq \sum_{c\in C} \pi_c(\boldsymbol{a'}).$$

Turning to strategic stability, it seems reasonable to require that individuals should be able to quit clubs if that increases their utility and clubs should be able to expel members if that raises their productivity. In addition, it seems reasonable to require that an individual and a club cannot coordinate on a deviation that makes them both strictly better off. i.e., no pair of individual i and club c can both benefit from a joint deviation where i is allowed to quit any clubs she is in, c is allowed to expel any members it has, and i joins c. We propose a notion of stability that reflects these ideas.

Formally, let $a_i = (a_{ic})_{c\in C}$ and $a_c = (a_{ic})_{i\in I}$ be the vectors recording the clubs i joins and the members c has, and let $a_{-i} = (a_{i'c})_{i'\neq i, c\in C}$ and $a_{-c} = (a_{ic'})_{i\in I, c'\neq c}$ denote the club joining of individuals other than i and member admission of clubs other than c. Moreover, we use $a_{-i,c} = (a_{i'c'})_{i'\neq i, c'\neq c}$ to represent the membership profile excluding individual i and club c, and we use $a_{-ic} = (a_{i'c'})_{i'c'\neq ic}$ to represent the membership profile excluding the relationship between individual i and club c. We write $a \geq a'$ if a is element-wise greater than or equal to a'.

Definition 2. A membership profile $\boldsymbol{a} \in A$ is *stable* if

1. $\forall i \in I, c \in C$: there is no $\boldsymbol{a'} \in A$ with $a'_i \leq a_i$ and $a'_{-i} = a_{-i}$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$, or $a'_c \leq a_c$ and $a'_{-c} = a_{-c}$ such that $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$, and

2. $\forall i \in I, c \in C$: there is no $\boldsymbol{a'} \in A$ with $a'_{ic} = 1$, $a'_{-ic} \leq a_{-ic}$, and $a'_{-i,c} = a_{-i,c}$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$ and $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$.

The definition of stability assumes that clubs have objectives that may be independent of the utility of individual members. Such an assumption seems appropriate in applications where the clubs have some "governing body" or an owner who decides on behalf of the club. In Section 6 we present three case studies where such an assumption is justified.

2.1 Comparison with the coalitions and networks approaches

Let us clarify the relation between our approach and the coalitions and networks approaches. In our model, each individual can join up to D clubs and every club can admit up to S members, where D and S can take arbitrary values. In a coalitions model, the outcome is a partition, so individuals can join only one club, that is, $D = 1$. Similarly, networks constitute a special case where every club can have exactly 2 members, roughly this means $S = 2$ and the payoff to the club from a single member is 0.

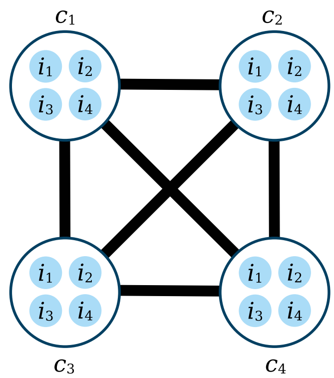
Example 1 (The relationship between our approach and that of coaltions and networks)**.** Suppose there are 8 individuals and 4 clubs, with individuals able to join 4 clubs and a club having a capacity of 4. In our model (so long as the utility function v is not too concave) the stable and welfare maximizing membership profile involves 4 clubs that are occupied by the same 4 members, as shown in Figure 1a. This means that 4 individuals are marginalized. In the coalition framework, the efficient and stable partition involves every individual joining one club each (thus two clubs are occupied by 4 members each), while the remaining 2 clubs remain unoccupied, as shown in Figure 1b. In contrast to our result there is no marginalization and the network of clubs is empty. In the networks framework, a relation is bilateral; so clubs consist of exactly 2 members. An efficient and stable network involves all 4 clubs being occupied by the same 2 members, making the clubs less connected and less productive, as shown in Figure 1c. $\square$

2.2 Comparison with the model of Fershtman and Persitz (2021)

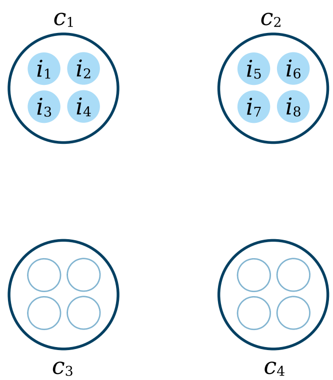
It is instructive to compare our model with that of Fershtman and Persitz (2021) (henceforce referred to as FD). As we already pointed out in the introduction the two models differ considerably. Here, we explain how the setups of the two models differ in more detail and use two examples to show how these differences lead to different incentives of agents and different stable structues.

(i) *number of agents*: In our model, there are two types of strategic agents: n individuals and m clubs, where n and m can be any positive integer. In FD, only the n individuals make decisions. FD assume that individuals can always form new clubs, which is equivalent to assuming that the number of clubs is large enough to always suit the needs of individuals ($m \geq n(n-1)/2$).

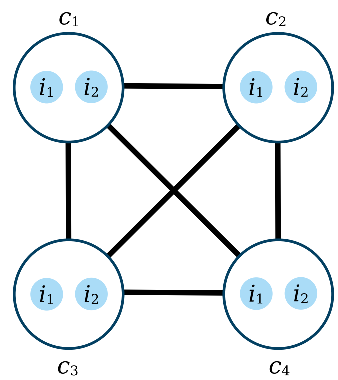
(ii) *cost of club joining and member admission*: We model the costs of club joining and member admission by assuming a capacity constraint D for individuals and a capcity constraint S for clubs. This can be viewed as imposing an extremely convex cost structure: an agent pays no cost when the constraint is not binding and an infinite amount of cost when breaking the constraint. By contrast, FD assume that individuals pay a



(a) stable and efficient membership profile



(b) stable and efficient coalition



(c) stable and efficient network

Figure 1: Comparison of our approach with coalitions and networks

constant cost for each club they join. They model the cost a club pays for admiting members by assuming that the strength of a link two individuals form by both joining the club decreases with the club size. Note that in the special case where the strength of link between two individuals is 0 when the club they join has more than S members, a club capcity constraint is essentially imposed.

(iii) *link formation technology*: We are interested in the network between clubs while FD study the network of individuals. We assume that two clubs have a link when they have a common member and the strength of the link is increasing in the number of common members. FD assume that two individuals have a link when thay join a same club and the strength of the link is decreasing in the size of the smallest club they are both in. (They assume that link strength does not depend on the number of clubs two individuals both join.)

(iv) *benefits from connections*: We assume that clubs only benefit from direct connections with other clubs; the amount of benefit to get can depend on the strength of the connection and various properties of the connected clubs.[9] FD, to the contrary, assume that individuals benefit from both direct and indirect connections. They specify the utility of individuals à la Jackson and Wolinsky (1996).

The differences listed above lead agents to have different incentives and result in diverse stable structures. Below are two examples that illustrates, respectively, how a membership profile can be stable under our (their) setup but never stable under theirs (ours).

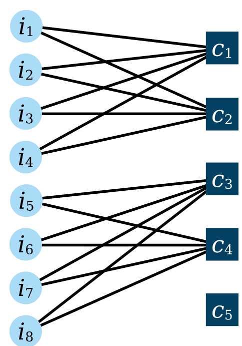
Example 2 (A membership profile that is stable under our setup but not that of FD)**.** Consider the membership profile depicted in Figure 2a, which leads to a club network and an individual network depicted in Figure 2b. This membership profile is both stable and efficient under our setup when $D = 2$, $S = 4$, and both h and f are convex. Nonetheless, it is never stable under the setup of FD since the individual network is not connected. More specically, all the individuals will want to quit one of the clubs they join as multiple joint memberships do not help individuals form stronger links, and any pair all unconnected inidviduals, i_1 and i_5, for example, will want to form a new club (join c_5) to access a new component.

□

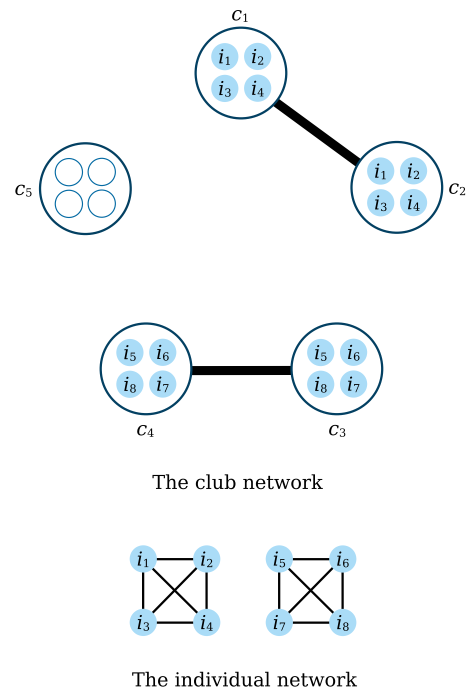
Example 3 (A membership profile that is stable under FD's setup but not ours)**.** Consider the membership profile depicted in Figure 3a, which leads to a club network and an individual network depicted in Figure 3b. This membership profile is stable when the membership fee lies in a medium range such that it is high enough to disincentivize peripheral individuals from forming a new club and low enough that it is beneficial for the central individual to maintain all her club memberships. (See Claim 2 of FD for the exact condition.) However, this membership profile is never stable under our setup since all the clubs would want to expel the member who joins only one club and admit a more active member instead. For example, c_1 will want to expel i_1 and admit i_2, and i_2 is willing to join c_1.

□

[9] See Section 5.1 for analysis of situations where the productivity of a club depends on the properties of its neighbouring clubs.



(a) the mebership profile



(b) the resulting networks

Figure 2: A membership profile that is stable under our setup but not that of FD

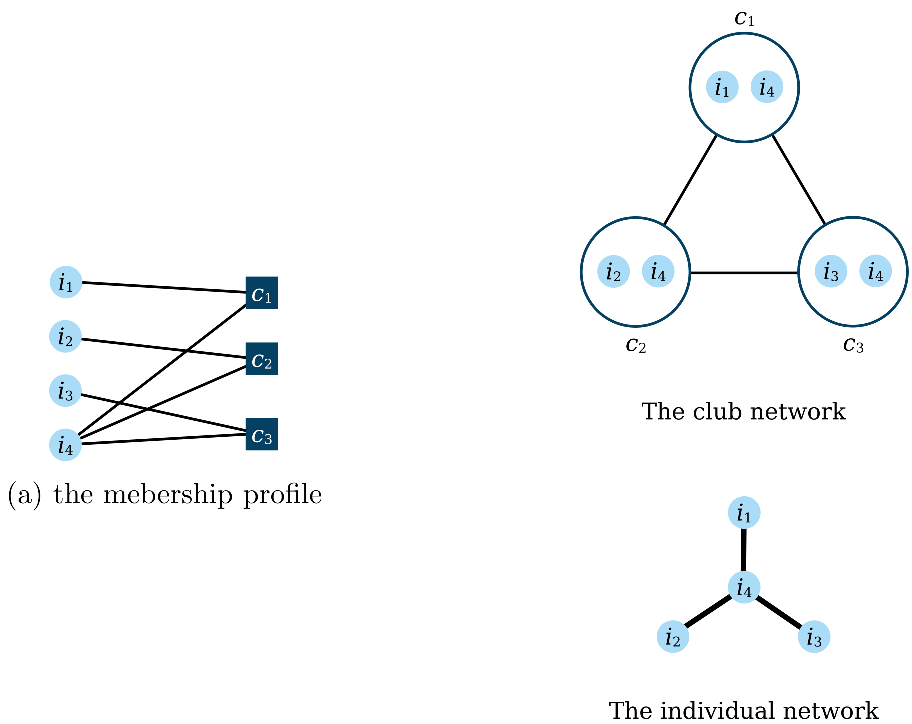
3 Marginalization

This section presents an analysis of a benchmark model in which returns from links take a linear form, $h(x) = \alpha x$, where $\alpha \geq 0$. So, there is a positive externality from links with other clubs when $\alpha > 0$. We take up non-linear functions $h(.)$ in Sections 3.1 and 4.

We first investigate stable membership profiles. Substituting the linear functional form for $h(\cdot)$ in the club productivity function in (1), we see that the productivity of a club $c \in C$ under a membership profile $\boldsymbol{a}$ is

$$\Pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \alpha \sum_{i \in C} a_{ic}(d_i(\boldsymbol{a}) - 1).$$

Observe that a club prefers an individual who is also a member of other clubs. Similarly, given their utility in (2), individuals prefer clubs with higher productivity. These two incentives press in the same direction: clubs like well-connected individuals and individuals prefer well-connected clubs. Thus, in this model, the incentives of clubs and individuals press toward marginalizing poorly connected clubs and poorly connected individuals.



(a) the mebership profile

(b) the resulting networks

Figure 3: A membership profile that is stable under FD's setup but not ours

To make this precise, let us define a partition of individuals and clubs. We partition the set of individuals I into three parts: a first group $I_1(\boldsymbol{a})$ that consists of individuals who join D clubs; a second group $I_2(\boldsymbol{a})$ that consists of individuals who join some but not D clubs; and a third group, $I_3(\boldsymbol{a})$, that consists of individuals who join no clubs. We also partition the set of clubs into three parts. The first group, $C_1(\boldsymbol{a})$, consists of clubs that have S members who join D clubs; the second group, $C_2(\boldsymbol{a})$, consists of clubs that have some members but either have less than S members or have members who join less than D clubs; and a third group, $C_3(\boldsymbol{a})$, that consists of clubs that are empty. Under the benchmark model, clubs in $C_1(\boldsymbol{a})$ achieve the highest productivity possible while clubs in $C_1(\boldsymbol{a})$ have zero productivity. With this partition, we can roughly evaluate how marginalized a membership profile is by looking at the cardinality of $I_2(\boldsymbol{a})$ and $C_2(\boldsymbol{a})$. If a membership profile marginalizes individuals (clubs), then $|I_2(\boldsymbol{a})|$ ($|C_2(\boldsymbol{a})|$) is small, so that most individuals (clubs) either join D or no clubs (achieve maximum or zero productivity). We can also develop a formal measure for marginalization, as explained below.

Notice that if total club capacity is less than the number of individuals, $mS < n$, then

$n - mS$ individuals are necessarily left out of clubs. We are interested in marginalization that results from an unfair assignment of club memberships to agents, that is in the number of individuals who are unnecessarily left out of clubs. This is the number of individuals that could be assigned to some clubs if club memberships were distributed more fairly. These are individuals who are *unfairly left out of clubs*. Let us say that a membership profile $\boldsymbol{a}$ exhibits *marginalization of individuals* if some individuals become members of clubs, some individuals are not members of any club, and it is possible to reassign club memberships increasing the number of agents in clubs and keeping the total number of club memberships unchanged. The total number of club memberships under $\boldsymbol{a}$ is equal to $\sum_{i\in I} d_i(\boldsymbol{a})$ and if there exist agents joining clubs then $\sum_{i\in I} d_i(\boldsymbol{a}) > 0$. If the total number of individuals $n \leq \sum_{i\in I} d_i(\boldsymbol{a})$ then each individual could be assigned a club membership keeping the total number of club memberships unchanged. Hence there are $|I_3(\boldsymbol{a})|$ agents who are unfairly left out of clubs and so in this case $\boldsymbol{a}$ exhibits marginalization of individuals if $|I_3(\boldsymbol{a})| > 0$. If the total number of individuals $n > \sum_{i\in I} d_i(\boldsymbol{a}) > 0$ then, keeping the total club membership unchanged, $n - \sum_{i\in I} d_i(\boldsymbol{a})$ individuals must be left out of clubs and only $|I_3(\boldsymbol{a})| - n + \sum_{i\in I} d_i(\boldsymbol{a})$ individuals who are out of clubs could be assigned a club membership. Thus in this case the latter is the number of individuals who are left out of clubs unfairly and $\boldsymbol{a}$ exhibits marginalization of individuals if $|I_3(\boldsymbol{a})| - n + \sum_{i\in I} d_i(\boldsymbol{a}) > 0$ (notice that the left-hand side of this inequality is always non-negative).

Notice that the minimal number of individuals needed to take $\sum_{i\in I} d_i(\boldsymbol{a})$ club memberships is equal to $\lceil \sum_{i\in I} d_i(\boldsymbol{a})/D \rceil$. Therefore, in the case of $\sum_{i\in I} d_i(\boldsymbol{a}) \geq n > 0$, the maximum number of individuals who are left out of clubs unfairly is equal to $n - \lceil \sum_{i\in I} d_i(\boldsymbol{a})/D \rceil$ and, in the case of $n > \sum_{i\in I} d_i(\boldsymbol{a}) > 0$, the maximum number of individuals who are left out of clubs unfairly is equal to $\sum_{i\in I} d_i(\boldsymbol{a}) - \lceil \sum_{i\in I} d_i(\boldsymbol{a})/D \rceil$. Based on these observations we define a *measure of marginalization for individuals* as

$$\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = \begin{cases} \frac{|I_3(\boldsymbol{a})| - n + \min\left(n, \sum_{i\in I} d_i(\boldsymbol{a})\right)}{\min\left(n, \sum_{i\in I} d_i(\boldsymbol{a})\right) - \left\lceil \frac{\sum_{i\in I} d_i(\boldsymbol{a})}{D} \right\rceil}, & \text{if } \sum_{i\in I} d_i(\boldsymbol{a}) > 0 \text{ and } I_3(\boldsymbol{a}) \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Notice that for any membership profile $\boldsymbol{a}$, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \in [0,1]$. The maximal value of $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ is attained when club memberships are assigned to the minimal number of agents needed to exhaust the total number of club memberships. In this case, we would say that $\boldsymbol{a}$ exhibits *extreme marginalization*. The minimal value of $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ is achieved when club memberships are allocated to as many individuals as possible. In this case we say that $\boldsymbol{a}$ is *egalitarian.*

Analogous to the measure of marginalization of individuals we define a *measure of marginalization of clubs* for a given membership profile $\boldsymbol{a}$:

$$\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = \begin{cases} \frac{|C_3(\boldsymbol{a})| - m + \min\left(m, \sum_{c\in C} s_c(\boldsymbol{a})\right)}{\min\left(m, \sum_{c\in C} s_c(\boldsymbol{a})\right) - \left\lceil \frac{\sum_{c\in C} s_c(\boldsymbol{a})}{S} \right\rceil}, & \text{if } \sum_{c\in C} s_c(\boldsymbol{a}) > 0 \text{ and } C_3(\boldsymbol{a}) \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

Let us work through some examples to illustrate how the measure of marginalization works and to develop a feel for the different issues at work in our model. Consider the following example. Suppose that $n \geq 7$, $m = 4$, $D = 3$ and $S = 4$. A membership profile $\boldsymbol{a}$ in which five individuals exhaust their membership availability while a sixth individual joins one club is stable. The total number of club memberships under $\boldsymbol{a}$ is equal to 16. Notice that $\lceil 16/3 \rceil = 6$ so the 16 club memberships are taken by the minimal number of individuals needed to take all of them. There are $|I_3(\boldsymbol{a})| = n - 6 \geq 1$ individuals left out of clubs and $\min(|I_3(\boldsymbol{a})|, |I_3(\boldsymbol{a})| - n + 16) = \min(n-6, 10) \geq 1$ of them are left out of clubs unfairly. Since the maximal number of individuals that can be left out of clubs unfairly is equal to $\min(n, \sum_{i\in I} d_i(\boldsymbol{a})) - \lceil \sum_{i\in I} d_i(\boldsymbol{a})/D \rceil = \min(n, 16) - 6 = \min(n-6, 10)$ so we have $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = \min(n-6, 10)/\min(n-6, 10) = 1$ and so $\boldsymbol{a}$ exhibits an extreme marginalization.

The example above might suggest that any stable club membership exhibits extreme marginalization. This, however, is not the case as the following example illustrates.

Example 4 (A stable membership profile that does not exhibit extreme marginalization)**.** Suppose that $m = 10$, $n \geq m+2$, and $D = S = 6$. Consider the following membership profile, $\boldsymbol{a}$. Let i_x, i_y and i_z be three individuals who join 4 clubs. For other individuals, let 8 of them, whom we denote by $i_1, \ldots, i_8$, join $D = 6$ clubs and the rest of them join no club. Allocate i_x, i_y, i_z, i_1, i_2, i_3 and i_4 to four clubs c_1, c_2, c_3 and c_4 in the way depicted in Figure 4. Also, let individuals i_1 to i_4 join any three other clubs and let individuals i_5 to i_8 join clubs $c_5, \ldots, c_{10}$. This membership profile is stable. To see how, under this membership profile, all clubs are full and clubs other than c_1 to c_4 reach the highest productivity possible and would not want any deviations. For clubs c_1 to c_4, they wish to make deviations. For example, c_1 wants to admit i_4 instead of i_x, i_y or i_z. If i_4 joins c_1, the productivity of c_1 would raise by 2α and be higher than that of c_2, c_3 and c_4 she is currently in. With this logic, it seems that i_4 would want to quit c_2, c_3 or c_4 and join c_1. However, note that with the deviation, the degree of i_x, i_y or i_z drops by 1, making the productivity of c_2, c_3 and c_4 drop by α. Although i_4 leaves one of c_2, c_3 and c_4, she is still in two of them. The aggregate productivity i_4 enjoys from clubs drops by 2α, which cancels out the productivity gain from c_1. Hence, i_4 has no incentive to make the deviation. Using the same logic, we can show that c_2, c_3 and c_4 cannot attract a higher-degree individual to replace i_x, i_y or i_z as well and the membership profile is stable. $\square$

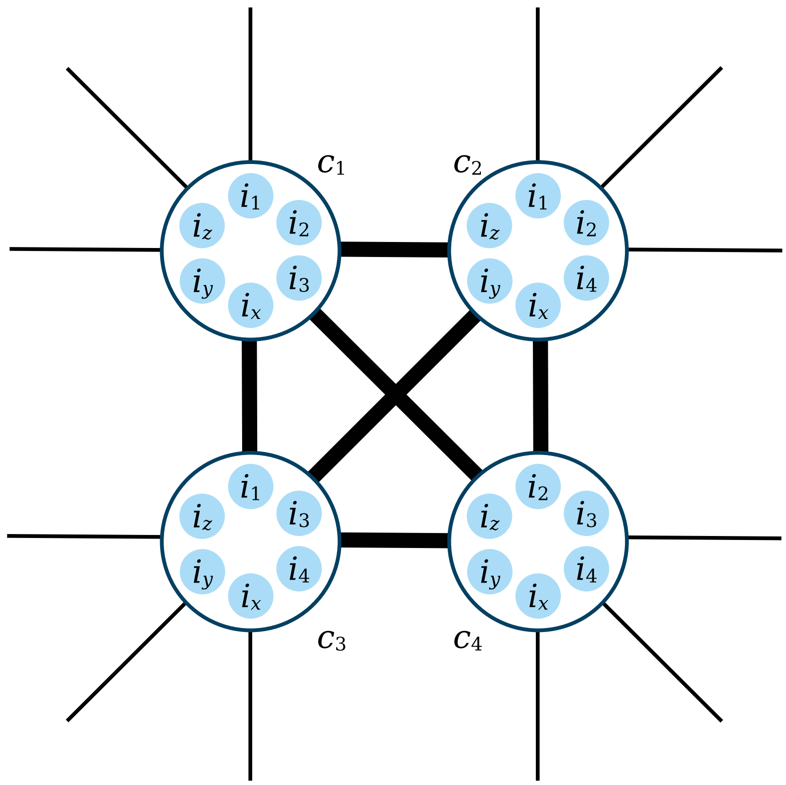


Figure 4: The coordination problem.

There are 60 total club memberships and $|I_3(\boldsymbol{a})| = n - 11 > 0$ individuals left out of clubs under $\boldsymbol{a}$. The number of individuals left out of clubs unfairly under $\boldsymbol{a}$ is $|I_3(\boldsymbol{a})| - n + \min(n, 60) = \min(n, 60) - 11 > 0$ and the maximal number of individuals who could be left out of clubs when there are 60 club memberships is $\min(n, 60) - \lceil 60/6 \rceil = \min(n, 60) - 10$. Thus the value of the measure of marginalization for membership profile $\boldsymbol{a}$ is $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 1 - \frac{1}{\min(n,60)-10} < 1$. Hence $\boldsymbol{a}$ does not exhibit an extreme marginalization of individuals. $\square$

This example draws attention to a coordination problem among individuals and clubs: note that the 10 clubs and individuals i_1 to i_8, i_x and i_y would be better off in the membership profile where the clubs are exactly filled by those individuals so that all those individuals have degree D, that is a membership profile featuring extreme marginalization. This coordination problem gives rise to a number of complications that inform the following partition of individuals and clubs in a stable membership profile.

Proposition 1. *Consider the benchmark model. There exists a stable membership profile. When $\alpha = 0$, an egalitarian membership profile is stable. When $\alpha > 0$, for a stable $\boldsymbol{a}$, $|I_2(\boldsymbol{a})| \leq S$ and $|C_2(\boldsymbol{a})| \leq D$. Hence,*

- *if $nD \geq mS$, then $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$.*
- *if $nD < mS$, then $\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) \geq 1 - \frac{S}{\min\left(\frac{mS-nD}{D},(S-1)n\right)}$.*

In the absence of network externalities, it is fairly straightforward to see that an egalitarian club profile is stable. For example, assume that $nD > mS$, a membership profile that assigns the mS club slots to distinct individuals is clearly stable as there is no advantage of having common membership in clubs.

Turning to the setting with positive externalities, let us comment on the expressions for the bounds. For $|I_2(\boldsymbol{a})|$, observe that for individuals in $I_2(\boldsymbol{a})$, if an individual i's degree is greater than or equal to the degree of another individual i', then the set of clubs i joins must be a superset of the set of clubs i' joins. Otherwise, i can crowd out i' and join one more club. Thus, for a club that hosts the individual with the lowest degree in $I_2(\boldsymbol{a})$, it must host all individuals in $I_2(\boldsymbol{a})$. Since a club can host at most S members, $|I_2(\boldsymbol{a})| \leq S$. We prove that the number of clubs in $C_2(\boldsymbol{a})$ is limited by D by showing that the member who has the highest degree in the least productive club of $C_2(\boldsymbol{a})$ must join all clubs in $C_2(\boldsymbol{a})$, as otherwise, she would deviate to join another $C_2(\boldsymbol{a})$ club and the club is willing to take her.

Now consider the bounds on $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ and $\mathcal{M}_{\mathcal{C}}(\boldsymbol{a})$. When $nD > mS$, given that $|I_2(\boldsymbol{a})|$ is bounded above, most individuals either join D or no clubs. Since the aggregate club capacity, mS, is lower than the aggregate individual availability, nD, only a limited number of individuals can have degree D and the rest individuals must be excluded by all clubs. This gives as a lower bound on $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$. To interpret this bound, consider a scenario where there are many clubs and even more individuals so that m is large and $n > mS$. In this case, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 1 - D/((D-1)m)$. Since m is large, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ is close to 1. A stable membership profile exhibits near-extreme marginalization of individuals. Similarly, when $nD < mS$, we can derive a lower bound on $\mathcal{M}_{\mathcal{C}}(\boldsymbol{a})$, which is close to 1 when n is large and $m > nD$, implying a near-extreme marginalization of clubs. Note that the argument above relies on the fact that the capacity constraints D and S are constant and independent of the number of clubs and the number of individuals. This assumption makes sense when such constraints can be interpreted as time limitations (in the case of individuals) or some physical space restrictions (in the case of clubs). We study a variant in Section 5.4 where clubs do not face capacity constraints and show that stable membership profiles still feature marginalization.

We next turn to the welfare properties of membership profiles. We have shown that in the presence of a connection externality, a stable membership profile marginalizes individuals or clubs. Are such membership profiles desirable? We show that the answer depends on whether we look at clubs-efficiency or at the utilitarian optimum. In our study of utilitarian optimum, we will make use of the following condition on the concavity of the utility function.

$$v\left(f(S)\right) - v(0) > (n-1)\left(v\left(f(S) + \frac{2\alpha S(D-1)}{n-1}\right) - v\left(f(S)\right)\right). \tag{4}$$

Proposition 2. *Consider the benchmark model and assume $\alpha > 0$. Suppose $nD \geq mS$ and that mS/D is an integer.*[10]

- *A membership profile is clubs-efficient if and only if mS/D individuals join D clubs and the remaining individuals join no clubs ($\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 1$).*

- *If $v''(\cdot) \geq 0$, then a membership profile is a utilitarian optimum if and only if it is clubs-efficient ($\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 1$). If $v''(\cdot) < 0$ and satisfies condition (4), then in any utilitarian optimum membership profile, either $d_i(\boldsymbol{a}) \leq 1$ for all $i \in I$ or $d_i(\boldsymbol{a}) \geq 1$ for all $i \in I$ ($\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 0$).*

Assume $nD < mS$ and that nD/S is an integer.

- *A membership profile is a utilitarian optimum if and only if nD/S clubs admit S members and the remaining clubs admit no members ($\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = 1$).*[11]

- *If $f''(\cdot) > 0$, then a membership profile is clubs-efficient if and only if nD/S clubs admit S members and the remaining clubs admit no members ($\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = 1$). If $f''(\cdot) < 0$, then a membership profile is clubs-efficient if and only if $(nD) \bmod m$ clubs admit $\lceil \frac{nD}{m} \rceil$ members and the remaining clubs admit $\lfloor \frac{nD}{m} \rfloor$ members ($\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = 0$).*

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10 In the Online Appendix, we provide characterizations of clubs-efficient and utilitarian optimal membership profiles without the integer condition.

11 If the integer condition (S divides nD) does not hold, then the utilitarian optimum characterization for when $v''(\cdot) \geq 0$ and when $v''(\cdot) < 0$ could be different. When $v''(\cdot) \geq 0$, there is one club that hosts some but less than S members. When $v''(\cdot) < 0$, the number of clubs that admit some but less than S members ranges from 1 to $S - 1$.

Consider first the case where $nD > mS$. Proposition 2 tells us that a membership profile that maximizes the aggregate output of the clubs exhibits extreme marginalization: a clubs-efficient profile allocates exactly mS/D individuals into memberships, all other individuals join no clubs. This is because this marginalization ensures maximal overlap of members between clubs.

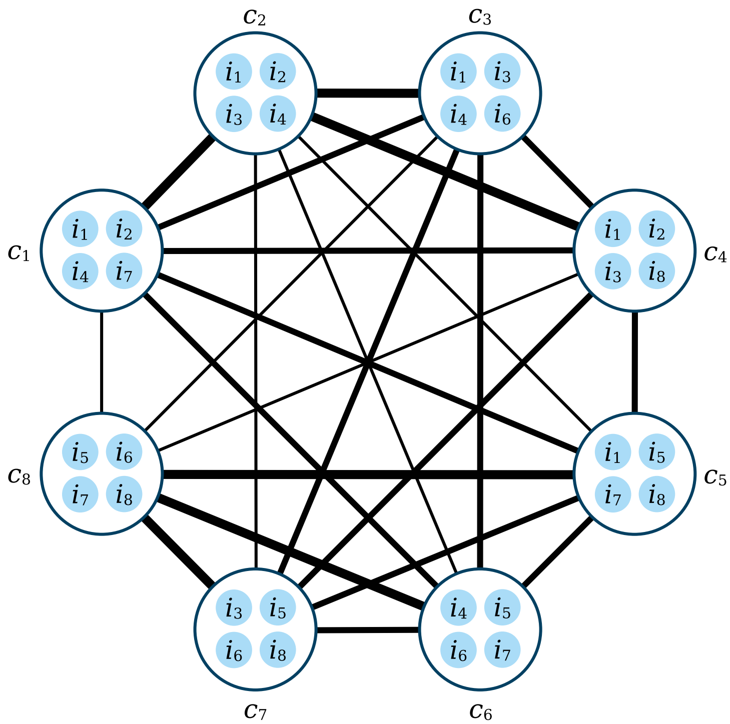
Turning to the utilitarian optimum, if the utility of individuals rises at an increasing or constant rate with the productivity of clubs they join, i.e., if $v''(\cdot) \geq 0$, then the profile that is utility-maximizing is the same as the profile that is productivity-maximizing. This is because when $v''(\cdot) = 0$, the aggregate utility of individuals is simply the number of individuals a club can admit, S, times the aggregate productivity of clubs, and when $v''(\cdot) > 0$, utilitarian optimality pushes toward marginalization of individuals, which coincides with the outcome generated by clubs-efficiency. If, on the other hand, the marginal utility is decreasing, i.e., $v''(\cdot) < 0$, then that opens up a potential trade-off: although a concentration of memberships maximizes the total output of clubs, it comes at the expense of entirely excluding $n - mS/D$ individuals from memberships. If the utility function is sufficiently concave – the marginal utility is declining sufficiently rapidly (a condition that is formalized in inequality condition (4), then the welfare benefit from picking more members outweighs the loss to aggregate productivity. We present an example that brings out the difference between clubs-efficiency and utilitarian optimum when we move from a convex/linear to a concave utility function.

Example 5 (The role of concave v on utilitarian optimal profiles)**.** Suppose $n = 16$, $D = 4$, $m = 8$ and $S = 4$. Figure 5a depicts a membership profile that is clubs-efficient and utilitarian optimum when $v(\cdot)$ is linear. Notice that in this membership profile, 8 individuals (i_1 to i_8) exhaust their membership availability while the other 8 individuals (i_9 to i_{16}) join no clubs. To appreciate the role of concave $v(\cdot)$ is concave, set $f(x) = x$, $\alpha = 1$, and

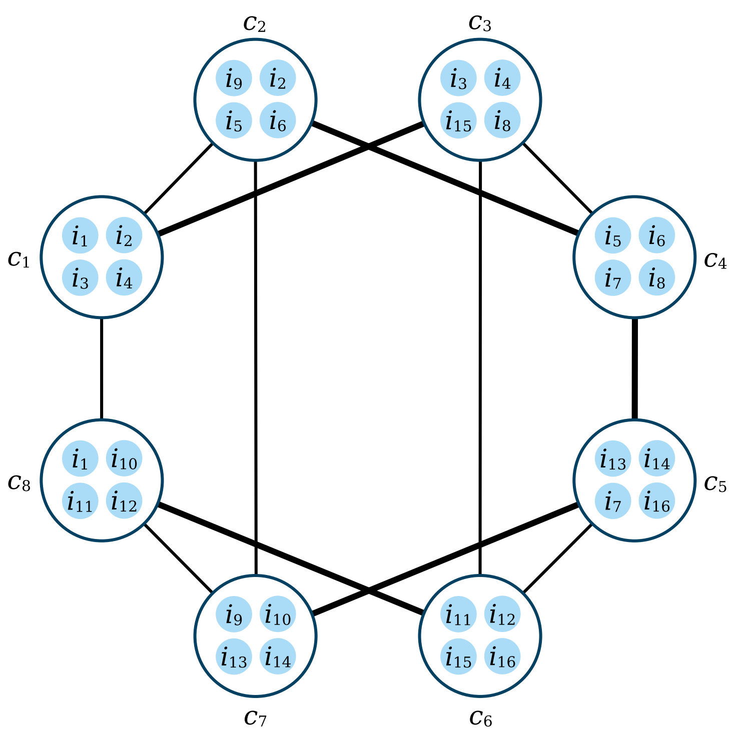
$$v(x) = \begin{cases} 10x & \text{when } x \leq 16, \\ 160 + 0.1\,(x - 16) & \text{when } x > 16. \end{cases}$$

In this case, the clubs-efficient outcomes remain unchanged and are as in Figure 5a, while the utilitarian optimal profile, which features all 16 individuals joining 2 clubs, is given in Figure 5b. □

Let us next take up the case where $nD < mS$. On club efficiency, note that there is enough club capacity to cover all individuals, so every person will join D clubs: keeping anyone out



(a) clubs-efficient and utilitarian optimal profile: convex or linear $v(\cdot)$



(b) utilitarian optimal profile: highly concave $v(\cdot)$

Figure 5: Efficient membership profiles.

of clubs is clearly inefficient. Moreover, as spillovers are linear, there is a constant spillover irrespective of how the individuals are allocated across clubs. So the issue of how to allocate individuals turns on the f function. If f is convex, then it is better to allocate individuals to fewer clubs, i.e., nD/S clubs; if on the other hand, f is concave then you allocate as evenly as possible across clubs, subject to integer constraints.

Regarding utilitarian optimum profiles, no matter what the f function, the optimal profile entails the marginalization of clubs. This is because to maximize the aggregate utility of individuals, it is clearly better to allocate more individuals to high-productivity clubs and fewer individuals to low-productivity clubs. This taken in tandem with the assumption that the productivity of a club rises with its size implies the marginalization of clubs.

When we compare Propositions 1 with 2, we see that there exists a tension between the incentives toward marginalization (created by the increasing club productivity from membership and from the strength of links with other clubs) and the demands of inclusiveness (created by the concave utility function and concave club production function).

We conclude our study of the benchmark model with a brief remark on stable and efficient membership profiles when spillovers across clubs are negative. This happens when $\alpha < 0$ in the benchmark model. Observe that when spillovers are negative, a club would like to only admit members who have no other memberships. So in a world with many individuals relative to club capacity, i.e., $n > mS$, any stable membership profile must involve exactly mS individuals filling the aggregate club capacity, i.e., every person joins at most one club and the resulting club network is an empty network. However, when the number of individuals is small the clubs face a trade-off: on the one hand, their productivity grows with membership (up to their capacity size). On the other hand, expanding membership may necessitate bringing in individuals who are already members of other clubs, and this lowers their productivity. We can apply the methods developed above to show that whatever the outcome of the tradeoff is, a stable profile and an aggregate productivity maximizing profile both feature an egalitarian membership profile where there does not exist two individuals i and i' with $|d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a})| > 1$. The argument goes as follows: suppose there exist two individuals i and i' where $d_{i'}(\boldsymbol{a}) \geq d_i(\boldsymbol{a}) + 2$. If so, then there exists a club c which i' joins but not i. Clearly, this club c would want to expel i' and recruit i. We show that i is also willing to join c. Since i' is willing to join c, it must be that $(d_{i'}(\boldsymbol{a}) - 1)|\alpha| \leq \pi_c(\boldsymbol{a})$, as otherwise i' would be better off leaving c, which makes the productivity of other clubs i' joins raise by $|\alpha|$. It follows that $(d_i(\boldsymbol{a'}) - 1)|\alpha| \leq \pi_c(\boldsymbol{a'})$, where $\boldsymbol{a'}$ is the profile where c admits i instead of i' but is otherwise the same as $\boldsymbol{a}$. This implies i is willing to join c. There is therefore a profitable deviation for the club-individual pair i and c. Turning to maximizing the aggregate productivity of clubs, note that the same deviation also improves the situation: it reduces the productivity of clubs i joins by α and raises the productivity of clubs i' joins by at least α. The result then follows given that i' is in more clubs than i does.

3.1 Non-linear returns from links and marginalization

We have so far considered the case where returns are linearly increasing in link strength. In some prominent instances, the returns from link strength are likely to be non-linear. For example, in case club links are used for information sharing, we would expect marginal returns to decline with link strength. On the other hand, if links help clubs coordinate strategies, then the marginal returns may be increasing in link strength. With these observations in mind, we examine the implications of non-linear returns from link strength. In this section, we study how robust the marginalization result is when we depart from the linear returns setting. We focus on the more natural case where the aggregate individual availability, nD, is greater than the aggregate club capacity, mS, here. We first show that when the marginal returns from links do not vary too much as links strengthen, then a stable membership profile always features marginalization that is asymptotically close to 1 (like in the case of linear h). We then turn to the case when h is very convex or concave. When h is very convex, we show that there exist stable membership profiles that feature the same level of marginalization as in the case when h is linear. However, there also exist stable profiles in which marginalization is bounded from above by a value smaller than 1 as long as each individual can join at least 3 clubs. When h is very concave, we show that a stable profile always features marginalization. Nonetheless, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ converges to 1 at a slower rate than when h is linear.

Proposition 3. *Assume that h is strictly increasing and $nD > mS$.*

(i) Suppose

$$\min_{x\in[0,S-1]}(h(x+1)-h(x)) > \frac{D-1}{D}\max_{x\in[0,S-1]}(h(x+1)-h(x)),$$

then for any stable membership profile $\boldsymbol{a}$, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$.

(ii) Suppose h is convex, then there exists a stable membership profile $\boldsymbol{a}$ with $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$. Suppose h is very convex in the sense that $h(S)-h(S-1) > 2(h(1)-h(0))$, if $n > mS$ and m is even then there exists a stable membership profile $\boldsymbol{a}$ with $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \leq \frac{D}{2(D-1)}$.

(iii) Suppose h is concave, then for any stable membership profile $\boldsymbol{a}$, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) > 1 - \frac{D^2 S}{(D-1)m}$.

We briefly explain why structures that do not feature marginalization can be stable when h is very convex but not when h is very concave. When h is sufficiently convex, a club wants to form links with certain other clubs that it currently has strong links with, and it is thus willing to keep an individual with a low degree as a member if the individual happens to be in the right clubs. A non-marginalized profile emerges when each individual who joins some clubs has a low degree and happens to be in clubs that are strongly linked to each other. Figure 6 provides an example of such a profile.

The same reasoning does not apply when h is concave, the number of clubs m is large, and the number of individuals n is large so that $n > mS$. In this case, suppose there are many individuals of low-degree individuals, for a club c that admits a low-degree individual i, even if i is in clubs that c has weak links with, there exists another individual i' who is in clubs that c does not have links with, which makes c want to expel i and admit i'. Therefore marginalization is robust when h is concave.

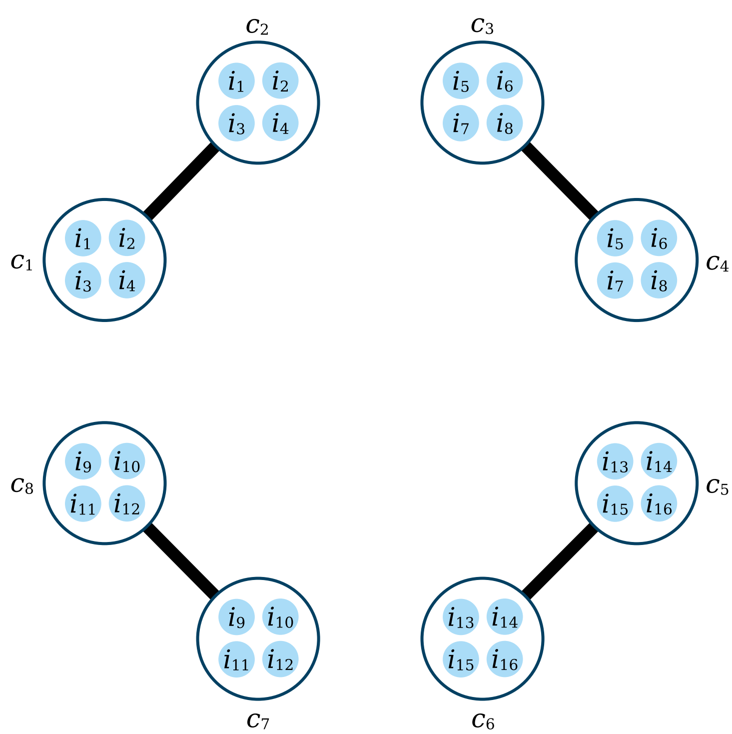


Figure 6: No marginalization when $h(\cdot)$ is very convex.

4 Small worlds, fragmented cliques, and strength of ties

In this section, we examine the network of clubs and the strength of ties that support this network.[12] We start by showing that if returns from link strength are linear then a variety of club networks are stable. We then turn to nonlinear returns and show that if the marginal return from link strength is increasing, then incentives of clubs and individuals push toward disconnected cliques of clubs with full-strength links. If, on the other hand, the marginal return from link strength is decreasing then the club network entails larger components that are connected through weak links.

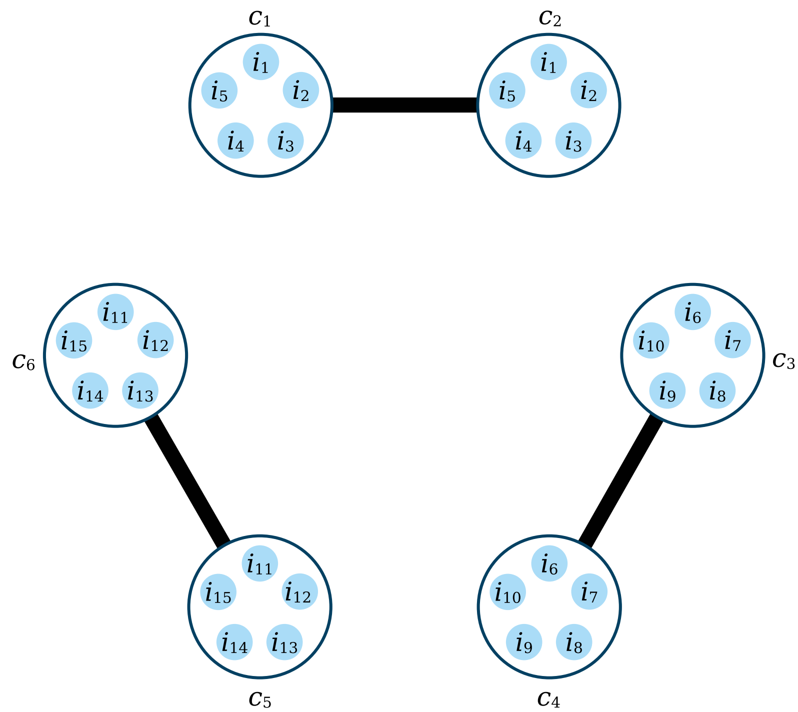
Example 6 (The curvature of h and club network)**.** Suppose that $n > 15$, $m = 6$, $D = 2$ and $S = 5$. Figure 7 depicts two clubs-efficiency and stable membership profiles when returns from links rise linearly. Note that the two profiles lead to the same degree distribution of individuals (the first 15 individuals all join two clubs while the others join no clubs) and the same aggregate link strength clubs have (each club shares five membership overlaps with other clubs). However, the resulting club networks take very different forms: one consists of three separate cliques where all links are of strength 5 while the other is a complete network where all links are of strength 1. This indicates that linear spillovers from links always lead to marginalization of individuals/clubs but the resulting club networks can be very different.

If $h(\cdot)$ is convex, there is a unique clubs-efficient membership profile which is depicted in Figure 7a. When $h(\cdot)$ is convex, the productivity of a club is maximized if the number of membership overlaps it has with other clubs is maximized and concentrated in as few clubs as possible. This can only be achieved when the club network takes the form depicted in Figure 7a.

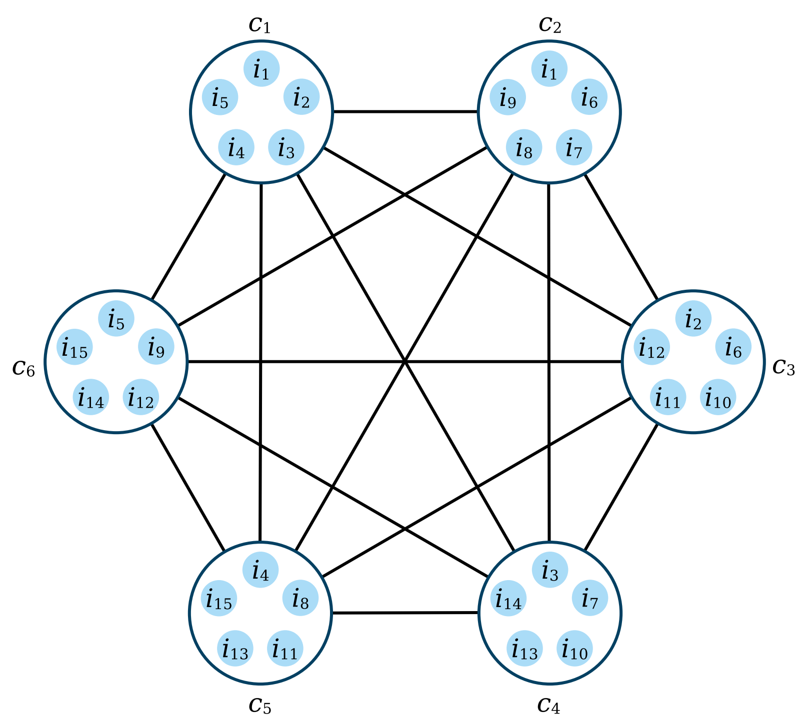
On the other hand, when $h(\cdot)$ is concave, Figure 7b depicts the unique clubs-efficient network. When $h(\cdot)$ is concave, clubs want to maximize their membership overlaps with other clubs and spread them as evenly as possible. In this example, for this to be the case, the club network has to be complete with all links being weak.

Turning to stability, the structure depicted in Figure 7a is stable when $h(\cdot)$ is convex since all clubs have reached the highest productivity possible and have no incentives to deviate. It

[12] A membership profile can be projected both into a network of clubs and a network of individuals. In this section, we focus on the club network. Nevertheless, the individual network, since originated from the same membership profiles as the club network, shares some important properties with the latter. For example, there exists a strong link (link with strength greater than 1) in the club network if and only if there exists a strong link in the individual network, and the club network is connected if and only if the individual network is connected. Therefore, we can infer the properties of the individual network with an analysis of the club network.



(a) Convex returns from link strength: $h''(\cdot) > 0$



(b) Concave returns from link strength: $h''(\cdot) < 0$

Figure 7: Clubs-efficient and stable membership profiles.

is not stable when $h(\cdot)$ is concave: there is a profitable deviation for individual i_6 and club c_1 where c_1 expels i_1 to admit i_6 and i_6 leaves c_3 to join c_1.

Similarly, the structure depicted in Figure 7b is stable when $h(\cdot)$ is concave but not so when $h(\cdot)$ is convex. Stability under a concave $h(\cdot)$ is obvious since all clubs have reached the highest productivity possible; instability under a convex $h(\cdot)$ can be verified by again considering the deviation by individual i_6 and club c_1 where c_1 expels i_2 to admit i_6 and i_6 leaves c_3 to join c_1. $\square$

The above example shows that the curvature of the returns from links has a significant influence on the structure of club networks. To get an intuition about the origin of that influence, notice that when $h(\cdot)$ is 0 at 0, strictly increasing and convex/concave then it is superadditive/subadditive.[13] If h is superadditive then a solution to the optimization problem

$$\max_{\boldsymbol{g}\in\mathbb{Z}_{\geq 0}^{C\times C}} \quad \sum_{c'\neq c} h\left(g_{cc'}(\boldsymbol{a})\right) \tag{5}$$

$$\text{s.t.} \quad \sum_{c'\neq c} g_{cc'}(\boldsymbol{a}) \leq T, \tag{6}$$

$$g_{cc'} \leq S, \tag{7}$$

where $T \in \mathbb{Z}_{\geq 0}$ is a constant, is any $\boldsymbol{g}$ such that $g_{cc'} = S$ for $\lfloor T/S \rfloor$ pairs $(c, c') \in C \times C$ with $c \neq c'$, $g_{cc'} = T \bmod S$ for one $(c, c') \in C \times C$ with $c \neq c'$, and $g_{cc'} = 0$ for all the remaining pairs $(c, c') \in C \times C$ with $c \neq c'$. On the other hand, if h is subadditive then a solution to the optimization problem (5) is any $\boldsymbol{g}$ such that $g_{cc'} = 1$ for T pairs $(c, c') \in C \times C$ with $c \neq c'$ and $g_{cc'} = 0$ for all the remaining pairs $(c, c') \in C \times C$ with $c \neq c'$. Suppose that $nD > mS$ and consider clubs-efficient structures. Clearly, any clubs-efficient membership structure must have S members in this case and the strength of a link between any two clubs is at most S. Each agent can contribute at most $D(D-1)/2$ to the total weight of links between clubs and so the total weight of links between clubs is at most $mSD(D-1)/2$. Hence finding a clubs-efficient structure amounts to solving the optimization problem (5) with $T = mSD(D-1)/2$ and subject to an additional constraint:

$$\boldsymbol{g} \text{ is a membership structure.} \tag{8}$$

[13] Recall that a function $h : \mathbb{R} \to \mathbb{R}$ is superadditive if for all $x, y \in \mathbb{R}$, $h(x + y) \geq h(x) + h(y)$ and it is subadditive if for all $x, y \in \mathbb{R}$, $h(x + y) \leq h(x) + h(y)$.

Constraint (8) makes the problem of characterising clubs-efficient membership structures difficult, especially in the case of subadditive (or even concave) h. For some values of parameters S, D, m and n the constraint is not binding and there exist membership structures attaining the maximum of problem (5) without constraint (5). In the case of superadditive h the constraint is not binding when D divides m and $n > mS/D$. In this case, an optimal membership profile can be constructed by partitioning clubs in m/D groups $(c_1^i, \dots, c_D^i)_{i=1}^{m/D}$, choosing m/D groups $(a_1^i, \dots, a_S^i)_{i=1}^{m/D}$ of size S of individuals, and then assigning all the individuals in group i to all clubs in group i, for each $i \in \{1, \dots, m/D\}$. In the case of subadditive h establishing the conditions for which the constraint is not binding is an open combinatorial problem (Chee et al., 2013). In such cases, an optimal membership structure is related to a so-called $(2, 1)$-packing of the clique over the set of all clubs. In the cases when constraint (8) is binding the problem of finding optimal membership structures is not easier. Given the difficulties with obtaining the characterization of optimal membership structures, we restrict attention to the cases of $D = 2$ and m even and provide a characterization of optimal structures then. Even these cases allow for contrasting the optimal structures under super- and subadditive functions h.

Formally, we say a club network $g = g(\boldsymbol{a})$ is clubs-efficient/utilitarian optimum/stable if it is created with a clubs-efficient/utilitarian optimum/stable membership profile $\boldsymbol{a}$. Additionally, we define a k-clique as a subnetwork that has k mutually linked clubs and a k-regular network as a network where all clubs have k links. The complete network is a special kind of regular network where all clubs are linked to each other ($k = m - 1$).

Proposition 4. *Assume $nD \geq mS$, $D = 2$, m is even, and $2 \leq S \leq m - 1$.*

- *When $h(\cdot)$ is superadditive, the clubs-efficient club network consists of m/D separate 2-cliques where all links are of strength S. This network is stable when $h(\cdot)$ is superadditive and unstable when $h(\cdot)$ is subadditive.*

- *When $h(\cdot)$ is subadditive, the clubs-efficient club network is an S-regular network (a complete network when $S = m - 1$) where all links are of strength 1. This network is stable when $h(\cdot)$ is subadditive and unstable when $h(\cdot)$ is superadditive.*

Assume $nD < mS$, $D = 2$, S divides n, and $2 \leq S \leq 2n/S - 1$.

- *When $h(\cdot)$ is superadditive, the utilitarian optimum club network consists of n/S separate 2-cliques where all links are of strength S. This network is stable when $h(\cdot)$ is superadditive and unstable when $h(\cdot)$ is subadditive.*

- *When $h(\cdot)$ is subadditive, the utilitarian optimum club network is an S-regular network (a complete network when $S = 2n/S - 1$) where all links are of strength 1. This network is stable when $h(\cdot)$ is subadditive and unstable when $h(\cdot)$ is superadditive.*

As mentioned earlier, the club network and the individual network generated by a membership profile share some important properties. The club network mentioned in Proposition 4 can be mapped into individual networks. When $h(\cdot)$ is convex, our characterization involves 2-cliques with strength S links for the club network; the corresponding individual network consists of S-cliques with strength 2 links. When $h(\cdot)$ is concave, our characterization features a S-regular club network with strength 1 links; the corresponding individual network is a $D(S-1)$-regular network with strength 1 links.

5 Extensions

5.1 Richer interdependence between clubs

In many applications, it is natural to assume that the properties of a club's neighboring clubs can affect the club's productivity. For example, there might be a greater value to a club when it is linked to large clubs as compared to small clubs. A club could also benefit more by linking to clubs that are better connected in the club network, which helps it form more indirect connections with other clubs. In the previous sections, we abstract away from this consideration. Here we investigate three variants to explore how introducing properties of neighbors into clubs' productivity function affects the structure of stable and efficient membership profiles. Let the productivity of a club be

$$\pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \alpha \sum_{c' \neq c} g_{cc'}(\boldsymbol{a}) q(p_{c'}(\boldsymbol{a})), \tag{9}$$

where $q(x) > 0$ for $x \geq 1$, $q'(\cdot) \geq 0$, and $p_c(\boldsymbol{a})$ can represent the size of club c, the aggregate strength of links c has, or the number of clubs c links with. We call the variants the *size model*, the *weight model*, and the *degree model* respectively when $p_c(\boldsymbol{a})$ represent different features of club c.[14] Note that when $q(x) = 1$ for $x = 1, ..., S$, we are back to the benchmark case where clubs only care about the aggregate strength of their links.

[14] We employ a slight abuse of notation here: the degree of an individual i denotes the number of clubs i joins and the degree of club c denotes the number of clubs c links with.

First, we show that in all three variants, a stable profile features marginalization as in the benchmark model.

Proposition 5. *Suppose the productivity of clubs follows* (9)*. When $\alpha = 0$, an egalitarian membership profile is stable. When $\alpha > 0$, for a stable $\boldsymbol{a}$, $|I_2(\boldsymbol{a})| \leq S$ and $|C_2(\boldsymbol{a})| \leq D$. Hence,*

- *if $nD \geq mS$, then $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$.*
- *if $nD < mS$, then $\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) \geq 1 - \frac{S}{\min\left(\frac{mS-nD}{D},(S-1)n\right)}$.*

Proposition 5 shows that even in richer setups of cross-club benefits, stable membership profiles exhibit marginalization. Intuitively, this is because clubs here, as they do in the benchmark model, have aligned preferences over individuals: they all prefer to admit individuals who are in bigger or better-connected clubs. Such aligned preferences drive marginalization.

We now turn to the possible tension between efficiency and stability. We have shown that, in the benchmark model, when v is sufficiently concave (f is concave), a utilitarian optimal (clubs-efficient) profile is egalitarian. This contrasts with the marginalization property present in all stable profiles. Now consider the size and weight model, since clubs benefit more from linking to bigger clubs, having bigger clubs generates a positive externality, whose recipients grow in number as clubs get larger. Hence, marginalizing clubs creates greater aggregate cross-club benefits and is more likely to be efficient. Additionally, since the weight of a club is increasing in its size and the degrees of its members, marginalizing individuals in the weight model produces a greater advantage in maximizing the productivity of clubs compared to the benchmark model. This makes a structure that marginalizes individuals more likely to be efficient under the weight model. We illustrate these two points with the following examples.

Example 7 (Efficient membership profiles that marginalize clubs)**.** Suppose that $\alpha > 0$, $nD < mS$ and nD/S is an integer. Proposition 2 shows that in the benchmark model, a clubs-efficient membership profile is egalitarian when f is concave. Now, first consider the size model. Let $f(x) = x^{\frac{1}{2}}$ so that f is concave and $q(x) = x$. The aggregate productivity of clubs under membership profile $\boldsymbol{a}$ is

$$\begin{aligned}\sum_{c\in C}\pi_c(\boldsymbol{a}) &= \sum_{c\in C} f_c(s_c(\boldsymbol{a})) + \alpha \sum_{i\in I, c\in C, c'\neq c} a_{ic}a_{ic'}q(s_{c'}(\boldsymbol{a})) \\ &\leq \sum_{c\in C}\left(f_c(s_c(\boldsymbol{a})) + \alpha(D-1)s_c(\boldsymbol{a})q(s_c(\boldsymbol{a}))\right) \\ &= \sum_{c\in C}\left(s_c(\boldsymbol{a})^{\frac{1}{2}} + \alpha(D-1)s_c(\boldsymbol{a})^2\right),\end{aligned}$$

where the equality is obtained when $d_i(\boldsymbol{a}) = D$ for all $i \in I$. The membership profile that maximizes $\sum_{c\in C}\left(s_c(\boldsymbol{a})^{\frac{1}{2}} + \alpha(D-1)s_c(\boldsymbol{a})^2\right)$ features nD/S clubs admitting S members and the remaining clubs admitting no members ($\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = 1$). For the weight model, note that when all individuals join D clubs, the weight of a club c is $(D-1)s_c(\boldsymbol{a})$. Hence, the aggregate productivity of clubs is bounded above by

$$\sum_{c\in C}\left(s_c(\boldsymbol{a})^{\frac{1}{2}} + \alpha(D-1)^2 s_c(\boldsymbol{a})^2\right).$$

Again, the membership profile that maximizes the above term features nD/S clubs admitting S members and the remaining clubs admitting no members ($\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = 1$). ☐

Example 8 (Efficient membership profiles that marginalize individuals)**.** Consider Example 5 again. With that example, We show that for the benchmark model, when v is sufficiently concave, a utilitarian optimal profile is egalitarian, as depicted in Figure 5b. Now, consider the weight model. Keep the assumptions on α, f and v unchanged and let

$$q(x) = \begin{cases} 1 & \text{when } x \leq 4, \\ 1 + 10\,(x-4) & \text{when } x > 4. \end{cases}$$

In this case, the utilitarian optimal profile features 8 individuals exhausting their membership availability while the other 8 individuals join no clubs, as depicted in figure 5a. This example shows that for situations when utilitarian optimal structures in the benchmark model are egalitarian, adding club weights into the production function can make utilitarian optimal structures feature extreme marginalization. ☐

The above examples show that due to additional forces towards marginalization, compared to the benchmark model, there is less tension between efficiency and stability in the size and weight model. Note that while introducing club sizes and weights to our model affects the welfare property of stable profiles, it does not change characterizations about stable club networks much. Consider the case where the aggregate individual availability is greater than the aggregate club capacity, that is, $nD > mS$. Since a stable membership profile exhibits marginalization, almost all clubs have size S and admit members with degree D, making the sizes and weights of most clubs the same. Hence, clubs have no preferences over what other clubs to link to and stable club networks can take many forms. However, this is not the case in the degree model, where clubs prefer to link with a club that has many weak links than a club with a handful of strong links. Such preference drives stable networks to have more links that are weak. We revisit Example 6 to demonstrate this effect. As mentioned in Example 6, both structures depicted in Figure 7 are clubs-efficient and stable under the benchmark model. Now consider the degree model. Assume that q is strictly increasing, then the unique clubs-efficient structure features a complete club network with weak links, as depicted in Figure 7b. This structure is also stable. The structure depicted in Figure 7, featuring a club network consisting of 3 strongly connected cliques, is no longer clubs-efficient nor stable. From this angle, taking degrees of neighboring clubs into account has similar effects on club networks as assuming decreasing marginal returns from links does.

5.2 Strong Stability

In the previous sections, our stability notion checks whether a membership profile is robust to deviations by a single agent (an individual or a club) and a pair of agents (an individual and a club). We can strengthen the stability notion by allowing deviations by a group of agents of any size.

Definition 3. A membership profile $\boldsymbol{a} \in A$ is strongly stable if $\forall I' \subseteq I$ and $C' \subseteq C$: there is no $\boldsymbol{a'} \in A$ with $a'_{ic} \leq a_{ic}$ for all $(i, c) \in I \times C$ where $i \in I', c \notin C'$ or $i \notin I', c \in C$ and $a'_{ic} = a_{ic}$ for all $i \notin I', c \notin C'$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$ for all $i \in I'$ and $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$ for all $c \in C'$.

In words, strong stability tests deviation by any group of individuals and clubs where agents within the group can freely add or terminate memberships with each other subject to capacity constraints and terminate memberships with agents outside the group.

We limit our analysis to the case where the aggregate individual availability is non-trivially greater than the aggregate club capacity ($nD > mD + DS$) and show that strong stability does not lead to stronger results than the stability notion we use in previous sections in terms of marginalization.[15]

Proposition 6. *Consider the benchmark model and assume $nD > mS + DS$. For any stable membership profile $\boldsymbol{a}$, there exists a strongly stable membership profile $\boldsymbol{a'}$ such that $|I_k(\boldsymbol{a'})| = |I_k(\boldsymbol{a})|$ for $k = 1, 2, 3$, implying that $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a'}) = \mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$.*

Proposition 6 shows that we cannot refine the result of the degree distribution of individuals by allowing more deviations from agents. In fact, adopting the solution concept of strong stability instead of stability may only sharpen the characterization of the club memberships of individuals who join D clubs, some of which belong to $C_2(\boldsymbol{a})$. This is reflected in the proof of Proposition 6 in the Appendix.

Note that both our stability notion and strong stability notion concern deviations that make all deviating agents strictly better off.[16] We define stability in this way to ensure the existence of stable membership profiles. The following example shows why an alternative stability notion that requires no deviations that make all deviating agents weakly better off and some deviating agents strictly better off can lead to the non-existence of stable profiles. Suppose there are one club that can admit one member and two individuals i_1 and i_2. There are three possible membership profiles here: the club admits no one; the club admits i_1; and the club admits i_2. The first profile is not stable. The second and third profiles are not stable if we adopt the alternative stability concept: suppose the club admits i_x, the deviation by the club and i_y where the club expels i_x to admit i_y makes i_y strictly better off and the productivity of the club unchanged.

5.3 Heterogeneous Individuals

Our model assumes homogeneous individuals so that the preference of clubs over individuals is completely based on the memberships of the individuals. In reality, individuals differ in other aspects that clubs could care about. To account for ths consideration, let $\theta_i \in \mathbb{R}_{>0}$

[15] The restriction of $nD > mS + DS$ ensures that all clubs are full in a stable membership profile which simplifies our analysis.

[16] There are two ways to define strong stability in network formation studies. One, proposed by Dutta and Mutuswami (1997), requires all participating members to be strictly better off for a deviation to be desirable; another, proposed by Jackson and Van den Nouweland (2005), only requires some participating members to be strictly better off.

be the type of individual i. We assume the productivity of club c in profile $\boldsymbol{a}$ to be

$$\pi_c(\boldsymbol{a}) = f\left(\sum_{i\in I} a_{ic}\theta_i\right) + \sum_{c'\neq c} h(g_{cc'}(\boldsymbol{a})). \tag{10}$$

Note that the production function of clubs under the specification of homogeneous individuals is a special case of function (10) when $\theta_i = 1$ for all $i \in I$. We do not change our assumption about individual utility, that is, individuals have the same utility function.

We focus on the more natural situation where $nD \geq mS$ and obtain the following results for stable membership profiles.

Proposition 7. *Suppose all individuals are different from each other: $\theta_i \neq \theta_j$ $\forall i \neq j$, and the productivity of clubs follows* (10)*. Assume that $h(x) = \alpha x$ where $\alpha \geq 0$ and $nD \geq mS$. Then for any stable profile $\boldsymbol{a}$, $|I_2(\boldsymbol{a})| \leq (D-1)S$ and $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{(D-1)D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$. Additionally,*

- *when $\alpha = 0$, $d_i(\boldsymbol{a}) \geq d_j(\boldsymbol{a})$ for all i, j with $\theta_i > \theta_j$ in a stable membership profile $\boldsymbol{a}$.*
- *when $\alpha > 0$, suppose $\theta_i - \theta_j < \alpha$ for all $i, j \in I$ and D does not divide $m + 1$, then there exists a stable membership profile $\boldsymbol{a}$ where $d_i(\boldsymbol{a}) \leq d_j(\boldsymbol{a})$ for all i, j with $\theta_i > \theta_j$.*

Propositioon 7 shows that in the presence of individual heterogeneity, a stable membership profile exhibits marginalization as long as $\alpha \geq 0$. This implies that connection benefit is not needed to arrive at marginalization. Nonetheless, connection externality influences who gets marginalized. Without it, a stable profile always sorts better types into (weakly) more clubs, so those marginalized must be the low types. When there is a connection externality and the difference between individuals is small, a stable profile could reverse the sorting completely and marginalize high types. This is because, with connection externality, club memberships are self-fulfilling: low types are admitted by clubs because of their memberships in other clubs. The condition of D not dividing $m + 1$ is imposed to ensure there does not exist a high type individual who joins only one club less than a lower type individual and thus will be wanted by a club the lower type is in for substitution.

5.4 Endogeneous club sizes

In this subsection, we relax the assumption that clubs face capacity constraints and allow them to choose how many members to admit. Let us consider an extension that keeps all assumptions of the benchmark model, except that instead of assuming a capacity constraint

S for clubs, we assume the returns from membership size, f, to be first increasing and then decreasing. Moreover, we assume that there is a $\overline{S}$ such that $f(\overline{S}) - f(\overline{S}+1) > \alpha(D-1)$. Since the maximum connection externality a member can bring to a club is $\alpha(D-1)$, a club would never admit more than $\overline{S}$ members. We focus on the case when $nD \geq mS$ and obtain the following result.

Proposition 8. *Consider the benchmark model with a change in the assumption that clubs do not face capacity constraints and there exists a $\overline{S}$ such that $f(\overline{S}) - f(\overline{S}+1) > \alpha(D-1)$. Assume $nD \geq mS$. When $\alpha = 0$, an egalitarian membership profile is stable. When $\alpha > 0$, for a stable membership profile $\boldsymbol{a}$, $|I_2(\boldsymbol{a})| \leq \overline{S}$ and $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$.*

Proposition 8 shows our marginalization characterization is not driven by the capacity constraints of clubs. In fact, the number of individuals who are not at the two extremes of the degree distribution is always bounded by the size of the largest club.

6 Case Studies

In this section, we present two case studies that map our theory onto empirical context of interlocking directorates and editorial boards of directors.

Interlocking Directorates: It is widely recognized that the board-to-board ties serve as a mechanism for the diffusion of corporate practices, strategies, and structures (Mizruchi (1996)). We may consider boards as clubs and directors as individuals; links between clubs raise productivity. In what follows, we discuss empirical studies on interlocking directorates and explain how our model sheds light on the understanding of the empirical findings.

Consider first the degree distribution of board directors. Conyon and Muldoon (2006) study the affiliations of board directors who hold positions in 1,733 firms in the United States in 2003. They find that 80.37% of the directors sit only on one board, 13.02% of them sit on two boards, and the remaining 6.61% of the directors sit on 8.6 boards on average. Thus most directors hold only one or two positions, but there is a small fraction of directors who occupy many positions. The authors show that similar patterns hold in Germany and the UK. This inequality in degrees of directors is in line with the marginalization result (Proposition 1)

Consider next the structure of board networks. Mizruchi (1982) provides a historical analysis of the US board network among 167 firms at seven points from 1904 to 1974, finding that almost all nodes were within distance 4. More recently, with the increased availability of data and advancement in analyzing techniques, Davis et al. (2003) study the largest manufacturing and service firms in the US over the period 1982 to 1999. They show that despite the major changes in the nature of economic activities, the structure of the board network remained relatively unchanged: the average geodesic distance between boards was 3.38, 3.46, and 3.46 in 1982, 1991 and 2001.

Turning finally to the strength of ties among boards: Battiston and Catanzaro (2004) investigate the board networks of the Fortune 1000 firms in 1999 and show that they consist mostly of weak links (the number of strength 1 links is about 10 times that the number of stronger links) and that they have a small world feature (the largest connected component includes 87% of all firms). Given that links between boards serve as information diffusion channels, the marginal returns from board-to-board ties are likely to be decreasing. Proposition 4 shows that in this case, the club network is likely to be held together by weak links. The empirical patterns are consistent with our theoretical analysis.

Interlocking directorates among Health Care Organizations: Willems and Jegers (2011) study the interlocking boards of 92 Belgian healthcare organizations. One of their main findings is that the board network is fragmented with strong links: the 92 organizations are divided into 23 components; 24 pairs of organizations share exactly the same set of board members and the heaviest link in the network is of strength 10.

Woo (2017) and Hansson et al. (2018) suggest that healthcare organizations often need to collaborate with each other to treat multi-diseased and vulnerable patients. To achieve smooth coordination, it is more efficient for organizations to have multiple shared directors with their partners. In the language of our model, this suggests that marginal returns from links are increasing in overlaps. In this case, the theory predicts that the resulting board network is fragmented with strong links. This is consistent with the empirical finding of Willems and Jegers (2011).

Boards of Journal Editors: The editorial board of a journal along with its set of referees shapes the research papers that are published in it. The collection of prestigious journals in a discipline taken together therefore can have a profound influence on the direction of research in that discipline. In economics, there has existed a concern for some time now that the leading journals are dominated by members from a few economics departments based in the United States. This concentration of editors has some to suggest that the discipline may be a risk of becoming too conformist and losing its innovativeness. This question has become more pressing over the last few decades as the profession has grown greatly and there has been a massive increase in the number of journals: this has resulted in a massive increase in the relative prestige of publishing in a few core journals. A leading economist has termed this phenomenon 'Top5ites' (see Serrano (2018) and Ductor and Visser (2021)) and in a recent paper, the emphasis on the top few journals in the career prospects of economists has been referred to as the 'Curse of the top-5' (Heckman and Moktan (2020)). We may view authors as individuals and boards of journals as clubs. In this case study, we draw on a recent paper by Ductor and Visser (2021) to document some facts about editorships and the relationship between the boards of leading journals and then relate them to our theoretical predictions.

Ductor and Visser (2021) study a set of 106 leading economics and finance journals over the period 1990-2011. They find that there were 79533 authors publishing in these journals but that only 6069 became editors, i.e. only 7.63%. Moreover, within the set of editors, over 75% were editors of just one journal but over 1.6% of these editors were editors at 4 or more journals. We recognize that the model assumes individuals are ex-ante homogenous while economics authors clearly differ in their abilities and productivity and their suitability for editorial roles. But, at a high level, these two facts are broadly consistent with the model's prediction on the marginalization of individuals (that can arise even if all individuals are similar).

Turning next to the links between the boards of different journals, for concreteness let us discuss the empirical situation in 2010. The network contains the 106 journals as nodes; a link between two journals reflects common editors. An inspection of this network reveals a number of interesting facts. The largest component contains 101 nodes, suggesting that it is more or less connected. The network is sparse with roughly 11% of all possible links being present. These links have uneven strength but the vast majority of the links are weak – over 82% have only one or two common editors. These facts suggest that the network is a small world that is held together with mostly weak ties.

To illustrate these patterns, we present the network of editorial boards of leading economics journals from the year 2010 in Figure 8. The network covers 28 leading economics journals.[17] We see that the network is connected and that most of the links are relatively

[17] These journals are Journal of Health Economics (JHE), Review of Economics and Statistics (REStat), Review of Economic Studies (REStud), Econometric Theory (ET), Journal of Monetary Economics (JME), Quarterly Journal of Economics (QJE), Journal of Economic Literature (JEL), Journal of Business and Economic Statistics (JBES), Econometrica (ECMA), Review of Financial Studies (RFS), RAND Journal of Economics (RAND), Economic Journal (EJ), Journal of Environmental Economics and Management (JEEM), Journal of Finance (JoF), Journal of Econometrics (JoE), Journal of International Economics (JIE), European Economic Review (EER), World Bank Economic Review (WBER), International Economic Review (IER), American Economic Review (AER), Journal of Human Resources (JHR), Journal

weak. Interestingly the network is held together through a hierarchical structure – the general interest journals share common editors with field journals; there are relatively few ties among the general interest journals and the field journals, respectively.

7 Conclusions

Empirical research has documented a tendency for decision-making power to be concentrated in a few persons at the head of large organizations. This phenomenon is termed the 'power elite' or the 'interlocking directorates', depending on the type of positions concerned. This paper proposed a simple model of club membership to study the circumstances that would lead to power elites and interlocking directorates and their welfare properties. There are two limitations of our approach to study club joining and member admission. First, we assume that there are capacity constraints for both individuals and clubs. While we relax the assumption on clubs' constraints in Section 5.4, we have not discussed the case when individuals can join any number of clubs, subject to some form of club joining cost. Second, while network formation studies typically assume that players can benefit from indirect connections, our benchmark model assumes that clubs only benefit from direct connections they have. In section 5.1, we extend our model by allowing clubs to benefit more from connecting to better-connected clubs. This assumption, though partially accounts for indirect benefits brought by links, is not based on well-defined micro foundations. We leave these interesting questions to future studies.

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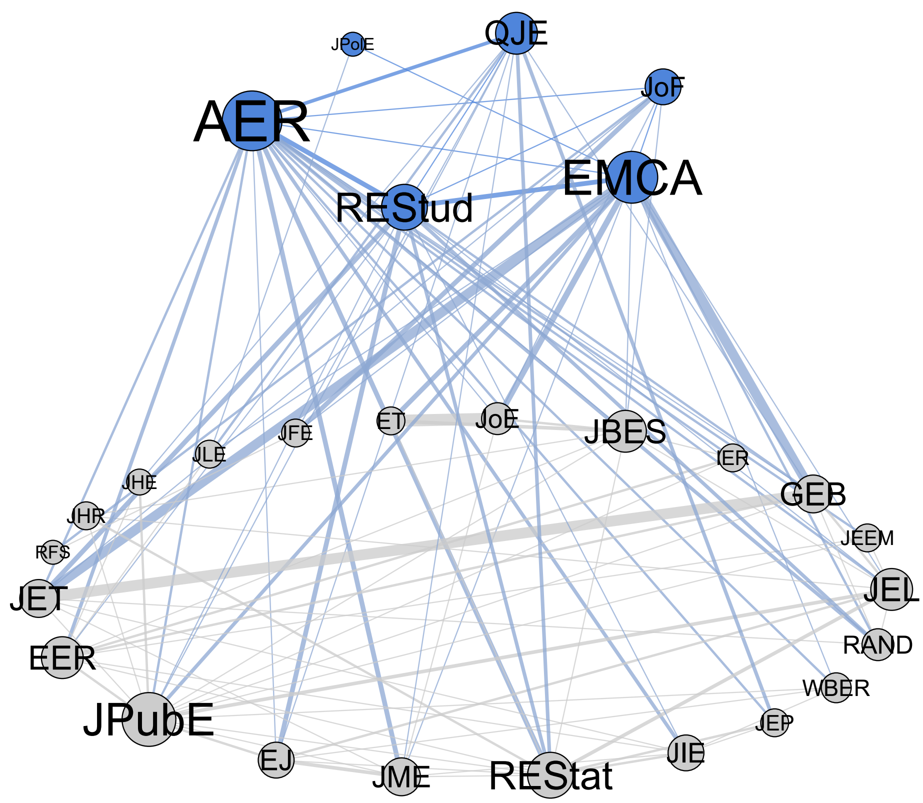

Figure 8: The editorial boards of economic journals 2010. Node size reflects the number of editors; link thickness indicates the number of common editors. Courtesy of Lorenzo Ductor and Bauke Visser

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8 Appendix: For Online Publication

Proofs

Proof of Proposition 1 To prove Proposition 1, we first prove the following lemma that lists the necessary and sufficient conditions for a membership profile to be stable.

Proposition 9. *Consider the benchmark model. There exists a stable membership profile. A membership profile* $\boldsymbol{a} \in A$ *is stable if and only if*

(i) for every individual $i \in I$ *and club* $c \in C$*, if* i *is not a member of* c*, then either* $d_i(\boldsymbol{a}) = D$ *or* $s_c(\boldsymbol{a}) = S$*,*

(ii) for every club c *with fewer than* S *members, every individual* i*, and every club* c' *that* i *joins, if* i *is not a member of* c*, then*

$$\pi_c(\boldsymbol{a}) + f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D - 1) \leq \pi_{c'}(\boldsymbol{a}),$$

In addition, if $\alpha > 0$*, then*

(iii) for every individual i *who joins fewer than* D *clubs, every club* c *that* i *does not join, every individual* i' *in club* c *must have with* $d_{i'}(\boldsymbol{a}) > d_i(\boldsymbol{a})$*, and*

(iv) for every individual i *who joins* D *clubs, every club* c *that* i *does not join and every individual* i' *that is a member of* c*, if* $d_i' < D$*, then*

$$\pi_c(\boldsymbol{a}) + \alpha(D - d_{i'}(\boldsymbol{a})) - \alpha \sum_{c'' \neq c'} a_{ic''} a_{i'c''} \leq \pi_{c'}(\boldsymbol{a}), \text{ for all } c' \text{ that } i \text{ joins.}$$

Proof. We first take up the characterization – the sufficient and necessary conditions – for stability. We then prove existence.

From the production function of clubs and the utility function of individuals, we know that there cannot be any $i \in I, c \in C$ and $\boldsymbol{a}' \in A$ with $a_i' \leq a_i$ and $a_{-i}' = a_{-i}$ such that $u_i(\boldsymbol{a}') > u_i(\boldsymbol{a})$, or $a_c' \leq a_c$ and $a_{-c}' = a_{-c}$ such that $\pi_c(\boldsymbol{a}') > \pi_c(\boldsymbol{a})$. Hence, the deviations we need to consider are joint deviations by i and c such that both of them are better off. Such deviation can be divided into four types: individual i joins club c and nothing else is changed; individual i quits some clubs and joins club c; club c dismisses some members and admits individual i; and individual i quits some clubs, club c dismisses some members, and i

joins c. Notice that for the last three kinds of deviations, if quitting two or more clubs and dropping two or more members is profitable, then quitting only one club and dropping only one member is also profitable given our utility and productivity specification. So, we only consider deviations with one quitting and (or) one dropping. We show that conditions (i)–(iii) are necessary and sufficient for the four kinds of deviations not to be jointly profitable.

For the necessity of condition (i), suppose it does not hold and there exists an individual $i \in I$ with $d_i(\boldsymbol{a}) < D$ and a club $c \in C$ with $s_c(\boldsymbol{a}) < S$, such that i is not a member of c. But then i joining c is strictly improving for both parties, which contradicts the stability of $\boldsymbol{a}$.

We also show that if condition (i) holds, then there is no jointly profitable deviation for i and c where i joins c and nothing else changes since such deviation is not feasible.

For the necessity of condition (ii), suppose, to the contrary, that there exists a club c with $s_c(\boldsymbol{a}) < S$, an individual $i \in I$ who is not a member of c, and a club $c' \in C$ that i joins, such that

$$\pi_c(\boldsymbol{a}) > \pi_{c'}(\boldsymbol{a}) - f(s_c(\boldsymbol{a}) + 1) + f(s_c(\boldsymbol{a})) - \alpha(D-1). \tag{11}$$

Notice that, by condition (i), $d_i(\boldsymbol{a}) = D$. Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i leaving c' and joining c and c accepting i. First, it is obvious that $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D-1), \tag{12}$$

so the difference in utility of i between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\begin{aligned} u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a}) =& v(\pi_c(\boldsymbol{a}) + f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D-1) + \sum_{c'' \neq c, c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\ & - v(\pi_{c'}(\boldsymbol{a}) + \sum_{c'' \neq c, c'} a_{ic''}\pi_{c''}(\boldsymbol{a})), \end{aligned} \tag{13}$$

which has the same sign as

$$\pi_c(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a}) + f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D-1), \tag{14}$$

which is positive since v is increasing. The deviation by individual i and club c from $\boldsymbol{a}$ to $\boldsymbol{a'}$ makes them both better off. A contradiction with the stability of $\boldsymbol{a}$.

We also show that if conditions (i) and (ii) hold, then there is no jointly profitable deviation for i and c where i quits a club to join c and nothing else changes. If there is such a deviation, it must be that $s_c(\boldsymbol{a}) < S$. Since i is not a member of c so, by condition (i), $d_i(\boldsymbol{a}) = D$. Let c' be the club that i leaves when joining c. Then, by (13) and (14) and condition (ii), the utility of i does not increase and so the deviation is not profitable to i.

For the necessity of condition (iii), suppose, to the contrary, that there exist individuals $i \in I$ and $i' \in I$ such that $D > d_i(\boldsymbol{a}) \geq d_{i'}(\boldsymbol{a})$ and a club $c \in C$ such that i' is a member of and i is not. Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i joining c and c accepting i and dropping i'. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = \alpha(d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a}) + 1), \tag{15}$$

which is positive if and only if $\alpha > 0$. Also, since v is increasing, both individual i and club c and strictly benefit deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$ when $\alpha > 0$. A contradiction with the stability of $\boldsymbol{a}$.

We also show that if conditions (i) and (iii) hold, then there is no jointly profitable deviation for i and c where c drops a member to admit i and nothing else changes. If there is such a deviation, it must be that $d_i(\boldsymbol{a}) < D$. Then from condition (i), it mush be $s_c(\boldsymbol{a}) = S$. Let i' be the individual that club c drops. Then, by (15) and condition (iii), the productivity of club c does not increase and so the deviation is not profitable to c.

For the necessity of condition (iv), suppose that $\alpha > 0$ and suppose, to the contrary, that there exist two individuals $i, i' \in I$ with $d_i(\boldsymbol{a}) = D$ and $d_{i'}(\boldsymbol{a}) < D$, a club $c \in C$ that has member i' but not i, and a club c' that i joins, such that

$$\pi_c(\boldsymbol{a}) > \pi_{c'}(\boldsymbol{a}) - \alpha\left(D - d_{i'}(\boldsymbol{a}) - \sum_{c'' \neq c'} a_{ic''}a_{i'c''}\right). \tag{16}$$

Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i joining c and leaving c', and c accepting i and dropping i'. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = \alpha(D - d_{i'}(\boldsymbol{a})), \tag{17}$$

which is positive if and only if $\alpha > 0$. The difference in utility of i between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$
\begin{aligned}
u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a}) =& v(\pi_c(\boldsymbol{a'}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) - v(\pi_{c'}(\boldsymbol{a}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\
=& v(\pi_c(\boldsymbol{a}) + \alpha\,(D - d_{i'}(\boldsymbol{a})) - \alpha \sum_{c''\neq c'} a_{ic''}a_{i'c''} + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\
&- v(\pi_{c'}(\boldsymbol{a}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})),
\end{aligned}
\tag{18}
$$

which has the same sign as

$$
\pi_c(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a}) + \alpha\left(D - d_{i'}(\boldsymbol{a}) - \sum_{c''\neq c'} a_{ic''}a_{i'c''}\right), \tag{19}
$$

which is positive since v is increasing. The deviation by individual i and club c from $\boldsymbol{a}$ to $\boldsymbol{a'}$ makes them both better off. A contradiction with the stability of $\boldsymbol{a}$.

We also show that if conditions (i)–(iv) hold, then there is no jointly profitable deviation for i and c where i leaves a club, c drops a member, and i joins c. Suppose there is such a deviation, if $d_i(\boldsymbol{a}) < D$ or $s_c(\boldsymbol{a}) < S$, since the deviation is profitable with i quitting a club and c dismissing a member, it is also profitable if i does not quit the club and c does not dismiss the member. We know conditions (i)–(iii) guarantee that there is no such mutually beneficial deviation. So, here we consider the deviations of i and c when $d_i(\boldsymbol{a}) = D$ and $s_c(\boldsymbol{a}) = S$. In this case, by (18) and (19) and condition (iv), the utility of i does not increase and so the deviation is not profitable to i.

We finally turn to the existence of stable membership profiles. We provide a proof by construction.

Suppose $nD \geq mS$. Let $m' \leq m$ and $n' \leq n$ be the largest integers such that $m'S = n'D$. Notice that since $m \geq D$ and $n \geq S$ so $m' \geq D$ and $n' \geq S$. Construct a membership profile $\boldsymbol{a}$ as follows. First, select n' individuals and m' clubs, let all selected individuals join D clubs we select so that all m' clubs have S members. This profile can be obtained by letting clubs admit individuals in sequence: make each club admit S individuals that have the smallest degree in its turn before moving to the next club. If $n - n' \geq S$, take S out of $n - n'$ remaining individuals and put each of them in each of $m - m'$ remaining clubs. Otherwise, put each of $n - n'$ remaining individuals in each of $m - m'$ clubs. It is easy to verify that this profile is stable.

Suppose $mS > nD$: consider a membership profile $\boldsymbol{a}$ where all individuals join D clubs, and $\lfloor \frac{nD}{S} \rfloor$ clubs have S members, one club has $(nD) \bmod S$ members, and the remaining clubs have 0 members. This profile can be obtained by letting clubs admit individuals in sequence. Make each club admit S individuals that have the smallest degree in its turn before moving to the next club. Stop when all individuals have degree D. This profile is always stable as it satisfies all four conditions in Proposition 9. Condition (i) is satisfied obviously. Conditions (iv) and (iii) are automatically satisfied as no individual joins less than D clubs. For condition (ii), if a club c has less than S members, then either it is the one club with $(nD) \bmod S$ members or it has 0 members. In both cases, for an individual i that is not in c and for any club that i is in, c' must have more members than c does and all members of c' join D clubs, making condition (ii) satisfied.

□

We now prove Proposition 1. Proposition 9 has established existence. Consider the egalitarian outcome result in the absence of network effects. When $\alpha = 0$, the membership profile generated with the following algorithm is stable. Let clubs admit individuals sequentially. Fill a club with S individuals that currently have the lowest degrees and then move to the next club. Stop until all clubs are full or all individuals have joined D clubs. Since $n \geq S$, this algorithm is feasible. If the algorithm terminates when all clubs have S members, then all clubs have productivity $f(S)$ which is the highest productivity a club can get. Hence the membership profile is stable. If the algorithm terminates when all individuals are in D clubs, then there are $\lfloor \frac{mS}{D} \rfloor$ clubs that have productivity $f(S)$, one club that has productivity $f((mS) \mod D)$, and the rest clubs have productivity 0. The only possible profitable deviation from one individual is to quit the club with productivity $f((mS) \mod D)$ and join a club with productivity $f(S)$, but no club with productivity $f(S)$ want to deviate. Hence the membership profile is stable. Given the way we construct the membership profile, we have $|d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a})| \leq 1$ for all $i, i' \in I$.

We now turn to the case when $\alpha > 0$. For the cardinality of $I_2(\boldsymbol{a})$, take any individual $i \in I_2(\boldsymbol{a})$ with minimal $d_i(\boldsymbol{a})$ and let $c \in C$ be any club that i members. By condition (iii) of Proposition 9, all individuals in $I_2(\boldsymbol{a})$ are members of c and, by condition (i) of Proposition 9, $s_c(\boldsymbol{a}) \leq S$. Hence $|I_2(\boldsymbol{a})| \leq S$.

For cardinality of $C_2(\boldsymbol{a})$, suppose that $C_2(\boldsymbol{a}) \neq \varnothing$, we will show that there exists an individual i that is a member of all clubs in $C_2(\boldsymbol{a})$. We consider the cases of $I_2(\boldsymbol{a}) = \varnothing$ and $I_2(\boldsymbol{a}) \neq \varnothing$ separately. If $I_2(\boldsymbol{a}) = \varnothing$, then members of the clubs in $C_2(\boldsymbol{a})$ are of degree D and, for any $c \in C_2(\boldsymbol{a})$, $s_c(\boldsymbol{a}) < S$ (as c does not achieve maximal productivity). Take any $c' \in C_2(\boldsymbol{a})$ with minimal productivity, $\pi_{c'}(\boldsymbol{a})$, and any member i of c'. Take any $c \in C_2(\boldsymbol{a}) \setminus \{c'\}$. Since $s_c(\boldsymbol{a}) < S$ and since $\pi_c(\boldsymbol{a}) \geq \pi_{c'}(\boldsymbol{a})$ so, by condition (ii) of Proposition 9, i is a member of c. Hence i is a member of all clubs in $C_2(\boldsymbol{a})$. If $I_2(\boldsymbol{a}) \neq \varnothing$ then take any $i \in I_2(\boldsymbol{a})$ with maximal degree. Take any club $c \in C_2(\boldsymbol{a})$. Since c does not achieve the highest productivity so either $s_c(\boldsymbol{a}) < S$ or c has a member in $I_2(\boldsymbol{a})$. In the first case, i is a member of c by condition (i) of Proposition 9. In the second case, i is a member of c by condition (iii) of Proposition 9. Hence i is a member of all clubs in $C_2(\boldsymbol{a})$. By condition (i) of Proposition 9, $d_i(\boldsymbol{a}) \leq D$. Hence $|C_2(\boldsymbol{a})| \leq D$.

We now use these derivations on the size of the different groups to derive bounds on the size of $I_1(\boldsymbol{a})$, $I_3(\boldsymbol{a})$, $C_1(\boldsymbol{a})$ and $C_3(\boldsymbol{a})$.

We begin with the case $nD \geq mS$ and show that $\frac{mS}{D} - \frac{S(D+3)}{2} \leq |I_1(\boldsymbol{a})| \leq \frac{mS}{D}$ and $n - \frac{mS}{D} - S \leq |I_3(\boldsymbol{a})| \leq n - \frac{mS}{D}$. Since the aggregate club capacity is mS, $|I_1(\boldsymbol{a})| \leq \frac{mS}{D}$. For the lower bound, suppose that in a stable membership profile $\boldsymbol{a}$, all clubs are full, then we have $|I_1(\boldsymbol{a})|D + \sum_{i \in I_2(\boldsymbol{a})} d_i(\boldsymbol{a}) = mS$, and hence $|I_1(\boldsymbol{a})|D + |I_2(\boldsymbol{a})|D > mS$. Since $|I_2(\boldsymbol{a})| \leq S$,

$$|I_1(\boldsymbol{a})| > \frac{mS}{D} - S. \tag{20}$$

Suppose that in a stable membership profile $\boldsymbol{a}$, not all clubs are full, then we know $|I_3(\boldsymbol{a})| = 0$ as otherwise there is a jointly profitable deviation for an individual in $I_3(\boldsymbol{a})$ and a club that is not full where the individual joins the club. Therefore,

$$|I_1(\boldsymbol{a})| + |I_2(\boldsymbol{a})| = n \geq \frac{mS}{D}, \tag{21}$$

and so $|I_1(\boldsymbol{a})| \geq \frac{mS}{D} - S$ given $|I_2(\boldsymbol{a})| \leq S$.

Regarding the bounds for $|I_3(\boldsymbol{a})|$. Since $|I_1(\boldsymbol{a})| + |I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| = n$, $|I_1(\boldsymbol{a})| \leq \frac{mS}{D}$, and $|I_2(\boldsymbol{a})| \leq S$, so $|I_3(\boldsymbol{a})| \geq n - \frac{mS}{D} - S$. Moreover, if $|I_3(\boldsymbol{a})| > n - \frac{mS}{D}$, then $|I_1(\boldsymbol{a})| + |I_2(\boldsymbol{a})| < \frac{mS}{D}$. The club capacity is not exhausted and there must exist a club c that is not full. There is a jointly profitable deviation for an individual i in $I_3(\boldsymbol{a})$ and club c where i joins c. A contradiction.

Next, consider the case when $nD < mS$. We show that $\frac{nD}{S} - D \leq |C_1(\boldsymbol{a})| \leq \frac{nD}{S}$ and $m - \frac{nD}{S} - D \leq |C_3(\boldsymbol{a})| \leq m - \frac{nD}{S}$. Since the aggregate individual availability is nD, $|C_1(\boldsymbol{a})| \leq \frac{nD}{S}$. To show the lower bound for $|C_1(\boldsymbol{a})|$ is $nD/S - D$, suppose that in a stable membership profile $\boldsymbol{a}$, all individuals exhaust their membership availability, then we have $|C_1(\boldsymbol{a})|S +$

$\sum_{c \in C_2(\boldsymbol{a})} s_c(\boldsymbol{a}) = nD$, and hence $|C_1(\boldsymbol{a})|S + |C_W(\boldsymbol{a})|S \geq nD$. Since $|C_2(\boldsymbol{a})| \leq D$, $|C_1(\boldsymbol{a})| \geq \frac{nD}{S} - D$. Suppose that in a stable membership profile $\boldsymbol{a}$, not all individuals exhaust their membership availability, then we know $|C_3(\boldsymbol{a})| = 0$ as otherwise there is a jointly profitable deviation for the individual who joins less than D clubs and a club in $C_3(\boldsymbol{a})$ where the individual joins the club. Therefore,

$$|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| = m \geq \frac{nD}{S}, \tag{22}$$

and so $|C_1(\boldsymbol{a})| \geq \frac{nD}{S} - D$ given $|C_2(\boldsymbol{a})| \leq D$.

Regarding the bounds for $|C_3(\boldsymbol{a})|$. Since $|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| + |C_3(\boldsymbol{a})| = m$, $|C_1(\boldsymbol{a})| \leq \frac{nD}{S}$, and $|C_2(\boldsymbol{a})| \leq D$, so $|C_3(\boldsymbol{a})| \geq m - \frac{nD}{S} - D$. Moreover, if $|C_3(\boldsymbol{a})| > m - \frac{nD}{S}$, then $|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| < \frac{nD}{S}$. The aggregate membership availability is not exhausted and there must exist an individual i that joins less than D clubs. There is a jointly profitable deviation for i and a club c in $C_3(\boldsymbol{a})$ and club c where i joins c. A contradiction.

Finally, we turn to the statements on $M_I(\boldsymbol{a})$ and $M_C(\boldsymbol{a})$. Consider the case of $nD > mS$ and let $\boldsymbol{a}$ be a stable membership profile. Suppose first that $nD > mS + DS$. By the lower bound on $I_3(\boldsymbol{a})$, $|I_3(\boldsymbol{a})| \geq (nD - mS - SD)/D > 0$. Hence there exists at least one individual without a club under $\boldsymbol{a}$. It follows that $\sum_{i \in I} d_i(\boldsymbol{a}) = mS$. For if $\sum_{i \in I} d_i(\boldsymbol{a}) < mS$ then there exists a club c with $s_c(\boldsymbol{a}) < S$ and, by $|I_3(\boldsymbol{a})| > 0$, there exists an individual i with $d_i(\boldsymbol{a}) = 0$. They would both benefit from i joining c, contradicting the stability of $\boldsymbol{a}$. Using $\sum_{i \in I} d_i(\boldsymbol{a}) = mS$ and $|I_3(\boldsymbol{a})| \geq n - \frac{mS}{D} - S$ we get

$$\begin{aligned}
\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) &= \frac{|I_3(\boldsymbol{a})| - n + \min(n, mS)}{\min(n, mS) - \lceil \frac{mS}{D} \rceil} \geq \frac{|I_3(\boldsymbol{a})| - n + \min(n, mS)}{\min(n, mS) - \frac{mS}{D}} \\
&\geq \frac{\min(n, mS) - \frac{mS}{D} - S}{\min(n, mS) - \frac{mS}{D}} = 1 - \frac{S}{\min(n, mS) - \frac{mS}{D}} \\
&= 1 - \frac{D}{\min\left(\frac{nD - mS}{S}, (D-1)m\right)}
\end{aligned} \tag{23}$$

Second, suppose that $nD \leq mS + DS$. Then

$$1 - \frac{D}{\min\left(\frac{nD - mS}{S}, (D-1)m\right)} \leq 1 - \frac{D}{\min\left(\frac{DS}{S}, (D-1)m\right)} \leq 0 \leq \mathcal{M}_{\mathcal{I}}(\boldsymbol{a}). \tag{24}$$

The upper bound on $\mathcal{M}_{\mathcal{C}}(\boldsymbol{a})$ can be obtained by analogous arguments.

■

Proof of Proposition 2

We prove a more general characterization of efficient membership profiles, without the parity conditions in Proposition 2.

Lemma 1. *Suppose $\alpha > 0$. Assume $nD \geq mS$.*

- *A membership profile is clubs-efficient if and only if there are $\lfloor \frac{mS}{D} \rfloor$ individuals that join D clubs, one individual that joins (mS) mod D clubs, and the remaining individuals join no clubs.*

- *If $v''(\cdot) \geq 0$, then a membership profile is utilitarian optimum if and only if it is clubs-efficient. If $v''(\cdot) < 0$ and satisfies condition (4), then in any utilitarian optimum membership profile, either $d_i(\boldsymbol{a}) \leq 1$ for all $i \in I$ or $d_i(\boldsymbol{a}) \geq 1$ for all $i \in I$.*

Assume $nD < mS$.

- *If $f''(\cdot) > 0$, then a membership profile is clubs-efficient if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits (nD) mod S members, and the remaining clubs admit no members. If $f''(\cdot) = 0$, then a membership profile is clubs-efficient if and only if each individual joins D clubs. If $f''(\cdot) < 0$, then a membership profile is clubs-efficient if and only if (nD) mod m admit $\lceil \frac{nD}{m} \rceil$ members and the remaining clubs admit $\lfloor \frac{nD}{m} \rfloor$ members.*

- *If $v''(\cdot) \geq 0$, then a membership profile is utilitarian optimum if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits (nD) mod S members, and the remaining clubs admit no members. If $v''(\cdot) < 0$ and (nD) mod $S = 0$, then the membership profile is utilitarian optimum if and only if nD/S clubs admit S members and the remaining clubs admit no members. If $v''(\cdot) < 0$ and (nD) mod $S > 0$, then in any utilitarian optimum membership profile, the number of clubs that admit some but less than S members is not more than $S - 1$.*

For the case when $nD \geq mS$. First, given a membership profile $\boldsymbol{a}$, the aggregate productivity of clubs is

$$\begin{aligned}\sum_{c \in C} \pi_c(\boldsymbol{a}) &= \sum_{c \in C} f(s_c(\boldsymbol{a})) + \alpha \sum_{c \in C} \sum_{i \in I} a_{ic}(d_i(\boldsymbol{a}) - 1) \\ &\leq mf(S) + \alpha \sum_{i \in I} d_i(\boldsymbol{a})(d_i(\boldsymbol{a}) - 1), \qquad (25)\end{aligned}$$

where the equality is obtained only when $s_c(\boldsymbol{a}) = S$ for all $c \in C$. Now we solve the following maximization problem:

$$\max \sum_{i \in I} d_i(\boldsymbol{a})(d_i(\boldsymbol{a}) - 1) \text{ s.t. } d_i(\boldsymbol{a}) \in \{0, 1, ..., D\} \text{ for all } i \in I \text{ and } \sum_{i \in I} d_i(\boldsymbol{a}) \leq mS.$$

Since $g(x) = x(x-1)$ is superadditive on the set of non-negative integers and this is strict on positive integers, the solution to the maximization problem is a vector $(d_i^*(\boldsymbol{a}))_{i \in I}$ such that $d_i^*(\boldsymbol{a}) = D$ for all $i \in I'$ where $I' \subset I$ and $|I'| = \lfloor \frac{mS}{D} \rfloor$, $d_i^*(\boldsymbol{a}) = (mS) \bmod D$ for some $i = k \in I \backslash I'$ (in the case of $(mS) \bmod D \geq 1$) and $d_i^*(\boldsymbol{a}) = 0$ for all $i \in I \backslash (I' \cup \{k\})$.

We now show that when $nD \geq mS$, there always exists a membership structure where there are $\lfloor \frac{mS}{D} \rfloor$ individuals that join D clubs, one individual that joins $(mS) \bmod D$ clubs, and the remaining individuals join no clubs (which makes $s_c(\boldsymbol{a}) = S$ for all $c \in C$), so that a structure $\boldsymbol{a} \in A$ is clubs-efficient if and only if it satisfies such a club joining pattern. Construct a membership structure as follows. Consider $\lfloor mS/D \rfloor$ individuals first, in a sequence. Make each such i join D clubs that have the smallest membership size at her turn before moving to the next individual. If $(mS) \bmod D \geq 1$ so that there are clubs that do not have S members at the end of the process, take one more individual and make him join those $(mS) \bmod D$ clubs. Since $\lfloor mS/D \rfloor D + (mS) \bmod D = mS$, the construction is valid and results in the desired membership structure.

For utilitarian optimal structures, given a membership profile $\boldsymbol{a}$, the aggregate utility of individuals is

$$\sum_{i \in I} u_i(\boldsymbol{a}) = \sum_{i \in I} v(\sum_{c \in C} a_{ic} \pi_c(\boldsymbol{a})). \tag{26}$$

We know that $\sum_{c \in C} a_{ic} \pi_c(\boldsymbol{a}) \leq D(f(S) + \alpha S(D-1))$ for all $i \in I$ and $\sum_{i \in I} \sum_{c \in C} a_{ic} \pi_c(\boldsymbol{a}) = \sum_{c \in C} s_c(\boldsymbol{a}) \pi_c(\boldsymbol{a}) \leq S \sum_{c \in C} \pi_c(\boldsymbol{a})$, where the equality is obtained only when $s_c(\boldsymbol{a}) = S$ for all $c \in C$. Given that a clubs-efficient structure that maximizes $\sum_{c \in C} \pi_c(\boldsymbol{a})$ features $s_c(\boldsymbol{a}) = S$ for all $c \in C$, $\sum_{i \in I} \sum_{c \in C} a_{ic} \pi_c(\boldsymbol{a})$ is maximized if and only if $\boldsymbol{a}$ is clubs-efficient. We also know that a clubs-efficient structure makes $\lfloor \frac{mS}{D} \rfloor$ individuals have utility $v(D(f(S) + S(D-1)))$, at most one individual have positive but less than $v(D(f(S) + S(D-1)))$ utility, and the rest individuals have zero utility. Hence, when $v''(\cdot) \geq 0$, the clubs-efficient membership profile is the solution to the maximization profile of $\max_{\boldsymbol{a} \in A} u_i(\boldsymbol{a})$. We have shown that when $v''(\cdot) \geq 0$, a membership profile is utilitarian optimum if and only if it is clubs-efficient.

Turning to when $v''(\cdot) < 0$ and satisfies

$$v(f(S)) - v(0) > (n-1)\left(v\left(f(S) + \frac{2\alpha S(D-1)}{n-1}\right) - v\left(f(S)\right)\right), \tag{27}$$

we show that suppose in a membership structure $\boldsymbol{a} \in A$, there exists two individuals $i, i' \in I$ such that $d_i(\boldsymbol{a}) > 1$ and $d_{i'}(\boldsymbol{a}) = 0$, then $\boldsymbol{a}$ cannot be utilitarian optimum. Suppose such a structure $\boldsymbol{a}$ is utilitarian optimum. Note first that it must be $s_c(\boldsymbol{a}) = S$ for all $c \in C$, as otherwise making individual i' join a club that is not full strictly raises aggregate welfare. Let $c \in C$ be a club where $a_{ic} = 1$. Consider another membership structure $\boldsymbol{a'}$ where c drops i and admits i'. The difference of aggregate utility between the two structures is

$$\sum_{i \in I}(u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a})) \geq v(f(S)) - v(0) + \sum_{i \neq i'}(u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a})), \tag{28}$$

since $u_{i'}(\boldsymbol{a'}) \geq v(f(S))$ and $u_{i'}(\boldsymbol{a}) = v(0)$. Given that i' replaces i in club c, the productivity of club c and clubs that i members decreases:

$$\pi_c(\boldsymbol{a}) - \pi_c(\boldsymbol{a'}) = \alpha(d_i(\boldsymbol{a}) - 1), \text{ and} \tag{29}$$

$$\pi_{c'}(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a'}) = \alpha \text{ for all } c' \neq c \text{ with } a_{ic'} = 1. \tag{30}$$

So, the aggregate productivity drop is at most $2\alpha(D-1)$, which is obtained when $d_i(\boldsymbol{a}) = D$. Since $v''(\cdot) < 0$ and the minimal utility an individual obtains when he is in a club is $v(f(S))$,

$$\begin{aligned}\sum_{i \neq i'}(u_i(\boldsymbol{a}) - u_i(\boldsymbol{a'})) &\leq \sum_{i \neq i'}\left[v\left(f(S) + \sum_{c \in C} a_{ic}(\pi_c(\boldsymbol{a}) - \pi_c(\boldsymbol{a'}))\right) - v(f(S))\right] \\ &\leq (n-1)\left[v\left(f(S) + \frac{2\alpha(D-1)S}{n-1}\right) - v(f(S))\right].\end{aligned} \tag{31}$$

Hence,

$$\sum_{i \in I}(u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a})) \geq v(f(S)) - v(0) - (n-1)\left[v\left(f(S) + \frac{2\alpha(D-1)S}{n-1}\right) - v(f(S))\right] > 0, \tag{32}$$

contradicting structure $\boldsymbol{a}$ being utilitarian optimum. This completes the proof.

For the case when $nD < mS$, given a membership profile $\boldsymbol{a}$, the aggregate productivity of clubs is

$$\sum_{c\in C}\pi_c(\boldsymbol{a}) = \sum_{c\in C} f(s_c(\boldsymbol{a})) + \alpha\sum_{c\in C}\sum_{i\in I} a_{ic}(d_i(\boldsymbol{a})-1)$$
$$\leq \sum_{c\in C} f(s_c(\boldsymbol{a})) + \alpha nD(D-1), \tag{33}$$

where the equality is obtained only when $d_i(\boldsymbol{a}) = D$ for all $i \in I$. Now we look at the problem of max $\sum_{c\in C} f(s_c(\boldsymbol{a}))$, s.t. $s_c(\boldsymbol{a}) \in \{0, 1, ..., S\}$ for all $c \in C$ and $\sum_{c\in C} s_c(\boldsymbol{a}) \leq nD$. When $f(\cdot)$ is convex, the solution to the maximization problem is a vector $(s_c^*(\boldsymbol{a}))_{c\in C}$ such that $s_c^*(\boldsymbol{a}) = S$ for all $c \in C'$ where $C' \subset C$ and $|C'| = \lfloor \frac{nD}{S} \rfloor$, $s_c^*(\boldsymbol{a}) = (nD) \bmod S$ for some $c = k \in C\backslash C'$ (in the case of $(nD) \bmod S \geq 1$) and $s_c^*(\boldsymbol{a}) = 0$ for all $c \in C\backslash(C' \cup \{k\})$. When $f(\cdot)$ is linear, the solution to the maximization problem is any $(s_c^*(\boldsymbol{a}))_{c\in C}$ where $s_c(\boldsymbol{a}) \in \{0, 1, ..., S\}$ for all $c \in C$ and $\sum_{c\in C} s_c(\boldsymbol{a}) = nD$. When $f(\cdot)$ is concave, the solution to the maximization problem is a vector $(s_c^*(\boldsymbol{a}))_{c\in C}$ such that $s_c^*(\boldsymbol{a}) = \lceil \frac{nD}{m} \rceil$ for all $c \in C'$ where $C' \subset C$ and $|C'| = (nD) \bmod m$, and $s_c^*(\boldsymbol{a}) = \lfloor \frac{nD}{m} \rfloor$ for all $c \in C\backslash C'$. This proves the characterization of clubs-efficient membership profiles.

For utilitarian optimum membership profiles, given a membership profile $\boldsymbol{a}$, we know the aggregate utility of individuals is $\sum_{i\in I} u_i(\boldsymbol{a}) = \sum_{i\in I} v(\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}))$ where $\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) \leq D(f(S) + S(D-1))$ for all $i \in I$ and

$$\sum_{i\in I}\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) = \sum_{c\in C} s_c(\boldsymbol{a}) f(s_c(\boldsymbol{a})) + \alpha\sum_{c\in C} s_c(\boldsymbol{a})\sum_{i\in I} a_{ic}(d_i(\boldsymbol{a})-1)$$
$$\leq \sum_{c\in C} s_c(\boldsymbol{a}) f(s_c(\boldsymbol{a})) + \alpha(D-1)s_c(\boldsymbol{a})^2, \tag{34}$$

where the equality is obtained only when $d_i(\boldsymbol{a}) = D$ for all $i \in I$. Since $g(x) = xf(x) + \alpha(D-1)x^2$ is superadditive on non-negative integers and strictly superadditive on positive integers, for any $\alpha \geq 0$, $D \geq 1$, and strictly increasing f with $f(0) = 0$, $\sum_{i\in I}\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a})$ is maximized if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members. When $v''(\cdot) \geq 0$, it is easy to see that this membership profile is also the solution to the maximization problem of $\max_{\boldsymbol{a}\in A} u_i(\boldsymbol{a})$. We have shown that when $v''(\cdot) \geq 0$, a membership profile is utilitarian optimum if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members.

Turning to when $v''(\cdot) < 0$, consider the utilitarian optimum structure where $v''(\cdot) \geq 0$.

Under this structure, the utility of $(nD) \bmod S$ individuals is

$$v[(D-1)(f(S)+\alpha S(D-1))+f\left((nD) \bmod S\right)+\alpha\left((nD) \bmod S\right)(D-1)], \tag{35}$$

while the utility of all other individuals is $v[(D)(f(S)+\alpha S(D-1))]$. If this structure is utilitarian optimum, we have finished the proof. If the structure is not utilitarian optimum, then $(nD) \bmod S \neq 0$ and in a utilitarian optimum membership profile, the lowest utility of an individual is greater than (35), implying that the smallest size of a club is greater than $(nD) \bmod S$. Suppose the smallest club size is $s_c(\boldsymbol{a}) = (nD) \bmod S + k$ where $k \in \{1, ..., S-(nD) \bmod S-1\}$. For the structure to be utilitarian optimal, the number of unfull clubs is at most k, where the bound k is reached when we reduce the club size of k clubs by 1 to increase the size of the smallest club. So, the number of clubs with size greater than 0 and lower than S is at most $1+k \leq S-(nD) \bmod S \leq S-1$.

■

Proof of Proposition 3

For point (i) we will show that if

$$\min_{x\in[0,S-1]}(h(x+1)-h(x)) > \frac{D-1}{D}\max_{x\in[0,S-1]}(h(x+1)-h(x)),$$

then the number of agents who join no clubs under a stable membership structure is bounded above by $|I_2(\boldsymbol{a})| \geq n-\frac{mS}{D}-S$, like in the case of linear returns from links. Then the derivation of the value of marginalization measure follows the same steps as in the proof of Proposition 1. We first show that the cardinality of $|I_2(\boldsymbol{a})|$ is not greater than S in any stable membership structure. Take any individual $i \in I_2(\boldsymbol{a})$ with the minimal degree and let $c \in C$ be any clubs that i members. We show that all individuals in $I_2(\boldsymbol{a})$ are members of c, and since $s_c(\boldsymbol{a}) \leq S$, $|I_2(\boldsymbol{a})| \leq S$. Suppose that there exists an $i' \in I_2(\boldsymbol{a})$ where $d_{i'}(\boldsymbol{a}) \geq d_i(\boldsymbol{a})$ and $a_{i'c} = 0$. There is a jointly profitable deviation for i' and c where c drops i and admits i'. The deviation improves the utility of i' since none of the clubs she is already in suffers from a loss of productivity and she is in a new club. The productivity change of club c is positive since

$$\begin{aligned}
\pi_c(\boldsymbol{a'}) = \pi_c(\boldsymbol{a}) &= f(s_c(\boldsymbol{a'})) - f(s_c(\boldsymbol{a})) + \sum_{c'\neq c} h(g_{cc'}(\boldsymbol{a'})) - \sum_{c'\neq c} h(g_{cc'}(\boldsymbol{a})) \\
&= \sum_{c\neq c, a_{ic}=1, a_{i'c}=0} [h(g_{cc'}(\boldsymbol{a})-1) - h(g_{cc'}(\boldsymbol{a}))] + \sum_{c\neq c, a_{ic}=0, a_{i'c}=1} [h(g_{cc'}(\boldsymbol{a})+1) - h(g_{cc'}(\boldsymbol{a}))] \\
&\geq -(d_i(\boldsymbol{a})-1) \max_{x\in[0,S-1]} (h(x+1)-h(x)) + d_{i'}(\boldsymbol{a}) \min_{x\in[0,S-1]} (h(x+1)-h(x)) \\
&> 0 \qquad (36)
\end{aligned}$$

given $\min_{x\in[0,S-1]}(h(x+1)-h(x)) > \frac{D-1}{D} \max_{x\in[0,S-1]}(h(x+1)-h(x))$ and $d_i(\boldsymbol{a}) \leq d_{i'}(\boldsymbol{a}) \leq D$.

Given that $|I_2(\boldsymbol{a})| \leq S$, we can prove that $\frac{mS}{D} - S \leq |I_1(\boldsymbol{a})| \leq \frac{mS}{D}$ and $n - \frac{mS}{D} - S \leq |I_3(\boldsymbol{a})| \leq n - \frac{mS}{D}$ following the same steps as in the proof of Proposition 1.

For the first part of point (ii), we show that there exists a stable membership profile $\boldsymbol{a}$ with $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$. Let $\boldsymbol{a}$ be a membership structure constructed as follows. If D divides m then equally divide clubs into m/D groups. Let all D clubs in the same group admit the same S individuals. If D does not divide m then divide the clubs into $\lceil m/D \rceil$ groups where the first $\lfloor m/D \rfloor$ groups have D clubs and the last group has m mod D clubs. Let all D clubs in the first $\lfloor m/D \rfloor$ groups admit the same S individuals; individuals who join these clubs have degree D. Let the m mod D clubs also admit the same S individuals; individuals who join these clubs have degree m mod D. It is easy to see that the structure is stable. The sum of degrees of individuals under $\boldsymbol{a}$ is $\sum_{i\in I} d_i(\boldsymbol{a}) = mS$. There are $\lceil m/D \rceil S$ individuals in clubs and $|I_3(\boldsymbol{a})| = n - \lceil m/D \rceil S$ individuals who join no club. Therefore

$$\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = \frac{\min(n, mS) - \lceil \frac{m}{D} \rceil S}{\min(n, mS) - \lceil \frac{mS}{D} \rceil} \geq \frac{\min(n, mS) - \frac{mS}{D} - S}{\min(n, mS) - \frac{mS}{D}} = 1 - \frac{D}{\min(\frac{nD-mS}{S}, (D-1)m)}. \quad (37)$$

For the second part of point (ii), assume that $h(S) - h(S-1) > 2(h(1) - h(0))$, $n > mS$ and m is even. Consider a membership structure $\boldsymbol{a}$ constructed as follows. Group clubs into $m/2$ pairs and select $m/2$ groups of individuals containing S individuals each. Match each pair of clubs with a unique group of individuals and make all the individuals assigned to each pair members of both clubs in the pair (an example of this constructed membership structure with $m = 8$ clubs having capacity $S = 4$ each is illustrated in Figure 6). The structure is stable because any deviation by a club weakens its existing link with another club from S to $S-1$ and creates at most two new links. Notice that the sum of degrees of individuals under

$\boldsymbol{a}$ is $\sum_{i\in I} d_i(\boldsymbol{a}) = 2\cdot mS/2 = mS$ and there are $|I_3(\boldsymbol{a})| = n - mS/2$ individuals who are not members of any club. Hence the value of marginalisation measure for $\boldsymbol{a}$ is

$$\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = \frac{\frac{mS}{2}}{mS - \lceil \frac{mS}{D} \rceil} \leq \frac{\frac{mS}{2}}{mS - \frac{mS+D-1}{D}} = \frac{D}{2(D-1)}. \tag{38}$$

For point (iii), first note that if $n > mS$ then all clubs are full in a stable profile $\boldsymbol{a}$, as otherwise there exists an individual $i \in I_3(\boldsymbol{a})$ and a club c that is not full, which implies that there is a profitable deviation by i and c where i joins c. We then provide the point by showing that in a stable profile $\boldsymbol{a}$, $|I_2(\boldsymbol{a})| < S^2 D$, which leads to

$$\begin{aligned}\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = &\frac{mS - (|I_1(\boldsymbol{a})| + |I_2(\boldsymbol{a})|)}{mS - \lceil \frac{mS}{D} \rceil} \geq \frac{mS - \frac{mS}{D} - S^2 D}{mS - \frac{mS}{D}} \\ &> 1 - \frac{S^2 D^2}{mS(D-1)} = 1 - \frac{D^2 S}{(D-1)m}.\end{aligned} \tag{39}$$

To derive the upper bound on $|I_2(\boldsymbol{a})|$, suppose $I_2(\boldsymbol{a}) \neq \varnothing$, let $i \in I_2(\boldsymbol{a})$ be the individual with the lowest degree in $I_2(\boldsymbol{a})$ and let c be a club that hosts i. The number of clubs c has links with is at most $(D-1)S$, since each member creates at most $D-1$ links for c and there are at most S members in c. The number of club memberships provided by c and its neighbouring clubs is thus at most $((D-1)S+1)S < S^2 D$. Suppose $|I_2(\boldsymbol{a})| \geq S^2 D$. Then there exists an individual $i' \in I_2(\boldsymbol{a})$ who is neither in club c nor in its neighboring clubs. Consider a deviation by c and i' where c expels i and admits i'. This deviation is feasible for i' because being in $I_2(\boldsymbol{a})$, i' is in less than D clubs under $\boldsymbol{a}$. It is also strictly profitable to i'. The deviation is also weakly profitable to c. This is because expelling i reduces $d_i(\boldsymbol{a})$ links which have strength at least 1 under $\boldsymbol{a}$. Admitting i' creates $d_{i'}(\boldsymbol{a}) \geq d_i(\boldsymbol{a})$ links between c and new clubs. Because h is 0 at 0, strictly increasing, and concave, the benefits from creating a new link are weakly higher than the losses from reducing the weight of a link. Hence replacing i with i' is weakly profitable to c, a contradiction with $\boldsymbol{a}$ being stable. ■

Proof of Proposition 4

First, we consider when $nD \geq mS$. When $h(\cdot)$ is superadditive, for any membership profile $\boldsymbol{a} \in A$, the productivity of a club $\pi_c(\boldsymbol{a})$ satisfies

$$\pi_c(\boldsymbol{a}) \leq f(S) + h(S(D-1)) = f(S) + h(S),$$

where equality is obtained only when the club has one strength-S link with another club. For every club to reach this highest level of productivity, the club network consists of m/D separate 2-cliques where all links are of strength S. We now show such a structure exists by construction: Allocate the first S individuals to clubs c_1 and c_2, the next S individuals to clubs c_3 and c_4,..., and the $\frac{m}{2}^{th}$ group of S individuals (individuals $i_{mS/2-S+1}$ to $i_{mS/2}$) to clubs c_{m-1} and c_m. Since all clubs have reached the highest productivity with the membership profile when $h(\cdot)$ is superadditive, it is also stable when $h(\cdot)$ is superadditive. To show the profile is not stable when $h(\cdot)$ is subadditive, consider a deviation by club c_1 and individual i_{S+1} where c_1 expels i_1 to admit i_{S+1} and i_{S+1} leaves c_3 to join c_1. It is straightforward to verify that the deviation benefits both c_1 and i_{S+1}.

When $h(\cdot)$ is subadditive, for any membership profile $\boldsymbol{a} \in A$, the productivity of a club $\pi_c(\boldsymbol{a})$ satisfies

$$\pi_c(\boldsymbol{a}) \leq f(S) + S(D-1)h(1) = f(S) + S \cdot h(1),$$

where equality is obtained only when the club has S strength-1 links with other clubs. For every club to reach this highest level of productivity, the club network is an S-regular network where all links are of strength 1. We now show such a structure exists by construction with the following algorithm: At each step, pick the club with the maximum number of empty slots, fill the slots with different individuals, and then allocate each of those individuals to a different club with the maximum number of empty slots. Stop when all clubs are full. Since all clubs have reached the highest productivity with the membership profile when $h(\cdot)$ is subadditive, it is also stable when $h(\cdot)$ is subadditive. Now we show the profile is not stable when $h(\cdot)$ is superadditive. In this profile, for each club c, it must have at least 2 strength-1 links with c' and c''. Let i_1 be the common member of c and c' and i'' be the common member of c and c''. There must also exist an individual, call him i_3, who is in c' and $c''' \neq c$. Consider the deviation by club c and individual i_3 where c expels i_2 to admit i_3 and i_3 leaves c''' to join c. This deviation benefits both c and i_3.

Turning to when $nD < mS$, let π^* be the highest productivity a club can obtain, note that for any membership profile $\boldsymbol{a} \in A$, the utility of an individual $u_i(\boldsymbol{a})$ satisfies

$$u_i(\boldsymbol{a}) \leq v(D \cdot \pi^*),$$

where the equality is obtained only when all clubs i joins have productivity π^*. For any individual to reach this highest level of utility, the subnetwork of clubs that contains all nonempty clubs must consist of n/S separate 2-cliques where all links are of strength S when $h(\cdot)$ is superadditive and be an S-regular network (a complete network when $S = 2n/S - 1$) where all links are of strength 1 when $h(\cdot)$ is subadditive. Such subnetworks can be constructed in the same way we construct we construct the clubs-efficient networks when $nD \geq mS$.

For the statements on stability, since all individuals have reached the highest level of utility, they have no incentives to deviate. We consider the same deviations examined for the case when $nD \geq mS$ to show that the utilitarian optimum club network under a superadditive (subadditive) $h(\cdot)$ is unstable when $h(\cdot)$ is subadditive (superadditive).

■

Proof of Proposition 5

For the egalitarian outcome result, when $\alpha = 0$, we are back to the benchmark model, and the proof then follows. To prove the rest of Proposition 5, we first prove the following Lemma.

Lemma 2. *Suppose the productivity of clubs follows* (9)*. When $\alpha > 0$, for a stable profile $\boldsymbol{a}$, if there exist two individuals i and i' in $I_2(\boldsymbol{a})$, then $d_i(\boldsymbol{a}) \geq d_{i'}(\boldsymbol{a})$ implies that the set of clubs i members is a superset of clubs i' members.*

Proof. First, note that the productivity of a club c under a membership profile $\boldsymbol{a}$ is

$$\pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \alpha \sum_{i \in I} a_{ic} \sum_{c' \neq c} a_{ic'} q(p_{c'}(\boldsymbol{a})). \tag{40}$$

We prove Lemma 2 by contradiction. Suppose there exist a stable profile $\boldsymbol{a}$, two individuals i and i' in $I_2(\boldsymbol{a})$ where $d_i(\boldsymbol{a}) \geq d_{i'}(\boldsymbol{a})$, and two clubs c and c' where $a_{ic} = a_{i'c'} = 1$ and $a_{ic'} = a_{i'c} = 0$. Since $\boldsymbol{a}$ is stable, $a_{ic} = 1$ and $a_{i'c} = 0$, club c does not find it profitable to replace i with i', which means

$$\sum_{c'' \neq c, c'} a_{i'c''} q(p_{c''}(\boldsymbol{a})) < \sum_{c'' \neq c, c'} a_{ic''} q(p_{c''}(\boldsymbol{a})). \tag{41}$$

Similarly, since $a_{i'c'} = 1$ and $a_{ic'} = 0$,

$$\sum_{c'' \neq c, c'} a_{ic''} q(p_{c''}(\boldsymbol{a})) < \sum_{c'' \neq c, c'} a_{i'c''} q(p_{c''}(\boldsymbol{a})), \tag{42}$$

a contradiction. □

We can prove the bound for $|I_2(\boldsymbol{a})|$ following the same steps as in the proof of Proposition 1. For cardinality of $C_2(\boldsymbol{a})$, suppose that $C_2(\boldsymbol{a}) \neq \varnothing$, we will show that there exists an individual i that is a member of all clubs in $C_2(\boldsymbol{a})$. We consider the cases of $I_2(\boldsymbol{a}) \neq \varnothing$ and $I_2(\boldsymbol{a}) = \varnothing$ separately. If $I_2(\boldsymbol{a}) \neq \varnothing$, the proof follows the same steps as in the proof of Proposition 1.

If $I_2(\boldsymbol{a}) = \varnothing$, then for any club in $C_2(\boldsymbol{a})$, its members are of degree D and $s_c(\boldsymbol{a}) < S$. We prove by contradiction that for two clubs $c, c' \in C_2(\boldsymbol{a})$, $s_c(\boldsymbol{a}) \geq s_{c'}(\boldsymbol{a})$ implies that the set of members of c is a superset of the set of members of c'. Suppose there exist a stable profile $\boldsymbol{a}$, two clubs $c, c' \in C_2(\boldsymbol{a})$ where $s_c(\boldsymbol{a}) \geq s_{c'}(\boldsymbol{a})$ and two individuals i and i' where $a_{ic} = a_{i'c'} = 1$ and $a_{ic'} = a_{i'c} = 0$. Since $\boldsymbol{a}$ is stable, $a_{ic} = 1$ and $a_{i'c} = 0$, i does not find it profitable to quit c and join c', which means

$$\pi_c(\boldsymbol{a}) + \alpha(D-1)q(p_c(\boldsymbol{a})) > \pi_{c'}(\boldsymbol{a}) + \alpha(D-1)q(p_{c'}(\boldsymbol{a})). \tag{43}$$

Similarly, since $a_{i'c'} = 1$ and $a_{ic'} = 0$,

$$\pi_c(\boldsymbol{a}) + \alpha(D-1)q(p_c(\boldsymbol{a})) < \pi_{c'}(\boldsymbol{a}) + \alpha(D-1)q(p_{c'}(\boldsymbol{a})), \tag{44}$$

a contradiction. Hence, take any $c \in C_2(\boldsymbol{a})$ with minimal size and any member i of c, i must be in all clubs of $C_2(\boldsymbol{a})$. Since $d_i(\boldsymbol{a}) \leq D$, $|C_2(\boldsymbol{a})| \leq D$.

The bounds on $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ and $\mathcal{M}_{\mathcal{C}}(\boldsymbol{a})$ are derived following the same steps as in the proof for Proposition 1.

■

Proof of Proposition 6

Before we prove Proposition 6, we introduce the following notions. Suppose there are k different levels of degrees for individuals in $I_2(\boldsymbol{a})$. Let $d_1, ..., d_k$, where $d_1 > ... > d_k$ be the levels of the degrees and let $I_2^l(\boldsymbol{a}) = \{i \in I_2(\boldsymbol{a}) : d_i(\boldsymbol{a}) = d_l\}$ for $l = 1, ..., k$. Moreover, let $C_2^l(\boldsymbol{a}) = \{c \in C_2(\boldsymbol{a}) : \pi_c(\boldsymbol{a})$ has the l^{th} highest level of productivity in $C_2(\boldsymbol{a})\}$.

Consider a stable membership profile $\hat{\boldsymbol{a}}$. Since $nD > mS + SD$, $s_c(\hat{\boldsymbol{a}}) = S$ for all $c \in C$. This is because when $nD > mS + SD$, $|I_3(\hat{\boldsymbol{a}})| > 0$, which means there exists an individual $i \in I$ with $d_i(\hat{\boldsymbol{a}}) = 0$ by Proposition 1. If there exists a club c that is not full, there is a blocking pair (i, c) where i joins c. A contradiction. This observation indicates that $C_3(\hat{\boldsymbol{a}}) = \emptyset$ and a club in $C_2(\hat{\boldsymbol{a}})$ must admit an individual with degree less than D. Additionally, we know from the proof of Proposition 1 that for any club $c \in C_2^l(\hat{\boldsymbol{a}})$, the set of members of c includes all individuals in $I_2^1(\hat{\boldsymbol{a}}) \cup ... \cup I_2^l(\hat{\boldsymbol{a}})$. If c is not full with individuals in $I_2(\hat{\boldsymbol{a}})$, then the rest of its members are elements of $I_1(\hat{\boldsymbol{a}})$.

Next consider a membership profile $\boldsymbol{a}$ where individuals in $I_2(\hat{\boldsymbol{a}})$ and $I_3(\hat{\boldsymbol{a}})$ join the same clubs as they do in membership profile $\hat{\boldsymbol{a}}$, and the memberships of individuals in $I_1(\hat{\boldsymbol{a}})$ are organized to satisfy the following condition: For an individual $i \in I_1(\boldsymbol{a})$ who is in $C_2^l(\boldsymbol{a})$, i is in all clubs in $C_2^1(\boldsymbol{a}) \cup ... \cup C_2^l(\boldsymbol{a})$. Note that $|I_k(\boldsymbol{a})| = |I_k(\hat{\boldsymbol{a}})|$ for $k = 1, 2, 3$. We prove Proposition 6 by showing that $\boldsymbol{a}$ is strongly stable.

Suppose, to the contrary, that $\boldsymbol{a}$ is not strongly stable, that is, there exists $I' \subseteq I$ and $C' \subseteq C$ such that there is $\boldsymbol{a'} \in A$ with $a'_{ic} \leq a_{ic}$ for all $(i, c) \in I \times C$, where $i \in I', c \notin C'$ or $i \notin I', c \in C$ and $a'_{ic} = a_{ic}$ for all $i \notin I', c \notin C'$, and $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$ for all $i \in I'$ and $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$ for all $c \in C'$. Since $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$ for all $c \in C'$, $C' \subseteq C_2(\boldsymbol{a})$. Define $\hat{I}_1(\boldsymbol{a}) \subset I_1(\boldsymbol{a})$ as the subset of individuals in $I_1(\boldsymbol{a})$ who join clubs in $C_2(\boldsymbol{a})$. Since $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$ for all $i \in I'$, $I' \subset \hat{I}_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})$. We show, by induction, that $C_2^l(\boldsymbol{a}) \cap C' = \emptyset$ for $l = 1, 2, \ldots, k$, hence completing the proof.

First, we show that if $c \in C_2^1(\boldsymbol{a})$ then $c \notin C'$. This is because if $c \in C_2^1(\boldsymbol{a})$, then c has all individuals in $\hat{I}_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})$. Those in $\hat{I}_1(\boldsymbol{a})$ already have the highest degree possible and those in $I_2^1(\boldsymbol{a})$ cannot attain a higher degree with a deviation that only involves clubs in $C_2(\boldsymbol{a})$ as those in $I_2^1(\boldsymbol{a})$ are members of all clubs in $C_2(\boldsymbol{a})$. This means that there is no way for c to improve its productivity by deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$.

Now, we show that if $(C_2^1(\boldsymbol{a}) \cup \ldots \cup C_2^l(\boldsymbol{a})) \cap C' = \emptyset$, then $C_2^{l+1} \cap C' = \emptyset$. With a deviation that only invokes clubs in $C_2^{l+1}(\boldsymbol{a}) \cup \ldots \cup C_2^k(\boldsymbol{a})$, individuals in $I_2^1(\boldsymbol{a}) \cup \ldots \cup I_2^{l+1}(\boldsymbol{a})$ cannot raise their degree, since they are already in all clubs in $C_2^{l+1}(\boldsymbol{a}) \cup \ldots \cup C_2^k(\boldsymbol{a})$, and individuals in $I_2^{l+2}(\boldsymbol{a}) \cup \ldots \cup I_2^k(\boldsymbol{a})$ cannot raise their degree to a level greater than d_{l+1}. This means that a club in $C_2^{l+1}(\boldsymbol{a}) \cup \ldots \cup C_2^k(\boldsymbol{a})$ must have raised its productivity by deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$ by replacing a member in $I_2(\boldsymbol{a})$ with a member in $\hat{I}_1(\boldsymbol{a})$. Let x be the number of individuals in $\hat{I}_1(\boldsymbol{a})$ that deviates and join c, there must exists an $y \in \{2, 3, \ldots, l+2\}$ such that

$$|I_2^y(\boldsymbol{a}) \cup I_2^{y+1}(\boldsymbol{a}) \cup \ldots \cup I_2^{l+1}(\boldsymbol{a})| < x \leq |I_2^{y-1}(\boldsymbol{a}) \cup I_2^y(\boldsymbol{a}) \cup \ldots \cup I_2^{l+1}(\boldsymbol{a}),$$

where we set $|I_2^y(\boldsymbol{a}) \cup I_2^{y+1}(\boldsymbol{a}) \cup \ldots \cup I_2^{l+1}(\boldsymbol{a})| = 0$ when $y = l + 2$.

Since $x > |I_2^y(\boldsymbol{a}) \cup I_2^{y+1}(\boldsymbol{a}) \cup \ldots \cup I_2^{l+1}(\boldsymbol{a})|$, one of the individual from $\hat{I}_1(\boldsymbol{a})$, say i, who joins c can only have memberships in clubs that have productivity not less than those in $C_2^{y-1}(\boldsymbol{a})$. Since $x \leq |I_2^{y-1}(\boldsymbol{a}) \cup I_2^y(\boldsymbol{a}) \cup \ldots \cup I_2^{l+1}(\boldsymbol{a})|$, the productivity of c does not exceed the productivity of a club in $C_2^{y-1}(\boldsymbol{a})$ under $\boldsymbol{a'}$, contradicting the requirement that i improves her utility by deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$. ■

Proof of Proposition 7

We start by deriving the upper bound on $|I_2(\boldsymbol{a})|$. Suppose there exist individuals with degree $0 < d < D$, let i be an individual with the lowest type whose degree is d. It can be shown that for any club c that i joins, c hosts all individuals with degree d. Suppose not so that there exists i' with $d_{i'}(\boldsymbol{a}) = d_i(\boldsymbol{a})$, $\theta_{i'} > \theta_i$ and $a_{i'c} = 0$. Then c would benefit by replacing i with i' and i' would want to join c. A blocking pair is formed. Since $s_c(\boldsymbol{a}) \leq S$, so $|\{i \in I : 0 < d_i(\boldsymbol{a}) < D\}| \leq S$. Given that there are at most $D-1$ degrees greater than 0 and less than D, $|I_2(\boldsymbol{a})| \leq (D-1)S$. The bound on $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ is then derived following the same steps as in the proof for Proposition 1.

Turning to the relationship between an individual's type and memberships. When $\alpha = 0$, suppose there exists individual i and j where $d_i(\boldsymbol{a}) < d_j(\boldsymbol{a})$ and $\theta_i > \theta_j$. Then there exists club c that admits j but not i. There is a jointly profitable deviation for i and c where c replaces j with i and i quits no club to join c, a contradiction.

When $\alpha > 0$ and $\theta_i - \theta_j < \alpha$, $\forall i \neq j$, consider a membership profile constructed in the following way. W.L.O.G., let $\theta_1 < \theta_2 < ... < \theta_n$. Assign individuals i_1 to i_S to clubs c_1 to c_D, i_{S+1} to i_{2S} to clubs c_{D+1} to c_{2D},..., and $i_{\lfloor m/D \rfloor (S-1)+1}$ to $i_{\lfloor m/D \rfloor S}$ to clubs $c_{\lfloor m/D \rfloor (D-1)+1}$ to $c_{\lfloor m/D \rfloor D}$. If $\lfloor m/D \rfloor = m/D$, the construction is completed. If $\lfloor m/D \rfloor \neq m/D$ and $n - \lfloor m/D \rfloor S \geq S$, let the rest $m \bmod D$ clubs admit individuals $i_{\lfloor m/D \rfloor S+1}$ to $i_{(\lfloor m/D \rfloor +1)S}$. If $\lfloor m/D \rfloor \neq m/D$ and $n - \lfloor m/D \rfloor S < S$, let the rest $m \bmod D$ clubs admit individuals admit individuals $i_{\lfloor m/D \rfloor S+1}$ to i_n. In this constructed profile, $d_i(\boldsymbol{a}) \geq d_j(\boldsymbol{a})$, $\forall i, j \in I$, where $\theta_i < \theta_j$. It is easy to verify that this profile is stable when $\theta_i - \theta_j < \alpha$, $\forall i \neq j$, and $m \bmod D \neq D-1$. ■

Proof of Proposition 8

First, we show that for a stable profile $\boldsymbol{a}$, if there exist two individuals i and i' where $D > d_i(\boldsymbol{a}) \geq d_{i'}(\boldsymbol{a}) > 0$, then the set of clubs i members is a superset of the clubs that i' members. Suppose, to the contrary, that there exist individuals $i \in I$ and $i' \in I$ such that $D > d_i(\boldsymbol{a}) \geq d_{i'}(\boldsymbol{a})$ and a club $c \in C$ such that i' is a member of and i is not. Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i joining c and c accepting i and dropping i'. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = \alpha(d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a}) + 1), \tag{45}$$

which is positive when $\alpha > 0$. Also, since v is increasing, both individual i and club c and strictly benefit deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$ when $\alpha > 0$. A contradiction with stability of $\boldsymbol{a}$.

For the cardinality of $I_3(\boldsymbol{a})$, take any individual $i \in I_3(\boldsymbol{a})$ with minimal $d_i(\boldsymbol{a})$ and let $c \in C$ be any club that i members. By the above finding, all individuals in $I_3(\boldsymbol{a})$ are members of c. Since $s_c(\boldsymbol{a}) \leq \overline{S}$, $|I_3(\boldsymbol{a})| \leq \overline{S}$. ■